System Characterization of Positive Real Conditions

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Cooperative Agreement NCC 9-16
Research Activity No. MS.4

NASA Johnson Space Center
Engineering Directorate
Navigation, Control, & Aeronautics Division

Research Institute for Computing and Information Systems
University of Houston - Clear Lake

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SYSTEM CHARACTERIZATION OF POSITIVE REAL CONDITIONS

Preface

This research was conducted under the auspices of the Research Institute for Computing and Information Systems by Q. Wang, J.L. Speyer, and H. Weiss of the University of California, Los Angeles. Dr. A. Glen Houston, Director of RICIS, served as RICIS research representative.

Funding has been provided by Navigation Control & Aeronautics Division, Engineering Directorate, NASA/JSC through Cooperative Agreement NCC 9-16 between NASA Johnson Space Center and the University of Houston-Clear Lake. The NASA technical monitor for this activity was David Geller, of the Navigation Section, Navigation and Gridance Systems Branch, Navigation Control & Aeronautics Division, Engineering Directorate, NASA/JSC.

System Characterization of Positive Real Conditions[1]

Q. Wang[2], J. L. Speyer[3], and H. Weiss[4]

Abstract

Necessary and sufficient conditions for positive realness in terms of state space matrices are presented under the assumption of complete controllability and complete observability of square systems with independent inputs. As an alternative to the positive real lemma and to the s-domain inequalities, these conditions provide a recursive algorithm for testing positive realness which result in a set of simple algebraic conditions. By relating the positive real property to the associated variational problem, the paper outlines a unified derivation of necessary and sufficient conditions for optimality of both singular and nonsingular problems.

1. Introduction

Positive real systems play a major role in control theory, especially in adaptive control, and in stability analysis. The impressive development of adaptive control and self-turning regulation over the last two decades [1,2] is hinged on satisfaction of some positive realness conditions. Alternatively, considerable initial knowledge about the controlled plant must be given. The prior knowledge is used to implement reference models, identifiers, or observer-based controllers of about the same order as the plant. Since the prior assumptions about the controlled plant may never be entirely satisfied, the stability properties of the related adaptive schemes are debatable. Therefore, a direct adaptive control procedure which does not use identifier or observer-based controllers in the feedback loop is preferred. The implementation of such an algorithm requires positive real controlled plants or alternatively, a synthesis of a positive real plant on the basis of the actual plant.

The existing tools for analysis and synthesis of positive real systems are based in the s-domain on complex variable inequalities which are inconvenient or in the state space requiring the positive real lemma equations. These tools are computationally complex and there is a need for an easily used complementary tool. In Sections 2 and 3, necessary and sufficient conditions for positive real systems with independent inputs are developed using optimal control theory for the associated partially singular problem. It is shown that in the totally singular case, these conditions are consistent with the generalized Legendre-Clebsch condition [3,4]. The new conditions are associated with the state space matrices of a minimal realization of a square system. The resulting test for positive realness reduces to recursively testing certain square matrices for positive definiteness and the solution to an algebraic Riccati equation. As an immediate result of the new necessary and sufficient conditions, we also show that the zeros of a positive real system lie in the closed left half complex plane. Some examples are given in Section 4 to illustrate the theory. Concluding marks are given in Section 5.

The derivation of the above results is related to dissipative systems. Basic definitions and physical characteristics are presented below.

1.1 Dissipative System

Consider the system input-output description $H: U \rightarrow Y$ where $U = L_2^l(R_+)$ and $Y = L_2^m(R_+)$. The notation $L_2^l(R_+)$ is used to denote the space of square integrable functions $f: R_+ \rightarrow R^l$ where $R_+ = [t_0, \infty)$. The supply rate associated with this system is defined as a function $w: R^l \times R^m \rightarrow R$ where

$$w(u,y) = y'Qy + 2y'Su + u'Ru \qquad (1.1)$$

and $Q \in R^{m \times m}$, $S \in R^{m \times l}$, $R \in R^{l \times l}$ are constant matrices, with Q and R symmetric.

Definition 1.1 [5]: A dynamical system H is <u>dissipative</u> with respect to the supply rate w(u,y) if and only if

$$\int_{t_0}^{t_1} w[u(t), y(t)]\, dt \geq 0 \qquad (1.2)$$

for all $t_1 \geq t_0$ and all $u \in L_2^l$, whenever the initial state satisfies $x(t_0) = 0$.

1. This work was partially sponsored by Eglin AFB under Contract F08635-87-K0417 and by NASA Johnson Space Center through the RICIS Program of the University of Houston at Clear Lake.
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Remark 1.1: Passivity corresponds to dissipativeness where $Q = R = 0, l = m, S = \frac{1}{2} I_m$ and I_m is m×m identity matrix.

Remark 1.2: Positive realness corresponds to passivity where the dynamical system is linear and time invariant.

Remark 1.3: The concept of a supply rate is related in the general case to the "stored energy" for the system. As an example, suppose that the system under consideration is an electrical network, whose elements are constants, and y(t) the vector of corresponding port voltage. Then the system is dissipative with respect to the supply rate $w(u,y) = u'y$ provided that all the resistances, inductances and capacitance are non-negative.

1.2 Energy, Power and Information Relationships in Dissipative Systems

The class of dissipative systems which has a finite dimensional internal state is completely described in terms of energy storage and power dissipation. Considering this class, the various facets of the standard state space model can be associated with the concepts of energy, power and information.

Assume that the system under consideration is described by a linear, time-invariant system

$$\dot{x} = Ax + Bu \qquad (1.3)$$

$$y = Cx + Du \qquad (1.4)$$

where $x \in R^n$, $u \in R^l$, $y \in R^m$ and A, B, C and D are constant matrices with appropriate dimension. Then, following [6], the system matrices can be regarded as representing:

1. an energy-transformation and dissipation map, associated with the matrix A.

2. a power injection map, associated with the matrices B and D.

3. an information-extraction map, associated with the matrix C.

Figure 1 describes the energy-power-information maps associated with the system matrices.

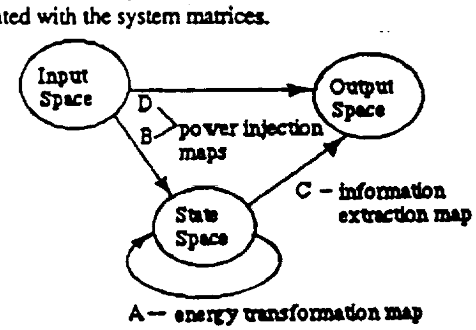

Fig. 1 Energy-Power-Information maps associated with the System Matrices

The matrix B represents the input coupling between the information represented by the applied input signals and the power available for injection into the system states. The matrix C represents the output coupling between the energy in the system states and the information in the available output signals. The matrix D represents the output coupling between the information represented by the applied input signals and the injected power into the available output signals.

1.3 Review of the Positive Real Property

The positive real property is related directly to the transfer function matrix description of the system. The positive real lemma, presented in Section 2, connects the positive realness to the parameters of a system realization with complete controllability and complete observability.

The Positive Real Property [7]: Let G(s) be an m × m matrix of functions of a complex variable s. then G(s) is termed positive real if the following conditions are satisfied:

(i) All the elements of G(s) are analytic in Re[s] > 0.

(ii) G(s) is real for real positive s.

(iii) $G^*(s) + G(s) \geq 0$ for Re [s] > 0 .

where $(\cdot)^*$ denotes complex conjugate transpose.

Remark 1.4: If G(s) is a real rational matrix of functions of s, then necessary and sufficient conditions for the positive real property to hold are given by the following theorem.

Theorem 1.1 [7]: Let G(s) be a real rational matrix of functions of s. Then, G(s) is positive real if and only if:

(i) No element of G(s) has a pole in Re[s] > 0.

(ii) $G^*(j\omega) + G(j\omega) \geq 0$ for all real ω, with jω not a pole of any element of G(s).

(iii) If $j\omega_0$ is a pole of any element of G(s), it is at most a simple pole, and the residue matrix,

$$k_0 = \lim_{s \to j\omega_0} (s - j\omega_0)\, G(s) \qquad \text{if } j\omega_0 \text{ is finite},$$

$$k_\infty = \lim_{s \to j\omega_0} G(s)/s \qquad \text{if } j\omega_0 \text{ is infinite},$$

is nonnegative definite Hermitian.

Following Definition 1.1, if the system is positive real, the angle between the output vector y(t) and the input vector u(t) is bounded below by - 90 deg. and above by + 90 deg.

2. Relations Between Optimal Control and Positive Realness

2.1 The Related Variational Problem

Consider the cost functional

$$V[x_0,t_0,u(\cdot)] = \int_{t_0}^{t_1} w[u(t), y(t)]\, dt \qquad (2.1)$$

where the supply rate

$$w(u,y) = y'u = u'D'u + x'C'u \qquad (2.2)$$

is associated with system (1.3) and (1.4), where the dimensions of u and y are m. The problem is to find necessary and sufficient conditions for optimality of $u^*(\cdot) \in U$ to minimize $V[x_0,t_0,u(\cdot)]$, denoted $V^*[x_0,t_0]$, subject to the dynamic equation of (1.3) where $x(t_0) = x_0$ is prescribed.

Remark 2.1: Since only the symmetric part of D contributes to w(u,y), then

$$w(u,y) = \frac{1}{2} \left(u'Ru + 2x'C'u \right) \tag{2.3}$$

where

$$R = D + D' \tag{2.4}$$

Remark 2.2: If $R \geq 0$, and rank $(R) = r < m$, there exists an orthogonal transformation $\Gamma = [\Gamma_1, \Gamma_2]$ such that

$$\begin{bmatrix} \Gamma_1' \\ \Gamma_2' \end{bmatrix} R [\Gamma_1, \Gamma_2] = \begin{bmatrix} R_r & 0 \\ 0 & 0 \end{bmatrix} \tag{2.5}$$

where R_r is positive. For instance, Γ_1 and Γ_2 may consist of normalized eigenvectors of R associated with nonzero and zero eigenvalues, respectively [8]. There is a natural partitioning of the control vector associated with this transformation, a r-dimensional nonsingular control and an (m-r)-dimensional singular control.

2.2 Positive Real Lemma Equations

Necessary and sufficient condition for $V^*[x_0,t_0]$ to be bounded below over a finite time interval $[t_0, t_1]$ are presented in Theorem II.3.3 of [9]. The required positive real conditions are obtained via the extension of the optimality condition to the time-invariant, infinite-time case [10].

Under the complete controllability and complete observability assumption of system (1.3), necessary and sufficient conditions for the nonnegativity of $V[0, t_0, u(\cdot)]$ are that there exist $\pi < 0$, L, and W such that

$$\begin{bmatrix} \pi A + A'\pi & \pi B + C' \\ B'\pi + C & R \end{bmatrix} = \begin{bmatrix} L' \\ W' \end{bmatrix} [L, W] \geq 0, \tag{2.6}$$

where W and L are matrices with proper dimension.

By identifying $P = -\pi$, the positive real Lemma is stated.

The Positive Real Lemma [7]: Let G(s) be an m×m matrix of real rational functions of a complex variable s, with $G(\infty) < \infty$. Let {A, B, C, D} be a minimal realization of G(s). Then, G(s) is positive real if and only if there exist real matrices P, L, and W with P positive definite and symmetric, such that:

$$PA + A'P = -L'L \tag{2.7}$$

$$B'P = C - W'L \tag{2.8}$$

$$W'W = D + D' \tag{2.9}$$

Remark 2.2: The generalized Legendre-Clebsch condition, which is a necessary condition for $V^*[x_0,t_0] > -\infty$ in the totally singular case, given in [3] for a linear time-invariant system can be written as

$$\frac{\partial}{\partial u}(\dot{H}_u) = CB - (CB)' = 0 \tag{2.10}$$

$$\frac{\partial}{\partial u}(\ddot{H}_u) = CAB + (CAB)' \leq 0 \tag{2.11}$$

where H is the variational Hamiltonian and $\lambda \in R^n$ is the associated Lagrange multiplier

$$H = u'Cx + \lambda'(Ax + Bu), \quad \dot{\lambda}' = -H_x.$$

By letting R = 0, the necessary conditions (2.10) and (2.11) are also obtained from the positive real lemma.

3. Positive Real Conditions in Terms of State-Space Matrices

Necessary and sufficient conditions for the nonnegativity of $V[0, t_0, u(\cdot)]$ are given by the existence of $\pi < 0$, L, and W which satisfy (2.6). Let G(s) be an m×m matrix of degree n. Consider a minimal realization {A, B, C, D} representing the finite-dimensional linear time-invariant dynamic equations given by (1.3) and (1.4). In terms of state space matrices A, B, C, and D, (2.6) gives necessary and sufficient conditions for a positive real system. In this section, new necessary and sufficient conditions are developed.

3.1 Standard Formulation of the Partially Singular Problem

Assume that G(s) is a square matrix of proper rational function with independent columns. For any realization, the matrices C and B are full rank. Without loss of generality, we consider a minimal realization {A, B, C, D} of the form that A is an n×n matrix which is partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is a k×k matrix, and A_{22} is an (n-k)×(n-k) matrix, where k is the dimension of the singular control,

$$B = [B_r, B_s], \quad C = \begin{bmatrix} C_r \\ C_s \end{bmatrix},$$

and

$$D + D' = \begin{bmatrix} R_r & 0 \\ 0 & 0 \end{bmatrix}$$

where R_r is a r×r nonsingular matrix corresponding to the nonsingular control, B_r is an n×r matrix, B_s is an n×k matrix related to the singular control, C_r is a r×n matrix, and C_s is a k×n matrix, where r = m-k is the dimension of the nonsingular control. If n > k, then C_s has the following form

$$C_s = [C_{s1}, 0],$$

where C_{s1} is a nonsingular matrix. Correspondingly, B_s is written as $\begin{bmatrix} B_{s1} \\ B_{s2} \end{bmatrix}$. We define this as a standard realization.

Notice that the realization can be obtained by choosing suitable bases for the state space and the input/output space. For example, suppose $\{\bar{A}, \bar{B}, \bar{C}, G(\infty)\}$ is a minimal realization of G(s). Let the column vectors of Γ, where Γ is described in Remark 2.2, be a basis of the input/output space, then the following transformation $y = \Gamma\eta$, $u = \Gamma v$ is defined. Furthermore, let $q_1, q_2, \ldots, q_{n-k}, q_{n-k+1}, \ldots, q_n$ be a basis of the state space, where $q_{n-k+1}, \ldots, q_n$ span the null space of $\Gamma_2'\bar{C}$, and $q_1, q_2, \ldots, q_{n-k}$ are arbitrary vectors such that $Q = [q_1, q_2, \ldots, q_{n-k}, q_{n-k+1}, \ldots, q_n]$ is nonsingular. This defines a transformation $x = Q\xi$. The resulting dynamic equation can be written as

$$\dot{\xi} = A\xi + Bv \tag{3.1}$$

$$\eta = C\xi + Dv, \tag{3.2}$$

where $A = Q^{-1}\bar{A}Q$, $B = Q^{-1}\bar{B}\Gamma$, $C = \Gamma'\bar{C}Q$, and $D = \Gamma'G(\infty)\Gamma$. The transfer function matrix of this system is $\Gamma'G(s)\Gamma$, the positive realness of $G(s)$ is equivalent to the positive realness of $\Gamma G(s)\Gamma$. The application of (2.6) and development of the new necessary and sufficient conditions for the partially singular problem will be discussed under assumption of a standard realization as discussed.

3.2 Derivation of New Necessary and Sufficient Conditions

Necessary and sufficient condition for nonnegative of $V[0,t_0,u(\cdot)]$ as given by condition (2.6) can be restated in the following equivalent forms: There exist a $\pi < 0$ and a matrix V such that

$$\begin{bmatrix} \pi A + A'\pi & \pi B + C' \\ B'\pi + C & R \end{bmatrix} = V'V. \tag{3.3}$$

Furthermore, R being positive semi-definite is a necessary condition for satisfying (3.3). If $R > 0$, then (3.3) can be reduced to a condition based upon a Reccati equation. That is, there exists a negative definite solution π to the algebraic Riccati equation

$$\pi(A - BR^{-1}C) + (A - BR^{-1}C)'\pi - \pi BR^{-1}B'\pi - C'R^{-1}C = 0. \tag{3.4}$$

If R is singular, (3.3) can be written as

$$\begin{bmatrix} \pi A + A'\pi & \pi B_r + C_r' & \pi B_s + C_s' \\ B_r'\pi + C_r & R_r & 0 \\ B_s'\pi + C_s & 0 & 0 \end{bmatrix} = V'V$$

or, equivalently, there exist a $\pi < 0$ and a matrix V_r such that

$$\pi B_s + C_s' = 0 \tag{3.5}$$

and

$$\begin{bmatrix} \pi A + A'\pi & \pi B_r + C_r' \\ B_r'\pi + C_r & R_r \end{bmatrix} = V_r'V_r. \tag{3.6}$$

If the dimension of the state is less than or equal to the dimension of the singular control, i.e., $n \le k$, π can be determined from equation (3.5). If and only if a $\pi < 0$ is solvable from (3.5) and the same π satisfies (3.6), the system is positive real. If $n > k$, the fact $\pi < 0$ and equation (3.5) imply that

$$C_sB_s = (C_sB_s)' = -B_s'\pi B_s > 0. \tag{3.7}$$

Since $C_s = [\,C_{s1},\ 0\,]$, and C_{s1} is nonsingular. Equation (3.7) also implies that B_{s1} is nonsingular. Furthermore, (3.5) provides a linear constraint on π which is discussed in Lemma 3.1 below.

Lemma 3.1: $\pi < 0$, $\pi B_s + C_s' = 0$ if and only if $C_sB_s > 0$ and

$$\pi = \begin{bmatrix} -(B_{s1}')^{-1}C_{s1} + (B_{s1}')^{-1}B_{s2}'\pi_1 B_{s2}(B_{s1})^{-1} & -(B_{s1})^{-1}B_{s2}'\pi_1 \\ -\pi_1 B_{s2}(B_{s1})^{-1} & \pi_1 \end{bmatrix} \tag{3.8}$$

for some $\pi_1 < 0$.

Proof: Denote π as $\pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{12}' & \pi_1 \end{bmatrix}$. To prove sufficiency, we assume that $\pi_1 < 0$, $C_sB_s > 0$, and

$$\pi_{11} = -(B_{s1}')^{-1}C_{s1} + (B_{s1}')^{-1}B_{s2}'\pi_1 B_{s2}(B_{s1})^{-1} \tag{3.9}$$

$$\pi_{12} = -(B_{s1}')^{-1}B_{s2}'\pi_1. \tag{3.10}$$

Define $F = \begin{bmatrix} I & -\pi_{12}(\pi_1)^{-1} \\ 0 & I \end{bmatrix}$, then F is nonsingular and

$$F\pi F = \begin{bmatrix} I & -\pi_{12}(\pi_1)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{12}' & \pi_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ -(\pi_1)^{-1}\pi_{12}' & I \end{bmatrix}$$
$$= \begin{bmatrix} \pi_1 - \pi_{12}(\pi_1)^{-1}\pi_{12}' & 0 \\ 0 & \pi_1 \end{bmatrix} = \begin{bmatrix} -(B_{s1}')^{-1}C_{s1} & 0 \\ 0 & \pi_1 \end{bmatrix}.$$

Since $C_sB_s = C_{s1}B_{s1} > 0$,

$$-(B_{s1}')^{-1}C_{s1} = -(B_{s1}')^{-1}C_{s1}B_s(B_{s1})^{-1} < 0.$$

Therefore $F\pi F' < 0$, and it also implies that $\pi < 0$. Furthermore, by using π_{11} and π_{12} defined in (3.9) and (3.10), we get

$$\pi B_s + C_s' = \begin{bmatrix} \pi_{11}B_{s1} + \pi_{12}B_{s2} \\ \pi_{12}'B_{s1} + \pi_1 B_{s2} \end{bmatrix} + \begin{bmatrix} C_{s1}' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.11}$$

Next, we prove the necessity. If $\pi < 0$, then $\pi_1 < 0$. From $\pi B_s + C_s' = 0$ we get

$$\pi_{11}B_{s1} + \pi_{12}B_{s2} + C_{s1}' = 0 \tag{3.12}$$

$$\pi_{12}'B_{s1} + \pi_1 B_{s2} = 0 \tag{3.13}$$

By solving (3.12) and (3.13), the expressions of π_{11} and π_{12} are obtained which are the same as shown in equations (3.9) and (3.10).

Q.E.D.

Let the matrix shown in (3.6) be denoted as $M(\pi, R_r)$

$$M(\pi, R_r) = \begin{bmatrix} \pi A + A'\pi & \pi B_r + C_r' \\ B_r'\pi + C_r & R_r \end{bmatrix}.$$

For any nonsingular matrix T, (3.6) is equivalent to $T'M(\pi, R_r)T = V_T' V_T$, where V_T is a matrix with proper dimension. By defining

$$T_r = \begin{bmatrix} 0 & B_{s1} & 0 \\ I & B_{s2} & 0 \\ 0 & 0 & I \end{bmatrix}$$

and using π defined in (3.8) as a function of π_1, then T_r is nonsingular, and

$$T_r'M(\pi, R_r)T_r = \begin{bmatrix} M_{11} & M_{12} & M_{12} \\ M_{12}' & M_{22} & M_{23} \\ M_{13}' & M_{23}' & M_{33} \end{bmatrix}$$

where

$$M_{11} = [\,0\ I\ 0\,]\,M(\pi,R)\begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}$$
$$= \pi_1(A_{22} - B_{s2}(B_{s1})^{-1}A_{12}) + (A_{22} - B_{s2}(B_{s1})^{-1}A_{12})'\pi_1$$

$$M_{12} = [\,0\ I\ 0\,]\,M(\pi,R)\begin{bmatrix} B_{s1} \\ B_{s2} \\ 0 \end{bmatrix} = \pi_1(A_{21}B_{s1} + A_{22}B_{s2} -$$
$$B_{s2}(B_{s1})^{-1}A_{11}B_{s1} - B_{s2}(B_{s1})^{-1}A_{12}B_{s2}) - C_{s1}A_{12}$$

$$M_{22} = [\,B_{s1}',\ B_{s2}',\ 0\,]\,M(\pi,R)\begin{bmatrix} B_{s1} \\ B_{s2} \\ 0 \end{bmatrix}$$
$$= -(C_sAB_s + B_s'A'C_s')$$

$$M_{23} = [\,B_{s1}',\ B_{s2}',\ 0\,]\,M(\pi,R)\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$$
$$= -C_sB_r + B_s'C_r$$

$$M_{33} = [\,0,\ 0,\ I\,]\,M(\pi,R)\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} = R_r.$$

By defining

$$A_1 = A_{22} - B_{s2}(B_{s1})^{-1}A_{12} \quad (3.14)$$

$$B_1 = [\ A_{21}B_{s1} + A_{22}B_{s2} - B_{s2}(B_{s1})^{-1}A_{11}B_{s1} - B_{s2}(B_{s1})^{-1}A_{12}B_{s2},\ \ B_{r2}\ ] \quad (3.15)$$

$$C_1 = [\ -C_{s1}A_{12},\ 0] \quad (3.16)$$

$$R_1 = \begin{bmatrix} -(C_sAB_s + B_s'A'C_s') & -C_sB_r + B_s'C_r' \\ -B_r'C_s' + C_rB_s & R_r \end{bmatrix}, \quad (3.17)$$

a condition which is equivalent to (3.6) can be stated as the follows: There exist a $\pi_1 < 0$ and a matrix V_1 such that

$$\begin{bmatrix} \pi_1A_1 + A_1'\pi_1 & \pi_1B_1 + C_1' \\ B_1'\pi_1 + C_1 & R_1 \end{bmatrix} = V_1'V_1. \quad (3.18)$$

According to the positive real lemma, Equation (3.18) implies that $\{\ A_1, B_1, C_1, \frac{R_1}{2}\ \}$ is positive real.

3.3 Necessary and Sufficient Conditions for Positive Realness

The results in Section 3.2 are summarized in the next theorem as an alternative necessary and sufficient condition for testing positive realness of a square system.

Theorem 3.1: The necessary and sufficient condition for $\{A, B, C, D\}$ to be positive real is that

(i) $R \geq 0$;

(ii) If $R > 0$, there exists a positive definite solution P to the following algebraic Riccati equation

$$P(A - BR^{-1}C) + (A - BR^{-1}C)'P + PBR^{-1}B'P + C'R^{-1}C = 0;$$

(iii) If rank $R = r < m$, and $n \leq m-r$, there exists $P = C_sB_s'(B_sB_s')^{-1} > 0$ satisfying $PB_s = C_s'$ and

$$\begin{bmatrix} -PA - A'P & -PB_r + C_r' \\ -B_r'P + C_r & R_r \end{bmatrix} \geq 0;$$

(iv) If rank $R = r < m$, and $n > m-r$, then $C_sB_s = (C_sB_s)' > 0$ and $\{\ A_1, B_1, C_1, \frac{R_1}{2}\ \}$ is positive real, where A_1, B_1, C_1, and R_1 are defined in equations (3.14) to (3.17).

Condition (ii) is obtained by identifying P with $-\pi$ in equation (3.4). Condition (iii) is the interpretation of (3.5) and (3.6) for the case $n \leq (m - r)$. If $P = -\pi > 0$ exists, then $PB_s = C_s'$, $PB_sB_s' = C_s'B_s$, and $P = C_sB_s'(B_sB_s')^{-1} > 0$. Condition (iv) corresponds to the situation we discussed through (3.7) to (3.13)

Remark 3.1: Alternative transformation approaches to the singular problem using the Kelley transformation for the linear quadratic problem are given in [9] for the matrix case. The approach here is different via the structure of π given by Lemma 3.1.

Remark 3.2: If $\{A, B, C, D\}$ is a minimal realization, then it is required for a positive real system that there exists a positive definite matrix P such that

$$PA + A'P \leq 0$$

Therefore, it is required that Re $\lambda_i\,[A] \leq 0$ and the Jordan form of A has no blocks of size greater than 1×1 with pure imaginary diagonal elements.

Remark 3.3: If G(s) is strictly proper, the minimal realization is totally singular, then the characteristic polynomial of $A_1 = A_{22} - B_{s2}(B_{s1})^{-1}A_{12}$ is equal to the zero polynomial of the system up to a nonzero scalar factor.

Proof: Let $\det G(s) = \det(\ C(Is - A)^{-1}B) = \frac{\psi(s)}{\Delta(s)}$, where

$$\Delta(s) = \det(\ Is - A\ )$$

and $\psi(s)$ is the zero polynomial of the system. Since state feedbacks do not change the numerator of the transfer function matrix, for any matrix K,

$$\det(G_k(s)) = \det(\ C(Is - A - BK\ )^{-1}B) = \frac{\psi(s)}{\Delta_k(s)}$$

where $\Delta_k(s) = \det(\ Is - A - BK\ )$

Let $K = [\ 0,\ (B_{s1})^{-1}A_{12}\ ]$, then

$$A + BK = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - B_{s2}(B_{s1})^{-1}A_{12} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_1 \end{bmatrix}$$

$$\Delta_k(s) = \det(\ Is - A - BK\ ) = \det \begin{bmatrix} Is-A_{11} & 0 \\ -A_{21} & Is-A_1 \end{bmatrix} = \det(\ Is-A_{11}) \det(Is-A_1)$$

$$\det(G_k(s)) = \det(\ C(Is - A - BK\ )^{-1}B) = \det(C_{s1}(Is-A_{11})^{-1}B_{s2}) = \frac{\det(C_{s1})\det(B_{s1})}{\det(Is-A_{11})}$$

Therefore,

$$\psi(s) = \Delta_k(s)\det(G_k(s)) = \det(C_{s1})\det(B_{s1}\ )\det(Is-A_1)$$

Q.E.D.

Remark 3.4: From (3.18) and Remark 3.3, we conclude that there are n - m finite zeros for a positive real system and all the zeros lie in the closed left half complex plane. In other words, the system is minimum phase.

4. Examples

Theorem 3.1 introduces a recursive procedure for testing positive real systems, requests only for testing a series of matrices $C_{is}B_{is} > 0$, for $i = 0, 1, 2, \ldots, l$, and the solution to a algebraic Riccati equation $P_l > 0$, where i is the index associated with the new system obtained from the i-th iteration, and i = 0 corresponds to B_s, C_s, and P. The testing stops when R_l becomes nonsingular, or the dimension of the state is less or equal to the dimension of the singular control.

The following examples illustrate the application of Theorem 3.1.

Example 4.1: Given $G(s) = \frac{(s+2)^2}{s(s+1)(s+3)}$, an observable realization of G(s) is

$$A = \begin{bmatrix} -4 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}, \quad C = [\ 1, 0, 0\ ], \quad D = 0.$$

First iteration:

$R = 0$

$CB = (CB)' = 1 > 0$

$$A_1 = \begin{bmatrix} -4 & 1 \\ -4 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = [\,-1, 0\,], \quad R_1 = 0.$$

Second iteration:

$R_1 = 0$

$C_1B_1 = (C_1B_1)' = -1 < 0$

Therefore, the system is not positive real.

Example 4.2: Given $G(s) = \dfrac{(s+1)^2}{s\,(s+2)(s+4)}$, an observable realization of G(s) is

$$A = \begin{bmatrix} -6 & 1 & 0 \\ -8 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad C = [\,1, 0, 0\,], \quad D = 0.$$

First iteration:

$R = 0$

$CB = (CB)' = 1 > 0$

$$A_1 = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad C_1 = [\,-1, 0\,], \quad R_1 = 8.$$

Second iteration:

$R_1 = 8 > 0$, the algebraic Riccati equation is

$$P_1 \begin{bmatrix} -15 & 8 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} -15 & -4 \\ 8 & 0 \end{bmatrix} P_1 + P_1 \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix} P_1 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

which has a positive definite solution

$$P_1 = \begin{bmatrix} 0.0394 & -0.0225 \\ -0.225 & 0.1557 \end{bmatrix} > 0$$

Therefore, the system is positive real.

Example 4.3: $G(s) = \dfrac{s^2 + z^2}{s\,(s^2 + p^2)}$. A minimal realization of G(s) is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -p^2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ z^2 \end{bmatrix}, \quad C = [\,1, 0, 0\,], \quad D = 0.$$

First iteration:

$R = 0$

$CB = (CB)' = 1 > 0$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -z^2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} z^2 - p^2 \\ 0 \end{bmatrix}, \quad C_1 = [\,-1, 0\,],$$

$R_1 = 0.$

Second iteration:

$R_1 = 0,$

$C_1B_1 = (C_1B_1)' = p^2 - z^2 > 0$ if and only if $p^2 > z^2$

$A_2 = 0, \quad B_2 = -z^2(z^2 - p^2), \quad C_2 = 1, \quad R_2 = 0.$

Third iteration:

$R_2 = 0.$

$P_2 = C_2B_2'(B_2B_2')^{-1} = \dfrac{1}{z^2(p^2 - z^2)} > 0$ if $p^2 > z^2$

$-P_2A_2 - A_2'P_2 = 0.$

Therefore, the system is positive real if and only if $p^2 > z^2$.

5. Summary and Conclusions

This paper reviews positive real system as a subclass of dissipative systems and states the positive real lemma equations. By using the variational problem associated with the partially singular problem, necessary and sufficient conditions for a system to be positive real are derived. These conditions are particularly transparent by using Lemma 3.1 which provides a uniquely structure for the matrix π. These positive realness conditions are expressed in terms of the state space matrix inequalities and algebraic Riccati equations and do not deal with inequalities in the s domain or with solutions of the positive real lemma equations. These tests are direct, and a system either satisfies these conditions or not. There is no requirement to search over all matrices to determine if a condition can be satisfied as in the positive real lemma. Examples are given which demonstrate the power of this approach.

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