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Abstract. In this paper we show that there exists a family of supersonic neutral modes for a compressible mixing layer in an unbounded domain. These modes have zero wavenumber and frequency with non-zero phase speed. They are analogous to the supersonic neutral modes of the compressible vortex sheet found by Miles. The results presented here give a more complete picture of the spectrum of the disturbances in this flow.

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1. Introduction. In recent studies (Jackson and Grosch, 1989, 1990), we considered the inviscid spatial stability problem for the compressible mixing layer in an unbounded domain. The key to understanding the stability characteristics of this flow is the understanding of the different parameter regions for which various types of instability modes can exist. In Figure 1 we show a plot of the sonic phase speeds $c_\pm$ versus the Mach number, where

$$c_+ = 1 - \frac{1}{M}, \qquad c_- = \beta_U + \frac{\sqrt{\beta_T}}{M}.$$

Here, M is the Mach number of the fast stream, β_U the ratio of the speed of the slow stream to that of the fast stream, and β_T the ratio of the temperature of the slow stream to that of the fast stream. These curves divide the phase speed-Mach number plane into four regions. If a disturbance exists with a Mach number and phase speed in region 1, the disturbance is subsonic at both boundaries, and we classify it as a subsonic mode. In region 3, the disturbance is supersonic at both boundaries, and we classify it as supersonic-supersonic mode. In region 2, the disturbance is subsonic in the fast stream and supersonic in the slow stream, and we classify it as a fast mode. Finally, in region 4, the disturbance is supersonic in the fast stream and subsonic in the slow stream, and we classify it as a slow mode. We found that there is only a single subsonic neutral mode for two dimensional waves. Beyond a critical Mach number M_s, the Mach number at which the phase speed equals that of a sonic wave, the subsonic neutral modes are transformed into supersonic neutral modes which are subsonic in one freestream and supersonic in the other. In addition, another supersonic neutral mode appears at $M_* \geq M_s$, the Mach number at which the sonic speeds of the fast and slow streams are equal. This supersonic neutral mode has the opposite behavior than the previous. That is, if the continuation of the subsonic neutral mode is supersonic in the fast stream and subsonic in the slow stream, this new mode is subsonic in the fast stream and supersonic in the slow stream.

In these studies we did not find any neutral or unstable modes in region 3 which were supersonic in both streams. In this paper we now show that there exists a family of neutral modes in region 3 having zero wavenumber and frequency with non-zero phase speed[1]. The approach taken here follows that of Papageorgiou and Smith (1989) and Papageorgiou (1990), who considered long waves in the wake behind a flat plate. We show that these modes are analogous to the supersonic neutral modes of the compressible vortex sheet found by Miles (1958). The results presented here give a more complete picture of the spectrum of the disturbances in a compressible mixing layer.

2. Formulation and Results. We consider here a compressible mixing layer in an unbounded domain with mean velocity $U(\eta)$ and temperature $T(\eta)$, where η is the similarity variable in terms of the downstream distance and the Howarth-Dorodnitzyn variable. With suitable normalization, U varies between 1 in the fast stream ($\eta = \infty$), and $\beta_U < 1$ in

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This possibility was suggested to us by R. Klein.

the slow stream ($\eta = -\infty$). The temperature T varies between 1 in the fast stream and $\beta_T > 0$ in the slow stream. The results presented here are independent of the detailed form of U and T.

The mean flow is perturbed by introducing two-dimensional wave disturbances. The disturbance equation for the normal velocity perturbation ϕ is given by (Lees and Lin, 1946)

$$\left[\frac{(U - c)\phi' - U'\phi}{G} \right]' - \alpha^2 (U - c)\phi = 0, \tag{1}$$

where primes denote differentiation with respect to the similarity variable η, and G is given by

$$G = T\,[T - M^2 (U - c)^2]. \tag{2}$$

Here, α is the wavenumber of the disturbance in the downstream direction and c is the corresponding phase speed.

Since we are interested in long wavelength perturbations ($\alpha \to 0$), the appropriate expansions are found to be

$$\phi = \phi_0 + \alpha\phi_1 + \alpha^2\phi_2 + \cdots \tag{3}$$

$$c = c_0 + \alpha c_1 + \alpha^2 c_2 + \cdots \tag{4}$$

$$\frac{1}{G} = \frac{1}{G_0} + \alpha G_1 + \alpha^2 G_2 + \cdots \tag{5}$$

where

$$G_0 = T\,[T - M^2 (U - c_0)^2], \tag{6}$$

$$G_1 = -2c_1 M^2 G_0^{-2} (U - c_0) T. \tag{7}$$

Let us now define the linear operator L:

$$L(\Psi) = (U - c_0)\Psi' - U'\Psi \equiv (U - c_0)^2 \left[\frac{\Psi}{U - c_0} \right]'. \tag{8}$$

Substituting the above expansions into (1) gives the following three leading-order problems:

$$\frac{d}{d\eta}(G_0^{-1}\,L(\phi_0)) = 0, \tag{9}$$

$$\frac{d}{d\eta}(G_0^{-1}\,[L(\phi_1) - c_1\phi_0']) + \frac{d}{d\eta}(G_1 L(\phi_0)) = 0, \tag{10}$$

$$\frac{d}{d\eta}(G_0^{-1}\,[L(\phi_2) - c_2\phi_0' - c_1\phi_1']) + \frac{d}{d\eta}(G_1\,[L(\phi_1) - c_1\phi_0']) + \frac{d}{d\eta}(G_2 L(\phi_0)) - (U - c_0)\phi_0 = 0. \tag{11}$$

The appropriate solution to (9) that is bounded for large η is given by

$$\phi_0 = A_0(U - c_0), \tag{12}$$

where A_0 is a constant. Based on this solution the following three remarks can be made:

[1] Since solutions in region 2 of Figure 1 are subsonic in the fast stream and supersonic in the slow stream, we see that ϕ_0 is a solution if we take the leading-order wave speed to be

$$c_0 = 1. \tag{13}$$

[2] Since solutions in region 4 of Figure 1 are subsonic in the slow stream and supersonic in the fast stream, we see that ϕ_0 is a solution if we take the leading-order wave speed to be

$$c_0 = \beta_U. \tag{14}$$

[3] Since solutions in region 3 of Figure 1 are supersonic in both the fast and slow streams, we see that ϕ_0 is a possible solution for some leading-order wave speed lying in the range

$$c_- < c_0 < c_+, \tag{15}$$

where $c_\pm$ are the phase speeds of the sonic neutral modes. Thus, the actual value of c_0 is not determined at this order, and can only be found by using the higher orders terms in the α expansion.

To determine the leading-order wave speed c_0 for region 3, we consider the expansions at the higher orders. The solutions to (10) and (11) are given by

$$\phi_1 = -c_1 A_0 + A_1(U - c_0)\int \frac{G_0}{(U - c_0)^2}\,d\eta, \tag{16}$$

$$\phi_2 = -c_2 A_0 + c_1 (U - c_0) \int \frac{\phi_1'}{(U - c_0)^2} d\eta$$

$$- A_1 (U - c_0) \int \frac{G_0^2 G_1}{(U - c_0)^2} d\eta + A_2 (U - c_0) \int \frac{G_0}{(U - c_0)^2} d\eta$$

$$+ A_0 (U - c_0) \int \frac{G_0}{(U - c_0)^2} \left[\int (U - c_0)^2 ds \right] d\eta. \tag{17}$$

We note here that both ϕ_1 and ϕ_2 could contain another term that is proportional to ϕ_0, which can be ignored by a suitable renormalization of A_0.

Inspection suggests that the solution diverges as $\eta \to \pm\infty$. Thus, these expansions will break down when $|\eta| = O(\alpha^{-1})$. There are two outer regions, one in the fast stream and one in the slow stream, which will be considered separately. For the outer region in the fast stream, the appropriate scalings are

$$\eta = \alpha^{-1} z \tag{18}$$

for the independent variable, and

$$\phi_F = \tilde{\phi}_0 + \alpha \tilde{\phi}_1 + \cdots \tag{19}$$

for the dependent variable. Substituting (18) and (19) into our original equation (1) yields, to leading order, the equation

$$\tilde{\phi}_0'' + G_0^+ \tilde{\phi}_0 = 0, \qquad G_0^+ = M^2 (1 - c_0)^2 - 1. \tag{20}$$

The solution for outgoing waves is found to be

$$\tilde{\phi}_0 = B_0 e^{-i \sqrt{G_0^+}\, z}. \tag{21}$$

The next order equation is given by

$$\tilde{\phi}_1'' + G_0^+ \tilde{\phi}_1 = g_1^+ \tilde{\phi}_0, \qquad g_1^+ = 2 c_1 M^2 (1 - c_0), \tag{22}$$

whose solution for outgoing waves is

$$\tilde{\phi}_1 = B_1 e^{-i \sqrt{G_0^+}\, z} + i \frac{g_1^+ B_0}{2\sqrt{G_0^+}} z\, e^{-i \sqrt{G_0^+}\, z}. \tag{23}$$

Expanding the solution as $z \to 0$, and matching with the inner solution (3) as $\eta \to \infty$, yields the following relationships

$$B_0 = A_0(1 - c_0), \tag{24}$$

$$B_1 = -c_1 A_0, \tag{25}$$

$$A_1 = i\, A_0(1 - c_0)^2 (G_0^+)^{-1/2}, \tag{26}$$

$$A_2 = i\, \frac{1 - c_0}{(G_0^+)^{3/2}} c_1 (2 - M^2 (1 - c_0)^2) A_0. \tag{27}$$

For the outer region in the slow stream, the appropriate scalings are

$$\phi_S = \hat{\phi}_0 + \alpha \hat{\phi}_1 + \cdots, \tag{28}$$

and z is defined as before. Substituting into our original equation (1) yields, to leading order, the equation

$$\hat{\phi}_0'' + G_0^- \hat{\phi}_0 = 0, \qquad G_0^- = \beta_T [M^2 (\beta_U - c_0)^2 - \beta_T]. \tag{29}$$

The solution for outgoing waves is found to be

$$\hat{\phi}_0 = D_0 e^{-i \sqrt{G_0^-}\, z}. \tag{30}$$

The next order equation is given by

$$\hat{\phi}_1'' + G_0^- \hat{\phi}_1 = g_1^- \hat{\phi}_0, \qquad g_1^- = 2 c_1 M^2 \beta_T (\beta_U - c_0), \tag{31}$$

whose solution for outgoing waves is

$$\hat{\phi}_1 = D_1 e^{-i \sqrt{G_0^-}\, z} + i\, \frac{g_1^- D_0}{2\sqrt{G_0^-}} z\, e^{-i \sqrt{G_0^-}\, z}. \tag{32}$$

Expanding the solution as $z \to 0$, and matching with the inner solution (3) as $\eta \to -\infty$, yields the following relationships

$$D_0 = A_0(\beta_U - c_0), \tag{33}$$

$$D_1 = -c_1 A_0, \tag{34}$$

$$A_1 = i\, A_0(\beta_U - c_0)^2 (G_0^-)^{-1/2}, \tag{35}$$

$$A_2 = i \frac{\beta_U - c_0}{(G_{\bar{0}})^{3/2}} \beta_T c_1 (2\beta_T - M^2 (\beta_U - c_0)^2) A_0. \tag{36}$$

The two conditions (26) and (35) on A_1 yields the following eigenvalue equation for c_0:

$$\beta_T [M^2 (\beta_U - c_0)^2 - \beta_T](1 - c_0)^4 = [M^2 (1 - c_0)^2 - 1](\beta_U - c_0)^4. \tag{37}$$

This equation is identical to (5.3a) of Miles (1958) if we re-express his result in our notation. Miles showed (in our notation), that for region 3:

[1] A single root of (37) exists for

$$M \geq M_* \equiv \frac{1 + \sqrt{\beta_T}}{1 - \beta_U}, \tag{38}$$

with phase speed

$$c_0 = \frac{\beta_U + \sqrt{\beta_T}}{1 + \sqrt{\beta_T}}, \tag{39}$$

which we classify as a constant speed supersonic-supersonic neutral mode. Note that this solution is independent of Mach number, and corresponds to the phase speed at which the two sonic speeds $c_\pm$ are equal.

[2] A double root first appears at

$$M = \frac{(1 + \beta_T^{1/3})^{3/2}}{1 - \beta_U}, \tag{40}$$

with phase speed

$$c_0 = \frac{\beta_U + \beta_T^{1/3}}{1 + \beta_T^{1/3}}. \tag{41}$$

[3] There are three distinct real roots for

$$M > \frac{(1 + \beta_T^{1/3})^{3/2}}{1 - \beta_U}. \tag{42}$$

One of the roots is given by (39), while the other two roots must be found numerically. For the special case of $\beta_T = 1$, these roots are given by

$$c_0 = \frac{1+\beta_U}{2} \pm \frac{1}{2M}\left[M^2(1-\beta_U)^2 + 4 - 4\sqrt{M^2(1-\beta_U)^2+1} \right]^{1/2}. \quad (43)$$

The root which corresponds to the (+) sign we classify as a fast supersonic-supersonic neutral mode, while that which corresponds to the (-) sign we classify as a slow supersonic-supersonic neutral mode.

In Figure 2 we display the real roots of (37) which lie in region 3 for $\beta_U = 0$ and various values of β_T. The roots are the neutral phase speeds. For completeness, we also display the neutral phase speeds which lie in the other regions (Jackson and Grosch, 1989). The classification scheme is given in the figure caption.

The two conditions (27) and (36) on A_2 yield the result that $c_1 = 0$. Thus, we have $A_2 = B_1 = D_1 = 0$, and the final solution in region 3 is now given by

$$\phi = A_0(U-c_0)\left[1 + i\,\alpha \frac{(1-c_0)^2}{\sqrt{G_0^+}} \int \frac{G_0}{(U-c_0)^2}\,d\eta + \right.$$

$$\left. \alpha^2 \int \frac{G_0}{(U-c_0)^2} \left[\int (U-c_0)^2 ds \right] d\eta + \cdots \right] \quad (44)$$

3. Conclusions. We have shown that the compressible mixing layer has a family of supersonic neutral modes in region 3 of Figure 1, which have phase speeds which are identical to those of the compressible vortex sheet (Miles, 1958). For the mixing layer the wavenumbers of these modes are zero, while those of the vortex sheet are arbitrary. One might expect this since the vortex sheet has no natural length scale, while the mixing layer has a non-zero thickness. We note that these results can be extended to three dimensional disturbances by rescaling the Mach number by $\cos\theta$, where θ is the angle of propagation. Finally, Artola and Majda (1987, 1989) showed that the nonlinear interaction of the three supersonic neutral modes of the compressible vortex sheet can lead to instabilities of the flow. We surmise that similar behavior might be exhibited by the interaction of the supersonic-supersonic neutral modes presented in this paper. We are currently investigating this numerically.

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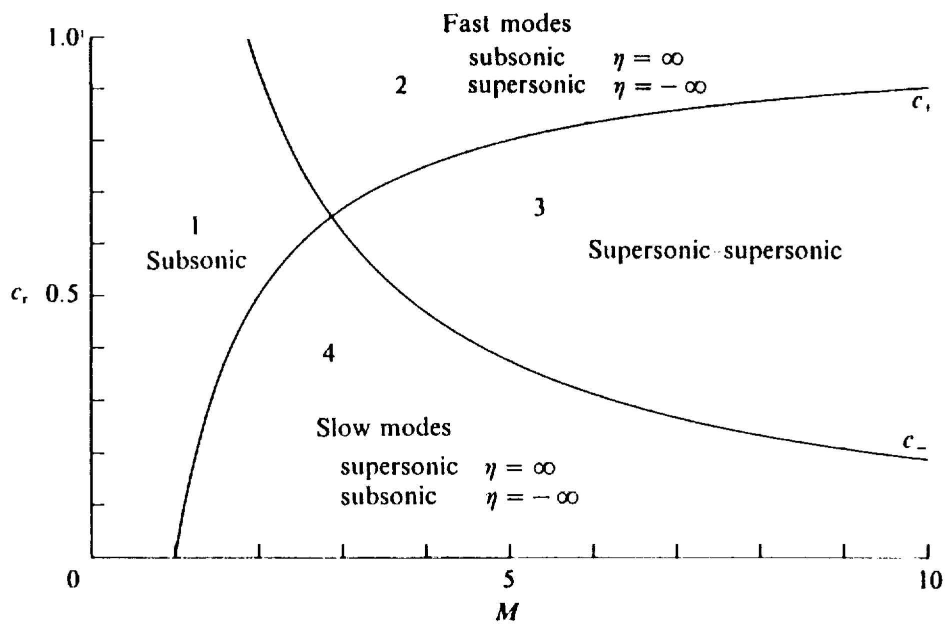

Figure 1. Plots of the sonic speeds $c_\pm$ versus Mach number showing the four regions in which different types of disturbances can exist.

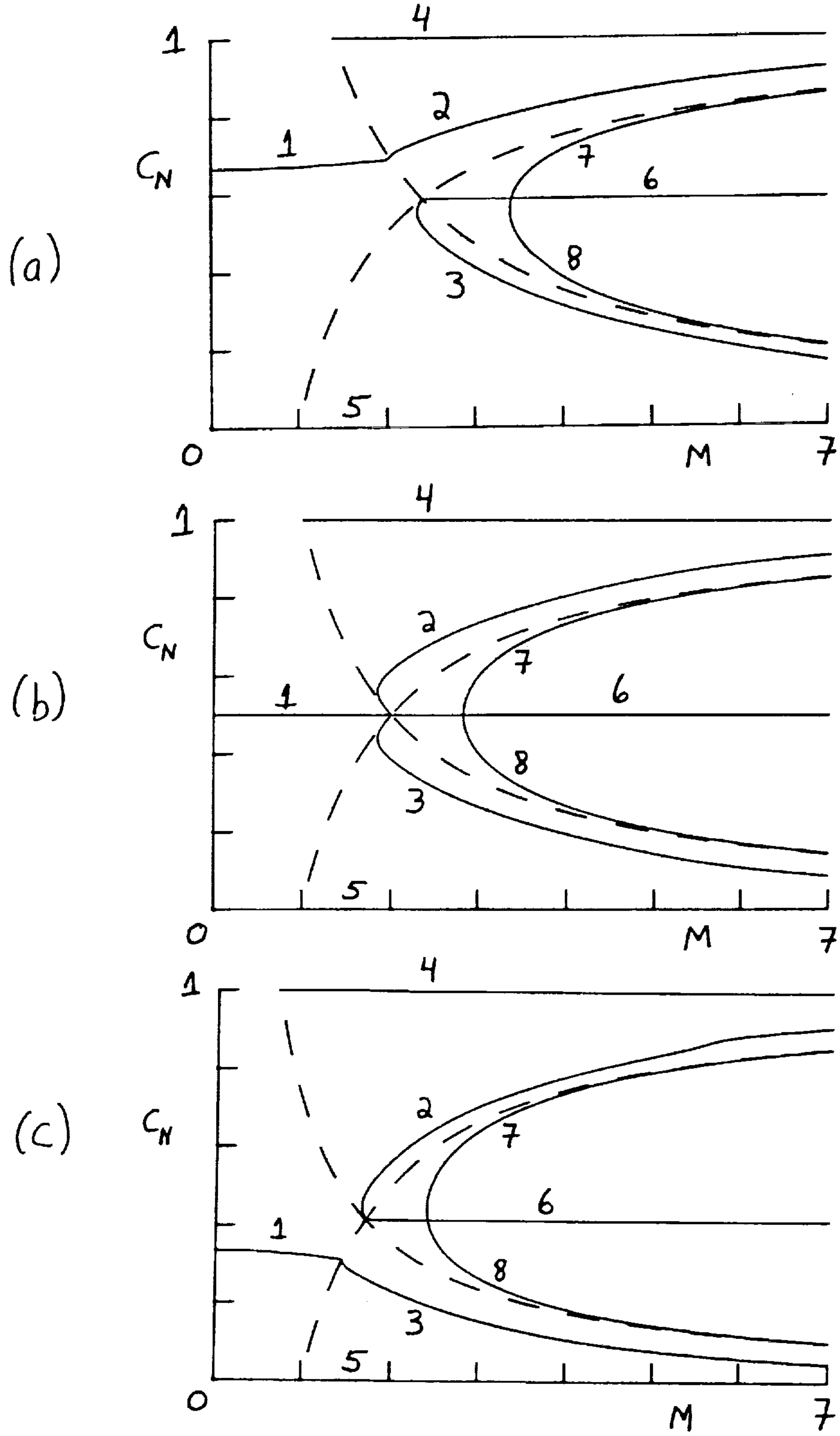

Figure 2. Plot of the neutral phase speeds as a function of Mach number for $\beta_U = 0$ and (a) $\beta_T = 2$, (b) $\beta_T = 1$, and (c) $\beta_T = 0.5$. The neutral mode classification is: (1) subsonic, (2) fast (supersonic), (3) slow (supersonic), (4) $c_N = 1$, (5) $c_N = 0$, (6) constant speed supersonic-supersonic, (7) fast supersonic-supersonic, and (8) slow supersonic-supersonic. The sonic curves are shown as dashed.

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16. Abstract

In this paper we show that there exists a family of supersonic neutral modes for a compressible mixing layer in an unbounded domain. These modes have zero wavenumber and frequency with non-zero phase speed. They are analogous to the supersonic neutral modes of the compressible vortex sheet found by Miles. The results presented here give a more complete picture of the spectrum of the disturbances in this flow.

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