COMBINED STRUCTURES-CONTROLS OPTIMIZATION OF LATTICE TRUSSES

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Research supported in part under NASA Task Assignment 49: NAS1-18585.

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ABSTRACT

The purpose of this paper is to demonstrate concretely the role that distributed parameter models can play in CSI, in particular in combined structures-controls optimization problems of importance in preliminary design. Closed form solutions can be obtained for performance criteria such as rms attitude error, making possible analytical solutions of the optimization problem. This is in contrast to the need for numerical computer solution involving the inversion of large matrices in traditional finite element model use. Another advantage of the analytic solution is that it can provide much needed "insight" into phenomena that can otherwise be obscured — or difficult to discern from numerical computer results.

As a compromise in level of complexity between a toy laboratory model and a real space structure we have chosen the lattice truss used in the EPS (Earth Pointing Satellite). The optimization problem chosen is a generic one: of minimizing the structure mass subject to a specified stability margin and to a specified upper bound on the rms attitude error ("tip response"), using co-located controller and sensors. Standard FEM treating each bar as a truss element is used, while the continuum model is anisotropic Timoshenko beam model. Performance criteria are derived for either model, except that for the distributed parameter model we obtain explicit closed form solutions. Numerical results obtained by the two models show complete agreement. Based on the continuum model we obtain a solution to the problem of optimal placement of actuators to minimize mean square attitude error. A canonical optimization problem is examined and shown to be trivial, and even capable of analytical solution, using the continuum model performance criteria formulas in contrast to the complex computer solutions based on FEM or truncated modal models currently in vogue.

Introduction

The most voiced criticism against the use of continuum models for structures is that they are (a) impossible to derive for a realistic structure and (b) even if it could be done, calculations using the model are equally impossible. We shall show that both statements are false — at least in the CSI optimization problem — in particular in preliminary design.

In combined controls-structures optimization, the optimization is the least difficult — the real challenge is to derive expressions for the chosen performance criteria in terms of the controls/structures parameters. We shall show that such formulas are much simpler when continuum models are used — moreover in many cases we can derive explicit closed form expressions in terms of elementary function which can actually trivialize the optimization problem. In particular the techinques of optimization need no longer dominate.

The purpose of this paper is to demonstrate concretely the role that distributed parameter models can play in CSI, in particular in combined structures-controls optimization problems of importance in preliminary design. Closed form solutions can be obtained for performance criteria such as rms attitude error, making possible analytical solutions of the optimization problem. This is in contrast to the need for numerical computer solution involving the inversion of large matrices in traditional finite element model use. Another advantage of the analytic solution is that it can provide much needed "insight" into phenomena that can otherwise be obscured — or difficult to discern from numerical computer results.

As a compromise in level of complexity between a toy laboratory model and a real space structure we have chosen the lattice truss used in the EPS (Earth Pointing Satellite). This is described in Section 1. The optimization problem chosen is a generic one: of minimizing the structure mass subject to a specified stability margin and to a specified upper bound on the rms attitude error ("tip response"). The mathematical statement of the performance criteria is given in Section 2. The first step is to evaluate the performance criterion for a given control configuration — we consider co-located sensor/controls only. The finite element model is described in Section 3, and the continuum model in Section 4. The dynamic state space model is seen to be identical in both cases except for state space dimension. Section 5 derives the performance criteria valid for either model, except that for the distributed parameter model we obtain explicit closed form solutions. Section 6 compares the numerical results obtained by the two models, showing complete agreement. As a byproduct of our analysis, we obtain a solution to the problem of optimal placement of actuators to minimize mean square attitude error — in Section 7. An optimization problem *per se* — a canonical one — is treated in Section 8 and by virtue of our explicit formulas for performance indices in terms of structure/control parameter, shown to be "trivial" and even capable of analytical solution — in contrast to computer solutions using FEM or truncated modal models as in [9, 10].

We should note that structural engineers (Noor, *et al.* [1, 2] and Renton [3]) have already voiced the advantages of continuum models in preliminary structure design — what is new here is the application to control design, to the Controls-Structures Interaction problem.

1. The Physical Article

The physical structure (Figure 1) is a lattice of rectangular bays, each single-laced single-bay. Offset at each end is an antenna. The controllers are force and moment actuators with co-located attitude as well as rate sensors stationed at arbitrary locations along the structure. Table 1 is a breakdown of the parameters describing each bay.

TABLE 1
Element Properties

	Battens	Longitudinal Bars	Diagonal Bars	Cross Bracing in Battens
Length L	b	ℓ	d	δ
Sectional Area A	A_b	A_ℓ	A_d	A_δ
Elastic Modulus E	E_b	E_ℓ	E_d	E_δ
Mass Density m	ρ_b	ρ_ℓ	ρ_d	ρ_δ
Element Mass = ρAL	m_b	m_ℓ	m_d	m_δ
Element Stiffness = EA/L	S_b	S_ℓ	S_d	S_δ

The beam geometry is shown in Figure 2. By the "nominal" structure, we shall mean the following choice of structural parameters:

$$b = \ell ; \qquad d = \delta = \sqrt{2}\,\ell$$

$$A_b = A_\ell = A_d = A_\delta = A$$

$$= E_b = E_\ell = E_d = E_\delta = E$$

$$= \rho_b = \rho_\ell = \rho_d = \rho_\delta = \rho$$

$$L = n\ell ;$$

Nominal values:

$$n = 9$$

$$\ell = 3 \text{ m}$$

$$\rho = 3250$$

$$E = 2.759 \times 10^{11}$$

$$A = 2.468 \times 10^{-4} \text{m}^2 .$$

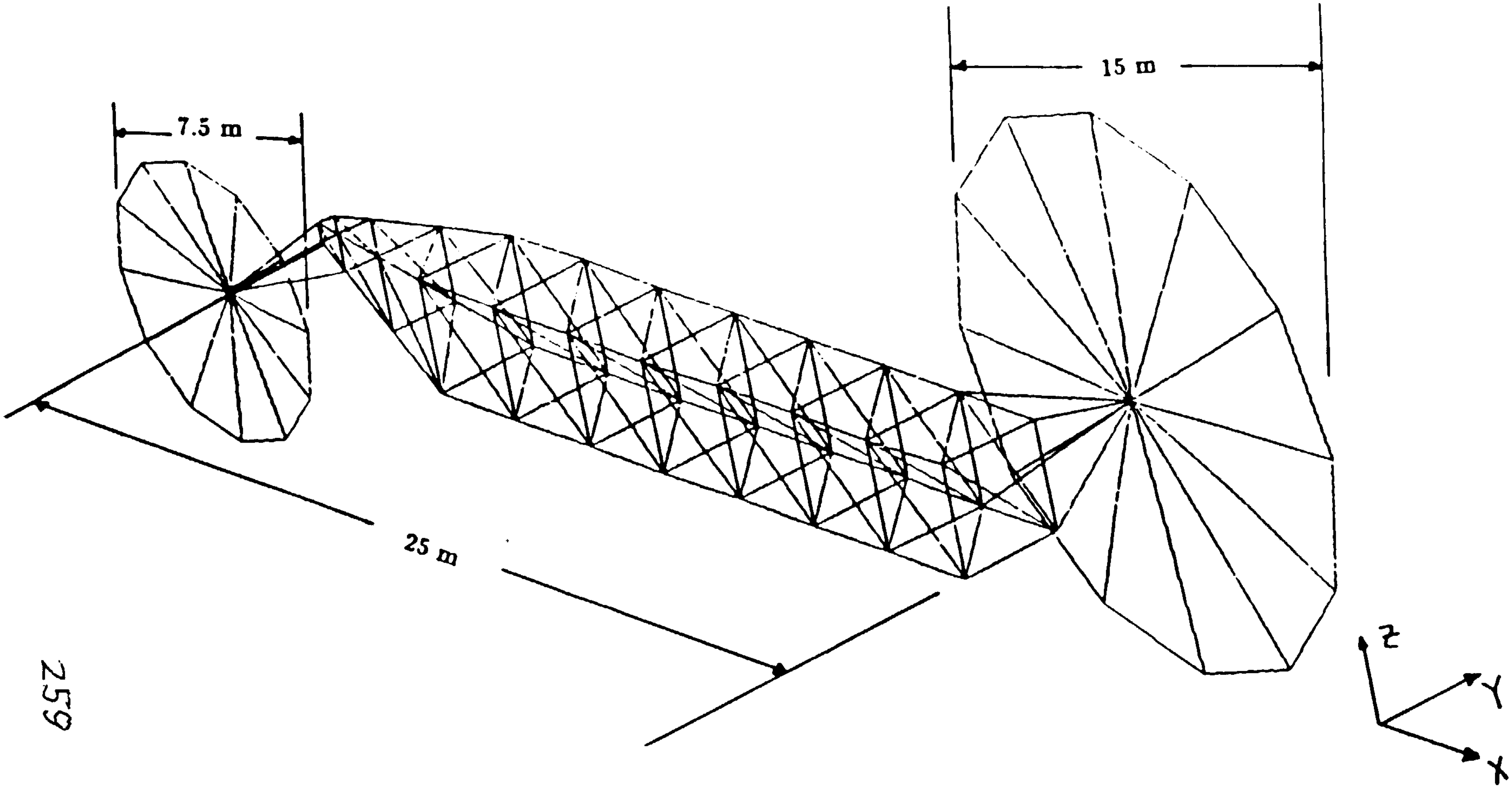

Figure 1. Earth Pointing Satellite (EPS)

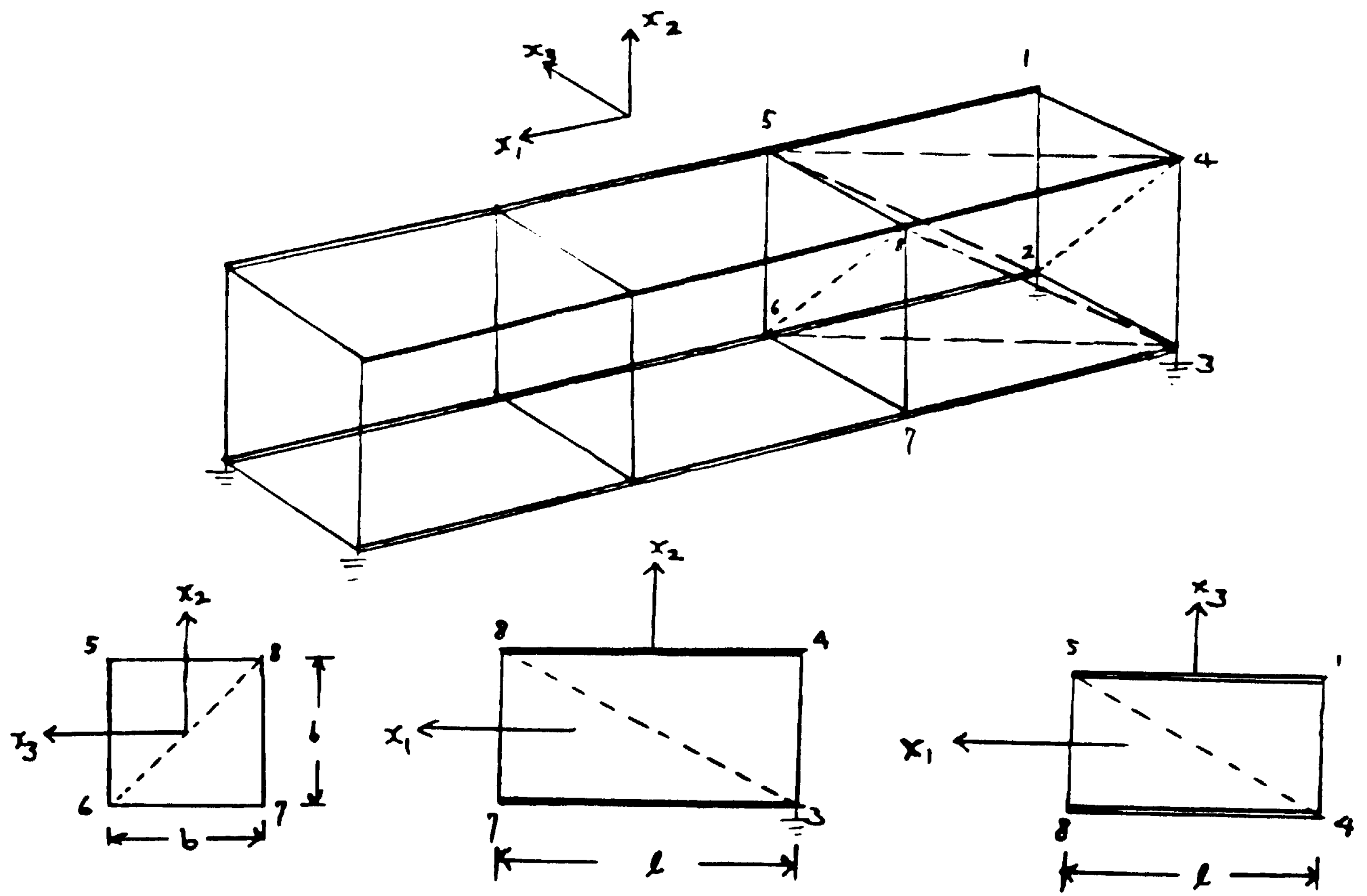

Figure 2. Beam Geometry

2. Performance Criteria

As in previous CIS optimization studies (see [9] and the references therein) the objective is to minimize the total mass of the structure, including the controller mass, subject to meeting specified performance requirements, which we shall now describe in mathematical terms.

The performance criteria chosen are:

(i) the mean-square attitude error due to sensor noise (using co-located sensors/ actuators). As we shall see, this actually depends on the steady state "tip" response to step inputs so that "noise" notions can be eschewed if necessary.

(ii) the "stability margin": defined as the sum of the absolute values of the real parts of the closed-loop eigenvalues. This is one measure of stability among very many (see, e.g., [10]). We choose this one because it is essentially equivalent to any other one but has the advantage that we can derive a simple closed-form expression for it.

Let us now define the criteria more precisely. First of all we assume the control law to be PD ("proportional plus derivative") as in classical servo design. Let $v_p(t)$ denote the "displacement" or "attitude" vectors at the sensor locations and let $v_r(t)$ the rate. Then $U(t)$ the control is defined to be

$$U(t) = \alpha v_p(t) + \gamma v_r(t) \tag{2.1}$$

where α and γ are (positive) scalar "gains." This is not altogether a "simplifying" assumption — that the scalar rate feedback is actually optimal is shown in [4]. The beam-axis being the x_1-axis, we have, with L denoting the beam-length:

$$0 \leq x_1 \leq L .$$

Let $f(0), f(L)$ denote the 6-DOF displacement vectors at the ends. Let

$$f(0) = \begin{vmatrix} u(0) \\ v(0) \\ w(0) \\ \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{vmatrix} ; \quad f(L) = \begin{vmatrix} u(L) \\ v(L) \\ w(L) \\ \phi_1(L) \\ \phi_2(L) \\ \phi_3(L) \end{vmatrix} \tag{2.2}$$

where u, v, w are the 3-DOF displacements, ϕ_1 the torsion angle about the beam axis, and ϕ_2, ϕ_3 about the mutually perpendicular axes. Then the mean square attitude error is defined by

$$\sigma_a^2 = \overline{u(0)^2 + u(L)^2} + \overline{v(0)^2 + v(L)^2} + \overline{w(0)^2 + w(L)^2}$$
$$+ |r_0|^2 \overline{\phi_1(0)^2} + |r_L|^2 \overline{\phi_1(L)^2} , \tag{2.3}$$

where the bars denote time averages, $|r|$ being the length of the moment arm as required.

Under our feedback law (often referred to as "positive-definite" feedback) the closed-loop system is guaranteed to be stable. Assuming no damping in th structure (as we shall indeed do), the real parts of the closed-loop eigenvalues are guaranteed negative (see Section 5) — or if we assume the structure is already damped we have stability enhance ment corresponding to the increment, the real parts being now more negative. Let σ_i denote the real part increment corresponding to the ith closed-loop mode. Then the infinite series

$$\sum_i^{\infty} -\sigma_i \tag{2.4}$$

converges. Denote the sum by σ_s. We shall take this to be the "stability index" — the higher the index, the more stable.

3. The Finite Element Model

Since most of the techniques in developing the FEM are standard, we shall only present the relevant numerical data. Each bar is taken as a truss element with 6 DOF. There are $(13+5)$ elements per bay, and hence $(13\times 9 + 5) = 122$ elements for 9 bays. There are 40 nodes with 3 DOF each, so that the stiffness matrix A and the mass/inertia matrix M are 120×120. The state vector is thus 120×1. The displacements along the axis of the truss are then expressed (or, rather, extrapolated):

$$u(k\ell) = u(k\ell, 0, 0) = \frac{u(k\ell,0-b/2) + u(k\ell,0+b/2) + u(k\ell,-b/2,0) + u(k\ell,+b/2,0)}{4}$$

$$v(k\ell) = v(k\ell, 0, 0) = \frac{v(k\ell,0-b/2) + v(k\ell,0+b/2) + v(k\ell,-b/2,0) + v(k\ell,+b/2,0)}{4}$$

$$w(k\ell) = w(k\ell, 0, 0) = \frac{w(k\ell,0-b/2) + w(k\ell,0+b/2) + w(k\ell,-b/2,0) + w(k\ell,+b/2,0)}{4}$$

$$\phi_1(k\ell) = \frac{1}{2}\left[\frac{w(k\ell,+b/2,0) - w(k\ell,-b/2,0)}{b} - \frac{v(k\ell,0,+b/2) - v(k\ell,0,-b/2)}{b}\right]$$

$$\phi_2(k\ell) = \frac{u(k\ell,0,+b/2) - u(k\ell,0,-b/2)}{b}$$

$$\phi_3(k\ell) = \frac{u(k\ell,+b/2,0) - u(k\ell,-b/2,0)}{b}$$

where k is an integer and $0 \le k\ell \le L$. Allowing for m controllers at $k = k_1, k_2, \dots, k_m$, the corresponding relations can be represened by a $6m\times 120$ matrix acting on the state vector. (We consider in this paper $m = 1$, 2 or 3.) Let B denote the transpose of this matrix. Then the state space dynamics with co-located sensors can be described by:

$$M\ddot{x} + Ax + BU + BN_a = 0 \tag{3.1}$$

with sensor data:

$$v_p(t) = B^*x(t) + N_p(t) \tag{3.2}$$

$$v_r(t) = B^*x(t) + N_r(t) \tag{3.3}$$

where $U(\cdot)$ denotes the control; $N_a(\cdot)$, $N_p(\cdot)$, $N_r(\cdot)$ model additive noise taken as (mutually independent) white Gaussian with spectral density matrices d_aI, d_pI, d_rI respectively, I being the identity matrix.

4. The Continuum Model

As we noted in Section 1, the problem of producing an "exact," "three-D" continuum model for a real-world structure like the truss we are considering can be a formidable one — although research in this area looks promising [5]. One way out of this difficulty is to exploit where possible the special nature of the truss — in our case it is a lattice of bays along the same axis numerous enough so that it is even visually "beam-like." In that case there are many ways to approximate by "one-D" beams — without going into the details of this theory, suffice it to say that the approach by Noor and Russell [2] is the one adapted here. We thus create an "equivalent" (referring to [2] for the precise sense) one-dimensional *anisotropic* Timoshenko beam as follows.

u denoting axial (longitudinal) displacement (along the x_1-axis);
ϕ_1 the torsion angle about this axis ,

w, ϕ_2 denoting the transverse bending displacement in the x_1-x_3 plane and torsion angle about the x_2-axis ,

v, ϕ_3 denoting the transverse bending displacement in the x_1-x_2 plane and torsion angle about the x_3-axis ,

the three axes being mutually perpendicular; $0 \le x_1 \le L$,
L being the beam length $= n\ell$; n = number of bays

The Timoshenko equations (valid between control nodes) are:

$$m_{11}\ddot{u} - c_{11}u'' - c_{14}v'' - c_{15}w'' - c_{15}\phi_2' + c_{14}\phi_3' = 0$$

$$m_{22}\ddot{v} - c_{44}v'' - c_{14}u'' + c_{44}\phi_3' = 0$$

$$m_{33}\ddot{w} - c_{55}w'' - c_{15}u'' - c_{55}\phi_2' = 0$$

$$m_{44}\ddot{\phi}_1 - c_{66}\phi_1'' - c_{36}\phi_2'' - c_{26}\phi_3'' = 0$$

$$m_{55}\ddot{\phi}_2 + m_{56}\ddot{\phi}_3 + c_{15}u' + c_{55}w' - c_{36}\phi_1'' + c_{55}\phi_2 - c_{33}\phi_2'' - c_{23}\phi_3'' = 0$$

$$m_{66}\ddot{\phi}_3 + m_{56}\ddot{\phi}_2 - c_{14}u' - c_{44}v' - c_{26}\phi_1'' - c_{23}\phi_2'' + c_{44}\phi_3 - c_{22}\phi_3'' = 0$$

where the superdots denote time-derivatives and the primes, space derivatives. The coefficients of these dynamic equations are related to the truss parameters as follows: (cf. [2]):

The mass coefficients are given by:

$$m_{11} = m_{22} = m_{33} = \frac{4m_b + 4m_\ell + 4m_d + m_\delta}{\ell}$$

$$m_{44} = 2m_{55} = 2m_{66} = \frac{\ell(8m_b + 12m_\ell + 8m_d + m_\delta)}{6\mu^2}$$

$$m_{56} = \frac{\ell m_\delta}{12\mu^2}$$

The stiffness (flexibility) c_{ij} are given by:

$$c_{11} = 4\ell S_\ell + \frac{4\ell S_b S_d \mu^2}{S_d + S_b(\ell + \mu^2)}$$

$$c_{44} = \frac{c_{14}}{\mu} = c_{55} = \frac{c_{15}}{\mu} = \frac{2\ell S_b S_d}{S_d + S_b(\ell + \mu^2)}$$

$$c_{22} = c_{33} = \frac{\ell^3 S_\ell}{\mu^2} + \frac{\ell^3 S_b S_d}{4(S_d + S_b(\ell + \mu^2))}$$

$$c_{23} = \frac{\ell^3 S_b S_d}{4(S_d + S_b(\ell + \mu^2))}$$

$$c_{66} = -2c_{26} = -2c_{36} = \frac{\ell^3 S_b S_d}{\mu^2(S_d + S_b(\ell + \mu^2))}$$

where

$$\mu = \frac{\ell}{b}\ .$$

In order not to complicate matters unduly in this demonstration, we shall freeze all parameters except the cross-sectional area A which will then be the structural parameter to be optimized. In this case

$$c_{11} = \frac{(40 + 24\sqrt{2})EA}{9 + 4\sqrt{2}} \qquad \text{(Newton)}$$

$$c_{14} = -c_{15} = c_{44} = c_{55} = \frac{2EA}{1 + 2\sqrt{2}} \qquad \text{(Newton)}$$

$$c_{22} = c_{33} = \frac{(2725 + 1476\sqrt{2})EA\ell^2}{2628 + 1336\sqrt{2}} \qquad \text{(Newton)m}^2$$

$$c_{23} = -\left[\frac{(97 + 140\sqrt{2})EA\ell^2}{2628 + 1336\sqrt{2}}\right] \qquad \text{(Newton)m}^2$$

$$c_{26} = -c_{36} = \frac{1}{2} c_{66} = \frac{(16 + 33\sqrt{2})EA\ell^2}{296 + 130\sqrt{2}} \qquad \text{(Newton)m}^2$$

$$m_{11} = m_{22} = m_{33} = (8 + 5\sqrt{2})A\rho \quad \text{kg/m}$$

$$m_{44} = 2m_{55} = 2m_{66} = \frac{(20 + 9\sqrt{2})A\ell^2\rho}{6} \quad \text{kg} \cdot \text{m}$$

$$m_{56} = -\frac{(A\ell^2\rho)}{6\sqrt{2}} \quad \text{kg} \cdot \text{m} .$$

Once the coefficients c_{ij}, m_{ij} are defined (on whatever basis), we can develop the generic state space dynamic model analogous to the FEM formulas (3.1), (3.2), (3.3):

$$M\ddot{x} + Ax + Bu + BN_a = 0 \tag{4.1}$$

$$\left.\begin{aligned} v_p(t) &= B^*x(t) + N_p(t) \\ v_r(t) &= B^*\dot{x}(t) + N_a(t) \end{aligned}\right\} \tag{4.2}$$

where $N_a(\cdot)$, $N_p(\cdot)$, $N_r(\cdot)$ are white Gaussian noise with spectral density $d_a I$, $d_p I$, $d_r I$ respectively.† Only, the dimension of the state $x(t)$ is not finite. The technique of derivation is also different, in particular in the role of the energy. See [6, 7] for details. Here we can only summarize the basic results.

Case 1: One Controller

We begin with constructing the state space model for one controller ("midcontroller"), and offset masses at each end, referring to [4, 6] again for more details and to [7] where the general case of distributed control is treated. Thus the state $x(t)$ at time t is defined by

$$x(t) = \begin{vmatrix} f(\cdot, t) \\ f(0, t) \\ f(s_2, t) \\ f(L, t) \end{vmatrix}$$

† See [7] for generalization to arbitrary diagonal matrices.

where s parametrizes the beam axis, zero denoting one end and L the other, L being the total beam length, and s_2 denotes the location of the mid-controllers and $f(\cdot, t)$ denotes a (6×1 vector) function of s, $0 \le s \le L$, representing displacements and angles:

$$f(s,t) = \begin{vmatrix} u(s,t) \\ v(s,t) \\ w(s,t) \\ \phi_1(s,t) \\ \phi_2(s,t) \\ \phi_3(s,t) \end{vmatrix} .$$

The stiffness operator A is defined as follows:

$$Ax = \begin{vmatrix} g \\ v \end{vmatrix} ; \qquad x = \begin{vmatrix} f(\cdot) \\ f(0) \\ f(s) \\ f(L) \end{vmatrix}$$

$$g(s) = -A_2 f''(s) + A_1 f'(s) + A_0 f(s) , \qquad 0 < s < s_2 \; ; \; s_2 < s < L .$$

The derivative $f'(\cdot)$ has a discontinuity at $s = s_2$, and

$$v = \begin{vmatrix} -L_1 f(0) - A_2 f(0) \\ -A_2(f'(s_2+) - f'(s_2-)) \\ L_1 f(L) + A_2 f'(L) \end{vmatrix}$$

and thus defined, the potential energy

$$= [Ax, x] = \int_0^L \left[H \begin{vmatrix} f'(s) \\ f(s) \end{vmatrix} , \begin{vmatrix} f'(s) \\ f(s) \end{vmatrix} \right] ds \ge 0$$

where

$$H = \begin{vmatrix} C_1 & 0 & 0 & -C_2 \\ 0 & C_3 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{A_0}} \\ -C_2^* & 0 & & \end{vmatrix}$$

$$A_2 = \begin{vmatrix} C_1 & 0 \\ 0 & C_3 \end{vmatrix}$$

$$C_1 = \begin{vmatrix} c_{11} & c_{14} & c_{15} \\ c_{14} & c_{44} & 0 \\ c_{15} & 0 & c_{55} \end{vmatrix}$$

$$C_3 = \begin{vmatrix} c_{66} & c_{36} & c_{26} \\ c_{36} & c_{33} & c_{23} \\ c_{26} & c_{23} & c_{22} \end{vmatrix}$$

$$A_1 = \begin{vmatrix} 0 & C_2 \\ -C_2^* & 0 \end{vmatrix}$$

$$C_2 = \begin{vmatrix} 0 & -c_{15} & c_{14} \\ 0 & 0 & c_{44} \\ 0 & -c_{55} & 0 \end{vmatrix}$$

$$A_0 = \text{Diag. } [\, 0,\ 0,\ 0,\ 0,\ c_{55},\ c_{44} \,]$$

$$L_1 = \begin{vmatrix} 0 & -C_2 \\ 0 & 0 \end{vmatrix} .$$

B is defined by

$$BU = \begin{vmatrix} 0 \\ 0 \\ U \\ 0 \end{vmatrix} , \qquad U \text{ is } 6\times 1$$

and

$$B^*x = f(s_2) \qquad (6\times 1) .$$

The mass/inertia operator M is defined

$$Mx = \begin{vmatrix} M_0 f(\cdot) \\ M_{b,0} f(0) \\ M_c f(s_2) \\ M_{b,L} f(L) \end{vmatrix}$$

where

$$M_0 = \begin{vmatrix} m_{11} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & m_{22} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & m_{33} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & m_{44} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & m_{55} & m_{56} \\ \cdot & \cdot & \cdot & \cdot & m_{56} & m_{66} \end{vmatrix}$$

$$\text{Diag. } M_{b_0} = (m_0 , \ m_0 , \ m_0 , \ \text{Diag. } I_0)$$

where m_0 is the offset mass at $s = 0$ and I_0 is its moment of inertia about zero, and similarly

$$\text{Diag. } M_{b_L} = (m_L , \ m_L , \ m_L , \ \text{Diag. } I_L) .$$

See also [6] for more on M_{b_0}, M_{b_L}.

$$M_c = \begin{vmatrix} m_c & 0 & 0 & 0 & 0 & 0 \\ 0 & m_c & 0 & 0 & 0 & 0 \\ 0 & 0 & m_c & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & I_c & \\ 0 & 0 & 0 & & & \end{vmatrix}$$

where m_c is the force actuator (rotor) mass and I_c the moment of inertia of the moment actuator about its center of gravity. The m.s. attitude error matrix is defined by

$$\lim_{T \to \infty} \left\{ \frac{1}{T} \int_0^T f(0, t) f(0,)^* \, dt + \frac{1}{T} \int_0^T f(L, t) f(L, t)^* \, dt \right\}$$

and σ_a^2 is the sum of the diagonal terms as defined.

Case 2: Two Controllers

Next we consider the case of two controllers, one at each end. Here, since there is no mid-controller, we may delete that entry in the state. Thus

$$x(t) = \begin{vmatrix} f(\cdot, t) \\ f(0, t) \\ f(L, t) \end{vmatrix}$$

$$Mx = \left| \begin{array}{c} M_0 f(\cdot, t) \\ M_{b,0} f(0, t) \\ M_{b,L} f(L, t) \end{array} \right| .$$

where the end-masses m_0, m_L must now include the actuator moving masses, and similarly for the moment of inertia matrices. We shall use the notation

$$M_c = \left| \begin{array}{cc} M_{b,0} & 0 \\ 0 & M_{b,L} \end{array} \right|$$

With $U(\cdot)$ denoting the control vector, (12×1), we have

$$BU = \left| \begin{array}{c} 0 \\ U \end{array} \right|$$

$$B^*x = \left| \begin{array}{c} f(0, t) \\ f(L, t) \end{array} \right| .$$

Finally

$$Ax = \left| \begin{array}{c} g \\ \mathrm{v} \end{array} \right|$$

$$\mathrm{v} = \left| \begin{array}{c} -L_1 f(0) - A_2 f'(0) \\ L_1 f(L) + A_2 f'(L) \end{array} \right|$$

$$g(s) = -A_2 f''(s) + A_1 f'(s) + A_0 f(s) , \qquad 0 < s < L .$$

Here the mean square attitude error-matrix

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T (B^*x(t))(B^*x(t))^* \, dt .$$

Case 3: Three Controllers

In this case we have a mid-controller at $s = s_2$ as well as a controller at each end. Here

$$x(t) = \left| \begin{array}{c} f(\cdot, t) \\ f(0, t) \\ f(s_2, t) \\ f(L, t) \end{array} \right| ,$$

in other words the same state vector as in Case 1.

$$Mx(t) = \begin{vmatrix} M_0 f(\cdot, \; t) \\ M_{b,0} f(0, \; t) \\ M_{b,2} f(s_2, t) \\ M_{b,L} f(L, \; t) \end{vmatrix},$$

where

$$M_{0,2} = \begin{vmatrix} m_c & 0 & 0 & 0 & 0 & 0 \\ 0 & m_c & 0 & 0 & 0 & 0 \\ 0 & 0 & m_c & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & I_c & \\ 0 & 0 & 0 & & & \end{vmatrix}$$

where m_c is the force actuator moving mass and I_c is the moment of inertia about its center of gravity, corresponding to the "mid-controller."

$$BU = \begin{vmatrix} 0 \\ U \end{vmatrix}$$

$$B^*x = \begin{vmatrix} f(0, \; t) \\ f(s_2, t) \\ f(L, \; t) \end{vmatrix}.$$

We can calculate σ_a^2 from the diagonal terms of the mean square error matrix:

$$\text{Diag.} \lim_{T \to \infty} \frac{1}{T} \int_0^T (B^*x(t))(B^*x(t))^* \, dt \, .$$

which now has 18 entries. Once again we adopt the notation:

$$M_c = \begin{vmatrix} M_0 & \cdot & \cdot & \cdot \\ \cdot & M_{b,0} & \cdot & \cdot \\ \cdot & \cdot & M_{b,2} & \cdot \\ \cdot & \cdot & \cdot & M_{b,L} \end{vmatrix}.$$

5. Formulas for Performance Criteria

We shall now develop formulas for the Performance Critera. First the mean square attitude error: Using either model FEM (3.1)-(3.3) or Continuum (4.1)-(4.2) we have the state space form:

$$M\ddot{x} + Ax + BU + BN_a = 0 \tag{5.1}$$

$$v_p = B^*x + N_p \tag{5.2}$$

$$v_r = B^*\dot{x} + N_r \tag{5.3}$$

and

$$U(t) = \alpha v_p(t) + \gamma v_r(t) . \tag{5.4}$$

Substituting this control law into the state equations we have:

$$M\ddot{x} + (A + \alpha BB^*)x + \gamma BB^*\dot{x} + B(N_a + \alpha N_p + \gamma N_r) = 0 . \tag{5.5}$$

The steady-state output covariance matrix

$$R_a = E((B^*x(t))(B^*x(t))^*) = \left(\frac{\alpha^2 d_p + d_a + \gamma^2 d_r}{2\gamma}\right)(B^*(A + \alpha BB^*)^{-1}B) \tag{5.6}$$

where the matrix part:

$$B^*(A + \alpha BB^*)^{-1}B$$

is recognized as the steady-state input-output response matrix: it is the value at $\omega = 0$ of the input-output transfer-function:

$$B^*((i\omega)^2 M + A + \alpha BB^* + (i\omega)\gamma BB^*)^{-1}B .$$

The scalar factor

$$d = \frac{\alpha^2 d_p + d_a + \gamma^2 d_r}{2\gamma} \tag{5.7}$$

consolidates the "noise" part. For the case of two controllers the mean square attitude error σ_a^2 can be calculated from

$$\text{Diag. } R_a .$$

For the third case of three controllers it is given by

$$\text{Diag. } LR_a L^*$$

where

$$LU = \begin{vmatrix} f(0) \\ f(L) \end{vmatrix} .$$

For Case 1 with one controller only, we have to calculate

$$E(f(0, t)\, f(0, t)^*)$$

$$E(f(L, t)\, f(L, t)^*) .$$

Separately, expressing each as a transformation of the state:

$$L_1 x = f(0)$$

$$L_2 x = f(L) .$$

In the FEM version, we have thus to invert the matrix

$$(A + \alpha BB^*)$$

which in our case is 120×120 — and of course can be done only numerically. For the continuum model however we have to invent the operator

$$(A + \alpha BB^*)^{-1}$$

but — and this is the main point of departure — this can be done *analytically*. Referring to [7] for details, here we shall simply enumerate the formulas below.

Case 1: One Controller Only

$$E[f(0, t)\, f(0, t)^*] = \text{Diag.} \left(\frac{1}{\alpha}, \frac{1+s_2^2}{\alpha}, \frac{1+s_2^2}{\alpha}, \frac{1}{\alpha}, \frac{1}{\alpha}, \frac{1}{\alpha} \right) d$$

$$E[f(L, t)\, f(L, t)^*] = \text{Diag.} \left(\frac{1}{\alpha}, \frac{1 + (L-s_2)^2}{\alpha}, \frac{1 + (L-s_2)^2}{\alpha}, \frac{1}{\alpha}, \frac{1}{\alpha}, \frac{1}{\alpha} \right) d$$

(each is a diagonal matrix!). Steady state step response corresponding to step input U:

$$
f(0,\infty) = \frac{1}{\alpha}\left|\begin{array}{cc} I_3 & \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -s_2 \\ 0 & s_2 & 0 \end{array} \\ 0 & I_3 \end{array}\right| U
$$

$$
f(L,\infty) = \frac{1}{\alpha}\left|\begin{array}{cc} I_3 & \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & L-s_2 \\ 0 & s_2-L & 0 \end{array} \\ 0 & I_3 \end{array}\right| U
$$

Note: Controller at $s = s_2$.

Case 2: Two End Controllers

$$
\text{Diag. } E[f(0,t)\, f(0,t)^*] = \text{Diag. } E[f(L,t)\, f(L,t)^*]\ .
$$

And the first four diagonal terms in either matrix are given in order by:

$$
= \left[1 + \frac{\mu\beta}{\lambda\beta - 2}\right]\left(\frac{1}{2\alpha}\right) d
$$

$$
= \left[1 + \frac{\mu(\lambda\beta - 1)}{\beta(\lambda\beta - 2)}\right]\left(\frac{1}{2\alpha}\right) d
$$

$$
= \left[1 + \frac{\mu(\lambda\beta - 1)}{\beta(\lambda\beta - 2)}\right]\left(\frac{1}{2\alpha}\right) d
$$

$$
= \left[\frac{1}{2\alpha} + \frac{L}{4}\,\frac{c_{33} + \frac{\alpha L}{2} + \frac{c_{66}}{4}}{\left(c_{33} + \frac{\alpha L}{2} + \frac{c_{66}}{4}\right)\left(c_{66} + \frac{\alpha L}{2}\right) - \frac{c_{66}^2}{2}}\right] d
$$

where

$$
\mu = \frac{\alpha L}{2c\delta} + \frac{L^2}{4}
$$

$$
\beta = \mu + 1
$$

$$
\lambda = \mu + 2 + \frac{(c_{11} - 2c)(2\alpha + c\delta L)}{2\alpha c\delta}
$$

$$c = c_{44}$$

$$\delta = \frac{1}{1 + \frac{L^2\gamma}{12}}$$

$$\gamma = \frac{c_{44}}{c_{33} - \frac{c_{66}}{4}} .$$

Case 3: Three Controllers

Here we have to express the answers in terms of 6×6 matrices. Thus let

$$D_{11} = \frac{E[f(0,\ t)\ f(0,\ t)^*]}{d}$$

$$D_{22} = \frac{E[f(s_2,\ t)\ f(s_2,\ t)^*]}{d}$$

$$D_{33} = \frac{E[f(L,\ t)\ f(L,\ t)^*]}{d} .$$

Then

$$D_{11} = d_{11}(s_2) - d_{12}(s_2)\left(d_{22}(s_2) + \left(d_{11}(L-s_2)^{-1} - \alpha\right)^{-1}\right)^{-1} d_{12}^*(s_2)$$

$$D_{22} = \left(d_{22}^{-1}(s_2) + d_{11}(L-s_2)^{-1} - \alpha\right)^{-1}$$

$$D_{33} = (\alpha I + m_3(L-s_2))^{-1}\{\alpha I + m_3(L-s_2) + m_{21}(L-s_2)(D_{22})m_{21}^*(L-s_2)\} \times (\alpha I + m_3(L-s_2))^{-1}$$

where

$$d_{11}(s) = ((\alpha + m_1(s)) - m_{21}(s)^*(\alpha + m_3(s))^{-1} m_{21}(s))^{-1}$$

$$d_{22}(s) = ((\alpha + m_3(s)) - m_{21}(s)(\alpha + m_1(s))^{-1} m_{21}^*(s))^{-1}$$

$$d_{12}(s) = -d_{11}(s)m_{21}(s)^*(\alpha I + m_3(s))^{-1}$$

$$m_1(s) = \begin{vmatrix} \frac{c\delta(s)}{s} A_{11} & \frac{c\delta(s)}{2} A_{12} \\ \frac{c\delta(s)}{2} A_{12}^* & \frac{C_3}{s} + \frac{cs\delta(s)}{4} A_{22} \end{vmatrix}$$

$$m_3(s) = \begin{vmatrix} \frac{c\delta(s)}{s} A_{11} & -\frac{c\delta(s)}{2} A_{12} \\ -\frac{c\delta(s)}{2} A_{12}^* & \frac{C_3}{s} + \frac{cs\delta(s)}{4} A_{22} \end{vmatrix}$$

$$m_{21}(s) = \begin{vmatrix} -\frac{c\delta(s)}{s} A_{11} & -\frac{c\delta(s)}{2} A_{12} \\ \frac{c\delta(s)}{2} A_{12}^* & -\frac{C_3}{s} + \frac{cs\delta(s)}{4} A_{22} \end{vmatrix}$$

where

$$\delta(s) = \frac{1}{1 + \frac{\gamma s^2}{12}}$$

$$A_{11} = \begin{vmatrix} \frac{1}{\delta(s)} \left(\frac{c_{11}}{c} - 2(1-\delta(s)\right) & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$A_{12} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$A_{22} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Stability Margin Formula

From (5.5) it is clear that the closed-loop eigenfunctions are specified by

$$\lambda^2 M\phi + (A + \alpha BB^*)\phi + \gamma\lambda BB^*\phi = 0$$

(again, irrespective of whether we are using the FEM version (3.1)–(3.3) or the continuum version (4.1), (4.2)) where ϕ denotes the eigenvector ("mode shape"), with the eigenvalue λ specified by

$$\lambda^2[M\phi, \phi] + [(A + \alpha BB^*)\phi, \phi] + \lambda\gamma[B^*\phi, B^*\phi] = 0$$

where we may normalize the mode shape ϕ so that $[M\phi, \phi] = 1$. Since $\alpha \geq 0$ and $\gamma \geq 0$, it follows that the eigenvalue λ is given by:

$$\lambda = -\sigma \pm i\omega$$

where

$$\sigma = -\gamma \frac{\|B^*\phi\|^2}{2}$$

$$\omega^2 = \omega_0^2 - \sigma^2$$

$$\omega_0^2 = [(A + \alpha BB^*)\phi, \ \phi] .$$

Thus the closed-loop eigenvalues have strictly negative real parts.

It is shown in [8] that the sum of the absolute values of the closed-loop eigenvalues is given by

$$\gamma \, \mathrm{Tr.} \, M_c^{-1} \tag{5.8}$$

in all cases, where in M_c only moving parts of the actuators must be considered (as opposed to the stationary mass such as the armature mass). The simplicity of this formula is striking when compared with taking the sum of the inverses of the absolute values of the real parts of the closed loop eigenvalues for a finite number of modes as in [10]. We may note that for zero natural damping (or with damping, if we consider only the increment),

$$\sum_1^\infty \frac{1}{|\sigma_i|} = \infty$$

for any continuum model. Again, (5.8) applies for the FEM version (3.1)-(3.3), as well as for the continuum model (4.1), (4.2).

6. Numerical Results

In this section we compare the numerical results by the two methods for the mean square attitude errors (and equivalently, tip response), for each of the three cases: one, two and three controllers — using the nominal values (cf. Section 1) for the truss parameters.

Case 1: One Controller

FEM

The tip response $f(0, \infty)$ and $f(L, \infty)$ was calculated for:

$$\alpha = c_{44} = 35571851.2\,; \qquad s_2 = 0\,, \frac{L}{9}, \frac{L}{3}, \frac{L}{2},$$

$$\alpha = 10{,}000\,; \qquad s_2 = 0\,, \frac{L}{9}, \frac{L}{3}, \frac{L}{2},$$

$$\alpha = 100\,; \qquad s_2 = 0\,, \frac{L}{9}, \frac{L}{3}, \frac{L}{2},$$

both $f(0, \infty)$ and $f(L, \infty)$ values were exactly the same as the values predicted by the continuum model and hence are not displayed.

Case 2: Two Controllers

$$b_{ii} = \text{Diag. } B^*(A + \alpha I)^{-1} B\,, \qquad i = 1, \ldots, 12$$

		FEM	Continuum
$\alpha = 1$	$b_{11} = b_{77}$	0.4998	0.5
	$b_{22} = b_{33} = b_{88} = b_{99}$	0.99727	0.99727
	$b_{44} = b_{10,10}$	0.50028	0.5
$\alpha = 50$	$b_{11} = b_{77}$	0.01	0.01
	$b_{22} = b_{33} = b_{88} = b_{99}$	0.019945	0.019945
	$b_{44} = b_{10,10}$	0.01	0.01
$\alpha = c_{44} = 3.557\times10^7$	$b_{11} = b_{77}$	2.2978×10^{-8}	2.2978×10^{-8}
	$b_{22} = b_{33} = b_{88} = b_{99}$	2.80545×10^{-8}	2.80546×10^{-8}
	$b_{44} = b_{10,10}$	2.478×10^{-8}	2.478×10^{-8}

Case 3: Three Controllers

$$\alpha = 3.557 \times 10^7$$

$$b_{ii} = i = 1, \ldots, 18$$

Actuator Position	$b_{11} \times 10^8$ FEM	$b_{11} \times 10^8$ Continuum
$s = \frac{L}{9}$	1.5119	1.5127
$s = \frac{3L}{9}$	1.8366	1.8370
$s = \frac{4L}{9}$	1.9461	1.9465
$s = \frac{5L}{9}$	2.0330	2.0333
$s = \frac{6L}{9}$	2.1031	2.1033
$s = \frac{7L}{9}$	2.1602	2.1603

In other words the FEM and the continuum gave exactly the same numerical results within (the SUN-386i) computer accuracy in all cases.

SUM OF M.S. AXIAL DISPLACEMENTS AT BOTH ENDS

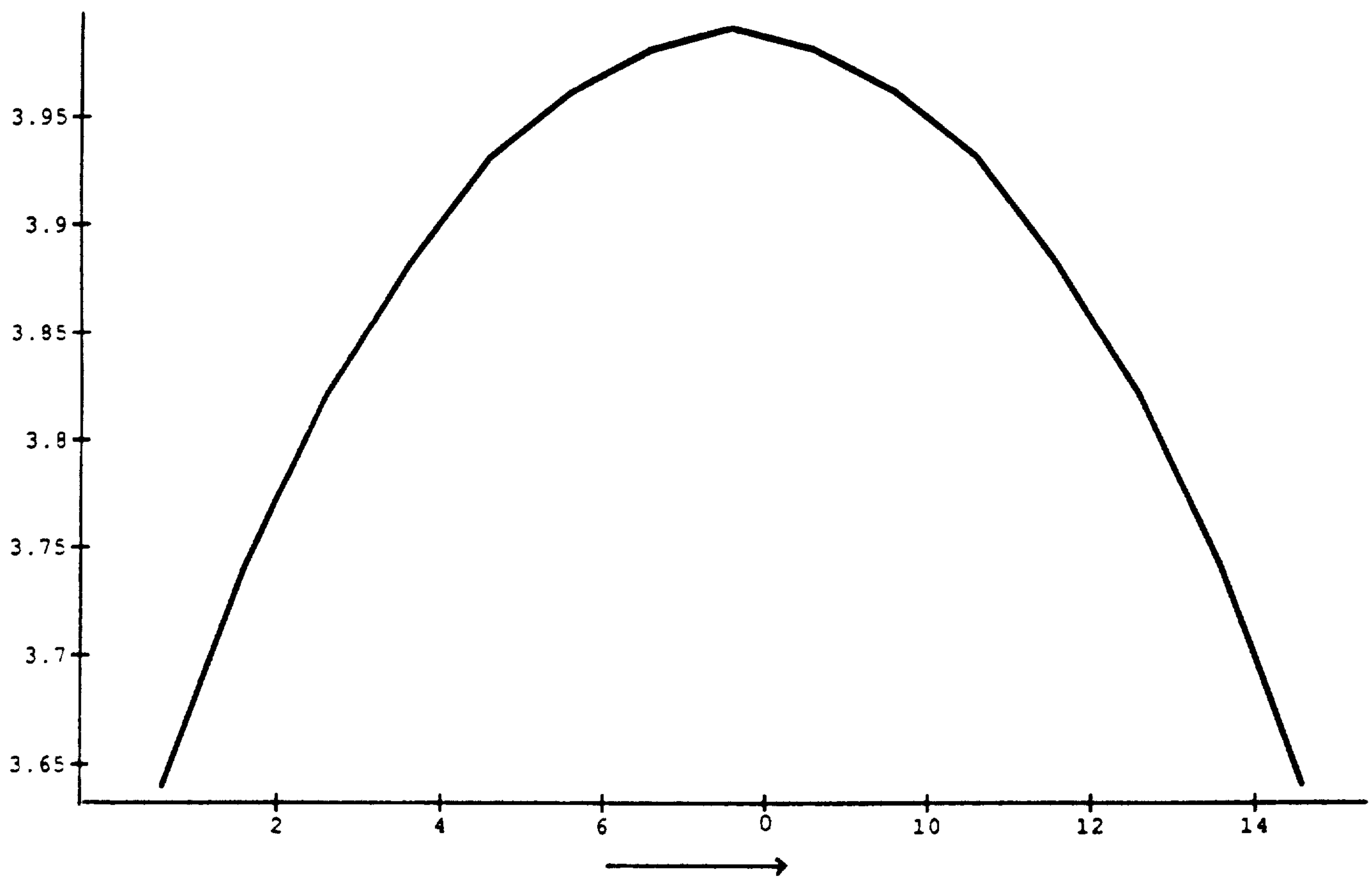

FIGURE 3

MIDACTUATOR POSITION: INCREMENTS L/16

SUM OF M.S. BENDING DISPLACEMENTS AT BOTH ENDS

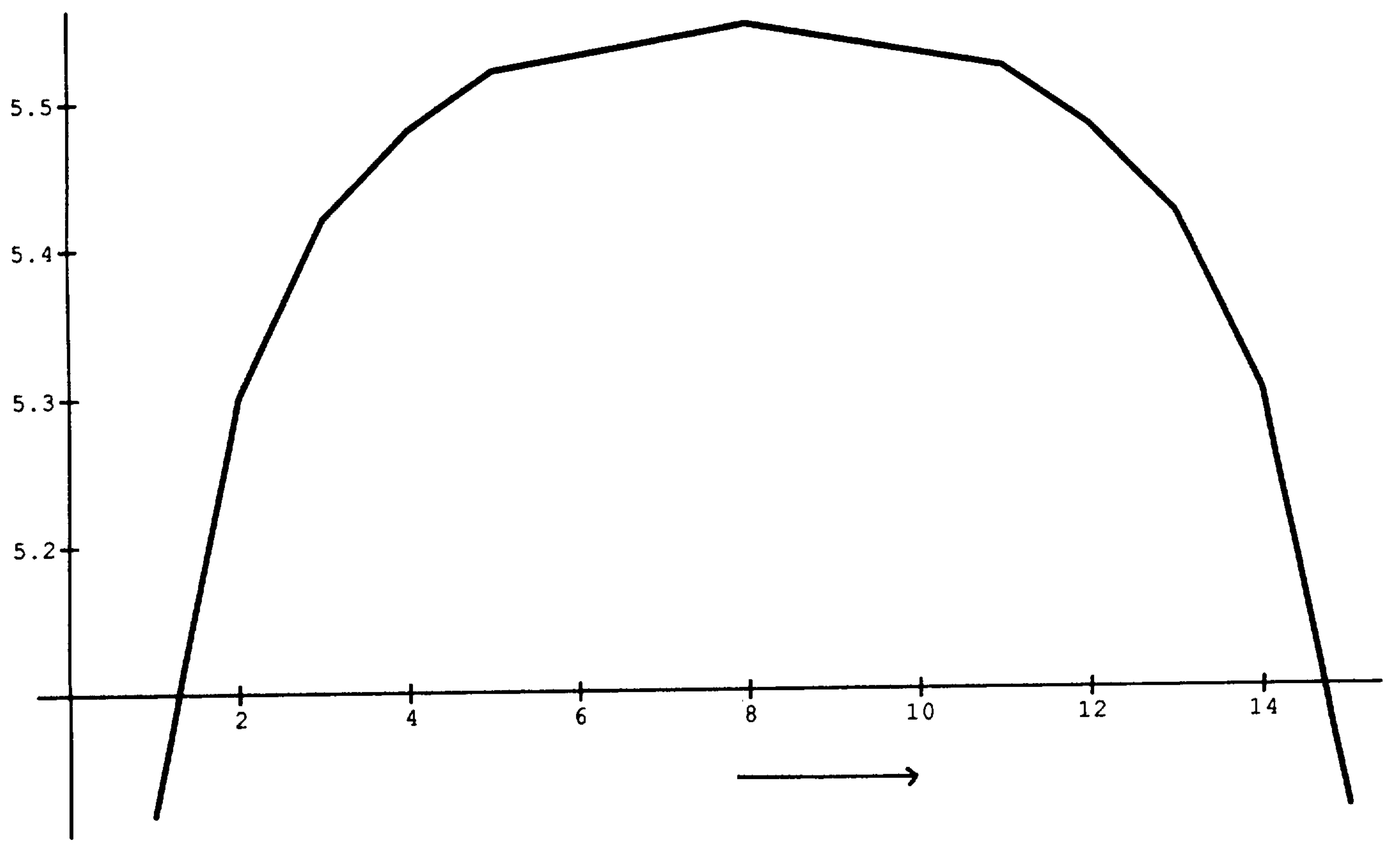

FIGURE 4

MIDACTUATOR POSITION: INCREMENTS L/16

SUM OF M.S. TORSION AT BOTH ENDS

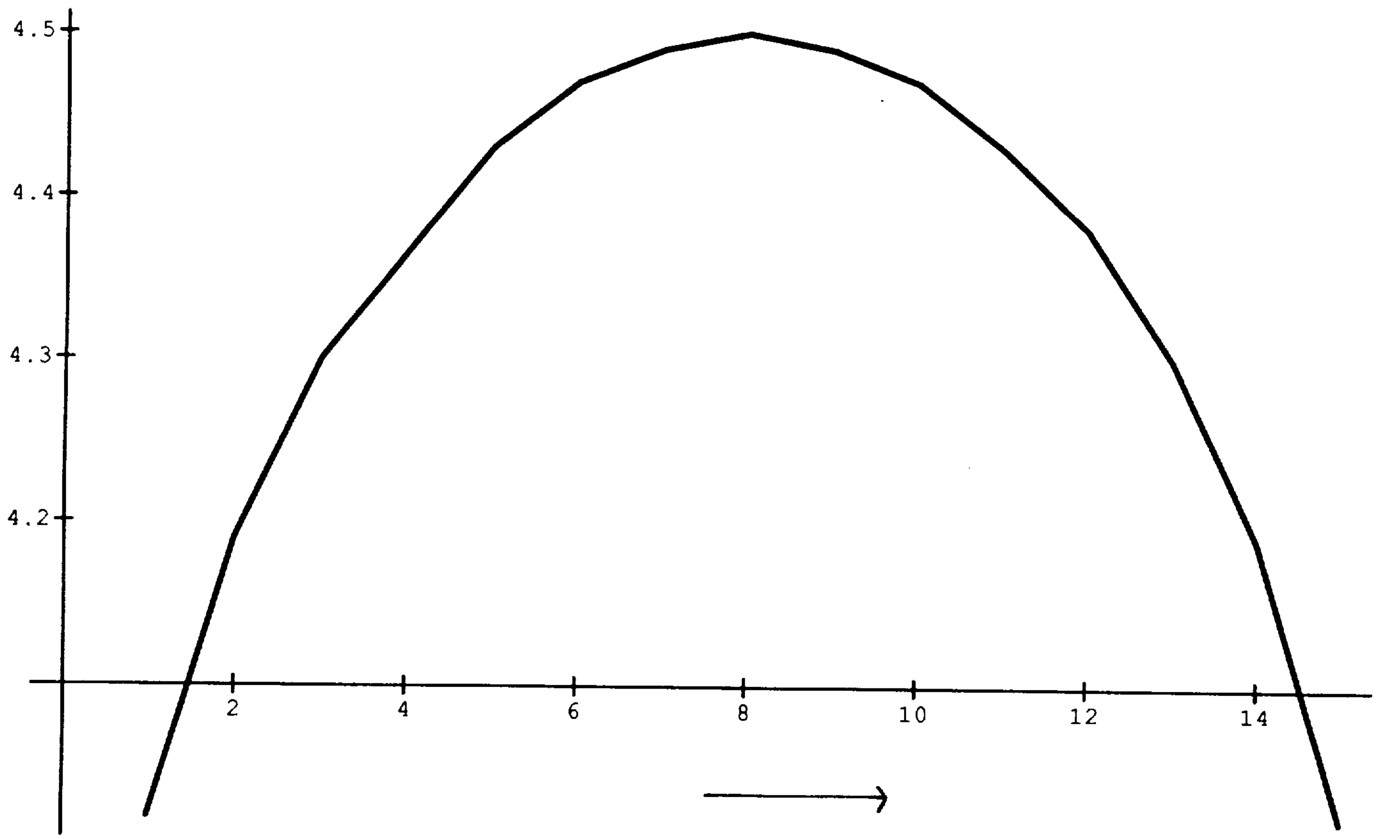

FIGURE 5

MIDACTUATOR POSITION: INCREMENTS L/16

M.S. AXIAL DISPLACEMENT AT ACTUATOR LOCATION

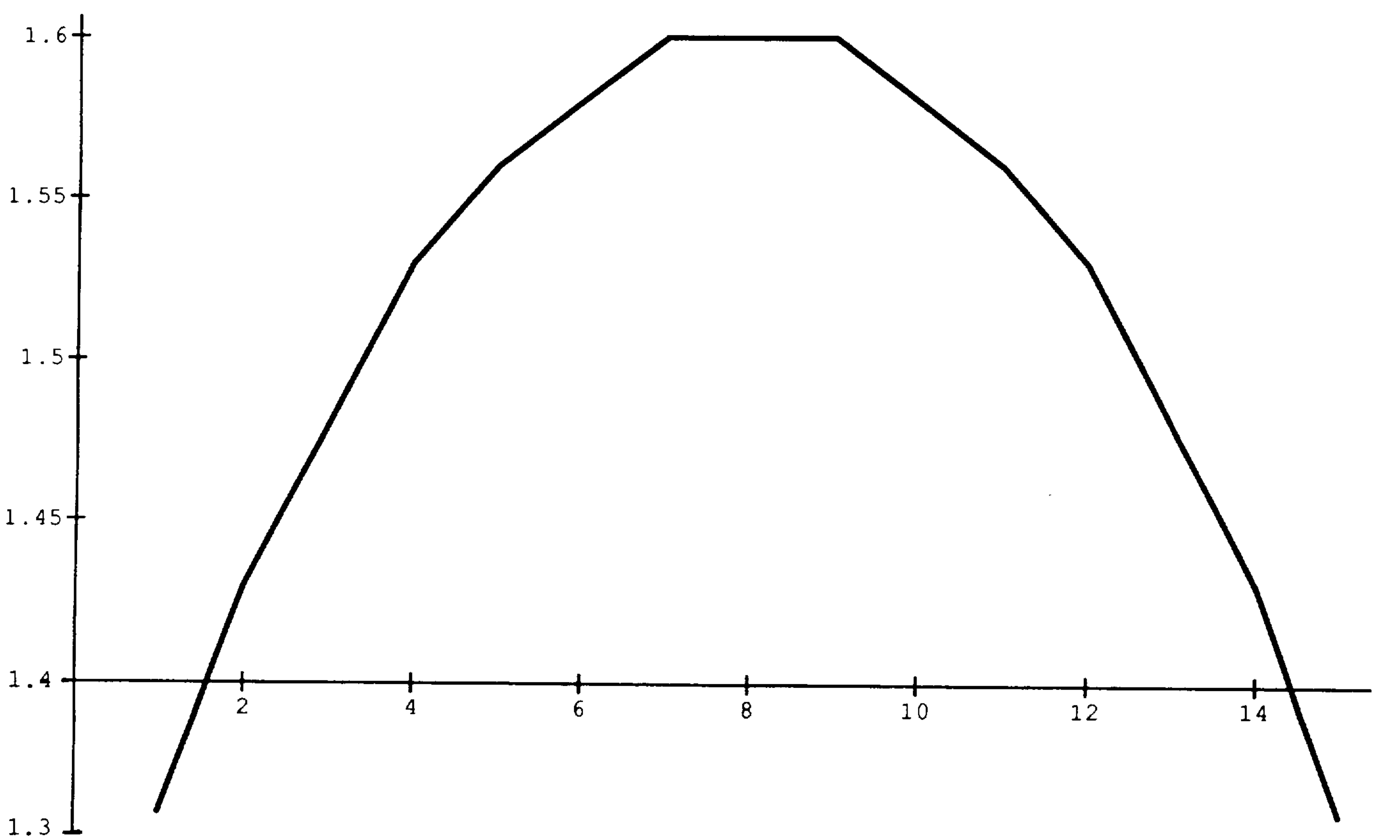

FIGURE 6

MIDACTUATOR POSITION: INCREMENTS L/16

7. Optimal Location of Controllers

As a byproduct of our theory, we can examine the problem of the optimal actuators placement that minimizes the mean square attitude for any given choice of control gains and structure parameters.

Case 1: One Actuator

Here we take the criterion as the sum of the mean square displacements at both ends. We calculate explicitly that

$$\frac{[E[f(0,t)\ f(0,t)^*]\ +\ E[f(L,t)\ f(L,t)^*]]}{d}$$

$$= \text{Diag.}\left[\frac{2}{\alpha},\ \frac{2+s^2+(L-s)^2}{\alpha},\ \frac{2+s^2+(L-s)^2}{\alpha},\ \frac{2}{\alpha},\ \frac{2}{\alpha},\ \frac{2}{\alpha}\right]$$

from which it is clear that the optimal placement is in the middle:

$$s = \frac{L}{2}.$$

Only the bending-displacement is affected by actuator position.

For Case 2 of two actuators there is no placement problem since one is required to be at each end.

Case 3: Three Actuators

Here we can consider the optimal placement of the midactuator while the other two are fixed one at each end. The behavior of the sum of the mean square errors (setting $\alpha = 1$) at both ends for the axial mode is shown in Figure 3; and the bending mode is shown in Figure 4; and the axial torsion in Figure 5. In all cases we see that the worst position is at the middle! The best place is at either end. Finally Figure 6 shows the mean square axial displacement at the actuator location. Again the worst place is the middle.

8. Optimization

We shall now treat a canonical optimization problem currently studied by FEM and truncated modal models [9, 10]. The objective is to minimize the total mass — structure and control — subject to meeting specified indices of performance. Here we take them to be:

(a) mean square attitude error due to sensor noise less than or equal to fixed value

(b) stability index: sum of the absolute values of the real parts of the closed loop eigenvalues to be not less than a fixed value.

We shall see that the problem can be solved analytically by virtue of the formulas we have developed using continuum models.

The structural parameter we shall use is the cross-sectional area A of the longerons (assumed to be the same for battens and cross-bars). Other parameters being fixed, the structure mass is then proportional to A. (The extension to the case of nonequal areas only complicates the algebra, as can be seen from the expressions (cf. Section 3) for the flexibility coefficients.) The control mass has to be subdivided into a stationary mass (armature mass, for example) and a moving mass (rotor mass, for example) since only the latter is involved in the stability index formula. The stationary mass is of course related to the moving mass — for simplicity we shall take it to be inversely proportional to the rotor mass. The control parameters are the attitude and rate gains α and γ. These of course will need to be constrained not to exceed prescribed limits. Thus we have the following formulation (nominal values for all structure parameters except A):

$$\text{Structure mass} = N\ell\rho A$$

$$\text{Control stationary mass} = \frac{k}{m}$$

$$\text{Moving mass} = m$$

$$\text{Total mass} = N\ell\rho A + \frac{k}{m} + m$$

$$\text{Stability index} = \frac{\gamma}{M + m}$$

(where M denotes the contribution of the end-masses). For the truss considered,

$$N = (76 + 46\sqrt{2}) .$$

Finally the mean square attitude error — to be specific, we shall consider the case of two controllers, one at each end; and take the sum of the mean square displacements at either end. First we express these in terms of the structural parameter A — we have thus to use the expressions we have derived for the flexibility coefficients $\{c_{ij}\}$ in terms of A in Section 5 and substitute them into the formulas for mean square errors for two controllers in Table 3 under Case 2. In doing so we shall also take advantage of the simplification possible by noting that for the nominal value of $L = 27$ meters, we can readily calculate that

$$\lambda\beta \geq 2$$

so that we may replace both $(\lambda\beta - 1)$ and $(\lambda\beta - 2)$ by $\lambda\beta$. Thus the first four diagonal terms in

$$2 \text{ Diag. } E[f(0, t)\, f(0, t)^*]$$

$$(= 2 \text{ Diag. } E[f(L, t)\, f(L, t)^*])$$

are given in order by:

$$f(\alpha, \gamma, A) = \frac{1}{\alpha}\left[1 + \frac{\frac{L^2}{4} + \frac{L}{2c\delta}\frac{\alpha}{EA}}{2 + \frac{L^2}{4} + \frac{L}{2c\delta}\frac{\alpha}{EA} + \left(\frac{c_{11} - 2c}{2c\delta}\right)\left(\frac{2\alpha + Lc\delta EA}{\alpha}\right)}\right] d$$

$$\frac{1}{\alpha}\left[1 + \frac{\frac{L^2}{4} + \frac{L}{2c\delta}\frac{\alpha}{EA}}{1 + \frac{L^2}{4} + \frac{L}{2c\delta}\frac{\alpha}{EA}}\right] d$$

$$\frac{1}{\alpha}\left[1 + \frac{\frac{L^2}{4} + \frac{L}{2c\delta}\frac{\alpha}{EA}}{1 + \frac{L^2}{4} + \frac{L}{2c\delta}\frac{\alpha}{EA}}\right] d$$

$$\left[\frac{1}{\alpha} + \frac{L}{2EA}\,\frac{c_{33} + \frac{c_{66}}{4} + \frac{L}{2}\frac{\alpha}{EA}}{\left(c_{33} + \frac{c_{66}}{4} + \frac{L}{2}\frac{\alpha}{EA}\right)\left(c_{66} + \frac{L}{2}\frac{\alpha}{EA}\right) - \frac{c_{66}^2}{2}}\right] d$$

where now

$$c_{11} = \frac{40 + 24\sqrt{2}}{9 + 4\sqrt{2}}$$

$$c = \frac{2}{1 + 2\sqrt{2}}$$

$$c_{33} = n\left(\frac{2725 + 1476\sqrt{2}}{2628 + 1336\sqrt{2}}\right)$$

$$c_{66} = n\left(\frac{32 + 66\sqrt{2}}{296 + 130\sqrt{2}}\right)$$

$$n = \text{number of bays}\ ; \qquad L = n\ell$$

$$\delta = \frac{1}{1 + \frac{L^2}{12} \cdot \frac{c}{(c_{33} - c_{66}/4)}} \ .$$

with d as given by (5.7). These formulas enable us to draw conclusions concerning the dependence on the cross-sectional area A without resorting to numerical computer calculations. We see that all the errors decrease as A increases. The axial error decreases from $\frac{2d}{\alpha}$ at $A = 0$ to $\frac{d}{\alpha}$ at $A = \infty$; similarly the torsion error. The bending error is least affected, decreasing from $\frac{d}{\alpha}$ at $A = 0$ to $\left(\frac{L^2}{4 + L^2}\right) \frac{d}{\alpha}$ at $A = \infty$. In all cases the minimal mean square error is at most 3 db less than the maximum!

For the optimization let us fix on the mean square bending error as being the easiest analytically: let

$$f(\alpha, \gamma, A) = \frac{d}{\alpha}\left[1 - \frac{1}{1 + \frac{L^2}{4} + \frac{L}{2c\delta}\frac{\alpha}{EA}}\right] .$$

Thus the optimization problem is that of minimizing

$$N\ell\rho A + \frac{k}{m} + m$$

subject to:

$$\frac{\gamma}{M + m} \geq \sigma_s^2$$

$$f(\alpha, \gamma, A) \leq \sigma_a^2 \ .$$

The first inequality can clearly be reversed to read

$$\frac{M + m}{\gamma} \leq \frac{1}{\sigma_s^2} .$$

We note that the objective functional

$$N\ell\rho A + \frac{k}{m} + m$$

is infinitely smooth and trivially convex, and the constraints are also infinitely smooth and convex. Hence we are assured of the existence of a minimum (which is further verified to be unique). Moreover we can go to the Langrange parameter formulation and minimize:

$$\left(N\ell\rho A + \frac{k}{m} + m\right) + \lambda_1\left(\frac{M + m}{\gamma}\right) + \lambda_2 f(\alpha, \gamma, A)$$

where $\lambda_1, \lambda_2 \geq 0$ are the Lagrange parameters. See [11] for the standard results that are applicable here.

Compared to the FEM versions [9, 10] this is a "trivial" problem and complete "analytical" solution is possible. We omit the details since our primary aim in this paper is to demonstrate the simplicity of the optimization problem relative to the FEM versions.

Acknowledgement

Grateful acknowledgement is made of several informal discussions with E. Armstrong and S. Joshi of the CSI Analytical Design Methods Team at NASA Langley FRC on the problem; and of the FEM calculations by H-C. Wang.

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