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PARAMETER IDENTIFICATION FOR NONLINEAR AERODYNAMIC SYSTEMS

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1. Introduction

This annual technical report covers the first year period of NASA grant NAG 1-1065 commencing October 23, 1989 and ending October 22, 1990. The research results were partially described in the semiannual report dated April 26, 1990 and also in the renewal proposal dated July 20, 1990. However, this report is inclusive of the results obtained during the one year period and, in addition, has appended four papers giving details of the algorithms and developments forming the background for the current work.

2. List of Scientific Collaborators

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* Received partial support under NAG-1-1065

3. Completed and Continuing Research

3.1. Modeling Considerations

Our proposed research into the parameter identification for nonlinear aerodynamic systems presumes that the underlying model can be arranged into an input/output (I/O) differential operator equation of the generic form [1]

$$\sum_{i,j} g_i(\theta) F_{ij}(u(t), y(t)) P_{ij}(p) E_j(u(t), y(t)) = 0 \tag{1}$$

where $[u(t), y(t)]$ denote input/output variables assumed to be available as measured data on some time interval, θ denotes a vector of parameters whose value is to be estimated based on the given data, $P_{ij}(p)$ is a polynomial in the differential operator $p = d/dt$ for each index pair (i,j), $i=1, \cdots n_1$, $j=1, \cdots n_2$, and the $[E_{ij}(u,y), F_{ij}(u,y), g_i(\theta)]$ are given (sufficiently smooth) functions of their arguments that depend on the specified model. Additional data-related smoothness conditions pertain to the functions $g_{ij}(t) = F_{ij}(u(t), y(t))$ for *inexact* models in which the F_{ij}'s are not all constants.[2] In the case of *exact* differential operator models, the F_{ij}'s are all constants, i.e., (1) reduces to the simpler form

$$\sum_{i,j} g_i(\theta) P_{ij}(p) E_j(u(t), y(t)) = 0, \tag{2}$$

and the algorithm for parameter estimation is especially efficient for this case since the equation error can be integrated exactly given any I/O pair to obtain an algebraic function of the parameters. (As detailed in Section 3 of Appendix A.)

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[1] Although scalar valued, vector versions of this equation may be developed to accommodate multivariable system models. Also, extensions to models which are *nonseparable* in some of the parameters is potentially possible, e.g., differential delay equation models with unknown time delays.

[2] Definitions and illustrations of the terms *exact, inexact, separable*, etc., are given in the attached Appendix A paper together with an algorithm for the model (1).

The augmented linearized equations for aircraft discussed in Section 4 of Klein [3] (augmented by aerodynamic derivative coefficients modeled as parametrized functions of aerodynamic inputs and responses) appear to be representable by the model (2), albeit with a large number of parameters. Part of our future work will be directed towards these models. Although our simulation experience with nonlinear models is not extensive, we can assert with some degree of confidence that the algorithm has good noise rejection properties in the case of linear system models with additive noise on the data. As detailed in the attached Appendix C paper, the noise rejection properties can be explained via the frequency domain interpretation of the Fourier based modulating functions. Therein also can be found an extension of the algorithm, via the maximum likelihood technique, to stochastic linear models with additive white gaussian noise. These results for linear systems show that the bias incurred by deterministic least squares can be effectively removed in a high noise-to-signal situation. However, with the exception of linear systems, i.e., systems describable by a linear differential operator equation like

$$\sum_{i=0}^{n} a_{n-i} p^{i} y(t) = \sum_{i=0}^{n-1} b_{n-1-i} p^{i} u(t), \quad a_0 = 1 \tag{3}$$

it is recognized that the I/O models (1) or (2) may appear vague and somewhat formidable because the more familiar state vector equations like

$$\dot{x}(t) = f(x(t), u(t), \theta) \tag{4}$$

$$y(t) = h(x(t), u(t), \theta)$$

are almost always the starting point into methods for the parameter identification of deterministic systems modeled by nonlinear ordinary differential equations. This is understandable and, therefore, part of our effort has gone into the relationship between the models (1) and (4) insofar as parameter identification is concerned. Although this effort is ongoing, one such relationship we have investigated is that of the "identifiability" property of these models. This notion has been the subject of a number of papers relative to the state equation model (4), and we have shown in the attached Appendix B paper that single-valuedness of the $g(\theta)$ function appearing in (1) is a necessary property else the ensuing parameter estimation problem will be ill-posed. Thus, defining the vector function $g(\theta) = (g_1(\theta), g_2(\theta), \cdots, g_{n_1}(\theta))$ relative to the model (1), the following "injective" property:

$$g(\theta) = g(\theta^*) \quad \text{if and only if} \quad \theta = \theta^* \tag{5}$$

is a necessary condition else nonuniqueness will plague the parameter estimation problem. This conclusion is based on an examination of the metric properties of the I/O model (1) when viewed as a function on the parameter space for θ. As pointed out in Appendix B, this conclusion is consistent with the results of other investigators when applied to specific state equation models that possess an equivalent I/O representation, but the latter model makes the determination of this property more transparent than the test for nonidentifiability of state equation models.

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[3] Klein, V., "Estimation of Aircraft Aerodynamic Parameters from Flight Data," *Prog. Aerospace Sci.*, Vol. 26, pp. 1-77, 1989.

3.2. Structure Determination

We have extended the algorithm for parameter identification described in the Appendix C paper to the order determination problem for linear differential systems, i.e., the model (3) where the order n is unknown in addition to the parameters $(a_1 \cdots a_n, b_0 \cdots b_{n-1})$, by making use of an important property of modulating functions, namely: any modulating function of order N is also a modulating function of order n for any $n < N$. Thus, by using a set of Fourier based modulating functions (as defined by Eq. (4) in Appendix C) of preselected order $N \geq \max n$, the algebraic equation counterpart to Eq. (10) of Appendix C is the equation:[4]

$$\sum_{i=0}^{n} a_{n-i} CD^{i+k} Y = \sum_{i=0}^{n-1} b_{n-1-i} CD^{i+k} U, \quad a_0=1 \tag{6}$$

$$k=0,1 \cdots N-n$$

Defining the vectors $W(m)$ and $V(m)$ by

$$W(m)=CD^m Y, \quad V(m)=CD^m U$$

Eq. (6) can be rewritten as

$$(\sum_{i=0}^{n} a_i q^{-i}) W(m) = (\sum_{i=0}^{n-1} b_i q^{-i}) V(m-1), \quad m=n, n+1 \cdots N \tag{7}$$

where q^{-1} is the unit delay operator, i.e., $q^{-1}W(m)=W(m-1)$. Although arriving at (7) is merely a redefinition of previously defined quantities, i.e., calculating the sequence pair $(V(m), W(m))$ is easy once the finite set of Fourier series coefficients of the input/output data has been calculated, Equation (7) is now in a form that can utilize well known discrete system algorithms if desired. However, here we employ this equivalent form because it facilitates iterating on the order n.

Without going into further algebraic details (full coverage of which will be put in a forthcoming paper), we use the parsimony principle in finding the simplest model, i.e., the lowest order, that adequately fits the data. The algorithm minimizes the least squares criterion:

$$E_n = \sum_{k=n}^{N} e_n'(k) e_n(k) \tag{8}$$

over the $2n$ parameters $(a_1 \cdots a_n, b_0 \cdots b_{n-1})$ iteratively for each n starting from $n=1$, where e_n is defined in terms of the equation error for (7). The decision rule for stopping the iteration is:

$$\hat{n} = \min \left\{ n \mid D_n < \delta, \ 1 \leq n \leq N \right\} \tag{9}$$

where

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[4] The precomputable matrix pair (C,D) and the vector pair (U,Y) of finite Fourier series coefficients of the I/O data are defined in Section 2 of Appendix C.

$$D_n = \frac{E_n - E_{n+1}}{E_1}.$$

Thus, the iterations are stopped when the rate of improvement in the least squares criterion falls below a user chosen threshold δ.

The above described algorithm has been applied to several examples, including the following fourth order Chebyshev filter system:

$$G(s) = \frac{0.0438}{s^4 + 0.6192s^3 + 0.614s^2 + 0.2038s + 0.0492} \tag{10}$$

With the threshold choice δ=0.1 in (9), the result of one simulation for the system (10) is summarized by the graphs in Fig. 1. Shown here is the input/output data on a 40 sec time interval, a plot of D_n verses n, and frequency response plots comparing the magnitudes of the transfer functions for the original and estimated system. The T=40sec time interval is approximately double the settling time for the system (10); this results in a resolution frequency of $\omega_0 = 2\pi/T = 0.157$ rad/sec which is adequate to resolve the modes in the frequency response for (10) as seen by the magnitude plot in Fig. 1. The output data included about 10% RMS additive white noise which accounts for the deviations in the frequency response plots. These and additional examples confirm that the algorithm has the potential to correctly determine the system order under moderate noise-to-signal ratios. Further research is planned to extend the above structure determination algorithm to polynomial I/O differential operator models.
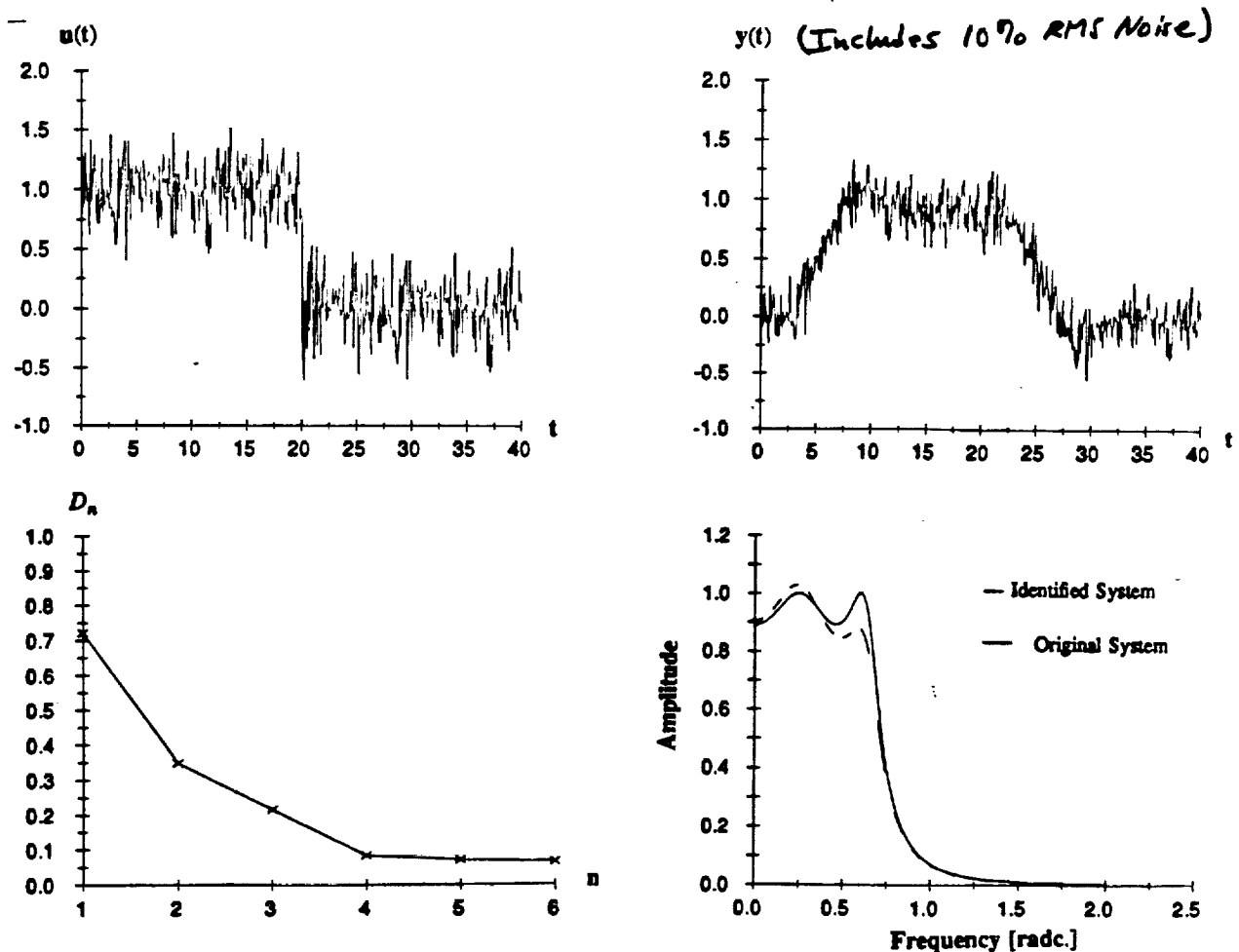
Fig. 1

3.3. Data Collinearity

Another problem we have addressed during the past year is the degeneracy in a least squares estimate caused by feedback. The problem for aircraft has been described in Klein [5] and is most obvious in the case of linear system identification since linear feedback will directly cause collinearity between the input/output regressors in the absence of external inputs during the observation interval. Using the algorithm of the Appendix C paper and the setup of Fig. 2, we have been studying the tradeoff between estimation accuracy in the parameters verses the degree of collinearity between the I/O regressor vectors and the RMS noise level of the contaminating noises (v, w). The degree of collinearity is controlled by inserting the external signal $d(t)$, i.e., perfect collinearity results (hence complete degeneracy) under the condition: $d(t)=0$.
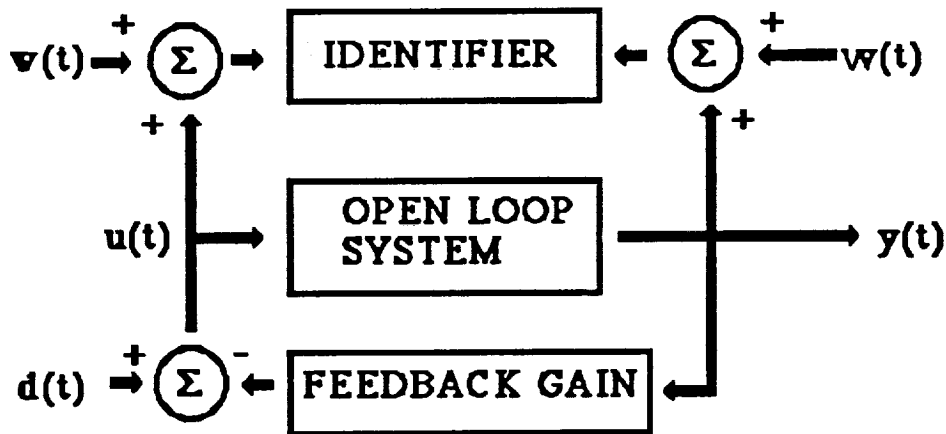

Fig. 2 : Closed Loop System Identification

An example simulation result is shown in Fig. 3 for the system with the forward (open) loop transfer function:

$$G(s)=\frac{20}{s^2+5s-5}. \tag{11}$$

The top of Fig. 3 shows the input/output data for one particular external signal which resulted in the normalized correlation coefficient $Nyu=-.9963$, where Nyu is defined by:

$$Nyu=\frac{Y'U}{|Y||U|}$$

and (U,Y) are the vectors of finite Fourier series coefficients of the I/O data defined in

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footnote

[5] Klein, V., "On Parameter Estimation of Highly Augmented Aircraft," AIAA Paper No. 89-3356. Presented at the 1989 Atmospheric Flight Mechanics Conf., Boston, MA, August 1989.

Time : Output <-> : -Input <--> :   Nyu = -.9963

Parameter Identification Under Near

Perfect Data Collinearity Conditions

Fig. 3

$$|\theta - \hat{\theta}| = \frac{max}{i} \left| \frac{\hat{\theta}_i - \theta_i}{\theta_i} \right| \approx 5\%$$
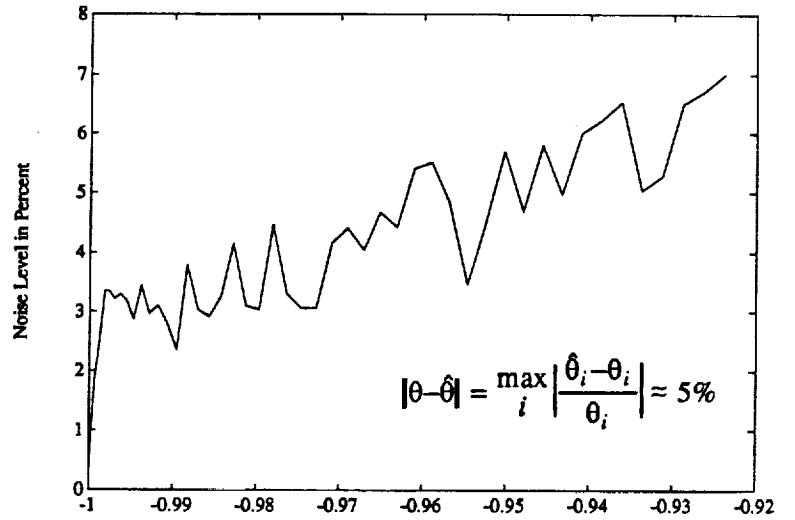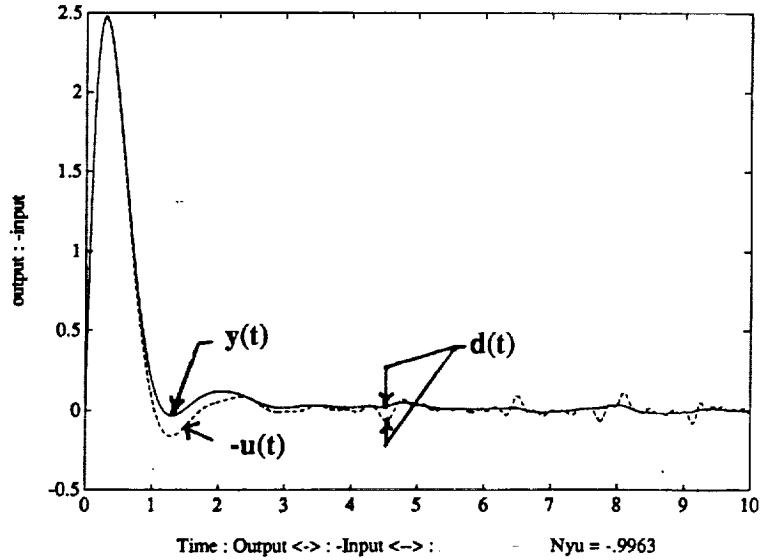
Noise Levels in $(v(t), w(t))$ Verses *Nyu* for 5% Error in $\hat{\theta}$

Noise Level : Error in Predicted Output : MF <-> : ARX <-->
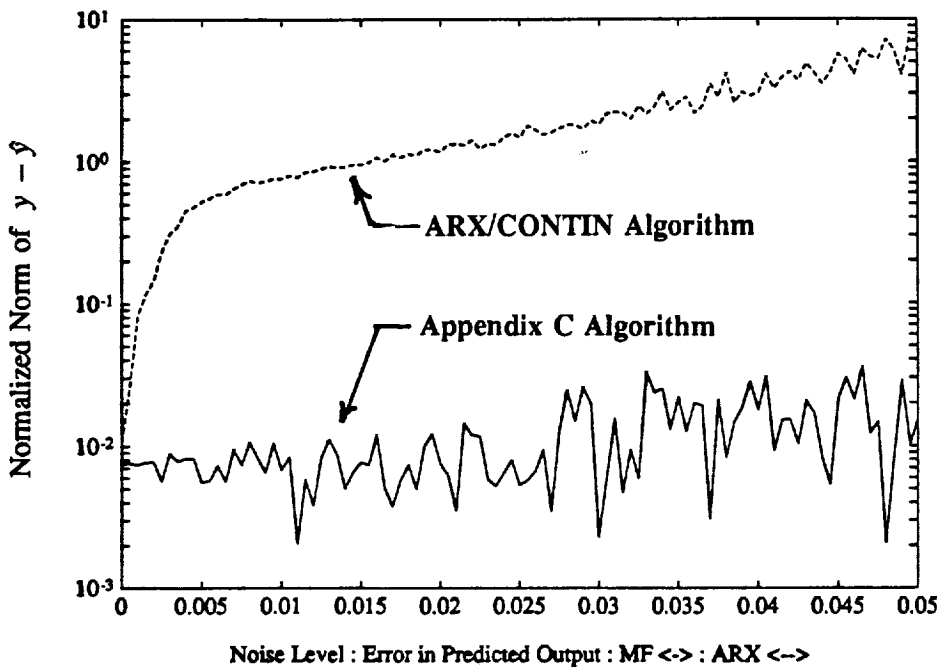
Error in Output Estimate Verses

RMS Noise Levels in (v(t),w(t))

Fig. 4

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Section 2 of the Appendix C paper. The lower portion of Fig. 3 is the result of numerous simulations under different $d(t)$ signals to achieve various values of Nyu while increasing the levels of the contaminating noises until a threshold 5% accuracy level is attained in the parameter estimates using the least squares algorithm of Appendix C. This graph shows that starting from complete degeneracy, i.e., $Nyu=-1.0$, zero noise level can be tolerated, but modest nonzero noise levels can be present in increasing amounts as collinearity collapses while still maintaining the accuracy level in the estimated parameters of approximately 5%.

The above results give some idea of the tradeoff between degree of collinearity and noise levels in the data for the least squares algorithm of the Appendix C paper, but the question arises as to how this compares with other approaches. This is difficult to answer, especially since there are not many algorithms available that are designed exclusively for differential systems. However, one easy comparison we have made is to apply the ARX parameter estimation algorithm of the System Identification Toolbox (by L. Ljung) in MATLAB.[6] We have been using MATLAB to carry out all simulations anyway, hence the ease in comparison. ARX identifies a model in discrete time which we then convert to a differential equation model using the MATLAB algorithm CONTIN designed for this purpose. One result of this comparison is shown in Fig. 4 which depicts the output error between the actual output and the predicted output (using the parameter values estimated by the two algorithms) versus increasing levels of RMS noises in the data. These simulations were carried out under non-collinearity conditions $d(t) \neq 0$ for the system (11) and show the superiority of the algorithm of Appendix C in relation to the ARX/CONTIN algorithm for this example. It must be admitted, however, that ARX is designed for discrete-time models and, therefore, such a comparison is somewhat biased.

The above study follows an earlier investigation into two approaches to alleviating the degeneracies of collinearity; one approach utilizes projection operators designed to zero out the collinear vectors known to cause the degeneracies thereby obtaining a least squares regression equation with fewer parameters and a better conditioned estimation problem. A second approach involves subdividing the total observation time interval into subintervals during which linear feedback is in effect, then redefining the modulating functions relative to one or more of these intervals in order to formulate the least squares estimation problem specific to the intervals causing the degeneracies. Neither of these approaches has shown any advantage to the straightforward application of the Appendix C algorithm, an example of which has been briefly discussed above for the system (11). An Sc.M. thesis is under preparation by A. Fullerton detailing these investigations.

3.4. Frequency Analysis

A method of frequency analysis for determining the transfer function $G(j\omega)$ from transient I/O data has been formulated in the Appendix D paper using *complex* valued Fourier based modulating functions in contrast with the trigonometric modulating functions used in the Appendix C paper for the parameter estimation problem. We started this investigation with the expectation that the more explicit representation of the complex form (compare Eq.

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[6] PC-MATLAB by The MathWorks, Inc.

(A.14) in Appendix C with Eq. (3) in Appendix D) might be valuable in answering questions about optimal inputs for parameter identification, optimal modulating functions for noise suppression, and the like. Although this work is ongoing, it became clear that the complex form is ideally suited to the frequency analysis problem since it facilitates a linear-in-parameters least squares formulation for frequency related parameters. Several variants of the formulation are possible, but all utilize a finite set of the Fourier series coefficients calculated from the I/O data over time intervals $[t_i, t_i + T_0]$, $i=1,2, \cdots$, each of duration $T_0 = 2\pi/\omega_0$, where ω_0 is a user selected "resolving" frequency.

A simulation result of applying the algorithm detailed in Section 2.2 of the Appendix D paper is given in Fig. 5 under noise-free conditions for the system with the low pass transfer function:

$$G(s) = \frac{1.7s^2 + 1736.8}{s^3 + 19.1s^2 + 257.48s + 1736.8}. \tag{12}$$

The nine seconds of I/O data shown was subdivided into nine $[0, T_0]$ intervals, i.e., $T_0 = 1$, and the algorithm produced essentially perfect estimation of the magnitude/phase plots for $G(j\omega)$ at the frequencies $k2\pi$, $k = 1,2 \cdots 6$, as shown by the rectangles. Also shown (by the $\times$ marks) are the magnitude/phase values that resulted from a direct computation of the ratio of the Fourier-type integrals:

$$\frac{\int_0^9 y(t)e^{-jk2\pi t}\,dt}{\int_0^9 u(t)e^{-jk2\pi t}\,dt}, \quad k = 1,2 \cdots 6 \tag{13}$$

The reason for the errors in the above "direct ratio" is due to the finite time data, i.e., the finite limits of integration in (13).

A similar comparison is shown in Fig. 6 relative to the high pass transfer function:

$$G(s) = \frac{s^3 + 22.02s}{s^3 + 22.24s^2 + 247.44s + 1943.23}. \tag{14}$$

This comparison shows that the error in the direct ratio will generally be more pronounced under nonzero initial conditions.

As the formulations in the Appendix D paper undergo further investigations, one modification that has been made is the removal of the presumed knowledge of the DC value $G(0)$. These and additional simulations under noise corrupted data conditions will be the focus of some our future work in this area.
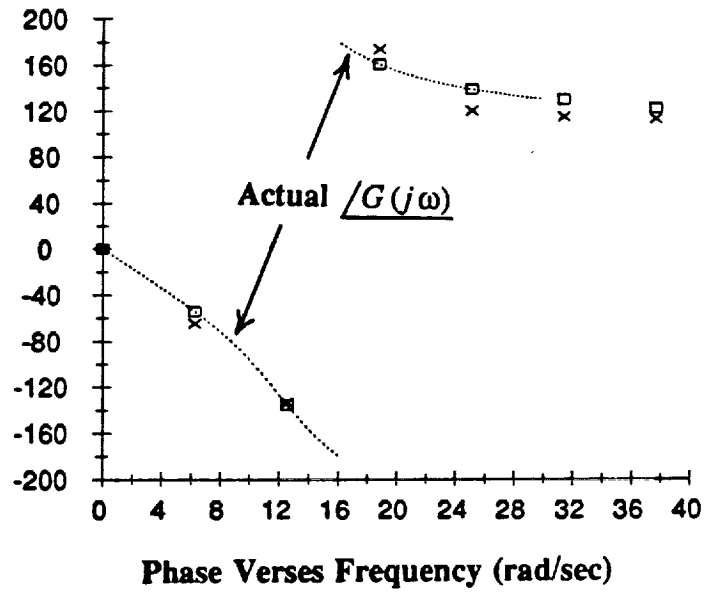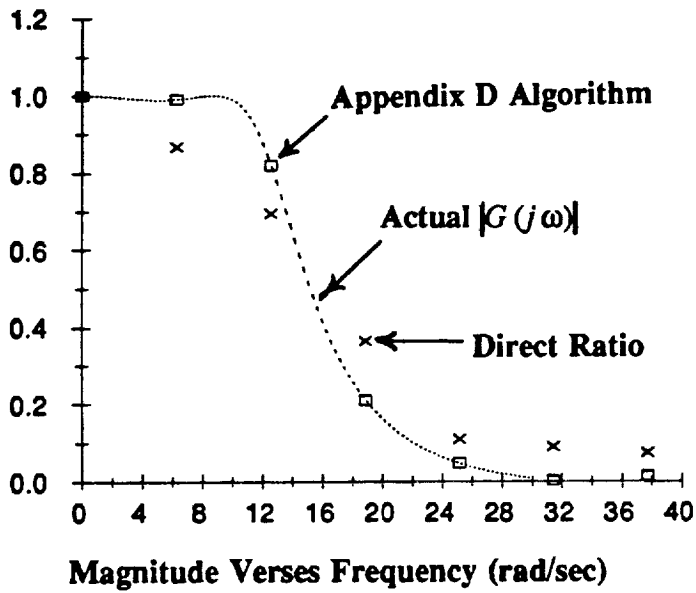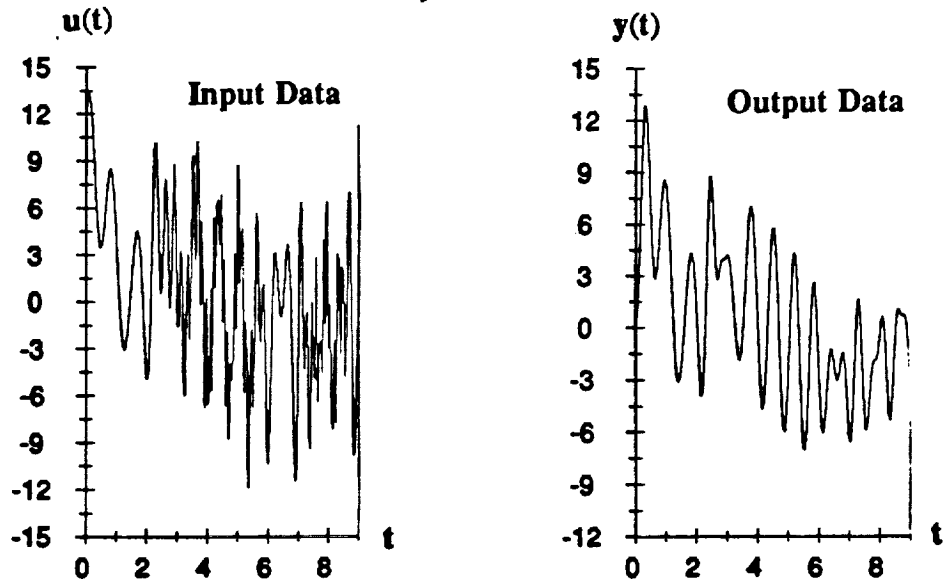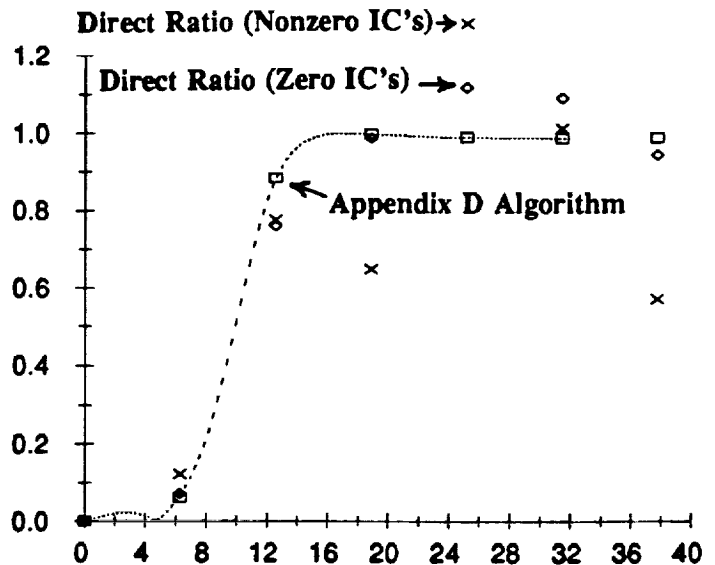
Fig. 5: Frequency Analysis for the Low Pass System of Eq. (12)

Magnitude Verses Frequency for the

High Pass System of Eq. (14)

Fig. 6

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Austin, Texas • December 1988

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Appendix A

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LEAST SQUARES PARAMETER IDENTIFICATION
OF NONLINEAR DIFFERENTIAL I/O MODELS [1]

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ABSTRACT

A least squares parameter identification technique is formulated for deterministic systems modeled by a class of input-output non-linear differential operator equations. Based on the notion of *exactness* in the calculus, a distinction is made on the basis of whether or not the equation error representation is an *exact* differential expression. It is shown how equation error models which are *exact* can be integrated for any given input-output data pair to yield an explicit function of the parameters that can be used for standard least squares estimation techniques. The formulation is then extended to apply to a class of *inexact* equation error system models. Also discussed is the notion of 'identifiability' as it relates to the class of systems under consideration.

1. Introduction

Consider a class of deterministic systems whose input-output (I/O) relation can be described implicitly by a differential operator equation of the form:

$$E(y(t),py(t),\cdots p^n y(t),u(t),pu(t),\cdots p^m u(t),\theta) = 0 \qquad (1)$$

where p denotes the differential operator d/dt so that $p^2=d^2/dt^2$, etc. It is assumed that E is a specified scalar valued nonlinear function of the output $y(t)$ and its first n time derivatives, the input $u(t)$ and its first m time derivatives, $m \leq n$, and a parameter vector θ; the latter is to be estimated given the input-output data $[u(t),y(t)]$, but not derivatives of the data, on some time interval. In the case of linear systems, i.e., $E(\cdot,\cdot,\cdot)$ linear in each of its arguments, equation error models have long been used in various parameter identification techniques (see for example Mendel [1]). A property often taken for granted with linear models is that they can be integrated, given continuous-time input-output data, or summed given sampled data for discrete-time system models, in order to obtain an explicit function of the coefficient parameters.

Such models seem not to have been explored very much for nonlinear systems, the emphasis instead being on techniques that deal with the normal form state vector equations:[2]

$$\dot{x}(t)=f(x(t),u(t),\theta)$$

$$y(t)=c(x(t),u(t),\theta). \qquad (2)$$

where (f,c) are given functions, generally nonlinear in x and u, parametrized by θ. Although these models apply to a broad class of physical systems, it seems difficult to ease the computational burden in parameter identification by exploiting any special structure like linearity in certain state variables or parameters. In addition, unknown initial conditions have to be appended to the parameter vector for time limited data unless the data is collected under special conditions.

The purpose of this paper is to show how special structure can play a role in easing the parameter identification problem for a class of nonlinear systems subsumed by (1). [3] The main assumptions are: (i) that the data is free of measurement noise, and (ii) for an arbitrary input-output data pair $[u(t),y(t)]$, observed over a time interval $0 \leq t \leq T$, there exists a parameter vector θ^* such that the model (1) is satisfied on $[0,T]$ with $\theta=\theta^*$. Thus, modeling errors are presumed small enough that deterministic least squares will be meaningful. In addition, standard causality and continuity conditions are tacitly assumed so that a unique bounded solution $y(t)$ exists satisfying (1) on any finite time interval $[0,T]$ given the system parameter vector θ and a bounded input $u(t)$ over the interval $[0,T]$, together with the appropriate initial conditions. However, being able to solve uniquely for $y(t)$ given $[\theta,u(t)]$, $0 \leq t \leq T$, and the initial conditions does not necessarily imply being able to integrate the parametrized equation error (1) relative to a data pair $[u(t),y(t)]$ on $[0,T]$ with the aim of obtaining a function of θ useful for parameter identification purposes.

As examples to illustrate the affect of structure, consider first the forced Van der Pol equation:

$$\ddot{y}(t)-\theta_1\left[1-y^2(t)\right]\dot{y}(t)+\theta_2 y(t)=u(t). \qquad (3)$$

Noting that $py^3=3y^2py$, (3) is equivalent to the differential operator equation:

$$(p^2-\theta_1 p+\theta_2)y(t)+\frac{1}{3}\theta_1 py^3(t)-u(t)=0 \qquad (4)$$

and therefore it is the second total differential of a function $z(t,\theta)$ defined implicitly by the solution to the following equation:

$$p^2 z(t,\theta)=(p^2-\theta_1 p+\theta_2)y(t)+\frac{1}{3}\theta_1 py^3(t)-u(t). \qquad (5)$$

With due consideration paid to unknown initial conditions, the solution $z(t,\theta)$ provides the basis for an explicit equation error function of θ given that $[u(t),y(t),y^3(t)]$ are regarded as forcing functions on some time interval $[0,T]$. It is noted that (4) can also be expressed in the following vector-matrix form:

$$(1,\theta_1,\theta_2)\begin{bmatrix} p^2 & 0 & 1 \\ 0 & p & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ \frac{1}{3}y^3(t)-y(t) \\ -u(t) \end{bmatrix}=0. \qquad (6)$$

Contrasting with the structure of the preceding example, the following differential operator equation does not share a similar property:

$$p^2 y(t)+\theta_1\left[py(t)\right]^2-\theta_2 u(t)=0. \qquad (7)$$

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[2] For example, the method of quasilinearization (Bellman and Kalaba [2]).

[3] The first four sections of this paper follow closely those of an earlier paper presented at the 1988 CISS meeting at Princeton University [3].

This equation, which models the position $y(t)$ of a particle subject to force $u(t)$ and drag proportional to the square of the velocity, cannot be integrated to yield an explicit function of θ when given the data $[u(t), y(t)]$ over $[0,T]$ because (7) is not the total second differential of any recognizable function $z(t,\theta)$ as in the case of the model (4). What is needed is some kind of "integrating factor" to handle the nonlinear drag term since velocity is not a measured signal. Utilizing the differential identity: $p^2(y^2)=2y(p^2y)+2(py)^2$, (7) can be equivalently represented by

$$p^2 \left[y(t) + \frac{1}{2}\theta_1 y^2(t) \right] - \theta_2 u(t) - \theta_1 y(t) \left[p^2 y(t) \right] = 0 \qquad (8)$$

which is in a form suitable for later reference.

The inherent difference in structure displayed by the above examples motivates the equation error identification techniques of this paper. The desired properties of any such technique can be described abstractly in terms of an operator B whose domain coincides with the range space of E and whose image element as a function of θ is expressed as: $e(\theta)=B(E(y,py,\cdots p^n y,u,pu,\cdots p^m u,\theta))$ for a given data pair $[u,y]$. This integral-type operator should possess the following properties: (i) $e(\theta)$ can be explicitly computed as a function of θ given the input-output data on a time interval $[0,T]$ without the need to estimate unknown initial conditions, i.e., time derivatives of the data are not available, and (ii) $e(\theta)$ provides a measure of error in the parameters such that $||e(\theta)||=0$ correctly reflects the true value of θ for a suitable norm $||\cdot||$ under ideal noise-free conditions. Hence, the minimization problem: $\min ||e(\theta)||$ facilitates the potential for obtaining a unique least squares estimate of the parameters under appropriate nondegeneracy conditions on the data. The ease with which such an operator B can be devised possessing these properties depends strongly on the nature of the model.

2. Structural Considerations

Motivated by the notion of an "exact differential" in the calculus, a system model of the input-output type (1) will be called *exact* if it admits to the representation:

$$E(y,py,\cdots p^n y,u,pu,\cdots p^m u,\theta)=\sum_{i=1}^{n_1} P_i(p)E_i(y,u,\theta) \qquad (9)$$

where the $P_i(p)$, $i=1,2,\cdots n_1$, are polynomials of degree $\leq n$ in the differential operator p and the E_i are nonunique but sufficiently smooth nonlinear functions of the triple (y,u,θ). If such a representation does not exist, then the model (1) is said to be *inexact*. The significance of the model being *exact* is that it is the total differential of order n with respect to time t of some function $z(t,\theta)$, i.e., there exists a sufficiently smooth function $z(t,\theta)$ such that for each fixed value of θ:

$$p^n z(t,\theta)=\sum_{i=1}^{n_1} P_i(p)E_i(y(t),u(t),\theta). \qquad (10)$$

Another basic model property important for system identification is that of separability with respect to the parameters. Thus, the model (1) is said to be *separable* with respect to the parameters if there exist scalar-valued functions $h_i(\theta)$ and $E_i(y,py,\cdots p^n y,u,pu,\cdots p^m u)$, $i=1,2,\cdots n_2$, such that

$$E(y,py,\cdots p^n y,u,pu,\cdots p^m u,\theta)$$

$$=\sum_{i=1}^{n_2} h_i(\theta)E_i(y,py,\cdots p^n y,u,pu,\cdots p^m u). \qquad (11)$$

In this case it is assumed that the vector function $h(\theta)$ defined by $h(\theta)=\text{col}(h_1(\theta),\cdots h_{n_2}(\theta))$ satisfies the single-valued property:

$$h(\theta)=h(\theta^*) \quad \text{if and only if } \theta=\theta^*. \qquad (12)$$

All linear system models are clearly *exact* and *separable* with respect to the parameters. If a nonlinear system model of the form (1) is both *exact* and *separable* with respect to the parameters, then consistent with the above definitions it will admit to the representation:

$$E(y,py,\cdots p^n y,u,pu,\cdots p^m u,\theta)=\sum_{i,j} h_i(\theta)P_{ij}(p)E_j(y,u) \qquad (13)$$

for some polynomials $P_{ij}(p)$ in p, each of degree $\leq n$, and nonlinear functions $E_j(y,u)$, none of which depend on the parameters θ, together with a vector function $h(\theta)$ which is assumed to satisfy (12). Model (4) provides such an example via the equivalent vector-matrix representation in (6).

Needless to say, *inexact* and *nonseparable* parameter models are inherently the most difficult to handle. The following sections will indicate ways of devising an operator B with the desired equation error properties for *exact* and a class of *inexact* models. [4] For the most part these methods will be efficient only for models which are separable in the parameters, or models which possess at most one or two nonseparable parameters. Letting α denote the nonseparable parameter(s), the model (1) can be referred to as *partially separable* with respect to the parameters (α,θ) if it admits to the representation (cf. (11)):

$$E(y,py,\cdots p^n y,u,pu,\cdots p^m u,\alpha,\theta)$$

$$=\sum_i h_i(\theta)E_i(y,py,\cdots p^n y,u,pu,\cdots p^m u,\alpha). \qquad (14)$$

Time lag systems with unknown delay parameters provide examples of such models. Furthermore, if the model (1) is both *exact* and *partially separable* with respect to parameters (α,θ) (cf. (13)):

$$E(y,py,\cdots p^n y,u,pu,\cdots p^m u,\alpha,\theta)$$

$$=\sum_{i,j} h_i(\theta)P_{ij}(p)E_j(y,u,\alpha). \qquad (15)$$

As before, the vector function $h(\theta)$ comprised of the $h_i(\theta)$'s in (14) and (15) is presumed to satisfy (12).

3. Exact Differential and Separable Parameter Models

Consider the class of *exact* and *separable* models represented by (13). In this case there exist at least two different ways of defining a linear operator B such that the norm of $e(\theta)$:

$$e(\theta)=\sum_i h_i(\theta)B\left(\sum_j P_{ij}(p)E_j(y,u)\right) \qquad (16)$$

has the properties of a metric function given certain nondegeneracy conditions on the data $[u(t),y(t)]$, $0\leq t \leq T$. In general terms, nondegeneracy is guaranteed if the set of functions $\{\sum_j P_{ij}(p)E_j(y(t),u(t))\}$, $i=1,2\cdots$, are linearly independent on $[0,T]$ and do not lie in the null space of the operator B. Then the abstract equation $e(\theta)=0$ has a unique solution $\theta=\theta^*$ which is the desired value for the parameters under ideal noise-free conditions. Further, $\min ||e(\theta)||$ provides a least squares estimate under nonideal conditions.

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[4] Also discussed in [3] is an intermediate class of *provisionally exact* system models that reduce to *exact* systems when subjected to special inputs over finite time intervals.

3.1. Operator B Via Moment Functionals

One way to specify an operator B with the desired properties is via the classical Shinbrot method of moment functionals, also called the modulating function approach, which converts linear differential expressions on a finite time interval into algebraic equations (see [4]). Using Fourier based modulating functions, this approach has been used in Pearson and Lee [5,6] to formulate parameter identification of both linear and polynomial input-output differential operator models. The basic ingredient here is a vector $\Phi(t)$ of modulating functions defined as follows:

$$\Phi(t) = C f(t), \quad 0 \le t \le T \qquad (17)$$

where $f(t)$ is defined by the $(2L+1) \times 1$ column vector of commensurable sinusoids

$$f(t) = \text{col}\left[1; \cos\omega_0 t, \sin\omega_0 t; \cdots ; \cos L\,\omega_0 t, \sin L\,\omega_0 t\right] \qquad (18)$$

$$0 \le t \le T, \quad \omega_0 = 2\pi/T$$

and C is a $(2L+1-n) \times (2L+1)$ matrix whose rows are determined by the end point conditions:

$$\Phi^{(i)}(0) = \Phi^{(i)}(T) = 0, \quad i = 0,1,2 \cdots (n-1). \qquad (19)$$

Here $\Phi^{(i)}(t)$ means $p^i \Phi(t)$. It can be seen from (17) and (18) that the time derivatives of $\Phi(t)$ have the representation

$$(-1)^i \Phi^{(i)}(t) = C D^i f(t), \quad i = 0,1,2,\cdots \qquad (20)$$

where D is an operational matrix defined by the block diagonal structure:

$$D = \omega_0 \, \text{diag}\left[0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 & L \\ -L & 0 \end{bmatrix}\right]. \qquad (21)$$

To invoke the method of moment functionals for the model (1) which possesses the representation (13): Multiply both sides of (1) by $\Phi(t)$, integrate over the time interval $[0,T]$ and interchange integration and summation over i, then use integration-by-parts while noting (19) and (20). The result is the following vector algebraic equation which essentially defines the operator B:

$$e(\theta) = \sum_{i,j} h_i(\theta) C P_{ij}(D) V_j. \qquad (22)$$

Note that the differential operator p in the polynomials $P_{ij}(p)$ has been replaced by the block diagonal matrix D of (21). The V_j are $(2L+1) \times 1$ vectors defined by

$$V_j = \int_0^T E_j(y(t), u(t)) f(t)\, dt. \qquad (23)$$

As discussed in [5,6], these data-related vectors can be computed by well known DFT/FFT techniques since they are finite Fourier series coefficient vectors for the functions $v_j(t) = E_j(y(t), u(t))$ on $[0,T]$. The Euclidean norm of $e(\theta)$ in (22) then provides a metric function $\rho(\theta, \theta^*)$ on the space of adjustable parameters, and the square of this norm defines a suitable function $J(\theta)$ for least squares minimization:

$$J(\theta) = h'(\theta) \Omega h(\theta) \qquad (24)$$

where the (i,j) component of the symmetric nonnegative definite matrix Ω is given by

$$\{\Omega\}_{ij} = \sum_{k,l} V_k' P_{ik}(D') C' C P_{jl}(D) V_l. \qquad (25)$$

Hence, the major computations to set up the least squares problem in this approach is to calculate the vectors of finite Fourier series coefficients for the data related functions $E_j(y(t), u(t))$ followed by the inner products involving these vectors in (25) to obtain Ω. The null space of the operator B is essentially all functions which are orthogonal to the sinusoids comprising $f(t)$ in (18). Hence, choosing a sufficiently large integer L shrinks the null space so

that from a practical standpoint nondegeneracy for the least squares problem means linear independence of the functions $\{\sum_j P_{ij}(p) E_j(y(t), u(t))\}$, $i = 1,2 \cdots$, on $[0,T]$. However, based on finite bandwidth considerations, choosing L large will necessitate a smaller 'resolving frequency' ω_0 and therefore a longer $[0,T]$ interval, as seen from (18). These issues are more fully discussed in [5].

3.2. Operator B Via State Variable Filters

Another such B operator is basically a projected state variable filter (see [7,8]). Its specification can be summarized as follows. Let $F(s)$ be a polynomial of degree larger than n chosen by the user so that $F^{-1}(s)P(s)$ is a strictly proper and stable transfer function matrix with the polynomial matrix $P(p)$ defined by the $P_{ij}(p)$ in (13). Let the filtered signals $z_i(t)$, $0 \le t \le T$, be defined implicitly by zero state solutions to the differential operator equations:

$$F(p) z_i(t) = \sum_j P_{ij}(p) E_j(y(t), u(t)). \qquad (26)$$

In order to obviate dealing with all unknown initial conditions define a projection of these signals on $[0,T]$ by the relation:

$$\bar{z}_i(t) = z_i(t) - c_o e^{At} W^{-1} \int_0^T e^{A'\tau} c_o' z_i(\tau)\, d\tau, \quad 0 \le t \le T \qquad (27)$$

where (A, c_o) is an observable realization for the homogeneous equation: $F(p)z(t) = 0$, and W^{-1} is the inverse of the observability Gramian for this realization over $[0,T]$, i.e.,

$$W = \int_0^T e^{A't} c_o' c_o e^{At}\, dt.$$

Then the integral squared norm of the following function

$$e(t, \theta) = \sum_i h_i(\theta) \bar{z}_i(t)$$

has the desired metric properties in that the square of this norm defines a suitable function for least squares minimization:

$$J(\theta) = h'(\theta) \left[\int_0^T \bar{z}(t) \bar{z}'(t)\, dt \right] h(\theta). \qquad (28)$$

Here $\bar{z}(t)$ stands for the column vector of functions with components $\bar{z}_i(t)$. Hence this specification of the operator B essentially involves integrating the linear differential equations (26) with forcing functions $v_j(t) = E_j(y(t), u(t))$, performing the projections in (27) which strip away the affect of unknown initial conditions, and calculating the Gram matrix for these projections in order to define the least squares function (28).

3.3. Partially Separable Parameter Models

Consider the class of models which are *exact* and *partially separable* with respect to parameters (α, θ) so that the representation (15) holds. In this case the least squares functions of (24) and (28) will take the forms

$$J(\alpha, \theta) = h'(\theta) \Omega(\alpha) h(\theta). \qquad (29)$$

and

$$J(\alpha, \theta) = h'(\theta) \left[\int_0^T \bar{z}(t, \alpha) \bar{z}'(t, \alpha)\, dt \right] h(\theta) \qquad (30)$$

respectively. It is seen that the dependence on the nonseparable parameter α comes through the V_j vectors for Ω, i.e.,

$$V_j(\alpha)=\int_0^T E_j(y,u,\alpha)f(t)dt,$$

while the filter signals $z_i(t)$ depend on α via

$$z_i(t,\alpha)=F^{-1}(p)\sum_j P_{ij}(p)E_j(y,u,\alpha).$$

If the source of the nonseparability is a delay parameter, e.g., $E_j(y(t),u(t),\alpha)=E_j(y(t),u(t-\alpha))$, then as shown in Pearson and Wuu [9] for differential-delay equation models, it is possible to derive the 'variable projection' functional using nonlinear least squares theory [10] in order to first estimate the α parameters, after which the separable parameters may be estimated as before. In this way, it is possible to decouple the difficult-to-estimate α parameters from the more straightforward θ parameters.

4. A Class of Inexact Differential System Models

Suppose the model (1) admits to the representation

$$E(y,py,\cdots p^n y,u,pu,\cdots p^m u,\theta)$$
$$=\sum_{i,j}h_i(\theta)F_{ij}(y,u)P_{ij}(p)E_j(y,u) \qquad (31)$$

where again the $P_{ij}(p)$ are polynomials in p of degree $\leq n$ and the vector function $h(\theta)$ comprised of the $h_i(\theta)$'s satisfies (12). Model (7) is an example of this class when expressed in the equivalent form (8). It will now be demonstrated that under certain smoothness conditions pertaining to the data-related functions $g_{ij}(t)$ defined by

$$g_{ij}(t)=F_{ij}(y(t),u(t)), \; 0\leq t\leq T \qquad (32)$$

the method of moment functionals of the previous section can be used as a kind of 'integrating functional' for the inexact model (31). In order to do so, it will first be noted that the multiplication of any modulating function $\phi(t)$ of order n over $[0,T]$ by an arbitrary n-times continuously differentiable function $g(t)$ defined on $[0,T]$ is again a modulating function of order n on $[0,T]$. That is, given that a n-times continuously differentiable function $\phi(t)$ satisfies the end point conditions:

$$\phi^{(i)}(0)=\phi^{(i)}(T)=0, \; i=0,1,2,\cdots(n-1),$$

the function $\bar\phi(t)=g(t)\phi(t)$ satisfies the same end point conditions for any sufficiently smooth function $g(t)$ on $[0,T]$. Relative to the functions $g_{ij}(t)$ of (32), it will be assumed that each such function can be represented to any desired degree of accuracy by a finite Fourier series, i.e., for a sufficiently large integer M,

$$g_{ij}(t)\approx\sum_{m=0}^M [a_{ij}(m)\cos m\omega_0 t +b_{ij}(m)\sin m\omega_0 t], \; 0\leq t\leq T \qquad (33)$$

where the $(a_{ij}(m),b_{ij}(m))$ coefficients can be computed via well known DFT/FFT techniques for each $g_{ij}(t)$ function.

Now invoke the method of moment functionals for the model (1) satisfying (31) using the Fourier based modulating functions defined in (17) and noting the aforementioned property of such functions. Thus, multiply both sides of (1) with the representation (31) by $\Phi(t)$ while using the expansions (33) for each $F_{ij}(y(t),u(t))$ function in (32), integrate over the time interval $[0,T]$ and interchange integration and summations, use integration-by-parts while noting (19) and (20), and take into account the trigonometric identities:

$$2\cos kx\cos mx=\cos(k-m)x+\cos(k+m)x$$

$$2\sin kx\cos mx=\sin(k-m)x+\sin(k+m)x$$

$$2\sin kx\sin mx=\cos(k-m)x-\cos(k+m)x.$$

The result is the vector algebraic equation: [5]

$$e(\theta)=\sum_{i,j}h_i(\theta)W(i,j)\bar V_j \qquad (34)$$

where the matrices $W(i,j)$ and vectors $\bar V_j$ are defined as follows:

$$W(i,j)=C\sum_{m=0}^M [a_{ij}(m)Q_m+b_{ij}(m)R_m]P_{ij}(\bar D) \qquad (35)$$

$$\bar V_j=\int_0^T E_j(y(t),u(t))\bar f(t)dt. \qquad (36)$$

The matrices Q_m and R_m in (35) are quasi-banded structures that arise from the interactions between the basis functions in (33) and the commensurable sinusoids comprising the Fourier based modulating functions in (17)-(18). Their precise representations can be found in Equations (25) and (26) of Pearson and Lee [11] where they arose in a different context. The tilde over the various expressions in (34)-(36), in particular $\bar D$ in (35) and $\bar f(t)$ in (36), is meant to indicate that the harmonic frequencies in these terms extend out to $(L+M)\omega_0$; again, this is due to the interactions between the sinusoids in (33) and Fourier based modulating functions. The fact that the Fourier series coefficients for these higher order harmonics have to be computed is of minor importance from a computational viewpoint since the DFT/FFT algorithm will yield many more frequencies than are actually retained if a high degree of accuracy is employed. (Refer to the discussion in Section 2.2 of [5].) However, noise in the data is important which may necessitate a longer time interval for the data, i.e., a shorter resolving frequency $\omega_0=2\pi/T$, in order to cut off high frequency noise in the higher harmonics.

The equation error expression in (34) essentially defines a B operator with the previously outlined desired properties, together with a kind of 'integrating factor' property to handle the *inexact* aspect of the model (31). As before, the Euclidean norm of $e(\theta)$ in (34) can be used to define a positive definite function $J(\theta)$ suitable for least squares minimization.

5. Uniqueness of a LS Estimate for Separable Models

The single-valued property assumed for the function $h(\theta)$ in (12) makes the statement of conditions for a unique least squares estimate more transparent than the conditions attending other models and approaches. To elaborate on this a bit more in relation to the separable model (11) let the functions $v_i(t)$ be defined by

$$v_i(t)=E_i(y(t),py(t),\cdots p^n y(t),u(t),\cdots p^m u(t))$$

and define $\varepsilon(t,\theta)$ by

$$\varepsilon(t,\theta)=\sum_{i=1}^{n_2}h_i(\theta)v_i(t).$$

Suppose θ^* is a value of the parameter vector such that $\varepsilon(t,\theta^*)=0$ for any input-output pair $[u(t),y(t)]$ on $[0,T]$. Thus for any particular input-output pair leading to a particular sequence of functions $v_i(t)$:

$$\varepsilon(t,\theta)=\sum_{i=1}^{n_2}\left[h_i(\theta)-h_i(\theta^*)\right]v_i(t).$$

Hence uniqueness of any least squares estimate is predicated on linear independence of the data-related functions $v_i(t)$, $i=1,2\cdots n_2$, on $[0,T]$. In turn, this dictates linear independence conditions on the input-output data and its derivatives in order to guarantee uniqueness. Of course, it is the projections of these functions via

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[5] It is assumed that $L\geq M$; if not, simply add more modulating functions until $L\geq M$.

the operator B that must be used to test uniqueness since derivatives of the data are not presumed to be available.

As an example for comparison with other models and approaches, consider a two compartmental model from chemical kinetics (Example 7.1 in Walter [12]):

$$\dot{y}(t)=-\theta_1 y(t)-\theta_2(1-\theta_3 x_2(t))y(t)+u(t) \qquad (37)$$

$$\dot{x}_2(t)=\theta_2(1-\theta_3 x_2(t))y(t)-\theta_4 x_2(t)$$

where $(\theta_1,\theta_2,\theta_3,\theta_4)$ represent parameters to be determined given the data $[u(t),y(t)]$ over some time interval. Under the assumption that $y(t)$ is positive for all t of interest, i.e., $y(t)>0$, $0 \le t \le T$, the following differential operator equation can be derived by eliminating the state $x_2(t)$ from the above equations:

$$h'(\theta) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p & 1 & 0 & 0 & 0 \\ 0 & 0 & p & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y(t) \\ -u(t) \\ \ln(y(t)) \\ -u(t)/y(t) \\ 1 \end{bmatrix}$$

$$+ \left[(p+1)(\frac{1}{y(t)}) \right] \left[p^2+p , p+1 \right] \begin{bmatrix} y(t) \\ -u(t) \end{bmatrix} = 0 \qquad (38)$$

where the components of the vector function $h(\theta)$ are defined by:

$$h_1=\theta_1\theta_2\theta_3, \quad h_2=\theta_2\theta_3, \quad h_3=\theta_4, \quad h_4=(\theta_1+\theta_2)\theta_4.$$

Bearing in mind the previous discussion for the model (31), first approximate the function $1/y(t)$ by a finite Fourier series as in (33) with coefficients $(a(m),b(m))$ and then replace the pairs $(a(m),b(m))$ in (35) by the pairs $(a(m)+m\omega_0 b(m),b(m)-m\omega_0 a(m))$, thereby achieving an approximation to the function $(p+1)(1/y(t))$ which plays the role akin to $g_{ij}(t)$ in (32).

The validity of extending the model (31) to the example (37) will require a greater degree of smoothness on the part of the data since the differential expression $(p+1)(1/y(t))$ is evaluated in terms of the approximation for $1/y(t)$. However, apart from the question of approximation, it is noted that since there is a one-to-one map between h and θ (except for a set of measure zero) the necessary and sufficient conditions for the existence and uniqueness of solutions to the parameter identification problem entail linear independence of the four signals: $[y(t), py(t)-u(t), p(\ln y(t))-u(t)/y(t), 1]$ on $[0,T]$. This should be compared with the 'generating series' analyses given in [12] for deciding the issue of 'structural identifiability' of the model (37). That is, not only is the issue of 'identifiability' resolved by inspection of (38), but the conditions for the uniqueness of a least square estimate can be related to linear independence of the projections of the appropriate signals.

6. Conclusions

Input-output nonlinear differential operator equation models have been classified according to the calculus notion of *exactness* for differential expressions. It has been shown that *exact* differential models can be integrated to yield a positive definite function suitable for least squares parameter identification given the I/O data on a finite time interval. This is not possible for *inexact* differential models. However, under a smoothness condition on the data, a kind of 'integrating functional' has been found via the classical method of moment functionals that permits integration of the parametrized equation error for a class of *inexact* models. The formulations also facilitate determining conditions on the I/O data that guarantee uniqueness of a least squares estimate.

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Identifiability and Well-Posedness in Nonlinear System I/O Modeling

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Abstract

If a specified set of parametrized nonlinear state differential equations possesses an equivalent input-output differential operator model, then the latter model has the potential for deciding the issue of the *identifiability* property of the former model in a straightforward manner. This property is discussed in relation to the property of *well-posedness* for the least squares parameter identification of a class of input-output models that are separable in the parameters.

Introduction

State differential equations of the form

$$\dot{x} = f(x, u, \theta)$$

$$y = h(x, u, \theta) \tag{1}$$

constitute the most commonly used starting point for investigations into the parameter identification of deterministic nonlinear differential systems. Given the input-output data $[u(t), y(t)]$ on some time interval $0 \leq t \leq T$, together with the functions $f(x, u, \theta)$ and $h(x, u, \theta)$, a number of methods are available for estimating the parameter vector θ such as quasilinearization, invariant embedding and the like. A theoretical issue considered by several investigators, e.g. [1-3], is that of the "identifiability" of the model (1). Although the definitions vary somewhat from one investigator to another, failure of the model (1) to be identifiable usually means a kind of over-parametrization or degeneracy in that it cannot be demonstrated that there exists a suitable input signal $u(t)$ and/or initial state $x_0 = x(0)$ such that the integration of the state equations corresponding to any two distinct parameter vectors θ_a and θ_b, $\theta_a \neq \theta_b$, gives rise to solutions $y_a(t)$ and $y_b(t)$ that can be distinguished from one another on $[0, T]$. An equivalent input/output differential operator model of the generic form

$$E(y, py, \cdots p^n y, u, pu, \cdots p^m u, \theta) = 0 \tag{2}$$

where p denotes the differential operator d/dt so that $p^2 = d^2/dt^2$, etc., has the potential for making this determination more transparent provided such an equivalence exists.[1] A simple example illustrating this situation is the scalar bilinear system (Example 2 in [2]):

$$\dot{x} = 1 + x + (1 + x)u$$

$$y = \theta(1 + x). \tag{3}$$

which has the equivalent input/output (I/O) relation

$$\dot{y} = y + yu. \tag{4}$$

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[1] The motivation for this study stems from the development in [4] of a least squares parameter estimation algorithm which is applicable to certain subclasses of the generic model (2).

The latter model clearly fails to be identifiable no matter what definition is employed since it is completely independent of the parameter θ. A less obvious example is the following second order system (this is Example 1 in [3]):

$$\dot{x}_1 = \theta_1 x_1^2 + \theta_2 x_1 x_2 + u$$

$$\dot{x}_2 = \theta_3 x_1^2 + \theta_4 x_1 x_2 \tag{5}$$

$$y = x_1.$$

If it is assumed that $y(t)$ is of fixed sign for all $t \in [0, T]$, e.g., $y(t) > 0$, $0 \leq t \leq T$, then (5) has the following equivalent I/O relation:

$$p^2(\ln y) - p\left(\frac{u}{y}\right) - \theta_1 py + (\theta_1 \theta_4 - \theta_2 \theta_3)y^2 + \theta_4(u - py)) = 0 \tag{6}$$

where $p(\ln y) = \dot{y}/y$. The identifiability property of this example (actually the nonidentifiability) will be made evident after a discussion of a subclass of the model (2) which includes (6).

Separable I/O Models

Consider the generic model (2) which is *separable* in the parameters, i.e., there exist scalar functions $g_i(\theta)$ and $E_i(y, py, \cdots p^n y, u, pu, \cdots p^m u)$, $i = 1, 2, \cdots n_1$, such that the function E in (2) admits to the representation

$$E(y, py, \cdots p^n y, u, pu, \cdots p^m u, \theta)$$

$$= \sum_{i=1}^{n_1} g_i(\theta) E_i(y, py, \cdots p^n y, u, pu, \cdots p^m u). \tag{7}$$

For any input-output pair $[u(t), y(t)]$ on $[0, T]$, let the functions $v_i(t)$ be defined by

$$v_i(t) = E_i(y(t), py(t), \cdots p^n y(t), u(t), \cdots p^m u(t)) \tag{8}$$

$$i = 1, 2 \cdots n_1$$

and define $e(\theta, t)$ by

$$e(\theta, t) = \sum_{i=1}^{n_1} g_i(\theta) v_i(t). \tag{9}$$

If θ^* is any value of the system parameters distinct from θ and $[u(t), y(t)]$ is any valid input-output pair, then the function $\delta(\theta, \theta^*, t)$ defined by

$$\delta(\theta, \theta^*, t) = e(\theta, t) - e(\theta^*, t) \tag{10}$$

$$= \sum_{i=1}^{n_1} \left[g_i(\theta) - g_i(\theta^*) \right] v_i(t)$$

makes clear the necessity of the single-valuedness of the vector function $g(\theta)$ comprised of the $g_i(\theta)$'s, i.e., the condition:

$$g(\theta) = g(\theta^*) \text{ if and only if } \theta = \theta^* \tag{11}$$

is necessary else any parameter estimation scheme based on the I/O model (2) which is separable in the parameters according to (7) is doomed to failure. This is clear from the consideration that $\delta(\theta, \theta^*, t)$ can provide for the basis of a metric function in

the parameter space only if (11) holds.[2] Stated in an alternative way, any least squares estimation scheme which is based on input-output data for the model (7) will not be *well-posed* if (11) fails.

Example 2 Continued: The (g_i, E_i) for the model (6) can be defined as

$$g_1 = 1, \qquad g_2 = \theta_1, \quad g_3 = \theta_1\theta_4 - \theta_2\theta_3, \quad g_4 = \theta_4 \qquad (12)$$

$$E_1 = p^2(\ln y) - p(\frac{u}{y}), \quad E_2 = py, \quad E_3 = y^2, \quad E_4 = u - py. \qquad (13)$$

Since both I/O models (4) and (6) possess $g(\theta)$ functions that fail to satisfy condition (11), i.e. $g(\theta)$ is degenerate in (4) while (12) makes evident the over-parametrization of (6) with respect to the (θ_2, θ_3) parameters, there does not exist an input function $u(t)$ which would facilitate a unique determination of the θ parameters given the input-output data.[3]

Assuming that condition (11) holds for an I/O model of the generic form (7), it seems reasonable that the least squares estimation problem is *well-posed*. However, the question of uniqueness of an estimate is data-dependent and is more difficult to answer, especially when only input-output data (no derivatives) is available. The following example illustrates this point. (This is Example 7.1 in [1].)

Example 3: A chemical kinetics model is defined by

$$\dot{x}_1 = -\theta_1 x_1 - \theta_2(1 - \theta_3 x_2)x_1 + u$$

$$\dot{x}_2 = \theta_2(1 - \theta_3 x_2)x_1 - \theta_4 x_2$$

$$y = x_1.$$

Eliminating the state x_2, the equivalent I/O model is found to be

$$p^2(\ln y) - p(\frac{u}{y}) + \theta_1\theta_2\theta_3 y + \theta_2\theta_3(py - u)$$

$$+ \theta_4\left[p(\ln y) - \frac{u}{y}\right] + (\theta_1 + \theta_2)\theta_4 = 0.$$

Here the representation (7) holds with

$$g_1 = \theta_1\theta_2\theta_3, \quad g_2 = \theta_2\theta_3, \quad g_3 = \theta_4, \quad g_4 = (\theta_1 + \theta_2)\theta_4$$

$$E_1 = y, \quad E_2 = py - u, \quad E_3 = p(\ln y) - \frac{u}{y}, \qquad E_4 = 1.$$

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[2] A physically realizable metric function $\rho(\delta, \delta^*)$, which is characterized by any particular input-output pair $[u(t), y(t)]$ on $[0,T]$, can be defined only if a way can be found to overcome the necessity of dealing with the derivatives of the data; such a way is discussed in [4] through the use of projected state variable filters or moment functionals for special subclasses of the model (2).

[3] Nonidentifiability of these examples was also concluded by the authors in [2] and [3].

The above function $g(\theta)$ is single-valued and moreover is one-to-one (except for a set of measure zero); hence, the least squares problem is not only *well-posed* but it is also basically linear in the $\hat{\theta} = g(\theta)$ parameters. A least squares estimate will be unique only if the corresponding regressor functions are linearly independent on $[0,T]$ which for this example means linear independence of the functions:

$$v_1(t) = y(t), \qquad v_2(t) = \dot{y}(t) - u(t)$$

$$v_3(t) = \frac{\dot{y}(t) - u(t)}{y(t)}, \quad v_4(t) = 1.$$

Concluding Remarks

The above examples make clear the necessity of a single-valuedness property of an equivalent input-ouput differential operator model in testing the identifiability of the state equation model (1) for the class of I/O models in (7). Stated another way, the least squares parameter estimation problem will not be well-posed for this class unless property (11) holds. The conclusion reached in each such example is consistent with the results obtained by the cited investigators but seems easier to reach when an equivalent I/O model exists. The question of uniqueness of an estimate will depend on linear independence of the data-dependent regressor functions. The latter question is more difficult to answer owing to the necessity of projecting these functions down into a subspace (using for example state variable filters) in order to obviate dealing with unknown initial and boundary conditions.

Acknowledgement

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Invited Paper

PARAMETER IDENTIFICATION OF LINEAR DIFFERENTIAL SYSTEMS VIA FOURIER BASED MODULATING FUNCTIONS*

A.E. PEARSON[1] AND F.C. LEE[2]

Abstract. The parameter identification of linear differential systems is considered from the viewpoint of Shinbrot's classical method of moment functionals using commensurable sinusoids as the modulating functions. This facilitates a least squares formulation in which the underlying computations require calculating a finite set of Fourier series coefficients of time limited input-output data while avoiding the necessity to estimate unknown initial conditions for a one-shot estimate, or unknown boundary conditions at each stage for sequential least squares. It is noted that a fast Fourier transform algorithm can be utilized for these calculations, thus providing a "fast algorithm" for the identification of continuous-time systems. It is shown that the frequency domain interpretation can be useful in enhancing the signal to noise ratio of the modulated data in the presence of noisy measurements. A maximum likelihood estimate is developed for the stochastic case of additive white gaussian noise in the data which effectively removes the bias when the parameter identification is considered in a recursive mode. Simulation results are included to illustrate the developments.

Key Words—Parameter identification, continuous time linear differential systems, least squares estimate, maximum likelihood estimate, Fourier modulating functions, fast Fourier transform algorithm.

1. Introduction

The identification of linear differential systems can be undertaken in a deterministic vein using the classical steady state frequency domain approach for estimating the system transfer function, or using a variety of methods based on a differential equation model in the time domain which would include quasilinearization, state variable filters, model reference techniques and adaptive observers (Young's survey, 1981). In a stochastic vein, the known methods would include generalized least squares, instrumental variables, maximum likelihood and extended Kalman filtering techniques (Young's survey, 1981; Åström, 1981). The deterministic methods are computationally simpler but may incur significant biases in the presence of noise. The stochastic methods, while promising to remove the biases asymptotically, are computationally demanding to a degree that they are more likely to be found discussed and used in a

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discrete-time format. Although a continuous-time model can be constructed from a discrete-time counterpart, the methods for accomplishing this are unwieldy and potentially unreliable due to such difficulties as nonuniqueness. Hence, a direct attack on the problem is clearly preferred if a continuous-time model is desired. Such is presumed to be the case here.

Another classical method that can be applied to linear differential systems is Shinbrot's method of moment functionals, also called the modulating function approach, which facilitates converting a differential equation on a finite time interval into an algebraic equation in the parameters (Shinbrot, 1957). In this regard let p denote the differential operator d/dt so that $p^2 = d^2/dt^2$, etc., and consider the following nth order differential equation model relating an input-output pair $(u(t), y(t))$ for a single input-single output system:

$$\sum_{i=0}^{n} a_{n-i}\, p^i y(t) = \sum_{i=0}^{n-1} b_{n-i}\, p^i u(t), \quad a_0 = 1. \tag{1}$$

As introduced by Shinbrot, $\phi(t)$ is a modulating function of order n relative to a fixed time interval $[0, T]$ if it is sufficiently smooth and possesses the property that

$$\phi^{(i)}(0) = \phi^{(i)}(T) = 0, \quad i = 0,1,\dots,(n-1), \tag{2}$$

where $\phi^{(i)}(t)$ means $p^i\phi(t)$, i.e., $\phi(t)$ and its first $(n-1)$ derivatives vanish at both end points of the time interval $[0, T]$. The significance of this property for system identification stems from the fact that if $(u(t), y(t))$ is presumed to satisfy the model (1) on $[0, T]$ then the multiplication or modulation of both sides of (1) with $\phi(t)$ followed by integration-by-parts over $[0, T]$, while noting (2), leads to the relation,

$$\sum_{i=0}^{n}(-1)^i\, a_{n-i} \int_0^T y(t)\phi^{(i)}(t)dt$$
$$= \sum_{i=0}^{n-1}(-1)^i\, b_{n-i} \int_0^T u(t)\phi^{(i)}(t)dt, \quad a_0 = 1. \tag{3}$$

Furthermore, if $\{\phi_i(t)\}$, $i = 1, 2, \dots, K$, is a set of linearly independent modulating functions of order n on $[0, T]$, a vector algebraic equation results which can be used to obtain a least squares estimate of the parameters (a_i, b_i), $i = 1, 2, \dots, n$, provided some nondegeneracy conditions are upheld. It is noted that the prime reasons for using such modulating functions are to avoid differentiating the data and to avoid estimating unknown initial conditions for time limited data.

The above idea has been pursued using modulating functions which stem from Hermite polynomials, as in Takaya (1968), the Poisson process, as in Fairman and Shen (1970) and Saha and Rao (1979; 1980; 1982; 1983), spline type functions, as in Maletinsky (1979), and trigonometric or Fourier type functions, as in the authors' (1983)[†]. However, with the exception of Shinbrot (1957), Maletinsky (1979) and the authors' works (1983; 1985), it will be found

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† The extension to other models has also been considered such as linear time varying systems with polynomial coefficients (Loeb and Cohen, 1965; Fairman and Shen, 1970; Saha and Rao, 1983) and certain types of nonlinear models (Shinbrot, 1957; Saha and Rao, 1983; Pearson and Lee, 1985).

that certain anomalies exist in the other formulations cited in that either a very long time interval is assumed, or the initial conditions are constrained in some way, etc. Notwithstanding these anomalies, the modulating function method has remained relatively obscure due in large measure to the rather severe computational burden associated with the linear functionals on the data in (3). Thus, with the current emphasis on recursive methods in system identification, computing these linear functionals at each stage appears cumbersome unless some type of "fast algorithm" is available. Since the fast Fourier transform (FFT) is such an algorithm for the discrete Fourier transform (DFT), and since the latter can be used to calculate the Fourier series coefficients of a time limited signal with great accuracy, it appears as though the modulating function approach should be re-examined for continuous system identification using Fourier based modulating functions. Although some aspects of the formulation have been developed in the works cited earlier, a number of issues remain to be examined even in the linear case. These include the handling of noise, structure determination procedures, optimal inputs for parameter identification, as well as the experience to be gained via simulation studies.

The contribution of this paper is to show how the frequency domain interpretation can be used to advantage in ameliorating the effects of random noise even in a deterministic setting and, further, to develop a maximum likelihood estimate within the context of the modulating function approach in order to eliminate the bias in the sequential least squares estimate when the data is corrupted by additive white gaussian noise. The maximum likelihood estimate will be a modification of the Levin (1964) algorithm for discrete systems identification, later analyzed in detail by Aoki and Yu (1970 a; b), tailored to fit the formulation of this paper for differential systems identification. Computer simulations will be presented to illustrate each of the developments.

2. Least squares estimate

A deterministic least squares estimate is formulated in this section given the input-output data $(u(t), y(t))$ over a fixed time interval $[0, T]$ for a one-shot estimate, or over a sequence of time intervals $[t_i, t_{i+1}]$, $i = 0, 1, ...,$ each of duration T, for a recursive estimate. Consider the set of commensurable sinusoids $\{\cos m\omega_0 t, \sin m\omega_0 t\}$, $m = 0, 1, ..., M$, where $\omega_0 = 2\pi/T$ plays the role of a "resolving frequency" in the identification problem. This role will be discussed in the section on computational considerations along with guidelines for choosing (T, M). It could be expected that for a specified order n in the model (1) there can be found any desired number of linearly independent modulating functions simply by choosing M sufficiently large and subjecting appropriate linear combinations of these sinusoids to the end point conditions (2). Although there are many ways of doing this, a systematic procedure is to separately form linear combinations of functions from the two sets $\{\cos m\omega_0 t\}$ and $\{\sin m\omega_0 t\}$, $m = 0, 1, 2, ..., M$, and subject each combination to the end point conditions (2), the only stipulation being that M satisfy $(2M+1-n)>0$. As shown in the Appendix, this (offline) procedure leads to $(2M+1-n)$ linearly independent modulating functions, $\phi_i(t)$, $i = 1, 2, ..., (2M+1-n)$, which can be put in the vector-matrix representation:

$$\Phi(t) = Cf(t), \quad 0 \le t \le T, \tag{4}$$

where $f(t)$ is defined by the column vector of sinusoids:

$$f(t) = \text{col} \, [1, \, \cos\omega_0 t, \, \sin\omega_0 t, \, \cos 2\omega_0 t, \, \sin 2\omega_0 t,$$

$$\ldots, \, \cos M\omega_0 t, \, \sin M\omega_0 t],$$

$$0 \leq t \leq T, \quad \omega_0 = 2\pi/T, \tag{5}$$

and the rows of the $(2M+1-n) \times (2M+1)$ matrix C are determined by the solution to Vandermonde type matrix equations. Further shown in the Appendix is the important fact that the matrix C has full rank. The role of this matrix as a projector for the least squares problem will be clarified below. Since by construction $\Phi(t)$ is a vector of modulating functions of order n it satisfies the end point conditions (2):

$$\Phi^{(i)}(0) = \Phi^{(i)}(T) = 0, \quad i = 0, 1, 2, \ldots, (n-1). \tag{6}$$

It can be seen from (4) and (5) that the time derivatives of $\Phi(t)$ have the representation,

$$(-1)^i \Phi^{(i)}(t) = CD^i f(t), \quad i = 0, 1, 2, \ldots, \tag{7}$$

where D is an operational matrix defined by the block diagonal structure:

$$D = \omega_0 \, \text{diag} \left[0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & M \\ -M & 0 \end{bmatrix} \right]. \tag{8}$$

Multiplying both sides of the model equation (1) with $\Phi(t)$ followed by integration-by-parts over $[0, T]$, while noting (6), there results the vector analog to (3):

$$\sum_{i=0}^{n} (-1)^i a_{n-i} \int_0^T \Phi^{(i)}(t) y(t) dt$$

$$= \sum_{i=0}^{n-1} (-1)^i b_{n-i} \int_0^T \Phi^{(i)}(t) u(t) dt, \quad a_0 = 1. \tag{9}$$

Taking note of (4), (5) and (7), the preceding equation is equivalent to

$$\sum_{i=0}^{n} a_{n-i} CD^i Y = \sum_{i=0}^{n-1} b_{n-i} CD^i U, \quad a_0 = 1, \tag{10}$$

where (U, Y) represent finite Fourier series coefficient vectors of the data defined by

$$U = \int_0^T u(t) f(t) dt, \quad Y = \int_0^T y(t) f(t) dt. \tag{11}$$

Rearranging (10) in terms of the parameter vector θ defined by

$$\theta = \text{col} \, [-a_1 \ldots -a_n, \, b_1 \ldots b_n] \tag{12}$$

and the partitioned matrix Γ defined by

$$\Gamma = [D^{n-1}Y \ldots Y, D^{n-1}U \ldots U], \tag{13}$$

there results the least squares regression equation in standard form:

$$C\Gamma\theta = CD^nY. \tag{14}$$

Hence, forming the normal equations for (14) and assuming that $C\Gamma$ has rank $(2n)$, the one-shot least squares estimate is given by (prime denoting transpose)

$$\hat{\theta} = [\Gamma'C'C\Gamma]^{-1}\Gamma'C'CD^nY. \tag{15}$$

Here, it is assumed that a sufficient number of modulating functions have been chosen so that

$$(2M+1-n) \geq 2n, \text{ i.e., } 2M \geq 3n-1 \tag{16}$$

else there are fewer algebraic equations than unknowns upon which to base the one-shot estimate.

In the case of sequential least squares, (14) is replaced by

$$C\Gamma(i)\theta = CD^nY(i), \; i = 0, 1, 2, \ldots, \tag{17}$$

where at each stage the underlying quantities are supplied by the finite Fourier series coefficient vectors of the input-output data taken over sequential time intervals $[t_i, t_{i+1}]$, $i = 0, 1, 2, \ldots$, each of duration T. Standard sequential least squares theory can then be applied to (17) in constructing a recursive solution $\{\theta(i)\}$. (See, for example, Mendel (1973).) In this case the number of modulating functions need be chosen subject only to the previously stated inequality: $(2M+1-n)>0$, c.f. (16).

2.1 Aspects of the least squares estimate   The parallelism between the differential equation model (1) and the algebraic equation derived in (10) may suggest the interpretation that the latter is a transformed version of the former. However, it is important to underscore the facts that (i) (U, Y) are *finite* Fourier series coefficient vectors extracted from the infinite dimensional transient data on $[0, T]$, and (ii) the matrix C plays the role of a *projector* on the finite dimensional space to which the computed vectors (U, Y) belong because this matrix acts to "strip away" the explicit influence of the unknown initial and boundary conditions, i.e., without this matrix the unknown boundary conditions on derivatives of the data would have to be appended to (10). These two projection aspects can be used to distinguish the Fourier based modulating function approach from other methods[†].

Conditions on the initial data and forcing function for (1) which would guarantee the existence and uniqueness of the one-shot estimate in (15), or convergence of the recursive estimate based on (17), have not yet been

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[†] An alternative method which employs state variable filters and a projection operator to annihilate initial condition effects of an integrated equation error function on $[0, T]$ is discussed in Pearson (1976). Comparatively, it is believed that the Fourier based modulating function approach is superior due to the computational advantage of employing the FFT algorithm and the potential for ameliorating noise.

determined but will surely involve the notions of minimality of the model, excitation of system modes, and the like. As necessary conditions, it is straightforward to show that the system must not be in steady state operation and that the input must not be a linear (zero memory) feedback on the output, else some columns of $C\Gamma$ will be linearly dependent. Likewise, each of the Fourier coefficient data vectors (U, Y) must contain at least n nonzero components, i.e., at least n frequencies must be present, else linear dependence will occur. In this connection it can be noted from (10) and (13) that order determination for the model (1) (within the context of the Fourier based modulating function technique) relates to finding the smallest integer n for which the columns of the following matrix become linearly dependent:

$$C \; [D^n Y, \; D^{n-1} Y, \; ..., \; Y, \; D^{n-1} U, \; ..., \; U \;].$$

Procedures for testing such linear dependence will also reflect back on conditions involving the initial data and forcing function for (1). Hence, effective order determination procedures and conditions for the existence and uniqueness of the least squares estimate are important interrelated topics for future investigation.

2.2 Computational considerations   The choice of (T, M) can be guided in reference to the amplitude plot of a system transfer function sketched in Fig. 1.
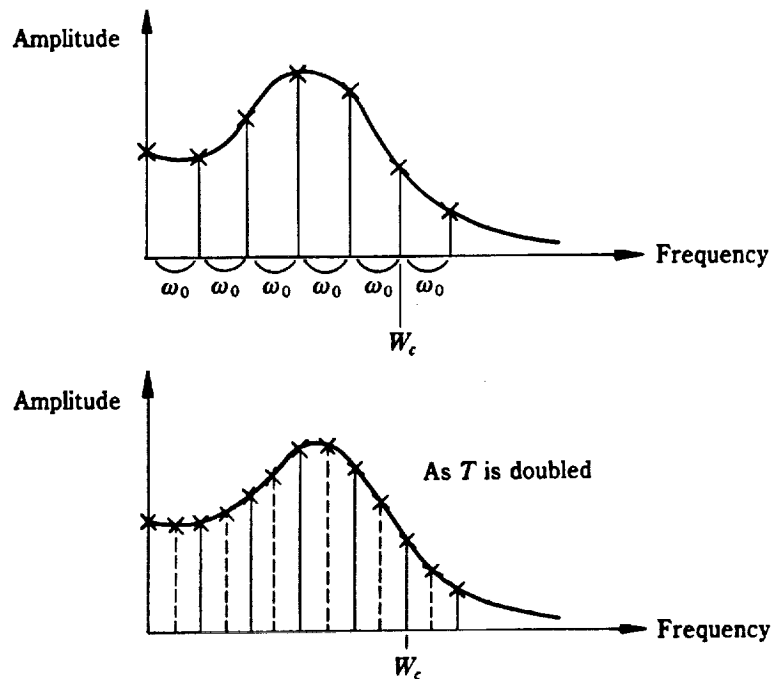

Fig. 1. A frequency domain interpretation.

As a first consideration, which is based on the desire that the frequencies retained in the pair (U, Y) should cover the system bandwidth while excluding higher frequency noise, it is clear that the highest frequency in the modulating functions, $M\omega_0$, should be comparable to the system bandwidth W_c, say 25% higher. Assuming W_c is approximately known, a quantitative statement of this is

$$M\omega_0 \cong 1.25W_c. \tag{18}$$

Likewise as shown in Fig. 1, the value of T should be chosen sufficiently large so as to assure reasonable resolution in distinguishing system modes of the transfer function, via the resolving frequency $\omega_0 = 2\pi/T$. However, this consideration is rather qualitative. A more quantitative measure of choosing T can be based on the bandwidth relation (18) and the number of algebraic equations formed from the modulating functions, i.e., $(2M+1-n)$, together with the total number of such algebraic equations one wishes to use in constructing the least squares estimate. Thus, considering the case of a one-shot estimate, if it is desired that the number of such algebraic equations should approximately equal double the number of unknowns, then M should satisfy $(2M+1-n)\cong 4n$ which together with $\omega_0 = 2\pi/T$ and (18) implies the relation ($2n$ unknowns presumed):

$$T \cong \frac{4n\pi}{W_c}. \tag{19}$$

Based on approximate knowledge of the system modes and *a priori* knowledge of contaminating noises in the frequency domain, it will be advantageous to be more selective in choosing the frequencies used in defining the modulating functions[†]. If $m_i\omega_0$, $i = 1, 2, \ldots, M$ represent such frequencies where the m_i are integers satisfying $m_i < m_{i+1}$ and M satisfies $(2M-n>0)$, the formulation is easily modified to reflect this selection by changing $f(t)$ in (5) to be defined as

$$f(t) = \text{col } [\cos m_1\omega_0 t, \sin m_1\omega_0 t, \ldots, \cos m_M\omega_0 t, \sin m_M\omega_0 t] \tag{20}$$

and redefining D in (8) as

$$D = \omega_0 \text{ diag} \left[\begin{bmatrix} 0 & m_1 \\ -m_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & m_2 \\ -m_2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & m_M \\ -m_M & 0 \end{bmatrix} \right]. \tag{21}$$

The simulation results will illustrate the advantage of this flexibility[‡].

The most important computational aspect of the Fourier based modulating function approach is the direct frequency domain interpretation afforded by the

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[†] For example, the zero frequency can be deleted if there is an unreliable DC value in the measurements.

[‡] It should be noted that the simple structure of the operational matrix D in (8) or (21) makes easy the computing of powers D^i, $0 \le i \le n$, as required in (13)–(14). Although alternative rearrangements of the sinusoids comprising $f(t)$ in (5) or (20) are readily accomodated by rearranging columns of C, any such rearrangements will generally alter the convenient block diagonal structure of D.

vectors (U, Y) and the efficiency with which these vectors can be computed by an FFT algorithm. In order to clarify this point, let $z(t)$ denote a data function on $[0, T]$ and assume uniform sampling in generating the discrete samples $z_i = z(ih)$, $h = T/N$, $i = 0, 1, ..., N$. Then (5) and (11) imply determining the following integrals (complex form):

$$\int_0^T z(t)\exp(jm\omega_0 t)dt, \quad m = 0, 1, ..., M.$$

Although the real and imaginary parts of the above integral can be evaluated by passing $z(t)$ through a bank of appropriately tuned harmonic oscillators, greater flexibility is offered by using well known digital approximations. For example, the standard parabolic rule yields

$$\int_0^T z(t)\exp(jm\omega_0 t)dt$$

$$= \frac{h}{3}\left[z_0 + z_N + 4\sum_{i=1,3,...}^{N-1} z_i W^{m_i} + 2\sum_{i=2,4,...}^{N-2} z_i W^{m_i} \right] + o(h^4),$$

where $W = \exp(j2\pi/N)$ and $o(\cdot)$ is the order of the error as a function of the sampling interval h. Assuming N is a power of 2, the usual FFT algorithm can be used to evaluate the DFT of the sum on the RHS of the above yielding the Fourier series coefficients for $m = 0, 1, ..., (N-1)$, i.e.,

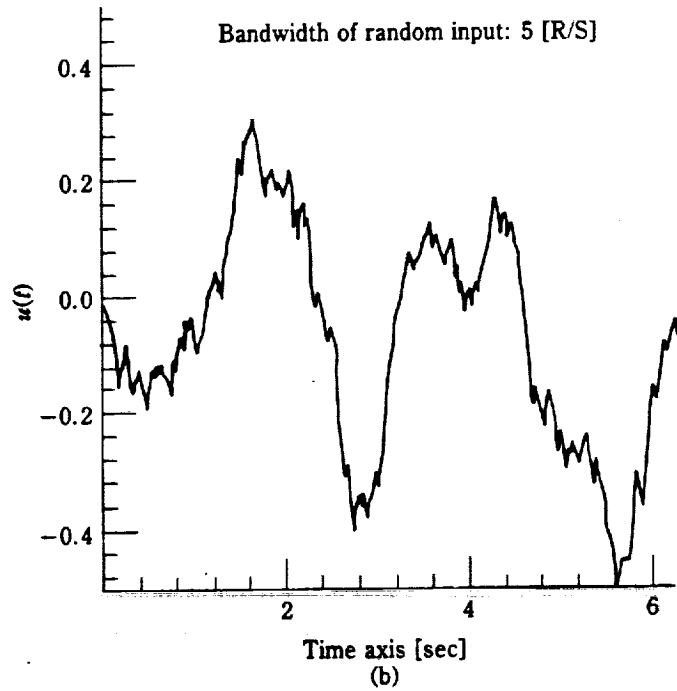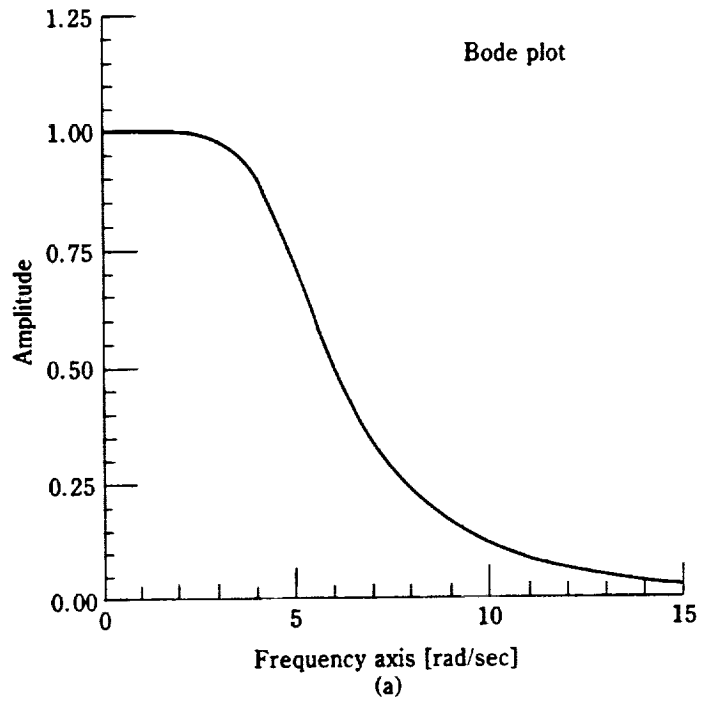$$Z = \frac{h}{3}\text{FFT}\left[z_0 + z_N, 4z_1, 2z_2, ..., 4z_{N-1} \right]. \tag{22}$$

The computational savings of this algorithm for large N are well known. However, a special FFT-type algorithm can be devised in consideration of the fact that only M Fourier coefficients are needed and that N is likely to be chosen much larger than M for good accuracy in the approximation. The efficiency of such a partial FFT algorithm can be shown to be $\log_2 M/M$ (Markel, 1971; Lee, 1984).
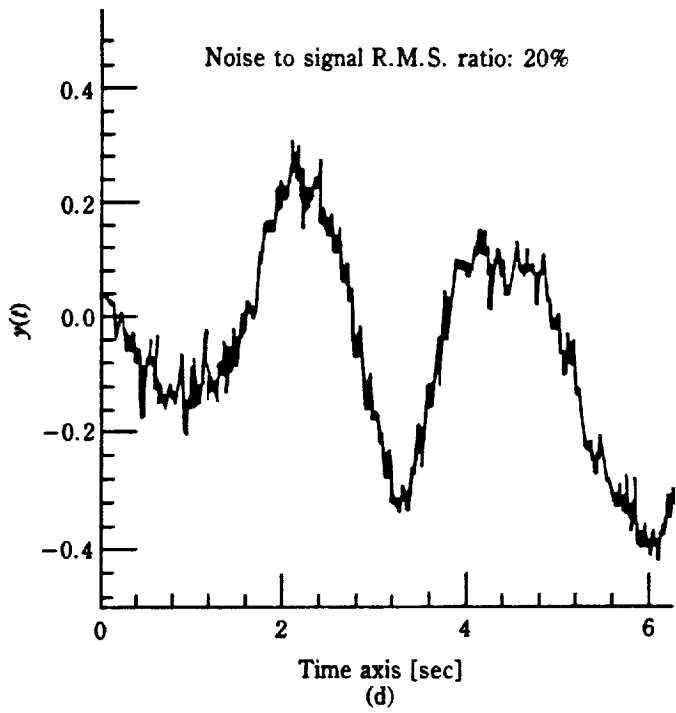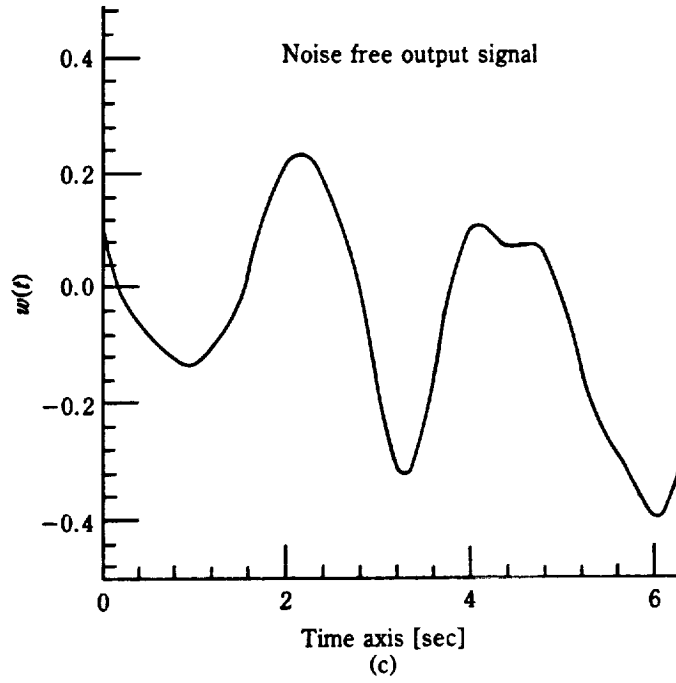
2.3 Simulation results for the one shot LSE

2.3.1 Low pass system   Consider first the low pass system defined by a Butterworth filter with bandwidth 5 [rad/sec] (see Fig. 2(a)) and the transfer function,

$$H_1(s) = \frac{125}{s^3 + 10s^2 + 50s + 125}. \tag{23}$$

The objective is to identify the four unknown system parameters $\theta = \{10, 50, 125, 125\}$ using the one-shot least squares estimate (15) based on time limited input-output data over a $T = 2\pi$ [sec] time interval with the initial conditions arbitrary and the input drawn from a colored gaussian random signal generator with bandwidth 5 [rad/sec]. A sample of the input, the corresponding noise-free output, and the output contaminated by an additive white gaussian noise with a 20% RMS noise-to-signal ratio is shown in Figs. 2(b), (c) and (d), respectively. The Fourier series coefficients for the first M modes in the data were calculated using the first M components of the parabolic approximation (22) with either
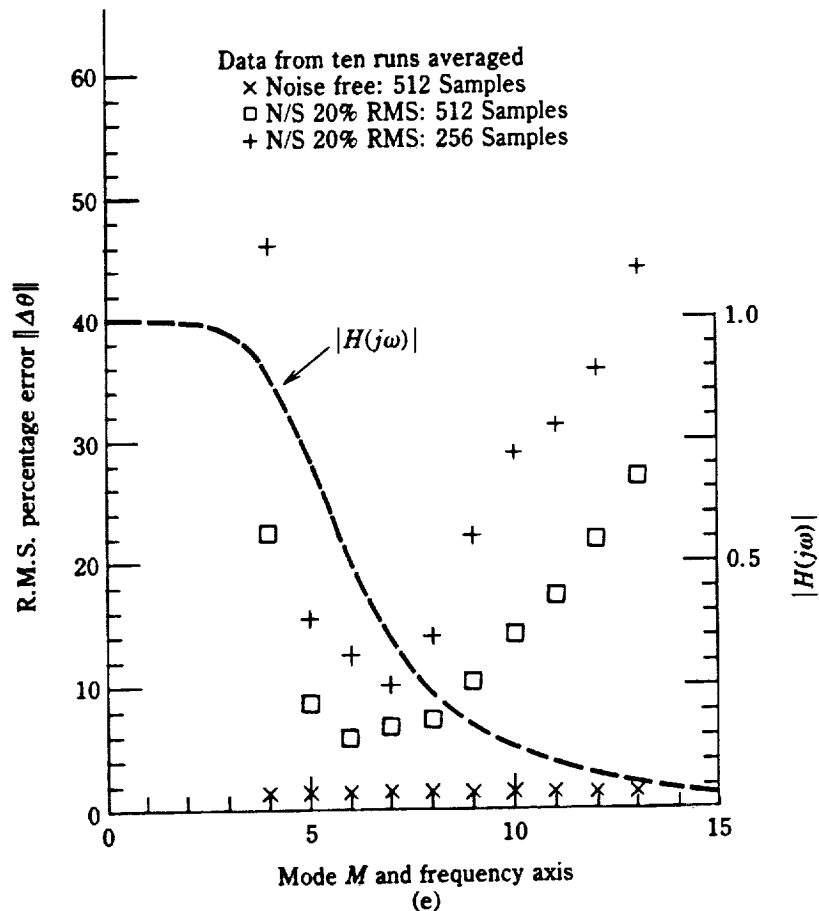
Bode plot

Frequency axis [rad/sec]
(a)

Bandwidth of random input: 5 [R/S]

Time axis [sec]
(b)

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Noise free output signal

Time axis [sec]

(c)

Noise to signal R.M.S. ratio: 20%

Time axis [sec]

(d)

Fig. 2.   Least squares identification of $H_1(s)$.

$N = 256$ or $N = 512$, as indicated in Fig. 2(e)[†]. Since the number of unknowns is 4, the minimum number M needed for a one-shot estimate is based on $(2M+1-n) \geq 4$, i.e., $M \geq 3$. Due to the random nature of the signals, ten separate runs have been made for each M from 4 to 13 and the results averaged to yield the curves plotted in Fig. 2(e). The normalized error criterion for the estimated parameters is defined by

$$\| \Delta \theta \| = \left[\frac{1}{K} \sum_{i=1}^{K} \left[\frac{\hat{\theta}_i - \theta_i{}^*}{\theta_i{}^*} \right]^2 \right]^{\frac{1}{2}} \cdot 100\% \qquad (24)$$

where $\theta_i{}^*$ is the true parameter value and K is the number of parameters.

[†] The IMSL (1982) was used to provide the integration routine (DVERK) for generating the "continuous" data and as the source for an FFT algorithm to compute the DFT's of the "sampled" data.

Except in the noise-free data case, which gives essentially zero error for any $M \geq 3$ and either $N = 256$ or $N = 512$, the results summarized in Fig. 2(e) show that a minimum error is reached around $M = 6$ or 7. This is explained by the consideration that as M increases from the minimal value needed to satisfy the modulating function constraints (2), i.e., $M = 3$, the estimation error decreases because more information in the data is used until somewhat beyond the system bandwidth at which point more noise than signal will be picked up causing a deterioration in the identification accuracy. Taking account of the Bode plot (the dotted curve in Fig. 2(e)), the results are consistent with the frequency domain interpretation, although the identification scheme is formulated in the time domain. The difference between the $N = 256$ and $N = 512$ curves in the noisy case reflects the increased accuracy due to a smaller sampling interval h.

2.3.2 Band pass system   Consider as a second example the band pass system with the transfer function,

$$H_2(s) = \frac{14.0625s^2}{s^4 + 5.325s^3 + 189.84s^2 + 466.17s + 7724.7} \cdot \quad (25)$$

This system was excited with a colored gaussian random signal of bandwidth 15 [rad/sec] to obtain input-output data over a $T = 2\pi$ [sec] time interval with arbitrary initial conditions. The Bode plot and an example run of the input, the output, and the noise corrupted output signals are shown in Figs. 3(a)−(d), respectively. The RMS noise-to-signal ratio of the additive white gaussian noise at the output is 20%.
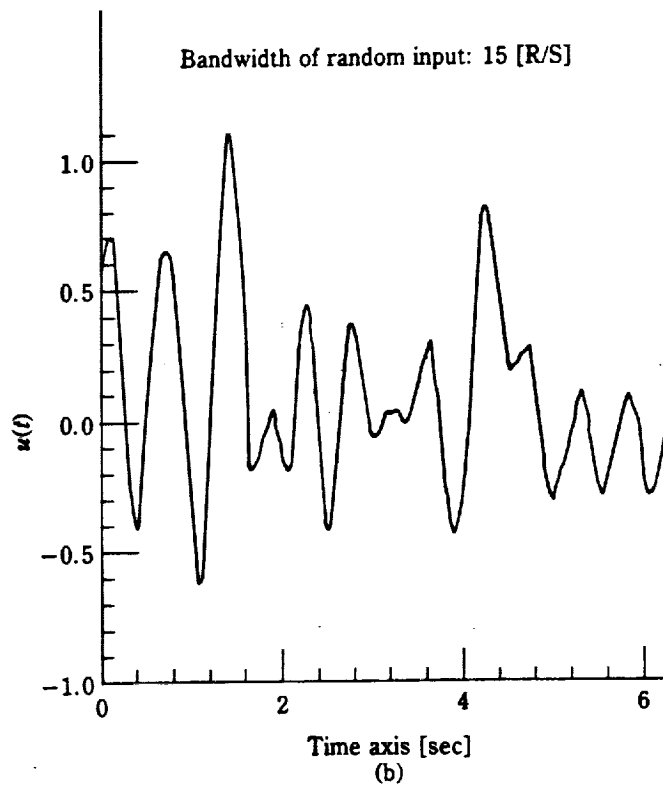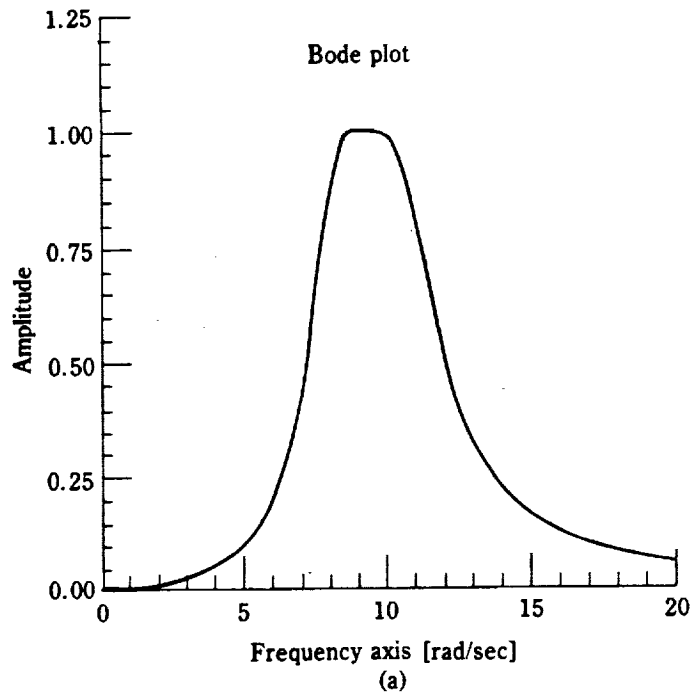
The simulation results summarized in Fig. 3(e) used two groups of modulating functions comprised of frequencies defined by the following:
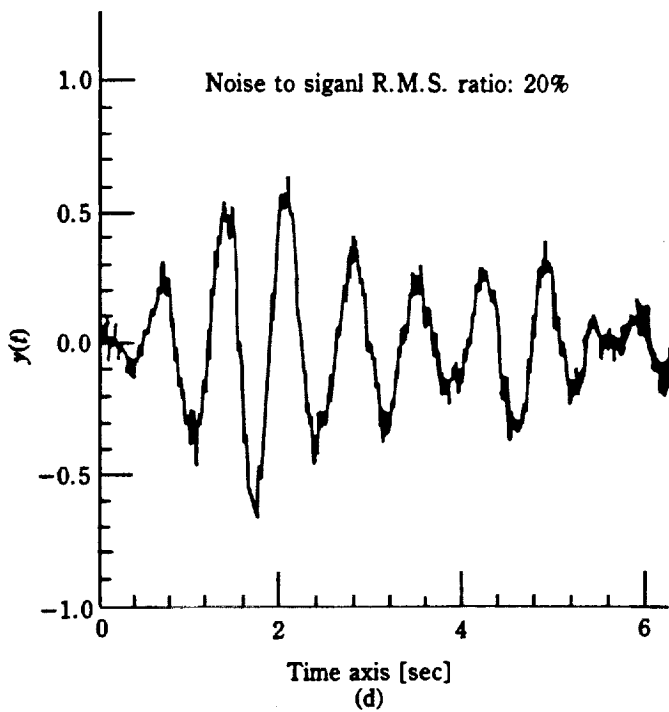
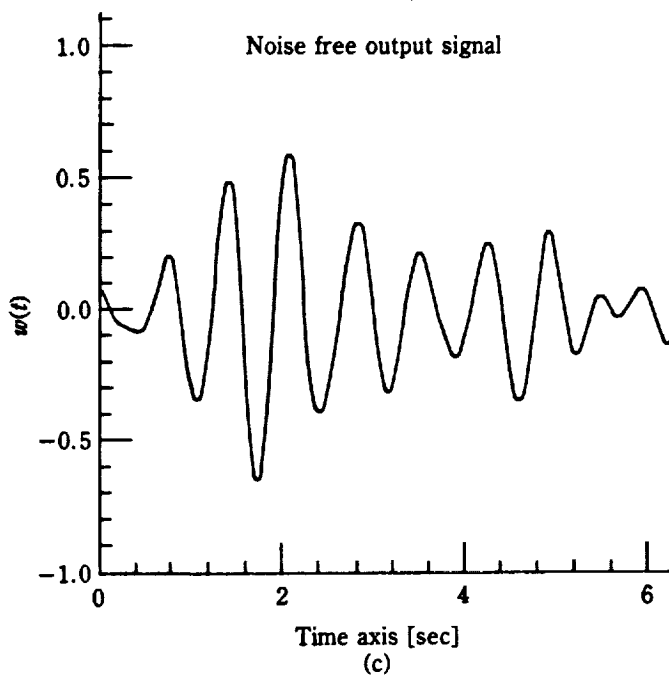Group One: $\{0, 1, 2, ..., M\}$ for $M = 11, 12, ..., 17$.

Group Two: $\{M, M + 1, ..., 14\}$ for $M = 0, 1, ..., 9$.

In each case the curves have been obtained by averaging the results over ten runs. The error criterion is the same as in (24) relative to the parameter vector $\theta = \{5.325, 189.84, 466.17, 7724.7, 14.0625\}$. The results for Group One are similar to those of the first example and can be explained analogously, i.e., the estimation error goes through a minimum as M increases from the value $M = 11$ (below the bandwidth) to the value $M = 17$ (above the bandwidth). The results for Group Two can be explained on the basis that as M decreases from the value $M = 9$ the estimation error decreases because more information in the signals is utilized by the least squares estimate. However, the curves for both $N = 256$ and $N = 512$ flatten out at the minimum because nothing is gained by including modes below the pass-band of $H_2(j\omega)$. Hence, the combined results are consistent with a frequency domain viewpoint in that the most significant information in the data has a band-pass nature like the system transfer function.

2.3.3 Nonminimum phase system   Consider as a third example the nonminimum phase system with the marginally stable transfer function,

Bode plot

Frequency axis [rad/sec]
(a)

Bandwidth of random input: 15 [R/S]

Time axis [sec]
(b)

Noise free output signal

$u(t)$

Time axis [sec]

(c)

Noise to siganl R.M.S. ratio: 20%

$y(t)$

Time axis [sec]

(d)
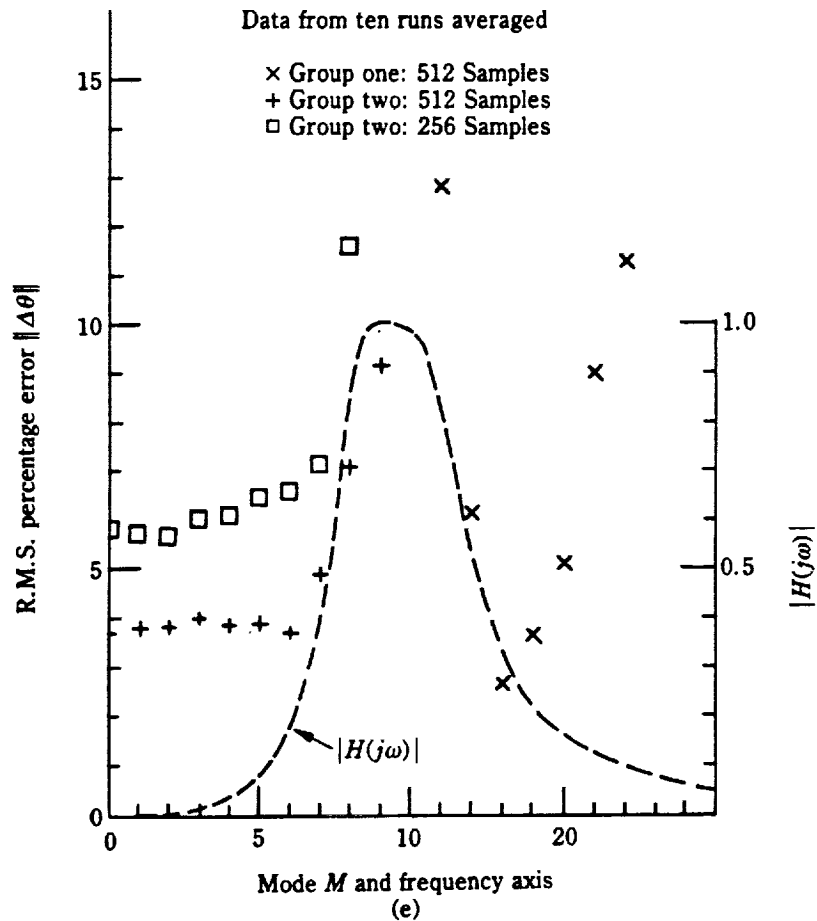
Fig. 3.   Least squares identification of $H_2(s)$.

$$H_3(s) = \frac{0.5s^3 + 2s^2 + 0s + 1}{s^4 + 0s^3 + 2.5s^2 + 0s + 0.5625} \cdot \qquad (26)$$

which has imaginary axis poles at ($\pm j0.5$, $\pm j1.5$) and zeros at (-4.118, $0.059 \pm j0.695$). The objective is to identify the parameters defined by: $\theta = \{0, 2.5, 0, 0.5625, 0.5, 2, 0, 1\}$. This system can be expected to be more difficult to identify owing to the larger number of unknown parameters, i.e., 8 for (26) verses 4 for (23) and 5 for (25).

Table 1. Least squares identification of $H_3(s)$

Input $u(t)$	$[0, T]$	DFT Order N	$(\frac{1}{8} \sum_{i=1}^{8} (\hat{\theta}_i - \theta_i)^2)^{\frac{1}{2}}$
$0.01t^3$	$[0, 2\pi]$	1024	0.06
	$[0, 4\pi]$	512	0.02
		256	0.33
		128	1.65
DC + Four sinusoids	$[0, 2\pi]$	512	0.07
		256	0.07
		128	0.08
		64	0.26
	$[0, 4\pi]$	512	0.003
		256	0.003
		128	0.004
		64	0.017

The simulation results summarized in Table 1 illustrate some of the trade-offs between frequency content of the input signal, the length of the time interval [0, T], and the order N of the DFT used in the parabolic approximation (22) for the one-shot identification of (26) under noise-free data conditions[†]. In the case of the single mode input signal, $u(t) = 0.01t^3$, over a [0, 2π] time interval, the identification accuracy is acceptable at $\|\Delta\theta\| \cong 0.06$, but at the expense of a rather high order DFT (N = 1024). Doubling the time interval to [0, 4π], thus having the resolving frequency (to 0.5 [rad/sec]) results in better accuracy for a lower order DFT (N = 512). The corresponding results show yet better accuracy when the multi-modal input signal is used consisting of a DC (constant) plus four sinusoids. In this case acceptable accuracy is attained for lower order DFT's, especially when the resolving frequency is 0.5 [rad/sec]. Notice that the guideline for T given by (19) yields: $T \cong 4n\pi/W_c = 8\pi$, which is conservative in light of the results obtained for this example.

3. Maximum likelihood estimate

If noise in the data cannot be effectively blocked by a judicious choice of modulating function frequencies then the least squares estimate may incur a significant bias, as is well known in regression analysis. In this case the equation error for (14), which is represented by

---

† The number of modes M in the vector of modulating functions (4)–(5) was also varied from $M = 7$ to 11 in order to cover the system bandwidth ($W_c \cong 2$ [rad/sec]). However, this affect on the identification accuracy is secondary under the noise-free conditions and hence is omitted from Table 1. Note the error criterion for $\Delta\theta$ in Table 1 is slightly different than (24) for this example.

$$\varepsilon = C\Gamma\theta - CD^nY, \qquad (27)$$

will not be negligible. A maximum likelihood estimate will be developed in this section[†] which is aimed at the asymptotic removal of the bias for the situation depicted in Fig. 4. Here the measured input-output data is denoted by $(x(t), y(t))$ respectively, while the noise-free input-output signals are designated by $(u(t), w(t))$. The noise signals $(v(t), n(t))$ corrupting the pair $(u(t), w(t))$ are assumed to be uncorrelated zero mean stationary gaussian random processes. Under these circumstances the formulation of the preceding section, which led to the algebraic equation (10) for data over a single $[0, T]$ interval, can be rewritten as follows for data given on K nonoverlapping time intervals, $[t_i, t_{i+1}]$, $i = 1, 2, \ldots, K$, each of duration T (c.f. (10)):

$$\begin{bmatrix} -1 & -a_1 & \ldots & b_n & 0 & 0 & \ldots & 0 & \\ 0 & 0 & \ldots & 0 & -1 & -a_1 & \ldots & b_n & \\ & & & & & -1 & -a_1 & \ldots & b_n \end{bmatrix} \begin{bmatrix} c_1'D^nW_1 \\ c_1'D^{n-1}W_1 \\ \vdots \\ c_1'U_1 \\ c_2'D^nW_1 \\ c_2'D^{n-1}W_1 \\ \vdots \\ c_2'U_1 \\ \vdots \\ c_m'D^nW_K \\ c_m'D^{n-1}W_K \\ \vdots \\ c_m'U_K \end{bmatrix} = 0 \quad (28)$$

where c_i' is the ith row of the matrix C, and (U_i, W_i) are the Fourier coefficient
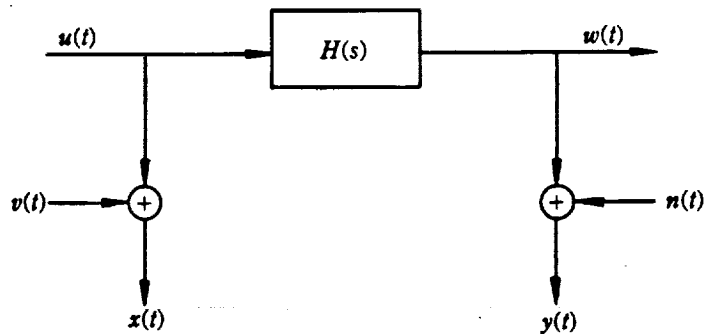

Fig. 4. Signals for the maximum likelihood estimate.

† The algorithm of Levin (1964) for discrete systems identification, originally due to Koopman (1937) and analyzed in detail by Aoki and Yu (1970 a; b), will be modified to fit the formulation of Sec. 2.

vectors for the noise-free input-output signals $(w(t),\ u(t))$, $t_i \le t \le t_i + T$. Similarly, $(X_i,\ Y_i)$ and $(V_i,\ N_i)$ will denote the Fourier coefficient vectors for $(x(t),\ y(t))$ and $(v(t),\ n(t))$ respectively. The matrix-vector equation (28) will henceforth be denoted by

$$G_K' p_K = 0. \tag{29}$$

With $(U_i,\ W_i)$ in the column vector p_K of (28) replaced by $(X_i,\ Y_i)$ (or $(V_i,\ N_i)$), this new column vector will be denoted as q_K (or r_K). From Fig. 4, it is easy to see that

$$r_K = q_K - p_K. \tag{30}$$

Since $(v(t),\ n(t))$ are assumed to be zero-mean gaussian processes, the Fourier coefficients vector r_K in (30) is a gaussian vector with a covariance matrix

$$R_K = \text{cov}(r_K)$$

which can be calculated as follows by assuming the auto- and cross-correlation functions of the noise processes are known:

$$
\begin{aligned}
E[(c'DV_i)(N_j'D'c)] \\
= c'D \cdot E\left\{ \int_{t_i}^{t_i+T} f(t-t_i)v(t)dt \int_{t_j}^{t_j+T} n(s)f'(s-t_j)ds \right\} \cdot D'c \\
= c'D\left\{ \int_{t_i}^{t_i+T} \int_{t_j}^{t_j+T} f(t-t_i)R_{v,n}(t,s)f'(s-t_j)dsdt \right\}D'c.
\end{aligned} \tag{31}
$$

where

$$R_{v,n}(t,s) = E[v(t)n(s)].$$

As a special case, when $(v(t),\ n(t))$ are uncorrelated white gaussian processes and $t_{i+1} - t_i \ge T$, the evaluation of (31) can be greatly simplified. By assuming that R_K is invertible, the conditional probability density function can be written as follows;

$$p(q_k \mid \xi, G_K^T p_K = 0) = \text{const}|R_K|^{-\frac{1}{2}}\exp(-\frac{1}{2}\|q_K - p_K\|_{R_K^{-1}}^2)$$

where ξ is defined by

$$\xi = \text{col}[-1,\ -a_1,\ \ldots,\ b_n]. \tag{32}$$

Then the maximum likelihood estimate of θ can be obtained by

$$\min_{\xi, p_K} \|q_K - p_K\|_{R_K^{-1}}^2, \tag{33}$$

subject to

$$G_K' p_K = 0.$$

Using a Lagrange multiplier vector λ, the above constrained optimization problem is equivalent to

$$\min_{\xi, p_K, \lambda} (\|q_K - p_K\|^2_{R_K^{-1}} + \lambda' G_K' p_K).$$

Taking as a first consideration the partial derivatives with respect to p_K and λ, and equating them to zero, the result is

$$-2R_K^{-1}(q_K - \hat{p}_K) + G_K \hat{\lambda} = 0, \qquad G_K' \hat{p}_K = 0.$$

The preceding equations yield

$$\hat{\lambda} = 2(G_K' R_K G_K)^{-1} G_K' q_K$$

$$\hat{p}_K = q_K - R_K G_K (G_K' R_K G_K)^{-1} G_K' q_K \qquad (34)$$

by assuming that $(G_K' R_K G_K)$ is invertible. Then, it follows that

$$\|q_K - p_K\|^2_{R_K^{-1}} = (q_K - \hat{p}_K)' R_K^{-1}(q_K - \hat{p}_K)$$

$$= (q_K' G_K)(G_K' R_K G_K)^{-1}(G_K' q_K)$$

$$= \|G_K' q_K\|^2_{(G_K' R_K G_K)^{-1}},$$

where the second equality is due to (34). Thus the optimization problem (33) is reduced to

$$\underset{\xi}{\text{minimize}} \|G_K' q_K\|^2_{(G_K' R_K G_K)^{-1}}. \qquad (35)$$

Although (35) is a nonlinear optimization problem for which the global optimum is in general difficult to find, an iterative algorithm will be suggested to find the local optimum by observing the following fact: If the weighting matrix $(G_K' R_K G_K)^{-1}$ is temporarily substituted by a given constant matrix W, the problem is equivalent to minimizing a quadratic cost. More specifically, to minimize with respect to ξ the quadratic form,

$$\|G_K' q_K\|^2_W, \qquad (36)$$

the vector $G_K' q_K$ can first be rewritten as follows (c.f. (28) and (13));

$$G_K' q_K = \begin{bmatrix} c_1' D^n Y_1 & c_1' D^{n-1} Y_1 & \dots & c_1' X_1 \\ c_2' D^n Y_1 & c_2' D^{n-1} Y_1 & \dots & c_2' X_1 \\ \vdots & \vdots & & \vdots \\ c_m' D^n Y_K & c_m' D^{n-1} Y_K & \dots & c_m' X_K \end{bmatrix} \begin{bmatrix} -1 \\ -a_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \begin{bmatrix} CD^n Y_1 & CD^{n-1} Y_1 & \dots & CX_1 \\ \vdots & \vdots & & \vdots \\ CD^n Y_K & CD^{n-1} Y_K & \dots & CX_K \end{bmatrix} \begin{bmatrix} -1 \\ -a_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \begin{bmatrix} CD^n Y_1 & C\Gamma_1 \\ \vdots & \vdots \\ CD^n Y_K & C\Gamma_K \end{bmatrix} \begin{bmatrix} -1 \\ \theta \end{bmatrix}.$$

Then (36) is equivalent to

$$\|G_K{}'q_K\|_w^2 = [-1,\ \theta']\begin{bmatrix} CD^nY_1 & C\Gamma_1 \\ \vdots & \vdots \\ CD^nY_K & C\Gamma_K \end{bmatrix}'W\begin{bmatrix} CD^nY_1 & C\Gamma_1 \\ \vdots & \vdots \\ CD^nY_K & C\Gamma_K \end{bmatrix}\begin{bmatrix} -1 \\ \theta \end{bmatrix} \qquad (37)$$

$$= [-1,\ \theta']\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}\begin{bmatrix} -1 \\ \theta \end{bmatrix}. \qquad (38)$$

Hence, it is evident that

$$\frac{\partial}{\partial\theta}\|G_K{}'q_K\|_w^2\Big|_{\hat\theta} = 0$$

leads to

$$\hat\theta = \Omega_{22}^{-1}\Omega_{21}, \qquad (39)$$

since the symmetry of W in (37) implies $\Omega_{21} = \Omega_{12}'$. Note that for $W = I$, $\hat\theta$ in (39) minimizes the squares of the equation error (27), and hence is exactly the same solution as suggested in Sec. 2. Based on these observations, the following iterative algorithm is suggested:

The iterative algorithm

Initialization: Letting $G_K{}'R_KG_K = I$, calculate $\hat\theta_0$ in (39).

Step 1: Calculate $\hat\theta_1$ in (39) by letting $W = (G_K{}'R_KG_K)^{-1}|_{\hat\theta_0}$.

Step 2: Check the convergence of the estimates. If it tends to converge and some more improvement is desired, replace the value of $\hat\theta_0$ by that of $\hat\theta_1$ and go to Step 1; otherwise stop.

The initialization step minimizes the squared norm of the equation errors; in Step 1, the weighted squared norms of the equation errors will be minimized; and so forth.

Special case 1:   When $(v(t), n(t))$ are uncorrelated stationary white gaussian processes and $t_{i+1} - t_i > T$, then

$$R_K = \text{diag}[R\ R\ \dots\ R],$$

where

$$R = \text{cov}[c_1{}'D^nN_1, c_1{}'D^{n-1}N_1,\ \dots,\ c_1{}'V_1,$$
$$\dots,\ c_m{}'D^nN_1, c_m{}'D^{n-1}N_1,\ \dots,\ c_m{}'V_1]. \qquad (40)$$

By denoting

$$\left.\begin{aligned} G_K{}' &\equiv \text{diag}[G'\ G'\ \dots\ G'], \\[4pt] q_K &\equiv \text{col}[q(1),\ q(2)\ \dots\ q(K)] \end{aligned}\right\}, \qquad (41)$$

and

where

$$q(i) = \text{col}[c_1{}'D^n Y_i, \ c_1{}'D^{n-1} Y_i, \ \ldots, \ c_1{}'X_i,$$
$$\ldots, \ c_m{}'D^n Y_i \ c_m{}'D^{n-1} Y_i, \ \ldots, \ c_m{}'X_i],$$

(35) can be reduced to

$$\min_{\xi} \sum_{i=1}^{K} \|G'q(i)\|^2_{(G'RG)^{-1}}. \tag{42}$$

The optimization algorithm is similar to the general case, but the dimension of the weighting matrix $(G'RG)^{-1}$ is reduced by a factor K compared to $(G_K{}'R_K G_K)^{-1}$ in (35).

Special case 2:   In addition to the assumptions made in the Special Case 1, assume that $m = 1$, i.e., only one algebraic equation will be extracted from each non-overlapping time interval. Now R in (40) becomes

$$R = \text{cov}[c_1{}'D^n N_1, \ c_1{}'D^{n-1}N_1, \ \ldots, \ c_1{}'V_1], \tag{43}$$

and G' in (41) becomes ξ'. Furthermore,

$$(G'RG)^{-1} = (\xi'R\xi)^{-1} \quad \text{and} \quad G'q(i) = \xi'q(i) \tag{44}$$

are scalars. Hence (42) can be reduced to an even simpler form as

$$\sum_{i=1}^{K} \|G'q(i)\|^2_{(G'RG)^{-1}} = \sum_{i=1}^{K} \frac{\xi'q(i)q(i)'\xi}{\xi'R\xi}$$

$$= \frac{\xi'(\sum_{i=1}^{K} q(i)q'(i))\xi}{\xi'R\xi}$$

$$= \frac{\xi'B\xi}{\xi'R\xi}, \tag{45}$$

where

$$B \equiv \sum_{i=1}^{K} q(i)q'(i).$$

As is well known in matrix theory, the minimum value and the optimal solution for (45) are the smallest eigenvalue and its corresponding eigenvector with the unit first element of the following generalized eigenvalue problem:

$$(B - \sigma R)\xi = 0. \tag{46}$$

The same solution has been used by Koopman (1937), Levin (1964), Aoki and Yue (1970 a; b) for the discrete system identification problem. In the statistics literatures, Sprent (1966; 1969) also suggested this solution and called it a "generalized least squares solution".

Recursive least squares

Recursive maximum likelihood

(a) Low pass system $H_1(s)$

Recursive least squares
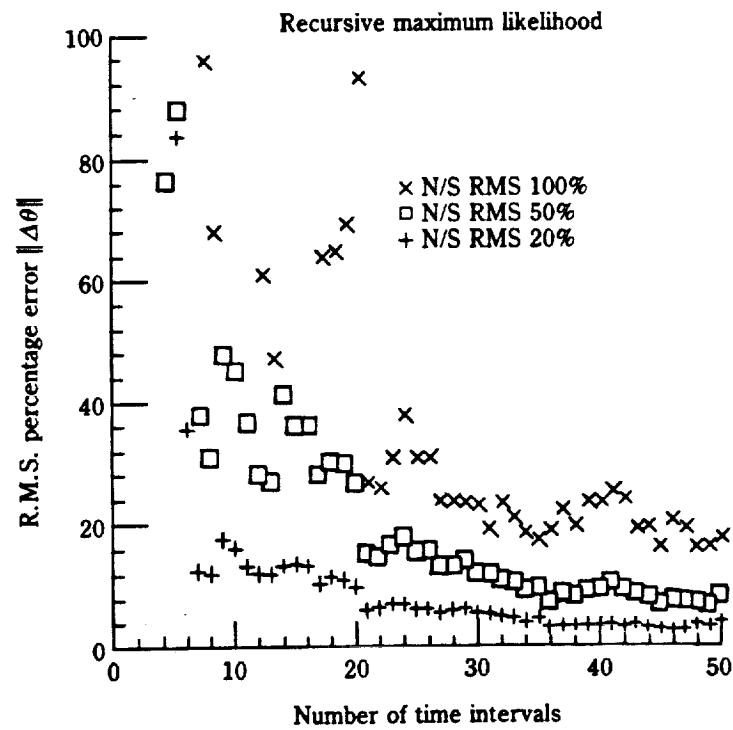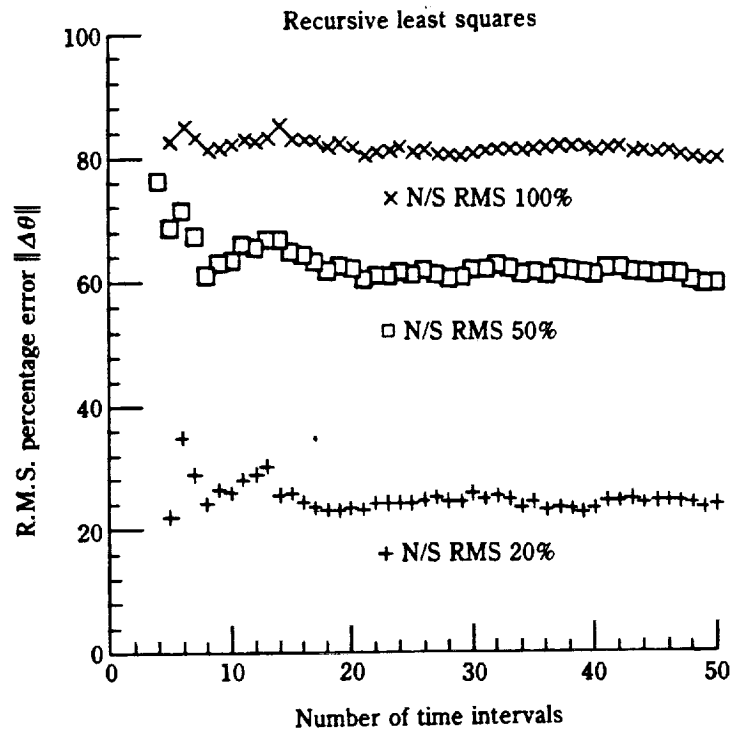
Recursive maximum likelihood
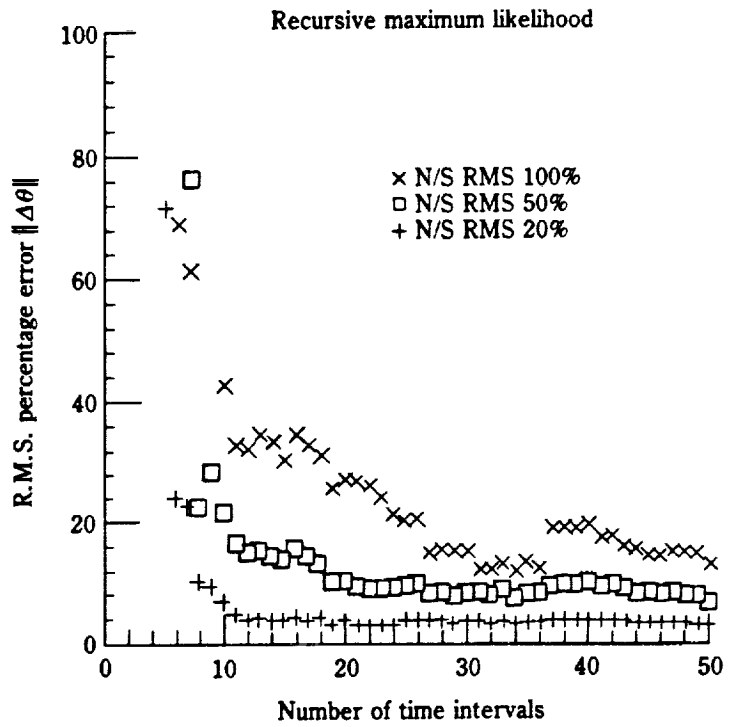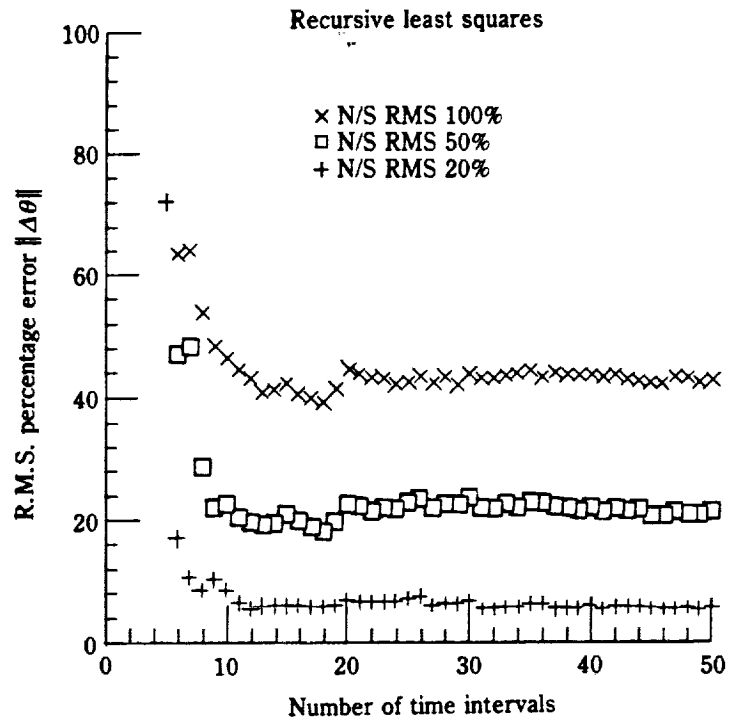
(b) Band pass system $H_2(s)$

Fig. 5.   Recursive identification of $H_1(s)$ and $H_2(s)$.

3.1. Simulation results for the recursive ML and LS estimates   The maximum likelihood estimate developed in the previous section was applied to the identification of the systems $H_1(s)$ and $H_2(s)$ given by (23) and (25), forced by the same colored gaussian random processes as previously described, i.e., bandwidths of 5 [rad/sec] and 15 [rad/sec] respectively. Both the input and output signals for each system were corrupted as shown in Fig. 4 by zero mean white gaussian noises, uncorrelated with each other. Recursive identification was carried out for each system using the data collected over the successive time intervals $[t_i, t_i+2\pi]$, $i = 0, 1, 2, ...,$ i.e., $\omega_0 = 1$ [rad/sec], and a 256 point DFT ($N = 256$) was used for the parabolic approximation (22) in calculating the Fourier coefficient vectors of the data on each $[0, 2\pi]$ time interval.

The modulating function modes were selected as (0, 1, 2, 3, 4, 5) and (7, 9, 11, 13) respectively for the systems $H_1(s)$ and $H_2(s)$, and the resulting algebraic equations, which numbered 8 and 4 respectively, were summed on each interval in order to meet the scalar equation requirement of Special Case 2. Hence, the optimization problem on each $[0, 2\pi]$ interval is to find the eigenvector corresponding to the smallest eigenvalue of the generalized eigenvalue problem (46). This was accomplished using a standard subroutine in the IMSL (1982).

The results of the recursive identification for each system are shown in the lower portions of Fig. 5 with the norm (24) plotted as a function of the number of time intervals. Also shown are the results of the standard recursive least squares algorithm applied to the same data (upper curves). The results clearly show that the recursive LSE incurs a bias at every noise level which can be effectively removed using the recursive MLE.

4. Conclusions

Depending on the nature and degree of completeness of *a priori* knowledge available relative to the system and noise spectral characteristics, a judicious choice of Fourier based modulating functions can be effective in ameliorating noise effects for a deterministic least squares identification based on time limited data. This has been verified via the simulation results for several examples. In the case of recursive identification involving noise corrupted data extracted from non-overlapping time intervals, the maximum likelihood estimate of Levin (1964), Aoki and Yue (1970 a; b) has been adapted to the modulating function formulation in order to remove the bias incurred by a standard least squares algorithm. Future problems include model order determination procedures and optimal inputs for system identification within the context of the modulating function technique.

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Appendix: A set of Fourier based modulating functions

Consider the set of commensurable sinusoids

$$\{1, \cos m_1 \omega_0 t, \sin m_1 \omega_0 t, \ldots, \cos m_M \omega_0 t, \sin m_M \omega_0 t\}, \qquad (A.1)$$

where $\omega_0 = 2\pi/T$ and $(m_1, m_2, \ldots, m_M)$ are selected positive integers satisfying $m_1 < m_2 < \ldots < m_M$. Within the $(2M+1)$ dimensional function space spanned by the functions in (A.1), there will exist a $(2M+1-n)$ dimensional subspace of modulating functions satisfying (2) which can be delineated as follows. In order to account for the different cases when the model order n is even or odd, the "integer part" notation $[n/2]$ is used: $[n/2]$ stands for $n/2$ when n is even, or $[(n-1)/2]$ when n is odd.

Cosine form: For each $k = 0, 1, \ldots, M-[(n+1)/2]$, let

$$\phi_{c,k}(t) = \sum_{j=0}^{[(n+1)/2]} a_{k,j} \cos m_{k+j} \omega_0 t \qquad (A.2)$$

define a modulating function of order n with the $a_{k,j}$ coefficients chosen such that

$$\phi_{c,k}^{(i)}(0) = \phi_{c,k}^{(i)}(T) = 0, \qquad i = 0, 2, 4, \ldots, 2[(n-1)/2] \qquad (A.3)$$

and

$$\sum_{j=0}^{[(n+1)/2]} a_{k,j}^2 = 1, \qquad a_{k,0} > 0. \qquad (A.4)$$

Sine form: For each $k = 1, 2, \ldots, M-[n/2]$, let

$$\phi_{s,k}(t) = \sum_{j=0}^{[n/2]} b_{k,j} \sin m_{k+j} \omega_0 t \qquad (A.5)$$

define a modulating function of order n with the $b_{k,j}$ coefficients chosen such that

$$\phi_{s,k}^{(i)}(0) = \phi_{s,k}^{(i)}(T) = 0, \qquad i = 1, 3, 5, \ldots, 2[n/2]-1 \qquad (A.6)$$

and

$$\sum_{j=0}^{[n/2]} b_{k,j}^2 = 1, \qquad b_{k,0} > 0. \qquad (A.7)$$

With respect to the end point conditions (2), notice that the vanishing of the odd and even derivatives was not included in (A.3) and (A.6) respectively, since these conditions are automatically satisfied by the forms assumed in (A.2) and (A.5). The existence and uniqueness of the above $a_{k,j}$ and $b_{k,j}$ coefficients can be

established by the following Lemma which also yields explicit expressions for their determination.

Lemma. The coefficients $a_{k,j}$ and $b_{k,j}$ specified in (A.2)–(A.7) are uniquely determined by Vandermonde type matrices.

Proof for $b_{k,j}$. For a fixed k in $1 \leq k \leq M - [n/2]$, condition (A.6) is equivalent to

$$
\begin{bmatrix}
m_k & m_{k+1} & \cdots & m_{k+[n/2]} \\
m_k^3 & m_{k+1}^3 & \cdots & m_{k+[n/2]}^3 \\
\vdots & \vdots & & \vdots \\
m_k^{2[n/2]-1} & m_{k+1}^{2[n/2]-1} & \cdots & m_{k+[n/2]}^{2[n/2]-1}
\end{bmatrix}
\begin{bmatrix}
b_{k,0} \\
b_{k,1} \\
\vdots \\
b_{k,[n/2]}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix},
$$

or

$$
\begin{bmatrix}
m_{k+1} & m_{k+2} & \cdots & m_{k+[n/2]} \\
m_{k+1}^3 & m_{k+2}^3 & \cdots & m_{k+[n/2]}^3 \\
\vdots & \vdots & & \vdots \\
m_{k+1}^{2[n/2]-1} & m_{k+2}^{2[n/2]-1} & \cdots & m_{k+[n/2]}^{2[n/2]-1}
\end{bmatrix}
\begin{bmatrix}
b_{k,1} \\
b_{k,2} \\
\vdots \\
b_{k,[n/2]}
\end{bmatrix}
= -b_{k,0}
\begin{bmatrix}
m_k \\
m_k^3 \\
\vdots \\
m_k^{2[n/2]-1}
\end{bmatrix}.
$$

The above equation will be denoted as

$$
M_k b_k = -b_{k,0} m_k.
$$

By letting $\lambda_{k+j} = m_{k+j}^2$, $j = 1, 2, \ldots, [n/2]$, M_k can be expressed as

$$
M_k =
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\lambda_{k+1} & \lambda_{k+2} & \cdots & \lambda_{k+[n/2]} \\
\vdots & \vdots & & \vdots \\
\lambda_{k+1}^{[n/2]-1} & \lambda_{k+2}^{[n/2]-1} & \cdots & \lambda_{k+[n/2]}^{[n/2]-1}
\end{bmatrix}
\begin{bmatrix}
m_{k+1} & 0 & \cdots & 0 \\
0 & m_{k+2} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & m_{k+[n/2]}
\end{bmatrix}. \quad (A.8)
$$

The first matrix on the right hand side of (A.8) is the well-known Vandermonde matrix; its determinant and inverse matrix can be expressed in explicit forms (Graybill, 1983). Hence, M_k^{-1} can be explicitly expressed as

$$
M_k^{-1} = \frac{1}{\prod\limits_{i>j}(m_{k+i}^2 - m_{k+j}^2)}
\begin{bmatrix}
\dfrac{1}{m_{k+1}} & 0 & \cdots & 0 \\
0 & \dfrac{1}{m_{k+2}} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \dfrac{1}{m_{k+[n/2]}}
\end{bmatrix}
\begin{bmatrix}
(i,j)\text{-element} \\
= \dfrac{c_{i,j}}{P_i(m_{k+i}^2)}
\end{bmatrix},
$$

where

$$P_i(x) \equiv \prod_{j=1, j \neq i}^{[n/2]} (x - m_{k+j}^2) \equiv \sum_{j=1}^{[n/2]} c_{i,j} \bar{x}^{j-1} \text{ for } i = 1, 2, \ldots, [n/2]. \quad \text{(A.9)}$$

So, b_k can be expressed as

$$b_k = -b_{k,0} M_k^{-1} m_k, \quad \text{(A.10)}$$

and the condition (A.7) implies that

$$b_{k,0}^2 (1 + m_k^T M_k^{-T} M_k^{-1} m_k) = 1, \text{ and } b_{k,0} > 0. \quad \text{(A.11)}$$

It is clear that the value of $b_{k,0}$ is uniquely specified as

$$b_{k,0} = (1 + m_k^T M_k^{-T} M_k^{-1} m_k)^{-\frac{1}{2}}. \quad \text{(A.12)}$$

Hence $\{b_{k,j}\}$ can be uniquely found by combining (A.9), (A.10), and (A.12). The derivation and proof for $a_{k,j}$ is similar and hence is omitted.

It is convenient for identification purposes to collect the $2M + 1 - n$ modulating functions into a single column vector,

$$\Phi(t) \equiv \begin{bmatrix} \phi_{c,0}(t) \\ \vdots \\ \phi_{c,M-[(n+1)+/2]}(t) \\ \phi_{s,1}(t) \\ \vdots \\ \phi_{s,M-[n/2]}(t) \end{bmatrix}. \quad \text{(A.13)}$$

It is easy to see that these $(2M + 1 - n)$ modulating functions are linearly independent, since in the construction procedure (A.2) to (A.7) a new sinusoid is added as k is increased. Hence $\Phi(t)$ consists of a set of basis functions for the $(2M + 1 - n)$ dimensional modulating function space which is a subspace of the $(2M + 1)$ dimensional trigonometric function space spanned by the functions in (A.1). Let $f(t)$ denote the column vector of $2M + 1$ sinusoids:

$$f(t) \equiv \text{col}[1, \cos m_1 \omega_0 t, \sin m_1 \omega_0 t, \ldots, \cos m_M \omega_0 t, \sin m_M \omega_0 t],$$

$$0 \leq t \leq T.$$

Then $\Phi(t)$ in (A.13) can be represented as

$$\Phi(t) = Cf(t), \quad \text{(A.14)}$$

where C is an $(2M + 1 - n) \times (2M + 1)$ matrix determined by the $\{a_{k,j}, b_{k,j}\}$ in (A.2)–(A.7).

The above results are summarized in the following Theorem.

Theorem. In the $(2M + 1)$-dimensional vector space spanned by the set in (A.1), there exits a $(2M + 1 - n)$-dimensional subspace of modulating functions of order n represented by the vector function $\Phi(t)$ in (A.14). The matrix C in (A.14) has rank $(2M + 1 - n)$.

Appendix D

FREQUENCY ANALYSIS VIA THE METHOD OF MOMENT FUNCTIONALS[1]

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Abstract

Several variants are presented of a linear-in-parameters least squares formulation for determining the transfer function of a stable linear system at specified frequencies given a finite set of Fourier series coefficients calculated from transient nonstationary input-output data. The basis of the technique is Shinbrot's classical method of moment functionals using complex Fourier based modulating functions to convert a differential equation model on a finite time interval into an algebraic equation which depends linearly on frequency-related parameters.

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1. INTRODUCTION

Methods for determining the transfer function of a stable linear system from input-output data include correlation and spectral analyses, as well as the direct sinusoidal measurements. Each of these "nonparametric" identification techniques require either a statistical stationarity assumption on the data, or a periodic steady state condition to be established, before initiating calculations of the transfer function at pertinent frequencies. Excellent summaries of these methods, as well as the analysis of noise effects and finite data lengths, can be found in Astrom [1], Ljung [2], Soderstrom and Stoica [3], and Unbehauen and Rao [4]. Notwithstanding noise considerations, long data lengths may be required in order to achieve good accuracy due to the stationarity or steady state assumption. By contrast, a method is proposed here that utilizes the frequency content in short data lengths in order to set up a least squares estimation of the transfer function at selected frequencies. Since short data lengths are used there is no assumption of steady state operation or stationarity of the data, though there must be present sufficient energy content in the data at the specified frequencies in order to avoid degeneracy in the least square estimate. The basis of the technique is the classical Shinbrot [5] method of moment functionals, also known as the modulating function technique, using *complex* Fourier based modulating functions. A forerunner of this approach can be found in Pearson and Lee [6] which utilized real valued Fourier based modulating functions, i.e., commensurable sinusoids, for the parameter estimation of linear differential systems. Therein also can be found a discussion of the background of this method with a listing of references. However, this paper appears to represent the first use of modulating functions in the context of the "nonparametric" system identification problem. Several variants of a deterministic least squares estimation of frequency-related parameters that underlie the transfer function will

be developed below.

2. Least Squares Formulations

Consider a stable linear system with input $u(t)$ and output $y(t)$ modeled on a finite time interval by the differential operator equation:

$$A(p)y(t) = B(p)u(t) + e(t) \tag{1}$$

where $(A(p), B(p))$ are polynomials in the differential operator $p = d/dt$ of degree less than or equal to an *a priori* integer n, and $e(t)$ represents the effect of modeling errors. The problem considered here is to estimate the transfer function $G(j\omega) = B(j\omega)/A(j\omega)$ at a finite set of frequencies $\{k\omega_0, k=1,2 \cdots M\}$, where ω_0 is a user selected "resolving" frequency and M a chosen integer, given the input-output data $[u(t), y(t)]$ over a finite set of time intervals $\{[t_i, t_i + T_0], i=1, \cdots N\}$.[1] These time intervals are each chosen of length $T_0 = 2\pi/\omega_0$ and need not necessarily be disjoint. However, a certain degree of independence in the data collected over the different $[0, T_0]$ time intervals is necessary in order to avoid degeneracies in the least squares estimate to be discussed below. Understandably these degeneracies are more likely to be avoided in normal operating records if the intervals are disjoint. In addition to the upper bound on the system order n, the DC value of the transfer function is assumed given or can be measured from the step response, i.e., $G(0) = B(0)/A(0)$ is presumed known *a priori*. If this is not the case, then the estimated transfer function can be scaled by the parameter $G(0)$.

The Shinbrot method of moment functionals is a technique for converting a differential equation on a finite time interval into an algebraic equation by the use of "modulating func-

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[1] If the system bandwidth ω_B is known, then choosing (M, ω_0) such that $M\omega_0 \simeq \omega_B$ will cover the bandwidth at the knots $k\omega_0$, $k=1,2 \cdots M$.

tions''. As introduced by Shinbrot [5], $\phi(t)$ is a modulating function of order n relative to a finite time interval $0 \leq t \leq T_0$ if it is sufficiently smooth and satisfies the end point conditions:

$$\phi^{(i)}(0) = \phi^{(i)}(T_0) = 0, \quad i = 0, 1 \cdots n-1 \tag{2}$$

where $\phi^{(i)}(t) = p^i \phi(t)$. Clearly, modulating functions can be constructed in many different ways.[2] Here a specific set of complex valued Fourier based modulating functions is defined in a way that is conducive to solving the problem at hand, viz., let

$$\phi_m(t) = e^{jm\omega_0 t}(1 - e^{j\omega_0 t})^n, \quad 0 \leq t \leq T_0 = 2\pi/\omega_0 \tag{3a}$$

$$m = 0, 1 \cdots M$$

define a set of modulating functions of order n with respect to the time interval $[0, T_0]$. Equivalently by the binomial expansion, each such function is representable by

$$\phi_m(t) = \sum_{k=0}^{n} b_k e^{j(m+k)\omega_0 t}. \tag{3b}$$

where b_k is defined in relation to the binomial coefficient by

$$b_k = (-1)^k \binom{n}{k}. \tag{4}$$

The first representation (3a) makes evident the fact that each $\phi_m(t)$ is indeed a modulating function of order n, i.e., (2) is satisfied,[3] while the second representation (3b) implies that calculating linear functionals defined by each $\phi_m(t)$ on a set of functions specified over $[0, T_0]$ will entail calculating the Fourier series coefficients of these functions at the frequencies $k\omega_0$, $k = m, m+1 \cdots m+n$, $m = 0, 1 \cdots M$. In turn, these coefficients can be calculated efficiently by DFT/FFT methods which provides an important motivating factor for this analysis. This will

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[2] See discussion in Pearson and Lee [6].

[3] Notice that any modulating function of a fixed order is automatically a modulating function of any lower order relative to the same time interval. This property facilitates the formulation for any system of order less than or equal to the upper bound n.

be discussed further below. The important property of the functions defined in (3) is contained in the following [4]

Modulation Property. Let $P(p)$ be a differential operator of order at most n, i.e., a polynomial in $p = d/dt$ of degree $\leq n$, and $z(t)$ any sufficiently smooth function defined on $[0,T_0]$. Then the modulation of $P(p)z(t)$ with $\phi_m(t)$ over $[0,T_0]$ satisfies

$$\int_0^{T_0} \phi_m(t) P(p) z(t) dt = \sum_{k=m}^{m+n} b_{k-m} P(-jk\omega_0) Z_{-k} \tag{5}$$

where Z_k is the kth harmonic Fourier series coefficient of $z(t)$, i.e.,

$$Z_k = \int_0^{T_0} z(t) e^{-jk\omega_0 t} dt. \tag{6}$$

Note that owing to the end point constraints (2) satisfied by each $\phi_m(t)$ function, none of the boundary point derivatives $z^{(i)}(0)$ or $z^{(i)}(T_0)$ appear in (5). This is crucial to the ensuing analysis and, in fact, represents a primary reason for employing the modulating function technique.

2.1. Formulation 1

A direct application of the above property to the problem posed involves rewriting the differential operator model (1) in the equation error form followed by the modulation with $\phi_m(t)$; thus,[5]

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[4] Proof of this property is given in the Appendix in order to proceed directly with the development.

[5] The process of going from the model (1) to equation (7) can be viewed as a projection from a space of functions on $[0,T_0]$ down into a finite dimensional space spanned by the modulating functions.

$$\int_0^{T_0} \phi_m(t) \left[A(p)y(t) - B(p)u(t) \right] dt = \int_0^{T_0} \phi_m(t)e(t)dt. \tag{7}$$

In view of (5) the preceding equation is equivalent to

$$\sum_{k=m}^{m+n} b_{k-m} \left[A(-jk\omega_0)Y_{-k} - B(-jk\omega_0)U_{-k} \right] = \sum_{k=m}^{m+n} b_{k-m} E_{-k}. \tag{8}$$

Define the real and imaginary parts of the polynomials $(A(jk\omega_0), B(jk\omega_0))$ as follows:

$$A(jk\omega_0) = \alpha_k + j\beta_k, \qquad B(jk\omega_0) = \gamma_k + j\delta_k \tag{9}$$

and collect these together to form the 4×1 "parameter" vector:

$$\theta_k = \begin{bmatrix} \alpha_k \\ \beta_k \\ \gamma_k \\ \delta_k \end{bmatrix}. \tag{10}$$

Also, define as follows the 2×4 data matrix $\psi_k(i)$ in terms of the real and imaginary parts of the kth harmonic Fourier series coefficients of the input-output data corresponding to the time interval $[t_i, t_i + T_0]$, $i = 1, 2 \cdot \cdot N$:

$$\psi_k(i) = \begin{bmatrix} Y_k^c(i) & Y_k^s(i) & -U_k^c(i) & -U_k^s(i) \\ -Y_k^s(i) & Y_k^c(i) & U_k^s(i) & -U_k^c(i) \end{bmatrix}. \tag{11}$$

The notation for the entries in (11) is explained by

$$Y_k^c(i) = \int_0^{T_0} y(t+t_i)\cos k\omega_0 t \; dt, \qquad Y_k^s(i) = \int_0^{T_0} y(t+t_i)\sin k\omega_0 t \; dt \tag{12}$$

and similarly for $(U_k^c(i), U_k^s(i))$. Then the real and imaginary parts of the equation error (8) can be collected into the following real valued 2×1 vector equation which serves as the starting point for a least squares estimation:

$$\sum_{k=m}^{m+n} b_{k-m} \psi_k(i)\theta_k = \varepsilon_m(i) \tag{13}$$

$$m = 0, 1 \cdot \cdot M, \qquad i = 1, 2 \cdot \cdot N.$$

The equation error vector in (13) is related to the Fourier coefficients of the original equation error by

$$\varepsilon_m(i) = \sum_{k=m}^{m+n} b_{k-m} \begin{bmatrix} E_k^c(i) \\ E_k^s(i) \end{bmatrix}.$$

Values of the transfer function $G(jk\omega_0)=B(jk\omega_0)/A(jk\omega_0)$ are seen from (9) and (10) to be related to the parameter vector θ_k by the real and imaginary part relations:

$$\text{Re}\,G(jk\omega_0)=\frac{\alpha_k\gamma_k-\beta_k\delta_k}{\alpha_k^2+\beta_k^2}, \quad \text{Im}\,G(jk\omega_0)=\frac{\alpha_k\delta_k-\beta_k\gamma_k}{\alpha_k^2+\beta_k^2} \tag{14}$$

or equivalently by the magnitude-phase relations:

$$|G(jk\omega_0)|^2 = \frac{\gamma_k^2+\delta_k^2}{\alpha_k^2+\beta_k^2}, \quad G(jk\omega_k) = \tan^{-1}\frac{\delta_k}{\gamma_k} - \tan^{-1}\frac{\beta_k}{\alpha_k}. \tag{15}$$

Starting from a presumed knowledge of the DC value $G(0)$, which implies that $\theta_0=[A(0),0,B(0),0]'$ is known, equation (13) can be rewritten in the standard regression equation format to estimate the parameters θ_k, $k=1,2\cdots M+n$ given the data over a sufficient number of $[t_i,t_i+T_0]$, intervals, $i=1,2\cdots N$. A consideration of this equation reveals the following:

1) The frequency range covered by the parameters in (13) is $(M+n)\omega_0$. Hence, if it is desired that the transfer function estimate cover a frequency range about 25% greater than the system bandwidth ω_B at a resolution ω_0, a choice in (M,ω_0) such that

$$(M+n)\omega_0 \approx 1.25\omega_B \tag{16}$$

reflects this objective.

2) Counting unknowns in (10) and (13), the total number is $4(M+n)$. Since each equation in (13) is of dimension 2, counting equations suggests that the total number N of $[t_i,t_i+T_0]$ intervals should satisfy: $2(M+1)N\geq4(M+n)$, i.e., $N\geq2(M+n)/(M+1)$.

However, the equations in (13) are partially decoupled with respect to the index m. Therefore, it seems best to estimate $(\theta_1, \cdot\cdot \theta_n)$ at the first stage, which corresponds to $m=0$. This means that there are $4n$ unknowns for the first stage requiring that N satisfy $N \geq 2n$. Thereafter, the number of unknowns is just 4 for each stage corresponding to $m=1,2 \cdot\cdot M$, which implies $N \geq 2$ assuming that the preceding estimates are used in each succeeding stage. This kind of "bootstrapping" of the least squares estimation facilitates keeping the number of unknowns to a modest level at each stage.

3) The two row vectors comprising $\psi_k(i)$ in (11) are seen to be mutually orthogonal for each k and i suggesting a maximal degree of independence for these equations in utilizing the information content in the data. This is a direct result of the Fourier nature of the underlying formulation.

Discussion: The above development shows that it is possible to formulate a linear-in-the parameters least squares estimation problem for parameters (10) that underlie (via (14) or (15)) the transfer function $G(jk\omega_0)$ at each kth harmonic frequency. The input-output data can be time-limited and transient, but must have sufficient energy content at the specified frequencies to avoid degeneracies in the least squares solution. Apart from being highly non-linear, the relations (14) and (15) involve the difference between parameter related quantities, e.g., $\alpha_k\gamma_k - \beta_k\delta_k$, whose values may be large for large k. This aspect of the formulation portends a potential source of error magnification which is alleviated by the formulation of the next section.

2.2. Formulation 2

Given that $[u(t),y(t)]$ satisfies the model (1) on a $[0,T_0]$ time interval, it follows that $[u(t),y(t)]$ also satisfies the model

$$A(-p)A(p)y(t)=A(-p)B(p)u(t)+A(-p)e(t).   \tag{17}$$

Choosing a set of modulating functions of order $2n$ to accommodate the upper bound on the highest degree differential operator in (17) and modulating this equation with the mth member of this set, the following projected equation error results which is analogous to (8):

$$\sum_{k=m}^{m+2n} \overline{b}_{k-m} \left[A(jk\omega_0)A(-jk\omega_0)Y_{-k} - A(jk\omega_0)B(-jk\omega_0)U_{-k} \right] = \sum_{k=m}^{m+2n} \overline{b}_{k-m}A(jk\omega_0)E_{-k}   \tag{18}$$

where $\overline{b}_k$ is defined by, cf. (4):

$$\overline{b}_k = (-1)^k \binom{2n}{k}.$$

Noting that $A(jk\omega_0)A(-jk\omega_0)$ is real while $A(jk\omega_0)B(-jk\omega_0)$ is complex, define real quantities (a_k,α_k,β_k) by the relations:

$$a_k=A(jk\omega_0)A(-jk\omega_0), \quad \alpha_k+j\beta_k=A(jk\omega_0)B(-jk\omega_0)   \tag{19}$$

and collect these into the 3×1 parameter vector $\overline{\theta}_k$ defined by

$$\overline{\theta}_k = \begin{bmatrix} a_k \\ \alpha_k \\ \beta_k \end{bmatrix}.   \tag{20}$$

Also, define the 2×3 data matrix $\overline{\psi}_k(i)$ by

$$\overline{\psi}_k(i)= \begin{bmatrix} Y_k^c(i) & -U_k^c(i) & U_k^s(i) \\ Y_k^s(i) & -U_k^s(i) & -U_k^c(i) \end{bmatrix}.   \tag{21}$$

where the notation for the entries in (21) is the same as defined in (12). Then the real and imaginary parts of the projected equation error (18) can be represented by the following real 2×1 vector equation:

$$\sum_{k=m}^{m+2n} \overline{b}_{k-m} \overline{\psi}_k(i) \overline{\theta}_k = \overline{\varepsilon}_m(i) \tag{22}$$

$$m=0,1 \cdot \cdot M, \quad i=1,2 \cdot \cdot N.$$

Equation (22) can be rewritten into the standard regression equation format for setting up

the least squares estimate of the parameters $(\overline{\theta}_1, \cdot \cdot \overline{\theta}_{M+2n})$ based on the data over $[t_i, t_i+T_0]$,

$i=1,2 \cdot \cdot N$. Again, presumed knowledge of the DC value implies that $\overline{\theta}_0 = A(0)[A(0),B(0),0]'$

is known or, if not, the resulting transfer function estimate can be scaled by the parameter

$G(0)$. Here the estimates of the transfer function are related to the parameters by the real and

imaginary part equations (as found from (19) and (20)):

$$\text{Re}G(jk\omega_0)=\frac{\alpha_k}{a_k}, \quad \text{Im}G(jk\omega_0)=-\frac{\beta_k}{a_k} \tag{23}$$

or equivalently by the magnitude-phase relations:

$$|G(jk\omega_0)|^2 = \frac{\alpha_k^2+\beta_k^2}{a_k^2}, \quad G(jk\omega_0) = -\tan^{-1}\frac{\beta_k}{\alpha_k}. \tag{24}$$

Consideration of the least squares formulation in this case leads to the following:

1) The frequency range covered by the parameters in (22) is $(M+2n)\omega_0$; hence, the

guideline (analogous to (16)) for choosing the pair (M,ω_0) in this case is

$$(M+2n)\omega_0 \approx 1.25\omega_B. \tag{25}$$

2) Counting unknowns in (20) and (22), the total number is $3(M+2n)$ which would

imply that the total number N of $[t_i, t_i+T_0]$ time intervals should satisfy:

$2N(M+1) \geq 3(M+2n)$. However, the partially decoupled nature of the equation (22)

with respect to the m index suggests bootstrapping the solution from the first stage.

Thus, for the initial stage $(m=0)$, N needs to satisfy $N \geq 3n$. In the succeeding stages,

N needs to satisfy $N \geq 2$ (since there are 3 unknowns and $2N$ equations at each such

stage); this assumes that the preceding estimates are used at each succeeding stage $(m=1,2 \cdot \cdot M)$.

Discussion: Comparing (23) and (24) with (14) and (15) reveals that the second formulation avoids the potential error magnification problem of differencing large estimated quantities in calculating the transfer function at high frequencies. However, the second formulation requires estimating $6n$ unknowns at the first stage, i.e., the $m=0$ stage, verses $4n$ unknowns for the first stage of the first formulation.

2.3. Formulation 2-Dual

The dual to the formulation of the preceding section is to observe that a given pair $[u(t),y(t)]$ satisfying the model (1) on a $[0,T_0]$ time interval also implies that it satisfies (cf. (17))

$$B(-p)A(p)y(t)=B(-p)B(p)u(t)+B(-p)e(t). \tag{26}$$

Again choosing a set of modulating functions of order $2n$, a development similar to that of the previous section leads to the real 2×1 vector equation

$$\sum_{k=m}^{m+2n} \bar{b}_{k-m} \tilde{\psi}_k(i)\bar{\theta}_k = \bar{\varepsilon}_m(i) \tag{27}$$

$$m=0,1 \cdot \cdot M, \quad i=1,2 \cdot \cdot N$$

where the data matrix $\tilde{\psi}_k(i)$ is defined by

$$\tilde{\psi}_k(i) = \begin{bmatrix} U_k^c(i) & -Y_k^c(i) & Y_k^s(i) \\ U_k^s(i) & -Y_k^s(i) & -Y_k^c(i) \end{bmatrix} \tag{28}$$

and the real 3×1 parameter vector $\bar{\theta}_k$ is defined by

$$\bar{\theta}_k = \begin{bmatrix} b_k \\ \alpha_k \\ -\beta_k \end{bmatrix} \quad\quad\quad (29)$$

with the entries in (29) defined by

$$b_k = B\,(jk\,\omega_0)B\,(-jk\,\omega_0), \quad \alpha_k + j\,\beta_k = A\,(jk\,\omega_0)B\,(-jk\,\omega_0). \quad (30)$$

Relations between the transfer function and the parameters in this case are found to be

$$\mathrm{Re}G\,(jk\,\omega_0) = \frac{\alpha_k b_k}{\alpha_k^2 + \beta_k^2}, \quad \mathrm{Im}G\,(jk\,\omega_0) = \frac{-\beta_k b_k}{\alpha_k^2 + \beta_k^2} \quad (31)$$

for the real and imaginary parts, or for the magnitude-phase:

$$|G\,(jk\,\omega_0)|^2 = \frac{b_k^2}{\alpha_k^2 + \beta_k^2}, \quad G\,(jk\,\omega_0) = -\tan^{-1}\frac{\beta_k}{\alpha_k}. \quad (32)$$

Comparing (21) and (28) verifies the duality of the two formulations by virtue of the interchange of input and output. Note that each formulation has the same total number of unknowns - in general. However, the dual formulation has the potential advantage of reducing the total number of unknowns in the event of *a priori* information on a lower degree numerator polynomial than denominator polynomial in the transfer function. For example, an "all pole" model means that $b_k = (B\,(0))^2$ is known for all k, i.e., a total of $2(M+n)$ unknowns verses $3(M+2n)$ for the previous case. The formulation leading to (27) can easily be modified to reflect this consideration.

3. Conclusions

Three formulations of a linear-in-the parameters least squares estimation have been presented for determining the transfer function of a linear system at specified frequencies given transient nonstationary input-output data. While some comparisons have been noted in the discussions following each formulation, a clear indication of the pitfalls and advantages of

each will have to await a thorough simulation study including the effects of noise. Such a study is currently underway.

4. References

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5. Appendix

To verify the modulation property (5) it is sufficient to consider the differential operator $P(p)=p^i$ for a fixed $i \leq n$. The result for a general n^{th} degree polynomial $P(p)$ then follows by superposition. Thus, for sufficiently smooth $z(t)$ on $[0,T_0]$ and $\phi_m(t)$ defined in (3), the left side of (5) in this case is

$$\int_0^{T_0} \phi_m(t)p^i z(t)dt = (-1)^i \int_0^{T_0} z(t)p^i \phi_m(t)dt \tag{33}$$

where integration-by-parts has been used i times taking into account the boundary conditions (2) possessed by each $\phi_m(t)$ function. Substituting the representation (3b) into the right side of (33), carrying out the indicated differentiation and changing the index of summation verifies (5) for $P(p)=p^i$ as purported.