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ON THE PROPAGATION OF PLANE WAVES ABOVE AN IMPEDANCE SURFACE

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ABSTRACT

The propagation of grazing incidence plane waves along a finite impedance boundary is investigated. A solution of the semi-infinite problem, where a harmonic motion, parallel to the boundary, is imposed along a line perpendicular to the boundary, is obtained. This solution consists of quasiplane waves, waves moving parallel to the boundary with amplitude and phase variations perpendicular to the boundary. Several approximations to the full solution are considered.

INTRODUCTION

Mathematical modeling of the propagation and reflection of harmonic plane waves above a finite impedance plane surface is a fundamental topic in acoustics. In the case where the angle between the normal to the wavefront and the surface is not zero an analytic solution is very easy to obtain. This solution consists of the incident plane wave propagating toward the surface plus a reflected plane wave propagating away from the surface at the same magnitude of the angle between its normal and the surface as the incident wave. The amplitude of the reflected wave is given by a reflection coefficient that is expressed in terms of incident angle and the specific impedance of the surface. However at zero incident angle (the wave normal parallel to the surface), complete cancellation of the incident and reflect waves occurs in this model and a zero solution results. Most acoustic texts claim that this situation is not possible [1-3].

McAninch[4] recently has investigated a related situation where a plane wave source is generating waves that would move parallel to a surface if its impedance was infinite but where the surface impedance is not infinite quasiplane waves result. McAninch's investigation, however, uses the parabolic approximation where only waves traveling in one direction are allowed. This paper approaches the same problem without the assumption of parabolic approximation.

FORMULATION OF THE PROBLEM

The governing acoustic wave equation for harmonic waves can be put in the form

$$(\nabla^2 + k^2)\,\phi = 0 \qquad (1)$$

where the time dependent part of the potential, $e^{-i\omega t}$, has been separated from the spatial part of the potential, $\phi(x,y)$. When an impedance boundary exists, the solution of equation (1) must also satisfy the boundary condition

$$\phi_y + \gamma\phi = 0 \qquad (2)$$

on y=0. Here the subscript y indicates a partial derivative with respect to y.

For a unique solution, some extra constraints must be introduced. One is to assume that ϕ will not be affected by the ground impedance as y approaches infinity, the second is that the acoustic pressure at x=0 is given by

$$\phi(0,y) = 1 \tag{3}$$

EXACT SOLUTION

We assume that the solution of (1) has the form of

$$\phi = e^{ikx} + f(x,y) \tag{4}$$

where e^{ikx} can be considered as a solution without the boundary condition given by (2). Substituting (4) into (1),(2) and (3) we get a new governing equation and set of boundary conditions

$$(\nabla^2 + k^2)\, f = 0 \tag{5}$$

$$f_y\,(x,0) + \gamma f\,(x,0) = -\,\gamma e^{ikx} \tag{6}$$

$$f\,(0,y) = 0 \tag{7}$$

and

$$\lim_{y\to\infty} f\,(x,y) = 0 \tag{8}$$

Equation (8) results from the first uniqueness condition listed above.

The sine transform,

$$F\,(\lambda,y) = \int_0^\infty f\,(x,y)\,\sin\,(\lambda x)\,dx \tag{9}$$

is equivalent to the Fourier transform of an even function and will be applied here. The inverse transform is given by

$$f\,(x,y) = \frac{2}{\pi}\int_0^\infty F\,(x,y)\,\sin\,(\lambda x)\,d\lambda \tag{10}$$

Applying (9) to (5) and (6) we have

$$\frac{\partial^2 F}{\partial y^2} - (\lambda^2 - k^2)\,F = 0 \tag{11}$$

and

$$F_y\,(\lambda,0) + \gamma F\,(\lambda,0) = -\,\gamma\frac{\lambda}{\lambda^2 - k^2} \tag{12}$$

Here, it is assumed that k is a complex number with a very small positive imaginary part.

Solving (11) and making use of the given boundary conditions, yields

$$F(\lambda,y) = A(\lambda,0)\, e^{-my} \tag{13}$$

$$A(\lambda,0) = -\frac{\gamma\lambda}{(\lambda^2-k^2)(\gamma-m)} \tag{14}$$

where $m = \sqrt{(\lambda^2-k^2)}$. Since the solution is required to remain finite, $Re\sqrt{(\lambda^2-k^2)} > 0$. Substituting (13) and (14) into (10) yields the inverse transform of $F(\lambda,y)$ as

$$f(x,y) = -\frac{2\gamma}{\pi}\int_0^\infty \frac{\lambda}{(\lambda^2-k^2)(\gamma-m)}\, e^{-my}\sin(\lambda x)\, d\lambda \tag{15}$$

For convenience, substitute the identity

$$\sin(\lambda x) = \frac{e^{i\lambda x} - e^{-i\lambda x}}{2i} \tag{16}$$

into (15), yielding

$$f(x,y) = -\frac{\gamma}{i\pi}(I_1 - I_2) \tag{17}$$

where

$$I_1 = \int_0^\infty \frac{\lambda e^{-my}}{(\lambda^2-k^2)(\gamma-m)}\, e^{i\lambda x}\, d\lambda \tag{18}$$

$$I_2 = \int_0^\infty \frac{\lambda e^{-my}}{(\lambda^2-k^2)(\gamma-m)}\, e^{-i\lambda x}\, d\lambda \tag{19}$$

In order to evaluate the above two integrals, introduce the complex variable $\Lambda = \lambda + is$ and define the contour integrals

$$I_{CI} = \int_{CI} \frac{\Lambda e^{-My}}{(\Lambda^2-k^2)(\gamma-M)}\, e^{i\Lambda x}\, d\Lambda \tag{20}$$

$$I_{CII} = \int_{CII} \frac{\Lambda e^{-My}}{(\Lambda^2-k^2)(\gamma-M)}\, e^{-i\Lambda x}\, d\Lambda \tag{21}$$

First evaluate the integral I_{CI} where the contour is shown in Figure 1 along with the branch lines which extend from the imaginary axis to the points $\Lambda = \pm k$. The value of this contour integral is determined by the residue within the contour. Writing

$$\gamma = \alpha + i\beta \tag{22}$$

it is clear a pole exists within the contour only when $\alpha > 0$ (since $\mathrm{Re}\sqrt{\Lambda^2 - k^2} > 0$, $\gamma - \sqrt{\Lambda^2-k^2} = 0$ only when $\mathrm{Re}(\gamma) = \alpha > 0$), and this pole is at $\Lambda = \sqrt{(k^2 + \gamma^2)}$. It is easy to determine the residue at this pole to be

$$\mathrm{Res}\,(\sqrt{k^2+\gamma^2}\,) = -\frac{e^{-\gamma y + i\sqrt{k^2+\gamma^2}\,x}}{\gamma} \tag{23}$$

and

$$I_1 = I_{C1} = -I_{C2} - I_{C3} - I_{C4} - I_{C5} - I_{C6} + \begin{cases} 2\pi i\,\mathrm{Res}\,(\sqrt{k^2+\gamma^2}\,) & \alpha > 0 \\ 0 & \alpha < 0 \end{cases} \tag{24}$$

when $R \to \infty$, I_{C2} will vanish, while

$$I_{C3} = -\int_0^\infty \frac{s}{(s^2+k^2)(\gamma - i\sqrt{s^2+k^2}\,)}\, e^{-i\sqrt{s^2+k^2}\,y}\, e^{-sx}\, ds \tag{25}$$

$$I_{C4} = \int_0^k \frac{\lambda}{(\lambda^2-k^2)(\gamma - i\sqrt{k^2-\lambda^2}\,)}\, e^{-i\sqrt{k^2-\lambda^2}\,y}\, e^{i\lambda x}\, d\lambda \tag{26}$$

$$I_{C5} = -\frac{\pi i}{\gamma}\, e^{ikx} \tag{27}$$

$$I_{C6} = -\int_0^k \frac{\lambda}{(\lambda^2-k^2)(\gamma + i\sqrt{k^2-\lambda^2}\,)}\, e^{i\sqrt{k^2-\lambda^2}\,y}\, e^{i\lambda x}\, d\lambda \tag{28}$$

Substituting the above integrals into (24), yields

$$\begin{aligned} I_1 = {} & \int_0^k \frac{\lambda}{(\lambda^2-k^2)} \left\{ \frac{e^{i\sqrt{k^2-\lambda^2}\,y}}{(\gamma + i\sqrt{k^2-\lambda^2}\,)} - \frac{e^{-i\sqrt{k^2-\lambda^2}\,y}}{(\gamma - i\sqrt{k^2-\lambda^2}\,)} \right\} e^{i\lambda x}\, d\lambda \\ & + \int_0^\infty \frac{s\,e^{-i\sqrt{s^2+k^2}\,y}}{(s^2+k^2)(\gamma - i\sqrt{s^2+k^2}\,)}\, e^{-sx}\, ds + \frac{i\pi}{\gamma}\, e^{ikx} \\ & + \begin{cases} -\dfrac{i2\pi}{\gamma}\, e^{-\gamma y}\, e^{i\sqrt{k^2+\gamma^2}\,x} & \alpha > 0 \\ 0 & \alpha < 0 \end{cases} \end{aligned} \tag{29}$$

The value of integral I_2 is much easier to evaluate. We choose a contour in the fourth quadrant, since there is no pole within the contour. I_2 can then be written as

$$I_2 = \int_0^\infty \frac{s\, e^{i\sqrt{s^2+k^2}\, y}}{(s^2+k^2)(\gamma + i\sqrt{s^2+k^2})}\, e^{-sx}\, ds \tag{30}$$

Here it must be recalled that there is a branch line along the imaginary axis. Subtracting I_2 from I_1 yields

$$I_1 - I_2 = \frac{i\pi}{\gamma}\, e^{ikx}$$

$$+ \int_0^k \frac{\lambda}{(\lambda^2 - k^2)} \left\{ \frac{e^{i\sqrt{k^2-\lambda^2}\, y}}{(\gamma + i\sqrt{k^2-\lambda^2})} - \frac{e^{-i\sqrt{k^2-\lambda^2}\, y}}{(\gamma - i\sqrt{k^2-\lambda^2})} \right\} e^{i\lambda x}\, d\lambda$$

$$+ \int_0^\infty \frac{s}{(s^2+k^2)} \left\{ \frac{e^{-i\sqrt{s^2+k^2}\, y}}{(\gamma - i\sqrt{s^2+k^2})} - \frac{e^{i\sqrt{s^2+k^2}\, y}}{(\gamma + i\sqrt{s^2+k^2})} \right\} e^{-sx}\, ds \tag{31}$$

$$+ \begin{cases} -\dfrac{i2\pi}{\gamma}\, e^{-\gamma y}\, e^{i\sqrt{k^2+\gamma^2}\, x} & \alpha > 0 \\ 0 & \alpha < 0 \end{cases}$$

By substitution of $\lambda = \sqrt{k^2-t^2}$ and $s = -i\sqrt{k^2-t^2}$, the above two integrals can be combined into one. The final result is

$$f(x,y) = \begin{cases} -e^{ikx} + P - K & \alpha > 0 \\ -e^{ikx} - K & \alpha < 0 \end{cases} \tag{32}$$

where

$$P = 2e^{-\gamma y}\, e^{i\sqrt{k^2+\gamma^2}\, x} \tag{33}$$

and

$$K = \frac{2\gamma}{\pi} \int_0^\infty \frac{(t\,\mathrm{Cos}\,(ty) - \gamma\,\mathrm{Sin}\,(ty))}{t(\gamma^2+t^2)}\, e^{i\sqrt{k^2-t^2}\, x}\, dt \tag{34}$$

P is called the surface wave, and it both decays with increasing height y, and also decays with the distance x due to the imaginary part of γ.

ASYMPTOTIC EXPANSION VALID FOR SMALL RECEIVER HEIGHTS

1. Soft boundary case

Integral K can be asymptotically evaluated for large x using the saddle-point method. This method is discussed by Morse and Feshbach[5] and will not be discussed here. Actually we can use some of conclusions from Wenzel [6] since we have the same factor $\sqrt{k^2-t^2}$ as occurred there.

The steepest-descent path has been shown in [6] to be given by

$$T = t + i\,s \tag{35}$$

where

$$s = -\frac{t}{\sqrt{1+\frac{t^2}{k^2}}} \tag{36}$$

and $t > 0$. Again using the residue theorem, integral K can be transformed into the integral L. Note that $i\gamma$ is not in the region of concern since $\beta \geq 0$. Thus

$$K = \begin{cases} P + L & -i\gamma \in D \\ L & -i\gamma \notin D \end{cases} \tag{37}$$

where D is the region between positive real axis and curve $s=-t\,(1+t^2/k^2)^{-1/2}$, P is the surface wave given in (33) and

$$L = \frac{2\gamma}{\pi} \int_{SDP} \frac{T\ \mathrm{Cos}\,(Ty) - \gamma\,\mathrm{Sin}\,(Ty)}{(\gamma^2 + T^2)\,T}\, e^{i\sqrt{k^2-T^2}\,x}\, dT \tag{38}$$

Substituting (37) into (32), we have

$$K = \begin{cases} P + L & -i\gamma \in \Gamma \\ L & -i\gamma \notin \Gamma \end{cases} \tag{39}$$

The region Γ in γ plane is bounded by the curve $\beta \geq 0$, $0 \leq \alpha = \beta(1+\beta^2/k^2)^{-1/2}$. The region Γ is called the surface wave region in the far field (shown in Figure 2), which is same as that of reference [6]. When $\gamma \in \Gamma$, we can easily show $\mathrm{Re}\sqrt{k^2+\gamma^2} \geq k$,that means, if the surface wave exists,its propagation speed is less than the speed of sound in free space. It is also found that $\mathrm{Im}\sqrt{k^2+\gamma^2}$ has a close relationship to the quantity $(\alpha\,\beta/k)$ so that a large imaginary part of γ and low frequency of the source can make the surface wave decay very quickly.

Expanding each expression in (38) around the saddle point T=0 and integrating each term, we get

$$L = \frac{1}{\gamma}\sqrt{\frac{2k}{i\pi x}}\, e^{ikx}\left\{ (1 - \gamma y) + \frac{ik}{\gamma^2 x}\left(1 - \gamma y + \frac{\gamma^2 y^2}{2} - \frac{\gamma^3 y^3}{6}\right) + O(x^{-2}) \right\} \tag{40}$$

This asymptotic expansion is not uniformly valid. The conditions for its validity are

$$|\gamma|\sqrt{\frac{x}{k}} \gg 1 \tag{41}$$

and

$$k\left|\frac{y^2}{2} - \frac{\gamma y^3}{6}\right|\frac{1}{x} \ll 1 \tag{42}$$

Extremely small $|\gamma|$ will not satisfy conditions (41) and (42), so another asymptotic method has to be developed.

The total solution under the condition of large x can be obtained by substituting (39) into (4) yielding

$$\phi = \begin{cases} P - L & \gamma \in \Gamma \\ -L & \gamma \notin \Gamma \end{cases} \tag{43}$$

where P and L are given in (33) and (40). If we neglect the surface wave, we can get an explicit equation for the wave above ground in the far field as

$$\phi = \frac{1}{|\gamma|}\sqrt{\frac{2k}{\pi x}\left((1-\alpha y)^2 + (\beta y)^2\right)}\; e^{i(kx + \frac{3\pi}{4} + \theta)} \tag{44}$$

where $\theta = -\arctan(\beta y / (1 - \alpha y))$. Furthermore, if the receiver is on the ground, the above expression can be written as

$$20 \operatorname{Log} \phi = 20 \operatorname{Log} a - 10 \operatorname{Log} x \tag{45}$$

with $a = (1/|\gamma|)(2k/\pi)^{1/2}$. This result shows that the acoustic pressure level drops 10dB when the distance increases 10 times or 3 dB per doubling of distance.

2. Hard boundary case

As mentioned before, the asymptotic expansion given in (40) is not uniformly valid in γ, with the method failing for small $|\gamma|$. An alternative method is developed in this section which is valid in the small $|\gamma|$ case. The method is almost the same as that used in evaluating L except the factor $1/(\gamma^2 + T^2)$ in (38) will not be expanded. After changing variables (38) becomes

$$L = \frac{\gamma(1 - \gamma y)}{\pi i}\sqrt{\frac{2x}{ik}}\, e^{ikx}\int_0^{\infty}\left(\frac{x\gamma^2}{2ik} - t^2\right)^{-1} e^{-t^2}\, dt \tag{46}$$

Making use of the formula

$$\int_0^{\infty} (z^2 - t^2)^{-1} e^{-t^2} dt = \begin{cases} \dfrac{\pi}{2iz} e^{-z^2} \operatorname{erfc}(-iz) & \operatorname{Im}(z) > 0 \\ \dfrac{\pi}{2iz} e^{-z^2} (\operatorname{erfc}(-iz) - 2) & \operatorname{Im}(z) < 0 \end{cases} \tag{47}$$

and neglecting the terms of order γ^2, yields

$$L = \begin{cases} -e^{ikx} [\, 1 + \gamma(\dfrac{2}{\sqrt{\pi}} \sqrt{\dfrac{ix}{2k}} - y)\,] & \operatorname{Im}(\dfrac{\gamma}{\sqrt{i}}) > 0 \\ -e^{ikx} [\, 1 + \gamma(\dfrac{2}{\sqrt{\pi}} \sqrt{\dfrac{ix}{2k}} - y) - 2(1 - \gamma y)\,] & \operatorname{Im}(\dfrac{\gamma}{\sqrt{i}}) < 0 \end{cases} \tag{48}$$

The conditions $\operatorname{Im}(\gamma / \sqrt{i}) > 0$ and $\operatorname{Im}(\gamma / \sqrt{i}) < 0$ can be identified as $\alpha < \beta$ and $\alpha > \beta$ respectively. $\alpha = \beta$ is the line which divides these two regions in γ plane. This is exactly the bounding curve $\alpha = \beta (1 + \beta^2/k^2)^{-1/2}$ obtained previously provided that $|\gamma| \to 0$. Recognizing this relation,we substitute (48) into (43) and rewriting surface wave approximately as $P = 2(1 - \gamma y) e^{ikx}$, finally get the total field expression as

$$\phi = e^{ikx} \left\{ 1 + \gamma \left[\frac{2}{\sqrt{\pi}} \sqrt{\frac{ix}{2k}} - y \right] \right\} \tag{49}$$

the condition for the validity of the above expansion is

$$|\gamma|\, y << 1 \tag{50}$$

and

$$|\gamma| \sqrt{\frac{x}{k}} << 1 \tag{51}$$

although x can't be small because of the nature of the saddle point method.

ANOTHER ASYMPTOTIC EXPANSION VALID FOR LARGE RECEIVER HEIGHT

The asymptotic expansions obtained above have their limitations in application. For example, they require the receiver's location to be near the ground. In this section we will derive a asymptotic expansion which is valid for large $R = \sqrt{x^2+y^2}$ (except for small y) . The idea is similar to that of Chien and Soroka [7].

By using the identity $\sin(\lambda x) = (e^{i\lambda x} - e^{-i\lambda x})/2i$ and transformation $\lambda = k \sin(z)$, (15) becomes

$$f(x,y) = \frac{\gamma}{i\pi} \int_C \frac{\operatorname{Tan}(z)}{\gamma + ik \operatorname{Cos}(z)} e^{ik(y \operatorname{Cos}(z) + x \operatorname{Sin}(z))} dz \tag{52}$$

The contour C is shown in Figure 3. In order to get an expansion in terms of the variable $R = \sqrt{x^2+y^2}$, we transform the Cartesian coordinate system into the polar coordinate system by

$$x = R \operatorname{Sin} \theta \tag{53}$$

and

$$y = R \operatorname{Cos} \theta \tag{54}$$

Substituting into (52), yields

$$f(x,y) = \frac{\gamma}{i\pi} \int_C \frac{\operatorname{Tan}(z)}{\gamma + i \operatorname{Cos}(z)} e^{i k R \operatorname{Cos}(z-\theta)} dz \tag{55}$$

the saddle point for the function i k R Cos(z-θ) is at z = θ and the path of steepest descent is found to be given by

$$\operatorname{Cos}(u - \theta) \operatorname{Cosh}(v) = 1 \tag{56}$$

where z = u + i v, by considering Im (i k R Cos (z - θ))=Im (i k R). This path, denoted as C' is shown in Figure (3). Deforming contour C into C', adding the possible poles (Cos z = i γ / k), we have

$$f(x,y) = Q + H\left(-\operatorname{Re}\left(1 - \frac{i\gamma}{k} \operatorname{Cos} \theta - \sqrt{1 + \frac{\gamma^2}{k^2}} \operatorname{Sin} \theta\right)\right) P \tag{57}$$

where Q is defined by (54) but with the contour C changed to C', H is Heavyside step function and P is the surface wave given in (33). The condition for the existence of pole is explained in reference [7], and will not be repeated here. In the limit of θ approaching π/2 the condition for the existence of the pole in the present case is equivalent to the condition for the existence of the pole in (39).

Q can be evaluated asymptotically with a method similar to that used in evaluating L, i.e. to expand each term around the saddle point θ and then integrate them with suitable transformation of the variable . The result is

$$Q = \sqrt{\frac{2}{i\pi k R}} \frac{\gamma \operatorname{Tan} \theta}{i(\gamma + i \operatorname{Cos} \theta)} e^{ikR} \left\{ 1 + \frac{1}{ikR} \left[1 + \frac{i k \operatorname{Cos} \theta}{2(\gamma + i k \operatorname{Cos} \theta)} + \frac{i k}{(\gamma + i k \operatorname{Cos} \theta) \operatorname{Cos} \theta} \right] \right\} \tag{58}$$

The conditions for the validity of the above expansion are

$$kR \gg 1 \tag{59}$$

and

$$\frac{\frac{\operatorname{Cos}^2 \theta}{2} + 1}{\left|\gamma + i \operatorname{Cos} \theta\right| R \operatorname{Cos} \theta} \ll 1 \tag{60}$$

It is clear that θ cannot be too close to $\pi/2$. This limitation is complimentary to the asymptotic expansions obtained previously (for small y). Substituting into (4) yields

$$\phi = e^{ikx} + Q + H\left(- \mathrm{Re}\left(1 - \frac{i\gamma}{k} \cos\theta - \sqrt{1 + \frac{\gamma^2}{k^2}} \sin\theta \right)\right) P \tag{61}$$

In the limit $R \to \infty$, Q and P will vanish , with the result that only the plane wave term remains.

Equation (38) can be evaluated accurately by numerical methods as well as by asymptotic expansions. Calculations show that the results match quite well when y is small. Figure 4 gives the amplitude of acoustic pressure on the ground versus the distance to the receiver obtained by numerical integration and from (44). Figures 5a, b and c show the amplitude of acoustic pressure versus the receiver height for several receiver locations as obtained from the asymptotic expansions, (40) and (61). These figures are similar to the results obtained by McAninch [4].

CONCLUSIONS

The acoustic field of a plane wave at grazing incident to a finite impedance has been theoretically investigated. Exact numerical and asymptotic expansions are developed, which are very similar to those found by Wenzel [6] for a point source and by McAninch [4] using the parabolic approximation to the wave equation. When y is small, the incident wave is indeed canceled, but the result is not zero due to the existence of a surface wave and the wave denoted as L. Near the ground, the acoustic pressure decays as $x^{-1/2}$ (assuming the surface wave is neglected). The asymptotic expansion for large distance R shows that the acoustic pressure decays as $R^{-1/2}$ when $R \to \infty$ and when the receiver is not close to the surface only incident wave exists.

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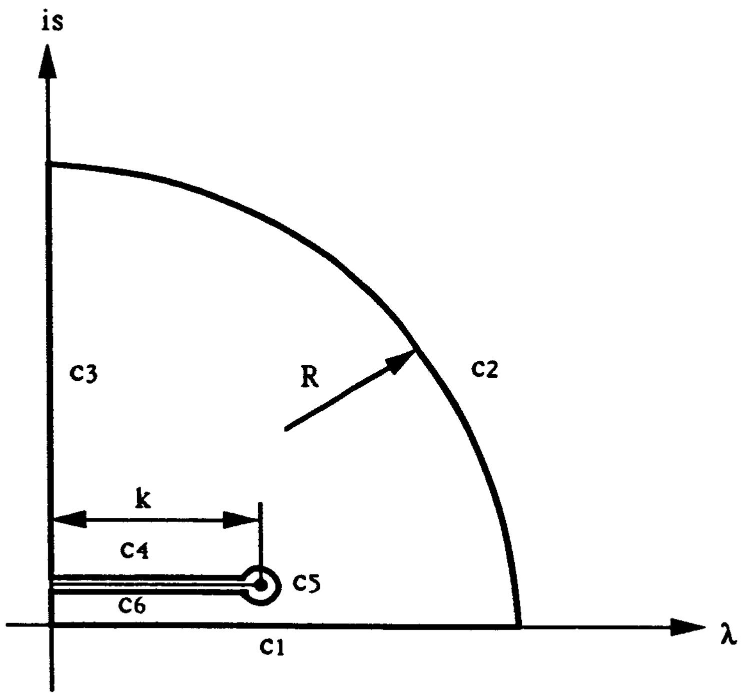

Figure 1. Integration path for the integral I_1 in the complex Λ plane.

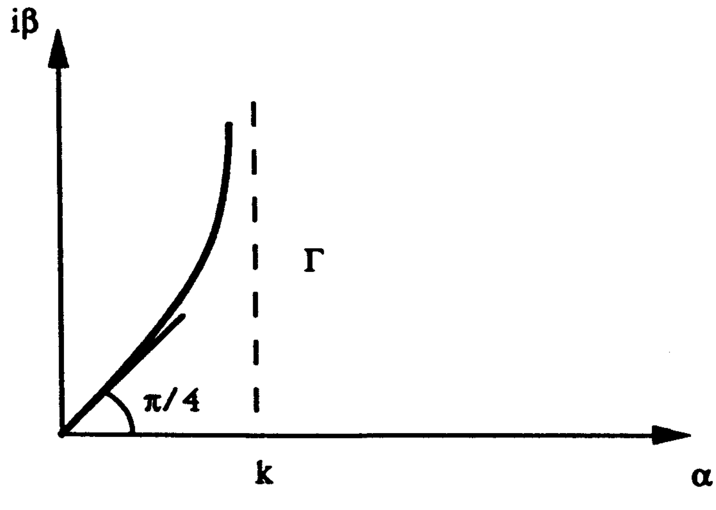

Figure 2. The region Γ in the γ plane.

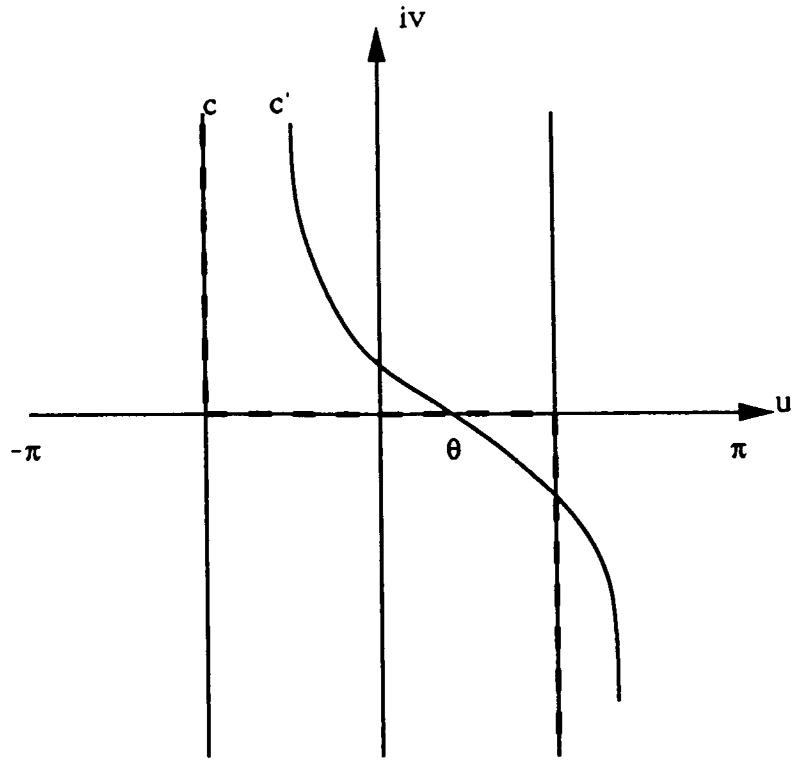

Figure 3. The u-v plane, showing the integration contour C and the steepest descent path C'.

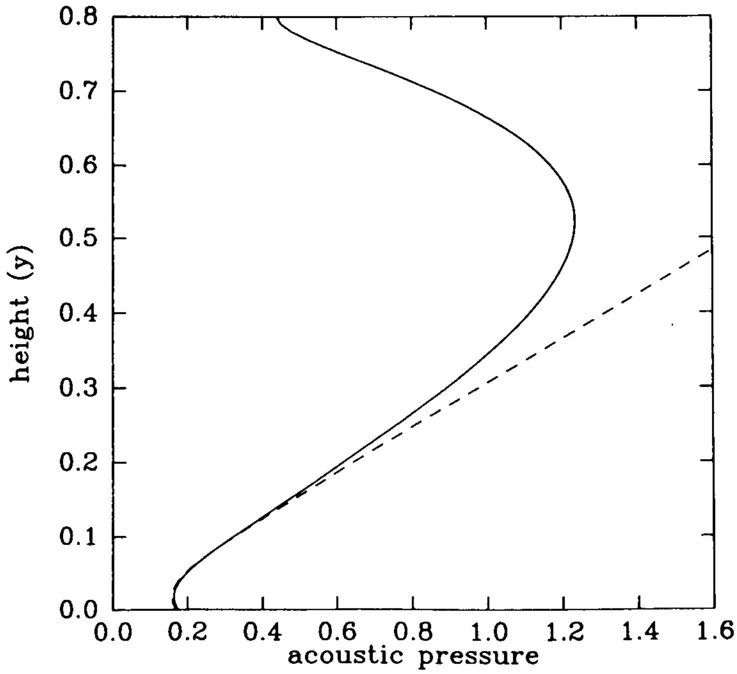

Figure 4. Comparison between a numerical result and the asymptotic result, (44), for the parameters, $k = 18.313$ and $\gamma = 6.935 + i\ 19.015$.

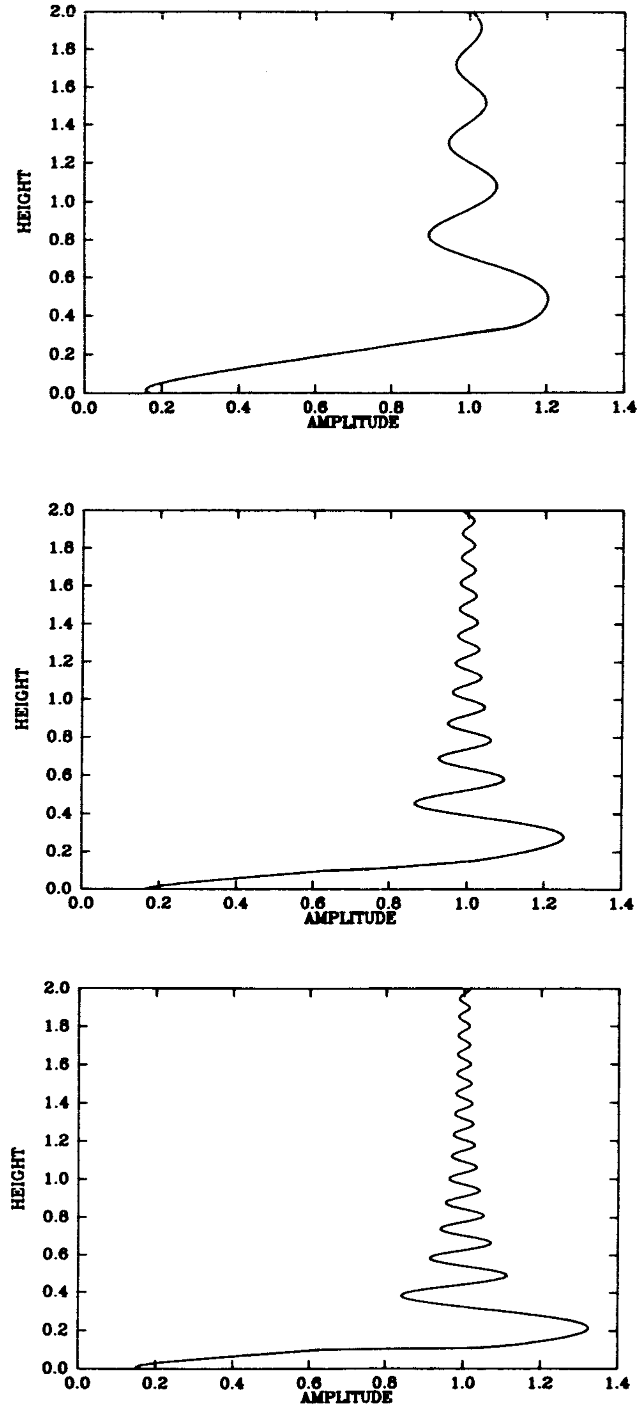

Figure 5. Amplitude of the acoustic pressure versus the receiver height for different parameters:
(a) f = 1000 Hz; γ = 6.935 + i 19.015.
(b) f = 2000 Hz; γ = -5.031 + i 35.932.
(c) f = 4000 Hz; γ = 18.038 + i 37.002.