The Gravitomagnetic Interaction and its Relationship to other Relativistic Gravitational Effects

A Theoretical Analysis by:

Kenneth Nordtvedt
Northwest Analysis
118 Sourdough Ridge
Bozeman, MT 59715
(406) 586-3263

Prepared for:

National Aeronautics and Space Administration

In order to explore the relationship between future observations of the precession of orbiting gyroscopes (GP-B) and existing observations of post-Newtonian, relativistic effects in gravity, we have formulated a phenomenological gravitational equation of motion for matter which presupposes the very minimum possible concerning the gravitational interaction.

The motional part of the gravitational equation of motion --- proportional to $(v/c)^2$ times the Newtonian accelerations --- is responsible for the interaction of gyroscopes with other bodies, but also plays a role in many other observables in the solar system which have been measured already. The set of all observations is a redundant --- completely determining but to date consistent --- set, and they predict the gyroscope precession rates with better than ten percent accuracy. A number of anomalous gyroscope precession terms are calculated within the general model, but their existence would be in conflict with other non-gyroscope observations.

If the gyroscope precession rates are found to differ from the predictions which can be uniquely made from other observations of relativistic effects in the solar system, a major crisis and opportunity!) would face gravitational theory: it would then appear that no motional, post-Newtonian gravitational equation of motion can be constructed which reconciles the total set of post-Newtonian observations in the solar system.

(NASA-CR-187736) THE GRAVITOMAGNETIC INTERACTION AND ITS RELATIONSHIP TO OTHER RELATIVISTIC GRAVITATIONAL EFFECTS Final Report, 1 Dec. 1990 - 30 Apr. 1991 (Northwest Analysis) 54 p CSCL 08G G3/46
N91-21649
Unclas
0003628

IN-46

3628

Report Documentation Page

National Aeronautics and Space Administration

1. Report No	2. Government Accession No.	3. Recipient's Catalog No.
	-	0 5 4

4. Title and Subtitle
The Gravitomagnetic Interaction And Its Relationship To Other Relativistic Gravitational Effects

5. Report Date April, 1991

6. Performing Organization Code

7. Author(s)
Kenneth Nordtvedt

8. Performing Organization Report No.

10. Work Unit No.

9. Performing Organization Name and Address
Northwest Analysis
118 Sourdough Ridge
Bozeman, MT 59715 (406) 586-3263

11. Contract or Grant No.
H-11215D

13. Type of Report and Period Covered
Final 12/1/90 to 4/30/91

12. Sponsoring Agency Name and Address
National Aeronautics and Space Administration
George Marshall Space Flight Center
Marshall Space Flight Center, Alabama 35812

14. Sponsoring Agency Code

15. Supplementary Notes
Modified version of this report to be submitted to Physical Review for publication.

16. Abstract
To better understand the relationship between the expected precession rates of an orbiting gyroscope (GP-B) and other observable consequences in the solar system of relativistic, post-Newtonian gravity, we have developed a phenomenological model of post-Newtonian gravity which presupposes the very minimum possible concerning the nature and foundations of the gravitational interaction. Solar system observations --- primarily interplanetary ranging --- fix all the parameters in the phenomenological model to various levels of precision. This permits prediction of gyroscope precession rates to better than 10 percent accuracy. A number of new precession terms are calculated which would exist if gravity were not a metric field phenomenon, but this would clash with other empirical observations of post-Newtonian effects in gravity. It is shown that gravitomagnetism --- the post-Newtonian gravitational corrections to the interactions between moving matter --- plays a ubiquitous role in determining a wide variety of gravitational effects, including the precession of orbiting gyroscopes.

17. Key Words (Suggested by Author(s))
Gravitomagnetism, GP-B, relativistic effects in gravity

18. Distribution Statement
unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
none	none	54	

NASA FORM 1626 OCT 86

I. Introduction

Even before Einstein's general relativity theory was created,[1] there was informed speculation that the inertial properties of matter were determined by matter's cosmological distribution, itself.[2] Such points of view became known as "Mach's Principle", but often such concepts were stated without a specific quantitative model or theory for proper and complete formulation.

It was soon noticed that in general relativity theory some of these Machian ideas were manifest. These concepts soon acquired the descriptive interpretation -- "the dragging of inertial frames" due to proximate moving matter.

In general relativity (and other metric theories of gravity) gravity is a consequence of the metric field $g_{\mu\nu}(\bar{r},t)$. At any space-time location a coordinate transformation can be found that eliminates all first space and time partial derivatives of the metric field; and a simple linear transformation then can set $g_{\mu\nu}$ equal to special relativity's Minkowski metric;

$$g_{\mu\nu}(\bar{r},t) \rightarrow \eta_{\mu\nu} + O(x^{\mu} - x^{\mu}_{o})^2 \qquad 1.1$$

The locality is now a freely falling, inertial frame with only tidal gravitational fields. The transformation which accomplishes (1.1) is non-linear;

$$x^{\mu\prime} = (x^{\mu} - x^{\mu}_{o}) + \tfrac{1}{2}\, \Gamma^{\mu}_{\alpha\beta}(x^{\alpha} - x^{\alpha}_{o})\, (x^{\beta} - x^{\beta}_{o}) \qquad 1.2$$

$\Gamma^{\mu}_{\alpha\beta}$ are the Chrisstofel symbols formed from $g_{\mu\nu}$ and its first partial derivatives, evaluated at the space-time locality x^{μ}_{o};

$$\Gamma^{\mu}_{\alpha\beta} = \tfrac{1}{2}\, g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \qquad 1.3$$

For the space-time indices μ, α, β taking various types of values, quasi-physical descriptions can be made of the various terms which occur in the transformation (1.2) to the locally freely falling inertial frame.

(μ, α, β all spatial coordinates) These terms indicate a non-linear "warping" of the spatial coordinates.

(μ spatial, α and β time coordinates) These terms indicate an accelerative dragging of the inertial frame;

$$\delta x^k \sim \tfrac{1}{2}\,\Gamma^k_{oo}\,\delta t^2 \quad k = x,y,z$$

(μ spatial, α spatial and β temporal) These terms indicate a dragging of the inertial frame going linear in time;

$$\delta x^k \sim \Gamma^k_{lo}\ \delta x^l\ \delta t$$

The Chrisstofel symbols Γ^k_{lo} can be viewed as a 3x3 spatial matrix with dimensions of frequency (1/t). The antisymmetric part of Γ^k_{lo} then plays the special role of interest to this study;

$$\Gamma^k_{lo} = s_{lk} + \sum_i e_{ilk}\,\omega_i \qquad 1.4$$

with s_{lk} representing the symmetric part of Γ^k_{lo} and the frequency vector ω_i being a convenient way to express the three independent components of the antisymmetric part of Γ^k_{lo}. ε_{ilk} is the antisymmetric permutation tensor.

Whereas s_{lk} represents a time dependent rescaling of the inertial coordinates (both a stretching and warping, in general), the interpretation of the effect of the antisymmetric part of Γ^k_{lo} is unambiguous and familiar -- a pure rotation of the local inertial frame;

$$\delta\bar{r}(t) = -\,\bar{\omega} \times (\bar{r} - \bar{r}_o)\,\delta t$$

with $\bar{\omega}$ giving the rotation axis and rate of rotation. In weak gravity and the first post-Newtonian order approximation;

$$\Gamma^{k}{}_{lo} = \tfrac{1}{2}(g_{lo,k} - g_{lk,o} - g_{ok,l}) \qquad 1.5a$$

with the antisymmetric part being simply;

$$\Gamma^{l}_{ko} - \Gamma^{k}_{lo} = g_{ko,l} - g_{lo,k} \qquad 1.5b$$

The three components g_{ko} of the complete metric field are commonly called the gravitomagnetic vector potential in analogy with electromagnetic theory;

$$g_{ko} \equiv \bar{h} \quad k = x,y,z$$

then

$$\bar{\omega} = -\bar{\nabla} \times \bar{h} \qquad 1.6$$

The source of the gravitomagnetic vector potential is moving matter; in general relativity (in a common gauge);

$$\bar{h} = (7/2)\frac{m_s}{r}\bar{v}_s + (1/2)\frac{m_s}{r}\bar{v}_s \cdot \hat{r}\hat{r} \qquad 1.7$$

for a source m_s moving at velocity $\bar{v}_s$. The acceleration of a test particle in the presence of a gravitomagnetic vector potential is

$$\delta\bar{a} = \partial\bar{h}/\partial t - \bar{v} \times (\bar{\nabla} \times \bar{h}) \qquad 1.8$$

with $\partial\bar{h}/\partial t$ being an "electric" type acceleration and $\bar{v} \times (\nabla \times \bar{h})$ being the "magnetic" type acceleration;

$$\delta\bar{a}(mag) = 4m_s/r^3\,\bar{v} \times (\bar{r} \times \bar{v}_s) \qquad 1.9$$

Just as two electric currents produce a magnetic force between themselves, two mass currents produce a gravitational force (1.9) whose unique signature is that it is a gravitational acceleration proportional to the velocity of both interacting masses. (1.9) can be considered a different way of viewing the dragging of inertial frames. This interaction does not require spinning mass -- simply moving mass.

Superimposing the acceleration (1.9) for a spinning body mass source and a spinning body test object, it then produces a torque acting on the two gyroscopes which causes precession of the gyroscope spin axes: from the geometrical point of view this precession of interacting gyroscopes is simply the following of rotating inertial frames.

But, in fact, a gravitomagnetic contribution to the post-Newtonian gravitational interaction plays a ubiquitous role in contributing to observable phenomena.[3] It is present in any configuration of matter where there is mutual motion of two interacting bodies. The spin-spin interaction of two gyroscopes is just a special case and configuration for measuring the nature of this part of the complete post-Newtonian structure of gravity. It is not necessary to use gyroscopes in order to measure gravitomagnetism.

There are other motional corrections to the static, Newtonian interaction. Just as (1.9) is an acceleration proportional to $(\overline{v})(\overline{v}_s)$, there are accelerations proportional to v^2 or v_r^2. The entire package of motional corrections will determine the properties of the gravitational interaction under Lorentz transformations. Gravity need not be Lorentz invariant.[4] If the underlying field theory of gravity were to be based on two (or more) tensor fields, or a vector field as well as a tensor field, then the cosmos could produce preferred inertial frames in its gravitational interaction: there would be phenomena resulting from the gravitational interaction which could depend on the velocity of the system relative to the cosmological preferred frames. The gravitomagnetic interactions are part of the whole package of motional corrections to the static Newtonian interaction, which as a package determines whether gravity has preferred inertial frames or not.

This study takes as phenomenological a point of view as possible. We build the post-Newtonian gravitational interaction structure from empirical observations, and with a minimum of a-priori theoretical presuppositions. Several assumptions which we will not make a-priori are that:

1. Gravity is metric field based;
2. Gravity has any special properties under Lorentz transformations;
3. Gravity fulfills any conservation laws or is derivable from a post-Newtonian many body Lagrangian.

What is assumed is simply that there are post-Newtonian equations of motion for;

1. mass particles in the presence of other matter;
2. photons in proximity to matter;
3. non-gravitational clocks near matter.

Isolated, but general N body gravitational systems are assumed to be surrounded by asymptotic inertial frames, in which non-gravitational rulers and clocks fulfill the transformation laws of special relativity. The coordinate speed of light is assumed to be a function of space-time position relative to matter, and therefore it can be expressed by a phenomenological expansion;

$$c(\bar{r},t) = 1 + \Gamma_1 \sum_i m_i/|\bar{r} - \bar{r}_i| + \Gamma_1' \sum_i m_i((\bar{r} - \bar{r}_i)\cdot\hat{c})^2/|\bar{r} - \bar{r}_i|^3 + \ldots \qquad 1.10$$

The dynamical evolution of the light propagation vector $\hat{c}$ can be based on (1.10) and a "least time" principle, or it can be assumed to be based on an independent phenomenological expansion;

$$d\hat{c}/ds = \Gamma_2 \sum_i m_i(\bar{r}_i - \bar{r} - (\bar{r}_i - \bar{r})\cdot\hat{c}\hat{c})/|\bar{r}_i - \bar{r}|^3 + \ldots \qquad 1.11$$

Since there have been independent observations in the solar system of the effect of proximate matter on light propagation times (1.10)[5] and on deflection of light (1.11),[6] we can empirically conclude that;

$$\Gamma_2 = -\Gamma_1 = 2 \pm 10^{-3} \qquad 1.12a$$

and

$$\Gamma_1' = 0 \pm 10^{-3} \qquad 1.12b$$

Experiments on the rate at which non-gravitational clocks run when in proximity to matter (and in motion) find first that different types of non-gravitational clocks all behave in a universal fashion;[7] and secondly that in lowest order (in powers of $1/c^2$) their proper rate τ is given by:[8]

$$d\tau/dt = 1 - \alpha \sum_i m_i/|\bar{r} - \bar{r}_i| - \tfrac{1}{2}v^2 + \ldots \qquad 1.13$$

with

$$\alpha = 1 \pm 10^{-4} \qquad 1.14$$

and t is the time kept by a non-gravitational clock which is at rest in the inertial frame and located asymptotically far from matter.

Non-gravitational clocks and photons, governed by the empirically determined expressions (1.10), (1.11) and (1.13), are then the chief probes with which to look at the dynamics of bodies in the solar system. It remains to assume a most general, post-Newtonian gravitational equation of motion for an N body system of particles. The coordinate accelerations of particle i in the presence of other bodies j,k will be assumed to be of the form;

$$d^2\bar{r}_i/dt^2 = \sum_j Gm_j \bar{r}_{ji}/r_{ji}^3 + \sum_j \bar{g}_{ij}(\text{motional}) + \sum_{jk} \bar{g}_{ijk}(\text{non-linear}) \tag{1.15}$$

The first term in (1.15) is the Newtonian approximation to the gravitational interaction; the two post-Newtonian contributions to (1.15) are proportional to $1/c^2$: they differ in that the two body motional terms are proportional to $(v/c)^2$ --- the square of body velocities --- times Newtonian accelerations, while the non-linear accelerative terms in (1.15) have an extra power of Newtonian gravitational potential (Gm/c^2r) multiplying Newtonian acceleration.

The dynamical equation of motion (1.15) can only be assumed to hold in one cosmological inertial frame. Until the detailed structure of $\bar{g}_{ij}$(motional) is specified by empirical observation, it can not be assumed that (1.15) has any special Lorentz transformation properties.

As seen in (2.1) $\bar{g}_{ij}$(motional) has ten arbitrary coefficients in its general expression, of which one is a free gauge or coordinate system parameter. This leaves nine coefficients to be specified by nine (or more) independent empirical observations concerning the shape and rate of solar system body orbits. If more than nine independent observations are available, but which depend only on the nine free motional coefficients, redundant constraint will exist on the model

(1.15). These redundant observations could conceivably clash and not all be consistent with any choice of the nine coefficients in the expression for $\bar{g}_{ij}$(motional). We would view this as a fundamental crisis (and perhaps opportunity!) in gravitational theory, as it is difficult to see how any theory could be constructed which does not lead to motional post-Newtonian gravitational accelerations of the form (2.1)

In particular, consider the future plans to orbit a precision gyroscope and measure the precession of its spin axis with respect to a reference frame defined by lines of sight to the distant stars.[9] There are expected secular precessions of the gyroscope's axis due to two sources:

1. It's orbital motion about the Earth (geodetic precession);
2. It's interaction with the gravitomagnetic field of the spinning Earth (inertial frame dragging).

From the point of view of our phenomenological gravitational equation of motion (1.15), both types of gyroscopic precession are simply additional manifestations of the motional, post-Newtonian corrections to Newtonian gravity: the geodetic precession is viewed as a consequence of acceleration terms proportional to the square of the velocity of the test body mass elements, while the precession related to the dragging of inertial frames is considered a result of acceleration terms proportional to the velocity of source masses and the velocity of the test body masses. We will derive expressions for the general secular gyroscope precessions which can occur in an equation of motion of the type (1.15) (see actually (2.1)); and also we will obtain the many other observables which result from $\bar{g}_{ij}$(motional) but are not gyroscope observables. There will be more independent observables than there are free coefficients in (1.15) --- redundancy! -

-- and therefore the possibility of inconsistency of gyroscope observations with the other solar system observations.

There are also some other non-secular (but nevertheless possibly measurable) gyroscope precessions which result from the phenomenological model (1.15). Some of these non-secular precessions would imply non-metric gravity. The solar system is traveling relative to the cosmos at a speed of order $|\bar{w}| \sim 2\ 10^{-3}c$, while an earth orbiting gyroscope is carried around the Sun at speed $|\bar{v}| \sim 10^{-4}c$. The equation of motion (2.1) with its $\bar{w}$-dependent terms (2.3) can therefore possibly produce additional perturbations of the spin axis of a gyroscope which are dimensionally of the form;

$$\delta\bar{s} = g_1 \bar{s} \cdot \bar{v}\bar{v} \qquad 1.16a$$

and

$$\delta\bar{s} = g_2 \bar{s} \cdot \bar{w}\bar{v} + g_3 \bar{s} \cdot \bar{v}\bar{w} + g_4 \bar{s}\bar{w} \cdot \bar{v} \qquad 1.16b$$

and in which the coefficients g_1 , ... g_4 would be linear combinations of the phenomenological coefficients c_i in (2.1). The term (1.16a) is only a few milliarcseconds in magnitude; however the preferred frame terms (1.16b) vary about 80 milliarcseconds with an annual period.

The format of this study is as follows. In section II. we state the general phenomenological form that the motional, post-Newtonian gravitational acceleration terms can have in an N body gravitational system. Then a set of "observables" are derived which are attributes of orbits and dynamics of planets and test bodies in the solar system which are routinely measured by ranging experiments and/or telescopic observations. The observables form a redundant set --- they more than uniquely determine the free coefficients in the equation of motion. The observables could, in fact, possibly be inconsistent with the general model. Precession rates for orbiting gyroscopes are calculated as some of the observables. All possible gyroscope precessions are determined by coefficients measured in other non-gyroscope observations.

In section III. additional observations are derived which depend also on the non-linear structure of the post-Newtonian gravitational interaction. Some of these observations produce additional constraint on the motional coefficients $c_1 \ldots c_{10}$.

In section IV. we review the accuracy with which the various observables have been measured by past solar system observations, and how well this then permits us to predict the magnitude of the gyroscope precession terms.

Section V. discusses some of the conservation laws in physical theory. The existence of conservation laws is shown to be related to constraints on the structure of the gravitational equation of motion, *i.e.* it requires relationships to hold among the coefficients c_i in these equations, and therefore is predictive concerning some of the observables of gravity.

Section VI. derives the constraints on the gravitational equation of motion which indicate that post-Newtonian gravity is derivable from a metric field. If gravity is a metric field based interaction, then several of the possible gyroscope precession terms vanish as a consequence of this principle, alone.

In section VII. it is shown that there is a family of coordinate transformations which makes one of the ten coefficients in the gravitational equation of motion arbitrary. But it is shown how all ten coefficients change under this transformation, and this permits confirmation that all observables calculated in this study are invariant under this coordinate transformation.

Finally, in section VIII. we collect all the gyroscope precession terms derived in our general model of gravity. We discuss their magnitude, their relationship to the structure of the gravitational interaction, and the implications for theory if the observed gyroscope precessions were to clash with the predictions.

II. Motional Post-Newtonian Corrections to the Gravitational Interaction.

Since our goal is to specify the post-Newtonian gravitational interaction solely from experimental observations -- including null observations -- the starting point for analysis of observation is a general, phenomenological expression for the equation of motion for N particles (bodies of negligible gravitational self-energy or binding energy) which gravitationally interact with each other.

A-priori this equation of motion can be assumed to hold in only one special asymptotic ($r \to \infty$) inertial frame (the preferred frame); Lorentz invariance of the gravitational interaction is not presumed but rather is to be discovered, if it exists, from the experimental observations. A system of bodies which moves as a whole when viewed in the special inertial frame will be analyzed by the N-body equation of motion, and the dynamical consequences of the system's motion determined for comparison with observation. The solar system, for example, moves relative to the universe's local preferred inertial frame at a speed of order $10^{-3}c$, yet no orbital effects related to this motion are evident within experimental accuracy.

Post-Newtonian corrections to the Newtonian acceleration are of order $1/c^2$ and take two forms. There are motional corrections which are of strength $(v/c)^2$ relative to the Newtonian acceleration. The second type of post-Newtonian corrections are the non-linear terms proportional to the square of mass sources ($m_j m_k$) and therefore dimensionally proportional to the inverse cube power of interbody lengths ($1/r^3$).

Writing a most general expression for the motional corrections, the phenomenological post-Newtonian equation of motion takes the form;

$$\bar{a}_i = -\sum_j \frac{m_j \bar{r}_{ij}}{r_{ij}^3}\left[1 + c_1 v_i^2 + c_2 \bar{v}_i \cdot \bar{v}_j + c_3 v_j^2 + c_4 (\bar{v}_i \cdot \bar{r}_{ij})^2 + c_5 \bar{v}_i \cdot \hat{r}_{ij} \bar{v}_j \cdot \hat{r}_{ij} + c_6 (\bar{v}_j \cdot \bar{r}_{ij})^2\right]$$
$$+ \sum_j \frac{m_j}{r_{ij}^3} \bar{r}_{ij} \cdot \left[c_7 \bar{v}_i \bar{v}_i + c_8 \bar{v}_i \bar{v}_j + c_9 \bar{v}_j \bar{v}_i + c_{10} \bar{v}_j \bar{v}_j\right] + \sum_{jk} m_j m_k \bar{\rho} ijk \qquad 2.1$$

Units of $G = c = 1$ are used. $\bar{\rho}_{ijk}$ is a general vector expression of dimensions $1/r^3$ and is composed from the inner-body vectors $\bar{r}_{ij}$, $\bar{r}_{jk}$ and $\bar{r}_{ki}$. Consideration of the non-linear part of the post-Newtonain interaction will take place in section III of this paper. c_1, c_2 ... c_{10} are ten dimensionless coefficients expressing the most general motional structure of the post-Newtonian interaction. The coordinate (or variable) freedom of the equation of motion discussed in section VII indicates that a "gauge" can be chosen to eliminate any one of the ten c_i. Our choice will be to set $c_4 = 0$; however this calculation of observables in this section will include all ten c_i; this explicitly reveals the gauge or coordinate invariance of the calculated observables.

In this section, experimental observables dependent only on ten c_i of the motional structure of the equations of motion will be derived; section III will derive additional observables also dependent on the non-linear structure of the equation of motion.

A system of bodies collectively moving at velocity $\bar{w}$ relative to the preferred inertial frame in which (2.1) is valid is considered. The coordinate position of each body then takes the form;

$$\bar{r}_i \rightarrow \bar{r}_i + \bar{w}t \tag{2.2a}$$

and

$$\dot{\bar{r}}_i \equiv \bar{v}_i \rightarrow \bar{v}_i + \bar{w} \tag{2.2b}$$

For such a system the equations of motion given by (2.1) develop additional terms proportional to $\bar{w}$ and $(\bar{w})^2$;

$$\begin{aligned}\delta\bar{a}_i = &-\sum_j \frac{m_j\bar{r}_{ij}}{r_{ij}^3}\Big[(2c_1 + c_2)\bar{w}\cdot\bar{v}_i + (2c_3 + c_2)\bar{w}\cdot\bar{v}_j \\ &+ (2c_4 + c_5)\bar{w}\cdot\hat{r}_{ij}\bar{v}_i\cdot\hat{r}_{ij} + (2c_6 + c_5)\bar{w}\,\hat{r}_{ij}\bar{v}_j\cdot\hat{r}_{ij}\Big] \\ &+\sum_j \frac{m_j\bar{r}_{ij}}{r_{ij}^3}\cdot\Big[(c_7 + c_8)\bar{v}_i\bar{w} + (c_7 + c_9)\bar{w}\bar{v}_i + (c_9 + c_{10})\bar{v}_j\bar{w} + (c_8 + c_{10})\bar{w}\bar{v}_j\Big]\end{aligned} \tag{2.3}$$

$$-\sum_j \frac{m_j\bar{r}_{ij}}{r_{ij}^3}\Big[(c_1 + c_2 + c_3)w^2 + (c_4 + c_5 + c_5)(\bar{w}\cdot\hat{r}_{ij})^2\Big] + (c_7 + c_8 + c_9 + c_{10})\sum_j \frac{m_j\bar{r}_{ij}\cdot\bar{w}\bar{w}}{r_{ij}^3}$$

Post-Newtonian orbital effects in a system moving in the preferred inertial frame therefore naturally divide into $\bar{w}$ and $(\bar{w})^2$ effects; the latter are considered first.

(Observable 1) Consider a two body orbiting system which moves in a direction $\bar{w}$ perpendicular to the orbital plane defined by $\bar{r}_{ij}$ and $\bar{v}_{ij} \equiv d\bar{r}_{ij}/dt$.

$$\bar{a}_1 = -\frac{m_2\bar{r}_{12}}{r_{12}^3}\Big[1 + (c_1 + c_2 + c_3)w^2\Big] \tag{2.4}$$

and a similar equation for body (2). The solutions to (2.4) are simply the Keplerian orbits but with periods altered by the factor;

$$T(w) = T(o)\left[1 - \frac{1}{2}(c_1 + c_2 + c_3)w^2\right] \tag{2.5}$$

Atomic clocks far from the system and moving along with the system (*i.e.*, clocks in system's rest frame) will be running slow by the special relativistic factor $(1 - ½ w^2)$. If gravitational clocks (orbits) are to show no dependence of their rate on their motion relative to the cosmos, the following constraint must hold:

$$c_1 + c_2 + c_3 = -1 \tag{2.6}$$

(Observable 2) Now let the velocity vector $\bar{w}$ lie in the orbital plane. The equation of motion is now

$$\begin{aligned} \bar{a}_1 = & -\frac{m_2\bar{r}_{12}}{r_{12}^3}\left[1 + (c_1 + c_2 + c_3)w^2 + (c_4 + c_5 + c_6)(\bar{w}\cdot\hat{r}_{12})^2\right] \\ & + (c_7 + c_8 + c_9 + c_{10})\frac{m_2\bar{r}_{12}\cdot\bar{w}\bar{w}}{r_{12}^3} \end{aligned} \tag{2.7}$$

Applying this to a nominal circular orbit or radius r_o, there results a radial perturbation

$$\delta r(t) = -\frac{r_o w^2}{3}\left[c_7 + c_8 + c_9 + c_{10} - ½(c_4 + c_5 + c_6)\right]\cos 2\omega t \tag{2.8a}$$

But an observer at rest in the cosmos must "see" or measure a Lorentz contraction of the orbit

$$\delta r(t)_{Lc} = -¼\, r_o w^2 \cos 2\omega t \tag{2.8b}$$

Otherwise a co-moving distant observer will see contractions of the orbit proportional to the square of the speed of the orbit through the cosmos. Comparing (2.8a,b) leads to the constraint;

$$c_7 + c_8 + c_9 + c_{10} - ½(c_4 + c_5 + c_6) = 3/4 \qquad 2.9$$

(Observable 3) The period of this preceding orbit can be calculated;

$$\left[\frac{2\pi}{T(w)}\right]^2 = \frac{m_2 + m_2}{r_o^3}\left[\begin{array}{c} 1 + (c_1 + c_2 + c_3)w^2 \\ +1/2(c_4 + c_5 + c_5 - c_7 - c_8 - c_9 - c_{10})w^2 \end{array}\right] \qquad 2.10$$

But r_o is the mean radius of the Lorentz contracted orbit; the transverse radius of the orbit is (using (2.8a));

$$r_\perp = r_0\left[1 + \frac{2(c_7 + c_8 + c_9 + c_{10}) - c_4 - c_5 - c_6}{6} w^2\right] \qquad 2.11$$

Rewriting (2.10) in terms of $r_\perp$ yields;

$$\left[\frac{2\pi}{T(w)}\right]^2 = \frac{m_1 + m_2}{r_\perp^3}\left[1 + \frac{2(c_1 + c_2 + c_3) + c_7 + c_8 + c_9 + c_{10}}{2} w^2\right] \qquad 2.12$$

For these gravitational clocks to be properly time dilated as well, a constraint exists;

$$c_1 + c_2 + c_3 + ½(c_7 + c_8 + c_9 + c_{10}) = -1 \qquad 2.13$$

A very accurate way to observationally test for the independence of orbital period dilation on the orientation of the orbit relative to the cosmos is to measure the isotropy of the Newtonian interaction in celestial bodies moving in the cosmos. This will be discussed further in the section on empirical observations. From (2.6) and (2.13) the condition for isotropy of the strength of the Newtonian interaction is just;

$$c_7 + c_8 + c_9 + c_{10} = 0 \qquad 2.14$$

(Observable 4) Consider the time rate of change of the angular motion for a two body system;

$$\frac{d}{dt}(\bar{r}_{12} \times \bar{v}_{12}) = (c_7 + c_9)\frac{m_2 - m_1}{r_{12}^3}\bar{r}_{12} \cdot \bar{w}\,\bar{r}_{12} \times \bar{v}_{12} + (c_7 + c_8)\frac{m_2 - m_1}{r_{12}^3}\bar{r}_{12} \cdot \bar{v}_{12}\,\bar{r}_{12} \times \bar{w} \qquad 2.15$$

The first term is a periodic perturbation, but the second term gives a secular perturbation to angular motion for eccentric orbits. The failure to observe such orbital effects requires the constraint:

$$c_7 + c_8 = 0 \qquad 2.16$$

(Observable 5) Consider the acceleration of a two body system's center of mass;

$$\frac{d^2}{dt^2}\left(\frac{m_2\bar{r}_1 + m_2\bar{r}_2}{m_1 + m_2}\right) = -\frac{m_1 m_2}{mr^3}\bar{r}\begin{bmatrix} 2(c_1 - c_3)\bar{w} \cdot \bar{v} \\ + 2(c_4 - c_6)\bar{w} \cdot \hat{r}\bar{v} \cdot \hat{r} \end{bmatrix} \qquad 2.17$$

$$+\frac{m_1 m_2}{mr^3}\begin{bmatrix} (c_7 + c_8 - c_9 - c_{10})\bar{r} \cdot \bar{v}\,\bar{w} \\ + (c_7 + c_9 - c_8 - c_{10})\bar{r} \cdot \bar{w}\,\bar{v} \end{bmatrix}$$

For a circular orbit the projection of the center of mass acceleration in the direction $\bar{w} \times (\bar{r} \times \bar{v})$

is;

$$\hat{w} \times (\hat{r} \times \hat{v}) \cdot \frac{d^2\bar{R}}{dt^2} = -\frac{m_1 m_2 wv}{mr^2}\begin{bmatrix} 2(c_1 - c_3)(\hat{w} \cdot \hat{v})^2 \\ + (c_7 + c_9 - c_8 - c_{10})(\hat{w} \cdot \hat{r})^2 \end{bmatrix} \qquad 2.18$$

Both terms in (2.18) produce secular accelerations; they cancel however if there is the constraint;

$$2(c_1 - c_3) + c_7 + c_9 - c_8 - c_{10} = 0 \qquad 2.19$$

This constraint results in elimination of secular center of mass acceleration for eccentric orbits as well. The other two terms in (2.17) produce periodic accelerations of the center of mass. (Observable 6) The complete equation of motion for the interbody vector $\bar{r} = \bar{r}_1 - \bar{r}_2$ of a two body system is obtained from (2.3); including the $\bar{w}$ terms it is;

$$\frac{d^2\bar{r}}{dt^2} = -\frac{m\bar{r}}{r^3} - \frac{m_2 - m_1}{r^3}\bar{r}\begin{bmatrix} (2c_1 + c_2)\bar{w} \cdot \bar{v} \\ + (2c_4 + c_5)\bar{w} \cdot \hat{r}\hat{r} \cdot \bar{v} \end{bmatrix} \qquad 2.20$$

$$+ \frac{m_2 - m_1}{r^3}\left[(c_7 + c_8)\bar{r} \cdot \bar{v}\bar{w} + (c_7 + c_9)\bar{r} \cdot \bar{w}\bar{v}\right]$$

The non-secular contribution to angular motion variation is:

$$\frac{d}{dt}(\bar{r} \times \bar{v}) = (c_7 + c_9)\frac{m_2 - m_1}{r^3}\bar{r} \cdot \bar{w}\,\bar{r} \times \bar{v} \qquad 2.21$$

The equation for radial perturbation x(t) of a circular orbit is then;

$$\ddot{x} + \omega^2 x = \frac{m_2 - m_1}{r^3} w^* v \left[2c_1 + c_2 + 2c_7 + 2c_9\right] \sin\omega t \qquad 2.22$$

$t = 0$ corresponds to $\bar{r}(t)$ traversing the in-plane component of $\bar{w}$ (w^* is the magnitude of $\bar{w}$ lying in the orbital plane). (2.22) has a runaway solution --- $x(t) \sim t\cos\omega t$ --- contrary to observation, requiring the constraint;

$$2c_1 + c_2 + 2(c_7 + c_9) = 0 \qquad 2.23$$

(Observable 7) The angular motion equation (2.21) then results in an angular position perturbation;

$$r\delta\theta(t) = \frac{m_1 - m_2}{m} r(c_7 + c_9) wv \cos\omega t \qquad 2.24$$

For this longitudinal perturbation to be unseen after a Lorentz transformation to the inertial frame of a co-moving observer, a constraint is required;

$$c_7 + c_9 = 1 \qquad 2.25$$

(Observable 8) From (2.20) the total radial perturbation is;

$$\delta\bar{a} \cdot \hat{r} = \frac{m_1 - m_2}{r^2} \left[\begin{array}{c} (2c_1 + c_2)\bar{w} \cdot \bar{v} \\ + (2c_4 + c_5 - 2c_7 - c_8 - c_9)\bar{w} \cdot \hat{r}\hat{r} \cdot \bar{v} \end{array} \right] \qquad 2.26$$

This radial perturbation along with the angular perturbation given by (2.21) can be applied to the general Keplerian elliptical orbit. The orbital perturbation is obtained most easily in the $u(\theta) = 1/r(\theta)$ representation of the orbit;

$$\frac{\delta u}{u_o} = \frac{e}{6}\frac{(m_1 - m_2)}{\ell} w(2c_4 + c_5)\sin(2\theta - \nu) \qquad 2.27$$

with the Keplerian orbit being given by;

$$1/r = u = u_o\{1 + e\cos\theta\}$$

and

$$\ell = |\bar{r} \times \bar{v}|$$

The angle ν is the orbital longitude of the in-plane component of $\bar{w}$. Failure to see such orbital bulges requires the constraint;

$$2c_4 + c_5 = 0 \qquad 2.28$$

(Observable 9) Consider a three body system consisting of a massive central body of mass M, with another body of mass m in circular orbit. A third body orbits the body of mass m. From (2.1) there will be a difference between the acceleration of the test body m by the massive central body, because of the difference in the two bodies' velocities. Collecting the terms linear in $\bar{u}$, the velocity of the test body relative to body m, the relative acceleration is;

$$\delta\bar{a} = -2c_1 \frac{M\bar{R}}{R^3}\bar{V}\cdot\bar{u} + c_7 \frac{M}{R^3}\bar{R}\cdot\bar{u}\,\bar{V} \qquad 2.29$$

$\bar{R}$ is the position of body m from the central body M, $\bar{V} = d\bar{R}/dt$ and $\bar{u} = d\bar{r}/dt$; $\bar{r}$ is the position of the test body from body m. The secular part of the acceleration which results from (2.29) is;

$$\delta\bar{a} = -\left(\frac{c_7}{2} + c_1\right)\frac{M}{R^2}\bar{u} \times \bar{L} \tag{2.30}$$

with $\bar{L}$ being the angular motion of body m;

$$\bar{L} = \bar{R} \times \bar{V}$$

Applying this perturbation to a near-circular test body orbit which lies in the plane of the circular orbit of body m, the orbital frequency ω and natural frequency for radial oscillations (eccentric motion) ω_o are altered;

$$\omega^2 = \frac{m}{r^3} - \left(\frac{c_7}{2} + c_1\right)\frac{M}{R^2}V\omega \tag{2.31a}$$

$$\omega_o^2 = \frac{m}{r^3} - (c_7 + 2c_1)\frac{MV}{R^2}\omega \tag{2.31b}$$

The difference produced between these frequencies is a contribution to the precession rate of the test body's orbital periastron;

$$\delta(\omega_o - \omega) = -½\,(c_1 + ½\,c_7)\,MV/R^2 \tag{2.32}$$

This is commonly called the "geodetic precession" of an orbit. Observations of the geodetic precession contribution to the Moon's orbit produced by the motion of the Earth-Moon system about the sun produces a weak constraint;[10]

$$c_1 + \tfrac{1}{2}c_7 = 3 \qquad 2.33$$

(Observable 10) A test body is in a circular orbit about a body of mass m which gravitationally free falls toward another body M (M >> m) which is at rest; the test body's orbit around m is perpendicular to the direction toward M ($\bar{R} \cdot \bar{r} = 0$). The equation of motion for the test body can then be written as;

$$\begin{aligned} \bar{a} = & -\frac{m\bar{r}}{r^3}\left[1 + (c_1 + c_2 + c_3)V^2\right] + c_7\frac{M}{R^3}\bar{R}\cdot\bar{V}\bar{u} \\ & -\frac{M\bar{R}}{R^3} + \chi\frac{M}{R}\frac{m\bar{r}}{r^3} \end{aligned} \qquad 2.34$$

with $\bar{V} = d\bar{r}/dt$, $\bar{u} = d\bar{r}/dt$: $\bar{R}$ is the position of m relative to M, and $\bar{r}$ is the position of the test body relative to m. The χ term is a necessary non-linear acceleration term which will drop out of the final expression for an observable. The gravitational freefall of m fulfills the Newtonian energy constraint;

$$\tfrac{1}{2}V^2 - M/R = \text{const}$$

The test body's angular motion can then be obtained from (2.34); it evolves in time during the freefall;

$$|\bar{r}\times\bar{u}| = \ell = \ell_{\infty}\left(1 - c_7\frac{M}{R}\right) \qquad 2.35$$

This, in turn, produces an evolution in the radius and angular frequency of the test body's orbit;

$$r = r_\infty\left[1 + \left(\chi - 2c_7 - 2(c_1 + c_2 + c_3)\right)\frac{M}{R}\right] \qquad 2.36a$$

and

$$\dot{\theta} = \frac{\ell}{r^2} = \dot{\theta}_\infty\left[1 + \left(3c_7 - 2\chi + 4(c_1 + c_2 + c_3)\right)\frac{M}{R}\right] \qquad 2.36b$$

in which the relationship $V^2 = 2M/R$ was used. Atomic clocks freefalling in the gravitational field of body M will change their ticking rate according to;[8]

$$\omega = \omega_\infty\left(1 - (1 + \alpha)\frac{M}{R}\right) \qquad 2.37$$

while the proper radius of the orbit is given by

$$r_p = r\left(1 - (\alpha + \Gamma)\frac{M}{R}\right) \qquad 2.38$$

In (2.37) and (2.38) we have introduced two additional phenomenological parameters related to non-gravitational clock behavior and the light propagation rate in a gravitational environment;

$$\frac{d\tau}{dt} = 1 - \tfrac{1}{2}v^2 - \alpha U(\bar{r}) \qquad 2.39$$

and

$$c(\vec{r}) = 1 + \Gamma U(\vec{r}) \tag{2.40}$$

Observationally $\alpha = 1 \pm 10^{-4}$ and $\Gamma = -2 \pm 10^{-3}$.

A combination of (2.36a,b) can be formed to produce an observable dependent only on the motional coefficient c_7 along with coefficients related to the gravitational influences on light propagation and non-gravitational clocks;

$$2\left(\frac{r_p}{r_{p\infty}} - 1\right) + \left(\frac{\dot{\theta}_p}{\dot{\theta}_{p\infty}} - 1\right) = (1 - \alpha - 2\Gamma - c_7)\frac{M}{R} \tag{2.41}$$

in which the proper frequencies $\dot{\theta}_p$ are measured by co-falling nongravitational clocks, and the proper orbital radius is measured in terms of locally measured light propagation times across the orbit. In the solar system no variation of satellite orbital radii or periods are observed as these satellites follow their planets on the eccentric motion about the Sun: this requires the constraint:

$$c_7 = 1 - \alpha - 2\Gamma \tag{2.42}$$

A constraint on non-linear aspects of the equation of motion follows from the separate absences of anomalous size or frequency evolution of the orbits;

$$\left(\frac{r_p}{r_{p\infty}} - 1\right) = \left(\frac{\dot{\theta}_p}{\dot{\theta}_{p\infty}} - 1\right) = 0 \tag{2.43}$$

requires the constraint;

$$\chi = 2(c_1 + c_2 + c_3) + \frac{3}{2}c_7 + \frac{1+\alpha}{2} \tag{2.44}$$

But this constraint will be considered along with the other constraints on non-linear aspects of the gravitational interaction in section III ($\chi = c_{11} + c_{12}$ in the notation of that section).

(Observable 11) Consider an extended body in motion at velocity $\bar{w}$. The matter distribution in the body will be Lorentz contracted with respect to the proper coordinates, the contraction being;

$$\delta\bar{r}_j = -\tfrac{1}{2}\bar{r}_j \cdot \bar{w}\bar{w} \tag{2.45}$$

This alters the Newtonian potential of that body;

$$\delta U = -\bar{\nabla} \cdot \sum_j \frac{m_j}{|\bar{r} - \bar{r}_j|} \delta\bar{r}_j \tag{2.46}$$

And the gradient of (2.46) in an altered acceleration field;

$$\delta\bar{a} = \bar{\nabla}\delta U = \tfrac{1}{2}\bar{\nabla}\bar{w} \cdot \bar{\nabla}\bar{w} \cdot \sum_j \frac{m_j}{|\bar{r} - \bar{r}_j|} \bar{r}_j \tag{2.47}$$

From (2.3) there is one term which generates a similar type acceleration field;

$$
\begin{aligned}
\delta\bar{a} &= -(c_4 + c_5 + c_6)\sum_j \frac{m_j}{|\bar{r} - \bar{r}_j|^5}\left(\bar{w}\cdot(\bar{r} - \bar{r}_j)\right)^2(\bar{r} - \bar{r}_j) \\
&= -(c_4 + c_5 + c_6)\left\{\frac{m}{r^3}(\bar{w}\cdot\hat{r})^2\,\bar{r} - \tfrac{1}{3}\,\bar{\nabla}\bar{w}\cdot\bar{\nabla}\bar{w}\cdot\sum \frac{m_j}{|\bar{r} - \bar{r}_j|}\,\bar{r}_j\right\}
\end{aligned}
\qquad 2.48
$$

For a spherical body

$$
\sum \frac{m_j}{|\bar{r} - \bar{r}_j|}\,\bar{r}_j = \frac{I}{3}\,\frac{\bar{r}}{r^3} \qquad 2.49
$$

with

$$
I = \sum_j m_j r_j^2
$$

The anomalous acceleration field from (2.47) plus (2.48) is then;

$$
\delta\bar{a} = \frac{1}{9}\left(\frac{3}{2} + c_4 + c_5 + c_6\right)\frac{I}{r^5}\left\{\begin{matrix} 15(\bar{w}\cdot\hat{r})^2\bar{r} - 6\bar{w}\;\bar{r}\bar{w} \\ -3w^2\bar{r} \end{matrix}\right\} \qquad 2.50
$$

Absence of observed perturbations of low satellite orbits or gravimeter anomalies on the Earth surface requires the constraint;

$$
c_4 + c_5 + c_6 = -3/2 \qquad 2.51
$$

(Observable 12) Let the extended body be spinning;

$$
\bar{v}_j = \bar{\Omega}\times\bar{r}_j \qquad 2.52
$$

Because of the time component of the Lorentz transformation, at a simultaneous preferred frame time there is unequal proper time across the extended body;

$$\tau_j = t - \bar{w} \cdot \bar{r}_j \qquad 2.53$$

The rotational motion (2.52) along with the time transformation (2.53) then produces a displacement of each mass element relative to its proper position;

$$\delta\bar{r}_j = \bar{v}_j(\tau_j - t) = -\bar{v}_j \bar{w} \cdot \bar{r}_j \qquad 2.54$$

which alters the Newtonian potential of the body by;

$$\begin{aligned} \delta U &= -\bar{\nabla} \cdot \sum \frac{m_j}{|\bar{r} - \bar{r}_j|} \delta\bar{r}_j \\ &= \bar{\nabla} \cdot \sum \frac{m_j}{|\bar{r} - \bar{r}_j|} \bar{\Omega} \times \bar{r}_j \bar{w} \cdot \bar{r}_j \end{aligned} \qquad 2.55$$

and the acceleration field by;

$$\delta\bar{a} = \bar{\nabla} \delta U$$

A number of terms in (2.3) produce a similar type of acceleration field;

$$\begin{aligned} \delta\bar{a} = &-\sum \frac{m_j(\bar{r} - \bar{r}_j)}{|\bar{r} - \bar{r}_j|^3} \left[(2c_3 + c_2)\bar{w} \cdot \bar{v}_j + (c_5 + 2c_6)\bar{w} \cdot (\hat{r} - \hat{r}_j)\, \bar{v}_j \cdot (\hat{r} - \hat{r}_j)\right] \\ &+ \sum_j \frac{m_j}{|\bar{r} - \bar{r}_j|^3} \left[(\bar{r} - \bar{r}_j) \cdot \bar{v}_j \bar{w}(c_9 + c_{10}) + (\bar{r} - \bar{r}_j) \cdot \bar{w}\, \bar{v}_j(c_8 + c_{10})\right] \end{aligned} \qquad 2.56$$

with

$$\sum_j \frac{m_j \bar{\Omega} \times \bar{r}_j}{|\bar{r} - \bar{r}_j|} = \frac{1}{2} \frac{\bar{J} \times \bar{r}}{r^3}$$

$\bar{J}$ being the body's rotational angular momentum. The total anomalous acceleration field from (2.55) and (2.56) is then;

$$\begin{aligned} \delta\bar{a} = {} & \frac{1}{2}\left(1 + c_2 + 2c_3 + \frac{c_5 + 2c_6}{3}\right)\left(\frac{\bar{w} \times \bar{J}}{r^3} - 3\,\frac{\bar{r} \cdot \bar{w} \times \bar{J}\bar{r}}{r^5}\right) \\ & + \frac{1}{2}\left(c_8 + c_{10} - \frac{c_5 + 2c_6}{3}\right)\left(\frac{\bar{w} \times \bar{J}}{r^3} - 3\,\frac{\bar{r} \times \bar{J}\bar{r} \cdot \bar{w}}{r^5}\right) \end{aligned} \qquad 2.57$$

Absence of low satellite orbit perturbations or gravimeter anomalies on Earth which would be produced by (2.57) require the constraints;

$$1 + c_2 + 2c_3 + (c_5 + 2c_6)/3 = 0 \qquad 2.58a$$

$$c_8 + c_{10} - (c_5 + 2c_6)/3 = 0 \qquad 2.58b$$

(Observable 13) Consider the interaction terms in (2.1) proportional to $(\bar{v}_i \bar{v}_j)$ and the source consisting of a rotating extended body;

$$\bar{v}_j = \bar{\Omega} \times \bar{r}_j$$

From (2.1) this source produces the acceleration field;

$$\bar{a}_i = \frac{c_2 + c_8}{2}\frac{\bar{v}_i \times \bar{J}}{r_i^3} - \frac{3}{2}\left(c_2 + \frac{1}{3}c_5\right)\frac{\bar{v}_i \times \bar{J} \cdot \overline{r_i r_i}}{r_i^5}$$
$$-\frac{3}{2}\left(c_8 - \frac{1}{3}c_5\right)\frac{\bar{r}_i \times \bar{J}\bar{r}_i \cdot \bar{v}_i}{r_i^5} \tag{2.59}$$

$\bar{r}_i$ is the position of body i from the center of the source, $\bar{J}$ is the total rotational angular momentum of the source

$$\bar{J} = \sum_j m_j \bar{r}_j \times (\bar{\Omega} \times \bar{r}_j) \tag{2.60}$$

For an orbiting test body the time rate of change of the angular motion of the orbit is;

$$\frac{d\bar{L}}{dt} = \bar{r}_i \times \bar{a}_i = -\frac{c_2 + c_8}{4}\frac{\bar{J} \times \bar{L}}{r_i^3}$$
$$+\frac{1}{4}\left(c_2 - c_8 + \frac{2}{3}c_5\right)\frac{\bar{r}_i \cdot \bar{J}\bar{v}_i + \bar{v}_i \cdot \bar{J}\bar{r}_i}{r_i^3} \tag{2.61}$$

which results in a secular precession of the orbit about $\bar{J}$ at the rate;

$$\omega = -\frac{c_2 + c_8}{4}\frac{J}{a_o^3(1 - e^2)^{3/2}} \tag{2.62}$$

a_o and e being the orbit's semimajor axis and eccentricity, respectively. Such a precession is one manifestation of the so-called "dragging of inertial frames" by a rotating mass with angular momentum $\underline{J}$

(Observable 14) The acceleration field (2.59) produced by one spinning mass causes a second spinning mass to precess. Let the test "gyroscope" have its mass elements moving with velocities;

$$\bar{v}_i = \bar{\Omega} \times \bar{r}_i \qquad 2.63$$

with the body having total rotational angular momentum;

$$\bar{S} = \sum_i m_i \bar{r}_i \times (\bar{\Omega} \times \bar{r}_i) \qquad 2.64$$

The precession of $\bar{S}$ is then governed by;

$$\dot{\bar{S}} = \frac{c_2 + c_8}{8}\left[\frac{\bar{J} \times \bar{S}}{R^3} - 3\,\frac{\bar{J} \cdot \bar{R}\bar{R} \times \bar{S}}{R^5}\right] + \frac{3}{8}\left(c_2 - c_8 + \frac{2}{3}c_5\right)\frac{\bar{J} \times \bar{R}\bar{R} \cdot \bar{S} + \bar{J} \times \bar{R} \cdot \bar{S}\bar{R}}{R^5} \qquad 2.65$$

The first term in (2.65) gives an instantaneous precession rate;

$$\bar{\omega} = \frac{c_2 + c_8}{8}\,\frac{\bar{J} - 3\bar{J} \cdot \hat{R}\hat{R}}{R^3} \qquad 2.66$$

while the second term produces a secular contribution to $\dot{\bar{S}}$ of:

$$\dot{\bar{S}} = -\frac{3}{16}\left(c_2 - c_8 + \frac{2}{3}c_5\right)\frac{\bar{J} \times \hat{p}\hat{p} \cdot \bar{S} + \bar{J} \times \hat{p} \cdot \bar{S}\hat{p}}{a_o^3(1 - e^2)^{3/2}} \qquad 2.67a$$

$\hat{p}$ is the orbit's normal vector. (2.67) can have a non-zero component along $\bar{S}$ which would change the magnitude of $\bar{S}$. There are also a few non-secular (annual period) contributions to a gyroscope's precession which are proportional to the velocity $\bar{w}$ of the solar system with respect to the preferred inertial frame. Measured in the solar system rest frame the variation of a gyroscope's angular motion is given by:

$$\delta\bar{S} = \tfrac{1}{2}(4 - 3c_7 - c_8 - 2c_9 + 2c_1 + c_2 + c_4 + c_5)\,\bar{V}\cdot\bar{w}\,\bar{S} + \frac{c_7 + c_8}{2}\,\bar{S}\cdot\bar{w}\,\bar{V} - \frac{2 + 2c_1 + c_2}{2}\,\bar{S}\cdot\bar{V}\,\bar{w} \qquad 2.67b$$

plus the additional precession rate which is not integrable in closed form;

$$d\bar{S}/dt = \frac{2c_4 + c_5}{2}\,\frac{M}{R^3}\,\bar{R}\,\bar{w}\cdot\hat{R}\hat{R}\cdot\bar{S} \qquad 2.67c$$

These precessions have dimensional amplitude of 40 milliarcseconds (mas) for motion around the Sun on the Earth's orbit. But all these expressions (2.67b,c) have zero coefficients if gravity is Lorentz invariant (see (2.68)).

And generalizing the geodetic precession (2.32), a gyroscope in orbit will undergo the secular precession;

$$d\bar{S}/dt = \tfrac{1}{2}(c_1 + \tfrac{1}{2}c_7)\,\frac{M(\bar{R}\times\bar{V})}{R^3}\times\bar{S} \qquad 2.67d$$

along with a non-secular variation in spin axis in non-metric theories of gravity;

$$\delta\bar{s} = \tfrac{1}{2}\left(\frac{c_7}{2} - c_1 - \frac{2}{3}c_4 - 1\right)\bar{S}\cdot\bar{V}\bar{V} \qquad 2.67e$$

The placement of an accurate gyroscope in low orbit around the Earth would permit observing both the precession (2.66) and possible anomalous contributions (2.67a-e).[7]

Summarizing these results, there are the following constraints from various observables;

Equation	Constraint
2.6	$c_1 + c_2 + c_3 + 1 = 0$
2.9	$2(c_7 + c_8 + c_9 + c_{10}) - (c_4 + c_5 + c_6) - 3/2 = 0$
2.14	$c_7 + c_8 + c_9 + c_{10} = 0$
2.16	$c_7 + c_8 = 0$
2.19	$2(c_1 - c_3) + c_7 + c_9 - c_8 - c_{10} = 0$
2.23	$2c_1 + c_2 + 2(c_7 + c_9) = 0$
2.25	$c_7 + c_9 - 1 = 0$
2.28	$2c_4 + c_5 = 0$
2.51	$c_4 + c_5 + c_6 + 3/2 = 0$
2.58a	$1 + c_2 + 2c_3 + (c_5 + 2c_6)/3 = 0$
2.58b	$c_8 + c_{10} - (c_5 + 2c_6)/3 = 0$

All of the above observables are related to preferred frame ($\bar{w}$ dependent) effects. Two additional observables are independent of preferred frame effects;

2.33	$c_1 + \tfrac{1}{2}c_7 = 3$
2.42	$c_7 = 1 - \alpha - 2\Gamma$

The first eleven constraints are redundant, but self consistent; they are all fulfilled by the relationships;

$c_4 = 0$ (gauge or coordinate choice) 2.68a

$c_6 = -3/2 + c_4$ 2.68b

$c_5 = -2c_4$ 2.68c

$c_3 = 1 + c_1$ 2.68d

$c_2 = -2 - 2c_1$ 2.68e

c_1 arbitrary 2.68f

$c_8 = -c_7$ 2.68g

$c_9 = 1 - c_7$ 2.68h

$c_{10} = c_7 - 1$ 2.68i

c_7 arbitrary 2.68j

These constraints on the c_i guarantee that no physical effects proportional to $\bar{w}$ or $(\bar{w})^2$ will be observed in a system moving with respect to the cosmos, when that system is observed from its own rest frame.

Geodetic precession, as observed in the lunar orbit, then provides a constraint between c_1 and c_7 (2.33).[10] And absence of anomalous changes in the radius or period of satellite orbits as they follow the eccentric orbits of a planet around the Sun, specified c_7 in terms of the well measured speed of light function in a gravitational environment, and the gravitational time dilation of non-gravitational clocks (2.42). For $\alpha = 1$, $\Gamma = -2$ (2.33) and (2.42) yield the specific coefficient values;

$c_1 = 1$ 2.69a

$c_7 = 4$ 2.69b

Then all ten c_i are specified to have the numerical values as determined in Einstein's general relativity theory (in an appropriate gauge).

In particular, the coefficients which determine the observables related to the dragging of inertial frames --- (2.62), (2.66) and (2.67) --- are then numerically uniquely specified;

$$c_2 + c_8 = -2 - 2c_1 - c_7 = -8 \qquad \text{2.70a}$$

$$c_2 - c_8 + 2c_5/3 = 0 \qquad \text{2.70b}$$

From (2.70a) it is seen that the normal frame dragging coefficient ($c_2 + c_8$) is predicted from the geodetic precession coefficient (2.33) if in addition all preferred frame effects are observed to be absent.

III. Constraints on the Non-Linear Terms in the Gravitational Equation of Motion.

The last term of (2.1) represents the intrinsically non-linear, post-Newtonian corrections to linearized gravity, as well as acceleration dependent terms from linearized gravity in which the acceleration of various bodies have been set equal to their Newtonian values, thereby producing non-linear contributions to the equation of motion.

Expressing the non-linear part of (2.1) as a phenomenological expansion continues the philosophy of this investigation;

$$\begin{aligned}\sum_{jk} m_j m_j \bar{\rho}_{ijk} &= \sum_{jk} \frac{m_j m_k}{r_{ik}^3}\left[c_{11}\frac{\bar{r}_{ik}}{r_{ij}} + c_{12}\frac{\bar{r}_{ik}}{r_{jk}}\right] \\ &+ \sum_{jk} \frac{m_j m_k}{r_{jk}^3}\left[c_{13}\frac{\bar{r}_{jk}}{r_{ij}} + c_{14}\frac{\bar{r}_{jk}\cdot\hat{r}_{ij}\hat{r}_{ij}}{r_{ij}}\right] \\ &+ c_{15}\sum_{jk}\frac{m_j m_k}{r_{ik}^3}\bar{r}_{ik}\cdot\left[\frac{\hat{r}_{ij}\hat{r}_{ij}}{r_{ij}} - \frac{\hat{r}_{jk}\hat{r}_{jk}}{r_{jk}}\right] + \ldots\end{aligned} \qquad \text{3.1}$$

The sum over j and k can include body i, itself, in some cases. The only restriction is that the two body indices on one interbody vector can not be the same. A strong constraint on the type of terms that can be included in the phenomenological expansion (3.1) results from the observed isotropy of the Newtonian interaction. In (3.1) one can let body j or k become a distant body (spectator body) which then effectively renormalizes the strength of the two body interaction. (3.1) becomes in this spectator limit;

$$\delta\bar{a}_i = \frac{M_s}{R_s}(c_{11} + c_{12}) \sum_j \frac{m_j \bar{r}_{ij}}{r_{ij}^3} + \frac{M_s}{R_s} c_{15} \sum_j \frac{m_j \bar{r}_{ij}}{r_{ij}^3} \cdot \left(\frac{\bar{R}_s \bar{R}_s - \bar{R}_s \bar{R}_s}{R_s^2} \right) \tag{3.2}$$

If the c_{15} term was not the exact difference between two terms which canceled in the spectator limit, it would yield a Newtonian interaction between i and j which was dependent on the angle between $\bar{r}_{ij}$ and the spectator location $\bar{R}_s$. Considering the galaxy as the spectator body to the solar system ($M_s/R_s \approx 10^{-6}$), the observed isotropy of the Newtonian interaction to a part in 10^{13} strongly constrains (at the 10^{-7} level) the type of terms which can appear in (3.1).[11]

We first consider observables within the two body problem but which include non-linear contributions from (3.1). In this case (3.1) simplifies to;

$$\delta\bar{a}_i = c_{11} \frac{m_j^2 \bar{r}_{ij}}{r_{ij}^4} + (c_{12} - c_{13} - c_{14}) \frac{m_j m_i \bar{r}_{ij}}{r_{ij}^4} \tag{3.3}$$

The solar system mass is so dominated by the Sun's mass, the second term in (3.3) plays no role in any Sun-planet observables; c_{11} is the only non-linear coefficient measurable to good accuracy from observations of post-Newtonian corrections to planetary orbits about the Sun. From (3.3) and the rest of the equation of motion (2.1) planets or test bodies orbiting the Sun are governed by the equation of motion ($c_4 = 0$ gauge);

$$\bar{a} = -\frac{M\bar{r}}{r^3}\left[1 + c_1 v^2\right] + c_7 \frac{M\bar{r}\cdot\bar{v}\bar{v}}{r^3} + c_{11}\frac{M^2\bar{r}}{r^4} \qquad 3.4$$

(Observable 15) For a circular orbit (3.4) yields a modified Kepler's third law relationship between orbital period and orbital coordinate radius;

$$\omega^2 = \frac{M}{r^3}\left[1 + (c_1 - c_{11})\frac{M}{r}\right] \qquad 3.5$$

Observations by radar ranging between the inner planets will give the weak constraint;

$$c_1 - c_{11} = -3 \qquad 3.6$$

(Observable 16) For eccentric orbits (3.4) is most transparently solved in the $u(\theta) \equiv 1/r(\theta)$ representation, in which the orbital solutions are given by;

$$u(\theta) = u_o\{1 + e\cos\alpha\theta\} \qquad 3.7$$

with

$$\alpha = 1 - (c_1 + c_7 - \tfrac{1}{2}c_{11})mu_o \qquad 3.8$$

(3.7) represents a precessing elliptical orbit with post-Newtonian contribution to orbital precession of (per orbital period);

$$\delta\theta = (2c_1 + 2c_7 - c_{11})\pi m u_o \tag{3.9}$$

(3.9) can be best constrained by observation of Mercury's perihelion precession;[12]

$$2c_1 + 2c_7 - c_{11} = 6 \tag{3.10}$$

Taking the difference between (3.10) and (3.6) yields a relationship between the two key motional coefficients;

$$c_1 + 2c_7 = 9 \tag{3.11}$$

(Observable 17) We now turn to observables which intrinsically require three separate mass elements in their construction. The gravitational (passive) to inertial mass ratio of a celestial body (particularly the Earth) is an observable which can be measured to good precision.[13,14,15]

Considering a celestial body as a gas of particles in internal equilibrium, the gravitational self energy contributions to this ratio (gravitational mass/inertial mass) are calculable from (2.1) plus the non-linear interaction given by (3.1). Let a body of mass elements m_i be accelerated by external bodies M_k which produce a Newtonian acceleration field;

$$\bar{g} = \sum_k M_k \bar{R}_k / R_k^3 \tag{3.12}$$

The weighted sum

$$\sum_i m_i \bar{a}_i, \quad \bar{a}_i = \bar{a}_i(\text{internal}) + \bar{a}$$

is performed over the mass elements of the celestial body, and the tensor and scalar virial relations are used which are a consequence of internal equilibrium;

$$\sum_i m_i \bar{v}_i \bar{v}_i - \frac{1}{2} \sum_{ij} \frac{m_i m_j}{r_{ij}^3} \bar{r}_{ij} \bar{r}_{ij} = 0 \qquad 3.13a$$

$$\sum_i m_i v_i^2 - \frac{1}{2} \sum_{ij} \frac{m_i m_j}{r_{ij}} = 0 \qquad 3.13b$$

The body's collective acceleration is then given by;

$$\bar{a} = \left\{ 1 + \frac{1}{M} \left[\begin{array}{c} \left(\frac{c_1}{2} - c_{11} - c_{13} - c_{15}\right) \sum_{ij} \frac{m_i m_j}{r_{ij}} \\ \left(-\frac{c_7}{2} + \frac{c_{11}}{2} - \frac{c_{12}}{2} + c_{14} - 2c_{15} \right) \sum_{ij} \frac{m_i m_j}{r_{ij}^3} \bar{r}_{ij} \bar{r}_{ij} \end{array} \right] \right\} \bar{g} \qquad 3.14$$

$$+ 3c_{15} \sum_k \frac{m_k \bar{R}_k}{R_k^3} \sum_{ij} \frac{m_i m_j}{r_{ij}^3} (\bar{r}_{ij} \cdot \hat{R}_k)^2$$

The constraint required for the gravitational to inertial mass ratio of Earth to be one as observed in lunar laser ranging is then;[16,17]

$$3c_1 - c_7 - 5c_{11} - c_{12} - 6c_{13} - 2c_{14} - 4c_{15} = 0 \qquad 3.15a$$

Additionally there are the two weaker constraints required for that ratio to be one for a rotating celestial body;

$$-c_7 + c_{11} - c_{12} - 2c_{14} - 4c_{15} = 0 \qquad 3.15b$$

and

$$c_{15} = 0 \qquad 3.15c$$

(Observable 18) The active gravitational mass of a celestial body can also be evaluated from the phenomenological expansions (2.1) plus (3.1). For a gas model of a celestial body, the active mass is given by;

$$\begin{aligned} M_A = & \sum_j m_j \left[1 + \tfrac{1}{2} v_j^2 - \tfrac{1}{2} \sum_k \frac{m_k}{r_{jk}} \right] \\ & + \sum \frac{m_j m_k}{r_{jk}} (c_3 + c_{14} - 2c_{12} + \tfrac{1}{2}) \\ & + \tfrac{1}{2} \sum \frac{m_j m_k}{r_{jk}^3} \bar{r}_{jk} \bar{r}_{jk} (c_{14} + 2c_{15} - c_{10} - c_{13}) \\ & + \tfrac{1}{2} \sum \frac{m_j m_k}{r_{jk}^3} (\bar{r}_{jk} \cdot \hat{R})(c_6 - 3c_{14}) \end{aligned} \qquad 3.16$$

in which $\hat{R}$ is the unit vector toward the test body measuring the active mass. If the active mass is to equal the energy content (first line of (3.16)) of the body, three constraints are required;

$$c_3 + c_{14} - 2c_{12} + \tfrac{1}{2} = 0 \qquad 3.17a$$

$$c_{14} + 2c_{15} - c_{10} - c_{13} = 0 \qquad 3.17b$$

and

$$c_6 - 3c_{14} = 0 \qquad 3.17c$$

Unfortunately, there are no solar system observations accurate enough to enforce these constraints.

(Observable 19) The perturbations of the orbit of an earth satellite are of interest. The Earth's orbit about the Sun and the satellite's orbit about the Earth are assumed to be nominally circular and coplanar. The relevant perturbations of the satellite relative to Earth are then;

$$\delta\bar{a} = -(2c_1 + c_2)\frac{m\bar{r}\bar{u}\cdot\bar{v}}{r^3} + (c_7 + c_9)\frac{m\bar{r}\cdot\bar{v}u}{r^3}$$
$$- c_1\frac{M\bar{R}u^2}{R^3} + c_7\frac{M\bar{R}\cdot\bar{u}\bar{u}}{r^3} + (c_{11} + c_{13} + c_{15})\frac{m}{r}\frac{M\bar{R}}{R^3} \tag{3.18}$$
$$- (c_{11} - c_{14} - 2c_{15})\frac{Mm\bar{R}\cdot\bar{r}\bar{r}}{R^3r^3} - 3c_{15}\frac{M\bar{R}}{R^3}\frac{m(\bar{r}\cdot\hat{R})^2}{r^3}$$

The resulting radial perturbations of the satellite orbit are then given by;[18,19]

$$\delta r(t) = \frac{\cos(\omega - \Omega)t}{(1 - \Omega/\omega)(\omega_o^2 - (\omega - \Omega)^2)} \cdot \left[\begin{array}{c} -(2c_1 + c_2 + 2c_7 + 2c_9)\dfrac{m}{r^2}uv \\ + \left[\begin{array}{c}(3c_{13} + 2c_{11} + c_{14} + \dfrac{5}{4}c_{15} - c_1 + c_2 + 2c_7) \\ -\dfrac{\Omega}{\omega}(c_{13} + c_{14} - c_1 + \dfrac{3}{4}c_{15})\end{array}\right]\dfrac{m}{r}\dfrac{M}{R^2}\end{array}\right] \tag{3.19}$$

$\bar{R}$ and $\bar{r}$ are the positions of Earth from the Sun and satellite from the Earth, respectively. $v = R\Omega$ and $u = r\omega$ are the speeds of the Earth relative to the Sun and the satellite relative to the Earth, respectively. Ω and ω are the orbital angular frequencies of the Earth and satellite, respectively. ω_o is the frequency of perigee of the satellite orbit. M and m are the masses of the Sun and Earth, respectively.

The two largest contributions to (3.19) are dimensionally larger than the experimental accuracy with which lunar laser ranging (and ranging to low earth satellite orbits) data is fitted by the general relativistic model; hence there are the two constraints;

$$2c_1 + c_2 + 2c_7 + 2c_9 = 0 \tag{3.20a}$$

and

$$3c_{13} + 2c_{11} + c_{14} + 5c_{15}/4 - c_1 + c_2 + 2c_7 = 0 \tag{3.20b}$$

The first of the above constraints is the same as (2.23).

All of the constraints presented in this section on the non-linear gravitational interaction are satisfied with the coefficients in (3.1) taking the general relativistic values;

$$c_{11} = 4 \tag{3.21a}$$

$$c_{12} = 1 \tag{3.21b}$$

$$c_{13} = -7/2 \tag{3.21c}$$

$$c_{14} = -1/2 \tag{3.21d}$$

$$c_{15} = 0 \tag{3.21e}$$

along with the first ten c_i taking the values given in section II, which are also the general relativistic values. However several of the constraints of this section are too weakly specified by observation, so that all five of the non-linear coefficients are not determined by solar system observations alone. We do obtain the weak constraint (3.11) between c_1 and c_7 though through consideration of these observations involving non-linear aspects of the gravitational interaction.

IV. Experimental Accuracy of Constraints on Observables

The models used to fit solar system observations are not as general as (2.1); they will generally assume at least that gravity is a metric field based interaction. Therefore, a number of the observables calculated in this study will not explicitly have been fitted in past analysis of observational data. So in this section we will sometimes have to infer what experimental accuracy would be if existing data were fit for the observables within our model.

We give an overview of the experimental accuracy of the various observable constraints derived in this study. We first consider two very high accuracy observations which give us the strong constraints;

$$c_7 + c_8 + c_9 - c_8 = 0 \pm 10^{-7} \qquad 4.1$$

and

$$2(c_1 - c_3) + c_7 + c_9 - c_8 - c_{10} = 0 \pm 10^{-6} \qquad 4.2$$

(4.1) results from the observed isotropy of the Newtonian gravitational interaction between the mass elements of the Sun: if (4.1) was not fulfilled, the Sun's spin axis would have precessed out of solar system alignment over the past 4.5 10^9 years.[11] (4.2) is the result of a possible self-acceleration of a spinning celestial body;

$$\bar{a} = \frac{1}{3}\left(\frac{U}{M}\right)\bar{w} \times \bar{\Omega}\left[2(c_1 - c_3) + c_7 + c_9 - c_8 - c_{10}\right] \qquad 4.3$$

U is the body's gravitational self energy, $\bar{\Omega}$ is its rotational angular frequency. Application of this to the Earth, as observed with lunar laser ranging data, leads to the constraint (4.2).[20]

The constraint (2.23) is fairly strong;

$$2c_1 + c_2 + 2c_7 + 2c_9 = 0 \pm 10^{-3} \qquad 4.4$$

and results from absence of anomalous range oscillations in low Earth satellite orbits (3.19) as measured by laser ranging.

The absence of gravimeter anomalies on Earth larger than about 10^{-7} gal give the moderately strong constraints;[20,21]

$$c_4 + c_5 + c_6 + 3/2 = 0 \pm 10^{-3} \qquad 4.5a$$

$$1 + 2c_2 + 2c_3 + (c_5 + 2c_6)/3 = 0 \pm 10^{-2} \qquad 4.5b$$

$$c_8 + c_{10} - (c_5 + 2c_6)/3 = 0 \pm 10^{-2} \qquad 4.5c$$

The constraint (2.16) is quantified by the upper limits on anomalous secular or semisecular changes in Earth's orbital period;

$$c_7 + c_8 = 0 \pm 10^{-3} \tag{4.6}$$

While (2.25) is required to suppress longitudinal anomalies in the inner planet position greater than a few tenths of a kilometer, and as measured by radar ranging;

$$c_7 + c_9 = 1 \pm 10^{-2} \tag{4.7}$$

Finally (2.6) is quantified by the absence of anomalous period changes in earth satellite orbits;

$$c_1 + c_2 + c_3 + 1 = 0 \pm 10^{-3} \tag{4.8}$$

The solution (2.68) for the motional coefficients is therefore accurate to about 1 percent, with some of the relationships much stronger.

The geodetic precession observation of the Moon's orbit is, however, only about 10 percent accurate.[10] So without the use of other observables involving non-linear aspects of the gravitational interaction, the dragging of inertial frames proportional to $c_2 + c_8$ can be predicted to only about 10 percent accuracy. In fact, the inclusion of the non-linear observables in section III does not improve the accuracy with which c_1 and c_7 can be determined. This is because a number of new coefficients must be introduced to express the general non-linear interaction (3.1); and the number of independent observables of high accuracy in the solar system does not increase sufficiently to both constrain the new coefficients in (3.1) as well as c_1 and c_7.

V. Conditions on the Equation of Motion for Momentum Conservation.

The phenomenological gravitational many body equation of motion (2.1) as it stands in its generality does not possess a conservation law for total momentum of an isolated system of bodies. However, certain constraint relationships among the coefficients c_i will indicate a momentum conservation law. Let an isolated system's total conserved momentum be given by an expression of the general form;

$$\bar{P} = \sum_i \left\{ m_i + p_1 m_i v_i^2 + p_2 \sum_j \frac{m_j}{r_{ij}} + p_3 \sum_j \frac{m_j}{r_{ij}^3} (\bar{r}_{ij})(\bar{r}_{ij}) \right\} \bar{v}_i \qquad 5.1$$

with $p_{1,2,3}$ being three dimensionless constants. Then $d\bar{P}/dt = 0$ yields;

$$\begin{aligned} 0 = & \sum_i m_i \bar{a}_i + p_1 \sum_i (2m_i \bar{v}_i \cdot \bar{a}_i \bar{v}_i + m_i v_i^2 \bar{a}_i) \\ & + p_2 \sum_{ij} \left(\frac{m_i m_j}{r_{ij}} \bar{a}_i - \frac{m_i m_j}{r_{ij}^3} \bar{r}_{ij} \cdot \bar{v}_{ij} v_i \right) \\ & + p_3 \sum_{ij} \frac{m_i m_j}{r_{ij}^3} \left(\bar{r}_{ij} \bar{r}_{ij} \cdot \bar{a}_i + r_{ij} \bar{v}_{ij} \cdot \bar{v}_i + \bar{v}_{ij} \bar{r}_{ij} \cdot \bar{v}_i - 3\bar{r}_{ij} \bar{v}_i \cdot \hat{r}_{ij} v_{ij} \cdot \hat{r}_{ij} \right) \end{aligned} \qquad 5.2$$

Using (2.1) and (3.1) to express the first term in (5.2), and elsewhere using the Newtonian acceleration field to represent $\bar{a}_i$ in post-Newtonian terms of (5.2), *i.e.*;

$$\bar{a}_i = \sum_k m_k \bar{r}_{ki}/r_{ki}^3$$

some of the coefficients c_i in (2.1) and (3.1) are then constrained to relationships involving the three constants p_i.

$$c_1 - c_3 = -p_1 + p_3 \tag{5.3a}$$

$$c_4 - c_6 = -3p_3 \tag{5.3b}$$

$$c_7 - c_{10} = 2p_1 + p_2 - p_3 \tag{5.3c}$$

$$c_8 - c_9 = p_2 + p_3 \tag{5.3d}$$

$$c_{11} - c_{12} + c_{13} = p_2 \tag{5.3e}$$

$$c_{14} + 2c_{15} = p_3 \tag{5.3f}$$

If a center of energy exists which is to move most generally at constant velocity;

$$\bar{R} = \frac{\sum_i m_i\left(1 + \frac{1}{2} v_i^2 - \frac{1}{2}\sum_j \frac{m_j}{r_{ij}}\right)\bar{r}_i}{\sum_i m_i\left(1 + \frac{1}{2} v_i^2 - \frac{1}{2}\sum_j \frac{m_j}{r_{ij}}\right)} \tag{5.4}$$

with

$$d\bar{R}/dt = \bar{P}/E$$

and therefore

$$d^2\bar{R}/dt^2 = 0$$

then

$$p_1 = -p_2 = -p_3 = 1/2 \tag{5.5}$$

and the constraints (5.3) are then even more restrictive. Not surprisingly, the momentum conservations constraints (5.3) automatically lead to the vanishing of the observable (2.19) associated with the possible self-acceleration of the center of mass of a gravitational system.

VI. Conditions on the Gravitational Equations of Motion to be Metric Field Derivable.

The phenomenological gravitational many body equations of motion (2.1) plus (3.1) are more general than what can be obtained from a metric theory of gravity. In metric theories there are several potential functions --- a symmetric $g_{\mu\nu}(\bar{r},t)$ --- which for test bodies yield the equation of motion;

$$d(\partial L/\partial \bar{v})/dt - \partial L/\partial \bar{r} = 0 \qquad 6.1a$$

with

$$L(\bar{r},t) = -\sqrt{g_{\mu\nu}(\bar{r},t)\, dx^{\mu}/dt\, dx^{\nu}/dt} \qquad 6.1b$$

A general expansion for the metric field potential contains seven free parameters;

$$g_{oo} = 1 - 2U + M_1 \sum_j \frac{m_j v_j^2}{|r - r_j|} + M_2 \sum_j \frac{m_j((\hat{r} - \hat{r}_j) \cdot \bar{v}_j)^2}{|\bar{r} - \bar{r}_j|} + M_6 U^2 + M_7 \sum_{jk} \frac{m_j m_k}{|\bar{r} - \bar{r}_j|} \frac{1}{r_{jk}} + \ldots \qquad 6.2a$$

$$g_{oi} \equiv \bar{h} = M_3 \sum_j \frac{m_j \bar{v}_j}{|\bar{r} - \bar{r}_j|} + M_4 \sum_j \frac{m_j}{|\bar{r} - \bar{r}_j|^3} (\bar{r} - \bar{r}_j)(\bar{r} - \bar{r}_j) \cdot \bar{v}_j \qquad 6.2b$$

$$g_{ij} = -(1 - M_5 U)\delta_{ij} \qquad 6.2c$$

with

$$U(\bar{r},t) = \sum_j \frac{m_j}{|\bar{r} - \bar{r}_j|} \qquad 6.2d$$

Four coordinate gauge choices were available to set some of the metric terms generally equal to zero: the spatial g_{ij} is kept diagonal and isotropic by a spatial coordinate transformation;

$$\bar{r} \rightarrow \bar{r} + \xi \sum_k m_k(\bar{r} - \bar{r}_k)/|\bar{r} - \bar{r}_k| \qquad 6.3a$$

while an acceleration dependent term in g_{oo} is generally eliminated by a time transformation;

$$t \rightarrow t + \chi \sum_k m_k(\bar{r} - \bar{r}_k) \cdot \bar{v}_k / |\bar{r} - \bar{r}_k| \qquad 6.3b$$

Since there are fewer free parameters in (6.2) than there are coefficients c_i in (2.1) plus (3.1), constraints on the c_i are necessary in order that the equation of motion is derivable from a metric field of general form. Using (6.2) in the Euler-Lagrange equation (6.1) yields an equation of motion whose terms can be compared to the phenomenological equation of motion (2.1) plus (3.1). The constraints on the c_i necessary for a metric field based equation of motion are then;

$$c_1 = -M_5/2 \qquad 6.4a$$

$$c_2 = -(M_3 + M_4) \qquad 6.4b$$

$$c_3 = M_4 - M_1/2 \qquad 6.4c$$

$$c_4 = c_5 = 0 \qquad 6.4d$$

$$c_6 = -3(M_2/2 + M_4) \qquad 6.4e$$

$$c_7 = 2 - M_5 \qquad 6.4f$$

$$c_8 = -(M_3 + M_4) \qquad 6.4g$$

$$c_9 = M_5 - 1 \qquad 6.4h$$

$$c_{10} = M_3 - M_4 - M_2 \qquad 6.4i$$

$$c_{11} = M_6 - M_5 + 1 \qquad 6.4j$$

$$c_{12} = M_7/2 \qquad 6.4k$$

$$c_{13} = -M_3 \qquad 6.4l$$

$$c_{14} = -M_4 \qquad 6.4m$$

$$c_{15} = 0 \qquad 6.4n$$

Several of the observables are determined by the fulfillment of the above metric constraints. For example, (2.25) and (2.28) are automatically fulfilled if (6.4) hold. Also the anomalous gyroscope precession (2.67a) vanishes if gravity is derivable from a metric field.

VII. Invariance of Motional Observables Under Coordinate (Gauge) Transformations.

The N body coordinates $\bar{r}_i$ used in (2.1) can be combined to form N new, independent coordinates associated with the motion of the N bodies. The equations of motion (2.1) and (3.1) would of course be altered under these coordinate transformations. In particular, if the transformation;

$$\bar{r}_i' = \bar{r}_i + \xi \sum_{k \neq i} m_k \bar{r}_{ik}/r_{ik} \qquad 7.1$$

is made, the ten c_i in (2.1) are altered by;

$$c_1 \rightarrow c_1 + \xi \qquad 7.2a$$

$$c_2 \rightarrow c_2 - 2\xi \qquad 7.2b$$

$$c_3 \rightarrow c_3 + \xi \qquad 7.2c$$

$$c_4 \rightarrow c_4 - 3\xi \qquad 7.2d$$

$$c_5 \rightarrow c_5 + 6\xi \qquad 7.2e$$

$$c_6 \rightarrow c_6 - 3\xi \qquad 7.2f$$

$$c_7 \rightarrow c_7 - 2\xi \qquad 7.2g$$

$$c_8 \rightarrow c_8 + 2\xi \qquad 7.2h$$

$$c_9 \rightarrow c_9 + 2\xi \qquad 7.2i$$

$$c_{10} \rightarrow c_{10} - 2\xi \qquad 7.2j$$

The form of $\bar{\rho}_{ijk}$ changes also. The coordinate speed of light function is altered, a time derivative of (7.1) giving in first order approximation;

$$c(\bar{r},t,\hat{c}) \rightarrow c(\bar{r},t,\hat{c}) + \xi\left(U - \sum_k m_k((\bar{r} - \bar{r}_k) \cdot \hat{c})^2 / |\bar{r} - \bar{r}_k|^3\right)$$

One is therefore always free to choose a gauge or set of coordinates which sets one of the $c_i = 0$. However, the observables which were derived in section II are all invariant under this transformation. Setting $c_4 = 0$ is the conventional choice of coordinates, as this leads to a simple scaling relationship between the spatial global coordinates and the local proper spatial coordinates measured, in particular, by light ray propagation;

$$\delta\bar{r}_p = (1 + U(\bar{r},t))\delta\bar{r} \tag{7.3}$$

It is important to note that the speed of light function is isotropic (independent of $\hat{c}$ --- $\Gamma' = 0$)

$$c(\bar{r},t) = 1 - 2\sum_j m_j / |\bar{r} - \bar{r}_j| + \Gamma' \sum_j m_j((\bar{r} - \bar{r}_j)\cdot\hat{c})^2 / |\bar{r} - \bar{r}_j|^3$$

in the same coordinate system that (7.3) is valid.

VIII. Gyroscope Precession.

In our general, phenomenological model for the gravitational equation of motion of matter, we derived several possible contributions to the precession rate of the spin axis of an orbiting gyroscope. There are the two well known secular terms which are non-zero in general relativity;

$$d\bar{S}/dt = \tfrac{1}{2}(c_1 + \tfrac{1}{2}c_7)\, m\, (\bar{r} \times d\bar{r}/dt)/r^3 \times \bar{S} \tag{8.1}$$

$$+ (c_2 + c_8)/8\, (\bar{J} - 3\bar{J}\,\hat{r}\hat{r})/r^3 \times \bar{S} \tag{8.2}$$

(8.1) is the geodetic precession contribution, while (8.2) is the "frame dragging" precession due to the Earth's spin angular momentum $\bar{J}$. (In general relativity $c_1 = 1$, $c_7 = -c_2 = -c_8 = 4$).

There is an additional secular precession in <u>non-metric</u> theories of gravity;

$$d\bar{S}/dt = -\frac{3}{16}\left(c_2 - c_8 + \frac{2}{3}c_5\right)\frac{\bar{J}\times\hat{p}\hat{p}\cdot\bar{S} + \bar{J}\times\hat{p}\cdot\bar{S}\hat{p}}{a_o^3(1 - e^2)^{3/2}} \tag{8.3}$$

a_o, e and $\hat{p}$ being the gyroscope orbit's semi-major axis, eccentricity and unit polar vector, respectively.

Non-secular precessions of sufficient magnitude to be measured are also of interest. In gravitational theories which have a preferred inertial frame (absence of Lorentz invariance), there are some possible changes in the gyroscope spin axis;

$$\begin{aligned}\delta\bar{S} &= \tfrac{1}{2}(4 - 3c_7 - c_8 - 2c_9 + 2c_1 + c_2 + 2c_4 + c_5)\,\bar{V}\cdot\bar{w}\,\bar{S} \\ &+ (2 + 2c_1 + c_2 + c_7 + c_8)/4\,(\bar{w}\times\bar{V})\times\bar{S} \\ &+ (c_7 + c_8 - 2 - 2c_1 - c_2)/4\,(\bar{S}\cdot\bar{V}\,\bar{w} + \bar{S}\cdot\bar{w}\,\bar{V})\end{aligned} \qquad 8.4a$$

and the additional precession rate (not integrable in closed form);

$$d\bar{S}/dt = \tfrac{1}{2}(2c_4 + c_5)\,M\,\bar{S}\cdot\hat{R}\,\hat{R}\cdot\bar{w}\,\bar{R}/R^3 \qquad 8.4b$$

$\bar{w}$ is the velocity of the solar system with respect to the cosmological preferred inertial frame, and $\bar{V} = d\bar{R}/dt$ is the velocity of the gyroscope with respect to the Sun, which has mass M. These precessions (8.4a,b) are absent in general relativity, but dimensionally are of magnitude 80 milliarcseconds and have an annual period.

Finally there is a small non-secular variation in gyroscope spin axis which only exists in non-metric theories of gravity;

$$\bar{S} = \tfrac{1}{2}(\tfrac{1}{2}c_7 - c_1 - 2c_4/3 - 1)\,\bar{S}\cdot\bar{V}\,\bar{V} \qquad 8.5$$

which is dimensionally only of order 2 milliarcseconds.

It should be pointed out that the geodetic precession (8.1), though dominated by its earth orbiting contribution ($\sim 3\ 10^3$ mas/yr), also has a contribution (~ 13 mas/yr) from the orbital motion of the gyroscope about the Sun.

As was shown in previous sections, all the coefficients in the motional, post-Newtonian gravitational interaction --- including the gravitomagnetic coefficients --- are measured to some precision by other observations in the solar system which do not involve gyroscope precession. Therefore the precession rates can be predicted with accuracies of about 10 percent, or perhaps slightly better. One could then view the gyroscope precession observations as a way of substantially improving the measurement of the coefficient combination in (7.1) --- $c_t + \frac{1}{2}c_7$.

If the gyroscope precessions are found to be different than predicted, we believe this would present a major crisis for gravitational theory. The model (2.1) is quite general, especially with regard to the motional, two-body post-Newtonian interaction which includes gravitomagnetism. Given two mass elements m_i, m_j located at relative coordinates $\bar{r}_{ij} = \bar{r}_i - \bar{r}_j$, and each having velocity $\bar{v}_i, \bar{v}_j$, respectively, we have found it impossible to generalize a post-Newtonian acceleration expression beyond the form given by (2.1). Yet this would seem to be necessary if redundant observations could not all be predicted by the nine coefficient model (2.1).

Since the spin angular momentum of the gyroscopes to be placed into orbit is motional angular momentum --- not quantum spin angular momentum --- there would appear to be no way to explain unpredicted precession by means of some anomalous coupling of gravity to spin. For macroscopic gyroscopes, spin angular momentum is simply a manifestation of particle motion, superimposed over all the matter in the gyroscope.

REFERENCES

1. Einstein, A., 1916, Ann. Phys., Lpz. 49.
2. Mach, E., (1893) in "The Sciences of Mechanics" 6th English Ed. 1960 (Open Court, Lasalla, Illinois).
3. Nordtvedt, K., 1988, Int. J. Theor. Phys. 27, 1395.
4. Will, C. and Nordtvedt, K., 1972, Astrophys. J. 177, 757.
5. Reasenberg, R.D., 1979, Astrophys. J. 234, L219.
6. Fomalout, E.B., 1976, Phys. Rev. Lett. 36, 1475.
7. Turneaure, J. Petal, 1983, Phys. Rev. D 27, 1705.
8. Vesttot, R.F.C., 1982, Phys. REv. Lett., 45, 26.
9. Everitt, C.W.F., 1988, in "Near Zero", (W.H. Freeman, San Francisco), Section VI.3.
10. Bertotti, B., Ciufolini, I., Bender, P., 1987, Phys. Rev. Lett. 58, 1062.
11. Nordtvedt, K., 1987, Astorphys. J. 320, 871.
12. Shapiro II, *et al*, 1976, Phys. Rev. Lett. 36, 555.
13. Nordtvedt, K., 1968, Phys. Rev. 170, 1186.
14. Williams, J., *et al*, 1976, Phys. Rev. Lett. 36, 551.
15. Shapiro, I.I., *et al*, 1976, Phys. Rev. Lett. 36, 555.
16. Nordtvedt, K., 1982, Rep. Prog. Phys., 45, 631.
17. Nordtvedt, K., 1968, Phys. Rev. 169, 1017.
18. Nordtvedt, K., 1988, Phys. Rev. Lett. 61, 2647.
19. Nordtvedt, K., 1991, Phys. Rev. (in press).
20. Nordtvedt, K. and Will, C., 1972, Astrophys. J. 177, 775.
21. Warburton, R.J., and Goodkind, J.M., 1976, Astrophys. J. 208, 881.