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Los Angeles

Randomly Sampled-Data Control Systems

A dissertation submitted in partial satisfaction of the

requirements for the degree

Doctor of Philosophy in Electrical Engineering

By

Kuoruey Han

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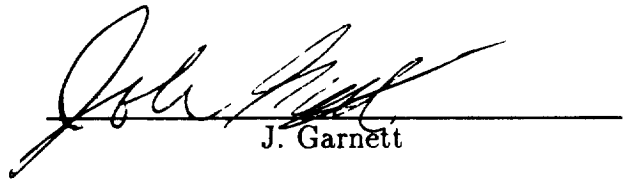
The dissertation of Kuoruey Han is approved.



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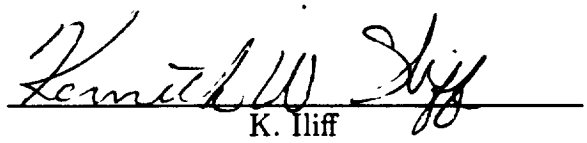

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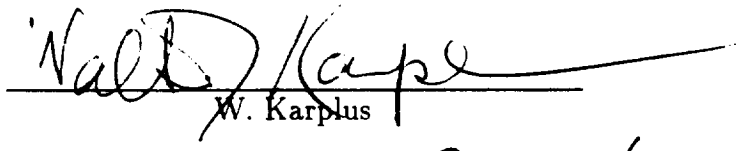

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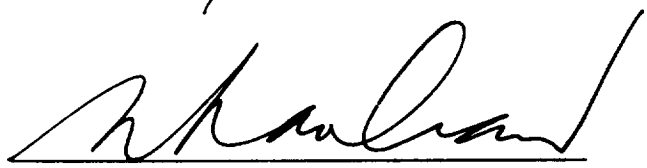

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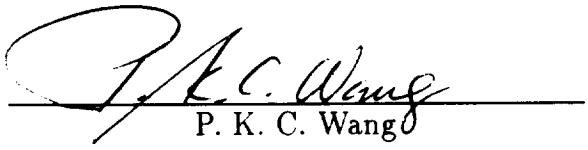

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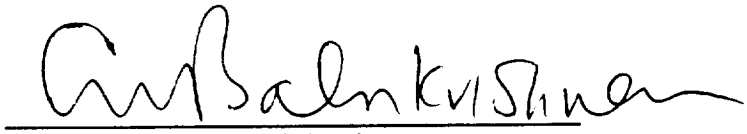

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1990

Dedication

To my parents, Wei-Chou and Yu-Suan, for their constant support and encouragement,

to my wife, Lih-An, for her love and patience,

and to my daughter, Yvonne, for bringing me the happiness.

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ABSTRACT OF THE DISSERTATION

Randomly Sampled-Data Control Systems

by

Kuoruey Han

Professor A. V. Balakrishnan, Chair

The purpose of this dissertation is to solve the LQR problem with random time sampling. Such a sampling scheme may arise from imperfect instrumentation as in the case of sampling jitter. It can also model the stochastic information exchange among decentralized controller to name just a few. The original continuous time control problem can be transformed into an equivalent discrete time stochastic control problem. However,the transformed discrete plant is a system with random parameters. Thus,the original control problem reduces to finding the optimal control for systems with random parameters.

However the control system,while discrete, is nonlinear and hence the optimal control law is difficult to obtain. In particular, the "separation principle" or "certainty equivalence principle" or "variance neutrality condition" does not hold in general. In fact, one finds only approximation techniques yielding suboptimal controls in the literature such as the self tuning control, dual control and OLFO (open loop feedback optimal) etc.

In this dissertation,a practical suboptimal controller is proposed with the nice property of mean square stability. The proposed controller is suboptimal in the sense that the control structure is limited to be linear. Because of i. i. d. assumption,this does not seem unreasonable. Once the control structure is fixed ,the

stochastic discrete optimal control problem is transformed into an equivalent deterministic optimal control problem with dynamics described by the matrix difference equation. The N-horizon control problem is solved using the Lagrange's multiplier method. The infinite horizon control problem is formulated as a classical minimization problem. Assuming existence of solution to the minimization problem ,the total system is shown to be mean square stable under certain observability conditions. Computer simulations are performed to illustrate these conditions.

Chapter 1

Introduction

1.1 Randomly sampled data control system

The advent of computer era has made the use of digital signal processing prevalent in many research fields. In control field,we cannot avoid this trend and therefore the sampling of continuous time signal has become a very important process. The optimal control theory for sampled data linear system has advanced rapidly since some of the early contribution. However, most of the systems considered in the past emphasized the use of uniform sampling scheme so that the information update interval is known deterministically. Of course uniform sampling scheme is inadequate in the diversified control field. For example,we cannot use that scheme to model the sampling jitter in the system. Nor can we use uniform sampling scheme to model the indeterministic information update process present in the distributed control system. In distributed control system,the local controller does not have a complete knowledge of the system description and have to rely on the information exchange among themselves in order to achieve satisfactory operation. Therefore,distributed

control system may have a stochastic information pattern where the data are randomly sampled. The above discussion lead us to consider the optimal control theory of systems with random time sampling.

1.2 Orgonization of dissertation

This dissertation is divided into six chapters. Chapter 1 is the introduction. In chapter 2, I formulate the continuous time control problem with stochasticly sampled data,where an equivalent discrete-time control problem with random parameters is derived. After reviewing some previous work in section 2, the fixed configuration approach is proposed to solve the control problem in section 3. The recursive least mean-square estimator is derived in chapter 3. Where it is shown that the estimator error covariance matrices depend on the particular control sequences. That is why the separation principle does not hold in this case. Certainly the certainty equivalence principle does not hold either. In chapter 4,necessary conditions of optimal control for N-horizon problem is derived. Although the control structure is fixed,the control is of closed loop type,i. e. ,it has the properties of dual control proposed by Feldbaum [15]. Next the formulation of ∞-horizon control problem is proposed in order to obtain a linear time invariant system. It is shown that under some proper conditions the total system is mean-square stable assuming existence of optimal control. The infinite horizon optimal control problem for general one dimensional system is solved in chapter 5. An analytic expression for performance index as a function of gain is derived. The existence of optimal control and mean square stability of the total system are established for one dimensional system. The simulation results is presented ,in which the performance of the optimally controlled

system is compared to the system with the so-called certainty equivalent control. All in all the optimal control system is shown to perform better in the presence of sampling uncertainties. The summaries and conclusions of the dissertation are given in chapter 6.

Chapter 2

Formulation of control problem

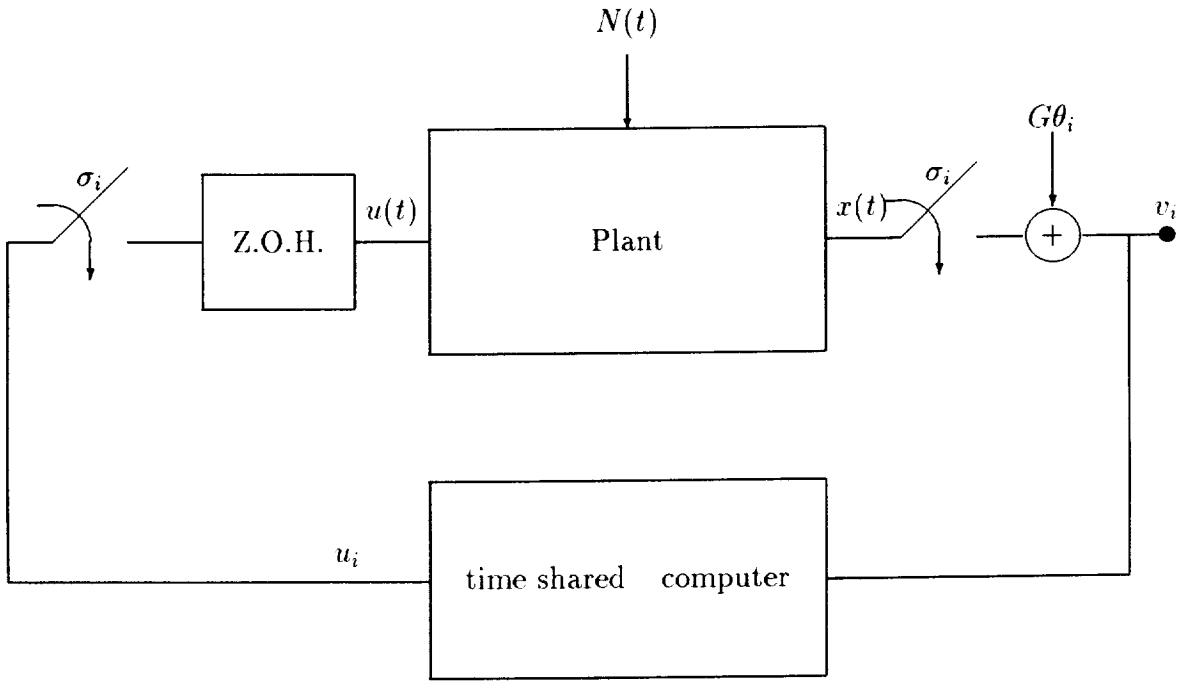
2.1 Stochastically sampled-data control system

Consider the control problem shown in fig 1. The plant,which is a linear time-invariant system,is controlled by a computer. A digital sensor is used to collect data $\{v_i\}$ for controlling the plant. The computer analyzes the data $\{v_i\}$ and generates control signal $\{u_i\}$ which is then fed back through zero order hold device to the controlled plant. Here the sampling process $\{\sigma_i\}$ is modeled by a stochastic process. This kind of stochastic sampling phenomenon is due to the imperfection in the sampling instrumentation. The stochastic sampling may even be used to model the time-sharing behavior among the centralized computer control system. The control signal is designed so that a quadratic cost functional is minimized.

To be more specific, consider the following formulation of control problem. Define:

$$v_i = v(\sigma_i)$$

$$v^i = (v_1, \cdots, v_i)$$



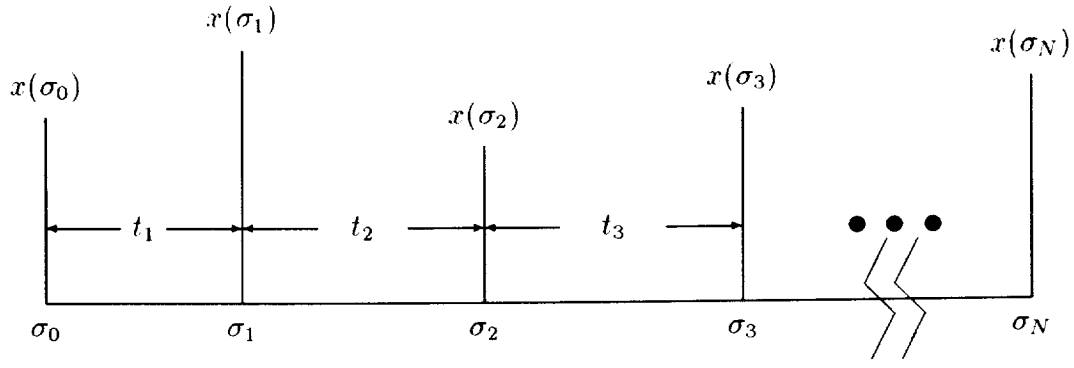


Figure 1: Randomly sampled data control systems.

$$t_{i+1} = \sigma_{i+1} - \sigma_i$$

plant:

$$\dot{x}(t) = Ax(t) + Bu(t) + \tilde{F}N(t) \tag{1}$$

$$v(\sigma_i) = Cx(\sigma_i) + G\theta_i \tag{2}$$

admissible control

$$\{u(\cdot)|u(t) = u_i \text{ for } t \in [\sigma_i, \sigma_{i+1}) \text{ and } u_i \text{ is measurable } v^i\}$$

performance index

$$\dot{J} = E\int_{\sigma_0}^{\sigma_N}\{[x(t), Qx(t)] + [u(t), Ru(t)]\}dt \tag{3}$$

The above formulation is a stochastic continuous-time control problem. In the next section,the above control problem will be transformed into an equivalent discrete-time optimal control problem. In the sequel,we will call the above control problem SCCP. The controller has access of the measured data at discrete time σ_k. Furthermore,the control $u(t)$ is assumed to be constant within the sample interval (σ_i, σ_{i+1}) and measurable v^i,i. e. it is *nonanticipative.* The state x is a $n \times 1$ vector and the input u is a $m \times 1$ vector. In general,the state noise $N(t)$ is assumed to be a white Gaussian stochastic process with zero mean and unit spectral density. The inter-sample process $\{t_k\}$ is assumed to be independent of the white noise process $\{N(t)\}$. Of particular interest is when t_k is either a Macov process (especially a Marcov chain) or a i. i. d. process (independent identically distributed) which will be discussed in latter chapters. The control objective is represented by a quadratic cost functional. Notice that N in the cost functional 3 is fixed. The SCCP control problem may be stated as follows:

SCCP: Given system 1 and 2 with constrained control find a control sequence $u_i, i = 0, \cdots, N-1$ such that the cost functional 3 is minimal where u_i is only a function of observations v^i and past controls u^{i-1}.

2.2 Equivalent discrete-time control problem

It is easy to show the problem formulated above can be transformed into an equivalent discrete-time stochastic control problem (see [12]). Let's define

$$x_i = x(\sigma_i)$$

$$v_i = v(\sigma_i)$$

We then obtain the following equivalent control problem:

Discrete plant

$$x_{i+1} = \Phi(t_{i+1})x_i + \Gamma(t_{i+1})u_i + \xi_i \tag{4}$$

$$v_i = Cx_i + G\theta_i \tag{5}$$

Performance index

$$\begin{aligned} J = \sum_{k=0}^{N-1} E\{&[\Psi(t_{k+1})x_k, x_k] + [R(t_{k+1})u_k, u_k] \\ &+[W(t_{k+1})u_k, x_k] + [W(t_{k+1})^* x_k, u_k]\} \end{aligned} \tag{6}$$

Admissible control u_i is measurable v_i.

where

$$\xi_i = \int_0^{t_{i+1}} e^{At}\hat{F}N(\sigma_{i+1} - t)\, dt \tag{7}$$

$$\Phi(t) = e^{At} \tag{8}$$

$$\Gamma(t) = \int_0^t e^{A\delta} B d\delta \tag{9}$$

$$\Psi(t_{k+1}) = \int_0^{t_{k+1}} \Phi(t)^* Q \Phi(t) dt \tag{10}$$

$$R(t_{k+1}) = \int_0^{t_{k+1}} (\Gamma(t)^* Q \Gamma(t) + R) dt \tag{11}$$

$$W(t_{k+1}) = \int_0^{t_{k+1}} \Phi(t)^* Q \Gamma(t) dt \tag{12}$$

The formulation above is a discrete time stochastic control problem with unconstrained control. The plant is described by a random parameter difference equation. It is well-known that system (4 , 5) and criterion 6 can be transtromed to another equivalent problem without the cross term $W(t_{k+1})$. In the sequel, we will assume $W(t_{k+1}) = 0$ without loss of generality. Furthermore, it is usually assumed that the future sample interval t_{k+1} is independent of past state and control $\{x_k, u_k\}$. we have the following result:

$$\begin{aligned} E[\Psi(t_{k+1})x_k, x_k] &= E[E[\Psi(t_{k+1})]x_k, x_k] \\ = E[\overline{\Psi(t_{k+1})}x_k, x_k] &= E[Q_k x_k, x_k] \end{aligned} \tag{13}$$

Similarly,we have

$$\begin{aligned} E[R(t_{k+1})u_k, u_k] &= E[\overline{R(t_{k+1})}u_k, u_k] \\ = E[H_k u_k, u_k] \end{aligned} \tag{14}$$

where $\overline{\Psi(t_{k+1})} = Q_k$ and $\overline{R(t_{k+1})} = H_k$. Note,in the case $\{t_{k+1}\}$ process is i. i. d. (independent identically distributed), $Q_k = Q$ and $H_k = H$. In the sequel,we will also assume $\{N(t)\}$,$\{\theta_i\}$ and $\{t_i\}$ are mutually independent process. If $N(t)$ is a Gaussian process with mean zero ,then so is ξ_i when conditioned on $\{t_i\}$ process.

Actually, we have the following result:

$$E[\xi_i \xi_i^* | t_{i+1}] = \int_0^{t_{i+1}} e^{At} \tilde{F} \tilde{F}^* e^{A^* t} dt \tag{15}$$

Let's define

$$F(t_{i+1}) = [\int_0^{t_{i+1}} e^{At} \tilde{F} \tilde{F}^* e^{A^* t}]^{1/2} \tag{16}$$

Threfore we can define

$$\xi_i = F(t_{i+1}) \eta_i \tag{17}$$

where $\{ \eta_i \}$ is a Gaussian Process with mean zero and independent of $\{ t_i \}$ process.

From the above consideration, let's summarize the formulation of control problem as follows. In general,we are actually facing a control problem with random parameters both in dynamical system and peformance index.

Plant

$$x_{i+1} = \Phi(t_{i+1}) x_i + \Gamma(t_{i+1}) u_i + F(t_{i+1}) \eta_i \tag{18}$$

$$v_i = C x_i + G \theta_i \tag{19}$$

where $\{ \eta_i \}$, $\{ t_i \}$ and $\{ \theta_i \}$ are mutually independent random processes. $\{ \theta_i \}$ and $\{ \eta_i \}$ are white Gaussian random processes with unity spectral desity. $\{ t_i \}$ is independent identically distributed (i. i. d.).

Performance index

$$J = E \sum_0^{N-1} \{ [x_k, Q x_k] + [u_k, H u_k] \} + E[x_N, \hat{Q} x_N] \tag{20}$$

where $Q \geq 0, H > 0$ and $\hat{Q} \geq 0$.

Admissible control u_i measurable $\{ v^i \}$.

control problem find $\{ u_i \}_0^{N-1}$ such that the performance index (20) is minimized.

2.3 Historical Perspective

There has been a fair amount of work on randomly sampled data system. The stability of such system was studied by Kushner and Tobias (1969) [20], Agniel and Jury (1969,1971) [1] and Beutler (1972) [8]. The optimal control problem was studied among the others by Kalman (1961) [19],Montgomery and Lee (1980) [21] and De Koning [10, 14, 13, 12, 11].

Assuming perfect observation (i. e. $C = I, G = 0$),no control cost ($H = 0$) and i. i. d. sampling ($\{t_k\}$ i. i. d.), Kalman succeeded in finding the optimal control by applying dynamic programming principle:

$$u_t = -E[\Gamma^* P(t+1)\Gamma]^\dagger E[\Gamma^* P(t+1)\Phi]x_t \tag{21}$$

where $P(t)$ satisfies nonlinear equation of Riccati type. He also solved the infinite-horizon problem in this case.

Under the assumption that sample interval $\{ t_i \}$ is a Marcov process, u_i is measurable v^i and $t_i = (t_i, \cdots, t_i)$ and perfect observation,Montgomery was able to derive the optimal control by applying the dynamical programming theory. The optimal control is as follows :

$$u_i^* = K_{i|t_i} x_i \tag{22}$$

$$K_{i|t_i} = -\{E_{t_i}[H + \Gamma^*(t_{i+1})P_{N-i-1}\Gamma(t_{i+1})]\}^{-1} E_{t_i}[\Gamma^*(t_{i+1})P_{N-i-1}\Phi(t_{i+1})] \tag{23}$$

where P_{N-i-1} satisfies Riccati type equation. The optimal control depends on past sample interval and past observation. It is preferred that the optimal control depends only on the past observation since sometimes $\{t_i\}$ is not available.

De Koning still assumes perfect observation ($C = I, G = 0$),extended Kalman's result to a general continuous time system and quadratic cost in the infinite horizon

case. Using the terminology of mean square stabilizability and detectability, he successfully showed the convergence of Riccati equation and existence of optimal control. He also proved the closed-loop system is mean square stable. His result can be summarized as follows:

$$u_i^* = -L_{B_*^{N-i-1}\hat{Q}} x_i \ , i = 0, \cdots, N-1 \tag{24}$$

$$J^*(x_0) = x_0^* B_*^N \hat{Q} x_0 \ \forall x_0 \tag{25}$$

$$\text{where } L_X = (\overline{\Gamma^* X \Gamma} + H)^\dagger \overline{\Gamma^* X \Phi} \tag{26}$$

$$B_* X = \overline{\Phi^* X \Phi} - \overline{\Phi^* X \Gamma}(\overline{\Gamma^* X \Gamma} + R)^\dagger \overline{\Gamma^* X \Phi} + Q \tag{27}$$

L_X is the gain operator whereas B_* is the associated Riccati operator.

De Koning's result beautifully extends the famillar LQR theory to random parameter system. However, the assumption of perfect observation is not available in practice. The purpose of this paper is thus trying to find a practical control under imperfect observation.

The problem become extremely difficult under imperfect observation. The optimal solution will be nonlinear and analytical solution is yet to be found. The famillar *separation principle* does not hold in general. However under variance neutrality condition, the separation principle does hold.

Definition: Variance Neutrality

If E $[(x_i - \hat{x}_i)(x_i - \hat{x}_i)^* | v^i, u^{i-1}]$ is independent of v^i and u^{i-1} ,the system is called to have the property of variance neutrality.

Assuming variance neutrality, it is easy to derive the optimal control by applying

dynamic programming. See appendix A for the following formulas.

$$u_i = -K_i \hat{x}_i, \ 0 \leq i \leq N-1 \tag{28}$$

$$K_i = [\overline{\Gamma^*(Q + P^c_{N-i-1})\Gamma} + H]^{-1} \overline{\Gamma^*(Q + P^c_{N-i-1})\Phi} \tag{29}$$

$$P^c_{N-i} = \overline{\Phi^*(Q + P^c_{N-i-1})\Phi} - K_i^* \overline{\Gamma*(Q + P^c_{N-i-1})\Phi} \tag{30}$$

Even though the separation principle principle hold, $\hat{x}_i = E(x_i|v^i)$ is not linear in general and analytical solution is not possible.

2.4 Fixed Configuration Approach

With all the difficulty considered above, it is natural to obtain a linear optimal solution, i. e. restricting the control structure to be linear and trying to find an optimal control in linear class. Once limited to linear configuration as shown in fig. 2, the control problem is simplified into finding the optimal linear dynamic matrices which minimizes the perfermance index given in equation 20.

Dynamics of Controller

$$u_n = K_n \hat{x}_n \tag{31}$$

$$\hat{x}_n = \hat{A}_{n-1}\hat{x}_{n-1} + \hat{B}_{n-1}v_n \tag{32}$$

Control Problem $\min J_N \ w.\ r.\ t.\ \{\hat{K}_n, \hat{A}_n, \hat{B}_n\}_0^{N-1}$

To further simplify the problem, we can assume that the controller is composed of a linear Kalman filter and a gain matrix as shown in fig. 3. This is equivalent to requiring:

$$E[(x_n - \hat{x}_n)v_i^*] = 0 \text{ for } 1 \leq i \leq n \tag{33}$$

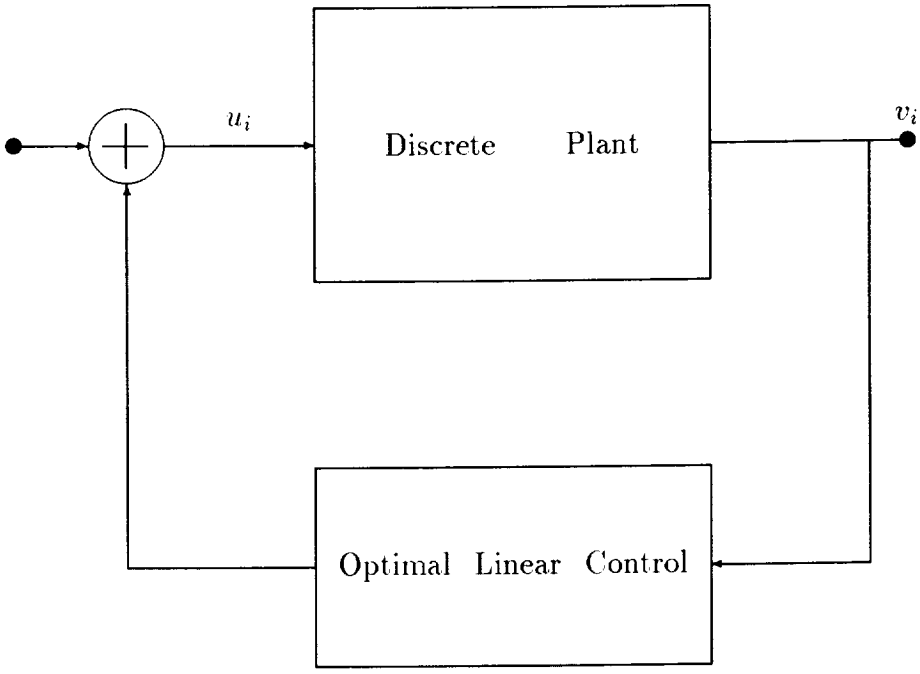


Figure 2: proposed linear control structure

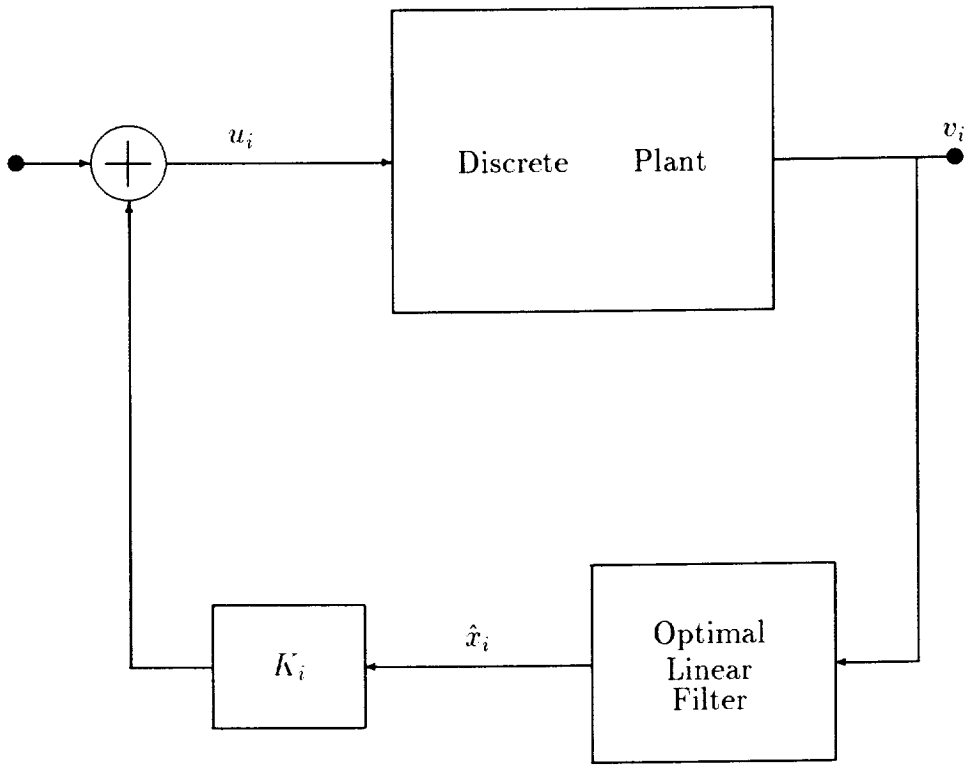


Figure 3: proposed linear separable control structure

Therefore $\hat{x}_n$ is just the optimal linear mean-square estimate of state x_n . It turns out that the Kalman filter is a function of { K_n },i. e. ,$\hat{A}_n$ *and* $\hat{B}_n$ in equation 32 depends on K_n. furthermore, the error covariance matrix $[E(x_n - \hat{x}_n)(x_n - \hat{x}_n)^*]$ is a function of K_n also. This is why the separation principle does not hold in this case. However,in the usual SLQR problem, the error covariance matrix for the Kalman filter is independent of K and then the controller and estimator can be designed separately.

Chapter 3

Optimal Linear Estimator

In this chapter, the orthogonality principle is applied to derive the optimal linear least mean-square estimator of the following dynamic systems.

$$x_{k+1} = \Phi(t_{k+1})x_k + \Gamma(t_{k+1})u_k + F(t_{k+1})\eta_k \tag{34}$$

$$v_k = Cx_k + G\theta_k \tag{35}$$

let's define:

$$\bar{E}[x_k|v^k] = \text{optimal linear estimate of } x_k \text{ in terms of } v^k = \{v_1, \cdots, v_k\}$$

$$\hat{x}_k = \bar{E}[x_k|v^k]$$

$$\bar{x}_k = \bar{E}[x_k|v^{k-1}]$$

$$\mathcal{L}(v^k) = \text{linear subspace spanned by } v^k$$

where optimality refers to minimization of the mean square error $E[\|x - \hat{x}_k\|^2]$. $\bar{x}_k$ is the optimal linear one-step predictor of x_k. The Kalman filter is derived under the assumption that $u_k = K_k\hat{x}_k$. It turns out that the error covariance matrices ,hence the filter itself,is dependent on the particular control sequence $\{u_k\}$.

3.1 Orthogonality Principle

Let's prove the following version of orthogonality principle.

Theorem 1 Let x, y be $n \times 1$ and $m \times 1$ random vectors respectively. A, A_* and $\tilde{A} \in R^{n \times m}$ matrices. Then

$$\bar{E}[x|y] = A_* y \text{ iff } E[x - A_* y, \tilde{A}y] = 0, \ \forall \tilde{A} \in R^{n \times m} \tag{36}$$

Proof:(sufficiency) If equation 36 holds,then we have

$$\forall A : E\|x - Ay\|^2 = E\|x - A_* y + A_* y - Ay\|^2$$

$$= E\|x - A_* y\|^2 + E\|(A_* - A)y\|^2 \geq E\|x - A_* y\|^2$$

The minimum occurs when $A = A_*$.

(necessity) let's compute

$$\frac{d}{d\lambda} E[x - (A_* + \lambda\tilde{A})y, x - (A_* + \lambda\tilde{A})y] \ \|_{\lambda=0}$$

$$= -2E[\tilde{A}y, x - A_* y] = 0$$

$$\Rightarrow E[x - A_* y, \tilde{A}y] = 0, \ \forall \tilde{A} \in R^{n*m} \ \square$$

3.2 Derivation of least mean-square estimator

To derive the Kalman filter equations, first notice that the following identities hold:

$$(i) \ \bar{v}_k = \bar{E}[v_k|v^{k-1}] = C\bar{x}_k \tag{37}$$

proof: Obviously $C\bar{x}_k \in \mathcal{L}(v^{k-1})$.

$$\forall i \leq k-1 : E[(v_k - C\bar{x}_k)v_i^*]$$

$$= E[(Cx_k + G\eta_k - C\bar{x}_k)v_i^*]$$

$$= E[C(x_k - \bar{x}_k)v_i^* + E[G\eta_k v_i^*] = 0, \ \square$$

$$(ii)\ \breve{v}_k = v_k - \bar{v}_k$$

$$= Cx_k + G\eta_k - C\bar{x}_k \text{ (from equation 37)}$$

$$= C\breve{x}_k + G\eta_k \tag{38}$$

$$(iii)\ \bar{x}_{k+1} = \overline{\Phi(t_{k+1})}\hat{x}_k + \overline{\Gamma(t_{k+1})}u_k \tag{39}$$

where $\overline{\phi(t_{k+1})} = E[\Phi(t_{k+1})]$ and $\overline{\Gamma(t_{k+1})} = E[\Gamma(t_{k+1})]$. Since $\{t_{k+1}\}$ is assumed i. i. d. ,we will simply write $\overline{\Phi(t_{k+1})} = \bar{\Phi}$ and $\overline{\Gamma(t_{k+1})} = \bar{\Gamma}$ hereafter.

proof: Obviously, the RHS $\in \mathcal{L}(v^k)$. Furthermore, $\forall i \leq k$,we have: [1]

$$E\{[x_{k+1} - \bar{\Phi}\hat{x}_k - \bar{\Gamma}u_k]v_i^*\}$$

$$= E\{[\Phi_k x_k + \Gamma_k u_k + F_k\eta_k - \bar{\Phi}\hat{x}_k - \bar{\Gamma}u_k]v_i^*\}$$

$$= E\{[\Phi_k x_k - \bar{\Phi}x_k]v_i^*\} + E\{[\Gamma_k u_k - \bar{\Gamma}u_k]v_i^*\}$$

$$+E\{F_k\eta_k v_i^*\} \tag{40}$$

The first term in equation 40 is equal to :

$$E[\Phi_k]E[x_k v_i^*] - E[\bar{\Phi}x_k v_i^*]$$

$$= \bar{\Phi}E[x_k v_i^*] - \bar{\Phi}E[x_k v_i^*] = 0$$

Since t_{k+1} is independent of x_k and v_i^*. Similarly, the second term is null. The third term is equal to :

$$E[F_k]E\{\eta_k v_i^*\} = 0$$

[1] In the sequel,we will simply write

$$\Phi(t_{k+1}) = \Phi_k, \Gamma(t_{k+1}) = \Gamma_k \text{ and } F(t_{k+1}) = F_k$$

Since $F_k\eta_k$ is independent of v_i^*. □

$$(iv)\breve{x}_k = x_k - \bar{x}_k = \text{one step predictor error} \tag{41}$$

$$(v)\hat{x}_k = \bar{x}_k + A_k\breve{v}_k \tag{42}$$

proof: We only need to prove that $\hat{x}_k - \bar{x}_k - A_k\breve{v}_k$ is orthogonal to $\{v^k\}$ for certain appropriate A_k matrices. For $i \leq k-1$, we obtain:

$$E[(x_k - \bar{x}_k - A_k\breve{v}_k)v_i^*]$$

$$= E[(x_k - \bar{x}_k)v_i^*] - E[A_k\breve{v}_k v_i^*]$$

$$= 0, \text{ by orthogonality principle}$$

The following equation has also to be satisfied:

$$E[(x_k - \bar{x}_k - A_k\breve{v}_k)v_k^*] = 0$$

$$\Rightarrow E[(x_k - \bar{x}_k)v_k^*] = A_k E[\breve{v}_k v_k^*] \tag{43}$$

The RHS of equation 43 is equal to:

$$A_k E\{(C\breve{x}_k + G\eta_k)(Cx_k + G\eta_k)^*\} \text{ (by equation 38)}$$

$$= A_k\{E(C\breve{x}_k x_k^* C^*) + GG^*\}$$

$$= A_k(CB_kC^* + GG^*) \tag{44}$$

provided we define matrix B_k as:

$$B_k = E[\breve{x}_k x_k^*] = E[\breve{x}_k \breve{x}_k^*] \tag{45}$$

The above identity follows from:

$$\begin{aligned} & E[\breve{x}_k x_k^*] = E[(x_k - \bar{x}_k)(x_k - \bar{x}_k + \bar{x}_k)^*] \\ = \; & E[(x_k - \bar{x}_k)(x_k - \bar{x}_k)^*] + E[(x_k - \bar{x}_k)\bar{x}_k^*] \\ = \; & E[(x_k - \bar{x}_k)(x_k - \bar{x}_k)^*] = E[\breve{x}_k \breve{x}_k^*] \end{aligned} \tag{46}$$

So, actually B_k is the one-step predictor error covariance matrix. In the sequel,we will assume $GG^* > 0$ for convenience. In that case $(CB_kC^* + GG^*)^{-1}$ always exists. The left hand side of equation 43 is equal to:

$$\begin{aligned} & E[(x_k - \bar{x}_k)(Cx_k + G\eta_k)^*] \\ = \; & E[(x_k - \bar{x}_k)x_k^* C^*] + 0, \text{ (by independence)} \\ = \; & E[\breve{x}_k \breve{x}_k^*]C^* \\ = \; & B_k C^* \end{aligned} \tag{47}$$

by the same argument as equation 46. Combining equation 44 and 47, we obtain the following identity.

$$(vi) A_k = B_k C^* (CB_kC^* + GG^*)^{-1} \tag{48}$$

The identity (v) is proved provided we can find the propagation equation for B_k. The propagation equation will be found in the next section. Combining the above identities, we can derive the Kalman filter equation.

$$\begin{aligned} \hat{x}_{k+1} &= \bar{x}_{k+1} + A_{k+1}\breve{v}_{k+1} \; (by \; v) \\ &= \bar{\Phi}\hat{x}_k + \bar{\Gamma}u_k + A_{k+1}(v_{k+1} - \bar{v}_{k+1}) \; (by \; ii \; and \; iii) \\ &= \bar{\Phi}\hat{x}_k + \bar{\Gamma}u_k + A_{k+1}(v_{k+1} - C\bar{x}_{k+1}) \; (by \; i) \\ &= \bar{\Phi}\hat{x}_k + \bar{\Gamma}u_k + A_{k+1}[v_{k+1} - C(\bar{\Phi}\hat{x}_k + \bar{\Gamma}u_k)] \; (by \; iii) \end{aligned} \tag{49}$$

Rearrange the above equation,we obtain the Kalman filter equation.

$$(vii)\hat{x}_{k+1} = (I - A_{k+1}C)\bar{\Phi}\hat{x}_k + (I - A_{k+1}C)\bar{\Gamma}u_k + A_{k+1}v_{k+1} \tag{50}$$

The above equation is fairly similar to the usual Kalman filter equation. However,one thing worth pointing out is that B_k depends on the particular control ($\{u_k\}$) sequence. The above filter equation holds true as long as $\{\ u_k\ \}$ is a linear function of $\{\ v^k\ \}$. In chapter four,we will combine the estimation and control problem together where we are interested in the linear optimal control of the form $\{u_k = K_k\hat{x}_k\}$. Therefore,in the following section we will derive B_k based on this particular control sequence.

3.3 Propagation equation for covariance matrices

In the derivation of B_k,we will assume $u_k = K_k\hat{x}_k$,i. e. u_k is the product of gain matrix K_k and optimal linear state estimator $\hat{x}_k$. The following notation will be used:

$$\begin{aligned} B_k &= E[\breve{x}_k\breve{x}_k^*] \\ &= \text{one step predictor error covariance matrix} \\ R_k &= E[x_k x_k^*] \qquad (51) \\ &= \text{second moment matrix of state} \\ \dot{\Phi}_k &= \Phi_k - \bar{\Phi} \qquad (52) \\ \dot{\Gamma}_k &= \Gamma_k - \bar{\Gamma} \qquad (53) \\ \Psi_k &= \Phi_k + \Gamma_k K_k \qquad (54) \\ \dot{\Psi}_k &= \Psi_k - \bar{\Psi} \qquad (55) \\ \breve{\Psi}_k &= \bar{\Phi} - \bar{\Gamma}_k K_k \\ &= \Phi + \bar{\Gamma} K_k - \Gamma_k K_k \\ &= \bar{\Psi} - \Gamma_k K_k \qquad (56) \end{aligned}$$

Let's begin with calculation of $\breve{x}_k$. By definition,we have:

$$\begin{aligned} \breve{x}_{k+1} &= x_{k+1} - \bar{x}_{k+1} \\ &= \Phi_k x_k + \Gamma_k u_k + F_k\eta_k - \bar{\Phi}\hat{x}_k - \bar{\Gamma}u_k \qquad (57) \\ &= \Phi_k x_k + F_k\eta_k - \breve{\Psi}_k\hat{x}_k \\ &= \dot{\Psi}_k x_k + \breve{\Psi}_k(I - A_k C)\breve{x}_k - \breve{\Psi}_k A_k G\theta_k + F_k\eta_k \qquad (58) \end{aligned}$$

where it is easy to show that:

$$\hat{x} = x_k - (I - A_k C)\breve{x}_k + A_k G\theta_k \tag{59}$$

Therefore,from equation 58 we obtain:

$$\begin{aligned} B_{k+1} &= \overline{\breve{\Psi} R_k \breve{\Psi}^*} + \overline{\breve{\Psi}(I - A_k C) B_k (I - A_k C)^* \breve{\Psi}^*} \\ &\quad + \mathcal{Z} \circ \overline{\breve{\Psi} B_k (I - A_k C)^* \breve{\Psi}^*} + \overline{F F^*} \\ &\quad + \overline{\breve{\Psi} A_k G G^* A_k^* \breve{\Psi}^*} \end{aligned} \tag{60}$$

where

$$\mathcal{Z} \circ X = X + X^* \tag{61}$$

For simplicity,we have omitted the t_{k+1} dependence in equation 60 due to i. i. d. assumption. It is easy to show that :

$$\begin{aligned} x_{k+1} &= \Psi_k x_k + \Gamma_k K_k A_k G\theta_k \\ &\quad - \Gamma_k K_k (I - A_k C)\breve{x}_k + F_k \eta_k \end{aligned} \tag{62}$$

$$\begin{aligned} R_{k+1} &= \overline{\Psi R_k \Psi^*} + \overline{\Gamma K_k A_k G G^* A_k^* K_k^* \Gamma^*} \\ &\quad - \mathcal{Z} \circ \overline{\Psi B_k (I - A_k C)^* K_k^* \Gamma^*} + \overline{F F^*} \\ &\quad \overline{\Gamma K_k (I - A_k C) B_k (I - A_k C)^* K_k^* \Gamma^*} \end{aligned} \tag{63}$$

Equation 60 and 63 constitute the propagation equation of covariance matrices. It is obviously that the covariance matrices depend on the $\{ K_k \}$ matrices. Since they depend on the past control,the variance neutrality condition does not hold in this case. The $\{ B_k \}$ matrices measure the accuracy of the estimate,and this information is used in deriving the suboptimal control in next chapter. Let's define:

$$W_k = E[\hat{x}_k \hat{x}_k^*] \tag{64}$$

From equation 42 43 and 44,we obtain:

$$\begin{aligned} W_k &= E[\bar{x}_k \bar{x}_k^*] + A_k E[\breve{v}_k \breve{v}_k^*] A_k^* \\ &= E[x_k x_k^*] - E[\breve{x}_k \breve{x}_k^*] + A_k E[\breve{v}_k \breve{v}_k^*] A_k^* \\ &= R_k - B_k + A_k (C B_k C^* + G G^*) A_k^* \end{aligned} \tag{65}$$

Now let's compute the error covariance matrix P_k defined as follows:

$$\begin{aligned} P_k &= E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^*] \\ &= E[x_k x_k^*] - E[\hat{x}_k \hat{x}_k^*] \\ &= R_k - W_k \end{aligned} \tag{66}$$

From equation 48 and 65 we obtain:

$$W_k = R_k - B_k + B_k C^* A_k^* \tag{67}$$

$$\Rightarrow R_k - W_k = P_k = B_k - B_k C^* A_k^*$$

$$\Rightarrow P_k = B_k (I - A_k C)^* = (I - A_k C) B_k \tag{68}$$

From equation 48,we have

$$A_k (C B_k C^* + G G^*) = B_k C^*$$

$$\Rightarrow (I - A_k C) B_k C^* = A_k G G^* = P_k C^* \tag{69}$$

$$\Rightarrow A_k = P_k C^* (G G^*)^{-1} \tag{70}$$

Substituting eq. 70 inte eq. 68, we have:

$$P_k = [I - P_k C^* (G G^*)^{-1} C] B_k \tag{71}$$

$$\Rightarrow P_k (I + C^* (G G^*)^{-1} C B_k) = B_k$$

$$\Rightarrow \quad P_k = B_k(I + C^*(GG^*)^{-1}CB_k)^{-1}$$
$$= (I + B_kC^*(GG^*)^{-1}C)^{-1}B_k \tag{72}$$

Combining equation 34 and 50,we obtain the total system equation (assuming $u_k = K_k\hat{x}_k$):

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi_k & \Gamma_kK_k \\ A_{k+1}C\Phi_k & J_{k+1}\bar{\Psi} + A_{k+1}C\Gamma_kK_k \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}$$
$$+ \begin{bmatrix} F_k\eta_k \\ A_{k+1}CF_k\eta_k + A_{k+1}G\theta_{k+1} \end{bmatrix} \tag{73}$$

$$\text{where } J_{k+1} = I - A_{k+1}C \tag{74}$$

It is easy to show the following equation from equation 73:

$$R_{k+1} = \overline{\Phi R_k\Phi^*} + \overline{FF^*} + \overline{\Gamma K_kW_k\Phi^*}$$
$$+ \overline{\Phi W_kK_k^*\Gamma^*} + \overline{\Gamma K_kW_kK_k^*\Gamma^*} \tag{75}$$

$$W_{k+1} = A_{k+1}CR_{k+1} + J_{k+1}\bar{\Psi}W_k\bar{\Psi}^* \tag{76}$$
$$\Rightarrow \quad W_{k+1} = A_{k+1}C(P_{k+1} + W_{k+1}) + J_{k+1}\bar{\Psi}W_k\bar{\Psi}^*$$
$$\Rightarrow \quad J_{k+1}W_{k+1} = A_{k+1}CP_{k+1} + J_{k+1}\Psi W_k\bar{\Psi}^* \tag{77}$$

It is easy to prove the following identity:

$$J_{k+1} = I - A_{k+1}C = (I + B_{k+1}C^*C)^{-1} \tag{78}$$

Therefore J_{k+1}^{-1} exists,and by equation 77 we obtain:

$$W_{k+1} = (I - A_{k+1}C)^{-1}A_{k+1}CP_{k+1} + \bar{\Psi}W_k\bar{\Psi}^* \tag{79}$$

From equation 72,we can show that the first term in the above equation is equal to $B_{k+1} - P_{k+1}$. Therefore,we obtain:

$$W_{k+1} = \bar{\Psi} W_k \bar{\Psi}^* + B_{k+1} - P_{k+1} \tag{80}$$

Furthermore,we can show:

$$x_{k+1} - \bar{x}_{k+1} = \Phi_k (x_k - \hat{x}_k) + \tilde{\Psi}_k \hat{x}_k + F_k \eta_k \tag{81}$$

$$B_{k+1} = \overline{\Phi P_k \Phi^*} + \overline{F F^*} + \overline{\tilde{\Psi} W_k \tilde{\Psi}^*} \tag{82}$$

Summary: We have obtained the following equations for relevant covariance matrices (P_k, B_k, W_k):

$$P_{k+1} = (I + B_{k+1} C^* C)^{-1} B_{k+1} \tag{83}$$

$$B_{k+1} = \overline{\Phi P_k \Phi^*} + \overline{F F^*} + \overline{\tilde{\Psi} W_k \tilde{\Psi}^*} \tag{84}$$

$$W_{k+1} = \bar{\Psi} W_k \bar{\Psi}^* + B_{k+1} - P_{k+1} \tag{85}$$

where we have assumed that $GG^* = I$. If not ,replace C by $(\sqrt{GG^*})^{-1} C$ and v_n by $(\sqrt{GG^*})^{-1} v_n$. One thing worth noting is that $B_{k+1} - P_{k+1} > 0$. This is because B_{k+1} is the one-step predictor error covariance matrix whereas P_{k+1} is the error covariance matrix [6].

3.4 Asymptotic behavior of Kalman filter

In the following consideration,we will assume $u_k = K \hat{x}_k$ since we are interested in obtaining a linear time invariant estimator. We hope to establish conditions for which P_i and W_i converge as $i \to \infty$. To that end,let's first give some definition (see [13]).

Consider the following system:

$$x_{i+1} = \Phi(t_{i+1})x_i + \Gamma(t_{i+1})u_i \tag{86}$$

If $u_i = Kx_i$,then we have

$$x_{i+1} = \Psi(t_{i+1})x_i \tag{87}$$

$$\text{where}\Psi(t_{i+1}) = \Phi(t_{i+1}) + \Gamma(t_{i+1})K \tag{88}$$

Definition 1: System 87 is called m. s. stable (mean square) if $\overline{\|x_i\|^2} \to 0$ for all x_0.

Definition 2: (Φ, Γ) is m. s. stabilizable if $\exists K$ such that system 87 is m. s. stable.

In view of m. s. observability consider the following system:

$$\begin{aligned} x_{i+1} &= \Phi(t_{i+1})x_i \\ v_i &= Cx_i \end{aligned} \tag{89}$$

Definition 3: (Φ, C) is called m. s. observable if $\overline{\|v_i\|^2} = 0, \forall i \Rightarrow x_0 = 0$.

Let S^n denote the linear space of real symmetric $n \times n$ matrices and define the transformation $A_\Phi : S^n \to S^n$ by:

$$A_\Phi S = \overline{\Phi S \Phi^*}, S \in S^n \tag{90}$$

It is easy to prove the following lemma.

Lemma 1: A_Φ is linear and monotone,i. e. ,

$$S \geq 0 \Longrightarrow A_\Phi^i S \geq 0, \ \forall \ i \tag{91}$$

Lemma 2: System 87 is m. s. stable iff

$$\rho(A_\Psi) < 1$$

Lemma 3: System 89 is m. s. observable iff

$$\sum_{i=0}^{p-1} A_{\phi^*}^i C^* C > 0, \; p = n(n+1)/2 \tag{92}$$

where $\rho(A_\phi)$ is the spectral radius of A_Φ.

3.5 Steady state Riccati equation

If $P_k \rightarrow P$ and $W_k \rightarrow W$,then from equations (84, 85) we have the following SSRE:

$$P = (I + BC^*C)^{-1}B \tag{93}$$

$$B = \overline{\Phi P \Phi^*} + \overline{\bar{\Psi} W \bar{\Psi}^*} + \overline{FF^*} \tag{94}$$

$$W = \bar{\Psi} W \bar{\Psi}^* + B - P \tag{95}$$

Let's assume $\rho(A_\Psi) < 1$,and therefore $\rho(A_{\bar{\Psi}}) < 1$. Equation 95 becomes:

$$W = (I - A_{\bar{\Psi}})^{-1}(B - P) \tag{96}$$

Substitute the above expression into equation 94,we obtain:

$$B = (I - A_{\bar{\Psi}})(I - A_\Psi)^{-1}(A_\Phi - A_\Phi A_\Psi - A_{\bar{\Psi}})$$

$$(I - A_{\bar{\Psi}})^{-1}P + (I - A_{\bar{\Psi}})(I - A_\Psi)^{-1}\overline{FF^*} \tag{97}$$

Define:

$$\acute{B} = (I - A_{\bar{\Psi}})^{-1}B \tag{98}$$

$$\acute{P} = (I - A_{\bar{\Psi}})^{-1}P \tag{99}$$

Then equation 93 and 97 become:

$$\acute{P} = [I + \acute{B}C^*C(I - A_{\bar{\Psi}})]^{-1}\acute{B} \tag{100}$$

$$\acute{B} = (I - A_{\Psi})^{-1}(A_{\Phi} - A_{\Phi}A_{\bar{\Psi}} - A_{\bar{\Psi}})\acute{P} + (I - A_{\Psi})^{-1}\overline{FF^*} \tag{101}$$

Note 1: $A_{\Phi} - A_{\Phi}A_{\bar{\Psi}} - A_{\bar{\Psi}} = (I - A_{\Phi})A_{\bar{\Psi}} + A_{\Phi} - A_{\Psi}$ is not necessarily monotonic.

Note 2: If $K = 0$,then $\Psi = \Phi$. We have:

$$\acute{P} = [I + \acute{B}C^*C(I - A_{\Phi})]^{-1}\acute{B} \tag{102}$$

$$\begin{aligned} \acute{B} &= A_{\Phi}\acute{P} + (I - A_{\Phi})^{-1}\overline{FF^*} \\ &= \bar{\Phi}\acute{P}\bar{\Phi}^* + (I - A_{\Phi})^{-1}\overline{FF^*} \end{aligned} \tag{103}$$

In this case,it is shown by De Koning [9] that error covariance matrices P_k and W_k converge and there exist a steady state Kalman filter provided $\rho(A_{\Phi}) < 1$,i. e. ,the system is mean square stable.

Chapter 4

Derivation of linear optimal control

4.1 Equivalent control problem

The control problem formulated in Chapter 2 is solved in this chapter. The control structure is limited to be a Kalman filter in series with the gain matrix as shown in fig. 3 of chapter 2. What remains to be done is the selection of the gain matrices $\{ K_n \}$ since we have already derived the Kalman filter equation associated with the control structure.

Let's write down the equations for generating the optimal control $\{ u_n \}$:

$$\hat{x}_{k+1} = (I - A_{k+1}C)\bar{\Phi}\hat{x}_k + (I - A_{k+1}C)\bar{\Gamma}u_k + A_{k+1}v_{k+1} \tag{104}$$

$$u_k = K_k\hat{x}_k \tag{105}$$

$$A_{k+1} = P_{k+1}C^*(GG^*)^{-1} = B_{k+1}C^*(CB_{k+1}C^* + GG^*)^{-1}$$

where A_{k+1} is the Kalman gain matrix. The control $\{ u_n \}$ is selected such that the

performance index J_N is minimized.

$$J_N = E\sum_1^N \{x_k^* Q x_k + u_{k-1}^* H u_{k-1}\} \tag{106}$$

$$= \sum_1^N \{trQR_k + trK_{k-1}^* H K_{k-1} W_{k-1}\} \tag{107}$$

where from equation 65 of chapter 4,we know:

$$W_k = R_k - B_k + A_k(CB_kC^* + GG^*)A_k^* \tag{108}$$

Let's define:

$$[A, B] = trAB^* \tag{109}$$

as the inner product in the space of $n \times n$ matrices. Equation 107 is then rewritten as:

$$J_N = \sum_1^N \{[Q, R_k] + [K_{k-1}^* H K_{k-1}, W_{k-1}]\} \tag{110}$$

The relevant progation equation for covariance matrices $\{R_k\}$ and $\{B_k\}$ are from equation 60 and 63 of chapter 4:

$$\begin{aligned} B_{k+1} &= \overline{\breve{\Psi} R_k \breve{\Psi}^*} + \overline{\breve{\Psi}(I - A_kC)B_k(I - A_kC)^*\breve{\Psi}^*} \\ &\quad + \mathcal{Z} \circ \overline{\breve{\Psi} B_k(I - A_kC)^*\breve{\Psi}^*} + \overline{FF^*} \\ &\quad + \overline{\breve{\Psi} A_k GG^* A_k^* \breve{\Psi}^*} \end{aligned} \tag{111}$$

$$\begin{aligned} R_{k+1} &= \overline{\Psi R_k \Psi^*} + \overline{\Gamma K_k A_k GG^* A_k^* K_k^* \Gamma^*} \\ &\quad - \mathcal{Z} \circ \overline{\Psi B_k(I - A_kC)^* K_k^* \Gamma^*} + \overline{FF^*} \\ &\quad + \overline{\Gamma K_k(I - A_kC)B_k(I - A_kC)^* K_k^* \Gamma^*} \end{aligned} \tag{112}$$

It is noticed that our *stochastic* control problem has been transformed into an equivalent *deterministic* control problem in which the state dynamics and performance

index J_N are given by equation 110 111 and 112 respectively. However,the equivalent control problem is nonlinear for J_N and state dynamics. Once the control structure is fixed,the sufficient statistics become $\{ R_k, B_k \}$. The optimal control sequence $\{ u_k = K_k \hat{x}_k \}$ is selected based on the information about the sufficient statistics-the error covariance matrix of state estimator B_k and the signal covariance matrix R_k.

There are many methods which can be used to solve this problem,such as dynamic programming and minimum principle to name just a few. Furthermore,the technique in solving the nonlinear optimization problem can be used to obtain numerical solution at least. Because of the nonlinearness involved,it is difficult to obtain analytical solution. In applying the minimum principle to get the necessary condition for optimal control,one difficulty remains. Notice that the state dynamics is described as a *matrix difference equation* in stead of a *vector difference equation*. The *matrix minimum principle* proposed by Athans (1967) (see [4]) is useful to derive the necessary conditions. In the following section the Lagrange's multiplier method is adopted to obtain the necessary condition which is basically the same as those for the matrix minimum principle.

4.2 N-horizon control problem

In this section the Lagrange's multiplier technique is used to derive the necessary condition for the optimal control of the N-horizon control problem formulated in the last section. Consider the auxiliary cost $\tilde{J}_N$ defined as follows:

$$\tilde{J}_N = J_N + \sum_{0}^{N-1} \{\alpha_k, B_{k+1} - \overline{\tilde{\Psi} R_k \tilde{\Psi}} - \overline{\tilde{\Psi} J_k B_k J_k^* \tilde{\Psi}^*}$$

$$- \mathcal{Z} \circ \overline{\tilde{\Psi} B_k J_k^* \tilde{\Psi}^*} - \overline{\tilde{\Psi} A_k G G^* A_k^* \tilde{\Psi}^*} - \overline{F F^*}\}$$

$$+\sum_{0}^{N-1}\{\gamma_k, R_{k+1} - \overline{\Psi R_k \Psi^*} - \overline{\Gamma K_k J_k B_k J_k^* K_k^* \Gamma^*}$$
$$+\mathcal{Z} \circ \overline{\Psi B_k J_k^* K_k^* \Gamma^*} - \overline{\Gamma K_k A_k G G^* A_k^* K_k^* \Gamma^*} - \overline{F F^*}\}$$
$$+\sum_{1}^{N}\{\delta_k, A_k - B_k C^*(C B_k C^* + G G^*)^{-1}\} \tag{113}$$

where $\{\alpha_k\}, \{\gamma_k\}$ and $\{\ \delta_k\ \}$ are Lagrange constant matrices. We are free to set

$$\alpha_k = \alpha_k^*, \gamma_k = \gamma_k^* \tag{114}$$

because $\{B_k\}$ and $\{R_k\}$ are symmetric matrices. Our task is then boiled down to:

$$\min \tilde{J}_N \quad w.\ r.\ t.\ \{[K_k]_0^{N-1}, [A_k]_1^{N-1}, [B_k]_1^{N-1}, [R_k]_1^N\} \tag{115}$$

After taking the gradient of $\tilde{J}_N$ with respect to the above matrices,we obtain the required necessary conditions for optimal control. The gradient of $\tilde{J}_N(X)$ with respect to X at X_* is defined as follows:

$$[\nabla_X \tilde{J}_N(X_*), \delta X] = \lim_{\lambda \to 0} \tilde{J}_N(X_* + \lambda \delta X)$$

$$\nabla_{R_N} \tilde{J}_N = Q + \gamma_{N-1} = 0$$

$$\nabla_{A_N} \tilde{J}_N = \delta_N = 0$$

$$\nabla_{B_N} \tilde{J}_N = \alpha_{N-1} = 0$$

From the above equations,we can obtain the following boundary conditions:

$$\gamma_{N-1} = -Q$$

$$\delta_N = 0$$

$$\alpha_N = 0$$

Taking the gradient matrix of $\tilde{J}_N$ with respect to R_i,we obtain the following:

$$\begin{aligned}
&\forall\, 1 \leq\ i \leq\ N-1 : \nabla_{R_i} \tilde{J}_N = \\
&\quad Q + K_i^* H K_i - \overline{\tilde{\Psi}_i^* \alpha_i \tilde{\Psi}_i} \\
&\quad + \gamma_{i-1} - \overline{\Psi_i^* \gamma_i \Psi} = 0 \\
&\Rightarrow \gamma_{i-1} = \overline{\Psi_i^* \gamma_i \Psi_i} + \overline{\tilde{\Psi}_i^* \alpha_i \tilde{\Psi}_i} - Q - K_i^* H K_i
\end{aligned} \tag{116}$$

Taking the gradient matrix of $\tilde{J}_N$ with respect to B_i,we obtain:

$$\begin{aligned}
&\forall\, 1 \leq\ i \leq\ N-1 : \nabla_{B_i} \tilde{J}_N = \\
&\quad - K_i^* H K_i + C^* A_i^* K_i^* H K_i A_i C + \alpha_{i-1} \\
&\quad - \overline{(I - A_i C)^* \tilde{\Psi}_i^* \alpha_i \tilde{\Psi}_i (I - A_i C)} \\
&\quad - \overline{\tilde{\Psi}_i^* \alpha_i \tilde{\Psi}_i (I - A_i C)} - \overline{(I - A_i C)^* \tilde{\Psi}_i^* \alpha_i \tilde{\Psi}_i} \\
&\quad - (I - A_i C)^* K_i^* \overline{\Gamma_i^* \gamma \Gamma_i} K_i (I - A_i C) \\
&\quad + \overline{\Psi_i^* \gamma_i \Gamma_i} K_i (I - A_i C) + (I - A_i C)^* K_i^* \overline{\Gamma_i^* \gamma_i \Psi_i} \\
&\quad - \delta_i (C B_i C^* + G G^*)^{-1} C \\
&\quad + C^* (C B_i C^* + G G^*)^{-1} C B_i \delta_i (C B_i C^* + G G^*)^{-1} C
\end{aligned} \tag{117}$$

Note: In the above derivation,we use the following identities.

1. $tr\ AB^* = tr\ B^* A \ \Rightarrow\ [A, B] = [B^*, A^*]$

2. If $A\ B$ both are real,then $tr\ AB^* = tr\ BA^* = tr\ B^* A$.

$$\Longrightarrow [A, B] = [B^*, A^*] = [B, A] = [A^*, B^*]$$

3. $-K_i^* \overline{\tilde{\Gamma}_i^* \gamma_i \tilde{\Psi}_i} = \overline{\tilde{\Psi}_i^* \alpha_i \tilde{\Psi}_i}$

Taking the gradient matrix of $\tilde{J}_N$ with respect to A_i,we have:

$$\begin{aligned}
&\forall\ 1 \leq\ i \leq\ N-1 : \nabla_{A_i} \tilde{J}_N \\
&= 2K_i^* H K_i A_i (C B_i C^* + G G^*) + 2\overline{\breve{\Psi}_i^* \alpha_i \breve{\Psi}_i}(I - A_i C) B_i C^* \\
&\quad + 2\overline{\breve{\Psi}_i^* \alpha_i \tilde{\Psi}_i} B_i C^* - 2\overline{\breve{\Psi}_i^* \alpha_i \breve{\Psi}_i} A_i G G^* \\
&\quad + 2K_i^* \overline{\Gamma_i^* \gamma_i \Gamma_i} K_i (I - A_i C) B_i C^* - 2K_i^* \overline{\Gamma_i^* \gamma_i \Psi_i} B_i C^* \\
&\quad - 2K_i^* \overline{\Gamma_i^* \gamma_i \Gamma_i} K_i A_i G G^* + \delta_i = 0
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\ \delta_i \\
&= 2[-K_i^* H K_i A_i (C B_i C^* + G G^*) - \overline{\breve{\Psi}_i^* \alpha_i \breve{\Psi}_i}(I - A_i C) B_i C^* \\
&\quad + K_i^* \overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Psi}_i} B_i C^* + \overline{\breve{\Psi}_i^* \alpha_i \breve{\Psi}_i} A_i G G^* \\
&\quad - K_i^* \overline{\Gamma_i^* \gamma_i \Gamma_i} K_i (I - A_i C) B_i C^* + K_i^* \overline{\Gamma_i^* \gamma_i \Psi_i} B_i C^* \\
&\quad + K_i^* \overline{\Gamma_i^* \gamma_i \Gamma_i^*} K_i A_i G G^*] \\
&= 2K_i^* (-H + \overline{\Gamma_i^* \gamma_i \Gamma_i} + \overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Gamma}_i}) K_i A_i (C B_i C^* + G G^*) \\
&\quad + 2\bar{\Phi}^* \alpha_i \bar{\Phi} A_i (C B_i C^* + G G^*) \\
&\quad + 2(K_i^* \overline{\Gamma^* \gamma_i \Phi} + K_i^* \overline{\tilde{\Gamma}^* \alpha_i \tilde{\Phi}} - \bar{\Phi}^* \alpha_i \bar{\Phi}) B_i C^*
\end{aligned} \tag{118}$$

Taking the gradient matrix of $\tilde{J}_N$ with respect to K_i,we obtain:

$$\begin{aligned}
&\forall\ 0 \leq\ i \leq\ N-1 : \nabla_{K_i} \tilde{J}_N \\
&= 2H K_i W_i - 2\overline{\tilde{\Gamma}_i^* \alpha_i \breve{\Psi}_i} R_i \\
&\quad + 2\overline{\tilde{\Gamma}_i^* \alpha_i \breve{\Psi}_i}(I - A_i C) B_i (I - A_i C)^* - 2\overline{\tilde{\Gamma}_i^* \alpha_i \breve{\Psi}_i}(I - A_i C) B_i \\
&\quad + 2\overline{\Gamma_i^* \alpha_i \tilde{\Psi}_i} B_i (I - A_i C)^* + 2\overline{\tilde{\Gamma}_i^* \alpha_i \breve{\Psi}_i} A_i G G^* A_i^* \\
&\quad - 2\overline{\Gamma_i^* \gamma_i \Psi_i} R_i - 2\overline{\Gamma_i^* \gamma_i \Gamma_i} K_i (I - A_i C) B_i (I - A_i C)^* \\
&\quad - 2\overline{\Gamma_i^* \gamma_i \Gamma_i} K_i A_i G G^* A_i^* + 2\overline{\Gamma_i^* \gamma_i \Gamma_i} K_i (I - A_i C) B_i \\
&\quad + 2\overline{\Gamma_i^* \gamma_i \Psi_i} B_i (I - A_i C)^* = 0
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \quad & HK_iW_i + \overline{\check{\Gamma}^*\alpha_i\check{\Gamma}}K_i(I - A_iC)B_i \\
& -\overline{\check{\Gamma}^*\alpha_i\check{\Gamma}}K_iR_i + \overline{\check{\Gamma}^*\alpha_i\check{\Gamma}}K_iB_i(I - A_iC)^* \\
& -\overline{\check{\Gamma}^*\alpha_i\check{\Gamma}}K_i(I - A_iC)B_i(I - A_iC)^* \\
& -\overline{\check{\Gamma}^*\alpha_i\check{\Gamma}}K_iA_iGG^*A_i^* - \overline{\Gamma^*\gamma_i\Gamma}K_iR_i \\
& -\overline{\Gamma^*\gamma_i\Gamma}K_i(I - A_iC)B_i(I - A_iC)^* \\
& -\overline{\Gamma^*\gamma_i\Gamma}K_iA_iGG^*A_i^* + \overline{\Gamma^*\gamma_i\Gamma}K_i(I - A_iC)B_i \\
& +\overline{\Gamma^*\gamma_i\Gamma}K_iB_i(I - A_iC)^* \\
= \quad & \overline{\check{\Gamma}^*\alpha_i\check{\Phi}}R_i - \overline{\check{\Gamma}^*\alpha_i\check{\Phi}}B_i(I - A_iC)^* \\
& +\overline{\Gamma^*\gamma_i\Phi}R_i - \overline{\Gamma^*\gamma_i\Phi}B_i(I - A_iC)^* \\
= \quad & HK_iW_i - \overline{\check{\Gamma}^*\alpha_i\check{\Gamma}}K_iW_i - \overline{\Gamma^*\gamma_i\Gamma}K_iW_i \qquad (119) \\
& \Longrightarrow \quad [H - \overline{\check{\Gamma}^*\alpha_i\check{\Gamma}} - \overline{\Gamma^*\gamma_i\Gamma}]K_iR_i \\
= \quad & (\overline{\check{\Gamma}^*\alpha_i\check{\Phi}} + \overline{\Gamma^*\gamma_i\Phi})(R_i - B_i + B_iC^*A_i^*) \\
= \quad & (\overline{\check{\Gamma}^*\alpha_i\check{\Phi}} + \overline{\Gamma^*\gamma_i\Phi})W_i \qquad (120)
\end{aligned}
$$

where we have use the following identities in deriving the above equation (see 65 and 48 of chapter 3):

$$W_i = R_i - B_i + A_i(CB_iC^* + GG^*)A_i^*$$

$$A_i = B_iC^*(CB_iC^* + GG^*)^{-1}$$

$$\Rightarrow \ \forall\, i: \ K_iW_i = (H - \overline{\check{\Gamma}^*\alpha_i\check{\Gamma}} - \overline{\Gamma^*\gamma_i\Gamma})^{-1}(\overline{\check{\Gamma}^*\alpha_i\check{\Phi}} + \overline{\Gamma^*\gamma_i\phi})W_i \qquad (121)$$

where we have assumed that $(H - \overline{\check{\Gamma}_i^*\alpha_i\check{\Gamma}_i} - \overline{\Gamma_i^*\gamma_i\Gamma_i})^{-1}$ exists. If not,the *Moore Penrose pseudo inverse* can be used. It can be shown later,that $(H - \overline{\check{\Gamma}_i^*\alpha_i\check{\Gamma}_i} - \overline{\Gamma_i^*\gamma_i\Gamma_i})$ is self adjoint and strictly positive definite provided H is. Rightnow,we will show

that K_i can be defined as anything on the null space of W_i. Let's give the following lemma.

Lemma :For the N-horizon control problem,K_i need only be defined on the range space of W_i(denoted by $\mathcal{R}(W_i)$. On the null space of W_i(denoted by $\mathcal{N}(W_i)$),K_i can be assumed anything.

proof: We know that

$$u_i = K_i \hat{x}_i, \quad W_i = E[\hat{x}_i \hat{x}_i^*]$$

$$x \in \mathcal{N}(W_i) \Rightarrow E x^* \hat{x}_i \hat{x}_i^* x = 0$$

$$\Rightarrow \hat{x}_i^* x = 0 \text{ w. p. 1}$$

In particular,if $\hat{x}_i \in \mathcal{N}(W_i)$,we obtain

$$\hat{x}_i^* \hat{x}_i = 0 \Rightarrow \hat{x}_i = 0 \text{ w. p. 1}$$

and therefore $u_i = 0$ with probability one and K_i can be assumed anything. In general,we know:

$$\hat{x}_i = P\hat{x}_i + Q\hat{x}_i$$

where P is the projection operator associated with $\mathcal{R}(W_i)$,while Q is the projection operator associated with $\mathcal{N}(W_i)$.

$$u_i = K_i \hat{x}_i = K_i P \hat{x}_i + K_i Q \hat{x}_i$$

Since $Q\hat{x}_i \in \mathcal{N}(W_i)$,we obtain:

$$[\hat{x}_i, Q\hat{x}_i] = 0 \text{ w. p. 1}$$

$$\Rightarrow [Q\hat{x}_i, Q\hat{x}_i] = 0$$

$$\Rightarrow \; Q\hat{x}_i = 0 \text{ w. p. } 1$$

$$\Rightarrow \; u_i = K_i P \hat{x}_i$$

Again in this case,K_i can assume anything on the null space of W_i. □

In the sequel,we will choose K_i equal to:

$$K_i = (H - \overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Gamma}_i} - \overline{\Gamma_i^* \gamma_i \Gamma_i})^{-1} (\overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Phi}_i} + \overline{\Gamma_i^* \gamma_i \Phi_i}) \tag{122}$$

on both the range and null space of W_i. Using the above expression ,equation 116 can be simplified into:

$$\begin{aligned} \gamma_{i-1} \;=\; & \overline{\Phi_i^* \gamma_i \Phi_i} + \overline{\tilde{\Phi}_i^* \alpha_i \tilde{\Phi}_i} + Q \\ & + \mathcal{Z} \circ K_i^* (\overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Phi}_i} + \overline{\Gamma_i^* \gamma_i \Phi_i}) \\ & - K_i^* (H - \overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Gamma}_i} - \overline{\Gamma_i^* \gamma_i \Gamma_i}) K_i \end{aligned} \tag{123}$$

In the similar fashion,equation 117 can be simplified into:

$$\begin{aligned} -\alpha_{i-1} \;=\; & -K_i^* (H - \overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Gamma}_i} - \overline{\Gamma_i^* \gamma_i \Gamma_i}) K_i \\ & + C^* A_i^* K_i^* (H - \overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Gamma}_i} - \overline{\Gamma_i^* \gamma_i \Gamma_i}) K_i A_i C \\ & - (I - A_i C)^* \bar{\Phi}_i^* \alpha_i \bar{\Phi}_i (I - A_i C) \\ & \mathcal{Z} \circ (I - A_i C)^* K_i^* (\overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Phi}_i} + \overline{\Gamma_i^* \gamma_i \Phi_i}) \\ & - \delta_i (C B_i C^* + G G^*)^{-1} C \\ & + C^* (C B_i C^* + G G^*)^{-1} C B_i \delta_i (C B_i C^* + G G^*)^{-1} C \end{aligned} \tag{124}$$

Equation 118 can be simplified into:

$$\begin{aligned} -\delta_i/2 \;=\; & -K_i^* (H - \overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Gamma}_i} - \overline{\Gamma_i^* \gamma_i \Gamma_i}) K_i B_i C^* \\ & + K_i^* (\overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Phi}_i} + \overline{\Gamma_i^* \gamma_i \Phi_i}) B_i C^* \\ =\; & 0 \text{ using equation 122} \end{aligned} \tag{125}$$

Let's define:

$$Z_i = (H - \overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Gamma}_i} - \overline{\Gamma_i^* \gamma_i \Gamma_i})$$

Equation 124 becomes:

$$\begin{aligned} -\alpha_{i-1} &= -K_i^* Z_i K_i + C^* A_i^* K_i^* Z_i K_i A_i C \\ &\quad -(I - A_i C)^* \Phi_i^* \alpha_i \bar{\Phi}_i (I - A_i C) \\ &\quad +\mathcal{Z} \circ (I - A_i C)^* K_i^* (\overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Phi}_i} + \overline{\Gamma_i^* \gamma_i \Phi_i}) \\ &= (I - A_i C)^* K_i^* Z_i K_i (I - A_i C) \\ &\quad -(I - A_i C)^* \Phi_i^* \alpha_i \bar{\Phi}_i (I - A_i C) \end{aligned} \tag{126}$$

Using equation 122,equation 123 becomes:

$$\begin{aligned} \gamma_{i-1} &= \overline{\Phi_i^* \gamma_i \Phi_i} + \overline{\tilde{\Phi}_i^* \alpha_i \tilde{\Phi}_i} - Q \\ &\quad \mathcal{Z} \circ K_i^* (\overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Phi}_i} + \overline{\Gamma_i^* \gamma_i \Phi_i}) - K_i^* Z_i K_i \\ &= \overline{\Phi_i^* \gamma_i \Phi_i} + \overline{\tilde{\Phi}_i^* \alpha_i \tilde{\Phi}_i} + 2K_i^* Z_i K_i - K_i^* Z_i K_i - Q \\ &= \overline{\Phi_i^* \gamma_i \Phi_i} + \bar{\Phi}_i^* \alpha_i \bar{\Phi}_i + K_i^* Z_i K_i - Q \end{aligned} \tag{127}$$

Remark: If $-\alpha_i$ and $-\gamma_i$ are assumed to be self-adjoint and positive semidefinite,then from equation 126 and 116 we know that $-\alpha_{i-1}$ and $-\gamma_{i-1}$ are again self-adjoint and positive semidefinite. Since the boundary conditions derived before are:

$$\alpha_N = 0 \text{ and } -\delta_N = Q$$

Therefore we know that $-\alpha_i$ and $-\delta_i$ are self-adjoint and positive semidefinite for all i. From the definition of Z_i ,we know it is self-adjoint and positive semidefinite. Therefore,the inverse of $(H - \overline{\tilde{\Gamma}_i^* \alpha_i \tilde{\Gamma}_i} - \overline{\Gamma_i^* \gamma_i \Gamma_i})$ exists provided H is self-adjoint and strictly positive definite. In the sequel,we will assume H is positive definite.

Since K_i can assume anything on the null space of W_i ,the solution to the optimal N-horizon control problem stated above is not unique in general. The optimal gain matrix K_i given by equation 122 is the most obvious and simple choice. We will call it the regular solution. As seen in equation 122 ,the optimal gain matrix has similar structure as those mentioned in section 3 of chapter2. If we assume that t_i is deterministic,then equation 122 becomes:

$$K_i = [H - \Gamma^* \gamma_i \Gamma]^{-1} [\Gamma^* \gamma_i \Phi]$$

because $\dot{\Gamma} = 0$ and $\dot{\Phi} = 0$. If t_i is a random process,then some correction must be made.

From the above discussion,the following theorem is given:

Theorem: Assuming existence,the necessary conditions for the optimal control for the N-horizon control problem stated before is that there exist auxiliary covariance matrices $-\alpha_i$ and $-\delta_i$ such that the following equations and boundary conditions hold.

$$K_i = [H - \overline{\dot{\Gamma}^* \alpha_i \dot{\Gamma}} - \overline{\Gamma^* \gamma_i \Gamma}]^{-1} [\overline{\dot{\Gamma}^* \alpha_i \dot{\Phi}} + \overline{\Gamma^* \gamma_i \Phi}] \tag{128}$$

$$Z_i = [H - \overline{\dot{\Gamma}^* \alpha_i \dot{\Gamma}} - \overline{\Gamma^* \gamma_i \Gamma}] \geq 0 \tag{129}$$

$$\begin{aligned} -\alpha_{i-1} &= (I - A_i C)^* K_i^* Z_i K_i (I - A_i C) \\ &\quad -(I - A_i C)^* \bar{\Phi}^* \alpha_i \bar{\Phi} (I - A_i C) \geq 0 \end{aligned} \tag{130}$$

$$\begin{aligned} -\gamma_{i-1} &= -\overline{\Phi^* \gamma_i \Phi} - \overline{\dot{\Phi}^* \alpha_i \dot{\Phi}} + Q \\ &\quad -K_i^* (H - \overline{\dot{\Gamma}^* \alpha \dot{\Gamma}} - \overline{\Gamma^* \gamma \Gamma}) K_i \geq 0 \end{aligned} \tag{131}$$

$$\delta_i = 0, \alpha_{N-1} = 0, \gamma_{N-1} = -Q \tag{132}$$

Let's define:

$$P_i^b = -\alpha_{i-1},\ P_i^r = -\gamma_{i-1} \tag{133}$$

then we have the following necessary conditions:

$$P_i^b = J_i^* K_i^* Z_i K_i J_i + J_i^* \bar{\Phi}^* P_{i+1}^b \bar{\Phi} J_i \tag{134}$$

$$P_i^r = \overline{\Phi^* P_{i+1}^r \Phi} + \overline{\dot{\Phi}^* P_{i+1}^b \dot{\Phi}} + Q - K_i^* Z_i K_i \tag{135}$$

$$P_N^b = 0, P_N^r = Q \tag{136}$$

where $J_i = I - A_i C$ and

$$A_i = B_i C^* (C B_i C^* + G G^*)^{-1} \tag{137}$$

$$Z_i = [H + \overline{\dot{\Gamma}^* P_{i+1}^b \dot{\Gamma}} + \overline{\Gamma^* P_{i+1}^r \Gamma}] \tag{138}$$

$$K_i = -Z_i^{-1} (\overline{\Gamma^* P_{i+1}^r \Phi} + \overline{\dot{\Gamma}^* P_{i+1}^b \dot{\Phi}}) \tag{139}$$

The above equations and equation 111 and 112 are the usual two point boundary value problem (TPBVP) which can be solved numerically using techniques such as shooting method and relaxation method etc. Let's now consider a special example where the sampling process is deterministic. In which case it is well known the optimal solution exist.

Special case: $\sigma(t_k) = 0$.

When the sampling process is deterministic, Φ and Γ are deterministic and therefore $\dot{\Phi}$ and $\dot{\Gamma} = 0$. By applying the necessary condition derived before,we obtain:

$$K_i = -(H + \Gamma^* P_{i+1}^r \Gamma)^{-1} (\Gamma^* P_{i+1}^r \Phi)$$

where

$$P_i^r = \Phi^* P_{i+1}^r \Phi + Q$$

$$-\Phi^* P_{i+1}^r \Gamma (H + \Gamma^* P_{i+1}^r \Gamma)^{-1} \Gamma^* P_{i+1}^r \Phi$$

Not surprisingly,the above equation are consistent with the known result. Furthermore it is not only necessary but also the sufficient condition for optimal control. The control is optimal in the nonlinear structure as well.

The calculation involved in the N-horizon problem is extremely huge. Furthermore,the total system derived is a time-varying one. Since our particular interest is in obtaining a stationary total system,we will consider the infinite horizon problem in the next section.

4.3 Infinite horizon control problem

To formulate the infinite horizon control problem,the performance index has to be changed as follows:

$$J = \lim_{N\to\infty} \frac{1}{N} E\{\sum_{1}^{N} (x_k^* Q x_k + u_k^* H u_k)\} \tag{140}$$

The control structure is assumed to be a linear time-invariant one, with one more restriction that the total system is m. s. stable. Assuming m. s. stability of the total system,all the second order covariance matrices converge and so the steady-state Kalman filter exists. Therefore, we restrict the control structure to be a steady-state Kalman filter in series with a gain matrix. The dynamics of the compensator is then written as follows:

$$\hat{x}_{k+1} = \bar{\Phi}\hat{x}_k + \bar{\Gamma}u_k + A[v_{k+1} - C\bar{\Phi}\hat{x}_k - C\bar{\Gamma}u_k] \tag{141}$$

$$u_k = K\hat{x}_k \tag{142}$$

$$\text{where } A = PC^*(GG^*)^{-1} \tag{143}$$

Since we assume the total system is m. s. stable,we have:

$$P_k \to P,\ B_k \to B$$

$$W_k \to W,\ A_k \to A$$

and P, B and W satisfies the SSRE (see equations 93 94 and 95 of chapter 3):

$$P = (I + BC^*C)^{-1}B \tag{144}$$

$$B = \overline{\Phi P \Phi^*} + \overline{\bar{\Psi} W \bar{\Psi}^*} + \overline{FF^*} \tag{145}$$

$$W = \bar{\Psi} W \bar{\Psi}^* + B - P \tag{146}$$

$$R = P + W \tag{147}$$

The performance index is equal to:

$$\begin{aligned} J &= \lim_{N\to\infty} \frac{1}{N} E\{\sum_{1}^{N}([Q, R_k] + [K^* H K, W_k])\} \\ &= [Q, R] + [K^* H K, W] \end{aligned} \tag{148}$$

The control problem is then minimize J with respect to $[R, K, W]$ under constraint equations 144 to 147. So the stochastic control problem has been transformed to a simple minimization problem. It is obviously that the optimal gain matrix K exists provided $\{t_i\}$ process is deterministic. In that case,from the LQG theory the total system is stable. In the present case,$\{t_i\}$ is assumed to be i. i. d. If the solution exist,we expect that the total system will be m. s. stable. Since equations 144 to 147 is actually a *generalized Liyapunov equation*,the total system will be m. s. stable if some form of m. s. observability conditions hold.

In Koning's paper [13],the following lemma regarding the generalized Liyapunov equation is given.

Lemma 1: Consider the transformation $\mathcal{A} : S^n \to S^n$ defined by:

$$\mathcal{A}X = \overline{A^* X A}, A \text{ random} \tag{149}$$

and the equation

$$X = \mathcal{A}X + \bar{B}, B \text{ random}, B \geq 0 \tag{150}$$

then $\exists$ solution $X \geq 0$,$(A, B^{1/2})$ m. s. observable $\Longrightarrow \mathcal{A}$ stable,$X > 0$.

Note: If $B > 0$ the above result will hold.

Using the above lemma,we want to prove that the total system will be m. s. stable provided there is a solution to the above control problem and $\overline{FF^*} > 0$ and $C^*C > 0$ — the simplest case of m. s. observability conditions. It is easy to show that the SSRE is just a version of generalized Liyapunov's equation. Let's define:

$$e_{k+1} = x_{k+1} - \hat{x}_{k+1} \tag{151}$$

Then using equation 73 of chapter 3,we can show:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = A_{xe} \begin{bmatrix} x_k \\ e_k \end{bmatrix} + F_{xe} \tag{152}$$

where($K_k = K$ by assumption):

$$A_{xe} = \begin{bmatrix} \Psi_k & -\Gamma_k K \\ J_{k+1}\hat{\Psi}_k & -J_{k+1}\Gamma_k K + J_{k+1}\bar{\Psi} \end{bmatrix} \tag{153}$$

$$F_{xe} = \begin{bmatrix} F_k \eta_k \\ J_{k+1}F_k\eta_k - A_{k+1}G\theta_{k+1} \end{bmatrix} \tag{154}$$

Let's define:

$$X_k = \begin{bmatrix} R_k & P_k \\ P_k & P_k \end{bmatrix} \tag{155}$$

then we have:

$$X_{k+1} = \overline{A_{xe}X_kA_{xe}^*} + \overline{F_{xe}F_{xe}^*} \tag{156}$$

Letting $k \rightarrow \infty$,we get the SSRE:

$$X = \overline{A_{xe}XA_{xe}^*} + \overline{F_{xe}F_{xe}^*} \tag{157}$$

where

$$\overline{F_{xe}F_{xe}^*} = \begin{bmatrix} \overline{FF^*} & \overline{FF^*}J^* \\ J\overline{FF^*} & J\overline{FF^*}J^* + AGG^*A^* \end{bmatrix} \tag{158}$$

$$A = PC^*(GG^*)^{-1}, J = I - AC \tag{159}$$

First,let's give the following two lemma:

Lemma 2: If $R \geq 0, P \geq 0$ and $R - P \geq 0$,then $X = \begin{bmatrix} R & P \\ P & P \end{bmatrix} \geq 0$.

proof:

$$\forall x,y: \begin{bmatrix} x \\ y \end{bmatrix}^* X \begin{bmatrix} x \\ y \end{bmatrix} = x^*Rx + y^*Py + x^*Py + y^*Px$$

$$= x^*(R-P)x + x^*Px + y^*Px + x^*Py + y^*Py$$

$$= x^*(R-P)x + (x+y)^*P(x+y) \geq 0 \quad \square$$

Lemma 3: If $C^*C > 0$ and $\overline{FF^*} > 0 \Longrightarrow \overline{F_{xe}F_{xe}^*} > 0$.

proof: Obviously $\overline{F_{xe}F_{xe}^*} \geq 0$. Furthermore;

$$\forall x,y: \begin{bmatrix} x \\ y \end{bmatrix}^* \overline{F_{xe}F_{xe}^*} \begin{bmatrix} x \\ y \end{bmatrix} = x^*\overline{FF^*}x + x^*\overline{FF^*}J^*y$$

$$+y^*J\overline{FF^*}x + y^*AGG^*A^*y + y^*J\overline{FF^*}J^*y$$

$$= (x+J^*y)^*\overline{FF^*}(x+J^*y) + y^*AGG^*A^*y = 0$$

$$\Longrightarrow A^*y = 0 \quad \text{and} \quad (x+J^*y) = 0 \tag{160}$$

$$\Longrightarrow (CBC^* + GG^*)^{-1}CBy = 0, \text{using eq. } 3-48$$

$$\Longrightarrow CBy = 0 \quad or \quad C^*CBy = 0$$

$$\Longrightarrow By = 0 \Longrightarrow y = 0, \text{since } B > 0 \text{ using eq. } 145$$

$$\Longrightarrow x = 0, \text{using eq. } 160$$

$$\Longrightarrow \overline{F_{xe}F_{xe}^*} > 0 \quad \square$$

Now,we are able to prove the following theorem.

Theorem 1: If the control problem has a solution,i. e. ,SSRE has solution ($R \geq 0, P \geq 0$ and $R - P \geq 0$) and $\exists K$ minimizes eq. 148, then the total system is m. s. stable (i. e. A_{xe} is m. s. stable) provided $C^*C > 0$ and $\overline{FF^*} > 0$.

proof: If $\exists$ solution ($R \geq 0, P \geq 0$ and $R - P \geq 0$) of SSRE, then $X \geq 0$ by lemma 2. Furthermore, $\overline{F_{x\epsilon}F_{x\epsilon}^*} > 0$ provided $C^*C > 0$ and $\overline{FF^*} > 0$ by lemma 3. Therefore, $A_{x\epsilon}$ is m. s. stable by lemma 1 $\square$.

Note:The assumption $\overline{FF^*} > 0$ is not unrealistic. If $(A, \tilde{F})$ is completely controllable and $t_i > 0$ with probability one,then we have:

$$\overline{FF^*} = E\left\{\int_0^{t_{i+1}} e^{At}\tilde{F}\tilde{F}^* e^{A^*t}dt\right\} > 0$$

However, the assumption $C^*C > 0$ is too restricted unless in one dimensional case. We hope it can be relaxed in the future. It is suspected that m. s. observability of (Φ, C) and (Φ^*, F^*) will be enough.

In the deterministic case,the control problem has a solution which is equivalent to the LQG control as the following example shows.

Special case: $\sigma(t_k) = 0$

For simplicity reason,consider the one dimensional case. We are facing the following minimization problem:

$$B = \Phi^2 P + F^2$$

$$P = (1 + BC^2)^{-1}B$$

$$W = \Psi^2 W + B - P, \Psi = \Phi + \Gamma K$$

$$J_\infty = QP + (Q + HK^2)W$$

$$= QP + (Q + HK^2)(1 - \Psi^2)^{-1}(B - P)$$

$$\min J_\infty \text{ w. r. t. } (P, W \text{and} K) \tag{161}$$

Obviously, B and P are independent of K. That is why the separation principle holds in this case. Setting $\frac{\partial J_\infty}{\partial K} = 0$,we have:

$$[HK + (Q + HK^2)(1 - \Psi^2)^{-1}\Psi\Gamma](1 - \Psi^2)^{-1}(B - P) = 0$$

$$\Longrightarrow -H\Gamma\Phi K^2 + (H - H\Phi + Q\Gamma^2)K + Q\Phi\Gamma = 0 \tag{162}$$

Let's define:

$$K = -(\Gamma^2 P_c + H)^{-1}\Gamma P_c \Phi \tag{163}$$

Substituting the above expression into equation 161,we have:

$$P_c = \Phi^2 P_c - \Phi^2\Gamma^2 P_c^2(\Gamma^2 P_c + H)^{-1} + Q \tag{164}$$

which is exactly the associated SSRE for the LQG problem as expected.

Assuming existence of optimal control,i.e.,assuming that the minimization problem formulated before has a solution,then we can derive a set of necessary conditions similar to those derived for the N-horizon optimal control problem. Applying the Lagrange's multiplier method as in section 4.2,the following necessary conditions is given.

Theorem 2: Assuming existence,the necessary conditions for the optimal control for the infinite-horizon control problem stated before is that there exist auxiliary covariance matrices $-\alpha$ and $-\delta$ such that the following equations hold.

$$K = [H - \overline{\tilde{\Gamma}^*\alpha\tilde{\Gamma}} - \overline{\Gamma^*\gamma\Gamma}]^{-1}[\overline{\tilde{\Gamma}^*\alpha\tilde{\Phi}} + \overline{\Gamma^*\gamma\Phi}] \tag{165}$$

$$Z = [H - \overline{\tilde{\Gamma}^*\alpha\tilde{\Gamma}} - \overline{\Gamma^*\gamma\Gamma}] \tag{166}$$

$$-\alpha = (I - AC)^*K^*ZK(I - AC) - (I - AC)^*\bar{\Phi}^*\alpha\bar{\Phi}(I - AC) \tag{167}$$

$$\begin{aligned} -\gamma &= -\overline{\Phi^*\gamma\Phi} - \overline{\tilde{\Phi}^*\alpha\tilde{\Phi}} + Q \\ &\quad -K^*(H - \overline{\tilde{\Gamma}^*\alpha\tilde{\Gamma}} - \overline{\Gamma^*\gamma\Gamma})K \end{aligned} \tag{168}$$

The above equations in terms of $-\alpha$ and $-\gamma$ are one variations of the familiar steady state Riccati type equations. Which together with equations 144 through 147 are essential to the solution of optimal control problem. Unfortunately,they are highly nonlinear as well as coupled set of equations. It is noticed that they are coupled through $\{ K \}$ and $\{ A \}$ matrices,the optimal control gain and optimal estimator gain respectively. All the difficulty in analyzing and solving the underlying optimal control problem comes from the coupling phenomenon,which in the case of deterministic uniform sampling problem the famous *separation principle* (instead of *coupled* effect) can be used to simplify the optimal control problem into separate control and estimation problems.

From the above discussion, it is obvious that one way to obtain the optimal gain matrices $\{ K \}$ and $\{ A \}$ is trying to find the solutions of the above mentioned nonlinear set of coupled Riccati type equations (144 through 147 ,165 through 168). There are many methods in the literature to solve the nonlinear systems of equation,e.g., the secant method ,the Newton-Ralphson method etc.

The alternate way is to solve the minimization problem directly. This way is preferrable because there are many efficient general techniques for finding the minimum of a function of many variables,e.g.,the simplex method,the steepest descent method,quadratic programming method,reduced gradient method to name just a few.

In the case where the sampling scheme is deterministic ,we know that both Ψ and $J\Phi$ are stable. It is interesting to know whether Ψ and $J\Phi$ are mean square stable. The result is given in the following theorem.

Theorem 3: If the optimal control problem has a solution,i.e., SSRE has a solution ($R \geq 0, P \geq 0, R - P \geq 0$) and $\exists$ K minimizing equation 148,then Ψ and $J\Phi$ are mean square stable provided $Q > 0$ and $\overline{FF^*} > 0$.

proof: First notice J defined as follows is nonsingular.

$$J = I - AC = I - PC^*C = (I + BC^*C)^{-1} \tag{169}$$

From equation 144,we have:

$$\begin{aligned} P &= (I - AC)B(I - AC)^* + (I - AC)BC^*A^* \\ &= (I - AC)B(I - AC)^* + PC^*CP \\ &= JBJ^* + PC^*CP \end{aligned} \tag{170}$$

Therefore from equation 145,we obtain:

$$P = J\overline{\Phi P \Phi^*}J^* + J\overline{\tilde{\Psi} W \tilde{\Psi}^*}J^* + J\overline{FF^*}J^* + PC^*CP \tag{171}$$

The latter three terms are strictly positive definite provided $J\overline{FF^*}J^*$ is.(which is true since J is nonsingular.) By lemma 1,we know that $J\Phi$ is mean square stable. Since we assume that the minimization has a solution,by applying theorem 2 we know there exist $-\gamma$ and $-\alpha$ covariance matrices (both of them positive semidefinite) satisfying equations 165 through 168. After some simple computation,we have:

$$-\gamma = -\overline{\Psi^* \gamma \Psi} - \overline{\tilde{\Psi}^* \alpha \tilde{\Psi}} + Q + K^* H K \tag{172}$$

Again the latter three terms are strictly positive definite provided Q is. By applying lemma 1 again ,we know that Ψ is mean square stable. □

Note: It is easy to prove that mean square stability imply mean stability. Therefore we also prove that Ψ and $J\bar{\Phi}$ are mean stable.

The generalized Lyapunov equation given in equation 157 is essential in the study of mean square stability of the total system. If (A^*_{xe} ,F^*_{xe}) (see equations 153 and 154) is mean square observable,i.e.,

$$\sum_0^\infty \mathcal{A}^i_{xe} \overline{F_{xe} F^*_{xe}} > 0 \tag{173}$$

where $\mathcal{A}_{xe}$ is defined as:

$$\forall\ X \in R^{2n \times 2n} : \mathcal{A}_{xe} X = \overline{A_{xe} X A^*_{xe}}$$

then A_{xe} is mean square stable by lemma 1. However,under what conditions is (A^*_{xe} ,F^*_{xe}) mean square stable ? Recall the dynamic equation for the one step predictor.

$$\bar{x}_{k+1} = \bar{\Psi} \hat{x}_k$$

$$
\begin{aligned}
&= \bar{\Psi}(\bar{x}_k + A\breve{v}_k) \\
&= \bar{\Psi}\bar{x}_k + \bar{\Psi}A(v_k - \bar{v}_k) \\
&= \bar{\Psi}(I - AC)\bar{x}_k + \bar{\Psi}Av_k \\
&= \bar{\Psi}J\bar{x}_k + \bar{\Psi}Av_k \qquad (174) \\
\hat{x}_{k+1} &= J\bar{\Psi}\hat{x}_k + Av_{k+1} \qquad (175)
\end{aligned}
$$

The above is the optimal linear filter. It is interesting to know whether $(\bar{\Psi}J,\bar{\Psi}A)$ or $(J\bar{\Psi},A)$ are completely controllable. The following lemma is given.

Lemma 2: Assuming $(\bar{\Psi})^{-1}$ exists then the following statements are equivalent.

1. $(\bar{\Psi}J,\bar{\Psi}A)$ is completely controllable.
2. $(J\bar{\Psi},A)$ is completely controllable.
3. $(\bar{\Psi},A)$ is completely controllable.
4. $(\bar{\Psi},\bar{\Psi}A)$ is completely controllable.

proof:(1) $\Longleftrightarrow$ (2) Notice the following identities.

$$(\bar{\Psi})^{-1}[\bar{\Psi}J]\bar{\Psi} = J\bar{\Psi} \quad , \quad (\bar{\Psi})^{-1}[\bar{\Psi}A] = A$$

Therefore $(\bar{\Psi}J,\bar{\Psi}A)$ and $(J\bar{\Psi},A)$ are *similar* state space representation of the same system and hence the proof is completed.

(3) $\Longleftrightarrow$ (4): Follow the same argument as before,we have

$$(\bar{\Psi})^{-1}[\bar{\Psi}]\bar{\Psi} = \bar{\Psi} \quad , \quad (\bar{\Psi})^{-1}[\bar{\Psi}A] = A$$

and hence the proof is completed.

(1) $\Longrightarrow$ (4): Assume

$$\forall\, i \;\geq\; 1 :\; A^{*}(\bar{\Psi})^{*i}x \;=\; 0$$

$$\Longrightarrow\; A^{*}(\bar{\Psi})^{*}x \;=\; 0$$

$$\Longrightarrow\; (\bar{\Psi}A)^{*}(\bar{\Psi}J)^{*}x \;=\; A^{*}\bar{\Psi}^{*}J^{*}\bar{\Psi}^{*}x$$

$$=\; A^{*}\bar{\Psi}^{*}(I - C^{*}A^{*})\bar{\Psi}^{*}x$$

$$=\; A^{*}\bar{\Psi}^{*2}x \;=\; 0$$

In general,by induction we have:

$$(\bar{\Psi}A)^{*}(\bar{\Psi}J)^{*i}x \;=\; 0, \forall\, i \;\geq\; 0$$

$$\Longrightarrow\; x \;=\; 0$$

$\Longrightarrow$ ($\bar{\Psi}$,$\bar{\Psi}A$) is completely controllable.

(4) $\Longrightarrow$ (1): Assume (4) is true.

$$\forall\, i \;\geq\; 0 :\; (\bar{\Psi}A)^{*}(\bar{\Psi}J)^{*i}x \;=\; 0$$

$$\Longrightarrow\; A^{*}\bar{\Psi}^{*}x \;=\; 0$$

$$\Longrightarrow\; A^{*}\bar{\Psi}^{*}J^{*}\bar{\Psi}^{*}x \;=\; 0$$

$$\Longrightarrow\; A^{*}\bar{\Psi}^{*}(I - C^{*}A^{*})\bar{\Psi}^{*}x \;=\; 0$$

$$\Longrightarrow\; A^{*}\,\bar{\Psi}^{*2}x \;=\; 0$$

By induction,we have:

$$\forall\, i \;\geq\; 1 :\; A^{*}(\bar{\Psi})^{*i}x \;=\; 0$$

$$\Longrightarrow\; x \;=\; 0$$

$\Longrightarrow (\bar{\Psi}J,\bar{\Psi}A)$ is completely controllable. □.

Theorem 4: (A^*_{xe},F^*_{xe}) is mean square observable if and only if $(\bar{\Psi},A)$ is completely controllable.

proof: (sufficiency) Assume

$$[x^* \; y^*]\mathcal{A}^i_{xe}(\overline{F_{xe}F^*_{xe}}) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \; \forall \, i \geq 0$$

where $\mathcal{A}_{xe}(X) = \overline{A_{xe}XA^*_{xe}}$.

$$\Longrightarrow i = 0, [x \; y]^* \overline{F_{xe}F^*_{xe}} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Longrightarrow A^*y = 0, \text{ and } (x + J^*y) = 0 (\text{by equation160})$$

$$\Longrightarrow J^*y = (I - C^*A^*)y = y$$

$$\Longrightarrow x + y = 0$$

$$\Longrightarrow y = -x, \; A^*x = 0, \; J^*x = x \tag{176}$$

Next,consider the case $i = 1$:

$$[x \; -x]^*[\mathcal{A}_{xe}(\overline{F_{xe}F^*_{xe}})] \begin{bmatrix} x \\ -x \end{bmatrix} = 0$$

$$= [x \; -x]^* A_{xe} \overline{F_{xe}F^*_{xe}} \begin{bmatrix} \Psi^* & \tilde{\Psi}^*J^* \\ -K^*\Gamma^* & -K^*\Gamma^*J^* + \bar{\Psi}^*J^* \end{bmatrix} \begin{bmatrix} x \\ -x \end{bmatrix} \tag{177}$$

The latter two terms are equal to:

$$\begin{bmatrix} \Psi^*x - \tilde{\Psi}^*J^*x \\ -K^*\Gamma^*x + K^*\Gamma^*J^*x - \bar{\Psi}^*J^*x \end{bmatrix} = \begin{bmatrix} \bar{\Psi}^*x \\ -\bar{\Psi}^*x \end{bmatrix}$$

by using equation 176.

Therefore,equation 177 imply the following:

$$\begin{bmatrix} \bar{\Psi}^* x \\ -\bar{\Psi}^* x \end{bmatrix}^* \overline{F_{xe} F_{xe}^*} \begin{bmatrix} \bar{\Psi}^* x \\ -\bar{\Psi}^* x \end{bmatrix} = 0$$

By the same argument as before,we can prove:

$$A^* \bar{\Psi}^* x = 0, \text{ and } J^* \bar{\Psi}^* x = \bar{\Psi}^* x \tag{178}$$

By induction,we can prove easily the following:

$$A^* \bar{\Psi}^{*i} x = 0, \forall\, i \;\geq\; 0$$

$$\Longrightarrow\; x = 0, \text{ and } y = 0$$

Since $(\bar{\Psi}, A)$ is completely controllable by assumption. The proof is therefore completed.

(necessity): Assume that (A_{xe}^*, F_{xe}^*) is mean square observable.

$$A^* \bar{\Psi}^{*i} x = 0, \forall\, i \;\geq\; 0$$

$$\Longrightarrow\; J^* \bar{\Psi}^{*i} = (I - C^* A^*) \bar{\Psi}^{*i} x = \bar{\Psi}^{*i} x$$

Following the same argument as before,we can prove:

$$\begin{bmatrix} x \\ -x \end{bmatrix}^* \mathcal{A}_{xe}^i \overline{F_{xe} F_{xe}^*} \begin{bmatrix} x \\ -x \end{bmatrix} \forall\, i \;\geq\; 0$$

$$= \begin{bmatrix} \bar{\Psi}^{*i} x \\ -\bar{\Psi}^{*i} x \end{bmatrix}^* \overline{F_{xe} F_{xe}^*} \begin{bmatrix} \bar{\Psi}^{*i} x \\ -\bar{\Psi}^{*i} x \end{bmatrix} = 0$$

$$\Longrightarrow \begin{bmatrix} x \\ -x \end{bmatrix} = 0, \text{ i.e., } x = 0$$

$\Longrightarrow$ The proof is complete. □

Note: From the theorem lemma 1 and lemma 2 ,it is obvious that the total system will be mean square stable provided the optimal linear estimator is completely controllable.

Corollary 1: Assume existence of optimal control $u_k = K\hat{x}_k$, the total system is mean square stable provided the dynamics of the estimator $(\bar{\Psi}J,\bar{\Psi}A)$ is completely controllable.

Note: Once the mimimization problem is solved ,the above corollary can be used to test whether the total system is mean square stable. However ,it is only a sufficient condition for m. s. stability.

Corollary 2: (Ψ,A) is completely controllable if both C^*C and $\overline{FF^*}$ are positive definite.

proof: We know $P > 0$ if $\overline{FF^*} > 0$ from equations 144 through 147. Furthermore,$A = PC^*C$ is full rank provided C^*C is. □

The above is a trivial case when the linear estimator is completely controllable,and hence the total system is mean square stable as is proved in theorem 1. In general,it is difficult to prove under what conditions the linear optimal estimator will be completely controllable even in the case where the sampling process is a deterministic uniform one.

Recall in the case of conventional LQG problem,in order for the total system to be stable it is enough to require the estimator to be *detectable*. Therefore to

get better result,somehow we need to introduce similar notion such as *mean square detectability.* This notion is first introduced by De Koning.(see [10] and [14])

Consider the following system:

$$x_{i+1} = \Phi_i x_i \tag{179}$$

$$v_i = C_i x_i \tag{180}$$

where $\{ \Phi_i \}$ $\{ C_i \}$ are sequence of independent random matrices with constant statistics.

Definition: (Φ_i, C_i) is called *mean detectable* if

$$\overline{v_i} = 0, \forall\, i \geq 0 \Longrightarrow \overline{x_i} \to 0$$

and *mean square detectable* if

$$\overline{\| v_i \|^2} = 0 \forall\, i \implies \overline{\| x_i \|^2} \to 0$$

It is easy to see that mean detectability means that unobservable (mean) modes are mean stable,and that the modes which are not mean square observable are mean square stable. The following lemma are proved by De Koning.

Lemma 3: (Φ, C) is mean square detectable if and only if

$$x_0^* \mathcal{A}_\Phi^i \overline{C^* C} x_0 = 0 \;\forall\, i \implies x_0^* \{\mathcal{A}_\Phi^i I\} x_0 \to 0$$

where $\mathcal{A}_\Phi$ is defined as:

$$\mathcal{A}_\Phi : S^n \to S^n : \; \mathcal{A}_\Phi X = \overline{\Phi^* X \Phi}$$

Lemma 4: Consider the transformation $\mathcal{A} : S^n \to S^n$ defined by

$$\mathcal{A}X = \overline{A^* X A},\ A \text{ random}$$

and the equation

$$X = \mathcal{A}X + B, B \text{ random }, B \geq 0$$

Then there exists a solution $X \geq 0$,and $(A,B^{1/2})$ mean square detectable $\Longrightarrow$ $\mathcal{A}$ stable.(i.e., A mean square stable)

Applying the above lemma,the total system will be mean square stable provided (A^*_{xe},F^*_{xe}) is mean square observable. First, the following theorem is given.

Theorem 5: If there exist $R \geq 0$ $P \geq 0$ and $R-P \geq 0$ satisfying the SSRE (equations 144 through 147) ,then (A^*_{xe},F^*_{xe}) is mean square detectable provided $\bar{\Psi}$ is stable and $\overline{FF^*} > 0$.

proof: define the linear monotonic transformation $\mathcal{A}$ as:

$$\mathcal{A}X = \overline{A_{xe}XA^*_{xe}}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^* \mathcal{A}^i(\overline{F_{xe}F^*_{xe}}) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \; \forall \; i \geq 0$$

$$\Longrightarrow \begin{bmatrix} x \\ y \end{bmatrix}^* \sum_{i=0}^{\infty} \mathcal{A}^i(\overline{F_{xe}F^*_{xe}}) \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Longrightarrow \begin{bmatrix} x \\ y \end{bmatrix}^* \begin{bmatrix} R & P \\ P & P \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Longrightarrow x^*(R-P)x + (x+y)^*P(x+y) = 0$$

$$\Longrightarrow x^*Wx + (x+y)^*P(x+y) = 0$$

$$\Longrightarrow Wx = 0, \; \& \; x+y = 0$$

because $P > 0$ provided $\overline{FF^*} > 0$. Since

$$W = \bar{\Psi}W\bar{\Psi}^* + B - P$$

We have

$$x^*Wx = x^*\bar{\Psi}W\bar{\Psi}^*x + x^*(B-P)x = 0$$

$$\Longrightarrow W\bar{\Psi}^*x = 0, \text{ and } (B-P)x = 0$$

Furthermore,we know:

$$B - P = B - (I + BC^*C)^{-1}B$$

$$= B - (I - PC^*C)B$$

$$= PC^*CB = BC^*CP$$

$$= BC^*A^*$$

$$\Longrightarrow BC^*A^*x = 0$$

$$\Longrightarrow C^*A^*x = 0$$

$$\Longrightarrow J^*x = (I - C^*A^*)x = x$$

In general,we have:

$$x \in \mathcal{N}(W) \Rightarrow \bar{\Psi}^*x \in \mathcal{N}(W) \text{ and } x \in \mathcal{N}(J^* - I)$$

Therefore, we have by induction:

$$\forall\, i : \bar{\Psi}^{*i}x \in \mathcal{N}(W) \text{ and } \bar{\Psi}^{*i}x \in \mathcal{N}(I - J^*)$$

$$\text{i.e. } \forall\, i : W\bar{\Psi}^{*i}x = 0 \text{ and } J^*\bar{\Psi}^{*i}x = \bar{\Psi}^{*i}x \tag{181}$$

Next,we want to prove:

$$\begin{bmatrix} x \\ -x \end{bmatrix}^* (\mathcal{A}^i I) \begin{bmatrix} x \\ -x \end{bmatrix} \longrightarrow 0 \text{ as } i \to \infty$$

$$\begin{bmatrix} x \\ -x \end{bmatrix}^* (\mathcal{A} I) \begin{bmatrix} x \\ -x \end{bmatrix}$$

$$= \begin{bmatrix} x \\ -x \end{bmatrix}^* \overline{A_{xe} \begin{bmatrix} \Psi^* & \tilde{\Psi}^* J^* \\ -K^*\Gamma^* & -K^*\Gamma^* J^* + \bar{\Psi}^* J^* \end{bmatrix}} \begin{bmatrix} x \\ -x \end{bmatrix}$$

$$= \begin{bmatrix} x \\ -x \end{bmatrix}^* \overline{A_{xe} \begin{bmatrix} \Psi^* x - \tilde{\Psi}^* J^* x \\ -K^*\Gamma^* x + K^*\Gamma^* J^* x - \bar{\Psi}^* J^* x \end{bmatrix}}$$

$$= \begin{bmatrix} \bar{\Psi}^* x \\ -\bar{\Psi}^* x \end{bmatrix}^* \begin{bmatrix} \bar{\Psi}^* x \\ -\bar{\Psi}^* x \end{bmatrix}$$
$$= 2x^* \bar{\Psi} \bar{\Psi}^* x$$

In general,we have by induction:

$$\begin{bmatrix} x \\ -x \end{bmatrix}^* (\mathcal{A}^i I) \begin{bmatrix} x \\ -x \end{bmatrix} = 2x^* \bar{\Psi}^i \bar{\Psi}^{*i} x \longrightarrow 0$$

The above is true provided $\bar{\Psi}$ is stable. □

Theorem 6: If the optimal control problem has a solution,i.e., SSRE has solution ($R \geq 0$, $P \geq 0$,and $R - P \geq 0$) and $\exists$ K minimize equation 148,then the total system is mean square stable provided $\overline{FF^*} > 0$ and $Q > 0$.

proof: From theorem 4,we know that Ψ is mean square stable and therefore is mean stable. (i.e.,Ψ is stable) From theorem 5, we know that ($A_{x\epsilon}^*$,$F_{x\epsilon}^*$) is mean square detectable and therefore $A_{x\epsilon}$ is mean square stable by lemma 4. □

From the above analysis,we know that the total system is mean square stable without requiring C^*C to be positive definite as long as Q and $\overline{FF^*}$ are positive definite. This is not surprising because it is the case in the coventional LQG problem (or when the sampling process is deterministic). In those two cases,the total system is stable provided (Φ,F) and (Φ^*,Q^*) are completely controllable.(or more loosely (Φ,F) is stabilizable and (Φ,Q) detectable) Moreover,the observability of (Φ,C) and stabilizability of (Φ,Γ) are only in connection with the existence of solutions of two famous isolated Riccati equation. (see [6]) In our problem,there is reason to believe some consistency will appear as the above theorem shows.

Chapter 5

Examples and simulation results:

In this chapter,the simulation results for randomly sampled data control is presented. Although,our techniques for solving the optimal control problem as described in chapter 5 does not limited to control of descrete plant obtained this way. It applies equally well to discrete plant with random parameter such as those encountered in chemical processes. However, my particular interest is in randomly sampled data control system design. The simulation of one dimensional system is presented thoroughly in section 5-1. In one dimensional case,the optimal control exists under proper mean square stabilizability and observability conditions of (Φ,Γ) and (Φ,c) respectively. Ferthermore, the total system is mean square stable (also stable w.p.1) provided $\overline{FF^*} > 0$ and $q > 0$. The simulation of one dimensional system turns out to be very successful. It validate all the results of previous chapters. The optimally controlled system perform better as compared to the system with certainty equivalent control. The main reason is that the optimal system can guarantee the mean square stability of the total system whereas the latter system can only achieve mean stability.

5.1 One Dimensional System Simulation

In this section ,we are dealing with the optimal control problem for the general one dimensional system. The continuous time plant with randomly sampled data control is described as the following one dimensional differential equation.

$$\dot{x}(t) = a_c x(t) + b_c u(t) + f_c n(t) \tag{182}$$

$$v_k = cx(\sigma_k) + g\theta_k \tag{183}$$

where all the variables are one dimensional scalars. $n(t)$ is the continuous time white Gaussian state noise. $\{\ \theta_k\ \}$ is the discrete time white Gaussian observation noise. $\{\ \sigma_k\ \}$ is the sampling time process which is modeled as a stochastic process with independent increment. The performance index is given as follows:

$$\hat{J} = \lim_{\sigma_N \to \infty} \frac{1}{\sigma_N - \sigma_0} E \int_{\sigma_0}^{\sigma_N} \{q_c x(t)^2 + h_c u(t)^2\} dt \tag{184}$$

where we assume the intersample process $t_i = \sigma_i - \sigma_{i-1}$ is strictly positive and therefore $\sigma_N \to \infty$ iff $N \to \infty$. Furthermore, we assume that in our simulation $\{\ t_i\ \}$ process is independent indentically distributed and t_i is uniformly distributed in the interval $[\Delta_1,\Delta_2]$,i.e.,

$$0 < \Delta_1 \ \leq \ t_i \ \leq \ \Delta_2 \ < \infty,\ i = 0, 1, \cdots$$

$$p_{t_i}(t_i) = \frac{1}{\Delta_2 - \Delta_1},\ t_i \in\ [\Delta_1, \Delta_2]$$

and therefore

$$E[t_i] = \frac{\Delta_1 + \Delta_2}{2} \tag{185}$$

$$E[(t_i - \bar{t}_i)^2] = \frac{(\Delta_2 - \Delta_1)^2}{12} \tag{186}$$

Let x_i and u_i denote respectively $x(\sigma_i)$ and $u(\sigma_i)$. We may transform system 182 and 183, given sampling process $\{ t_i \}$, to the equivalent discrete time system as described in chapter 2.

$$x_{k+1} = \Phi_k x_k + \Gamma_k u_k + F_k \eta_k \tag{187}$$

$$v_k = c x_k + g\theta_k \tag{188}$$

where $\{ \Phi_k \}$, $\{ \Gamma_k \}$ and $\{ F_k \eta_k \}$ are sequences of independent random variables. Moreover, Φ_j and Γ_j are independent of $F_k \eta_k, j \neq k$, and uncorrelated with $F_j \eta_j$. Let's compute Φ_k,Γ_k and F_k random variables and their associated statistics.

$$\Phi_k = e^{a_c t_k} \tag{189}$$

$$\Gamma_k = \int_0^{t_k} e^{a_c \delta} b_c d\delta = \frac{b_c}{a_c}(e^{a_c t_k} - 1) \tag{190}$$

$$F_k^2 = \int_0^{t_k} e^{2a_c t} f_c^2 dt = \frac{f_c^2}{2a_c}(e^{2a_c t_k} - 1) \tag{191}$$

Assume t_k is uniformly distributed in $[\Delta_1, \Delta_2]$, we can compute their first and second order moments as follows:

$$\bar{\Phi} = E[e^{a_c t_k}] = \int_{\Delta_1}^{\Delta_2} e^{a_c t} \frac{1}{\Delta_2 - \Delta_1} dt$$

$$= \frac{e^{a_c \Delta_2} - e^{a_c \Delta_1}}{(\Delta_2 - \Delta_1) a_c} \stackrel{\text{def}}{=} \Pi(a_c; \Delta_2, \Delta_1) \tag{192}$$

In the sequel,we will simply write $\Pi(a_c)$ for simplicity. Following the same procedure as above,we obtain:

$$\bar{\Gamma} = \frac{b_c}{a_c}[\Pi(a_c) - 1] \tag{193}$$

$$\overline{\Phi^2} = \Pi(2a_c) \tag{194}$$

$$\overline{\Gamma^2} = \frac{b_c^2}{a_c^2}[\Pi(2a_c) - 2\Pi(a_c) + 1] \tag{195}$$

$$\overline{\Phi\Gamma} = \frac{b_c}{a_c}[\Pi(2a_c) - \Pi(a_c)] \tag{196}$$

$$\overline{\tilde{\Phi}^2} = \overline{(\Phi - \bar{\Phi})^2} = \overline{\Phi^2} - (\bar{\Phi})^2 \tag{197}$$

$$\overline{\tilde{\Phi}\tilde{\Gamma}} = \overline{\Phi\Gamma} - \bar{\Phi}\bar{\Gamma} \tag{198}$$

$$\overline{\tilde{\Gamma}^2} = \overline{\Gamma^2} - (\bar{\Gamma})^2 \tag{199}$$

$$\overline{F^2} = \frac{f_c^2}{2a_c}[\Pi(2a_c) - 1] \tag{200}$$

The performance index 184 can also be transformed into an equivalent long term average sum criterion.

$$J_{eq} = \frac{1}{N}\sum_{k=0}^{N-1} E[\bar{q}x_k^2 + \bar{h}u_k^2 + 2\bar{w}x_k u_k] \tag{201}$$

In order to simplify the computation ,we will omit the cross term $2\bar{w}x_k u_k$ by setting $\bar{w} = 0$. This does not lose any generality since our techniques apply equally well in the case $\bar{w} \neq 0$. In the sequel,we will use the following performance index which is defined directly in terms of discrete time system variables.

$$J = \frac{1}{N}\sum_{k=0}^{N-1} E[qx_k^2 + hu_k^2] \tag{202}$$

where it is assumed that $q > 0$ and $h > 0$. In the following ,we will try to solve the infinite horizon discrete time optimal control problem as described in chapter 4. That is:

> Given system equations 187 and 188,find an admissible control that will minimize the performance index 202.

Admissible means that u_i is a function of $v^i = (v_1, \cdots, v_i)$ and $E[\|x_i^2\|]$ converge as $i \to \infty$ to the same constant value for all x_0. In essence,we try to seek an stationary

optimal control that will make the total system mean square stable. As in chapter 4,in order to get an practical control, we will limit the control structure to be linear and separable. To be exactly,our controller consists of an optimal linear estimator in series with a gain operator. The dynamics of the controller is as follows:(see chapter 4)

$$\hat{x}_{k+1} = \bar{\Phi}\hat{x}_k + \bar{\Gamma}u_k + a(v_{k+1} - c\bar{\Phi}\hat{x}_k - c\bar{\Gamma}u_k) \tag{203}$$

$$u_k = k\hat{x}_k \tag{204}$$

where $a = pc/g^2$. Once the control structure is limited ,the infinite horizon optimal control problem can be transformed into a simple classical minimization problem.

$$J = qr + k^2hw$$

$$= q(p + w) + k^2hw \tag{205}$$

$$p = (1 + bc^2g^{-2})^{-1}b \tag{206}$$

$$b = \overline{\Phi^2}p + \overline{\tilde{\Psi}^2}w + \overline{F^2} \tag{207}$$

$$w = (\bar{\Psi})^2w + b - p \tag{208}$$

where

$$\bar{\Psi} = \bar{\Phi} + \bar{\Gamma}k \tag{209}$$

$$\tilde{\Psi} = \tilde{\Phi} + \tilde{\Gamma}k \tag{210}$$

$$\overline{\tilde{\Psi}^2} = \overline{\tilde{\Phi}^2} + 2\overline{\tilde{\Phi}\tilde{\Gamma}}k + \overline{\tilde{\Gamma}^2}k^2 \tag{211}$$

$$\overline{\Psi^2} = \overline{\Phi^2} + 2\overline{\Phi\Gamma}k + \overline{\Gamma^2}k^2 \tag{212}$$

Our nonlinear optimization problem with equality constraint is then as follows:

$$\min_{b,p,w,k} J(b, p, w, k) = q(p + w) + hk^2w$$

subject to constraint equations 206 and 208.

By applying theorem 4 of chapter 4 ,we know it is necessary that Ψ is mean square stable if there is a solution to the above minimization problem. Since $\Psi = \Phi + \Gamma k$,this place additional constraint on k. Moreover,for the above minimization problem 5.1 to have a applicable solution,we must require S to be nonempty. Where S is defined as:

$$S = \{k \in \ R^1 : \overline{(\Phi + \Gamma k)^2} < 1\}$$

Basically,it is equivalent to require that (Φ,Γ) be mean square stabilizable. Let's give the following fact concerning mean square stabilizability of (Φ, Γ).

fact 1: Assume $\overline{\Gamma^2} \neq 0$, then we have:

(i) (Φ, Γ) is mean square stabilizable if and only if

$$\overline{\Phi^2\Gamma^2} - \overline{\Gamma^2} - (\overline{\Phi\Gamma})^2 < 0$$

(ii) If (Φ,Γ) is mean square stabilizable then

$$S = (-\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}} - \frac{\sqrt{(\overline{\Phi\Gamma})^2 - (\overline{\Phi^2} - 1)\overline{\Gamma^2}}}{\overline{\Gamma^2}}, -\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}} + \frac{\sqrt{(\overline{\Phi\Gamma})^2 - (\overline{\Phi^2} - 1)\overline{\Gamma^2}}}{\overline{\Gamma^2}}) \qquad (213)$$

proof: (Φ ,Γ) being mean square stable:

$$\Longrightarrow \ \exists \ k \text{ such that } \overline{(\Phi + \Gamma k)^2} < 1$$

$$\Longrightarrow \ \overline{\Phi^2} + 2\overline{\Phi\Gamma}k + \overline{\Gamma^2}k^2 < 1$$

$$\Longrightarrow \ k^2 + \frac{2\overline{\Phi\Gamma}}{\overline{\Gamma^2}}k + \frac{\overline{\Phi^2} - 1}{\overline{\Gamma^2}} < 0$$

$$\Longrightarrow \ \left(k + \frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}}\right)^2 + \frac{\overline{\Phi^2} - 1}{\overline{\Gamma^2}} - \left(\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}}\right)^2 < 0 \qquad (214)$$

Therefore,we have to require:

$$\frac{\overline{\Phi^2}-1}{\overline{\Gamma^2}}-\left(\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}}\right)^2<0$$

$$\Longrightarrow (\overline{\Phi^2}-1)\overline{\Gamma^2}-(\overline{\Phi\Gamma})^2<0$$

$$\Longrightarrow \overline{\Phi^2\Gamma^2}-\overline{\Gamma^2}-(\overline{\Phi\Gamma})^2<0$$

Solving equation 214,we obtain:

$$-\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}}-\frac{\sqrt{(\overline{\Phi\Gamma})^2-(\overline{\Phi^2}-1)\overline{\Gamma^2}}}{\overline{\Gamma^2}}<k<-\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}}+\frac{\sqrt{(\overline{\Phi\Gamma})^2-(\overline{\Phi^2}-1)\overline{\Gamma^2}}}{\overline{\Gamma^2}}\square$$

Let's define k_l and k_h as follows:

$$k_l=-\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}}-\frac{\sqrt{(\overline{\Phi\Gamma})^2-(\overline{\Phi^2}-1)\overline{\Gamma^2}}}{\overline{\Gamma^2}} \tag{215}$$

$$k_h=-\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}}+\frac{\sqrt{(\overline{\Phi\Gamma})^2-(\overline{\Phi^2}-1)\overline{\Gamma^2}}}{\overline{\Gamma^2}} \tag{216}$$

fact 2: For one dimensional sampled data control system where the sampling process $\{t_i\}$ is uniformly distributed ,we have:

(i) (Φ_k,Γ_k) is mean square stabilizable. (where Φ_k and Γ_k are defined in equations 189 through 191)

(ii) $S=(k_l,k_h)$ is nonempty and

$$k_l=\min(-\frac{a_c}{b_c},-\frac{a_c}{b_c}\left[\frac{\overline{\Phi^2}-1}{\overline{\Phi^2}-2\bar{\Phi}+1}\right]) \tag{217}$$

$$k_h=\max(-\frac{a_c}{b_c},-\frac{a_c}{b_c}\left[\frac{\overline{\Phi^2}-1}{\overline{\Phi^2}-2\bar{\Phi}+1}\right]) \tag{218}$$

proof: Recall from equations 193 through 196,we can deduce the following results:

$$\bar{\Gamma}=\frac{b_c}{a_c}(\bar{\Phi}-1) \tag{219}$$

$$\overline{\Gamma^2} = \frac{b_c^2}{a_c^2}(\overline{\Phi^2} - 2\bar{\Phi} + 1) \tag{220}$$

$$\overline{\Phi\Gamma} = \frac{b_c}{a_c}(\overline{\Phi^2} - \bar{\Phi}) \tag{221}$$

$$\begin{aligned} \Longrightarrow \quad & (\overline{\Phi\Gamma})^2 - (\overline{\Phi^2} - 1)\overline{\Gamma^2} \\ = \; & \frac{b_c^2}{a_c^2}(\overline{\Phi^2} - \bar{\Phi})^2 - \frac{b_c^2}{a_c^2}(\overline{\Phi^2} - 1)(\overline{\Phi^2} - 2\bar{\Phi} + 1) \\ = \; & \frac{b_c^2}{a_c^2}(\bar{\Phi} - 1)^2 \\ = \; & (\bar{\Gamma})^2 > 0 \end{aligned} \tag{222}$$

By fact 1,we know that (Φ, Γ) is mean square stabilizable. Moreover,we have:

$$\begin{aligned} & -\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}} \pm \frac{\sqrt{(\overline{\Phi\Gamma})^2 - (\overline{\Phi^2} - 1)\overline{\Gamma^2}}}{\overline{\Gamma^2}} \\ & = -\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}} \pm \frac{\sqrt{(\bar{\Gamma})^2}}{\overline{\Gamma^2}} \\ & = -\frac{\overline{\Phi\Gamma}}{\overline{\Gamma^2}} \pm \frac{\mid \bar{\Gamma} \mid}{\overline{\Gamma^2}} \\ & = \frac{-\frac{b_c}{a_c}(\overline{\Phi^2} - \bar{\phi}) \pm \frac{b_c}{a_c}(\bar{\Phi} - 1)}{\frac{b_c^2}{a_c^2}(\overline{\Phi^2} - 2\bar{\Phi} + 1)} \\ & -\frac{a_c}{b_c} \text{ or } -\frac{a_c}{b_c}\left[\frac{\overline{\Phi^2} - 1}{\overline{\Phi^2} - 2\bar{\Phi} + 1}\right] \end{aligned}$$

The second statement is thus proved. □

Note: (1) $\overline{\Phi^2} - 2\Phi + 1 = \overline{\Phi^2} - (\bar{\Phi})^2 + (\bar{\Phi} - 1)^2 \geq 0$. (2) If $\overline{\Psi^2} < 1$,i.e., Ψ is mean square stable,then we have $(\bar{\Psi})^2 < \overline{\Psi^2} < 1$ and therefore Ψ is also mean stable. (3) The nonlinear optimization problem 5.1 require one more constraint to have meaningful solution.

$$k_l < k < k_h$$

where k_l and k_h are given in equation 218. Therefore, our minimization problem becomes:

$$\min_{p,b,w,k} J = q(p+w) + hwk^2$$

subject to the following conditions :

$$\begin{aligned} p &= (1 + bc^2g^{-2})^{-1}b \\ b &= \overline{\Phi^2}p + \overline{\tilde{\Psi}^2}w + \overline{F^2} \\ w &= (\bar{\Psi})^2 w + b - p \\ k_l &< k < k_h \end{aligned} \tag{223}$$

In order to solve the above optimization problem,various methods can be used. First of all ,we can apply the Lagrange's multiplier method and try to solve a system of nonlinear equations. (see theorem 2 of chapter 4) Secondly ,we can use various search methods to solve it numerically. Before doing that,we will simplify the above optimization problem first. Because of the low dimension of the system, it is very easy to express J as a function of k only without any equality constraint. Once this is done, a simple but powerful search method such as the Nelder and Mead simplex method can be used to find the optimal gain k. In the following ,we will try to calculate the analytical expression for J as a function of k only.

Since $k_l < k < k_h$,we have $\overline{\Psi^2} \neq 1$ and $(\bar{\Psi})^2 \neq 1$. Therefore from equation 223 ,we can deduce the following: (like what we did in section 5 of chapter 3)

$$\begin{aligned} w &= [1-(\bar{\Psi})^2]^{-1}(b-p) \\ b &= \rho p + f^2 \qquad (224) \\ p &= (1+bc^2)^{-1}b \qquad (225) \end{aligned}$$

where we have assumed $g^2 = 1$ for simplicity and

$$\begin{aligned} \rho &= (1-\overline{\Psi^2})^{-1}[\overline{\Phi^2}(1-\bar{\Psi}^2) - \overline{\bar{\Psi}^2}] \qquad (226) \\ f^2 &= (1-\overline{\Psi^2})^{-1}(1-\bar{\Psi}^2)\overline{F^2} \qquad (227) \end{aligned}$$

Note: (1) Equations 224 and 225 are the traditional steady state Riccati equation for the Kalman filter ([6]). However, it is not always true that $\rho \geq 0$ in our case. (2) Both ρ and f^2 are function of k only.

Solving equations 224 and 225 ,we have:

$$\begin{aligned} p &= (1+f^2c^2)^{-1}f^2 \text{ ; if } \rho = 0 \\ p &= \frac{-(1+c^2f^2-\rho) \pm \sqrt{(1+c^2f^2-\rho)^2+4f^2\rho c^2}}{2\rho c^2}; \rho \neq 0 \qquad (228) \end{aligned}$$

$$b = \rho p + f^2 \tag{229}$$

As shown in equations 211 and 212,$\overline{\Psi^2}$ and $\bar{\Psi}^2$ both are function of k. Substituting the expressions of ρ and f^2 into equation 228,we have (after some lengthy computation):

$$p = \frac{\sum_{i=0}^{2} \tilde{a}_i k^i + \sqrt{\sum_{i=0}^{4} \tilde{b}_i k^i}}{\overline{\Phi^2}(1 - \bar{\Psi}^2) - \overline{\tilde{\Psi}^2}} \tag{230}$$

Note: (1)It is easy to prove that p is continuous at $\rho = 0$. (2) We have to choose "+" sign in equation 228 as the following fact shows.

We know that there exists positive semidefinite solution if $\rho > 0$ (see [6]) under appropriate conditions. For $\rho < 0$, we have the following fact.

fact 3: If $\rho = -|\rho| < 0$ and $c^2 \neq 0$ then $p \geq 0, b \geq 0$ and $w \geq 0$ if we use the positive sign in equation 228.

proof: If $\rho = -|\rho| < 0$,we have

$$p = \frac{1 + c^2 f^2 + |\rho| \mp \sqrt{(1 + c^2 f^2 + |\rho|)^2 - 4 f^2 |\rho| c^2}}{2|\rho| c^2}$$

$$(1 + c^2 f^2 + |\rho|)^2 - 4 f^2 |\rho| c^2$$

$$= (c^2 f^2 - |\rho| + 1)^2 + 4|\rho| \geq 0$$

Therefore, $p \geq 0$ for both sign using the above identities. However,we have

$$b = -\rho p + f^2$$

$$= \frac{-(1 + c^2 f^2 + |\rho|) \pm \sqrt{(1 + c^2 f^2 + |\rho|)^2 - 4 f^2 \rho c^2} + 2 c^2 f^2}{2 c^2 f^2}$$

We can prove that:

$$(1 + c^2 f^2 + |\rho|)^2 - 4 f^2 \rho c^2 - (c^2 f^2 - 1 - |\rho|)^2$$

$$= 4f^2c^2 \geq 0$$

Therefore $b \geq 0$ for positive sign only. Following the same argument as in chapter 4,we know $w \geq 0$ also. □

From equations 223 through 224 , we have:

$$\begin{aligned} w &= (1-\bar{\Psi}^2)^{-1}[(\rho-1)p+f^2] \\ J &= qp+(q+hk^2)(1-\bar{\Psi}^2)^{-1}[(\rho-1)p+f^2] \\ &= [q+(q+hk^2)(1-\bar{\Psi}^2)^{-1}(\rho-1)]p+(1-\bar{\Psi}^2)^{-1}f^2(q+hk^2) \end{aligned} \quad (231)$$

Substituting the expressions for p,ρ and f^2 in equations 226 227 and 230, we obtain the analytic expression for J as a function of k only.

$$J(k) = \frac{\sum_{i=0}^{6}\bar{a}_i k^i + \sum_{i=0}^{4}\bar{c}_i k^i(\sqrt{\sum_{i=0}^{4}\bar{b}_i k^i})}{(1-\bar{\Psi}^2)[\overline{\Psi^2}(1-\bar{\Psi}^2)-\overline{\tilde{\Psi}^2}](1-\overline{\Psi^2})} \quad (232)$$
$$\text{for } k_l < k < k_h$$

where k_l and k_h are given in equations 215 and 216. $(\bar{a}_i, \bar{b}_i, \bar{c}_i)$ are functions of statistics of system variables (Φ_k, Γ_k, F_k) (up to second moment) , system variables (c, g) and weighting indices of the performance criterion (q, h). Our minimization problem has been simplified into:

$$\min_k J(k)$$
$$k_l < k < k_h \quad (233)$$

Of course,we need to assume that (Φ, Γ) is mean square stabilizable for $S = (k_l, k_h)$ to be nonempty. c^2 is assumed to be nonzero,otherwise $b = p, w = 0$ and $\hat{x}_k = 0$ w. p. 1. Since $J(k)$ is an *analytic function* of k and $S = (k_l, k_h)$ is closed and bounded,we know that there exists an optimal $k^* \in S$ which minimize $J(k)$. From the above argument,the following facts are given:

fact 4: There exists an optimal solution (p^*, b^*, w^*, k^*) to the optimization problem 223 with the property $p^* > 0$, $b^* > 0$ and $w^* > 0$ provided (Φ, Γ) is mean square stabilizable and $c \neq 0$.

fact 5: The total system is mean square stable provided $\overline{F^2} > 0$ and $q > 0$.

The above is a direct application for theorem 6 of chapter 4. In the following,some examples will be given along with the simulation results. But ,first the total system matrix is given below:

fact 6: The *Kronecker product* of total system matrix A_{xe} and A_{xe} is equal to

$$\overline{A_{xe} \otimes A_{xe}} = \begin{bmatrix} \overline{\Psi^2} & -\overline{\Psi\Gamma}K & -\overline{\Psi\Gamma}K & \overline{\Gamma^2}k^2 \\ \cdots & \cdots & \cdots & \cdots \\ J\overline{\tilde{\Psi}^2} & -J\overline{\Psi\Gamma}K + J\bar{\Psi}^2 & -J\overline{\tilde{\Psi}\Gamma}K & J\overline{\Gamma^2}K^2 - J\bar{\Psi}\bar{\Gamma}K \\ \cdots & \cdots & \cdots & \cdots \\ J\overline{\tilde{\Psi}^2} & -J\overline{\tilde{\Psi}\Gamma}K & -J\overline{\Psi\Gamma}K + J\bar{\Psi}^2 & J\overline{\Gamma^2}K^2 - J\bar{\Psi}\bar{\Gamma}K \\ \cdots & \cdots & \cdots & \cdots \\ J^2\overline{\tilde{\Psi}^2} & -J^2\overline{\tilde{\Psi}\Gamma}K & -J^2\overline{\tilde{\Psi}\Gamma}K & J^2\overline{\Gamma^2}K^2 + J^2\bar{\Psi}^2 - 2J^2K\bar{\Psi}\bar{\Gamma} \end{bmatrix}$$

Example 1: Assume the continuous time system equation is given by:

$$\dot{x}(t) = 20x(t) + 2u(t) + 0.5n(t) \tag{234}$$

$$v_k = x(\sigma_k) + 0.3\theta_k \tag{235}$$

The continous time system is randomly sampled by $\{\sigma_k\}$ - the sampling process. It is assumed that $\{t_k\}$ is an independent identically distributed stochastic process. It is

assumed that $\{t_k = \sigma_k - \sigma_{k-1}\}$ is uniformly distributed in the interval $[\Delta_1, \Delta_2]$,where

$$\Delta_1 = 0.05; \quad \Delta_2 = 0.15$$

Therefore,we have:

$$E[t_k] = 0.1 \text{ and } Var[t_k] = 8.33 \times 10^{-3}$$

The performance index is given below:

$$J = \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{k=0}^{N-1} (0.1u_k^2 + 10x_k^2)\right\} \tag{236}$$

Therefore,we have defined:

$$q = 10; \quad h = 0.1$$

The equivalent discrete time system equations and associated statistics is given below after some simple calculations (see equations 193 through 200):

$$x_{k+1} = \Phi_k x_k + \Gamma_k u_k + F_k \eta_k \tag{237}$$

$$v_k = x_k + 0.3\theta_k \tag{238}$$

where from equations 189 through 191, we have

$$\Phi_k = e^{20t_k}$$

$$\Gamma_k = \frac{1}{10}(e^{20t_k} - 1)$$

$$F_k = \frac{0.5}{\sqrt{40}}\sqrt{e^{40t_k} - 1}$$

$$\bar{\Phi} = 8.6836; \quad \bar{\Gamma} = 0.7684$$

$$\overline{\Phi^2} = 99.0099; \quad \overline{\Gamma^2} = 0.8264$$

$$\overline{\Phi\Gamma} = 9.0326; \quad \overline{\tilde{\Phi}^2} = 23.6045$$

$$\overline{\tilde{\Gamma}^2} = 0.2360; \quad \overline{\tilde{\Phi}\tilde{\Gamma}} = 2.3605$$

$$\overline{F^2} = 0.6126$$

Using fact 2,we can compute $S = (k_l, k_h)$ as follows:

$$k_l = -11.8595; \quad k_h = -10$$

Recall that S is the interval in which Ψ is mean square stable. S is not empty as is guaranteed by fact 2. Our equivalent minimization problem is as follows:

$$\min_{p,b,w,k} 10(p+w) + 0.1wk^2$$

subject to the following constraints:

$$p = (1 + 11.111b)^{-1}b$$

$$b = 99.0099p + \overline{\tilde{\Psi}^2}w + 0.6126$$

$$w = \bar{\Psi}^2 w + b - p$$

$$-11.8595 < k < -10 \tag{239}$$

where we define:

$$\bar{\Psi} = 8.6836 + 0.7684k \tag{240}$$

$$\overline{\tilde{\Psi}^2} = 23.6045 + 4.721k + 0.236k^2 \tag{241}$$

$$\overline{\Psi^2} = 99.0099 + 18.0652k + 0.8264k^2 \tag{242}$$

As discussed before, $J(k)$ can be expressed as a function of k only in the form of equation 232. Figure 4 display $J(k)$ as a function of k in the interval $S = (k_l, k_h)$. In order to solve the optimal control problem,all we have to do is to find the minimum of function $J(k)$ in the interval S. In our simulation, the so-called Nelder and Mead

Figure 4: The performance index $J(k)$ as a function of k

simplex method is adopted to search for the minimum. It turn out that the optimal gain k^* is equal to:

$$k^* = -10.8856 \tag{243}$$

We also have

$$p^* = 0.0893$$

$$b^* = 11.8895$$

$$w^* = 13.1420$$

$$J^*(k^*) = 288.0407 \tag{244}$$

The dynamics of our controller is as follows:

$$\hat{x}_{k+1} = 8.6836\hat{x}_k + 0.7684u_k + 0.9925(v_{k+1} - 8.6836\hat{x}_k - 0.7684u_k) \tag{245}$$

$$u_k = -10.8856\hat{x}_k \tag{246}$$

We are now in a position to check the validity of theorem 2 of chapter 4. From that theorem,we know that there exist $-\alpha > 0$ and $-\gamma > 0$ satisfying the following equations:

$$k = (0.1 - \overline{\tilde{\Gamma}^2}\alpha - \overline{\Gamma^2}\gamma)^{-1}(\overline{\tilde{\Phi}\tilde{\Gamma}}\alpha + \overline{\Phi\Gamma}\gamma) \tag{247}$$

$$-\alpha = J^2k^2(0.1 - \overline{\tilde{\Gamma}^2}\alpha - \overline{\Gamma^2}\gamma) - J^2\bar{\Phi}^2\alpha \tag{248}$$

$$-\gamma = -\bar{\Phi}^2\gamma - \overline{\tilde{\Phi}^2}\alpha + 10 - (0.1 - \overline{\tilde{\Gamma}^2}\alpha - \overline{\Gamma^2}\gamma)k^2 \tag{249}$$

where

$$J = (1 - ac);\ a = pcg^{-2}$$

Therefore,we have

$$k = (0.1 - 0.236\alpha - 0.8264\gamma)^{-1}(2.3605\alpha + 9.0326\gamma) \tag{250}$$

$$-\alpha = J^2k^2(0.1 - 0.236\alpha - 0.8264\gamma) - 75.405J^2\alpha \quad (251)$$

$$-\gamma = -99.0099\gamma - 23.6045\alpha + 10 - (0.0236\alpha - 0.8264\gamma)k^2 \quad (252)$$

where

$$a = pcg^{-2} = 11.111p;\ J = 1 - a$$

We already know that

$$a = 0.9925;\ \text{and}\ J = 1 - a = 0.0075 \quad (253)$$

Solving equations 251 252 and 253, we obtain

$$-\alpha = 0.18549 > 0\ ;\ -\gamma = 31.31169 > 0$$

Substituting the above answer into equation 250,we have

$$k = -10.8864$$

which is the same as k^* as expected. □

Equations 240 and 242 give rise to the following result:

$$\Psi = 0.3195 < 1;\ \overline{\Psi^2} = 0.2872 < 1$$

It is noticed that Ψ is both mean and mean square stable as desired. The Kronecker product of total system matrix A_{xe} and A_{xe} is computed and shown below:

$$\overline{A_{xe} \otimes A_{xe}} = \begin{bmatrix} 0.2872 & 0.3971 & 0.3971 & 97.8275 \\ 0.0014 & 0.0038 & -0.0171 & 0.7558 \\ 0.0014 & -0.0171 & 0.0038 & 0.7558 \\ 0.00001 & -0.0001 & -0.0001 & 0.0058 \end{bmatrix}$$

The eigenvalues of $\overline{A_{xe} \otimes A_{xe}}$ are found to be:

$$\sigma(\overline{A_{xe} \otimes A_{xe}}) = \begin{bmatrix} 0.2940 \\ -0.0104 \\ -0.0039 \\ 0.0208 \end{bmatrix}$$

which clearly shows the mean square stability of the total system. Recall section 3 of chapter 4, the evolution equations of covariances $\{p_k, b_k, w_k\}$ is as follows:

$$p_{k+1} = (1 + b_{k+1}c^2g^{-2})^{-1} \tag{254}$$

$$b_{k+1} = \overline{\Phi^2}p_k + \overline{\check{\Psi}^2}w_k + \overline{F^2} \tag{255}$$

$$w_{k+1} = \check{\Psi}^2 w_k + b_{k+1} - p_{k+1} \tag{256}$$

Starting from $(p_0 = b_0 = w_0 = 0)$,figure 5 shows the evolution of covariances (p_k, b_k, w_k) for 100 runs. It is noticed that they converge very fast to the steady state value (p^*, b^*, w^*). This is expected since the total system is mean square stable. Next,let's compare the performance of the optimal control to that of the certainty equivalent control. If we didn't take into account the uncertainty in the sampling process,then the easiest thing we can do is to replace all the stochastic parameters by its mean values and then apply the traditional LQG theory to find the so called certainty equivalent control. The certainty equivalent gain is found to be:

$$k_c = -11.2177$$

Our optimal gain is $k^* = -10.8856$. Therefore,the optimal control is more *cautious* facing the uncertainty of the sampling process. Table 1 is a comparison of the cost components between the optimal control system and certainty equivalent control

Figure 5: evolution of covariances (p_k, b_k, w_k) for 100 runs

type	*state cost*	*control cost*	*total cost*
optimal control	132.313	155.728	288.041
c. c. control	146.1175	182.7440	328.8615

Table 1: Comparison of cost between optimal and certainty equivalent control system in example 1 of section 1 of chapter 5

system. As you can see, this cautiousness paid off since the performance index of them are :

$$J(k_c) = 328.8614;\ J(k^*) = 288.0407 \tag{257}$$

which shows a 14% degradation in performance index for certainty equivalent control. Moreover,in the calculation of equation 257 we have applied the optimal filter gain $a = 0.9925$ instead of the certainty equivalent filter gain a_c which is equal to:

$$a_c = 0.9828$$

The Monte carlo simulation is presented next. Figure 6 display the trajectories of the state variable x_k and optimal state estimate $\hat{x}_k$. Figure 7 displays the average cost as a function of time. It turns out that the simulated cost is equal to

$$J^*_{sim} = 224.1396$$

which is very close to the theorectical value 288.0407. The simulated average cost for the certainty equivalent system is (using $k_c = -11.2177$ and $a_c = 0.9878$):

$$J^c_{sim} = 281.8415$$

Again this is a 25% poor in forformance for the certainty equivalent strategy. All in all,the optimal system perform better than the certainty equivalent system.

Figure 6: simulation of trajectories of x_k and $\hat{x}_k$

Figure 7: simulation of average cost as a function of time

Example 2: The continuous time system is given by:

$$\begin{aligned} \dot{x}(t) &= 50x(t) + 2u(t) + 0.5n(t) \\ v_k &= x_k + 0.3\theta_k \end{aligned}$$

The sampling process $\{t_i\}$ is independent identically distributed. Furthermore,it is assumed to be uniformly distributed in the interval (Δ_1, Δ_2).

$$\Delta_1 = 0.05; \quad \Delta_2 = 0.15$$

The mean and variance of t_i is then:

$$E[t_i] = 0.1; \quad Var[t_i] = 8.33 \times 10^{-3}$$

The performance index is given by:

$$J = \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{k=0}^{N} (0.01u_k^2 + 100x_k^2)\right\}$$

We then have

$$S = (k_l, k_h); \; k_l = -25.0549; \; k_h = -25$$

The optimal gain k^* and the optimal cost J^* are found to be

$$k^* = -25.0273 \in S; \quad J^*(k^*) = 8.17 \times 10^6$$

The certainty equivalent gain k_c and its associated cost J_c is

$$k_c = -25.0698 \notin S; \quad J_c(k_c) \approx \infty$$

Therefore the certainty equivalent system is not mean square stable although it is mean stable. However,for the optimal system due to cautious in applying the gain

C.2

the total system is still mean square stable as is shown below by the spectrum of $\overline{A_{xe} \otimes A_{xe}}$ (the Kronecker product of total system matrix A_{xe} and A_{xe}):

$$\sigma(\overline{A_{xe} \otimes A_{xe}} = \begin{bmatrix} 0.6067 \\ -0.0004 \\ 0.0000 \\ 0.0004 \end{bmatrix}$$

Figure 8 plots the performance index $J(k)$ as a function of k. In general,the optimal system is guaranteed to be mean square stable under certain conditions as described in fact 5. Whereas the certainty equivalent can only guarantee mean stable. □

Figure 8: The performance index $J(k)$ for example 2 versus k.

5.2 Two Dimensional System Simulation: An Example

Because of the complexity of the problem, no general results for two dimensional systems are given. We do know that once there is a solution to the generalized Riccati equation,the total system is guaranteed to be mean square stable under appropriate conditions. In the following, a specific example is presented. Where the solution of the steady state Riccati equation is obtained through Newton's method. The total system is shown to be mean square stable. The performance of the optimal control system is compared to that of the certainty equivalent control system through simulation. The results are 54% better in favor of the optimal control system.

Example 1: The continuous time plant is given by

$$\dot{x} = Ax + Bu + \tilde{F}n \tag{258}$$

$$v_k = Cx(\sigma_k) + G\theta_k \tag{259}$$

where

$$A = \begin{bmatrix} 0.01 & -1 \\ 1 & 0.01 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \; \tilde{F} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$C = [\,1\,,\,1\,]; \; G = 1$$

It is assumed that $\{t_k = \sigma_k - \sigma_{k-1}\}$ is an independent identically distributed stochastic process. To be specific, we assume $\{t_k\}$ is uniformly distributed in the interval $[\Delta_1, \Delta_2]$, where

$$\Delta_1 = 2.4523 \; ; \; \Delta_2 = 3.5477$$

Therefore ,we have:

$$E[t_i] = 3 \; ; \; Var[t_i] = 0.1$$

The performance index is given by:

$$J = \lim_{N \to \infty} \frac{1}{N} E\{\sum_{i=1}^{N} [x_i^* Q x_i + H u_i^2]\}$$

where

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \; ; \; H = 1$$

The equivalent discrete time system equations are then given below:

$$x_{k+1} = \Phi(t_{k+1}) x_k + \Gamma(t_{k+1}) u_k + \sqrt{FF^*(t_{k+1})}\eta_k \tag{261}$$

$$v_k = C x_k + G\theta_k \tag{262}$$

where

$$\Phi(t) = e^{0.01t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$\Gamma(t) = \frac{1}{1.0001} \begin{bmatrix} e^{0.01t}(\cos t - 0.01 \sin t) - 1 \\ e^{0.01t}(0.01 \cos t + \sin t) - 0.01 \end{bmatrix}$$

$$FF^*(t) = \begin{bmatrix} f_7(t) + 4f_8(t) - 2f_6(t) & -\frac{3}{2} f_6(t) + 2f_5(t) \\ -\frac{3}{2} + 2f_5(t) & f_8(t) + 4f_7(t) + 2f_6(t) \end{bmatrix}$$

in which

$$f_5(t) = \frac{e^{0.02t}(0.01 \cos 2t + \sin 2t) - 0.01}{2.0002}$$

$$f_6(t) = \frac{1 - e^{0.02t}(\cos 2t - 0.01 \sin 2t)}{2.0002}$$

$$f_7(t) = \frac{1}{0.04}(e^{0.02t} - 1) + \frac{1}{4.0004}[e^{0.02t}(0.01 \cos 2t + \sin 2t) - 0.01]$$

$$f_8(t) = \frac{1}{0.01}(e^{0.02t} - 1) - \frac{1}{4.0004}[e^{0.02t}(0.01 \cos 2t + \sin 2t) - 0.01]$$

The associated statistics with $(\Phi_k, \Gamma_k, FF_k^*)$ are given below:

$$\overline{\Phi \otimes \Phi} = \begin{bmatrix} 0.9452 & 0.1186 & 0.1186 & 0.1166 \\ -0.1186 & 0.9452 & -0.1166 & 0.1186 \\ -0.1186 & -0.1166 & 0.9452 & 0.1186 \\ 0.1166 & -0.1186 & -0.1186 & 0.9452 \end{bmatrix} ; \overline{\Gamma \otimes \Gamma} = \begin{bmatrix} 3.8897 \\ -0.2181 \\ -0.2181 \\ 0.1119 \end{bmatrix}$$

$$\overline{\Phi \otimes \Gamma} = \begin{bmatrix} 1.9163 & 0.2570 \\ -0.0994 & -0.1141 \\ -0.2570 & 1.9163 \\ 0.1141 & -0.0994 \end{bmatrix} ; \overline{\Gamma \otimes \Phi} = \begin{bmatrix} 1.9163 & 0.2570 \\ -0.2570 & 1.9163 \\ -0.0994 & -0.1141 \\ 0.1141 & -0.0994 \end{bmatrix}$$

$$\bar{\Phi} = \begin{bmatrix} -0.97 & -0.1373 \\ 0.1373 & -0.97 \end{bmatrix} ; \bar{\Gamma} = \begin{bmatrix} -1.9712 \\ 0.1176 \end{bmatrix}$$

$$\overline{FF^*} = \begin{bmatrix} 7.7424 & -0.3657 \\ -0.3657 & 7.7221 \end{bmatrix}$$

$$\overline{\dot{\Phi} \odot \dot{\Phi}} = \overline{\Phi \otimes \Phi} - \bar{\Phi} \odot \bar{\Phi}$$

$$= \begin{bmatrix} 0.0042 & -0.0145 & -0.0145 & 0.0978 \\ 0.0145 & 0.0042 & -0.0978 & -0.0145 \\ 0.0145 & -0.0978 & 0.0042 & -0.0145 \\ 0.0978 & 0.0145 & 0.0145 & 0.0042 \end{bmatrix}$$

$$\overline{\dot{\Gamma} \odot \dot{\Gamma}} = \overline{\Gamma \otimes \Gamma} - \bar{\Gamma} \odot \bar{\Gamma}$$

$$= \begin{bmatrix} 0.0040 \\ 0.0136 \\ 0.0136 \\ 0.0981 \end{bmatrix}$$

$$\overline{\tilde{\Phi} \otimes \tilde{\Gamma}} = \overline{\Phi \otimes \Gamma} - \bar{\Phi} \otimes \bar{\Gamma}$$

$$= \begin{bmatrix} 0.0041 & -0.0136 \\ 0.0146 & -0.0979 \\ 0.0136 & 0.0041 \\ 0.0979 & 0.0146 \end{bmatrix}$$

where " $\otimes$ " is the *Kronecker product* of matrices.

Note: (1) $(A, \tilde{F})$ are controllable $\Rightarrow$ $\overline{FF^*} > 0$ (refer to notes for theorem 1 of section3 in chapter 4). Assuming the existence of the optimal control,$\overline{FF^*} > 0$ together with $Q > 0$ guarantee the mean square stability of the total system as will be shown later.

(2) It is easy to show that (Φ, Γ) is mean square stabilizable and that (Φ, C) is mean square observable. This may have something to do with the existence of optimal solution.

To find the optimal solution,we will applying theorem 2 of chapter 4 and try to find the solution of the coupled steady state Riccati equation. By using the Newton's method, we are able to obtain the following solutions:

$$P = \begin{bmatrix} 251.9681 & -251.9698 \\ -251.9698 & 252.9645 \end{bmatrix}$$

$$B = \begin{bmatrix} 251.9685 & -252.2166 \\ -252.2166 & 393.7223 \end{bmatrix}$$

$$W = \begin{bmatrix} 611.7 & 442.1 \\ 442.1 & 487.3 \end{bmatrix}$$

$$-\alpha = \begin{bmatrix} 1310.1 & 1.983 \\ 1.983 & 0.0081 \end{bmatrix}$$

$$-\gamma = \begin{bmatrix} 21.487 & -25.16 \\ -25.16 & 740.4 \end{bmatrix}$$

$$K^* = [H - \overline{\tilde{\Gamma}^* \alpha \tilde{\Gamma}} - \overline{\Gamma^* \gamma \Gamma}]^{-1} [\overline{\tilde{\Gamma}^* \alpha \tilde{\Phi}} + \overline{\Gamma^* \gamma \Phi}]$$

$$= [-0.7625 \ , \ 0.7156]$$

The certainty equivalent control is found to be:

$$K_c = [-0.5129 \ , \ 0.3210]$$

The optimal cost is found to be:

$$J(K^*) = 1.6162 \times 10^4$$

However,the cost associated with K_c is equal to :

$$J_c(K_c) = 2.4851 \times 10^4$$

This turns out to be 54% increase in cost which should be of no surprise. Next ,let's verify the mean square stability of the total system. The total system matrix is given below (see equation 153 of chapter 4):

$$A_{xe} = \begin{bmatrix} \Psi_k & -\Gamma_k K \\ J\tilde{\Psi}_k & -J\Gamma_k K + J\bar{\Psi} \end{bmatrix}$$

The following fact is well known and is given without proof: (see Kalman's paper)

fact: A_{xe} is mean square stable if and only if $\rho(\overline{A_{xe} \otimes A_{xe}}) < 1$,where $\rho(.)$ is the spectral radius of matrix.

It is easy to prove the following lemma and hence is given without proof:

lemma: $\overline{A_{xe} \otimes A_{xe}}$ is *similar* (not necessarily equal to) to the following matrix:

$$
\begin{bmatrix}
\overline{\Psi \otimes \Psi} & -\overline{\Psi \otimes \Gamma K} & -\overline{\Gamma K \otimes \Psi} & \overline{\Gamma K \otimes \Gamma K} \\
\cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots \\
\overline{\Psi \otimes J\tilde{\Psi}} & -\overline{\Psi \otimes J\Gamma K} + \bar{\Psi} \otimes J\bar{\Psi} & -\overline{\Gamma K \otimes J\tilde{\Psi}} & \overline{\Gamma K \otimes J\Gamma K} - \bar{\Gamma} K \otimes J\bar{\Psi} \\
\cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots \\
\overline{J\tilde{\Psi} \otimes \Psi} & -\overline{J\tilde{\Psi} \otimes \Gamma K} & -\overline{J\Gamma K \otimes \Psi} + J\bar{\Psi} \otimes \bar{\Psi} & \overline{J\Gamma K \otimes \Gamma K} - J\bar{\Psi} \otimes \bar{\Gamma} K \\
\cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots \\
\overline{J\tilde{\Psi} \otimes J\tilde{\Psi}} & -\overline{J\tilde{\Psi} \otimes J\Gamma K} & -\overline{J\Gamma K \otimes J\tilde{\Psi}} & \overline{J\Gamma K \otimes J\Gamma K} - J\bar{\Gamma} K \otimes J\bar{\Psi} - J\bar{\Psi} \otimes J\bar{\Gamma} K + J\bar{\Psi} \otimes J\bar{\Psi}
\end{bmatrix}
\tag{263}
$$

After some lengthy computation and by applying the above lemma, the eigenvalues

of $\overline{A_{xe} \otimes A_{xe}}$ are found to be:

$$\sigma(\overline{A_{xe} \otimes A_{xe}}) = \begin{bmatrix} 0.9823 \\ 0.5581 + 0.220i \\ 0.5581 - 0.220i \\ -0.4476 \\ -0.4179 \\ 0.2395 \\ 0.9037 \\ -0.4749 \\ -0.3565 \\ 0.0046 + 0.0034i \\ 0.0046 - 0.0034i \\ 0.0001 \\ -0.0028 \\ 0.0059 + 0.0031i \\ 0.0059 - 0.0031i \\ -0.0039 \end{bmatrix}$$

As shown above, all the eigenvalues lie inside the unit circle. Indeed, the total system is mean square stable as expected. □

Chapter 6

conclusion

Chapter 3 and 4 are devoted to solving the optimal LQR problem with i. i. d. sampling. The fixed configuration approach is adopted to obtain the suboptimal control which has the nice property of cautious control. The optimal linear least mean square estimator for system with random parameters is derived in chapter 3. The evolution equations of covariance matrices is derived where it is noticed that the covariance matrices are dependent on the particular control sequence. The necessary conditions for N-horizon optimal control problem is derived using Lagrange's multiplier method in chapter 4. The suboptimal solution is noticed to be the optimal solution provided the sampling is deterministic. Finally , the infinite horizon control problem is formulated as a classical minimization problem where assuming existence of solution the total system is shown to be mean square stable provided certain mean square observability conditions hold. Various theorem are given to test the mean square stability of the total system. Furthermore,it is shown that the total system is stable with probability one.

The simulation results presented in chapter 5 show the applicability of our approch. For one dimensional system,the existence problem of optimal control is solved. It is shown that the performance of the optimal system is much better as compared to that of the so called certainty equivalent control systems. The optimal system can guarantee the mean square stability due to the cautiousness facing the uncertainties of the sampling process, whereas the certainty equivalent system can only guarantee mean stability and may not be stable under certain conditions.

The techniques used in deriving the optimal control is not limited to systems with randomly sampled observations. It can be applied to derive the optimal control for all systems with random parameters as well. Furthermore,our theorey naturally provide a method to obtain the robust control-robust to parameter variations. For example,we can treat certain unknown parameters as random variables say with a uniform distribution in some reasonable interval and then apply our theorey to derive the optimal control. In so doing ,the total system is stable even with parameter variations.

The optimal control problem for systems with random parameters and imperfect observations is in general a very difficult problem. Because of the curse of dimensionality of sufficient statistics,no analytic solution can be found in general. Efficient suboptimal control is yet to be found. Our techniques do provide a beginning toward this direction.

Appendix A

Derivation of optimal control control under variance neutrality condition

The optimal control problem of concern is the following:

Discrete Plant

$$x_{k+1} = \Phi(t_{k+1})x_k + \Gamma(t_{k+1})u_k + F(t_{k+1})\eta_k \tag{264}$$

$$v_k = Cx_k + G\theta_k \tag{265}$$

where $\{\eta_k\}$, $\{\theta_k\}$ and $\{t_k\}$ are mutually independent random sequences. Furthermore, it is assumed that $\{t_k\}$ is independent identically distributed.

Performance Index

$$J = E\left\{\sum_{k=1}^{N}(x_k^* Q x_k + u_{k-1}^* H u_{k-1})\right\} \tag{266}$$

Admissible Control

$$u_k \text{ is measurable } v^k = \{v_1, v_2, \cdots, v_k\}$$

Goal find optimal admissible sequence $\{u_k\}_0^{N-1}$ such that performance index J is minimized subject to dynamic constraint 264 and 265.

In the following ,we will use *dynamic programming* techniques to derive the optimal control. Consider the last stage problem:

$$\lambda_N = E[x_N^* Q x_N + u_{N-1}^* H u_{N-1} \| v^{N-1}]$$

$$= E[(\Phi(t_N)x_{N-1} + \Gamma(t_N)u_{N-1} + F(t_N)\eta_{N-1})^* Q \times$$

$$(\Phi(t_N)x_{N-1} + \Gamma(t_N)u_{N-1} + F(t_N)\eta_{N-1}) \| v^{N-1}] + u_{N-1}^* H u_{N-1}$$

$$= E[x_{N-1}^* \Phi(t_N)^* Q \Phi(t_N) x_{N-1} + u_{N-1}^* \Gamma(t_N)^* Q \Gamma(t_N) u_{N-1}$$

$$+2x_{N-1}^* \Phi(t_N)^* Q \Gamma(t_N) u_{N-1} \| v^{N-1}]$$

$$+E[\eta_{N-1}^* F(t_N)^* Q F(t_N) \eta_{N-1} \| v^{N-1}] + u_{N-1}^* H u_{N-1}$$

Because of independence assumption,we have:

$$\lambda_N = E[x_{N-1}^* \overline{\Phi^* Q \Phi} x_{N-1} + u_{N-1}^* \overline{\Gamma^* Q \Gamma} u_{N-1} + 2u_{N-1}^* \overline{\Phi^* Q \Gamma} x_{N-1} \| v^{N-1}]$$

$$+u_{N-1}^* H u_{N-1} + tr\{Q\overline{FF^*}\}$$

By taking the gradient of λ_N with respect to u_{N-1},we obtain:

$$\frac{\partial \lambda_N}{\partial u_{N-1}} = H u_{N-1} + \overline{\Gamma^* Q \Gamma} u_{N-1} + \overline{\Gamma^* Q \Phi} E[x_{N-1} \| v^{N-1}] = 0$$

$$\Longrightarrow \quad u_{N-1} = (H + \overline{\Gamma^* Q \Gamma})^{-1} \overline{\Gamma^* Q \Phi} \hat{x}_{N-1}$$

$$\stackrel{\text{def}}{=} -K_{N-1} \hat{x}_{N-1}$$

where we have defined:

$$K_{N-1} \stackrel{\text{def}}{=} (H + \overline{\Gamma^* Q \Gamma})^{-1} \overline{\Gamma^* Q \Phi}$$

where we have assumed that the inverse exist,otherwise the Moore-Penrose pseudo inverse has to be used. Next, we will compute the optimal cost associated with the optimal control derived above.

$$\begin{aligned} r_N &\stackrel{\text{def}}{=} \min_{u_{N-1}} \lambda_N \\ &= E[x_{N-1}^* \overline{\Phi^* Q \Phi} x_{N-1} + u_{N-1}^* \overline{\Phi^* Q \Gamma} x_{N-1} \| v^{N-1}] \\ &+ E[u_{N-1}^* (\overline{\Gamma^* Q \Gamma} u_{N-1} + H u_{N-1} + \overline{\Gamma^* Q \Phi} x_{N-1}) \| v^{N-1}] \\ &+ tr\{Q \overline{F F^*}\} \\ &= E[x_{N-1}^* \overline{\Phi^* Q \Phi} x_{N-1} \| v^{N-1}] - \hat{x}_{N-1}^* K_{N-1}^* \overline{\Gamma^* Q \Phi} \hat{x}_{N-1} \\ &+ tr\{Q \overline{F F^*}\} \end{aligned} \tag{267}$$

From orthogonality principle,we have:

$$\begin{aligned} &E[(x_{N-1} - \hat{x}_{N-1})^* K_{N-1}^* \overline{\Gamma^* Q \Phi} (x_{N-1} - \hat{x}_{N-1}) \| V^{N-1}] \\ &\stackrel{\text{def}}{=} K_{N-1}^* \overline{\Gamma^* Q \Phi} P_1^f \\ &= E[(x_{N-1} - \hat{x}_{N-1})^* K_{N-1}^* \overline{\Gamma^* Q \Phi} x_{N-1} \| v^{N-1}] \\ &= E[x_{N-1}^* K_{N-1}^* \overline{\Gamma^* Q \Phi} x_{N-1} \| v^{N-1}] - \hat{x}_{N-1}^* \overline{\Phi^* Q \Gamma} K_{N-1} \hat{x}_{N-1} \end{aligned}$$

where we have defined:

$$P_1^f \stackrel{\text{def}}{=} E[(x_{N-1} - \hat{x}_{N-1})(x_{N-1} - \hat{x}_{N-1})^* \| v^{N-1}]$$

From variance neutrality assumption,we know that P_1^f is independent of v^{N-1} and u_{N-1}. Therefore, from equation 267 we have:

$$r_N = E[x_{N-1}^*(\overline{\Phi^* Q \Phi} - K_{N-1}^* \overline{\Gamma^* Q \Phi} x_{N-1} \| V^{N-1}]$$

$$+tr\ \{Q\overline{FF^*}\} + tr\ \{K_{N-1}^* \overline{\Gamma^* Q \Phi} P_1^f$$

$$= E[x_{N-1}^* P_1^c x_{N-1} \| v^{N-1}] + \nu_1$$

where we have defined:

$$P_1^c \stackrel{\text{def}}{=} \overline{\Phi^* Q \Phi} - K_{N-1}^* \overline{\Gamma^* Q \Phi}$$

$$\nu_1 \stackrel{\text{def}}{=} tr\ \{Q\overline{FF^*}\} + tr\ \{K_{N-1}^* \overline{\Gamma^* Q \Phi} P_1^f$$

Now,consider the last two stage:

$$r_{N-1} = \min_{u_{N-2}, u_{N-1}} E[x_{N-1}^* Q x_{N-1} + u_{N-2}^* H u_{N-2} + x_N^* Q x_N + u_{N-1}^* H u_{N-1} \| v^{N-2}]$$

$$= \min_{u_{N-2}} E\{x_{N-1}^* Q x_{N-1} + u_{N-2}^* H u_{N-2} + \min_{u_{N-1}} E[x_N^* Q x_N + u_{N-1}^* H u_{N-1} \| v^{N-1}] \| v^{N-2}\}$$

$$= \min_{u_{N-2}} E[x_{N-1}^* Q x_{N-1} + u_{N-2}^* H u_{N-2} + x_{N-1}^* P_1^c x_{N-1} + \nu_1 \| v^{N-2}]$$

$$= \min_{u_{N-2}} E[x_{N-1}^* (Q + P_1^f) x_{N-1} + u_{N-2}^* H u_{N-2} + \nu_1 \| v^{N-2}]$$

The above is the same one stage problem as before. By applying the same procedures as before ,we obtain:

$$\begin{aligned}
u_{N-2} &= -K_{N-2} \hat{x}_{N-2} \\
r_{N-1} &= E[x_{N-2}^* P_2^c x_{N-2} + \nu_2 \| v^{N-2}] \\
K_{N-2} &= [\overline{\Gamma^*(Q + P_1^c)\Gamma} + H]^{-1} \overline{\Gamma^*(Q + P_1^c)\Phi} \\
P_2^c &= \overline{\Phi^*(Q + P_1^c)\Phi} - K_{N-2}^* \overline{\Gamma^*(Q + P_1^c)\Phi} \\
\nu_2 &= tr(Q + P_1^c)\overline{FF^*} + tr K_{N-2}^* \overline{\Gamma^*(Q + P_1^c)\Phi} P_2^f + \nu_1
\end{aligned}$$

In general,by induction we obtain the following results:

$$\forall\, 0 \leq\ i\ \leq\ N-1$$

$$\begin{aligned}
u_i &= -K_i\hat{x}_i \\
r_{i+1} &= E[x_i^* P_{N-i}^c x_i + \nu_{N-i} \| v^i] \\
K_i &= [\overline{\Gamma^*(Q + P_{N-i-1}^c)\Gamma} + H]^{-1}\overline{\Gamma^*(Q + P_{N-i-1}^c)\Phi} \\
P_{N-i}^c &= \overline{\Phi^*(Q + P_{N-i-1}^c)\Phi} - K_i^*\overline{\Gamma^*(Q + P_{N-i-1}^c)\Phi} \\
\nu_{N-i} &= tr(Q + P_{N-i-1}^c)\overline{FF^*} + trK_i^*\overline{\Gamma^*(Q + P_{N-i-1}^c)\Phi}P_{N-i}^f + \nu_{N-i-1}
\end{aligned}$$

□.

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