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Methods of Applied Dynamics

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Contents

Methods of Applied Dynamics

Description:

This monograph is designed to give the practicing engineer a clear understanding of the principles of dynamics with special emphasis on their applications. Beginning with the basic concepts of kinematics and dynamics the course proceeds to the discussion of the dynamics of a system of particles. The analytical (Lagrangian) method of dynamic analysis is treated in full detail. Both classical and modern formulations of the Lagrange equations including constraints are discussed and applied to the dynamic modeling of aerospace structures using the modal synthesis technique. A list of references is given at the end of the monograph.

Chapter 1

Kinematics

Kinematics relates to the geometry of motion disregarding the forces causing the motion.

1.1 Vectors

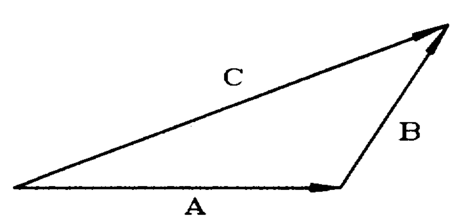
A vector has direction and magnitude (velocity, force, etc.)

Physical types of vectors:

1. free vector: velocity
2. sliding vector: force on rigid body
3. bound vector: position, force on elastic body

NOTE: All mathematical operations with vectors involve only their free vector properties of magnitude and direction.

Addition: The sum of two vectors is represented by the diagonal of a parallelogram formed by the two vector sides.



NOTE: Vectors are denoted by bolding the symbol.

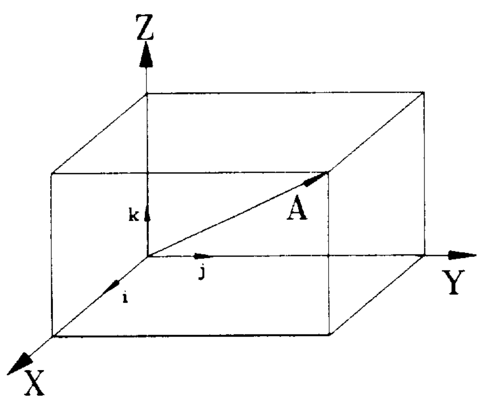
Vector addition is commutative and associative.

Unit Vector: A unit vector has unit magnitude (length).
Vectors are often conveniently expressed in terms of unit vectors:

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$$

where A_1, A_2, A_3 are known as (scalar) components of the vector **A**.

Often unit vectors are used which are (mutually) orthogonal: $\mathbf{i}, \mathbf{j}, \mathbf{k}$



$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$
Here the components A_x, A_y, A_z are the orthogonal projections of **A** onto the coordinate axes $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

NOTE: The unit vector in the direction of the vector **A** is identified as:

$$\mathbf{e}_A = \mathbf{A}/A \quad A = |\mathbf{A}|$$

P.S. #1: The German word for unit is "EINHEIT." This is the origin of the common use of the letter **e** for the unit vector.

P.S. #2: It is important to use suggestive symbols for physical quantities: m = small mass; M = large mass.

Scalar ("Dot") Product

$$\mathbf{A} \cdot \mathbf{B} = A\ B\ \cos\ \theta$$

where θ is the (smaller) angle between **A** and **B**. Expressed in terms of orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\mathbf{A} \cdot \mathbf{B} = (A_1\,\mathbf{e}_1 + A_2\,\mathbf{e}_2 + A_3\,\mathbf{e}_3) \cdot (B_1\,\mathbf{e}_1 + B_2\,\mathbf{e}_2 + B_3\,\mathbf{e}_3)$$

$$= A_1\,B_1 + A_2\,B_2 + A_3\,B_3$$

NOTE:

$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$ (UNIT LENGTH)

$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_3 = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0$ (ORTHOGONALITY)

Vector ("Cross") Product

$$\mathbf{A} \times \mathbf{B} = A\,B \sin\theta\,\mathbf{N} \qquad 0 \leq \theta \leq \pi$$

where θ is the smaller angle between **A** and **B**. **N** is perpendicular to **A** and **B** such that $\mathbf{A}, \mathbf{B}, \mathbf{N}$ form a right-handed system. (Right hand thumb rule)

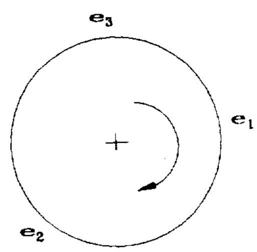
Expressed in terms of orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_1\,\mathbf{e}_1 + A_2\,\mathbf{e}_2 + A_3\,\mathbf{e}_3) \times (B_1\,\mathbf{e}_1 + B_2\,\mathbf{e}_2 + B_3\,\mathbf{e}_3) \\ &= \mathbf{e}_1(A_2\,B_3 - A_3\,B_2) + \mathbf{e}_2(A_3\,B_1 - A_1\,B_3) \\ &+ \mathbf{e}_3(A_1\,B_2 - A_2\,B_1) \end{aligned}$$

NOTE:

$$\begin{aligned} \mathbf{e}_1 \times \mathbf{e}_1 &= \mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_3 = 0 \\ \mathbf{e}_1 \times \mathbf{e}_2 &= -\mathbf{e}_2 \times \mathbf{e}_1 = \mathbf{e}_3 \\ \mathbf{e}_2 \times \mathbf{e}_3 &= -\mathbf{e}_3 \times \mathbf{e}_2 = \mathbf{e}_1 \\ \mathbf{e}_3 \times \mathbf{e}_1 &= -\mathbf{e}_1 \times \mathbf{e}_3 = \mathbf{e}_2 \end{aligned}$$

Mnemonic Code:



Arrange the three unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in clockwise order. The vector product of two unit vectors is equal to the third if the two follow each other clockwise and equal to the negative third if the two follow each other counterclockwise. This rule is especially useful if some components are zero; e.g.

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_1\,\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3) \times b_1\,\mathbf{e}_1 \\ &= -a_2\,b_1\,\mathbf{e}_3 + a_3\,b_1\,\mathbf{e}_2\end{aligned}$$

NOTE:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \qquad \text{(anticommutative)}$$

Scalar Triple Product:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

This product is geometrically equal to the volume of a parallelepiped of sides $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

Vector Triple Product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Moment of a Vector:

If the first vector in the vector product $\mathbf{A} \times \mathbf{B}$ represents a position vector $\mathbf{R}$ then the resultant vector product is called the moment of the vector $\mathbf{B}$:

$$\mathbf{M} = (\mathbf{R} \times \mathbf{B})$$

Velocity: It is the time derivative of the position vector $\mathbf{R}$:

$$\mathbf{V} = \tfrac{d\mathbf{R}}{dt} = \dot{\mathbf{R}} \qquad \begin{array}{l} |\mathbf{V}| = \text{Speed} \\ |\mathbf{R}| = \text{Distance} \end{array}$$

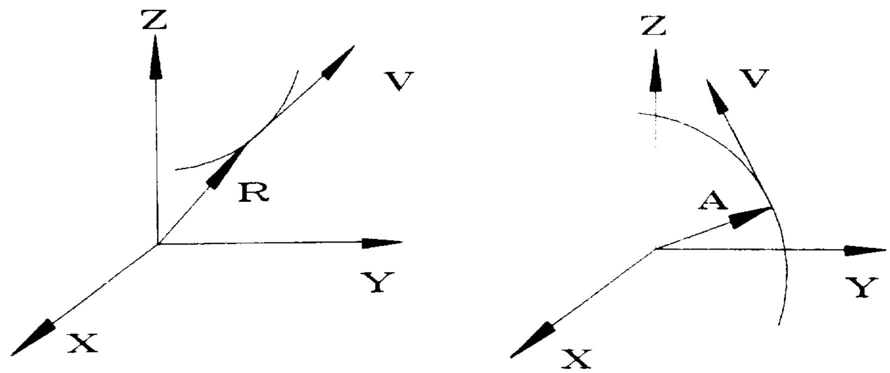
Acceleration: It is the time derivative of the velocity $\mathbf{V}$:

$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2} = \dot{\mathbf{V}} = \ddot{\mathbf{R}}$$

NOTE:

Sometimes we need the time derivatives of the scalar and vector products:

$$\begin{aligned} \tfrac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{A} \cdot \tfrac{d\mathbf{B}}{dt} + \tfrac{d\mathbf{A}}{dt} \cdot \mathbf{B} \\ &= \mathbf{A} \cdot \dot{\mathbf{B}} + \dot{\mathbf{A}} \cdot \mathbf{B} \\ \tfrac{d}{dt}(\mathbf{A} \times \mathbf{B}) &= \mathbf{A} \times \tfrac{d\mathbf{B}}{dt} + \tfrac{d\mathbf{A}}{dt} \times \mathbf{B} \\ &= \mathbf{A} \times \dot{\mathbf{B}} + \dot{\mathbf{A}} \times \mathbf{B} \end{aligned}$$

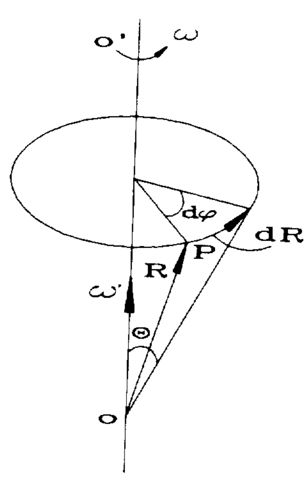


<u>TRAJECTORY</u>
$\mathbf{R} = \mathbf{R}(t)$

<u>HODOGRAPH</u>
$\mathbf{V} = \mathbf{V}(t)$

1.2 Angular Velocity

Finite angular rotations are not commutative and therefore cannot be treated as vectors. (Rotate a book 90° about the x and y axes and repeat the procedure in reverse order and observe the difference in final orientation). Angular velocity can be shown to be a vector. Consider the rotation of a point P about an axis OO′ called <u>instantaneous axis of rotation</u>.



$dR = R\ \sin\theta\ d\phi$
$\frac{d\mathbf{R}}{dt} = R\ \sin\theta\ \frac{d\phi}{dt}$
$= R\ \omega \sin\ \theta$
where $\omega = \frac{d\phi}{dt}$

We now assign a vector $\boldsymbol{\omega}$ in the direction of OO′ with the magnitude equal to ω. Then the linear velocity $\mathbf{v}$ of the point P due to this rotation is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R} \qquad \mathrm{R} = \mathrm{CONST} \tag{1.1}$$

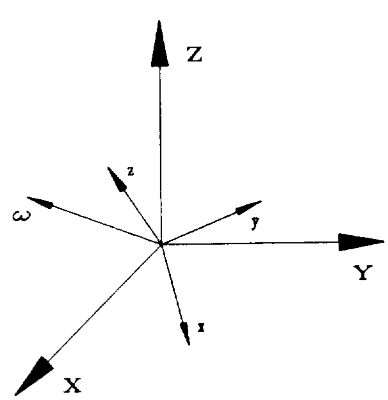
The magnitude of $\mathbf{v}$ is clearly $\omega\ R\ \sin\theta$ and its direction is normal to the plane spanned by $\boldsymbol{\omega}$ and $\mathbf{R}$. (Right-hand thumb rule: fingers indicate rotational sense, thumb is in direction of $\boldsymbol{\omega}$). If now a second axis is given through O, then the linear velocity to this rotational rate Ω is given by $\mathbf{V} = \Omega \times \mathbf{R}$. The total velocity of point P is then

$$\mathbf{V}_p = \mathbf{v} + \mathbf{V} = (\boldsymbol{\omega} \times \mathbf{R}) + (\boldsymbol{\Omega} \times \mathbf{R}) = (\boldsymbol{\omega} + \boldsymbol{\Omega}) \times \mathbf{R}$$

$$= \boldsymbol{\Omega}_p \times \mathbf{R} \ \text{ where } \ \boldsymbol{\Omega}_p = \boldsymbol{\omega} + \boldsymbol{\Omega} \qquad \text{Q. E. D.}$$

1.3 Vector Derivative in a Rotating Frame

A vector $\mathbf{A}$ is seen by an observer in a fixed frame X, Y, Z and also by another observer in a rotating frame x, y, z. Unit vectors in the fixed frame are denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and in the rotating frame by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The rotating frame has angular velocity $\boldsymbol{\omega}$ relative to the fixed frame.



$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$
$= A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$
$\mathbf{A}$ = generic vector

The rate of change of $\mathbf{A}$ as seen from the fixed frame:

$$\dot{\mathbf{A}} = \dot{A}_1\mathbf{e}_1 + \dot{A}_2\mathbf{e}_2 + \dot{A}_3\mathbf{e}_3 + A_1\dot{\mathbf{e}}_1 + A_2\dot{\mathbf{e}}_2 + A_3\dot{\mathbf{e}}_3$$

But:

$$\dot{\mathbf{e}}_1 = \boldsymbol{\omega} \times \mathbf{e}_1 \ , \ \dot{\mathbf{e}}_2 = \boldsymbol{\omega} \times \mathbf{e}_2 \ , \ \dot{\mathbf{e}}_3 = \boldsymbol{\omega} \times \mathbf{e}_3$$

Therefore:

$$\dot{\mathbf{A}} = (\dot{\mathbf{A}})_{rel} + \boldsymbol{\omega} \times \mathbf{A} \quad \text{where } (\dot{\mathbf{A}})_{rel} = \dot{A}_1\mathbf{e}_1 + \dot{\mathbf{A}}_2\mathbf{e}_2 + \dot{\mathbf{A}}_3\mathbf{e}_3 \tag{1.2}$$

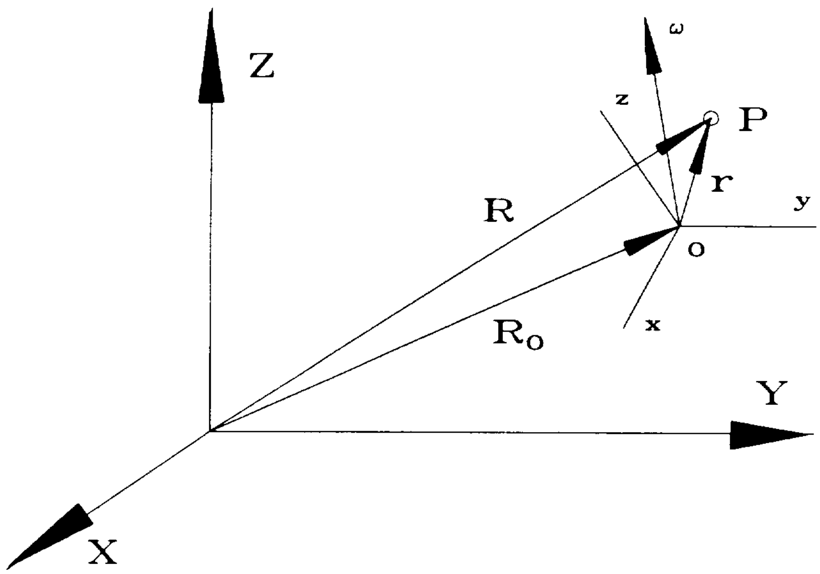
This relation holds for any two systems A and B:

$$(\dot{\mathbf{A}})_A = (\dot{\mathbf{A}})_B + \boldsymbol{\omega}_{BA} \times \mathbf{A}$$

NOTE: If $\mathbf{A} = \boldsymbol{\omega}_{BA}$ then $(\dot{\boldsymbol{\omega}}_{AB})_A = (\dot{\boldsymbol{\omega}}_{BA})_B$ where $\boldsymbol{\omega}_{BA}$ is the angular velocity of B as seen in A.

1.4 General Motion in a Moving Frame

The X, Y, Z frame is fixed (inertial frame) and the x, y, z frame rotates relative to it.



NOTATION

$\mathbf{R}_0$ = Position of Origin

$\mathbf{R} = \mathbf{R}_0 + \mathbf{r}$ Position of P

$\mathbf{v}$ = relative velocity of P in x, y, z

$\mathbf{a}$ = relative acceleration of P in x, y, z

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_0 + \dot{\mathbf{r}} = \dot{\mathbf{R}}_0 + (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}) \tag{1.3}$$

$$\ddot{\mathbf{R}} = \ddot{\mathbf{R}}_0 + \dot{\mathbf{v}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}$$

$$\ddot{\mathbf{R}} = \ddot{\mathbf{R}}_0 + (\mathbf{a} + \boldsymbol{\omega} \times \mathbf{v}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r})$$

$$\ddot{\mathbf{R}} = \ddot{\mathbf{R}}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\,(\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{a} \tag{1.4}$$

Alternate Form:

$$\ddot{\mathbf{R}} = \dot{\mathbf{V}}_0 + \boldsymbol{\omega} \times \mathbf{V}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\,(\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{a} \tag{1.5}$$

NOTE:

It is very important to have a thorough understanding of the physical meaning of the five acceleration terms on the right-hand side of Eq. (1.4).

1. $\ddot{\mathbf{R}}_0$ = acceleration of O of moving frame (D'Alembert/Einstein acceleration)
2. $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ = "slingshot" acceleration (Euler acceleration)
3. $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ = centripetal acceleration
4. $2(\boldsymbol{\omega} \times \mathbf{v})$ = Coriolis acceleration
5. $\mathbf{a}$ = relative acceleration as seen in moving frame

P. S.

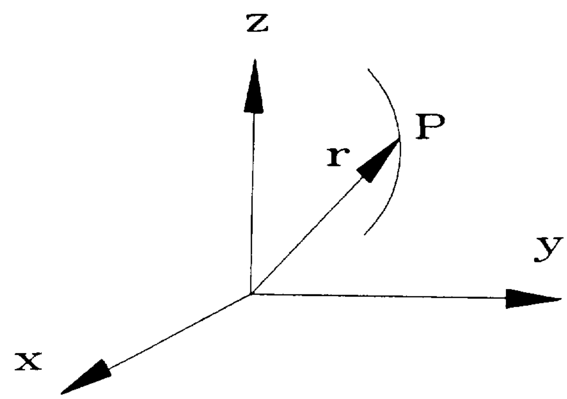
The Coriolis (1792-1843) acceleration is somewhat difficult to visualize. It is composed of two separate kinematical effects: one velocity change is due to a change in the direction of $\mathbf{v}$ due to $\boldsymbol{\omega}$ ("slingshot" effect), the other velocity change is due to a radial change of the point P position. Both changes of velocity are equal to $(\boldsymbol{\omega} \times \mathbf{v})$ resulting in the factor 2 in the Coriolis acceleration. Mathematically:

$$\mathbf{a}_1 = (\boldsymbol{\omega} \times \mathbf{v}) \qquad \text{2nd term of Equation 1.2}$$

$$\mathbf{a}_2 = \boldsymbol{\omega} \times (\frac{d\mathbf{r}}{dt})_{Rel} = (\boldsymbol{\omega} \times \mathbf{v})$$

Tangential and Normal Components

Consider the position of a point P as it moves along a curved path in space.



s = distance along curve

Velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{dt} = v \, \mathbf{e}_T$$

where

$$\mathbf{e}_T = \frac{d\mathbf{r}}{ds} = \underline{\text{tangent unit vector}}$$

Acceleration:

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{v}\mathbf{e}_T + v \, \dot{\mathbf{e}}_T = \dot{v}\mathbf{e}_T + v\frac{d\mathbf{e}_T}{ds} \cdot \frac{ds}{dt} = \dot{v}\mathbf{e}_T + v^2\frac{d\mathbf{e}_T}{ds}$$

Define:

$$\frac{d\mathbf{e}_T}{ds} \equiv \kappa \mathbf{e}_N \tag{1.6}$$

where

κ	=	curvature [rad/meter]
$\frac{1}{\kappa}$	=	ρ = radius of curvature
$\mathbf{e}_N$	=	normal unit vector

The acceleration is then given by

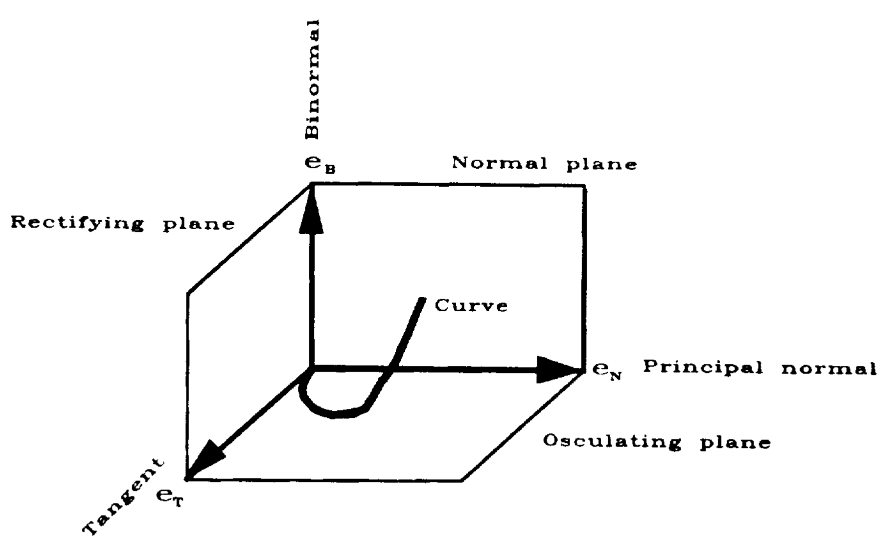
$$\mathbf{a} = a\ \mathbf{e}_T + v^2 \kappa \mathbf{e}_N = a\ \mathbf{e}_T + \frac{v^2}{\rho}\ \mathbf{e}_N$$

The first term is the tangential acceleration and the second term is the normal or centripetal acceleration.

Define a third unit vector to complete the orthogonal triad of unit vectors (Trihedron) at the point P:

$$\mathbf{e}_B \equiv \mathbf{e}_T \times \mathbf{e}_N$$

$\mathbf{e}_B$ = binormal unit vector



$$
\begin{aligned}
(\mathbf{e}_T, \mathbf{e}_N) &= \text{Osculating Plane} \\
(\mathbf{e}_B, \mathbf{e}_N) &= \text{Normal Plane} \\
(\mathbf{e}_T, \mathbf{e}_B) &= \text{Rectifying Plane}
\end{aligned}
$$

The curvature κ measures the rotation rate of the normal plane as the point P moves along the curve.

Now we differentiate the binormal unit vector $\mathbf{e}_B$ with respect to the distance along the curve:

$$\frac{d\mathbf{e}_B}{ds} = \mathbf{e}'_B = \mathbf{e}'_T \times \mathbf{e}_N + \mathbf{e}_T \times \mathbf{e}'_N = (\kappa \mathbf{e}_N) \times \mathbf{e}_N + \mathbf{e}_T \times \mathbf{e}'_N = \mathbf{e}_T \times \mathbf{e}'_N$$

Therefore, $\mathbf{e}'_B \perp \mathbf{e}_T$.

Since $\mathbf{e}_B$ is a unit vector, we also have $\mathbf{e}'_B \perp \mathbf{e}_B$. Therefore, $\mathbf{e}'_B$ must be parallel to $\mathbf{e}_N$.

Define:

$$\frac{d\mathbf{e}_B}{ds} \equiv -\tau\, \mathbf{e}_N \tag{1.7}$$

where τ = torsion [rad/meter], $\frac{1}{\tau} = \sigma$ = radius of torsion

A positive torsion ($\tau > 0$) corresponds to a clockwise rotation for a motion of P along the curve. The torsion $\tau = 0$ for a plane curve. The torsion measures the rotation rate of the osculating plane as the point P moves along the curve.

Since $\mathbf{e}_N = \mathbf{e}_B \times \mathbf{e}_T$ we obtain the spatial derivative of $\mathbf{e}_N$ as:

$$\mathbf{e}'_N = \mathbf{e}'_B \times \mathbf{e}_T + \mathbf{e}_B \times \mathbf{e}'_T = (-\tau \mathbf{e}_N) \times \mathbf{e}_T + \mathbf{e}_B \times (\kappa \mathbf{e}_N)$$

or

$$\frac{d\mathbf{e}_N}{ds} = \tau \mathbf{e}_B - \kappa \mathbf{e}_T \qquad \text{DARBOUX Vector} \tag{1.8}$$

The set of Equations 1.6, 1.7, and 1.8 is collectively called the Frenet-Serret formulas.

The angular velocity of the trihedron can be easily obtained in terms of the curvature κ, the torsion τ and the velocity v of the point P as:

$$\mathbf{w} = \kappa \, v \, \mathbf{e}_B - \tau \, v \, \mathbf{e}_T$$

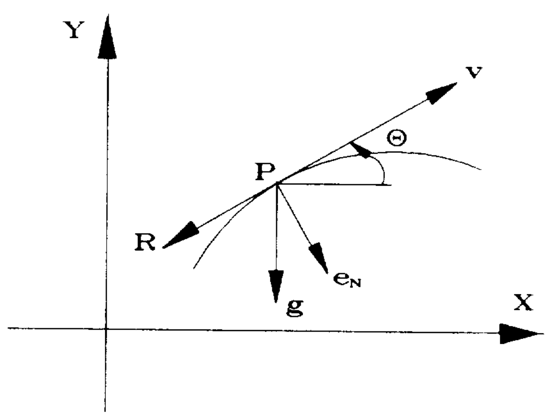
The first term represents the angular velocity of the tangential unit vector and the second term the angular velocity of the binormal unit vector.

Ballistics Equations

The motion of a projectile through the air is often analyzed using path variables, which are measurements made along the tangential and normal direction of the trajectory or path. Because of their convenience the (N-T) coordinates are referred to as natural coordinates.

The resistance (drag) is taken proportional to a power of the tangential velocity v

$$R = k \, f(v)$$



The equations of motion are then

$$m \, \dot{v} = -mg \, \sin \, \theta + k \, f(v) \tag{1.9}$$

$$\frac{mv^2}{\rho} = g \, m \, \cos \, \theta \tag{1.10}$$

The radius of curvature can be related to the path angle θ:

$$\frac{1}{\rho} = -\frac{d\theta}{ds} = -\frac{1}{v}\frac{d\theta}{dt} \tag{1.11}$$

The minus sign is taken to agree with our definition of the flight path angle θ in the figure.

Dividing by m and introducing Equation 1.11 in Equation 1.10 yields:

$$\dot{v} = -g \sin \theta - c\, f\,(v) \tag{1.12}$$

$$v\dot{\theta} = -g \cos \theta \tag{1.13}$$

where $c = \frac{k}{m}$ = <u>ballistic coefficient</u>

To obtain the position of the projectile as a function of time in the (x, y) coordinate system we have to use the inertial velocities:

$$\dot{x} = v \cos \theta \tag{1.14}$$

$$\dot{y} = v \sin \theta \tag{1.15}$$

Thus, Eqs. (1.12) - (1.15) represent the equations of motion of the mass center of the projectile in the plane. Their general solutions have to be obtained by numerical integration.

Chapter 2

Dynamics of a Particle

2.1 Newton's Laws

We introduce Newton's laws of motions as axioms. (Axiom = a self-evident or accepted principle.) The truth of these axioms is established by experimental verification or prediction.

Isaac Newton (1642-1727) published his laws in 1687 in Latin. Using modern language they are

1. Every body continues in its state of rest or of uniform motion in a straight line, unless compelled to change that state by forces acting on it.
2. The time rate of change of linear momentum of a body is proportional to the force acting upon it and occurs in the direction in which the force acts.
3. To every action there is an equal and opposite reaction; that is, the mutual forces of two bodies acting upon each other are equal in magnitude and opposite in direction.

In a rigorous sense these laws apply only to a mass point or single particle.

NOTE:

The first law is only a special case of the second law when there are no external forces. The third law will later allow the transition from the dynamics of a single particle to the dynamics of a system of particles.

Dimensions and Units

The world of dynamics can be described in terms of four dimensions: Length (L), Time (T), Force (F), and Mass (M). It is, however, customary to treat the dimension of mass as a primary dimension and derive the dimension of force via Newton's law $F = ma$ or vice versa.

NOTE: This contrivance is also common in other areas: distance expressed in carhours or lightyears. (Find other examples).

SI - System (Metric)

Basic Dimension	Unit
Length	Meter (m)
Time	Second (s)
Mass	Kilogram (kg)
Derived Dimension	**Unit**
Force	Newton ($kg\ m\ s^{-2}$)

Customary System (British)

Basic Dimension	Unit
Length	Foot (ft)
Time	Second (s)
Force	Pound (lb)
Derived Dimension	**Unit**
Mass	Slug ($lbfts^{-2}$)

NOTE:

$$Weight = Force$$

$$W = mg_0$$

$$g_0 = 9.81 \quad \frac{m}{s^2} = 32.2 \quad \frac{ft}{s^2}$$

<u>Question:</u>

a) How much does 1 slug weigh?

$$W = mg_0 = \frac{1\ lb\ sec^2}{ft} \times 32.2 \quad \frac{ft}{sec^2} = 32.2 \quad lbs$$

$$1 \text{ slug} \doteq 32.2 \text{ lb}$$

b) How much does 1 kg weigh?

$$W = mg_0 = 1kg \times 9.81 \quad \frac{m}{s^2} = 9.81 \quad \frac{kg\ m}{s^2}$$

$$1 \text{ kg} \doteq 9.81 \text{ Newton}$$

P. S. The metric system sometimes allows for the auxiliary unit of force called kilopond (kp).

$$1 \text{ kp} = 9.81 \text{ Newton}$$

The customary system is sometimes used with inches as the length unit. The unit of mass is then variously called SLINCH, SNAIL or MUG (lb sec^2/in).

$$1 \text{ SLINCH} \doteq 386.4 \text{lbs.}$$

2.2 D'Alembert's Principle (1747)

With a stroke of genius d'Alembert (1717-1783) wrote Newton's law in the form:

$$\mathbf{F} + (-m\,\ddot{\mathbf{R}}) = 0$$

It is exactly this apparent triviality which makes D'Alembert's principle such an ingenious step. D'Alembert principle introduces a new force, the force of inertia and makes possible the use of moving reference frame. The inertial forces act exactly like all the other forces. They cannot be distinguished in their nature from any other impressed (external) force. If an observer is not aware that he is in an accelerated system, then purely mechanical observations cannot reveal to him that fact. Einstein revised D'Alembert's principle to a general principle of nature in his gravitational theory (Equivalence principle).

Einstein's "box experiment" (Gedanken experiment)

A closed elevator is pulled upwards with constant acceleration g_0 while at the same time, the gravity field disappears. It is then impossible to distinguish between the following two hypotheses:

1. The elevator moves upward with constant acceleration g_0. No gravity field exists.

2. The elevator is at rest but there is a gravity field of magnitude g_0.

NOTE:

The name "apparent force" for the force of inertia is misleading, if it is interpreted as a force which is not as "real" as any other external force. Sometimes inertia forces are also called "fictitious" or "effective" forces.

Because of D'Alembert's principle it is possible to analyze all dynamic phenomena in making reference frames with strict rigor and consistency.

Staying consistently in a moving reference frame, Newton's law can be reformulated using Equation 1.4; notice that the interest lies now in the relative acceleration $\mathbf{a}$ and not in the absolute acceleration $\ddot{\mathbf{R}}$.

$$m\,\mathbf{a} = \mathbf{F} - m\,\ddot{\mathbf{R}}_0 - m\,(\dot{\boldsymbol{\omega}} \times \mathbf{r}) - m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] - 2\,m(\boldsymbol{\omega} \times \mathbf{v}) \qquad (2.1)$$

or

$$m\,\mathbf{a} = \mathbf{F} + \mathbf{A} + \mathbf{E} + \mathbf{C} + \mathbf{K} \qquad (2.2)$$

where

$\mathbf{A} = -m\,\ddot{\mathbf{R}}_0$	d'Alembert force
$\mathbf{E} = -m\,(\dot{\boldsymbol{\omega}} \times \mathbf{r})$	Euler force
$\mathbf{C} = -m\,[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$	Centrifugal force
$\mathbf{K} = -2\,m(\boldsymbol{\omega} \times \mathbf{v})$	Coriolis force or compound centrifugal force

P. S. #1

Sometimes it is stated that D'Alembert's principle transforms a problem in dynamics to one in statics. The Coriolis force does no work and is therefore called a workless ("wattless") force.

P. S. #2

D'Alembert's Principle may be called the Equal Rights Amendment (ERA) of dynamics because it declares the inertial forces to be equal to any other force.

2.3 Work; Kinetic and Potential Energy

The work done by a force **F** as it moves along a path from A to B is defined as the line integral:

$$W = \int_A^B \mathbf{F} \cdot d\,\mathbf{r} \qquad \underline{\text{Work}}$$

With Newton's law:

$$\mathbf{F} = m\,\ddot{\mathbf{r}}$$

we obtain:

$$W = \int_A^B \mathbf{F} \cdot d\,\mathbf{r} = \int_A^B m\,\ddot{\mathbf{r}} \cdot d\,\mathbf{r}$$

Remember:

$$\ddot{\mathbf{r}} \cdot d\,\mathbf{r} = \frac{1}{2}\frac{d}{dt}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})dt = \frac{1}{2}dv^2$$

Therefore:

$$\int_A^B m\,\ddot{\mathbf{r}} \cdot d\mathbf{r} = \frac{m}{2}\int_A^B d(v^2) = \frac{m}{2}\,(v_B^2 - v_A^2)$$

Introduce:

$$T = \frac{1}{2}\,m\,v^2 \qquad \text{Kinetic Energy}$$

$$W = T_B - T_A \tag{2.3}$$

The increase in the kinetic energy of a particle is equal to the work done by the external force.

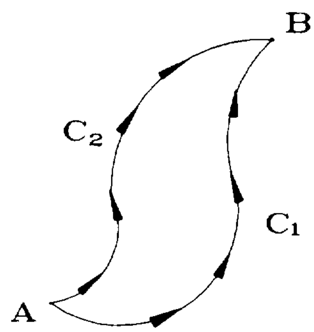
P.S.

It was assumed that the force field $\mathbf{F} = \mathbf{F}(\mathbf{r})$ is only a function of the position $\mathbf{r}$ and not of time also.

Potential Energy

Force Field $\mathbf{F} = \mathbf{F}(\mathbf{r})$

I. The line integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of path.



Work done by force

$W = \int_{C_1} \mathbf{F} \cdot d\,\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\,\mathbf{r}$

The following alternate statements are all equivalent to I:

II. The contour integral vanishes.

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = -\int_B^A \mathbf{F} \cdot d\mathbf{r} \Rightarrow \oint \mathbf{F} \cdot d\,\mathbf{r} = 0$$

III. $\mathbf{F} = -\text{grad } V$ where V = Potential Energy

IV. Curl $\mathbf{F} = \nabla \times \nabla V = 0$ ∇ = Nabla operator (Del-Operator)

NOTE:

In Electrodynamics: $\mathbf{E} = -$ grad V where V = electrical potential.
In Fluid Dynamics: $\mathbf{v} =$ grad ϕ where ϕ = velocity potential.

The work done by the external force is now:

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{r} = -\int_A^B \text{grad}V \cdot d\,\mathbf{r} = -\int_A^B dV = \int_B^A dV = V_A - V_B$$

Inserting in Equation 2.3 yields

$$V_A - V_B = T_B - T_A$$

or

$$V_A + T_A = V_B + T_B$$

This is the principle of mechanical energy conservation. Therefore such forces or force fields are called conservative.

NOTE:

Sometimes forces in nature are derivable from a time-dependent potential $V = V(\mathbf{r}, t)$. For these statements I to IV hold if the time is kept constant (t = CONST). These forces or force fields are called irrotational. Energy is not conserved.

P.S.

In earlier practice, it was customary to use the negative of the potential energy which was called the work function $U = -V$. In view of the above described conservation law it was an advantage to change this sign. The operational "vector" ∇ was introduced by Sir William Hamilton (1805-1865). The name "nabla" was coined by Oliver Heavyside (1850 - 1925) after an ancient Assyrian harp whose form it resembles. It is also called "atled." (Delta spelled backwards)

Consider now the work done against the external force:

$$W = -\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \text{grad } V \cdot d\mathbf{r} = \int_A^B dv = V_B - V_A = \Delta V$$

Therefore, the difference of the potential energy is the work I have to do against the force going from A to B.

Example 1:

Gravitational Force

$$\mathbf{F} = -\frac{mg\ R_0^2}{R^2}\,\mathbf{e}_r \qquad R_0 = \text{earth radius}$$

Work done against gravity:

$$W = -\int_{r_0}^{r} \mathbf{F} \cdot d\mathbf{r} = mg\ R_0^2 \int_{r_0}^{r} \frac{dR}{R^2} = mgR_0^2\left(\frac{1}{r_0} - \frac{1}{r}\right) = V(r) - V(r_0)$$

At the reference point r_0 we set the potential energy to zero: $V(r_0) = 0$

a) Reference point: $r_0 = R_0$ (sea level)

$$V(r) = mg\ R_0^2\left(\frac{1}{R_0} - \frac{1}{r}\right) = mgR_0^2\left(\frac{1}{R_0} - \frac{1}{R_0 + h}\right)$$

where h = height above ground.

$$V(h) = \frac{mg\ R_0^2\ h}{R_0(R_0 + h)}$$

If $h << R_0$ $\quad V(h) = mg\ h$

b) Reference Point: $r_0 = \infty$

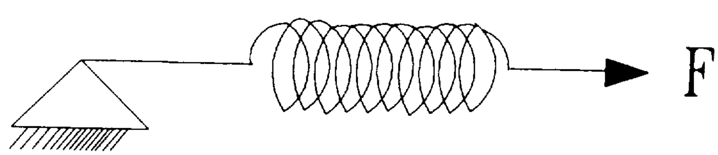
$$V(r) = -\frac{mg\ R_0^2}{r}$$

P. S.

The potential energy per unit mass is called the potential.

$$V^*(r) = \frac{V(r)}{m} = -\frac{gR_0^2}{r}$$

Example 2: Linear Spring Force



$$F = kx$$

$$V = \int_A^B k\ x\ d\ x = \frac{1}{2}kx^2\ |_A^B = \frac{1}{2}\ k\ (x_B^2 - x_A^2)$$

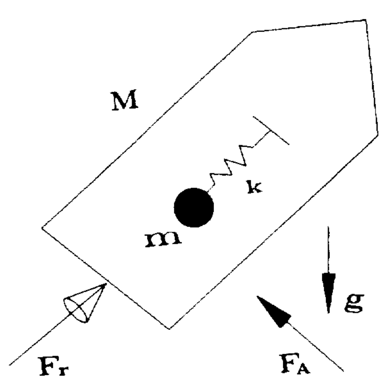
If the spring is initially unstretched, the potential energy of the spring for the elongation x is:

$$V = \frac{1}{2}kx^2$$

2.4 Applications of D'Alembert's Principle

Example 1

Dynamics of an accelerometer inside a rocket.



M = rocket mass

k = spring constant

m = mass of acceleration

r = displacement along sensitive axis

$\mathbf{F}_r$ = thrust force

g = gravity

$\mathbf{F}_A$ = aerodynamic force

Reference Frame is fixed in rocket with origin R_0.

$$m\,\mathbf{a} = -k\mathbf{r} + m\mathbf{g} - m[\ddot{\mathbf{R}}_0 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) + (\dot{\boldsymbol{\omega}} \times \mathbf{R}) + 2(\boldsymbol{\omega} \times \mathbf{v})]$$

where $\mathbf{R}$ = distance of m from origin O.

Dynamics of Rocket:

$$M\,\ddot{\mathbf{R}}_0 = M\mathbf{g} + \mathbf{F}_A + \mathbf{F}_r$$

$$m\,\mathbf{a} = -k\mathbf{r} + m\mathbf{g} - m\left[\mathbf{g} + \frac{F_A}{M} + \frac{F_r}{M} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) + (\dot{\boldsymbol{\omega}} \times \mathbf{R}) + 2(\boldsymbol{\omega} \times \mathbf{v})\right]$$

The gravity force mg cancels!

NOTE:

The (steady-state) surface of a liquid inside a rocket is perpendicular to the combined thrust and aerodynamic forces.

Example 2:

Particle on turntable which rotates with uniform angular velocity $\boldsymbol{\omega}$ (no friction).

$$\mathbf{a} = -2(\boldsymbol{\omega} \times \mathbf{v}) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$\boldsymbol{\omega} = \omega\mathbf{k} \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

(1) $\ddot{x} = 2\omega\dot{y} + \omega^2 x$

(2) $\ddot{y} = -2\omega\dot{x} + \omega^2 y$

Introduce:

$z = x + i\,y$ complex variable

Multiply Eq. (2) by i and add to (1):

$$\ddot{z} + 2\, i\, \omega\, \dot{z} - w^2 z = 0$$

Assume: $z = z_0\, e^{st}$

Characteristic Equation: $s^2 + 2\, i\, \omega\, s - \omega^2 = (s + i\, \omega)^2 = 0$

$s_1 = -i\, \omega \qquad s_2 = -i\, w \qquad$ Repeated Root!

$$z = (z_0 + z_1\, t)e^{-i\omega t}$$

Remember: Variation of Parameters.

Initial Conditions: $z(0) = 0 \qquad \dot{z}(0) = v_0$ (COMPLEX)

$$z = v_0 t\, e^{-iwt}$$

The particle moves radially outward with uniform velocity which is superimposed by clockwise angular velocity.

The path of the particle in (x, y) plane is:

Set $v_0 = \dot{x}_0 \rightarrow z = \dot{x}_0 t(\cos\ \omega t - i\ \sin\ \omega t)$,

$x = \dot{x}_0 t\ \cos \omega t$,

$y = -\dot{x}_0 t\ \sin \omega t$

Polar Coordinates:

$$x = r\ \cos \phi$$
$$y = r\ \sin \phi$$

$$x^2 + y^2 = \dot{x}_0^2\, t^2 = r^2 \rightarrow \; r \; = \dot{x}_0 t$$

Let $\phi = -\omega t$ $\qquad\qquad\qquad r = -(\frac{\dot{x}_0}{\omega})\phi$

Archimedian Spiral

<u>Kinetic Energy</u>:

$$T = \frac{1}{2}\, m\, v^2 = \frac{1}{2}\, m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}\, m\, \Big[(\dot{x}_0\; \cos\omega t - \dot{x}_0 t\; \omega\; \sin\omega t)^2$$

$$+(\dot{x}_0\; \sin\omega t + \dot{x}_0 t\; \omega \cos\; \omega t)^2\Big]$$

$$T = \frac{1}{2}\, m\, \dot{x}_0^2 + \frac{1}{2}\, m \dot{x}_0^2 t^2\, \omega^2$$

From Archimedian Spiral: $t^2 = \frac{r^2}{\dot{x}_0^2}$

$$T = \frac{1}{2}\, m\, \dot{x}_0^2 + \frac{1}{2}\, m\, \omega^2\, r^2$$

The second term is equal to the work done by the centrifugal force.

$$W = \int_0^r\, F dr = \int_0^r\; m\, \omega^2 r dr = \frac{1}{2}\, m\omega^2 r^2$$

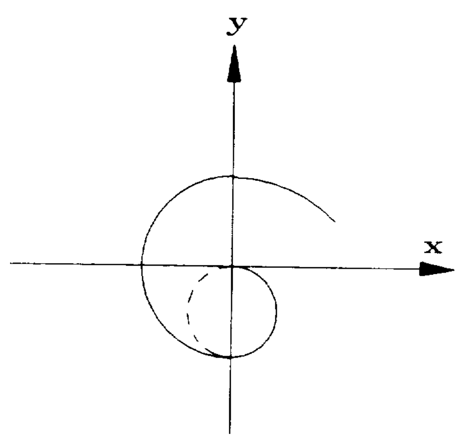
The centrifugal force for $\omega =$ const is a conservative force. For $\omega = \omega(t)$, it is an irrotational force.

NOTE:

The Coriolis force does no work because it is perpendicular to the velocity. It is a wattless force like the Lorentz force in electrodynamics. $[P = \mathbf{F}\cdot\mathbf{v} = -2m(\boldsymbol{\omega}\times\mathbf{v})\cdot\mathbf{v}]$

P.S.

If the centrifugal force had been neglected as a higher order term of ω then the path would be a circle (dashed line) in figure.



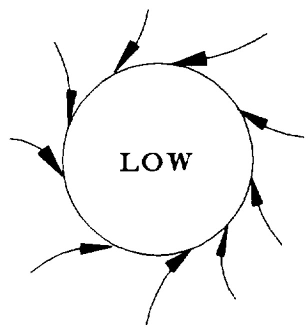
Historical Note:

During the British-German naval battle of the Falkland Islands (about 50^o latitude), the British gun shots landed almost one hundred yards to the left of the German ships, because the firing tables had been calculated for Britain's northern latitude.

NOTE:

For a projectile on the surface of the earth, the effective angular velocity is $\omega = \omega_0 \sin\ \phi$ where $\phi =$ latitude.

Example 3: Buys-Ballot Lane (1817-1890)



In a cyclone, the winds rotate about a center of low atmospheric pressure clockwise in the southern hemisphere and counterclockwise in the northern.

Example 4: Foucault Pendulum (1851)

$$\ddot{x} = 2\omega \sin\phi \dot{y} + (\omega^2 \sin^2\phi)x - g\frac{x}{l}$$

$$\ddot{y} = -2\omega \sin\phi \dot{x} + (\omega^2 \sin^2\phi)y - g\frac{y}{l}$$

Introduce: $z = x + iy$ and repeating the steps of example 2 yields:

$$\ddot{z} + 2i\,u\,\dot{z} + (\frac{g}{l} - u^2)z = 0$$

where $u = \omega \sin\phi$ (ϕ = latitude)

Assume: $z = Ae^{st}$

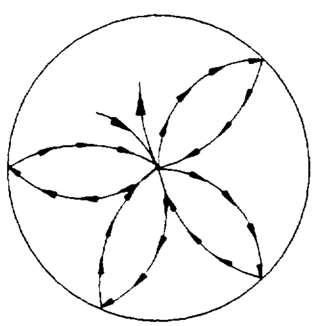
$$s^2 + 2\,i\,u\,s + \frac{g}{l} = 0$$

$$s_1 = (-u + \sqrt{\frac{g}{l}})i \qquad s_2 = (-u - \sqrt{\frac{g}{l}})i$$

General Solution:

$$z = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

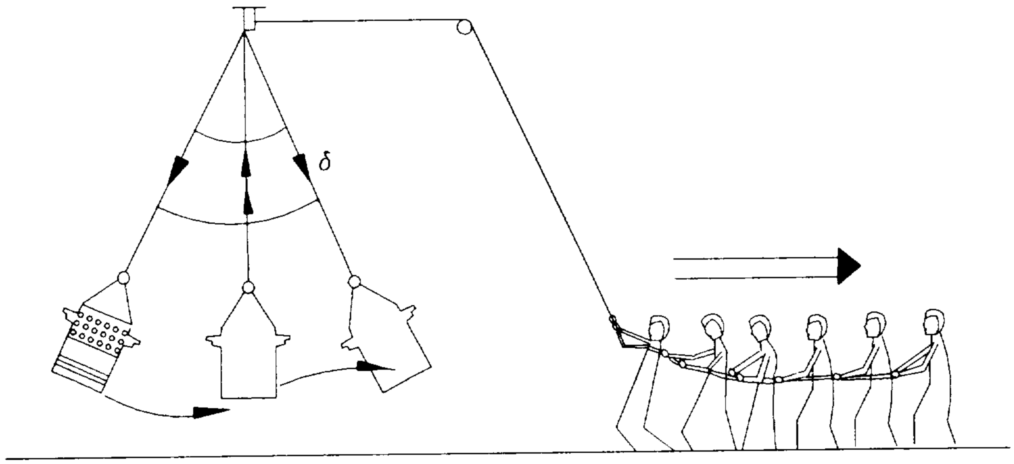
$$z = (A_1 e^{i\sqrt{\frac{g}{l}}t} + A_2 e^{-i\sqrt{\frac{g}{l}}t})e^{-i\omega t}$$



The complex vector z rotates with clockwise rotation in the northern hemisphere while performing a pendulous motion. The rotational angular velocity is $\omega \sin\phi$ and the pendulous motion has frequency of $\omega_p = \sqrt{\frac{g}{l}}$

<u>Example 5</u>: "Incense Swing"

Pilgrims to Santiago de Campostella, Spain, visit the shrine of St. James to burn incense. The incense and charcoal are held in a large silver brazier hung from the ceiling. The brazier is set swinging with a small amplitude, and then it is pumped by about six men until it is swinging through 180°. The swinging makes the charcoal burn energetically for the pilgrims. The pumping is the interesting part: they do it by shortening the rope by about a meter each time is passes through the vertical; they release the same amount of rope when the container reaches its maximum height. How does this shortening and lengthening of the rope increase the amplitude?



Each time the pendulum is suddenly shortened by δ and then lengthened by the same amount at the extreme position.

The motion of the pendulum is

$$\ddot{\phi} + \omega_0^2 \sin\phi = 0 \qquad \text{where } \omega_0^2 = g/l$$

and

$$\dot{\phi}^2 = 2\omega_0^2(\cos\phi - \cos\phi_0)$$

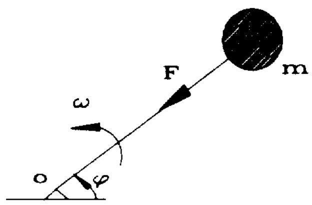
<u>Tension</u>: $T = m(\dot{\phi}^2\, l + g\cos\phi)$

$$\underline{\phi = 0:}\; W_1 = 2\delta m[2\omega_0^2(1 - \cos\phi_0)l + g]$$

$$\underline{\phi = \phi_0:}\; W_2 = 2\delta mg\cos\phi_0$$

Energy Increase: $\Delta E = W_1 - W_2 = \underline{6 \text{ mg } \delta \ (1 - \cos\phi_0)}$.

Example 6: "Twirling Ice Skater".



A whirling particle of mass m is pulled in by a string toward a fixed center at O.

$$m\mathbf{a} = \mathbf{F} - m(\dot{\omega} \times \mathbf{r}) - m\omega \times (\omega \times \mathbf{r}) - 2m(\omega \times \mathbf{v})$$

a) Angular Motion: (ϕ - direction)

$$m\dot{\omega}r + 2m\omega\dot{r} = 0$$

$$\dot{\omega}r + 2\omega\dot{r} = 0 \Rightarrow \int_{\omega_0}^{\omega} \frac{d\omega}{\omega} + 2\int_{r_0}^{r} \frac{dr}{r} = 0$$

$$ln(\frac{\omega}{\omega_0}) + ln(\frac{r}{r_0})^2 = 0$$

or

$$m\omega r^2 = m\omega_0 r_0^2 \qquad \text{(Angular Momentum is conserved)}$$

b) Energy

Work done against centrifugal force

$$W = -\int F dr = -m\int_{r_0}^{r} \omega^2 r dr = -m(\omega_0 r_0^2)^2 \int_{r_0}^{r} \frac{dr}{r^3}$$

$$W = \frac{1}{2}m(\omega_0 r_0^2)^2(\frac{1}{r^2} - \frac{1}{r_0^2}) \quad \text{where} \quad r_0 > r.$$

Difference of kinetic energy:

$$\Delta T = T - T_0 = \frac{1}{2}m(\omega^2 r^2 - \omega_0^2 r_0^2)$$

$$\Delta T = \frac{1}{2}m\left(\frac{(\omega_0^2 r_0^2}{r^2} - \omega_0^2 r_0^2\right) = \frac{1}{2}m\left(\frac{r_0^4\Omega_0^2}{r^2} - \omega_0^2 r_0^2\right)$$

$$\Delta T = \frac{1}{2}m(\omega_0 r_0^2)^2(\frac{1}{r^2} - \frac{1}{r_0^2}) = W \qquad \text{Q.E.D.}$$

Chapter 3

Dynamics of a System of Particles

The laws of motion will now be extended to a group of N particles which act on each other within a certain bounding envelope. The interaction forces between these mass particles are assumed to obey Newton's third law:

$$\mathbf{R}^*_{ik} = -\mathbf{R}^*_{ki}$$

3.1 Translation and Rotation

Summing up all the forces (external and internal) Newton's second law is:

$$m_i \, \ddot{\mathbf{R}}_i = \mathbf{F}_i + \sum_{k=1}^{N} \mathbf{R}^*_{ik} \tag{3.1}$$

where the sum extends over all N particles. Then summing over all N particles gives:

$$\sum m_i \, \ddot{\mathbf{R}}_i = \sum \mathbf{F}_i + \sum \sum \mathbf{R}^*_{ik} \tag{3.2}$$

Because of Newton's third law the total interaction between the particles becomes zero. Therefore:

$$\sum m_i \, \ddot{\mathbf{R}}_i = \sum \mathbf{F}_i \equiv \mathbf{F} \tag{3.3}$$

where **F** is the total external force.

Remember:

$\mathbf{R}_i$ is the position vector of the i-th particle in the fixed (inertial) frame. The associated acceleration $\ddot{\mathbf{R}}_i$ is called the inertial acceleration. ($\dot{\mathbf{R}}_i$ is the inertial velocity.)

Using Equation 1.4 of Section 1.4, Equation 3.3 can be written in the form:

$$\sum m_i \left[\ddot{\mathbf{R}}_0 + (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) + 2(\boldsymbol{\omega} \times \mathbf{v}_i) + \mathbf{a}_i\right] = \mathbf{F} \qquad (3.4)$$

This is the translation equation of motion referred to a moving frame.

Very often (but not always) it is of advantage to select the origin of the moving frame such that:

$$\sum m_i \, \mathbf{r}_i = 0 \qquad \text{at all times} \qquad (3.5)$$

This means also that:

$$\sum m_i \, \mathbf{v}_i = 0 \quad \text{and} \quad \sum m_i \, \mathbf{a}_i = 0 \qquad (3.6)$$

In other words, the origin of the moving frame is made to coincide with the center of mass (C.M.) location of the particle system. By definition the C.M. is:

$$\mathbf{R}_c = \frac{1}{m} \sum m_i \, \mathbf{R}_i \qquad \text{where} \quad m = \sum m_i \qquad (3.7)$$

In this case the translation equation (3.4) simplifies to:

$$m \, \ddot{\mathbf{R}}_0 = m \, \ddot{\mathbf{R}}_c = \mathbf{F} \qquad (3.8)$$

This is the center of mass law which states that the C.M. moves as if the sum of all external forces were acting on the total system mass concentrated at the C.M.

NOTE:

The total force **F** has to be determined for the actual system of particles and not for the total mass at the C.M.

Examples:

Exploding bomb shell; chair being thrown out of window.

Next consider taking moments of Equation 3.1 about the origin O of a moving frame:

$$\mathbf{r}_i \times (m_i \ddot{\mathbf{R}}_i) = \mathbf{r}_i \times (\mathbf{F}_i + \sum \mathbf{R}^*_{ik}) \tag{3.9}$$

Again summing over all N particles yields:

$$\sum \mathbf{r}_i \times (m_i \ddot{\mathbf{R}}_i) = \sum \left[\mathbf{r}_i \times (\mathbf{F}_i + \sum \mathbf{R}^*_{ik})\right] = \sum \mathbf{r}_i \times \mathbf{F}_i + \sum \sum \mathbf{r}_i \times \mathbf{R}^*_{ik} \tag{3.10}$$

Because of Newton's third law, it can be shown that the interaction effect again vanishes.

Therefore:

$$\sum \mathbf{r}_i \times (m_i \ddot{\mathbf{R}}_i) = \sum \mathbf{r}_i \times \mathbf{F}_i = \mathbf{L}_0 \tag{3.11}$$

where $\mathbf{L}_0$ is the total external moment of the forces (torque) taken about the original O of the moving frame.

NOTE:

The moment of a vector depends on the reference point of the position vector, i.e., changing the reference point changes the moment.

equation 3.11 can also be expressed in terms of a moving reference frame via Equation 1.4 of Section 1.4. The result is:

$$\sum m_i \mathbf{r}_i \times \left[\ddot{\mathbf{R}}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) + 2(\boldsymbol{\omega} \times \mathbf{v}_i) + \mathbf{a}_i\right] = \mathbf{L}_0 \qquad (3.12)$$

This is the rotation equation of motion.

If the origin of the moving frame is again made to coincide with the C.M. Equation 3.12 becomes:

$$\sum m_i \mathbf{r}_i \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i) + 2(\boldsymbol{\omega} \times \mathbf{v}_i) + \mathbf{a}_i] = \mathbf{L}_0 \qquad (3.13)$$

Comparing Equation 3.8 and Equation 3.13 it is seen that choosing the C.M. as reference point O leads to a dynamic decoupling of the translational and rotational motion.

The equations are however, in general, coupled through external forces like aerodynamic forces etc. The latter depends on the translational velocity and orientation (e.g., angle of attack) of the system. The forces (and therefore, the associated torques) are, in general, given as

$$\mathbf{F} = \mathbf{F}(\mathbf{R}, \mathbf{v}, \boldsymbol{\alpha}, \boldsymbol{\omega}) \qquad (3.14)$$

where symbolically, $\mathbf{R}$ = position, $\mathbf{v}$ = velocity, $\boldsymbol{\alpha}$ = orientation, and $\boldsymbol{\omega}$ = angular velocity of the system.

The following special cases can be encountered:

CASE A: $\mathbf{F} = \mathbf{F}(\mathbf{R}, \mathbf{v})$

Solve: first translation 3.8 then rotation 3.13

CASE B: $\mathbf{F} = \mathbf{F}\ (\boldsymbol{\alpha}, \boldsymbol{\omega})$

Solve: first rotation 3.13 then rotation 3.8 or 3.4

3.2 Linear and Angular Momentum

Revisiting the translational Equation 3.3 it is possible to reformulate it by introducing the concept of the linear momentum of a system. It is defined as:

$$\mathbf{P} = \sum \mathbf{m_i}\ \dot{\mathbf{R}}_\mathbf{i} = \sum \mathbf{m_i}\ \mathbf{v_i} \tag{3.15}$$

With it, Equation 3.3 becomes:

$$\dot{\mathbf{P}} = \mathbf{F} \qquad\qquad m_i = \text{CONSTANT} \tag{3.16}$$

The time rate of change of the linear momentum of a system is equal to the total external force acting on it. In particular, if the external face is zero, the linear momentum of a system remains constant, i.e.

$$\mathbf{P} = \mathbf{P}_0 \quad \text{for} \quad \mathbf{F} = 0 \tag{3.17}$$

This is the conservation law of the linear momentum.

A similar relationship can be established for the rotational equation by introducing the concept of the angular momentum (moment of linear momentum) of a system. It is defined as:

$$\mathbf{H}_0 = \sum \mathbf{r}_i \times (m_i\ \dot{\mathbf{R}}_i) = \sum \mathbf{r}_i \times (m_i\ \mathbf{v}_i) \tag{3.18}$$

Taking the time derivative we obtain

$$\dot{\mathbf{H}}_0 = \sum \dot{\mathbf{r}}_i \times (m_i\ \mathbf{v}_i) + \sum \mathbf{r}_i(m_i\ddot{\mathbf{R}}_i) \tag{3.19}$$

Because $\mathbf{v}_i = \mathbf{v}_0 + \dot{\mathbf{r}}_i$, it follows then that

P.S. From Eq. (3.11) $\rightarrow$ $\mathbf{L}_0 = \sum \mathbf{r}_i \times (m_i\ \ddot{\mathbf{R}}_i)$.

$$\dot{\mathbf{H}}_0 = \sum m_i\ \dot{\mathbf{r}}_i \times (\mathbf{v}_0 + \dot{\mathbf{r}}_i) + \mathbf{L}_0 \tag{3.20}$$

and finally:

$$\dot{\mathbf{H}}_0 = \sum (m_i\ \dot{\mathbf{r}}_i) \times \mathbf{v}_0 + \mathbf{L}_0 \tag{3.21}$$

The rate of change of the angular momentum about the origin O of the moving frame is equal to the total torque about this origin plus a term depending on the velocity of the origin and the center of mass of the system. This is slightly different from our previous find for the translational equation. However, the rate of change of the angular momentum is equal to the torque for the following cases:

CASE A: Origin is fixed $\mathbf{v}_0 = 0$

CASE B: Origin is C.M. $\sum m_i\ \dot{\mathbf{r}}_i = 0$

CASE C: $\mathbf{v}_0 || \sum m_i\ \dot{\mathbf{r}}_i = m\ \dot{\mathbf{r}}_c$

where $\dot{\mathbf{r}}_c$ is the velocity of the mass center relative to the moving frame as seen by an inertial observer. NOTE:

Some authors define the angular momentum of a system about the origin of a moving frame as:

$$\mathbf{H}_0 = \sum \mathbf{r}_i \times (m_i\ \dot{\mathbf{r}}_i) \tag{3.22}$$

where $\dot{\mathbf{r}}_i$ is the relative velocity of a particle as viewed by an inertial observer. Taking similar steps as above leads to the final result:

$$\dot{\mathbf{H}}_0 = \mathbf{L}_0 - \sum m_i \, \mathbf{r}_i \times \ddot{\mathbf{R}}_0 = \mathbf{L}_0 - \mathbf{r}_i \times (m\ddot{\mathbf{R}}_0) \tag{3.23}$$

Regardless of how the angular momentum is defined, the time rate of change of the angular momentum of the system is equal to the external torque when referred to the C.M. or a fixed point O. In particular, if the external torque about these two points is zero, the angular momentum of a system remains constant:

$$\mathbf{H} = \mathbf{H}_0 \quad \text{if} \quad \mathbf{L}_0 = 0 \tag{3.24}$$

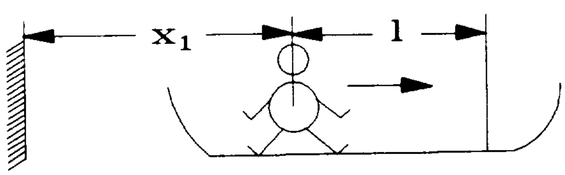
This is the conservation of angular momentum.

NOTE:

Linear and Angular Momentum are very useful concepts when properly understood and applied. However, force arguments based upon Equation 3.4and Equation 3.12 provide a more direct insight into the dynamic behavior of a system.

Example 1:

A person stands in a boat and walks in the boat a certain distance and then stops. How far will the boat move?



$$m_B \, V_B + m_p(V_B + v_p) = 0$$

$$V_B = -\frac{m_p v_p}{m_B + m_p}$$

$$\text{Set} \quad V_B = \frac{d\,x_B}{dt} \quad \text{and} \quad v_p = \frac{dl}{dt}$$

$$x_B = -\frac{m_p\, l}{m_B + m_p}$$

NOTE:

The final result does not depend on the time history of v_p.

Explain motion using force arguments via Equation 3.4: $\sum m_i(\ddot{\mathbf{R}}_0 + \mathbf{a}_i) = 0$.

P.S.:

The common way to solve this problem is to use the principle of motion of the mass center (see Equation 3.8) which states, that in the absence of external face, the mass center will remain at rest or in uniform motion:

$$M\, x_C = (\sum m_i x_i)_{BEFORE} = (\sum m_i x_i)_{AFTER}$$

$$m_p\, x_1 + m_B\, S = m_p\, x_2 + m_B(S + x_B)$$

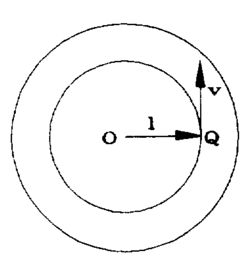
$$\text{Set :} \qquad x_2 = x_1 + x_B + l$$

$$m_p x_1 + m_B S = m_p(x_1 + x_B + l) + m_B(S + x_B)$$

$$\text{Solve for } x_B$$

Example 2:

A horizontal circular disk can rotate freely about its vertical axis. Along a circular track of radius l, a particle Q starts travelling with a constant speed v. What angular velocity will the disk acquire if it was initially at rest? What happens if the particle stops?



$\mathbf{H}_0 = \sum m_i\, \mathbf{r}_i \times \mathbf{v}_i = 0$
The velocity of the mass elements of the disk is $\mathbf{v}_i = \omega \times \mathbf{r}_i$
The angular momentum of the disk is then:
$H_1 = (\sum m_i\, r_i^2)\omega = I\, \omega$

where I = moment of inertia without small mass m.

The angular momentum of Q is:

$$h = m\, l(v + \omega l)$$

$$H_1 + h = I\omega + m\, l(v + \omega l) = 0$$

$$\omega = -\frac{mlv}{I + ml^2}$$

When the particle stops, the disk stops too, but it has rotated through a finite angle ϕ.

3.3 Kinetic Energy and Work

The concept of kinetic energy and work will now be extended to systems of particles.

The kinetic energy is defined as:

$$T = \frac{1}{2} \sum m_i\, \dot{\mathbf{R}}_i^2 = \frac{1}{2} \sum m_i (\dot{\mathbf{R}}_0 + \dot{\mathbf{r}}_i)^2$$

$$= \frac{1}{2} \sum m_i\, \dot{\mathbf{R}}_0^2 + \frac{1}{2} \sum m_i\, \dot{\mathbf{r}}_i^2 + \sum m_i\, \dot{\mathbf{r}}_i \cdot \dot{\mathbf{R}}_0 \tag{3.25}$$

or

$$T = \frac{1}{2}\, m\, \dot{\mathbf{R}}_0^2 + \frac{1}{2} \sum m_i\, \dot{\mathbf{r}}_i^2 + m\, \dot{\mathbf{R}}_0 \cdot \dot{\mathbf{r}}_C \tag{3.26}$$

Theorem of König

For the case where $\mathbf{R}_0 = \mathbf{R}_C$, Equation 3.26 reduces to

$$T = \frac{1}{2}\, m\, \dot{\mathbf{R}}_C^2 + \frac{1}{2} \sum m_i \dot{\mathbf{r}}_i^2 \tag{3.27}$$

To derive the expression for the total work done by all the forces acting on all the particles m_i of the system it is assumed that $\mathbf{R}_0 = \mathbf{R}_C$:

$$W = \sum W_i = \sum \int_{A_i}^{B_i} (\mathbf{F}_i + \sum \mathbf{R}_{ij}^*) \, (d\mathbf{R}_C + d\mathbf{r}_i) \tag{3.28}$$

or

$$W = \sum \int_{A_C}^{B_C} (\mathbf{F}_i + \sum \mathbf{R}_{ij}^*) \cdot d\mathbf{R}_C + \sum \int_{A_i}^{B_i} (\mathbf{F}_i + \sum \mathbf{R}_{ij}^*) \cdot d\mathbf{r}_i \tag{3.29}$$

Because of Newton's Law this simplifies to:

$$W = \int_{A_C}^{B_C} \mathbf{F} \cdot d\mathbf{R}_C + \sum \int_{A_i}^{B_i} (\mathbf{F}_i + \sum \mathbf{R}_{ij}^*) \cdot d\mathbf{r}_i \tag{3.30}$$

Now for each particle, the principle of work and kinetic energy applies. Therefore, using Newton's third law, we get

$$W = \frac{1}{2} \, m \, \dot{\mathbf{R}}_C^2 \, |_{A_C}^{B_C} + \frac{1}{2} \sum m_i \dot{\mathbf{r}}_i^2 \tag{3.31}$$

Comparing Equations 3.28 and 3.30 it follows from the center of mass law (Equation 3.8) that

$$\int_{A_C}^{B_C} \mathbf{F} \cdot d\mathbf{R}_C = \frac{1}{2} \, m \, \dot{\mathbf{R}}_C^2 \, |_{A_C}^{B_C} \tag{3.32}$$

and

$$\sum \int_{A_i}^{B_i} (\mathbf{F}_i + \mathbf{R}_{ij}^*) \cdot d\mathbf{r}_i = \sum \frac{1}{2} \, m \, \dot{\mathbf{r}}_i^2 \, |_{A_i}^{B_i} \tag{3.33}$$

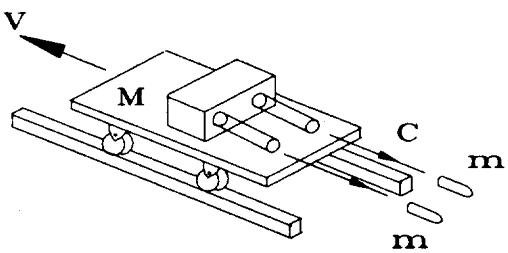
Therefore the work done by the external and internal forces is equal to the increase in the kinetic energy of relative motion. The velocities $\dot{\mathbf{r}}_i$ relative to the C.M. can arise from rigid body rotations in which the distances between the particles do not change, as well as the more obvious case of changing particle separations.

In cases where the external and also the internal forces can be derived from a potential energy, the total energy of the system is again conserved.

$$E = T_A + V_A = T_B + V_B \tag{3.34}$$

3.4 Variable Mass

Occasionally dynamic systems have to be dealt with whose mass varies with time. This is caused by mass particles either leaving or entering a certain boundary envelope (control volume). Consider for the following preliminary example.



Two bullets are fired from a vehicle which is rolling on rails without friction. Calculate the velocity of the vehicle after the bullets have been fired (a) simultaneously or (b) in sequence. The muzzle velocity is C relative to the barrel and M is the vehicle mass without the bullets. Applying the principle of linear momentum we obtain

$$(a) \qquad M V_1 + 2\, m(V_1 - C) = 0 \qquad V_1 = \frac{2mC}{M + 2m}$$

After first bullet is fired

$$(b) \qquad V_1^* = \frac{mC}{M + 2m}$$

After second bullet is fired

$$V_2 = mC(\frac{1}{M + m} + \frac{1}{M + 2m})$$

Note that $V_2 > V_1$. Consider now the case where N bullets of mass m are fired sequentially:

$$V = C \sum_{k=1}^{N} \frac{m}{M + km}$$

If the bullet size becomes infinitesimal small and $N \to \infty$ the summation can be replaced by integration to yield:

$$V = C \int_{M}^{M_0} \frac{dm}{m}$$

where M_0 is the total initial mass including the bullets.

and finally:

$$V = C \, ln \frac{M_0}{M}$$

This is the well-known velocity equation for a rocket in free space (no gravitational and aerodynamic forces). The velocity is seen to be proportional to the muzzle velocity (exhaust velocity). An interesting step can be taken by differentiating the velocity equation with respect to time:

$$\dot{V} = -C \frac{\dot{M}}{M}$$

or

$$M \, \dot{V} = -C \, \dot{M} \equiv F_T$$

F_T = thrust force.

This is the basic force equation for a rocket and thrust force

$$F_T = -C \, \dot{M}$$

Notice that the fuel flow $\dot{M}$ is negative because the system is losing mass.

A more general approach to the dynamics of a variable mass system can be taken by examining the translational Equation 3.4 and the rotational Equation 3.12

A) Translation (Thrust force)

To gain an understanding of the thrust producing mechanism we set $\omega = \dot{\omega} = 0$ and assume also that there are no external forces acting on the system. ($\mathbf{F} = 0$). Equation 3.4 reduces then to:

$$m \, \ddot{\mathbf{R}}_0 + \sum m_i \, \mathbf{a}_i = 0 \qquad m = m_0 + \sum m_i \tag{3.35}$$

where m is the total mass of the system at any point in time and m_0 represents the mass of the main body which remains constant. The origin O of the moving reference frame is assumed to be fixed in the main body. It is obvious that the thrust producing mechanism must originate from the second term of Equation 3.35. To show this, we introduce a modification in the notation to rewrite this term as:

$$\sum m_i \, \mathbf{a}_i = \lim_{\Delta t \to 0} \frac{\sum \Delta m_i \Delta \mathbf{v}_i}{\Delta t} \tag{3.36}$$

Consider now the case in which an abrupt change in velocity $\Delta \mathbf{v}$ is imparted to a large number of very small particles. Then Equation 3.35 becomes

$$\sum \, m_i \, \mathbf{a}_i = \sum \, \frac{dm_i}{dt} \, \Delta \, \mathbf{v}_i = \sum \dot{m}_i \, \Delta \mathbf{v}_i \tag{3.37}$$

Notice that $\dot{m}_i$ is the mass per unit time which undergoes the velocity change $\Delta \mathbf{v}_i$ relative to the moving frame. For steady state conditions this becomes the rate at which mass leaves the system. The acceleration which the main body attains is occurring during the short acceleration phase of the moving mass particles. It is important to realize that this acceleration is not caused by the fact that mass is leaving or entering the system. From Equation 3.35 we can now write:

$$m \, \ddot{\mathbf{R}}_0 = - \sum \, \dot{m}_i \, \Delta \mathbf{v}_i \equiv \mathbf{F}_T \qquad \text{(thrust force)} \tag{3.38}$$

It is seen that the term $(-\dot{m}\Delta \mathbf{v})$ acts as an effective force on the main body. By the nature of the above derivation the term $\dot{m}$ is always to be taken as positive regardless of whether mass is leaving or entering the system. The difference between the two is that particles leaving the system are experiencing a rapid relative acceleration, whereas particles entering the system are rapidly decelerated. Both of these processes result in a finite change $\Delta \mathbf{v}$ of the relative velocity.

In a rocket the thrust is produced by accelerating gas particles from zero relative velocity rearward until they reach a relative exhaust velocity C. The magnitude of the thrust is therefore,

$$F_T = \dot{m} C \tag{3.39}$$

It is common rocket engine practice in the US to characterize the performance of an engine by the specific impulse (specific thrust) which is defined as:

$$I_s = \frac{F_T}{\dot{w}} = \frac{F_T}{\dot{m} g_0} = \frac{C}{g_0} \quad \text{seconds} \tag{3.40}$$

The specific impulse is the thrust per propellant flow rate. It is really a measure of the effective exhaust velocity C and its only merit is that it has the same unit in the metric as in the customary system of units. Typical values of the specific impulse are in the range of 200-450 seconds.

Space Shuttle SSME : I_S = 410 seconds (average)

Space Shuttle SRB : I_S = 265 seconds

The effective exhaust velocity is simply $C = I_S g_0$. In the metric system g_0 = 9.81 m/s^2 $\approx$ 10 m/s^2 and the effective exhaust velocity is just ten times the specific impulse.

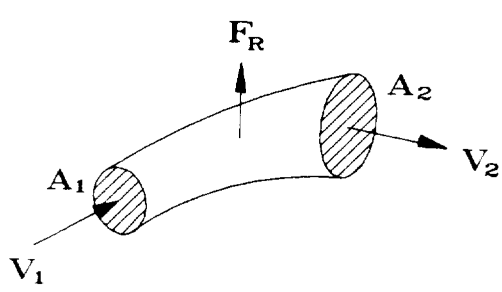
P.S.

The speed of sound in air is approximately 300 m/s. It is often useful to express velocities in terms of the Mach number. As an example, the specific impulse I_S = 410 seconds corresponds to an exhaust velocity of approximately 14 Mach. The speed of light is C = 300,000 km/s = 3 x $10^8 m/s$. It is equal to 10^6 Mach (1 Megamach!)

A very important case of mass flow is that of steady fluid flow in pipes. The continuity condition requires that the flow rate $\dot{m}$ be constant, i.e.

$$\dot{m} = \rho\, A_1 v_1 = \rho A_2 v_2 \text{ (kg/s)} \tag{3.41}$$

where A_1 is the entrance cross section and A_2 the exit cross section, and v_1 and v_2 the corresponding velocities.



According to Equation 3.38 the force exerted by the liquid flow on the pipe is:

$$\mathbf{F}_R = -\dot{m}(\mathbf{v}_2 - \mathbf{v}_1) \qquad \text{Euler's equation} \tag{3.42}$$

B) Rotation

If the dynamic system has an angular velocity internal mass flow generation produces an additional effect coming from the Coriolis term of Equation 3.4. Concentrating only on this effect, the acceleration of the main body is:

$$m\,\ddot{\mathbf{R}}_0 + 2\sum m_i(\boldsymbol{\omega} \times \mathbf{v}_i) = 0 \tag{3.43}$$

The Coriolis term can be modified again as previously to:

$$2\sum m_i(\boldsymbol{\omega} \times \mathbf{v}_i) = 2\sum \frac{\Delta m_i(\boldsymbol{\omega} \times \Delta\mathbf{r}_i)}{\Delta t} \tag{3.44}$$

Taking the limits $\Delta t \rightarrow dt$ and $\Delta m_i \rightarrow dm_i$, Equation 3.43 can formally be written as:

$$2\sum m_i(\boldsymbol{\omega} \times \mathbf{v}_i) = 2\sum \frac{dm_i}{dt}(\boldsymbol{\omega} \times \Delta\mathbf{r}_i) = 2\sum \dot{m}_i(\boldsymbol{\omega} \times \Delta\mathbf{r}_i) \tag{3.45}$$

The physical interpretation is that it represents the effect of an internal flow rate which extends over a distance $\Delta\mathbf{r}$. The resultant acceleration of the main body is therefore:

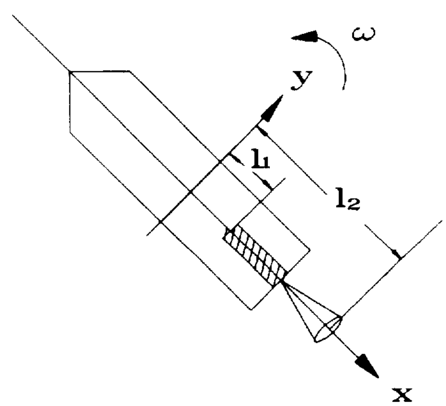
$$m\,\ddot{\mathbf{R}}_0 = -2\sum \dot{m}_i(\boldsymbol{\omega} \times \Delta\mathbf{r}_i) \tag{3.46}$$

In general, the flow rate will extend over a finite length, such that the total effective force of the flow rate will be obtained by integration. Therefore, we obtain finally:

$$m\,\ddot{\mathbf{R}}_0 = -2\boldsymbol{\omega} \times \sum \int_{\mathbf{r}_A}^{\mathbf{r}_B} \dot{m}_i d\mathbf{r}_i \tag{3.47}$$

where the integration extends over the stream lines. The effective Coriolis force stemming from internal flow rates is seen to be dependent on the geometry of the flow. It is therefore, quite difficult to make a statement as to its overall effect on the system acceleration.

These two effective forces caused by mass flow rates affect, of course, also the rotational motion as governed by Equation 3.12 by producing concomitant effective torques. Of particular interest is the effect of the Coriolis torque caused by a flow rate $\dot{m}$ because it gives rise to the so called jet damping effect. A simplified example is given to illustrate the situation. A rocket rotates about a transverse axis through the center of mass at O.



A uniform mass flow rate is assumed to exist along the x-axis only extending from $x = l_1$ to $x = l_2$ which is the nozzle exit.

The Coriolis torque is:

$$\mathbf{L}_C = -2 \int \dot{m}\ \mathbf{r} \times (\boldsymbol{\omega} \times d\mathbf{r}) \tag{3.48}$$

and with $\boldsymbol{\omega} = \omega\mathbf{k}$ and $\mathbf{r} = x\ \mathbf{i}$:

$$\mathbf{L}_C = -2\dot{m}\omega\frac{(\ell_2^2 - \ell_1^2)}{2} = -\dot{m}\omega(l_2^2 - l_1^2) \tag{3.49}$$

A few special cases are worthy of note:

a) $l_1 = l_2 \quad \mathbf{L}_C = 0$

b) $l_1 = 0 \quad \mathbf{L}_C = -\dot{m}\omega l_2^2$

c) $l_1 = -l_2 \quad \mathbf{L}_C = 0$

Case (b) is often given in textbooks as the general term for the jet damping using erroneous angular momentum arguments. To understand the jet damping effect, recall the example of the twirling ice skater.

NOTE:

The rotational equation of motion is

$$I\dot{\omega} = -\dot{m}\omega(l_2^2 - l_1^2)$$

It is seen that the Coriolis torque is causing an angular deceleration proportional to the angular velocity ω of the rocket. This can be physically interpreted as an effective damping. Since the Coriolis force is wattless, the energy dissipation has to be caused by the centrifugal force.

3.5 Impact Dynamics

During the impact of two bodies, very large forces act for a very short time. Such forces are called impulsive. Because they are so large, all other forces (e.g. gravity) can be neglected in their presence. When impulsive forces act on a body, the velocities undergo an instantaneous finite change ($\Delta\mathbf{v}\neq 0$) whereas its position and orientation remain unchanged ($\Delta\mathbf{r} = 0$). The impulsive forces are specified by their short duration time integral:

$$\dot{\mathbf{F}} = \int_0^{\epsilon} \mathbf{F}dt \quad , \quad \epsilon = \text{short time} \tag{3.50}$$

They are treated similarly to the Dirac delta function. The linear and angular velocity changes of a body during impact are obtained by integrating the equations of motion 3.4 and 3.12 with respect to time. Since only rigid bodies will be considered here, the relative velocity $\mathbf{v}$ and the relative acceleration $\mathbf{a}$ are zero. The resultant impulse equations are algebraic equations with the velocity changes as unknowns.

$$m\Delta\mathbf{V}_0 + \Delta\omega \times (m\mathbf{r}_C) = \int \mathbf{F}dt = \dot{\mathbf{F}} \tag{3.51}$$

$$(m\mathbf{r}_C) \times \Delta\mathbf{V}_0 + \sum m_i\mathbf{r}_i \times (\Delta\boldsymbol{\omega} \times \mathbf{r}_i) = \int \mathbf{L}_0 dt = \hat{\mathbf{L}}_0 \qquad (3.52)$$

These are two algebraic equations which have to be solved for the velocity changes $\Delta\mathbf{V}_0$ and $\Delta\boldsymbol{\omega}$. However, another equation is required because the impulsive force $\hat{\mathbf{F}}$ is also unknown. (The impulsive torque $\hat{\mathbf{L}}_0$ can be determined from the impulsive $\hat{\mathbf{F}}$ by taking its moment about the origin O). There will, of course, be a set of impact equations for each of the two colliding bodies.

The additional equation sets up a relationship between the normal components of the relative velocities of approach and separation of the two bodies.

$$e = \frac{v_{2N} - u_{1N}}{u_{1N} - u_{2N}} \qquad (0 \leq e \leq 1) \qquad (3.53)$$

where e is called the coefficient of restitution. It is notationally convenient to use the letter **u** for the velocity before impact and **v** for the velocity after impact. The coefficient of restitution depends on the material of the colliding bodies, on their geometries and also upon the impact velocity. All impact conditions lie between the two extremes $e = 0$ and $e = 1$.

Case A: $e = 1$ (elastic)

According to Equation 3.53 the velocity of approach is equal to the velocity of separation. No energy loss.

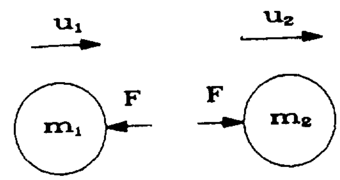
Case B: $e = 0$ (plastic)

The two bodies stick together after impact. Maximum energy loss.

Impact phenomena are almost always accompanied by energy losses. The higher the impact velocity the more energy loss occurs. The coefficient of restitution actually approaches UNITY (no energy loss) as the impact velocity goes to zero. Energy is lost through heat generation, generation and dissipation of internal vibrations (elastic stress waves) and sound energy.
NOTE:

The impact dynamics equations are a set of linear algebraic equations.

Example 1:



Consider first the central collision of two spheres.

$$\hat{F} = -m_1(v_1 - u_1) = m_2(v_2 - u_2)$$

1) Plastic Collision:

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$v_1 = v_2 = v$$

$$v = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$$

2) Elastic Collision:

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \qquad \text{MOMENTUM}$$

$$m_1 u_1^2 + m_2 u_2^2 = m_1 v_1^2 + m_2 v_2^2 \qquad \text{ENERGY}$$

$$m_1(v_1 - u_1) = -m_2(v_2 - u_2)$$

$$m_1(v_1^2 - u_1^2) = -m_2(v_2^2 - u_2^2)$$

$$v_1 + u_1 = v_2 + u_2$$

or

$$(v_2 - v_1) = (u_1 - u_2)$$

$$\text{SEPARATION VELOCITY} = \text{APPROACH VELOCITY}: 1 = \frac{v_2 - v_1}{u_1 - u_2}$$

Solve for the velocities after impact:

$$v_1 = u_1 - \frac{2m_2}{m_1 + m_2}(u_1 - u_2)$$

$$v_2 = u_2 + \frac{2m_1}{m_1 + m_2}(u_1 - u_2)$$

3) <u>Inelastic Collision</u>:

$$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{\text{SEPARATION VELOCITY}}{\text{APPROACH VELOCITY}}$$

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

$$v_1 - v_2 = -e(u_1 - u_2)$$

$$v_1 = u_1 - \frac{m_2(1 + e)}{m_1 + m_2}(u_1 - u_2)$$

$$v_2 = u_2 + \frac{m_1(1+e)}{m_1+m_2}(u_1 - u_2)$$

Loss of kinetic energy:

$$\Delta T = \frac{1}{2}\left[m_1(u_1^2 - v_1^2) + m_2(u_2^2 - v_2^2)\right]$$

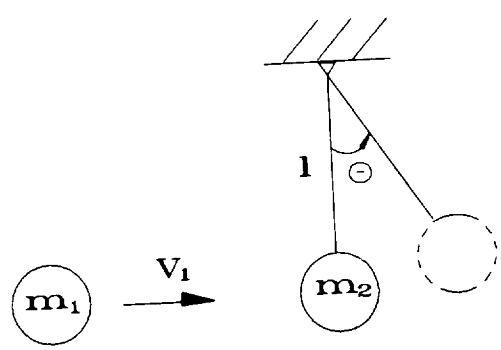
After some algebraic manipulation the final result is:

$$\Delta T = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$$

for $e = 0 : \Delta T = \frac{1}{2}\mu(u_1 - u_2)^2$

where $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$. μ = reduced mass. μ = Greek "mu"

Example 2: Ballistic Pendulum



$$\text{For} \quad m_1 : \hat{F} = m_1(v_1 - v_1') \tag{3.54}$$
$$\text{For} \quad m_2 : \hat{F} = m_2(v_2' - v_2) \tag{3.55}$$

Combining Equations 3.54 and 3.55 yields the conservation law of linear momentum.

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \tag{3.56}$$

The coefficient of restitution relation is:

$$e = \frac{v_2' - v_1'}{v_1 - v_2} = \frac{\text{SEPARATION VELOCITY}}{\text{APPROACH VELOCITY}} \tag{3.57}$$

Initial condition for $m_2 : v_2 = 0$

We obtain for the unknowns v_1' and v_2':

$$v_1' = \frac{m_1 - m_2 e}{m_1 + m_2} v_1$$

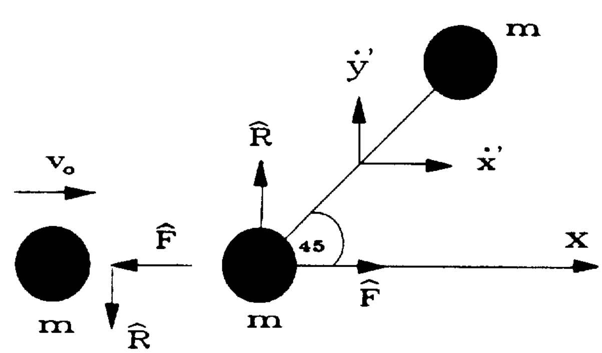
$$v_2' = \frac{(1+e)m_1}{m_1 + m_2} v_1$$

The angular velocity after impact is

$$\dot{\theta}' = v_2'/l = \frac{(1+e)m_1 v_1}{(m_1 + m_2)l}$$

<u>Example 3</u>

Mass m, moving along the x-axis with velocity v, hits m_2 and sticks to it. If all three particles are of equal mass m, and if m_2 and m_3 are connected by a rigid massless rod, find the motion after impact.



There are <u>5 Unknowns</u>: v_x' , v_y' , $\dot{x}'$, $\dot{y}'$, $\dot{\phi}'$.

Linear momentum in normal direction:

$$m\ v_0 = m\ v'_x + 2\ m\ \dot{x}' \tag{3.58}$$

Linear momentum in tangential direction:

$$0 = m\ v'_y + 2\ m\ \dot{y}' \tag{3.59}$$

Angular momentum about C.M. of dumbbell:

$$m(\frac{l}{2})\ \frac{v_0}{\sqrt{2}} = \frac{ml^2}{2}\ \dot{\phi}' + m(\frac{l}{2})\frac{v'_x}{\sqrt{2}} - m\ (\frac{l}{2})\ \frac{v'_y}{\sqrt{2}} \tag{3.60}$$

Coefficient of restitution:

$$e = \frac{(\frac{\dot{\phi}'}{\sqrt{2}}\frac{l}{2} + \dot{x}') - v'_x}{v_0} \tag{3.61}$$

<u>Case A</u>: Sliding Rebound (μ = coefficient of friction)

$$\dot{R} = \mu\ \dot{F} = -m\ v'_y$$

$$\dot{F} = m(v_0 - v'_x) \qquad \text{FOR PARTICLE}$$

Combining these two equations yields:

$$\mu(v_0 - v'_x) = -v'_y \tag{3.62}$$

Rearranging the last five equations yields

(1′) $v'_x + 2\ \dot{x}' = v_0$

(2′) $v'_y + 2\dot{y} = 0$

(3′) $v'_x + \dot{\phi}'\ell\sqrt{2}\ - v'_y = v_0$

(4′) $-v_x' + \frac{\dot{\phi}'\ell}{2\sqrt{2}} + \dot{x}' = e\ v_0$

(5′) $\mu\ v_x' - v_y' = \mu\ v_0$

Solving these equations for the unknowns results in:

$$v_x' = \frac{(3 - \mu - 4e)}{(7 - \mu)}\ v_0$$

$$v_y' = \frac{-4\mu(1 + e)}{(7 - \mu)}\ v_0$$

$$\dot{x}' = \frac{2(1 + e)}{(7 - \mu)}\ v_0$$

$$\dot{y}' = \frac{2\mu(1 + e)}{(7 - \mu)}\ v_0$$

$$\dot{\phi}' = \frac{2\sqrt{2}(1 - \mu)(1 + e)v_0}{(7 - \mu)\ell}$$

Case B: Normal Rebound

Equation 3.62 is replaced by normal rebound condition:
(5″) $v_y' = \dot{y}' - \frac{\dot{\phi}'\ell}{2\sqrt{2}}$

All other equations remain the same. Solving for the unknowns:

$$v_x' = \frac{5 - 7e}{12}\ v_0$$

$$v_y' = -\frac{1 + e}{12}\ v_0$$

$$\dot{x}' = \frac{7}{24}\,(1+e)\,v_0$$

$$\dot{y}' = \frac{1}{24}\,(1+e)\,v_0$$

$$\dot{\phi}' = \frac{(1+e)\,v_0}{2\sqrt{2}}$$

The coefficient of friction above which normal rebound occurs is obtained from calculating the two impulsive forces:

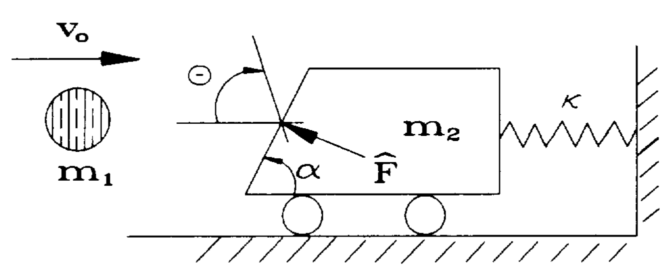
$$\hat{R} = \mu\,\hat{F} = -m\,v_y' = \frac{(1+e)}{12}\,m\,v_0$$

$$\hat{F} = m\,(v_0 - v_x') = \frac{7}{24}(1+e)\,m\,v_0$$

$$\mu = 2/7$$

Example 4:

A sphere m_1 is projected horizontally against a carriage m_2 which is backed up by a spring k. If the coefficient of restitution is e and the surface is perfectly smooth ($\mu = 0$). Calculate the rebound velocity v_1', the rebound angle θ and the maximum travel δ of the carriage after impact.



The prime indicates the condition after impact.

Impulse Equations:

$$m_1\,(\dot{x}_1' - \dot{x}_1) = -\hat{F}\ sin\ \alpha \tag{3.63}$$

$$m_1\,(\dot{y}_1' - \dot{y}_1) = \hat{F}\ \cos\ \alpha \tag{3.64}$$

$$m_2\,(\dot{x}_2' - \dot{x}_2) = \hat{F}\ \sin\ \alpha \tag{3.65}$$

$$\dot{y}_2' = 0 \qquad \text{carriage is on rail} \tag{3.66}$$

Normal Velocities:

$$v_{1N} = \dot{x}_1\ \sin\ \alpha \tag{3.67}$$

$$v_{1N}' = \dot{x}_1'\ \sin\ \alpha - \dot{y}_1'\ \cos\ \alpha \tag{3.68}$$

$$v_{2N} = 0 \tag{3.69}$$

$$v_{2N}' = \dot{x}_2'\ \sin\ \alpha \tag{3.70}$$

Restitution Equation:

$$e = \frac{v_{2N}' - v_{1N}'}{v_{1N} - v_{2N}} = \frac{\dot{x}_2' \sin\ \alpha - (\dot{x}_1'\ \sin\ \alpha - \dot{y}_1'\ \sin\ \alpha)}{\dot{x}\ \sin\ \alpha - 0} \tag{3.71}$$

Unknowns:

$\dot{x}_1'$, $\dot{y}_1'$, $\dot{x}_2'$, $\hat{F}$, $\dot{y}_2'$.

The following steps are taken:

A) Eliminate $\hat{F}$:

1) Combine Equations 3.63 and 3.65:

$$m_1\ \dot{x}_1' + m_2\ \dot{x}_2' = m_1\ \dot{x}_1 \qquad \text{linear momentum for } m_1 \text{ in x-direction} \tag{3.72}$$

2) Multiply Equation 3.63 by cos α and Equation 3.64 by $\sin\alpha$:

$$m_1\, \dot{x}_1'\, \cos\, \alpha + m_1\, \dot{y}_1'\, \sin\, \alpha = m_1 \dot{x}_1\, \cos\alpha \tag{3.73}$$

The linear momentum along the smooth surface is conserved for m_1.

B) Solve Equations 3.71, 3.72, 3.73 for $\dot{x}_1', \dot{y}_1'$, and $\dot{x}_2'$

C) Determine θ:

$$\tan\, \theta = \frac{\dot{y}_1'}{\dot{x}_1'} \tag{3.74}$$

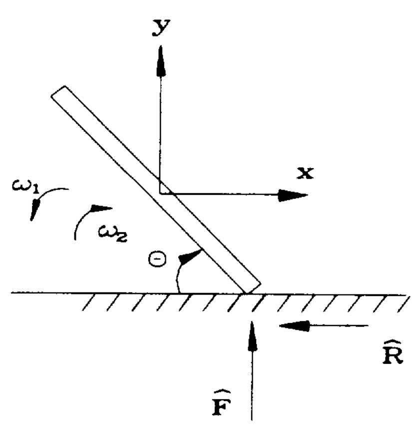
D) Determine v':

$$v' = \sqrt{(\dot{x}_1')^2 + (\dot{y}_1')^2} \tag{3.75}$$

E) Determine δ: $\frac{1}{2}\, m_2\, (\dot{x}_2')^2 = \frac{1}{2}\, k\, \delta^2 \qquad \delta = \sqrt{\frac{m_2}{k}}\, \dot{x}_2'$

Example 5:

A uniform bar of length ℓ and mass m falls on a horizontal floor with velocity $\mathbf{v}_0 = u_0\, \mathbf{i} - v_0 \mathbf{j}$. The bar falls without rotation. If a coefficient of restitution e and a coefficient of friction μ exists between the floor and the bar, determine the minimum friction coefficient for a normal rebound and the velocities after a sliding rebound.



Impulse Equations:

$$\hat{F} = m\, (\dot{y} + v_0) \tag{3.76}$$

$$\dot{R} = -m\ (\dot{x} - u_0) \tag{3.77}$$

$$\dot{F}\frac{\ell}{2}\ \cos\ \theta = I\omega_1 \tag{3.78}$$

$$\dot{R}\ \frac{\ell}{2}\ \sin\ \theta = I\omega_2 \tag{3.79}$$

The total angular velocity of the bar after impact is

$$\omega = \omega_1 - \omega_2$$

Restitution Equation:

$$e = \frac{\dot{y} + \frac{\ell}{2}\ \cos\ \theta\ (\omega_1 - \omega_2)}{v_0} \tag{3.80}$$

A) NORMAL REBOUND: $(\dot{R} < \mu\dot{F})$

$$\dot{x} + (\frac{\ell}{2}\ \sin\ \theta)\ (\omega_1 - \omega_2) = 0 \tag{3.81}$$

Substituting Equation 3.76 in 3.78 and 3.77 in 3.79:

$$m\ (\dot{y} + v_0)\ \frac{\ell}{2}\ \cos\ \theta = \frac{m\ell^2}{12}\ \omega_1 \tag{3.82}$$

or

$$\omega_1 = \frac{6\ (\dot{y} + v_0)\cos\theta}{\ell} \tag{3.83}$$

$$m(u_0 - \dot{x})\frac{\ell}{2}\ \sin\ \theta = \frac{m\ \ell^2}{12}\ \omega_2 \tag{3.84}$$

or

$$\omega_2 = \frac{6(u_0 - \dot{x})\sin\theta}{\ell}$$

Inserting 3.83 and 3.84 into Equation 3.80 and 3.81:

$$\epsilon\ v_0 = \dot{y} + 3\ \cos\ \theta\,[(\dot{y} + v_0)\ \cos\ \theta - (u_0 - \dot{x})\ \sin\ \theta] \tag{3.85}$$

$$\dot{x} = 3\sin\,\theta\,[(u_0 - \dot{x})\,\sin\,\theta - (\dot{y} + v_0)\,\cos\,\theta] \tag{3.86}$$

$$(3\sin\,\theta\,\cos\,\theta)\dot{x} + (1 + 3\cos^2\,\theta)\,\dot{y} = e\,v_0$$

$$-\,3v_0\,\cos^2\,\theta + 3u_0\,\sin\,\theta\,\cos\,\theta \tag{3.87}$$

$$(1 + 3\sin^2\,\theta)\,\dot{x} + (3\sin\,\theta\,\cos\,\theta)\,\dot{y} = 3\,\sin^2\,\theta u_0 - 3v_0\,\sin\,\theta\,\cos\,\theta \tag{3.88}$$

<u>Let $\theta = 45^o$</u>:

$$\cos\,\theta = \sin\,\theta = \frac{1}{\sqrt{2}} \tag{3.89}$$

$$\frac{3}{2}\,\dot{x} + \frac{5}{2}\,\dot{y} = (e - \frac{3}{2})\,v_0 + \frac{3}{2}\,u_0 \tag{3.90}$$

$$\frac{5}{2}\,\dot{x} + \frac{3}{2}\,\dot{y} = u_0\,(\frac{3}{2}) - \frac{3}{2}\,v_0 \tag{3.91}$$

Final results for velocities after impact are:

$$\dot{y} = \frac{(5e - 3)\,v_0 + 3\,u_0}{8} \qquad \dot{x} = \frac{3\,\{u_0 - (1 + e)v_0\}}{8} \tag{3.92}$$

$$\omega = \omega_1 - \omega_2 = \frac{3}{2\sqrt{2}\ell}\,\{2(1 + e)\,v_0 - u_0\} \tag{3.93}$$

The impulsive forces are:

$$\hat{R} = m\left\{\frac{5u_0 + (1 + e)\,v_0}{8}\right\} \tag{3.94}$$

$$\hat{F} = m\left\{\frac{3u_0 + 5(1 + e)\,v_0}{8}\right\} \tag{3.95}$$

<u>Normal Rebound Condition</u>: $\quad \hat{R} < \mu\hat{F}$

$$\mu > \frac{3\,\mu_0 + 5(1+e)\,v_0}{5\,\mu_0 + (1+e)\,v_0} \tag{3.96}$$

SLIDING REBOUND: $\hat{R} = \mu\,\hat{F}$

$$\hat{R} = \mu\, m(\dot{y} + v_0) \text{ from Equation 3.76} \tag{3.97}$$

From Equations 3.77, 3.78, and 3.79:

$$\mu\, m\,(\dot{y} + v_0) = -m(\dot{x} - u_0) = m(u_0 - \dot{x}) \tag{3.98}$$

$$\mu\, m\,(\dot{y} + v_0)\,\frac{\ell}{2}\,\sin\,\theta = \frac{m\ell^2}{12}\,\omega_2 \tag{3.99}$$

$$m\,(\dot{y} + v_0)\frac{\ell}{2}\,\cos\,\theta = \frac{m\ell^2}{12}\,\omega_1 \tag{3.100}$$

$$\omega_1 = \frac{6(\dot{y} + v_0)\cos\,\theta}{\ell} \qquad \omega_2 = \frac{6\,\mu(\dot{y} + v_0)\,\sin\,\theta}{\ell} \tag{3.101}$$

From Eq. (5):

$$e\,v_0 = \dot{y} + 3\,\cos\theta\,[(\dot{y} + v_0)\cos\,\theta - \mu\,(\dot{y} + v_0)\,\sin\,\theta] \tag{3.102}$$

$$e\,v_0 = \dot{y}\,[1 + 3\,\cos\,\theta(\cos\,\theta - \mu\,\sin\,\theta)] + 3v_0\,\cos\,\theta\,(\cos\,\theta - \mu\,\sin\,\theta) \tag{3.103}$$

$$\dot{y} = \frac{v_0\,[e - 3\,\cos\,\theta\,(\cos\,\theta - \mu\,\sin\,\theta)]}{1 + 3\cos\,\theta\,(\cos\,\theta - \mu\,\sin\,\theta)} \tag{3.104}$$

Let $\theta = 45^0$:

$$\dot{y} = \frac{v_0\,\{2e - 3(1-\mu)\}}{2 + 3(1-\mu)} \qquad \dot{x} = u_0 - \frac{2\,\mu(1+e)v_0}{2 + 3(1-\mu)} \tag{3.105}$$

$$\omega = \omega_1 - \omega_2 = \frac{12(1+e)\,(1-\mu)v_0}{\sqrt{2}\,[2 + 3\,(1-\mu)]\,\ell} \tag{3.106}$$

Chapter 4

Dynamics of a Rigid Body

This section is almost entirely treating the rotational dynamics of a rigid body. Later we will also include internal moving parts whose movements are often prescribed time functions. First the general equations of motion of Equation 3.12 are brought in a more suitable form.

4.1 Euler's Equations

For a rigid body the relative velocity $\mathbf{v}$ and the relative acceleration $\mathbf{a}$ in Equation 3.4 and Equation 3.12 of the preceding sections are zero. For the case of the general motion of a rigid body, it is customary to choose the center of man as the origin of the reference frame and have the reference frame fixed in the body (Body-fixed reference frame). If there is no external coupling between the translation and rotation, the stational motion can be analyzed separately.

With these assumptions Equation 3.12 is:

$$\Sigma\, m_i\, \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathrm{r}_i) + \Sigma\, m_i\, \mathbf{r}_i \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)] = \mathbf{L}_0 \tag{4.1}$$

The first term is seen to be the (negative) Euler torque, whereas the second term is the (negative) centrifugal torque. This equation can be rewritten by using the following vector identity:

$$\mathbf{r} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathrm{r})] = \boldsymbol{\omega} \times [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})] \tag{4.2}$$

The rotational Equation 4.1 becomes then:

$$\Sigma\, m_i\, \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathrm{r}_i) + \boldsymbol{\omega} \times \Sigma\, m_i\, \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \mathbf{L}_0 \tag{4.3}$$

At this point it would be possible to introduce an orthogonal reference frame and convert Equation 4.3 to three scalar first order differential equations for dynamics analysis and computer programming. However, it is possible and very useful to employ a notation known as vector-dyadic notation which, unlike the matrix notation, is independent of a reference frame in the same way as vectors are.

4.2 Vector-Dyadic/Matrix Notation

A new vector operation is introduced which is formed by the juxtaposition (side by side storage) of two vectors for the purpose of later taking scalar and vector products with an ordinary vector. This vector operator is called the dyadic product and is given by:

$$\mathcal{D} = \mathbf{A}\ \mathbf{B} \tag{4.4}$$

where $\mathbf{A}$ is called the antecedent and $\mathbf{B}$ is the consequent. The dyadic product is not commutative because $\mathbf{AB} \neq \mathbf{BA}$. However, the distributive law holds:

$$\begin{aligned} \mathbf{A}(\mathbf{B}_1 + \mathbf{B}_2) &= \mathbf{AB}_1 + \mathbf{AB}_2 \\ (\mathbf{A}_1 + \mathbf{A}_2)\mathbf{B} &= \mathbf{A}_1\mathbf{B} + \mathbf{A}_2\mathbf{B} \end{aligned} \tag{4.5}$$

The juxtaposition of two vectors $\mathbf{AB}$ is also called a dyad.

NOTE:

Dyads will be designated by capital script letters.

The sum of dyadic products is called a dyadic:

$$\mathcal{D} = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \ldots \mathbf{A}_n\mathbf{B}_n \tag{4.6}$$

The dyadic obtained by interchanging the order of the $\mathbf{A}_i$ and the $\mathbf{B}_i$ is called the conjugate of $\mathcal{D}$:

$$\mathcal{D}_c = \mathbf{B}_1\mathbf{A}_1 + \mathbf{B}_2\mathbf{A}_2 + \ldots \mathbf{B}_n\mathbf{A}_n \tag{4.7}$$

The dyadic which is equal to its conjugate is called self-conjugate or symmetric.

Because the distributive law holds for a dyadic product any dyad $\mathbf{AB}$ can be written in form of a dyadic as follows.

Expressing the vectors $\mathbf{A}$ and $\mathbf{B}$ in terms of orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, we get:

$$\begin{aligned} \mathcal{D} = \mathbf{AB} &= (A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3)(B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3) \\ &= A_1B_1\mathbf{e}_1\mathbf{e}_1 + A_1B_2\mathbf{e}_1\mathbf{e}_2 + A_1B_3\mathbf{e}_1\mathbf{e}_3 \\ &+ A_2B_1\mathbf{e}_2\mathbf{e}_1 + A_2B_2\mathbf{e}_2\mathbf{e}_2 + A_2B_3\mathbf{e}_2\mathbf{e}_3 \\ &+ A_3B_1\mathbf{e}_3\mathbf{e}_1 + A_3B_2\mathbf{e}_3\mathbf{e}_2 + A_3B_3\mathbf{e}_3\mathbf{e}_3 \end{aligned} \tag{4.8}$$

This is called the nonion form of the dyad because it contains nine components.

The following rules apply for the scalar and vector product of a vector with a dyad:

$$\begin{aligned} \mathcal{D}\cdot\mathbf{R} = (\mathbf{AB})\cdot\mathbf{R} = \mathbf{A}(\mathbf{B}\cdot\mathbf{R}) &\qquad \text{Vector} \\ \mathbf{R}\cdot\mathcal{D} = \mathbf{R}\cdot(\mathbf{AB}) = (\mathbf{R}\cdot\mathbf{A})\mathbf{B} &\qquad \text{Vector} \\ \mathcal{D}\times\mathbf{R} = (\mathbf{AB})\times\mathbf{R} = \mathbf{A}(\mathbf{B}\times\mathbf{R}) &\qquad \text{Dyadic} \\ \mathbf{R}\times\mathcal{D} = \mathbf{R}\times(\mathbf{AB}) = (\mathbf{R}\times\mathbf{A})\mathbf{B} &\qquad \text{Dyadic} \end{aligned} \tag{4.9}$$

The unit (identity) dyadic is defined as

$$\mathcal{E} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3 \tag{4.10}$$

Indeed: $\mathbf{A}\cdot\mathcal{E} = \mathcal{E}\cdot\mathbf{A} = \mathbf{A}$

We are now ready to go back to Equation 4.3 and cast it in vector-dyadic notation. Observing the above rule of forming the scalar product of a vector with a dyadic we obtain:

$$\begin{aligned} \Sigma\, m_i\mathbf{r}_i \times (\boldsymbol{\omega}\times\mathbf{r}_i) &= \Sigma\, m_i\left[(\mathbf{r}_i\cdot\mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i\cdot\boldsymbol{\omega})\mathbf{r}_i\right] \\ &= \Sigma\, m_i\left[r_i^2\mathcal{E} - (\mathbf{r}_i\mathbf{r}_i)\right]\cdot\boldsymbol{\omega} = \mathcal{I}\cdot\boldsymbol{\omega} \end{aligned} \tag{4.11}$$

where

$$\mathcal{I} = \Sigma\, m_i\left[r_i^2\mathcal{I} - (\mathbf{r}_i\mathbf{r}_i)\right]$$

$$\mathcal{I} = \underline{\text{inertia dyadic}}$$

The vector-dyadic form of Equation 4.1 is therefore

$$\mathcal{I}\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathcal{I}\cdot\boldsymbol{\omega} = \mathbf{L}_0 \tag{4.12}$$

It should be reiterated that this form of Euler's dynamical equations is independent of a reference frame.

To convert Equation 4.12 into scalar form we introduce an orthogonal body-fixed reference frame (body axes) whose origin is fixed in inertial space or coincides with the mass center of the rigid body. The position of a mass particle m_i is then given by:

$$\mathbf{r}_i = x_i\mathbf{e}_1 + y_i\mathbf{e}_2 + z_i\mathbf{e}_3 \tag{4.13}$$

and the angular velocity of the rigid body:

$$\boldsymbol{\omega} = \omega_x\mathbf{e}_1 + \omega_y\mathbf{e}_2 + \omega_z\mathbf{e}_3 \tag{4.14}$$

The ensuing scalar equations can be concisely expressed by using matrix notation.

To this end, define the moments of inertia as:

$$\begin{aligned} I_{xx} &= \Sigma\, m_i(y_i^2 + z_i^2) \\ I_{yy} &= \Sigma\, m_i(x_i^2 + z_i^2) \\ I_{zz} &= \Sigma\, m_i(x_i^2 + y_i^2) \end{aligned} \tag{4.15}$$

Also define the products of inertia as:

$$\begin{aligned} I_{xy} &= I_{yx} = -\Sigma\, m_i x_i y_i \\ I_{xz} &= I_{zx} = -\Sigma\, m_i x_i z_i \\ I_{yz} &= I_{zy} = -\Sigma\, m_i y_i z_i \end{aligned} \tag{4.16}$$

NOTE:

For continuous bodies the summation is replaced by appropriate integration e.g.:

$$I_{xx} = \int (y^2 + z^2)dm \qquad \text{etc.}$$

Caution:

Some authors define the products of inertia of Equation 4.16 using the opposite sign such that:

$$I_{xy} = \Sigma\, m_i x_i y_i \qquad \text{etc.}$$

The moments and products of inertia can be arranged in matrix form:

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{xz} \\ I_{zx} & I_{xy} & I_{zz} \end{bmatrix}$$

I = inertia matrix

Any vector product between two vectors can be cast in matrix form:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \tag{4.17}$$

Introduce the skew-symmetric matrix

$$\tilde{a} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \quad \tilde{a} = singular$$

Then the matrix notation for the vector product is

$$\mathbf{c} = \tilde{a}\mathbf{b} \tag{4.18}$$

where **b** and **c** are now column matrices.

NOTE:

No confusion is likely to occur by designating a column matrix by boldfacing the letter, which is the same convention used for vectors, because it should be obvious from the context whether an equation is written in matrix form or in vector-dyadic form.

The matrix form of Euler's dynamical equations is then

$$I\dot{\omega} + \tilde{\omega} I \omega = L_0 \tag{4.19}$$

It should be mentioned that in the dynamics area where only orthogonal (rotational) transformations are used a dyadic is essential identical to a matrix. For every vector-dyadic equation there exists an equivalent matrix equation. The advantage of the vector-dyadic equation is, of course, that it is independent of the reference frame.

Examples:

A) Angular Momentum of a Rigid Body

$$\mathbf{H} = \Sigma\, \mathbf{r}_i \times m_i \mathbf{v}_i = \Sigma\, \mathbf{r}_i \times m_i(\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_i) \tag{4.20}$$

If the origin O is fixed in inertial space or coincides with the mass center we have:

$$\mathbf{H} = \Sigma\, m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \tag{4.21}$$

or

$$\mathbf{H} = \mathcal{I} \cdot \boldsymbol{\omega} \qquad \mathcal{I} = \text{inertia dyadic} \tag{4.22}$$

The corresponding matrix equation is:

$$\mathbf{H} = I\, \boldsymbol{\omega} \tag{4.23}$$

B) Kinetic Energy of a Rigid Body:

$$\begin{aligned} T &= \tfrac{1}{2}\, \Sigma\, m_i(\mathbf{v}_i \mathbf{v}_i) \\ &= \tfrac{1}{2}\, \Sigma\, m_i(\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_i)^2 \\ &= \tfrac{1}{2}\, m\, v_0^2 + \mathbf{v}_0 \cdot (\boldsymbol{\omega} \times \Sigma\, m_i\, \mathbf{r}_i) + \tfrac{1}{2}\, \Sigma\, m_i(\boldsymbol{\omega} \times \mathbf{r}_i)^2 \end{aligned}$$

If the origin is fixed or at mass center, the middle term becomes zero.
The total kinetic energy is then composed of translational and rotational kinetic energy.

$$T = T_t + T_r \tag{4.24}$$

The expression $(\boldsymbol{\omega} \times \mathbf{r}_i)^2$ can be transformed as:

$$\begin{aligned} (\boldsymbol{\omega} \times \mathbf{r}_i)^2 &= (\boldsymbol{\omega} \times \mathbf{r})\ (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &= \boldsymbol{\omega} \cdot [\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] \end{aligned}$$

Therefore, the rotational kinetic energy of a rigid body is:

$$T_r = \frac{1}{2}\, \boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\omega} \tag{4.25}$$

The corresponding matrix equation is:

$$T_r = \frac{1}{2}\,\boldsymbol{\omega}^T\, I\, \boldsymbol{\omega} \tag{4.26}$$

where the superscript T denotes the transpose of a matrix.

4.3 Orientation Kinematics

An important aspect of rotational motion in space is the specification of the relative orientation between two reference frames (coordinate systems). In many applications the orientation to be defined is that between a moving reference frame (or rigid body) and a space-fixed (inertial) frame. There exist practically three schemes for doing this:

a) Direction Cosines

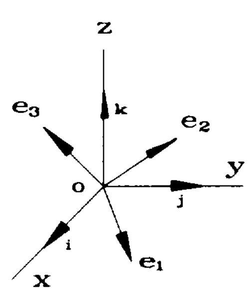
b) Euler Angles

c) Quaternoins (Euler parameters)

Other methods are sometimes used, but they are of little or no advantage over these three.

As a general rule the unit vectors of an inertial frame will be designated by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and those of a moving frame by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

a) Direction Cosines



In this method each unit vector of the moving frame $\mathbf{e}_i$ is defined in terms of the three angles each makes with the three axes of the inertial frame. They are obtained by the corresponding scalar products

$\mathbf{e}_1 \cdot \mathbf{i} = \cos\alpha_{11} = A_{11}$

$\mathbf{e}_1 \cdot \mathbf{j} = \cos\alpha_{12} = A_{12}$

$\mathbf{e}_1 \cdot \mathbf{k} = \cos\alpha_{13} = A_{13}$ etc.

The coefficients A_{ik} are called the direction cosines. The first index relates to the unit vector of the moving frame and the second to that of the inertial frame.

There are nine direction cosines and they completely define the relative orientation of two reference frames. We can write in vector form:

$$\mathbf{e}_1 = A_{11}\,\mathbf{i} + A_{12}\,\mathbf{j} + A_{13}\,\mathbf{k}$$

$$\mathbf{e}_2 = A_{21}\,\mathbf{i} + A_{22}\,\mathbf{j} + A_{23}\,\mathbf{k}$$

$$\mathbf{e}_3 = A_{31}\,\mathbf{i} + A_{32}\,\mathbf{j} + A_{33}\,\mathbf{k}$$

It can be easily seen that the nine direction cosines are not independent. The relationship between them can be obtained by observing the fact that the unit vectors $\mathbf{e}_i$ are of unit length and mutually orthogonal. Therefore the following six equations hold:

$$\begin{array}{lll} \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 & \mathbf{e}_2 \cdot \mathbf{e}_2 = 1 & \mathbf{e}_3 \cdot \mathbf{e}_3 = 1 \\ \mathbf{e}_1 \cdot \mathbf{e}_2 = 0 & \mathbf{e}_1 \cdot \mathbf{e}_3 = 0 & \mathbf{e}_2 \cdot \mathbf{e}_3 = 0 \end{array}$$

These yield six relationships between the A_{ik} which can be compactly written as:

$$\sum_{k=1}^{3} A_{ik}\,A_{jk} = \delta_{ij}$$

where δ_{ij} is the Kronecker Delta. This equation is known as the orthonormality condition of the direction cosines.

A more convenient form is obtained by using matrix notation. Introducing the direction cosine matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

the above orthonormality condition is then

$$A\,A^T = E \qquad (4.27)$$

where E = unit matrix.

Premultiplying Equation 4.27 by A^{-1} yields also:

$$A^{-1} = A^T \qquad (4.28)$$

Matrices having the property that the inverse matrix is equal to the transposed matrix are called orthogonal (rotational) matrices. There are nine more (not independent) relationships among the elements of A given by:

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$$

They can be summarized using the adjoint matrix adj A as

$$A^T = \text{adj } A$$

NOTE:

In many dynamics problems the moving reference frame originally coincides with the inertial frame. It is, therefore, often customary to refer to the moving frame as the "new" frame and the inertial frame as the "old" frame. Using this terminology one would say that the direction cosines express the new unit vectors in terms of the old unit vectors.

It is also necessary to know how the components of a vector change from one reference frame to the other.

Consider the arbitrary vector

$$\mathbf{R} = x\,\mathbf{e}_1 + y\,\mathbf{e}_2 + z\,\mathbf{e}_3 = X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k} \tag{4.29}$$

Introducing the unit vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ yields:

$$\begin{aligned} \mathbf{R} &= (A_{11}x + A_{21}y + A_{31}z)\mathbf{i} \\ &+ (A_{12}x + A_{22}y + A_{32}z)\mathbf{j} \\ &+ (A_{13}x + A_{23}y + A_{33}z)\mathbf{k} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} \end{aligned}$$

Equating components of both sides and using matrix notation we obtain

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = A^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ or } \mathbf{X} = A^T\mathbf{x}$$

This equation expresses the old components of a vector in terms of the new components. To reverse the situation we solve for the new components by using the orthonormality condition of the rotational matrix A and obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \text{ or } \mathbf{x} = A\mathbf{X}$$

We see that the components of a vector transform exactly like their unit vectors.

Since the orientation of the moving reference frame is constantly changing with time the direction cosine matrix is a function of time. It is possible to derive a first order matrix differential equation by determining the time rate of change of the unit vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ relative to inertial space.

Employing results from Section 1.3 we can write:

$$\begin{aligned} \dot{\boldsymbol{e}}_1 &= \boldsymbol{\omega} \times \boldsymbol{e}_1 = \dot{A}_{11}\,\mathbf{i} + \dot{A}_{12}\,\mathbf{j} + \dot{A}_{13}\mathbf{k} \\ \dot{\boldsymbol{e}}_2 &= \boldsymbol{\omega} \times \boldsymbol{e}_2 = \dot{A}_{21}\,\mathbf{i} + \dot{A}_{22}\,\mathbf{j} + \dot{A}_{23}\mathbf{k} \\ \dot{\boldsymbol{e}}_3 &= \boldsymbol{\omega} \times \boldsymbol{e}_3 = \dot{A}_{31}\,\mathbf{i} + \dot{A}_{32}\,\mathbf{j} + \dot{A}_{33}\mathbf{k} \end{aligned}$$

Expressing the angular velocity of the reference frame in terms of the moving reference frame we obtain

$$\boldsymbol{\omega} = \omega_1\,\mathbf{e}_1 + \omega_2\,\mathbf{e}_2 + \omega_3\,\mathbf{e}_3 \tag{4.30}$$

The above vector products can then be written as follows:

$$\boldsymbol{\omega} \times \mathbf{e}_1 = (\omega_1\mathbf{e}_1 + \omega_2\,\mathbf{e}_2 + \omega_3\,\mathbf{e}_3) \times \mathbf{e}_1 = -\omega_2\,\mathbf{e}_3 + \omega_3\,\mathbf{e}_2 \tag{4.31}$$

$$= -\omega_2(A_{31}\,\mathbf{i} + A_{32}\,\mathbf{j} + A_{33}\mathbf{k}) + \omega_3(A_{21}\,\mathbf{i} + A_{22}\,\mathbf{j} + A_{23}\,\mathbf{k}) \tag{4.32}$$

The other cross products can be transformed in like manner. Equating coefficients and solving for the time derivatives of the direction cosines we arrive at the following compact matrix differential equation:

$$\dot{A} = -\tilde{\omega}\,A \qquad \text{kinematical (Poisson) differential equations} \tag{4.33}$$

where $\tilde{\omega}$ is the skew symmetric matrix:

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

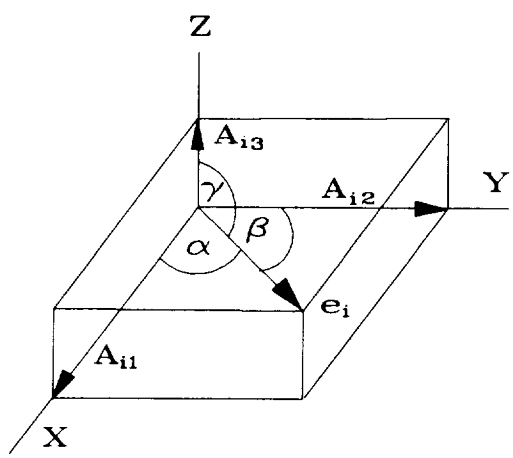
NOTE:

The angular velocity has to be expressed in the moving reference frame coordinates.

The kinematical matrix differential Equation 4.33 actually consists of nine first order linear differential equations for the nine direction cosines.

Visualization of Direction Cosines

To provide a visual aid for the rotational motion of a rigid body relative to an inertial reference frame, it is convenient to plot the projection of any suitable unit vector of the moving body frame on the three orthogonal planes which make up the inertial reference frame.



Z-X plane: Plot A_{i3} vs. A_{i1}

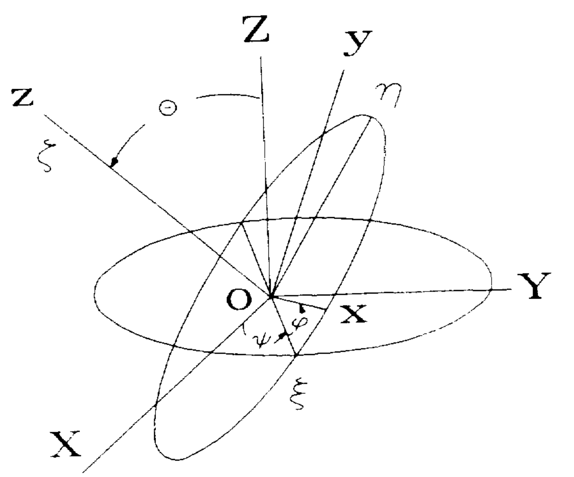
Z-Y plane: Plot A_{i3} vs. A_{i2}

Y-X plane: Plot A_{i2} vs. A_{i1}

a) Orientation Angles

It was seen in the preceding paragraph that only three independent parameters are required for specifying the orientation of a reference frame relative to another frame. The scheme to use three independent parameters to define the orientation of a reference frame is due to Leonard Euler (1707-1783). It consists of a specified sequence of three rotations about three noncoplanar (and nonorthogonal) axes. There are two basic types of rotation sequences.

I) Classical Euler Angles (Type 3-1-3)



1) Precession ψ about 3-axis

$$\begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = A_3(\psi) \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

NOTE:

There are actually 12 possible Euler rotation sequences considering only positive rotations.

2) Nutation θ about $1'$-axis

$$\begin{bmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix} = A_1(\theta) \begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix}$$

3) Spin ϕ about $3''$-axis

$$\begin{bmatrix} \mathbf{e}_1''' \\ \mathbf{e}_2''' \\ \mathbf{e}_3''' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{bmatrix} = A_3(\phi) \begin{bmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{bmatrix}$$

The matrices $A_3(\psi), A_1(\theta)$ and $A_3(\phi)$ are also known as canonical rotation matrices.

The rotation matrix for the combined sequence of rotations is given by:

$$A = A_3(\phi)\ A_1(\theta)\ A_3(\psi)$$

Multiplication of the three canonical matrices yields then:

$$A = \begin{bmatrix} (c\phi\ c\psi - s\phi c\theta\ s\psi) & (c\phi s\psi + s\phi c\theta c\psi) & (s\phi s\theta) \\ (-s\phi\ c\psi - c\phi c\theta\ s\psi) & (-s\phi s\psi + c\phi c\theta c\psi) & (c\phi s\theta) \\ (s\theta\ s\psi) & (-s\theta\ c\psi) & (c\theta) \end{bmatrix}$$

where $c\phi = \cos\phi \quad s\phi = \sin\phi$ etc.

NOTE:

The notation angle θ is usually restricted to the range $0 \le \theta \le \pi$.
As in the preceding paragraph it is again possible to establish a kinematical differential equation for the rate of change of the Euler angles. To this end we add the Euler angle rates vectorically.

$$\boldsymbol{\omega} = \dot{\boldsymbol{\psi}} + \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\phi}} \tag{4.34}$$

or in terms of the appropriate unit vectors:

$$\omega_1\ \mathbf{e}_1''' + \omega_2 \mathbf{e}\ \mathbf{e}_2''' + \omega_3\ \mathbf{e}_3''' = \dot{\psi}\ \mathbf{e}_3 + \dot{\theta}\mathbf{e}_1' + \dot{\phi}\ \mathbf{e}_3'' \tag{4.35}$$

The unit vectors on the right hand side can be easily converted by using the above canonical rotation matrices. We obtain then the desired relationship in matrix form as:

$$\boldsymbol{\omega} = A_3(\phi)\ A_1(\theta)\ A_3(\psi) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + A_3(\phi)\ A_1(\theta) \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} + A_3(\phi) \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \tag{4.36}$$

or in component form:

$$\omega_1 = \dot{\psi} \sin\theta \sin\phi + \dot{\theta} \cos\phi$$
$$\omega_2 = \dot{\psi} \sin\theta \cos\phi - \dot{\theta} \sin\phi$$
$$\omega_3 = \dot{\psi} \cos\theta + \dot{\phi}$$

These equations are known as Euler's kinematical equations.

To obtain the desired differential equations for the Euler angle rates the above equations are inverted and arranged in matrix form which yields:

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{\sin\theta} \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ (\sin\theta\cos\phi) & (-\sin\theta\sin\phi) & 0 \\ (-\cos\theta\sin\phi) & (-\cos\theta\cos\phi) & \sin\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{4.37}$$

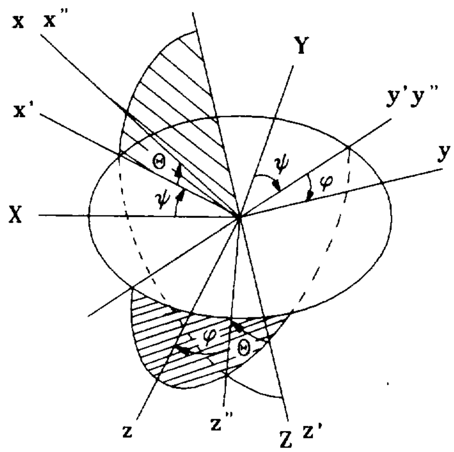
It is seen that for $\theta = 0$ the above equations become singular. This singularity has its physical origin from the fact that for $\theta = 0$ the angular velocities $\dot{\phi}$ and $\dot{\psi}$ (spin and precession rates) can no longer be distinguished. This situation corresponds to a condition of double gimbal gyroscopic suspensions known as gimbal lock. It turns out that the gimbal lock singularity is an intrinsic property of all three-parameter orientation schemes.

NOTE:

For double-gimbal gyroscopic suspension systems (Cardanic suspension) the nutation angle θ corresponds to the inner gimbal angle whereas the precession angle ψ corresponds to the outer gimbal angle.

P.S. A variety of Euler angle systems are in common use. Although there is no essential difference between them their end formulas (rotation matrix and kinematical equation) are difficult to compare. Therefore, it is better to derive each system from scratch if needed.

II. Modern Euler Angles (type 3-2-1)



1) Yaw ψ about 3-axis

$$\begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = A_3(\psi) \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

1) Pitch θ about $2'$-axis

$$\begin{bmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix} = A_2(\theta) \begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix}$$

1) Roll ϕ about $1''$-axis

$$\begin{bmatrix} \mathbf{e}_1''' \\ \mathbf{e}_2''' \\ \mathbf{e}_3''' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{bmatrix} = A_1(\phi) \begin{bmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{bmatrix}$$

The rotation matrix for the combined sequence of rotations is given by:

$$A = A_1(\phi)\, A_2(\theta)\, A_3(\psi) \tag{4.38}$$

Multiplication of the three canonical matrices yields then:

$$A = \begin{bmatrix} (c\theta\; c\psi) & (c\theta s\psi) & (-s\theta) \\ (s\phi\; s\theta c\psi - c\phi\; s\psi) & (s\phi s\theta s\psi + c\phi c\psi) & (s\phi c\theta) \\ (c\phi\; s\theta\; c\psi + s\phi s\psi) & (c\phi\; s\theta\; s\psi - s\phi c\psi) & (c\phi c\theta) \end{bmatrix}$$

NOTE:

The pitch angle θ is usually restricted to the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The yaw angle ψ is also known as leading or azimuth angle.

The pitch angle θ is also known as attitude or elevation angle.

The roll angle ϕ is also known as bank or clock angle.

To obtain the kinematic differential equations for the Euler angle rates we again add the angular velocities vectorially:

$$\boldsymbol{\omega} = \dot{\boldsymbol{\psi}} + \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\phi}} = \dot{\psi}\; \mathbf{e}_3 + \dot{\theta}\; \mathbf{e}_2' + \dot{\phi}\; \mathbf{e}_1'' \tag{4.39}$$

By taking the appropriate scalar products as above in Equation 4.35 we obtain the desired kinematical equations:

$$\omega_1 = \dot{\phi} - \dot{\psi}\; \sin\theta$$

$$\omega_2 = \dot{\theta}\cos\phi + \dot{\psi}\; \cos\theta\; \sin\phi$$

$$\omega_3 = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\; \sin\phi$$

The Euler angle rates are again obtained by inverting the above equation and arranging in matrix form with the result:

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{\cos\theta} \begin{bmatrix} 0 & \sin\phi & \cos\phi \\ 0 & (\cos\phi\cos\theta) & (-\sin\phi\cos\theta) \\ \cos\theta & (\sin\phi\sin\theta) & (\cos\phi\sin\theta) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{4.40}$$

It is seen that the intrinsic singularity now occurs for $\theta = 90^o$.

NOTE:

The names given to these orientation angles indicate their preferred applications. The classical Euler angles are generally chosen for gyroscopic (spinning) systems whereas the modern Euler angles find application in the flight dynamics of airplanes and missiles. There the interest lies often in small angle deviations (perturbations) about a nominal flight condition. It is easily seen that the classical Euler angles cannot be used for such situations because of the singularity at $\theta = 0$. In airplanes the body axes are selected such, that the x-axis points "forward", the y-axis is "to the right" and the z-axis "downwards."

C) Quaternions (1843)

It is possible to avoid the singularity of the three-parameter schemes by specifying the orientation of a reference frame using four parameters. This was first discovered by Euler in 1776. These four parameters are therefore also known as Euler parameters. When William R. Hamilton (1805-1865) formulated his quaternion algebra it turned out that when applied to rotational kinematics, the quaternions are essentially equivalent to the Euler parameters. In fact, the quaternion method proves to be a very convenient tool in the study of rotational dynamics.

The four-parameter scheme centers about Euler's theorem: Every rotation of a rigid body is equivalent to a single rotation about some axis **e** (eigen axis) through some angle δ.

It is obvious that any vector lying along this axis of rotation is unaffected by the rotation, i.e., it must have the same components in the new and old system. Using matrix notation for this vector Euler's theorem can be formulated as:

$$\mathbf{R}' = A\,\mathbf{R} = \mathbf{R} \qquad \text{or } (A - E)\,\mathbf{R} = 0 \tag{4.41}$$

where E = unit matrix and A = rotation matrix.

Recognizing Equation 4.41 as an eigenvalue problem, Euler's theorem can be restated as follows:

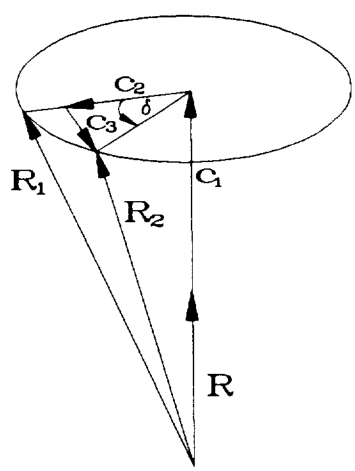
The rotational matrix A defining the orientation of a reference frame has the eigenvalue $+1$.

To prove Euler's theorem it is therefore necessary and sufficient to show that the coefficient determinant

$$|A - E| = 0 \tag{4.42}$$

This can be done by using the orthogonality condition of matrix A and the fact that the rotational matrix has the determinant +1. The latter fact can be deduced by observing that any rotation matrix must evolve continuously from the unit matrix assuming that the two reference frame are coincident prior to rotation. The unit matrix E has, of course, the determinant +1. A sudden change in sign from +1 to -1 would be incompatible with a continuous motion and represent a transition from a right-handed to a left-handed system.

To arrive at the four-parameter representation we take a geometric approach. Consider the rotation of a vector **T** about an axis **e** through the angle δ according to Euler's theorem.



The vector $\mathbf{R}_2$ can be expressed in terms of $\mathbf{R}_1$ and the unit vector **e** along the eigenaxis as:

$\mathbf{R}_2 = (\mathbf{R}_1 \cdot \mathbf{e})\mathbf{e}$
$+ [\mathbf{R}_1 - (\mathbf{R}_1 \cdot \mathbf{e})\mathbf{e}] \cos\delta$
$+ (\mathbf{e} \times \mathbf{R}_1) \sin\delta$
$\mathbf{R}_2 = \mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3$

The first term is obviously the projection of $\mathbf{R}_1$ along the axis of rotation **e**.

The second and third terms have the same magnitude, i.e., $\mathbf{R}_1 \sin(\mathbf{R}_1, \mathbf{e})$ which is in fact, the radius of the circle which is traversed by the tip of the vector $\mathbf{R}_1$ during its rotation.

The second vector lies in the plane of $\mathbf{R}_1$ and $\mathbf{e}$, and is perpendicular to $\mathbf{e}$. The third vector is perpendicular to the plane spanned by $\mathbf{R}_1$ and $\mathbf{e}$.

It is now possible to convert the above equations to vector-dyadic form by introducing the rotation dyadic:

$$\mathcal{D} = \mathcal{E}\cos\delta + (\mathbf{ee})(1-\cos\delta) + (\mathbf{e}\times\mathcal{E})\sin\delta \tag{4.43}$$

which results in

$$\mathbf{R}_2 = \mathcal{D}\cdot\mathbf{R}_1 \tag{4.44}$$

Notice that the rotation dyadic $\mathcal{D}$ is defined by four parameters, namely the three direction cosines of the rotation axis unit vector $\mathbf{e} = \cos\alpha\mathbf{e}_1 + \cos\beta\mathbf{e}_2 + \cos\gamma\mathbf{e}_3$ and the rotation angle δ.

To obtain the desired rotation matrix A from the rotation dyadic $\mathcal{D}$, we have to remember that the rotation matrix refers to the orientation of one reference frame relative to another. That means, the vector $\mathbf{R}$ is assumed to remain fixed and the changes in the components of $\mathbf{R}$ are due to this rotation of the reference frame. In the preceding discussion, it was assumed that the reference frame was fixed and that the vector $\mathbf{R}$ was rotated in the opposite direction. The rotation matrix, is therefore, obtained by simply changing the sign of the rotation angle δ in Equation 4.43. It is important to notice that the direction cosines of the rotation axis refer, of course, to the rotating reference frame when they appear in the rotation matrix A. Their instantaneous relation to the angular velocity is:

$$\cos\alpha = \frac{\omega_1}{\omega}\ ,\ \cos\beta = \frac{\omega_2}{\omega}\ ,\ \cos\gamma = \frac{\omega_3}{\omega}$$

The result is:

$$A = \mathcal{E}\ \cos\delta + \mathbf{ee}^T(1-\cos\delta) - \tilde{\mathbf{e}}\ \sin\delta \tag{4.45}$$

where

$$\mathbf{e} = \begin{bmatrix}\cos\alpha\\ \cos\beta\\ \cos\gamma\end{bmatrix} \quad \text{and} \quad \tilde{e} = \begin{bmatrix}0 & -\cos\gamma & \cos\beta\\ \cos\gamma & 0 & -\cos\alpha\\ -cos\beta & \cos\alpha & 0\end{bmatrix}$$

From Equation 4.45 we obtain the elements of the rotation matrix as follows:

$$A_{11} = \cos\delta + \cos^2\alpha(1 - \cos\delta) \tag{4.46}$$

$$A_{12} = \cos\alpha\cos\beta(1 - \cos\delta) + \cos\gamma\sin\delta \tag{4.47}$$

$$A_{13} = \cos\alpha\cos\beta(1 - \cos\delta) - \cos\beta\sin\delta \tag{4.48}$$

$$A_{21} = \cos\alpha\cos\beta(1 - \cos\delta) - \cos\gamma\sin\delta \tag{4.49}$$

$$A_{22} = \cos\delta + \cos^2\beta(1 - \cos\delta) \tag{4.50}$$

$$A_{23} = \cos\beta\cos\gamma(1 - \cos\delta) + \cos\alpha\sin\delta \tag{4.51}$$

$$A_{31} = \cos\alpha\cos\gamma(1 - \cos\delta) + \cos\beta\sin\delta \tag{4.52}$$

$$A_{32} = \cos\beta\cos\gamma(1 - \cos\delta) - \cos\alpha\sin\delta \tag{4.53}$$

$$A_{33} = \cos\delta + \cos^2\gamma(1 - \cos\delta) \tag{4.54}$$

These elements can be written in simpler form by introducing the Euler parameters:

$$q_1 = \cos\alpha\sin\frac{\delta}{2} \tag{4.55}$$

$$q_2 = \cos\beta\sin\frac{\delta}{2} \tag{4.56}$$

$$q_3 = \cos\gamma\sin\frac{\delta}{2} \tag{4.57}$$

$$q_4 = \cos\frac{\delta}{2} \tag{4.58}$$

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \tag{4.59}$$

$$-\pi \leq \delta \leq \pi \tag{4.60}$$

Recalling the trigonometric half-angle formulas

$$1 - \cos\delta = 2\sin^2\frac{\delta}{2} \tag{4.61}$$

$$\sin\delta = 2\sin\frac{\delta}{2}\cos\frac{\delta}{2} \tag{4.62}$$

the first element A_{11} can be written in terms of Euler parameters as:

$$\cos\delta + \cos^2\alpha(1-\cos\delta) = 1 - 2\sin^2\frac{\delta}{2} + 2\cos^2\alpha\sin\frac{\delta}{2}$$
$$= 1 - 2(1-q_4^2) + 2q_1^2 = 1 - 2(q_1^2+q_2^2+q_3^2) + 2q_1^2$$

$$= 1 - 2q_2^2 - 2q_3^2 = q_1^2 - q_2^2 - q_3^2 + q_4^2$$

The other elements can be written in similar form.

The notation matrix in terms of Euler parameters is then:

$$A = \begin{bmatrix} (q_1^2 - q_2^2 - q_3^2 + q_4^2) & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & (-q_1^2 + q_2^2 - q_3^2 + q_4^2) & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & (-q_1^2 - q_2^2 + q_3^2 + q_4^2) \end{bmatrix} \qquad (4.63)$$

NOTE:

The angle of rotation δ can be found easily by taking the trace of the rotation matrix A in the form given in Equations 4.46 thru 4.54:

$$\text{tr } A = A_{11} + A_{22} + A_{33} = 1 + 2\cos\delta \qquad (4.64)$$

It should be noted, that as expected, the four Euler parameters are not independent, because one needs only three parameters to define the orientation. They are related by the one constraint equation $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$.

It remains to derive the kinematical differential equations for the rate of change of the Euler parameters. Going back to Equation 4.33 we had

$$\dot{A} = -\tilde{\omega}A \qquad (4.65)$$

This matrix equation consists of nine differential equations of which only three are independent. For the kinematical differential equations we need four relationships. Taking the time derivatives of the diagonal elements of Equation 4.65 and the constraint equation associated with the Euler parameters we obtain:

$$q_1\dot{q}_1 - q_2\dot{q}_2 - q_3\dot{q}_3 + q_4\dot{q}_4 = \omega_3(q_1q_2 - q_3q_4) - \omega_2(q_1q_3 + q_2q_4) \qquad (4.66)$$

$$-q_1\dot{q}_1 + q_2\dot{q}_2 - q_3\dot{q}_3 + q_4\dot{q}_4 = -\omega_3(q_1q_2 + q_3q_4) + \omega_1(q_2q_3 - q_1q_4) \qquad (4.67)$$

$$-q_1\dot{q}_1 - q_2\dot{q}_2 + q_3\dot{q}_3 + q_4\dot{q}_4 = \omega_2(q_1q_3 - q_2q_4) - \omega_1(q_2q_3 + q_1q_4) \qquad (4.68)$$

$$q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3 + q_4\dot{q}_4 = 0 \tag{4.69}$$

Introducing the matrices

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \qquad \overline{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & -\omega_1 \\ \omega_3 & 0 & -\omega_1 & -\omega_2 \\ -\omega_2 & \omega_1 & 0 & -\omega_3 \\ \omega_1 & \omega_2 & \omega_3 & 0 \end{bmatrix} = \left[\begin{array}{c|c} \tilde{\omega} & -\boldsymbol{\omega} \\ \hline \boldsymbol{\omega} & 0 \end{array}\right]$$

the above equations can be written in matrix form as:

$$\dot{\mathbf{q}} = -\frac{1}{2}\,\overline{\omega}\,\mathbf{q} \tag{4.70}$$

Integration of Quaternions

The kinematical Equations 4.70 are not numerically integrated as such. A better integration procedure can be developed by assuming for a moment that the angular velocity $\boldsymbol{\omega}$ is constant. For this case the quaternions of Equations 4.55 thru 4.58 can be integrated in closed form, because the direction cosines remain constant, such that at all times:

$$\cos\,\alpha = \frac{\omega_1}{\omega} \qquad \cos\,\beta = \frac{\omega_2}{\omega} \qquad \cos\,\gamma = \frac{\omega_3}{\omega} \tag{4.71}$$

Setting the initial rotation angle $\delta_0 = \omega\, t_0$ and the final rotation angle $\delta = \omega\, t$ we have then:

$$\begin{aligned} q_1 &= \frac{\omega_1}{\omega}\,\sin\,\frac{\delta+\delta_0}{2} \\ q_2 &= \frac{\omega_2}{\omega}\,\sin\,\frac{\delta+\delta_0}{2} \\ q_3 &= \frac{\omega_3}{\omega}\,\sin\,\frac{\delta+\delta_0}{2} \\ q_4 &= \cos\,\frac{\delta+\delta_0}{2} \end{aligned}$$

To arrive at the relationship between the initial and final quaternions, let us also introduce:

$$q_{10} = \frac{\omega_1}{\omega} \sin \frac{\delta_0}{2}$$
$$q_{20} = \frac{\omega_2}{\omega} \sin \frac{\delta_0}{2}$$
$$q_{30} = \frac{\omega_3}{\omega} \sin \frac{\delta_0}{2}$$
$$q_{40} = \cos \frac{\delta_0}{2}$$

The next step will be illustrated using q_1 as an example. Expanding the trigonometric function, we obtain:

$$q_1 = \frac{\omega_1}{\omega} (\sin \frac{\delta}{2} \cos \frac{\delta_0}{2} + \cos \frac{\delta}{2} \sin \frac{\delta_0}{2})$$

$$q_1 = \frac{\omega_1}{\omega} \sin \frac{\delta}{2} (\cos \frac{\delta_0}{2}) + \cos \frac{\delta}{2} (\frac{\omega_1}{\omega} \sin \frac{\delta_0}{2})$$

$$q_1 = \frac{\omega_1}{\omega} \sin \frac{\delta}{2} q_{40} + \cos \frac{\delta_0}{2} q_{10}$$

The other components are manipulated in like manner. We obtain them:

$$q_1 = \cos \frac{\delta}{2} q_{10} + \frac{\omega_1}{\omega} \sin \frac{\delta}{2} q_{40}$$
$$q_2 = \cos \frac{\delta}{2} q_{20} + \frac{\omega_2}{\omega} \sin \frac{\delta}{2} q_{40}$$
$$q_3 = \cos \frac{\delta}{2} q_{30} + \frac{\omega_3}{\omega} \sin \frac{\delta}{2} q_{40}$$
$$q_4 = -\sin \frac{\delta}{2} (\frac{\omega_1}{\omega} q_{10} + \frac{\omega_2}{\omega} q_{20} + \frac{\omega_3}{\omega} q_{30}) + \cos \frac{\delta}{2} q_{40}$$

where $\delta = \omega t$,

or in matrix form:

$$\mathbf{q} = (E \cos \frac{\omega t}{2} - \frac{\overline{\omega}}{\omega} \sin \frac{\omega t}{2})\mathbf{q}_0$$

where $E = (4 \times 4)$ unit matrix and $\overline{\omega}$ defined as in Equation 4.70. While only an approximation for a time-varying angular velocity the error is extremely small for short time integration intervals. This scheme automatically satisfies the constraint equation and its computer efficiency is excellent.

NOTE:

Computer coding is more efficient when using the above components because the matrix equation involves some cancellation for the off-diagonal terms.

d) Interrelatedness

Quite often it is necessary to convert from one orientation scheme to another. The conversion from Euler angles or quaternions to direction cosines is, of course, directly given by the rotation matrix A expressed either in Euler parameters or Euler angles. The inverse problem of converting from the direction cosine (rotation) matrix to Euler angles or quaternions is more involved due to the inherent ambiguity resulting from having more equations than unknowns.

A) Classical Euler Angles from Rotation Matrix

$$\begin{array}{ll} \theta = \cos^{-1} A_{33} & 0 \le \theta \le \pi \\ \phi = \tan^{-1} A_{13}/A_{23} & -\pi < \phi < \pi \\ \psi = \tan^{-1} A_{31}/A_{32} & -\pi < \psi < \pi \end{array}$$

B) Modern Euler Angles from Rotation Matrix

$$\begin{array}{ll} \theta = \sin^{-1} (-A_{33}) & -\pi/2 < \theta < \pi/2 \\ \phi = \tan^{-1} A_{23}/A_{33} & -\pi < \phi < \pi \\ \psi = \tan^{-1} A_{12}/A_{11} & -\pi < \psi < \pi \end{array}$$

The above relationships can be easily verified by inspection of the rotation matrices expressed in terms of Euler angles. The ambiguities stemming from the inverse tangent functions can be resolved by observing the signs of the direction cosines in their arguments as numerators and denominators.

The following quadrants can then be determined:

I. Quadrant: +/+

II. Quadrant: +/-

III. Quadrant: -/-

IV. Quadrant: -/+

C) Quaternions from Classical Euler Angles

$$q_1 = \sin\frac{\theta}{2}\ \cos\,(\frac{\psi-\phi}{2})$$
$$q_2 = \sin\frac{\theta}{2}\ \sin\,(\frac{\psi-\phi}{2})$$
$$q_3 = \cos\frac{\theta}{2}\ \sin\,(\frac{\psi+\phi}{2})$$
$$q_4 = \cos\frac{\theta}{2}\ \cos\,(\frac{\psi+\phi}{2})$$

D) Quaternions from Modern Euler Angles

$$q_1 = \sin\frac{\phi}{2}\ \cos\frac{\psi}{2}\ \cos\frac{\theta}{2} - \sin\frac{\psi}{2}\ \sin\frac{\theta}{2}\ \cos\frac{\phi}{2}$$
$$q_2 = \sin\frac{\theta}{2}\ \cos\frac{\psi}{2}\ \cos\frac{\phi}{2} + \sin\frac{\psi}{2}\ \sin\frac{\phi}{2}\ \cos\frac{\theta}{2}$$
$$q_3 = \sin\frac{\psi}{2}\ \cos\frac{\theta}{2}\ \cos\frac{\psi}{2} - \sin\frac{\theta}{2}\ \sin\frac{\phi}{2}\ \cos\frac{\psi}{2}$$
$$q_4 = \sin\frac{\psi}{2}\ \sin\frac{\theta}{2}\ \sin\frac{\phi}{2} + \cos\frac{\psi}{2}\ \cos\frac{\theta}{2}\ \cos\frac{\phi}{2}$$

NOTE:

For small angles the quaternions are approximately:

$$q_1 = \frac{\phi}{2}$$
$$q_2 = \frac{\theta}{2}$$
$$q_3 = \frac{\psi}{2}$$
$$q_4 = 1$$

Notice also the ubiquitous presence of half angles which is typical for all four-parameter schemes.

The derivation of these formulas can be done taking the following steps: (Quaternion algebra would provide a more elegant way).

Take the diagonal terms of the rotation matrix A and the constraint equation of the quaternions:

$$\begin{aligned} q_1^2 - q_2^2 - q_3^2 + q_4^2 &= A_{11} \\ -q_1^2 + q_2^2 - q_3^2 + q_4^2 &= A_{22} \\ -q_1^2 - q_2^2 + q_3^2 + q_4^2 &= A_{33} \\ q_1^2 + q_2^2 + q_3^2 + q_4^2 &= 1 \end{aligned}$$

As an example we solve for q_2 by adding the second and fourth equation to obtain:

$$2\, q_2^2 + 2\, q_4^2 = 1 + A_{22} \tag{4.72}$$

Also adding the first and third equation:

$$-\,2\, q_2^2 + 2\, q_4^2 = A_{11} + A_{33} \tag{4.73}$$

Eliminating q_4^2 yields then:

$$4q_2^2 = 1 - A_{11} + A_{22} - A_{33} \tag{4.74}$$

Next we introduce the half-angle relations:

$$\sin\,\alpha_1 = 2\,\sin\frac{\alpha_1}{2}\,\cos\frac{\alpha_1}{2}$$

$$\cos\ \alpha_1 = 1 - 2\ \sin^2\ \frac{\alpha_1}{2}$$

To make the notation more concise we let $\sin\alpha_1 = \overline{s_1}$, $\cos\alpha_1 = \overline{c_1}, \sin\frac{\alpha_1}{2} = s_1$ and $\cos\frac{\alpha_1}{2} = c_1$.

Introducing in addition, the notation for the modern Euler angles: $\psi = \theta_1, \theta = \theta_2$ and $\phi = \theta_3$, we rewrite the direction cosines as:

$$\begin{aligned} A_{11} &= \overline{c_1}\,\overline{c_2} \\ A_{22} &= \overline{s_1}\,\overline{s_2}\,\overline{s_3} + \overline{c_1}\,\overline{c_3} \\ A_{33} &= \overline{c_2}\,\overline{c_3} \end{aligned}$$

The second quaternion can then be manipulated as follows:

$$\begin{aligned} 4q_2^2 &= 1 + \overline{s_1}\,\overline{s_2}\,\overline{s_3} + \overline{c_1}\,\overline{c_3} - \overline{c_1}\,\overline{c_2} - \overline{c_2}\,\overline{c_3} \\ &= 1 + 8\ s_1\ s_2\ s_3\ c_1\ c_2\ c_3 + (1 - 2\ s_1^2)(1 - 2\ s_3^2) \\ &- (1 - 2\ s_1^2)(1 - 2\ s_2^2) - (1 - 2\ s_2^2)(1 - 2\ s_3^2) \\ q_2^2 &= 2\ s_1\ s_2\ s_3\ c_1\ c_2\ c_3 + s_2^2 + s_1^2\ s_3^2 - s_1^2\ s_2^2 - s_2^2\ s_3^2 \\ &= 2\ s_1\ s_2\ s_3\ c_1\ c_2\ c_3 + s_2^2\ (s_1^2 + c_1^2)\ (s_3^2 + c_3^2) \\ &+ s_1^2\ s_3^2\ (s_2^2 + c_2^2) - s_1^2 s_2^2 (s_3^2 + c_3^2) - s_2^2 s_3^2 (s_1^2 + c_1^2) \\ &= 2 s_1\ s_2\ s_3\ c_1\ c_2\ c_3 + s_2^2\ c_1^2\ c_3^2 + s_1^2\ s_3^2\ c_2^2 \end{aligned}$$

Finally:

$$q_2^2 = (s_2\ c_1\ c_3 + s_1\ s_3\ c_2)^2 \tag{4.75}$$

Taking the square root we take the positive sign and that for small angles $q_2 = \frac{\theta}{2}$:

$$q_2 = s_2\ c_1\ c_3 + s_1\ s_3\ c_2 \tag{4.76}$$

or reverting back to the original notation:

$$q_2 = \sin\ \frac{\theta}{2}\ \cos\ \frac{\psi}{2}\ \cos\ \frac{\phi}{2} + \sin\ \frac{\psi}{2}\ \sin\ \frac{\phi}{2}\ \cos\frac{\theta}{2} \tag{4.77}$$

The other quaternions can be obtained in a similar fashion.

E) Quaternions from Rotation Matrix

Different algorithms have been suggested to solve this inverse problem. The most elegant an efficient one makes use of the inherent symmetry of the quaternion relationships. To reveal this symmetry we introduce the following definitions:

$$p = 2\ q, \quad A_{44} = T_R \quad \text{and} \quad T_R = A_{11} + A_{22} + A_{33} \tag{4.78}$$

Weakness:

The kinematical equations are transcendental and singular (gimbal look). As a consequence, computer coding is ineffective and awkward.

Quaternions:

Strength:

The small set of four parameters having linear nonsingular kinematical equations admits of highly efficient computer coding. Quaternion algebra affords concise derivation of multiple reference frame inter-relationships.

Weakness:

Poor visualization.

E) Direction (Aerodynamic) Angles

It was observed that the definition of an orientation (attitude) requires three independent parameters. By contrast a direction can be specified using only two independent parameters. This can be understood by realizing that a direction is defined by a unit vector whose three direction cosines have to satisfy the constraint equation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. This leaves two independent parameters for the direction specification. Direction angles find their foremost application in determining the aerodynamic forces acting on a body. This explains the terminology commonly used for the direction angles. There are two types of direction angles in common practice. It is assumed that prior to the rotation sequence the x-axis of the body is aligned with the desired direction. In aerodynamic terms, the x-axis is aligned with the nominal flight velocity $\mathbf{v}$.

Type I:

Here the direction angles are defined by the following rotation sequence:

1) Counterclockwise rotation β about z-axis

$$\begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

2) Clockwise rotation α about y'-axis

$$\begin{bmatrix} e_1'' \\ e_2'' \\ e_3'' \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} e_1' \\ e_2' \\ e_3' \end{bmatrix}$$

The total rotation matrix is obtained by multiplying the two canonical rotation matrices:

$$\begin{bmatrix} e_1'' \\ e_2'' \\ e_3'' \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

As assumed the flight velocity before the rotation was along the x-axis: ($\mathbf{v} = v\mathbf{i}$). Therefore:

$$\mathbf{v} = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}$$

The velocity components in the rotated reference frame are therefore:

$$v_x = v \cos\alpha \cos\beta$$
$$v_y = v \sin\beta$$
$$v_z = v \sin\alpha \cos\beta$$

The two orientation angles α and β can now be expressed in terms of these velocity components as:

Angle-of-attack	$\tan\alpha = \frac{v_z}{v_x}$	$-180^o \le \alpha \le 180^o$
Sideslip angle	$\sin\beta = \frac{v_y}{v}$	$-90^o \le \beta \le 90^o$

The restrictions on the ranges of the orientation angles are introduced to avoid ambiguities when taking the inverse trigonometric functions to obtain the direction angles. The quadrant of the angle of attack
is determined by the appropriate signs in the numerator and denominator of the argument.

Comparing the direction angles with the modern Euler angles, it is readily seen that the angle-of-attack corresponds to the pitch angle θ and the sideslip angle β to the negative (!) yaw angle ψ. A positive sideslip angle β obtains for a "nose left" situation. This unusual sign convention stems from the desire to have a positive sideslip angle when the wind comes from the right.

Type II:

Here the direction angles are defined by a slightly different rotation sequence which essentially corresponds to the modern Euler angle rotation sequence with $\psi = 0$.

1) Clockwise rotation α_T about y-axis

$$\begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} \cos\alpha_T & 0 & -\sin\alpha_T \\ 0 & 1 & 0 \\ \sin\alpha_T & 0 & \cos\alpha_T \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

2) Clockwise rotation ϕ_R about x'-axis

$$\begin{bmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_R & \sin\phi_R \\ 0 & -\sin\phi_R & \cos\phi_R \end{bmatrix} \begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix}$$

The total rotation matrix is then:

$$\begin{bmatrix} \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{bmatrix} = \begin{bmatrix} \cos\alpha_T & 0 & -\sin\alpha_T \\ \sin\alpha_T\sin\phi_R & \cos\phi_R & \cos\alpha_T\sin\phi_R \\ \sin\alpha_T\phi_R & -\sin\phi_R & \cos\alpha_T\cos\phi_R \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

Assuming again that the nominal flight velocity is along the original x-axis, the velocity components in the rotated frame are:

$$v_x = v \cos \alpha_T$$
$$v_y = v \sin \alpha_T \sin\phi_R$$
$$v_z = v \sin \alpha_T \cos \phi_R$$

The two orientation angles can then be defined as:

Total angle-of-attack	$\cos\alpha_T = \frac{v_x}{v}$	$0 \leq \alpha_T < 180^\circ$
Roll Angle	$\tan\phi_R = \frac{v_y}{v_z}$	$0 \leq \phi_R < 360^\circ$

Notice again the range restriction for the total angle of attack α_T which allows a unique inversion of the cosine function.

$$\cos \alpha_1 = 1 - 2 \sin^2 \frac{\alpha_1}{2}$$

To make the notation more concise we let $\sin\alpha_1 = \overline{s_1}$, $\cos\alpha_1 = \overline{c_1}, \sin\frac{\alpha_1}{2} = s_1$ and $\cos\frac{\alpha_1}{2} = c_1$.

Introducing in addition, the notation for the modern Euler angles: $\psi = \theta_1, \theta = \theta_2$ and $\phi = \theta_3$, we rewrite the direction cosines as:

$$\begin{aligned} A_{11} &= \overline{c_1}\,\overline{c_2} \\ A_{22} &= \overline{s_1}\,\overline{s_2}\,\overline{s_3} + \overline{c_1}\,\overline{c_3} \\ A_{33} &= \overline{c_2}\,\overline{c_3} \end{aligned}$$

The second quaternion can then be manipulated as follows:

$$\begin{aligned} 4q_2^2 &= 1 + \overline{s_1}\,\overline{s_2}\,\overline{s_3} + \overline{c_1}\,\overline{c_3} - \overline{c_1}\,\overline{c_2} - \overline{c_2}\,\overline{c_3} \\ &= 1 + 8\, s_1\, s_2\, s_3\, c_1\, c_2\, c_3 + (1 - 2\, s_1^2)(1 - 2\, s_3^2) \\ &- (1 - 2\, s_1^2)(1 - 2\, s_2^2) - (1 - 2\, s_2^2)(1 - 2\, s_3^2) \\ q_2^2 &= 2\, s_1\, s_2\, s_3\, c_1\, c_2\, c_3 + s_2^2 + s_1^2\, s_3^2 - s_1^2\, s_2^2 - s_2^2\, s_3^2 \\ &= 2\, s_1\, s_2\, s_3\, c_1\, c_2\, c_3 + s_2^2\,(s_1^2 + c_1^2)\,(s_3^2 + c_3^2) \\ &+ s_1^2\, s_3^2\,(s_2^2 + c_2^2) - s_1^2 s_2^2 (s_3^2 + c_3^2) - s_2^2 s_3^2 (s_1^2 + c_1^2) \\ &= 2 s_1\, s_2\, s_3\, c_1\, c_2\, c_3 + s_2^2\, c_1^2\, c_3^2 + s_1^2\, s_3^2\, c_2^2 \end{aligned}$$

Finally:

$$q_2^2 = (s_2\, c_1\, c_3 + s_1\, s_3\, c_2)^2 \tag{4.75}$$

Taking the square root we take the positive sign and that for small angles $q_2 = \frac{\theta}{2}$:

$$q_2 = s_2\, c_1\, c_3 + s_1\, s_3\, c_2 \tag{4.76}$$

or reverting back to the original notation:

$$q_2 = \sin\frac{\theta}{2}\cos\frac{\psi}{2}\cos\frac{\phi}{2} + \sin\frac{\psi}{2}\sin\frac{\phi}{2}\cos\frac{\theta}{2} \tag{4.77}$$

The other quaternions can be obtained in a similar fashion.

E) Quaternions from Rotation Matrix

Different algorithms have been suggested to solve this inverse problem. The most elegant an efficient one makes use of the inherent symmetry

of the quaternion relationships. To reveal this symmetry we introduce the following definitions:

$$p = 2\, q, \quad A_{44} = T_R \quad \text{and} \quad T_R = A_{11} + A_{22} + A_{33} \tag{4.78}$$

where T_R is the trace of the rotation matrix. The diagonal terms of the rotation matrix Equation 4.63 can be cast into a set of symmetrical equations:

$$
\begin{aligned}
p_1^2 &= 1 + 2\,A_{11} - T_R \\
p_2^2 &= 1 + 2\,A_{22} - T_R \\
p_3^2 &= 1 + 2\,A_{33} - T_R \\
p_4^2 &= 1 + 2\,A_{44} - T_R
\end{aligned}
\tag{4.79}
$$

The off-diagonal terms of Equation 4.63 furnish all combinations of cross product terms:

$$
\begin{aligned}
p_1\,p_2 &= A_{12} + A_{21} \qquad & p_3\,p_4 &= A_{12} - A_{21} \\
p_1\,p_3 &= A_{31} + A_{13} \qquad & p_2\,p_4 &= A_{31} - A_{13} \\
p_2\,p_3 &= A_{23} + A_{32} \qquad & p_1\,p_4 &= A_{23} - A_{32}
\end{aligned}
\tag{4.80}
$$

The first step in solving the above equations for the unknown quaternions is to select the largest p_i of the diagonal Equations 4.80.

This assures the highest possible numerical accuracy for the solutions. The next step is to choose the appropriate three off-diagonal equations from Equation 4.80 to calculate the remaining three unknown quaternions.

To avoid ambiguities the positive square root is taken when the first quaternion is calculated in step one. It is customary to restrict the range of the rotation angle to $-\pi \leq \delta \leq +\pi$. From the definition of the Euler parameters in Equation 4.58 it follows that q_4 is always positive. This requires that all the signs of the quaternions have to be changed if the above algorithm yields a negative p_4. A useful computer code for the solution of the above problem can be set up by taking the following steps:

Step 1: Define matrix

$$
\tilde{p} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{23} \\ A_{21} & A_{22} & A_{23} & A_{31} \\ A_{31} & A_{32} & A_{33} & A_{12} \\ -A_{32} & -A_{13} & -A_{21} & T_R \end{bmatrix}
$$

Step 2: Define matrix

$$p = \hat{p} + \hat{p}^T + (1 - T_R)E$$

where E is a 4 x 4 unit matrix.

Step 3: Select $k = \text{MAX } p_{ii}$

where k is the row of the largest diagonal element of p.

Step 4: $q_j^* = p_{kj}/2\sqrt{p_{kk}}$

Step 5: $q_j = q_j^* \; sgn \; q_4^*$

where sgn $= -1$ for $q_4^* < 0$

$+1$ for $q_4^* > 0$

Summary:

Direction Cosines:

Strength:

Linear nonsingular kinematical equations. Easy visualization makes them well suited for attitude display plots.

Weakness:

The unduly redundant set of nine parameters requires excessive computational effort.

Euler Angles:

Strength:

The set of three independent parameters makes them a natural tool for analytical studies. The classical system is particularly useful for analyzing

gyroscopic (spinning) systems whereas the modern system is applied to the stability and response analyses of systems which deviate only moderately from nominal operating conditions. For these cases the classical system would become singular.

Weakness:

The kinematical equations are transcendental and singular (gimbal look). As a consequence, computer coding is ineffective and awkward.

Quaternions:

Strength:

The small set of four parameters having linear nonsingular kinematical equations admits of highly efficient computer coding. Quaternion algebra affords concise derivation of multiple reference frame inter-relationships.

Weakness:

Poor visualization.

E) Direction (Aerodynamic) Angles

It was observed that the definition of an orientation (attitude) requires three independent parameters. By contrast a direction can be specified using only two independent parameters. This can be understood by realizing that a direction is defined by a unit vector whose three direction cosines have to satisfy the constraint equation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. This leaves two independent parameters for the direction specification. Direction angles find their foremost application in determining the aerodynamic forces acting on a body. This explains the terminology commonly used for the direction angles. There are two types of direction angles in common practice. It is assumed that prior to the rotation sequence the x-axis of the body is aligned with the desired direction. In aerodynamic terms, the x-axis is aligned with the nominal flight velocity $\mathbf{v}$.

Type I:

Here the direction angles are defined by the following rotation sequence:

1) Counterclockwise rotation β about z-axis

$$\begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

2) Clockwise rotation α about y'-axis

It is easily observed that the type II orientation angles correspond to the last two rotations of the modern Euler angle system. Therefore the total angle of attack α_T can be identified with the pitch angle θ. Whereas the roll angle ϕ_R is really nothing else but the role angle ϕ.

Both systems of orientation angles appear to be equally in use. However, the type II angles are not suitable for dynamic studies of small perturbations from a nominal flight condition because the roll angle approaches an indeterminate form ($v_y = v_z \approx 0$). It can be seen that the type I angles are actually well behaved for this condition. Notice also that in this case the x-component of the velocity v_x is nearly equal to the nominal flight velocity v ($v_x \approx v$).

The types of orientation angles are mathematically related as follows:

A) Conversion from type I to type II:

$$\cos\alpha_T = \cos\alpha\cos\beta$$
$$tg\phi_R = (\sin\alpha)^{-1} tg\beta$$

B) Conversion from type II to type I:

$$tg\alpha = \cos\phi_R tg\alpha_T$$
$$\sin\beta = \sin\phi_R \sin\alpha_T$$

Extreme caution has to be exercised when converting from one type to the other to make sure that the orientation angles lie in their correct quadrant.

For large orientation angles the type II system is easier to visualize than the type I system.

NOTE:

If the orientation angles have to be determined by starting from a given direction, the rotation sequence has to be reversed and the rotations be performed in the opposite clock sense. Thus the type I system would require first a counterclockwise rotation about the y-axis and then a clockwise rotation about the z-axis to align the given direction with the positive x-axis. Using the type II system, the reverse rotation sequence starts with a counter clockwise rotation (roll) about the x-axis followed by a counter clockwise rotation (pitch) about the y'-axis.

4.4 Moment of Inertia Properties:

The following properties of the moment of inertia matrix and its elements (moments and products of inertia) play a central role in dynamics studies.

A) Triangle inequality of the moments of inertia

The three moments of inertia about three rectangular axes are such that the sum of any two is greater than the third.

Let:

$$I_{11} = \Sigma(x_2^2 + x_3^2)m$$
$$I_{22} = \Sigma(x_1^2 + x_3^2)m$$
$$I_{33} = \Sigma(x_1^3 + x_2^2)m.$$

Then:

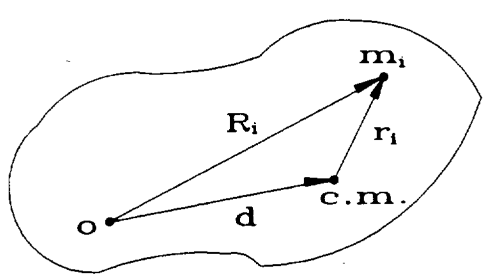
$$I_{11} + I_{22} - I_{33}$$
$$= \Sigma(x_2^2 + x_3^2 + x_1^2 + x_3^2 - x_1^2 - x_2^2)m = 2\pi x_3^2 m > 0$$

Therefore: $I_{11} + I_{22} > I_{33}$ Q.E.D.

Similar relationships are obtained by cyclic permutations of the indices.

B) Theorem of Parallel Axes (Steiner Theorem)

Let O be the origin of the reference frame about which the moment of inertia dyadic is defined and C.M. be the mass center of the rigid body.



In the figure:

$\mathbf{d}$ = distance of C.M. from O

$\mathbf{r}_i$ = location of mass element m_i.

Using vector-dyadic notation the inertia dyadic about O can be written as:

$$\mathcal{I}_0 = \Sigma \left[R_i^2\mathcal{E} - (\mathbf{R}_i\ \mathbf{R}_i)\right] m_i$$

$$\mathcal{I}_0 = \Sigma \left[(\mathbf{d} + \mathbf{r}_i)^2\mathcal{E} - (\mathbf{d} + \mathbf{r}_i)\ (\mathbf{d} + \mathbf{r}_i)\right]\ m_i$$

$$\mathcal{I}_0 = \Sigma \left[(\mathbf{d}^2 + 2\mathbf{d} \cdot \mathbf{r}_i + \mathbf{r}_i^2)\mathcal{E}\ -(\mathbf{d}\ \mathbf{d} + \mathbf{d}\mathbf{r}_i + \mathbf{r}_i\mathbf{d} + \mathbf{r}_i\ \mathbf{r}_i\right]\ m_i$$

For the mass center $\Sigma\mathbf{r}_i\ m_i = 0$ and $\Sigma\mathbf{d}\ m_i = M\mathbf{d}$ where $M = \Sigma\ m_i$.

Therefore:

$$\mathcal{I}_0 = \Sigma(\mathbf{r}_i^2\mathcal{E} - \mathbf{r}_i\ \mathbf{r}_i)\ m_i + M(\mathbf{d}^2\mathcal{E} - \mathbf{d}\ \mathbf{d})$$

or

$$\mathcal{I}_0 = \mathcal{I}_{C.M.} + M(d^2\mathcal{E} - \mathbf{d}\ \mathbf{d}$$

Converting to matrix form, we introduce:

$$\mathbf{d} = d_1\mathbf{e}_1 + d_2\mathbf{e}_2 + d_3\mathbf{e}_3$$

It can be easily verified that the dyadic $\mathcal{L} = \mathbf{a}^2\mathcal{I} - \mathbf{aa}$ corresponds to the matrix $L = -\tilde{a}^2$
where

$$\tilde{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Therefore the matrix form of the above inertia dyadic $\mathcal{I}_0$ is:

$$I_0 = I_{C.M.} - M\ \tilde{d}^2$$

In component form:

$$\begin{aligned} I_{11} &= I_{11}^c + M(d_2^2 + d_3^2) \\ I_{22} &= I_{22}^c + M(d_1^2 + d_3^2) \\ I_{33} &= I_{33}^c + M(d_1^2 + d_2^2) \end{aligned}$$

and also:

$$\begin{aligned} I_{12} &= I_{12}^c - M d_1 d_2 \\ I_{13} &= I_{13}^c - M d_1 d_3 \\ I_{23} &= I_{23}^c - M d_2 d_3 \end{aligned}$$

It can be seen that a translation of the axes away from the mass center always results in an increase of the moments of inertia. On the other hand, the products of inertia may increase or decrease depending on the particular situation.

C) Theorem of Rotated Axes

Here we establish the relation between the moment of inertia matrices expressed in two different reference frames. To do this we calculated the rotational kinetic energy for the two different systems whose common origin is

chosen at the mass center. For convenience sake we call one system the primed system and the other the unprimed system. It is apparent that the rotational kinetic energy, being a scalar, has to be the same for both coordinate systems.

$$T = \frac{1}{2}\, \boldsymbol{\omega}'^T I' \boldsymbol{\omega}' = \frac{1}{2} \boldsymbol{\omega}^T I \boldsymbol{\omega}$$

The rotational transformation is:

$$\boldsymbol{\omega}' = A\boldsymbol{\omega}$$

The kinetic energy is therefore:

$$T = \frac{1}{2} \boldsymbol{\omega}^T (A^T I' A) \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega}^T (I) \boldsymbol{\omega}$$

The two bracketed terms must be identical, i.e.,

$$A^T I' A = I$$

By proper pre-and post-multiplication with the rotation matrix A, we can solve for the primed moment of inertia matrix:

$$I' = A I A^T$$

This is the desired transformation law of the inertia matrix when going from an old (unprimed) to a new (primed) system by a rotation.

D) Principal Axes

The moment of inertia matrix is a real symmetric matrix. It is a theorem of matrix analysis that any real symmetric matrix can be reduced to diagonal form by means of an orthogonal transformation. This means that we can always find a reference frame in which all products of inertia are zero. This reference frame is known as principal axes system and the three mutually orthogonal coordinate axes as principal axes. The diagonal terms of this diagonal inertia matrix are called principal moments of inertia.

The problem of finding such a principal axes frame is equivalent to solving the eigenvalue problem

$$(I - \lambda E)\boldsymbol{x} = 0$$

The eigenvalues of this problem are the principal moments of inertia and their associated eigenvectors represent geometrically (unit) vectors along the principal axes of the reference frame. The eigenvectors can be grouped together to form a matrix Φ whose columns are the eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$.

Designating the components of the eigenvectors as

$$\mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix}$$

the matrix of the column vectors called modal matrix is

$$\boldsymbol{\Phi} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = [\mathbf{x}_1 | \mathbf{x}_2 | \mathbf{x}_3]$$

Let also $\lambda_1 = I_1$, $\lambda_2 = I_2$, $\lambda_3 = I_3$.

The diagonalization of the inertia matrix I using the modal matrix $\boldsymbol{\Phi}$ can then be performed as follows:

$$\begin{aligned} \boldsymbol{I}\,\boldsymbol{\Phi} &= [\boldsymbol{I}\boldsymbol{x}_1 \;|\; \boldsymbol{I}\boldsymbol{x}_2 \;|\; \boldsymbol{I}\boldsymbol{x}_3] \\ &= [I_1\boldsymbol{x}_1 \;|\; I_2\boldsymbol{x}_2 \;|\; I_3\boldsymbol{x}_3] \\ &= [\boldsymbol{x}_1 \;|\; \boldsymbol{x}_2 \;|\; \boldsymbol{x}_3] \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \boldsymbol{\Phi} I_D \end{aligned}$$

The second step in the above equation is a consequence of the eigenvalue solution $\boldsymbol{I}\boldsymbol{x}_1 = \lambda_1\,\boldsymbol{x}_1 = I_1\,\boldsymbol{x}_1$. Premultiplying by $\boldsymbol{\Phi}^T$ yields:

$$\boldsymbol{\Phi}^T\,\boldsymbol{I}\,\boldsymbol{\Phi} = \boldsymbol{\Phi}^T\,\boldsymbol{\Phi}\,I_D = I_D$$

or finally:

$$I_D = \boldsymbol{\Phi}^T\,\boldsymbol{I}\,\boldsymbol{\Phi}$$

Comparing this transformation law with the law governing the change of the inertia matrix under rotation we see that the modal matrix $\boldsymbol{\Phi}$ is related to the rotation matrix A simply as:

$$A = \boldsymbol{\Phi}^T$$

The rotation matrix is the transposed modal matrix. This rotation is also known as principal axes transformation.

Example:

For the Gemini spacecraft the inertia matrix referred to the control axis system with origin at the mass center was:

$$I = \begin{bmatrix} 4560.7 & 31.2 & -43.4 \\ 31.2 & 4545.0 & -270.7 \\ -43.4 & -270.7 & 1567.4 \end{bmatrix} \text{ slg ft}^2$$

We want to make a principal axis transformation:

1) Solve the eigenvector problem

$$(I - \lambda E)\boldsymbol{x} = 0$$

Eigenvalues:

$$\begin{aligned} I_1 &= 4530.11944 \\ I_2 &= 4600.531811 \\ I_3 &= 1542.448749 \end{aligned}$$

Modal Matrix:

$$\boldsymbol{\Phi} = \begin{bmatrix} 0.74693 & 0.66477 & 0.01339 \\ -0.66305 & 0.74319 & 0.08965 \\ 0.04964 & -0.07584 & 0.99588 \end{bmatrix}$$

NOTE:

The eigenvectors are normalized to unit length.

2) Check the sign of the determinant of $\boldsymbol{\Phi}$.

$$|\boldsymbol{\Phi}| = 0.9999997403$$

If $|\boldsymbol{\Phi}|$ turns out to be negative we change the sign on one row or column.

3) Find minimum rotation angle δ:

$$tr\, \boldsymbol{\Phi} = 1 + 2\cos\delta$$

To do this we have to change the sign on any two rows.

$$
\begin{aligned}
tr\ \boldsymbol{\Phi}_1 &= +0.74693 + 0.74319 + 0.99588 &\rightarrow\quad \delta = 42.01^o\\
tr\ \boldsymbol{\Phi}_2 &= -0.74693 - 0.74319 + 0.99588 &\rightarrow\quad \delta = 138.34^o\\
tr\ \boldsymbol{\Phi}_3 &= -0.74693 + 0.74319 - 0.99588 &\rightarrow\quad \delta = 178.88^o\\
tr\ \boldsymbol{\Phi}_4 &= +0.74693 - 0.74319 - 0.99588 &\rightarrow\quad \delta = 174.19^o
\end{aligned}
$$

4) The rotation matrix A is obtained by taking the transpose of the modal matrix having the minimum rotation angle δ :

$$
A = \boldsymbol{\Phi}_1^T = \begin{bmatrix} 0.74693 & -0.66305 & 0.04964 \\ 0.66477 & 0.74319 & -0.07584 \\ 0.01339 & 0.08965 & 0.99588 \end{bmatrix}
$$

The orientation of the principal axis system relative to the control axis system can be expressed in terms of the classical Euler angles or in terms of the modern Euler angles.

Example:

Determine the direction of the minimum principal moment of inertia axis $\mathbf{e}_3$ relative to the corresponding control axis. We obtain:

$$
\theta = \cos^{-1}\ A_{33} = \cos^{-1}\ 0.99588
$$

$$
\theta = 5.2^o
$$

$$
\phi = \tan^{-1}\ \frac{A_{13}}{A_{23}} = \tan^{-1}\ \frac{0.04964}{-0.07584}
$$

This angle lies in the second quadrant.

$$
\phi = (180 - 33.206)^o \rightarrow \phi = 146.8^o
$$

E) Ellipsoid of Inertia

The inertial properties of a rigid body can be conveniently depicted by an ellipsoidal surface which is in essence a plot of the moment of inertia of the body as a function of the rotation axis direction. Going back to the matrix form for the rotational kinetic energy of a rigid body we have

$$T = \frac{1}{2}\boldsymbol{\omega}^T I \boldsymbol{\omega} = \frac{1}{2} I_A \omega^2 \qquad (4.81)$$

where the scalar I_A is the moment of inertia about the instantaneous rotation axis. The single scalar expression for the kinetic energy on the right side can be verified by letting the angular velocity $\boldsymbol{\omega}$ coincide momentarily with one of the coordinate axes. Now let us define a vector $\boldsymbol{\rho}$ having the same direction as $\boldsymbol{\omega}$ such that:

$$\boldsymbol{\rho} = \frac{1}{\omega\sqrt{I_A}}\boldsymbol{\omega}$$

Inserting this in Equation 4.81 above, we obtain

$$\boldsymbol{\rho}^T \; I \; \boldsymbol{\rho} = 1 \text{ where } \boldsymbol{\rho} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Consider now $\boldsymbol{\rho}$ to be a positive vector drawn from the origin O to a point (x, y, z) then we can write the scalar equation:

$$I_{11}x^2 + I_{22}y^2 + I_{33}z^2 + 2I_{12}x_y + 2I_{13}x_z + 2I_{23}yz = 1$$

This ellipsoidal surface centered about O is called the ellipsoid of inertia.

The coordinate transformation which brings the inertia ellipsoid in its standard form is, of course, exactly the principal axis transformation previously discussed.

The moment of inertia about any rotation axis can be found directly from the magnitude of the vector $\boldsymbol{\rho}$.

$$I_A = \frac{1}{\sqrt{\rho}}$$

where ρ is the length of the straight line drawn from the origin O to a point on the surface of the inertia ellipsoid.

NOTE:

A quantity closely related to the moment of inertia is the radius of gyration k_A defined by

$$I_A = Mk_A^2$$

In terms of the radius of gyration $\boldsymbol{\rho}$ can be written as:

$$\boldsymbol{\rho} = \frac{1}{\omega k_A \sqrt{M}} \boldsymbol{\omega}$$

The inertia ellipsoid is fixed with the body and rotates with it. Two rigid bodies having equal mass can have quite different shapes and still have identical ellipsoids of inertia. They are called dynamically equivalent or equimomental. Although the inertia ellipsoid is a figure of revolution, the corresponding rigid body need not be. Roughly speaking the shape of the inertia ellipsoid is similar to the shape of the corresponding rigid body. For instance, a prolate (oblate) rigid body has a prolate (oblate) inertia ellipsoid. The standard form of the inertia ellipsoid is

$$I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2 = 1$$

where I_1, I_2 and I_3 are the principal moments of inertia.

4.5 Free Motion of a Rigid Body

The free motion is characterized by the absence of external moments. The origin of the reference frame is either fixed in space or at the mass center. Both cases are dynamically identical. There are two ways of treating this problem - the geometric (Poinsot method 1834) and the analytic (Euler case 1758) way.

To simplify the problem, the reference frame is chosen to be a principal axes system. There is, of course, no loss of generality involved in this particular choice of the reference frame.

The equations of motion are obtained from Equation 4.12 and read in scalar form:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0 \qquad (4.82)$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = 0$$

A) Poinsot Method (1834)

This method represents a geometric solution of the torque-free motion of an unsymmetrical rigid body. With no external torque acting on the body, the kinetic energy and the angular momentum are conserved.

Therefore:

$$2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \qquad (4.83)$$

This equation represents geometrically an ellipsoid known as Poinsot Ellipsoid or kinetic energy ellipsoid. It differs from the standard inertia ellipsoid only by the scale factor $\sqrt{2T}$.

Also:

$$H^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \equiv 2TD \qquad (4.84)$$

This ellipsoid is the angular momentum ellipsoid.

The quantity D is defined as:

$$D = \frac{H^2}{2T} = \text{CONSTANT} \tag{4.85}$$

It has the dimension of a moment of inertia and is used in the subsequent discussion as a matter of convenience.

The Poinsot method does not yield the angular velocity $\boldsymbol{\omega}$ as a function of time but only the path traced by the instantaneous rotation (spin) axis on the rigid body. This trace is called the polhode and is the curve obtained by the intersection of the Poinsot ellipsoid and the angular momentum ellipsoid. Mathematically speaking, it is given by the simultaneous solution of Equation 4.83 and Equation 4.84.

For the ensuing discussion, we assume without loss of generality that

$$I_1 < I_2 < I_3 \tag{4.86}$$

It can then be shown that for this case, the constant D must lie in the range

$$I_1 \leq D \leq I_3 \tag{4.87}$$

As a consequence, a polhode angular momentum ellipsoid is always more elongated than the corresponding polhode Poinsot ellipsoid. The following two figures illustrate the geometrical aspects of the Poinsot method.

It is observed that the polhodes form closed curves about the smallest and largest moment of inertia reflecting a stable motion. On the other hand, the polhodes in the vicinity of the intermediate moment of inertia I_2 have hyperbolic character indicating an unstable motion.

In the case of axial symmetry, the Poinsot ellipsoid and the angular momentum ellipsoid become ellipsoids of revolution (spheroids). The polhodes become circles perpendicular to the spin axis. The rotation about the axis of symmetry becomes more stable, whereas the rotational motion about a transverse principal axes is neutrally stable.

The polhodes furnish an argument for the stability behavior of the rotational motion about a principal axis in the presence of internal energy dissipation.

Since

$$I_1 \leq (D = \frac{H^2}{2T}) \leq I_3 \tag{4.88}$$

from Equation 4.87 it is observed that in this case the kinetic energy T must decrease while the angular momentum H remains constant. As a consequence, the quantity

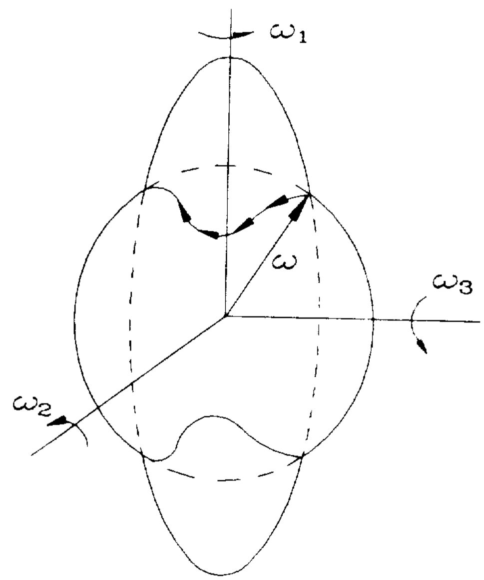


Figure 4.1: Intersection of Poinsot and Momentum Ellipsoid

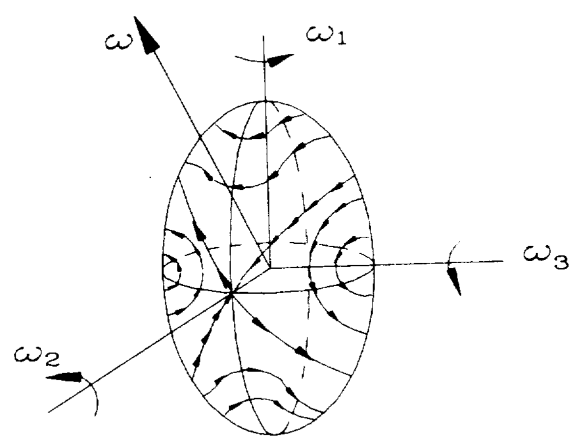


Figure 4.2: Pollodes on Inertia Ellipsoid

D is increasing which means that a motion in the vicinity of the principal axis of minimum moment of inertia (D $\approx I_1$), will gradually go over to a rotation about the principal axis of maximum moment of inertia (D $\approx I_3$).

Thus, the principal axis of minimum moment of inertia is one of unstable equilibrium in the presence of internal energy dissipation. This behavior was actually observed for some satellites notably the Explorer I satellite which was spin-stabilized about the longitudinal axis of minimum moment of inertia. The source for the energy dissipation can be provided by internal fluid motion or feasible antennas. The important question is how long a spinning satellite without large changes in orientation.

The Poinsot method can also be used to calculate the limits of the mutational or wobble motion of an unsymmetrical torque-free rigid body. To this end, we write the Poinsot ellipsoid (Equation 4.83) and the angular momentum ellipsoid (Equation 4.84) in terms of the angular momentum components. Thus,

$$\frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3} = 2T \qquad (4.89)$$

$$H_1^2 + H_2^2 + H_3^2 = H^2 \qquad \text{Momentum Sphere} \qquad (4.90)$$

Multiplying Equation 4.89 by D and observing that by definition $H^2 = 2TD$ we can combine both equations to obtain the path of the angular momentum vector on the momentum sphere:

$$H_1^2(1 - \frac{D}{I_1}) + H_2^2(1 - \frac{D}{I_2}) + H_3^2(1 - \frac{D}{I_3}) = 0 \qquad (4.91)$$

The trace of the momentum vector **H** on the momentum sphere is illustrated in the figure.

Case A: Spin about I_1-axis ($D < I_2$)

For θ_{MIN} : $H_2 = 0(\omega_2 = 0)$

$$\frac{H_3^2}{H_1^2} = -\frac{1 - D/I_1}{1 - D/I_3} = \frac{I_3}{I_1}\frac{(D - I_1)}{(I_3 - D)}$$

$$tg\ \theta_{MIN} = \sqrt{\frac{I_3}{I_1}(\frac{D - I_1}{I_3 - D})}$$

For θ_{MAX} : $H_3 = 0(\omega_3 = 0)$

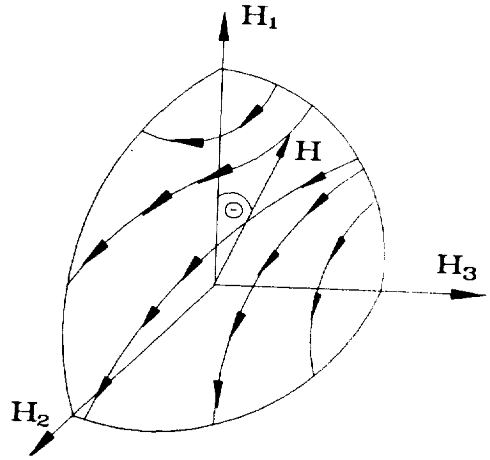


Figure 4.3: Trace of **H** on Momentum Sphere

$$\frac{H_2^2}{H_1^2} = -\frac{1 - D/I_1}{1 - D/I_2} = \frac{I_2}{I_1}\left(\frac{D - I_1}{I_2 - D}\right)$$

$$tg\ \theta_{MAX} = \sqrt{\frac{I_2}{I_1}\left(\frac{D - I_1}{I_2 - D}\right)}$$

<u>Case B:</u> Spin about I_3-axis ($D > I_2$)
For θ_{MIN} : $H_2 = 0 (\omega_2 = 0)$

$$\frac{H_1^2}{H_3^2} = -\frac{1 - D/I_3}{1 - D/I_1} = \frac{I_1}{I_3}\left(\frac{I_3 - D}{D - I_1}\right)$$

$$tg\ \theta_{MIN} = \sqrt{\frac{I_1}{I_3}\left(\frac{I_3 - D}{D - I_1}\right)}$$

For θ_{MAX} : $H_1 = 0 (\omega_1 = 0)$

$$\frac{H_2^2}{H_3^2} = -\frac{1 - D/I_3}{1 - D/I_2} = \frac{I_2}{I_3}\left(\frac{I_3 - D}{D - I_2}\right)$$

$$tg\ \theta_{MAX} = \sqrt{\frac{I_2}{I_3}\left(\frac{I_3 - D}{D - I_2}\right)}$$

B) <u>Analytical Method</u>

The analytical solution of the Equation 4.82 for the torque-free unsymmetrical body was first obtained by Euler in 1758. He showed that the angular velocity components become elliptic functions of time.

There are, of course, also three very simple particular solutions of Equation 4.82 namely:

$$\begin{aligned} \omega_1 &= \text{CONSTANT} \qquad & \omega_2 = \omega_3 = 0 \\ \omega_2 &= \text{CONSTANT} \qquad & \omega_1 = \omega_3 = 0 \\ \omega_3 &= \text{CONSTANT} \qquad & \omega_1 = \omega_2 = 0 \end{aligned}$$

These solutions represent steady rotations about the principal axes of inertia. These are the only axes about which the body will spin steadily.

The derivation of the Euler case will not be given only the solution.

The elliptic functions appearing in the solution are defined as follows:

The elliptical integral of the first kind is:

$$u = \int_0^y \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}$$

that means u is a function of y and k:

$$u = F(y, k)$$

The elliptic function is then the inverse function

$$y = F^{-1}(u, k) = \mathcal{S}n\ (u, k)$$

where k is called the modulus $(0 \leq k \leq 1)$

The solution for the angular velocity components can then be written as:

$$\omega_1 = H\sqrt{\frac{(D - I_3)}{I_1 D(I_1 - I_3)}}\,\mathcal{D}n\ (\lambda t, k)$$

$$\omega_2 = -H\sqrt{\frac{(I_1 - D)}{I_2 D(I_1 - I_2)}}\ \mathcal{S}n\ (\lambda t, k)$$

$$\omega_3 = H\sqrt{\frac{(I_1 - D)}{I_3 D(I_1 - I_3)}}\ \mathcal{C}n\ (\lambda t, k)$$

where the $\mathcal{C}n$ and $\mathcal{D}n$ functions are related to the $\mathcal{S}n$ function by:

$$\mathcal{C}n^2 x = 1 - \mathcal{S}n^2 x$$

$$\mathcal{D}n^2 x = 1 - k^2 \mathcal{S}n^2 x$$

The constants λ and k are given by:

$$\lambda = H\sqrt{\frac{(I_1 - I_2)(D - I_3)}{(I_1 I_2 I_3 D)}} \tag{4.92}$$

$$k = \sqrt{\frac{(I_2 - I_3)(I_1 - D)}{(I_1 - I_2)(D - I_2)}} \quad 0 < k < 1 \tag{4.93}$$

The above solution corresponds to a rotation about the I_1-axis with the initial condition $\omega_2(0) = 0$.

For small values of $k(I_2 \approx I_3)$ the elliptic functions approach the trigonometric functions:

$$\mathcal{S}n(\lambda t, 0) = \sin\ \lambda t$$

$$\mathcal{C}n(\lambda t, 0) = \cos\ \lambda t$$

$$\mathcal{D}n(\lambda t, 0) = 1$$

The determination of the angular velocity components as functions of time does, of course, not complete the solution of the problem, because we still have to find the orientation of the body relative to an inertial observer at any time. This can be done, in principle, by substituting the solutions into Euler's kinematical equations, which yields three differential equations for the three Euler angles. The solution of these equations in this general form presents a formidable problem and will not be pursued any further. However, it is important to notice, that although the angular velocities are periodic functions of time, the motion of the rigid body as viewed from an inertial observer is no longer periodic.

C) Perturbation Method

In many practical applications the rigid body motion must not deviate greatly from a nominal direction. In such situations valuable insight into the dynamic behavior of a system can be obtained by linearizing the equations of motion about a nominal condition.

To illustrate the method we consider an unsymmetrical rigid body which spins about the I_1-axis. If the asymmetry is rather small (i.e. $I_2 \approx I_3$) we observe from the Poinsot construction that the motion is confined to the vicinity of the spin axis. Therefore the first equation of Equation 4.82 is approximately

$$I_1\dot{\omega}_1 = 0 \quad \text{or} \quad \omega_1 = \omega_0 = \text{CONSTANT}$$

The second and third of these equations can then be written as:

$$\dot{\omega}_2 + \lambda_1\omega_3 = 0 \tag{4.94}$$

$$\dot{\omega}_3 - \lambda_2\omega_2 = 0 \tag{4.95}$$

where

$$\lambda_1 = \frac{I_1 - I_3}{I_2}\,\omega_0 \qquad\qquad \lambda_2 = \frac{I_1 - I_2}{I_3}\,\omega_0$$

Because of the triangle inequality rule of the moment of inertia both $|\lambda_1|$ and $|\lambda_2|$ are smaller than the nominal spin ω_0.

We assume in the ensuing discussing that I_2 is the intermediate moment of inertia.

In order to be able to compare the approximate solution with the exact solution obtained in the preceding section, we assume the same initial condition:

$$\omega_2(0) = 0 \text{ for } t = 0$$

The equations will be solved by the Laplace transformation method.

Denoting the Laplace transforms of the angular velocities by capital letters:

$$\mathcal{L}\{\omega(t)\} = \Omega(s)$$

we obtain:

$$s\Omega_2(s) + \lambda_1\Omega_3(s) = 0 \tag{4.96}$$

$$s\Omega_3(s) - \omega_3(0) - \lambda_2\Omega_2(s) = 0 \tag{4.97}$$

where $\omega_3(0)$ is the initial angular velocity along the I_3-axis.

Solving for the Laplace transforms of the angular velocities we obtain

$$\Omega_2(s) = -\frac{\lambda_1\omega_3(0)}{s^2+\lambda_1\lambda_2} \quad \Omega_3(s) = \frac{s\omega_3(0)}{s^2+\lambda_1\lambda_2} \tag{4.98}$$

The corresponding inverse transforms yield the angular velocities as a function of time:

$$\omega_2(t) = -\frac{\lambda_1\omega_3(0)}{\sqrt{\lambda_1\lambda_2}}\sin\sqrt{\lambda_1\lambda_2}t \tag{4.99}$$

$$\omega_3(t) = \omega_3(0)\cos\sqrt{\lambda_1\lambda_2}t \tag{4.100}$$

For a spin close to the I_1-axis the quantity $D \approx I_1$. Inserting this approximation in Equation 4.93 it is seen that $\lambda = \sqrt{\lambda_1\lambda_2}$, that means the angular frequencies of the angular velocities are in agreement with the exact solution. It can also be easily verified that the same agreement holds true for the amplitude ratio of ω_3 and ω_2.

It is also of interest to plot the path of the angular velocity in the equatorial $x_2 - x_3$ plane of the body. This is actually the projection of the polhode onto the equatorial plane. We obtain an elliptical path which is traversing the x_1-axis in a clockwise sense if $\lambda_1 > 0 (I_1 > I_3)$ and in a counterclockwise sense if $\lambda_1 < 0 (I_1 < I_3)$. This is also in agreement with the Poinsot construction.

The real interest lies, of course, in the motion of the body, especially the axis of symmetry or spin axis, as seen by an inertial observer. To this end we introduce the modern Euler angle system which is particularly suited for perturbation studies.

Repeating the corresponding kinematical equations we have:

$$\begin{aligned}\omega_1 &= \dot\phi - \dot\psi \sin\theta \\ \omega_2 &= \dot\theta\cos\phi + \dot\psi\cos\theta\sin\phi \\ \omega_3 &= \dot\psi\cos\theta\cos\phi - \dot\theta\sin\phi\end{aligned}$$

We now use the approximation $\omega_1 \approx \omega_0 = \dot\phi$ and restrict the motion to small angles θ and ψ. Thus:

$$\omega_2 = \dot\theta\cos\omega_0 t + \dot\psi\sin\omega_0 t$$
$$\omega_3 = -\dot\theta\sin\omega_0 t + \dot\psi\cos\omega_0 t$$

<u>CASE A:</u> Spin about Minimum Moment of Inertia (Rod: $\lambda_1 < 0; I_1 < I_2 < I_3$)

Set:

$$\lambda = \sqrt{\lambda_1\lambda_2} \qquad c_2 = \sqrt{\frac{\lambda_1}{\lambda_2}}\omega_3(0) \qquad c_3 = \omega_3(0)$$

The equations of motion are then:

$$\omega_2 = c_2\sin\lambda t = \dot\theta\cos\omega_0 t + \dot\psi\sin\omega_0 t$$

$$\omega_3 = c_3\cos\lambda t = -\dot\theta\sin\omega_0 t + \dot\psi\cos\omega_0 t \tag{4.101}$$

We now introduce a <u>complex cone angle:</u>

$$\alpha = \psi + i\,\theta \tag{4.102}$$

Adding the two equations of motion in quadrature we obtain:

$$\omega_3 + i\omega_2 = c_3\cos\lambda t + ic_2\sin\lambda t = (\dot\psi + i\dot\theta)e^{i\omega_0 t} = \dot\alpha e^{i\omega_0 t} \tag{4.103}$$

Solving for the complex cone angle yields:

$$\dot\alpha = (c_3\cos\lambda t + ic_2\sin\lambda t)e^{-i\omega_0 t} \tag{4.104}$$

To bring this equation into a more convenient form for integration we introduce:

$$c_3 = B_1 + B_2 \text{ and } c_2 = B_1 - B_2 \tag{4.105}$$

Thus:

$$\begin{aligned}\dot{\alpha} &= [(B_1 + B_2)\cos\lambda t + i(B_1 - B_2)\sin\lambda t]\, e^{-i\omega_0 t} \\ &= [(B_1(\cos\lambda t + i\sin\lambda t) + B_2(\cos\lambda t - i\sin\lambda t)]\, e^{-i\omega_0 t} \\ &= B_1 e^{-i(\omega_0-\lambda)t} + B_2 e^{-i(\omega_0+\lambda)t}\end{aligned} \tag{4.106}$$

Integrating with respect to time furnishes the complex cone angle:

$$\alpha = i\left[\frac{B_1}{(\omega_0 - \lambda)} e^{-i(\omega_0-\lambda)t} + \frac{B_2}{(\omega_0+\lambda)} e^{-i(\omega_0+\lambda)t}\right] \tag{4.107}$$

Converting back to the original constants by observing from 4.105 that:

$$B_1 = \frac{c_3 + c_2}{2} \quad \text{and} \quad B_2 = \frac{c_3 - c_2}{2}$$

we finally arrive at:

$$\alpha = i\left[A_1 e^{-i(\omega_0-\lambda)t} + A_2 e^{-i(\omega_0+\lambda)t}\right] \tag{4.108}$$

where

$$A_1 = \frac{c_3 + c_2}{2(\omega_0 - \lambda)} \quad \text{and} \quad A_2 = \frac{c_3 - c_2}{2(\omega_0 + \lambda)}$$

It is seen that the complex cone angle is represented by two rotating vectors having different amplitudes and frequencies.

In order to make a more definite statement about the motion we have first to prove the following two inequalities:

$$\lambda = \sqrt{\lambda_1\lambda_2} < \omega_0 \ \text{ and } \ \sqrt{\frac{\lambda_1}{\lambda_2}} > 1 \tag{4.109}$$

Proof # 1:

$$\lambda_1\lambda_2 = \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\,\omega_0 < \omega_0$$

$$(I_1 - I_3)(I_1 - I_2) < I_2 I_3$$

$$I_1^2 - I_1I_2 - I_1I_3 + I_2I_3 < I_2I_3$$

$$I_1(I_1 - I_2 - I_3) < 0 \quad \text{Q. E. D.}$$

The final inequality holds because of the triangle inequality for the moments of inertia. As a consequence both vectors in Equation 4.108 represent clockwise rotations.

Proof #2:

This inequality ensures us that $\mathcal{I}_2$ is indeed the intermediate moment of inertia.

$$\frac{\lambda_1}{\lambda_2} = \frac{(I_1 - I_3)I_3}{(I_1 - I_2)I_2} > 1$$

Because $I_1 < I_3$ and $I_1 < I_2$ the inequality can be rewritten as:

$$(I_3 - I_1)I_3 > (I_2 - I_1)I_2$$

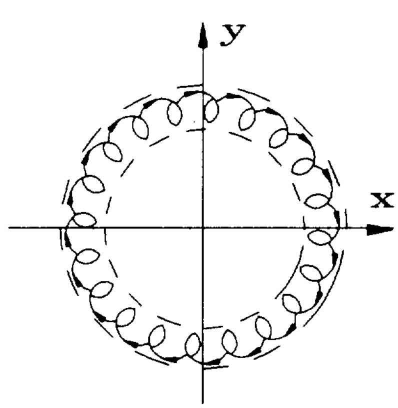
Adding the term I_2I_3 to both sides of the inequality gives:

$$(I_2 + I_3 - I_1)I_3 > (I_2 + I_3 - I_1)I_2$$

According to the triangle inequality rule $I_2 + I_3 - I_1 > 0$ and therefore,

$$I_3 > I_2 \qquad \text{Q. E. D.}$$

The motion can be made visible by projecting the tip of the unit vector along the body-fixed x_1-axis (spin axis) onto the inertial Y - Z plane which is normal to the nominal spin direction.



The coning motion is a superposition of a large amplitude small frequency rotation and a small amplitude high frequency rotation both having a clockwise rotation about the spin axis. The spin itself is, of course, also clockwise. The motion is confined within an annular ring. The minimum cone angle is the initial one:

$$A_1 + A_2 = \alpha_{MIN} = \frac{I_3\omega_3(0)}{I_1\omega_0}$$

The maximum value is obtained for ($A_2 < 0$!)

$$A_1 - A_2 = \alpha_{MAX} = \sqrt{\frac{I_2(I_3 - I_1)}{I_3(I_2 - I_1)}}\,\alpha_{MIN}$$

This is in agreement with the exact solution on page 115 and 116. The motion is, in general, not periodic.

Case B:

Spin about the Maximum Moment of Inertia (Disk: $\lambda_1 > 0; I_1 > I_2 > I_3$)

This case is obtained by changing the sign in the ω_2 equation of Equation 4.101 because $C_2 < 0$ for $\lambda_1 > 0$. All other steps remain the same.

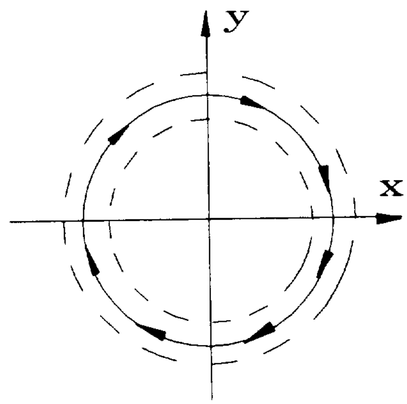
The final complex cone angle is:

$$\alpha = i\left[A_1\, e^{-i(\omega_0+\lambda)t} + A_2 e^{-(\omega_0-\lambda)t}\right] \tag{4.110}$$

where now

$$A_1 = \frac{c_3 + c_2}{2(\omega_0 + \lambda)} \text{ and } A_2 = \frac{c_3 - c_2}{2(\omega_0 - \lambda)}$$

The coning motion consists now of a superposition of large amplitude high frequency rotation and a small amplitude small frequency rotation both having again the same clockwise sense of rotation. This is graphically illustrated in the following figure



The motion is again confined to the annular region. Maximum and minimum cone angles are identical to the previous ones.

NOTE:

For a symmetric satellite $I_2 = I_3$ the coning motion becomes steady with a frequency of

$$\omega_p = \omega_0 + \lambda = \frac{I_1}{I_2}\omega_0 \qquad \text{Precession}$$

This precessional motion is always clockwise ("forward") about the spin axis regardless of whether the spin is about the maximum or minimum moment of inertia. One can obtain this result from the preceding steps by observing that for a symmetric body $I_2 = I_3$ which leads to $c_2 = c_3$. It is important to pay careful attention to the sign of λ which changes when going from a long slender rod to the case of a flat disk. It is negative for the former case and positive for the latter.

Sometimes the geometric axes or control axis system deviate slightly from the principal axis system. Let us assume that the control axis has a small angle β with the spin axis. This misalignment is strictly a geometric effect and does not enter the dynamics equations. It can be simply taken care of, by adding to the complex cone angle of Equation 4.108 or Equation 4.110 a vector of magnitude β which rotates clockwise with the spin frequency ω_0, is an example . Equation 4.108 would read for this case

$$\alpha = i\left[A_1 e^{-i(\omega_0-\lambda)t} + A_2 e^{-i(\omega_0+\lambda)t} + \beta e^{-i\omega_0 t}\right] \qquad (4.111)$$

4.6 Forced Motion of a Rigid Body

Unlike the case of the torque-free rigid body the equations of motion of a rigid body which is acted upon by external torques can only be solved analytically for very special cases. It was first shown by Lagrange (1788) that the rotational equations of motion can be integrated for a symmetric body in a uniform gravity field with a fixed point on its axis of symmetry ("heavy top"). We will not discuss this rather lengthy treatment but instead again will present perturbation methods which can be used to describe small deviations of a rigid body from a nominal condition. The mathematical models will be applicable for a variety of spinning ("gyroscopic") systems. The methods will differ depending on whether or not the external torques depend on the orientation of the body.

Method I (Body-fixed torques)

In this case the Euler equations of motion can be directly integrated to find the angular velocities as functions of time. The motion relative to inertial space is then obtained by subsequently integrating the kinematical equations. The method is essentially the same as the one used in the preceding paragraph for the torque-free case.

As an example we consider an axially symmetric spinning missile with a thrust misalignment. The principal axes system is aligned such that the thrust misalignment produces a constant torque about the x_2-axis.

The equations of motion are then

$$I_1 \, \dot{\omega}_1 = 0$$

$$I_2 \, \dot{\omega}_2 + (I_1 - I_3) \, \omega_1 \, \omega_3 = L_2 \qquad I_2 = I_3$$

$$I_3 \, \dot{\omega}_3 + (I_2 - I_1) \omega_1 \, \omega_2 = 0$$

From the first of these equations we see that the angular velocity component ω_1 about the spin axis is constant. We will denote it by ω_0. Introducing the quantity:

$$\lambda = \frac{I_1 - I_2}{I_2} \, \omega_0$$

we can write the other two equations as:

$$\dot{\omega}_2 + \lambda\,\omega_3 = \frac{L_2}{I_2}$$
$$\dot{\omega}_3 - \lambda\,\omega_2 = 0$$

This set of equations can be solved again by the method of Laplace transformation and yields the solution for the initial condition $\omega_2(0) = 0$:

$$\omega_2(t) = -\frac{\lambda\omega_3(0)}{\mid\lambda\mid}\sin\mid\lambda\mid t + \frac{L_2}{\mid\lambda\mid I_2}\sin\mid\lambda\mid t$$
$$\omega_3(t) = \omega_3(0)\cos\mid\lambda\mid t - \frac{L_2\lambda}{I_2\lambda^2}\cos\mid\lambda\mid t + \frac{\lambda L_2}{\lambda^2 I_2}$$

where $\omega_3(0)$ is the initial angular velocity about the I_3-axis.

For the symmetric case, there is no need to distinguish mathematically between a spin about the minimum and a spin about the maximum moment of inertia because the change in sign of λ can be automatically performed by the sine function. Therefore, we obtain for both cases:

$$\omega_2(t) = -\left[\omega_3(0) - \frac{L_2}{\lambda I_2}\right]\sin\lambda t$$
$$\omega_3(t) = \left[\omega_3(0) - \frac{L_2}{\lambda I_2}\right]\cos\lambda t + \frac{L_2}{\lambda I_2}$$

Using the approximate kinematical equations on page 114 and the complex cone angle introduced in Equation 4.102 we get:

$$\dot{\alpha} = \left[\omega_3(0) - \frac{L_2}{\lambda I_2}\right]e^{-i(\omega_0+\lambda)t} + \frac{L_2}{\lambda I_2}e^{-i\omega_0 t} \tag{4.112}$$

Upon integration the cone angle is obtained as

$$\alpha = i\frac{[\omega_3(0) - L_2/(\lambda I_2)]}{(\omega_0+\lambda)}e^{-i(\omega_0+\lambda)t} + \frac{iL_2}{\lambda I_2\,\omega_0}e^{-i\omega_0 t} \tag{4.113}$$

For a spin about the minimum moment of inertia ($\lambda < 0$) the resultant coning motion consists of the regular low frequency precession with frequency $\omega_p = (\omega_0 + \lambda)$ which is superimposed by a small amplitude high frequency oscillation. Both rotational vectors revolve about the spin axis in a clockwise sense.

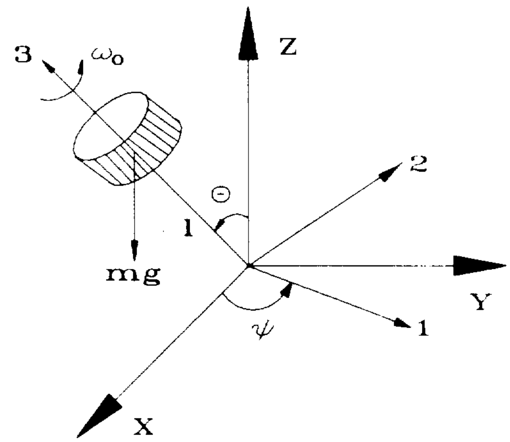


Figure 4.4: Floating Coordinate System

NOTE:

This method can also be used for an unsymmetric rigid body as long as the asymmetry is relatively small ($I_2 \approx I_3$). Otherwise, the deviation from the nominal spin condition do not remain small and the linearization process used in this method is no longer valid.

METHOD II (Modified Euler Equations)

For axially symmetric bodies a modification of Euler's equation is possible by introducing a reference frame which is aligned with the symmetry axis but does not rotate with the body ("floating" reference frame). Because of the symmetry this does not cause a time changing moment of inertia matrix. The floating coordinate system is defined by the first two rotation sequences of the classical Euler angle system. The corresponding rotation matrix and kinematical equations are therefore obtained by setting $\phi = 0$.

The floating coordinate system is also known as node-axis system. The nominal spin is about the I_3-axis.

We can perform the desired modification by going back to the vector-dyadic form of the Euler equations:

$$\mathcal{I} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathcal{I} \cdot \boldsymbol{\omega} = \mathbf{L}_0 \tag{4.114}$$

It is important to realize that the angular velocity appearing in this equation is that of the rigid body relative to inertial space. This is now expressed as the sum:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_F \tag{4.115}$$

where $\boldsymbol{\omega}_0$ is the spin rate of the rigid body and $\boldsymbol{\omega}_F$ is the angular velocity of the floating frame. In vector form, we have:

$$\boldsymbol{\omega}_0 = \dot{\phi}\ \mathbf{e}_3 \qquad \dot{\boldsymbol{\omega}}_0 = \ddot{\phi}\ \mathbf{e}_3 \tag{4.116}$$

$$\boldsymbol{\omega}_F = \dot{\theta}\ \mathbf{e}_1 + \dot{\psi}\ \sin\ \theta\ \mathbf{e}_2 + \dot{\psi}\ \cos\ \theta\ \mathbf{e}_3$$

Also for the moment of inertia dyadic we have:

$$\mathcal{I} = I_1\ \mathbf{e}_1\ \mathbf{e}_1 + I_2\ \mathbf{e}_2\ e_2 + I_3\ \mathbf{e}_3\ \mathbf{e}_3 \text{ with } I_1 = I_2 \tag{4.117}$$

Introducing these terms in the Euler equation yields:

$$\mathcal{I} \cdot (\dot{\boldsymbol{\omega}}_0 + \dot{\boldsymbol{\omega}}_{oF}) + (\boldsymbol{\omega}_0 + \boldsymbol{\omega}_F) \times \mathcal{I} \cdot (\boldsymbol{\omega}_0 + \boldsymbol{\omega}_F) = \mathbf{L}_0 \tag{4.118}$$

It is easily observed that the term:

$$\boldsymbol{\omega}_0 \times \mathcal{I} \cdot \boldsymbol{\omega}_0 = 0$$

Therefole the Equation 4.118 can also be written as:

$$\mathcal{I} \cdot (\dot{\boldsymbol{\omega}}_0 + \dot{\boldsymbol{\omega}}_F) + \boldsymbol{\omega}_F \times \mathcal{I} \cdot (\boldsymbol{\omega}_0 + \boldsymbol{\omega}_F) = \mathbf{L}_0 \tag{4.119}$$

This is the desired modification of the Euler equations. They can be expressed in terms of the Euler angles by substituting Equation 4.116 and Equation 4.117 into Equation 4.119 and carrying out the vector-dyadic operations.

Many cases of practical interest can be investigated using the modified Euler equations. Linearization is often possible about a steady state condition. When analyzing gyroscopic systems, it is usually assumed that the spin rate $\dot{\phi}$ is large in comparison to $\dot{\psi}$ and $\dot{\theta}$. It is then approximately equal to ω_0 which is the constant nominal spin rate.

<u>Example:</u>

Consider the case of the heavy top where the external moment is given by $L_1 = mgl \sin\theta$ (see figure).

Assuming a steady state condition of $\theta = \frac{\pi}{2}$ we obtain the set of differential equations:

$$I_1\,\ddot{\psi} - I_3\,\omega_0\,\dot{\theta} = 0$$
$$I_1\,\ddot{\theta} + I_3\,\omega_0\,\dot{\psi} = mgl = L_1$$

The solution of this set is most conveniently obtained by the Laplace transform method.

Introducing the notation

$$p = \frac{I_3\omega_0}{I_1}$$

the angles ψ and θ are then for $\psi(0) = 0$ and $\dot{\theta}(0) = 0$:

$$\psi = \frac{1}{p}\,(\dot{\psi}_0 - \frac{L_1}{I_3\omega_0})\ \sin\ p\,t + \frac{L_1}{I_3\ \omega_0}t$$

$$\theta = \frac{1}{p}\,(\dot{\psi}_0 - \frac{L_1}{I_3\omega_0})\ \cos\ p\,t + \frac{\pi}{2} - \frac{1}{p}\,(\dot{\psi}_0 - \frac{L_1}{I_3\omega_0})$$

The gyroscope rotates about the z-axis with average angular precession

$$\omega_p = \frac{L_1}{I_3\ \omega_0}$$

METHOD III: (Direction-dependent torque)

In many practical cases the external torques acting on a spinning body depend only on its direction relative to a nominal spin direction. The equations of motion are derived in terms of the deviation of the rigid body from the nominal steady state spin condition. The body-fixed reference frame is aligned with the principal axes and the nominal spin rate ω_0 is aligned with the inertially fixed x-axis having unit vector $\mathbf{i}$. The total angular velocity $\boldsymbol{\omega}$ of the rigid body is again considered to be the sum of two components:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_B$$

where $\boldsymbol{\omega}_0$ is the nominal spin rate and $\boldsymbol{\omega}_B$ the angular velocity of the rigid body associated with the deviation from the nominal reference frame which itself rotates with constant angular velocity $\boldsymbol{\omega}_0$ relative to inertial space.

The nominal spin vector is expressed in terms of the deviation angles of the rigid body from the nominal reference frame. Using modern Euler angles we obtain, therefore:

$$\boldsymbol{\omega}_0 = \omega_0 \mathbf{i} = \omega_0 [(\cos\psi \cos\theta)\mathbf{e}_1 + (-\sin\psi \cos\phi + \cos\psi \sin\theta \sin\phi)\,\mathbf{e}_2 \\ +(\sin\psi \sin\phi + \cos\psi \sin\theta \cos\phi)\mathbf{e}_3]$$

The angular velocity of the body frame relative to the nominal frame is given by the kinematical equations:

$$\boldsymbol{\omega}_B = (\dot{\phi} - \dot{\psi}\sin\theta)\mathbf{e}_1 + (\dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi)\mathbf{e}_2 \\ +(\dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi)\mathbf{e}_3$$

As mentioned before, the reference frame is aligned with the principal axis system which means the moment of inertia dyadic is given by

$$\mathcal{I} = I_1\,\mathbf{e}_1\,\mathbf{e}_1 + I_2\,\mathbf{e}_2\,\mathbf{e}_2 + I_3\,\mathbf{e}_3\,\mathbf{e}_3 \tag{4.120}$$

Euler's equations are then:

$$\mathcal{I}\cdot(\dot{\boldsymbol{\omega}}_0 + \dot{\boldsymbol{\omega}}_B) + (\boldsymbol{\omega}_0 + \boldsymbol{\omega}_B)\times\mathcal{I}\cdot(\boldsymbol{\omega}_0 + \boldsymbol{\omega}_B) = \mathbf{L} \tag{4.121}$$

Carrying out the various vector-dyadic operations and retaining only linear terms results in the following set of perturbation equations:

$$I_1\,\ddot{\phi} = L_1$$

$$I_2\,\ddot{\theta} + (I_1 - I_3)\,\omega_0^2\,\theta - (I_2 + I_3 - I_1)\,\omega_0\,\dot{\psi} = L_2$$

$$I_3\,\ddot{\psi} + (I_1 - I_2)\,\omega_0^2\,\psi + (I_2 + I_3 - I_1)\,\omega_0\,\dot{\theta} = L_3$$

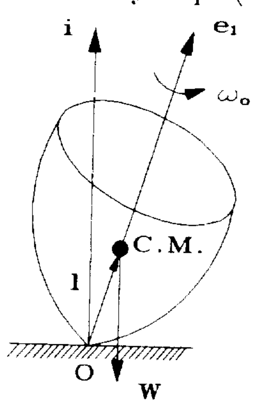
It was assumed that the deviations of the rigid body from the nominal spin state remain small. We can see from the first equation, that the roll angle ϕ will increase with time if there is a torque about the I_1-axis. For the equations to be valid we have to assume, therefore, that $L_1 = 0$. Furthermore, for an unsymmetrical body, the deviations of the spin axis from the nominal direction are only small if the asymmetry is small i.e., $I_2 \approx I_3$.

The following examples will be restricted to the symmetrical condition $I_2 = I_3$.

NOTE:

This method can also be used in the absence of external torque (torque-free rigid body). In this case, Method I is superior because it works with two pairs of first order differential equations rather than with a pair of second-order differential equations. The results are, of course, identical.

Example 1: Heavy Top. ($I_2 = I_3$)



The external torque is:
$\mathbf{L}_0 = \mathbf{l} \times \mathbf{W}$
$= l\, \mathbf{e}_1 \times (-mg\mathbf{i})$
$\mathbf{i} = \mathbf{e}_1 - \psi\, \mathbf{e}_2 + \theta\, \mathbf{e}_3$
$\mathbf{L}_0 = mgl\, \theta\, \mathbf{e}_2 + mgl\, \psi\, \mathbf{e}_3$

Introducing the notation:

$$K = (I_1 - I_2)\, \omega_0^2 - mgl$$
$$G = 2\, I_2 - I_1 > 0$$

we can write the perturbation equations as:

$$I_2\, \ddot{\theta} + K\, \theta - G\, \omega_0\, \dot{\psi} = 0$$
$$I_2 \ddot{\psi} + K\, \psi + G\, \omega_0\, \dot{\theta} = 0 \qquad (4.122)$$

Introducing also the complex deviation angle

$$\delta = \psi + i\theta \qquad (4.123)$$

we can add the Equation 4.122 in quadrature to obtain:

$$I_2\, \ddot{\delta} - i\, G\, \omega_0\, \dot{\delta} + K\, \delta = 0 \qquad (4.124)$$

Solving by Laplace transformation we have the characteristic equation

$$I_2\, s^2 - i\, G\, \omega_0\, s + k = 0 \tag{4.125}$$

The characteristic roots (eigenvalues) are:

$$s_{1,2} = i\left[\frac{G\omega_0}{2I_2} \pm \sqrt{(\frac{G\omega_0}{2I})^2 + \frac{K}{I_2}}\right] \tag{4.126}$$

or in terms of the original system parameters:

$$s_{1,2} = i\left[\frac{(2I_2 - I_1)\omega_0}{2I_2} \pm \sqrt{(\frac{I_1\omega_0}{2I_2})^2 - \frac{mgl}{I_2}}\right] = i\lambda_{1,2} \tag{4.127}$$

The complex deviation angle can perform a uniform rotation with frequency λ_1 or λ_2 (or both) as seen by an observer moving with the rigid body. Rotational motion exists, however, only as long as the radicand is positive. This means that for a stable motion, the spin rate has to be higher than

$$\omega_0 > \frac{2}{I_1}\sqrt{mgl\, I_2} \tag{4.128}$$

In this case, we have the so called "sleeping" top. Of greater interest is the rotational speed of the deviation vector as seen by an inertial observer. To do this we just have to add the clockwise rotation of the nominal reference frame. The complex deviation angle in the inertial frame is therefore:

$$\delta_I = \delta_0\, e^{i(\lambda - \omega_0)t} \tag{4.129}$$

where δ_0 is the initial condition. We obtain then two "precessional" frequencies, a low one and a high one.

$$\omega_1 = -\left[\frac{I_1\omega_0}{2I_2} - \sqrt{(\frac{I_1\omega_0}{2I_2})^2 - \frac{mgl}{I_2}}\right] \quad \text{LOW} \tag{4.130}$$

$$\omega_2 = -\left[\frac{I_1\omega_0}{2I_2} + \sqrt{(\frac{I_1\omega_0}{2I_2})^2 - \frac{mgl}{I_2}}\right] \quad \text{HIGH} \tag{4.131}$$

According to the definition of the complex deviation angle given in Equation 4.122 it can be observed that both precessional frequencies indicate a clockwise ("forward") precession.

The precession of the heavy top is usually the slow precession. It is interesting to notice, that the high precession rate emanates from the precession rate of the torque-free symmetric body as given on page 119.

If the spin rate is high we can approximate the low root by expanding the square root in a Taylor series and obtain

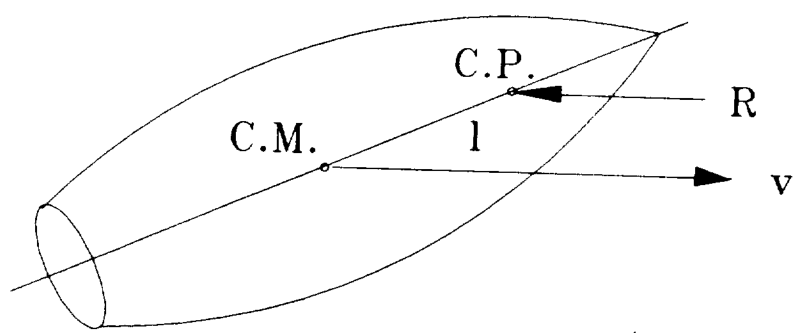
$$\omega_1 = \frac{mgl}{I_1\ \omega_0} \tag{4.132}$$

or in another form:

$$mgl = \omega_1\ I\ \omega_0 \tag{4.133}$$

Example 2: Spin-Stabilization

Artillery shells (bullets) and missiles which are exposed to destabilizing aerodynamic forces are often stabilized by giving them a high spin rate about the long axis.



In this case the destabilizing moment is due to the resultant aerodynamic force **R** acting at the center of pressure (C.P.) which is in front of the mass center (C.M.).

The torques exerted by this force about the transverse axes can be derived in a similar manner as in example 1 and yield:

$$\mathbf{L} = (q\, s\frac{\delta C_N}{\partial \alpha})\ell\, \theta\, \mathbf{e}_2 + (qs\frac{\partial C_N}{\partial \alpha})\, \ell\, \psi\, \mathbf{e}_3$$

where $q = \frac{1}{2}\, \rho\, v^2$ is the dynamic pressure and s a reference area, and $\frac{\partial C_N}{\partial \alpha}$ the normal force coefficient.

It is seen that this example is identical with the heavy top. All we have to do is to replace

$$mgl \rightarrow (qs\frac{\partial C_N}{\partial \alpha})$$

Example 3: Rotor Dynamics.
An important aspect of rotor dynamics is the gyroscopic motion in which the rotor system is precessing in its bearings. Assuming a symmetric rotor and bearing support, the elastic and dynamic characteristics depend only on the direction of the spin axis relative to its nominal state. The effect of the bearing stiffness is considered to be equivalent to a linear spring, with spring constant k. Unlike the previous two examples the spring forces are restoring forces.

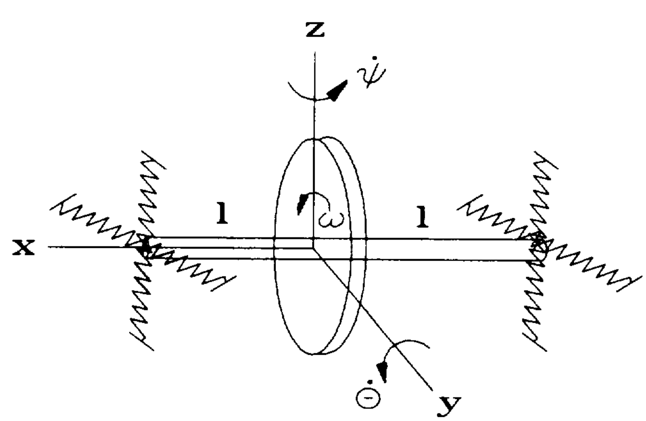


Figure 4.5: Rotor Dynamics

The equations of motion are again identical with Equation 4.122 if we set

$$k = (I_1 - I_2)\,\omega_0^2 + 2\,k\,\ell^2$$

In rotor dynamics jargon the precessional motion of the rotor is referred to as "subsynchronous whirl." The two whirl frequencies are then:

$$\omega_1 = -\left[\frac{I_1\omega_0}{2I_2} - \sqrt{(\frac{I_1\omega_0}{2I_2})^2 + \frac{2k\ell^2}{I_2}}\right] \quad \text{LOW} \tag{4.134}$$

$$\omega_2 = -\left[\frac{I_1\omega_0}{2I_2} + \sqrt{(\frac{I_1\omega_0}{2I_2})^2 + \frac{2k\ell^2}{I_2}}\right] \quad \text{HIGH} \tag{4.135}$$

The low frequency precession is always counterclockwise ("backward whirl") whereas the high frequency precession is clockwise or forward. Since the inevitable unbalances in a rotor have the tendency to excite a forward precession, the backward whirl is usually not observed. For the heavy top both precessions are forward and only the low frequency is usually observed because it requires less energy for it being excited.

NOTE:

The present model allows a quick and rigorous assessment of the stability behavior of a gyroscopic system in the presence of damping by invoking the Kelvin-Tait Theorem (1921). According to it internal damping destroys gyroscopic stability if the stiffness matrix of the linearized differential equations written in matrix form is negative definite. Therefore, a torque-free satellite spinning about the minimum moment of inertia is unstable when damping is present. For the rotor dynamics example inspection of the stiffness matrix shows that the system becomes unstable if the rotor speed is

$$\omega_0^2 > \frac{2\,k\,\ell^2}{I_2 - I_1}$$

This phenomenon of instability due to internal friction (damping) is well attested.

Method IV: Approximate Theory

If the spin rate ω_0 of a symmetric rigid body is very high it is possible to derive a simplified model of its gyroscopic behavior which is sufficiently accurate to explain many practical applications. Assuming that the spin ω_0 is practically constant the Euler torque $\mathcal{I} \cdot \dot{\boldsymbol{\omega}}$ in Euler's equation can be neglected. We obtain then:

$$\mathbf{L}_0 = \boldsymbol{\omega} \times \mathbf{H} \quad \text{where } \mathbf{H} = \mathcal{I} \cdot \boldsymbol{\omega} \tag{4.136}$$

We also assume that the angular momentum is essentially along the spin axis, i.e.

$$\mathbf{H} = \mathbf{I}_1 \, \boldsymbol{\omega}_0 \tag{4.137}$$

Since the magnitude of the angular momentum is constant, it can only change its direction which, of course, is now also identically with the directional change of the spin axes. This directional change is called the "precession" of the gyroscope and is governed by Equation 4.136, where the angular velocity becomes the precession rate ω_p. We also still observe that the angular momentum change is equal to the applied torque if the reference point is fixed or at the mass center of the gyroscope. Equation 4.136 is then written in the form:

$$\mathbf{L}_0 = \frac{d\mathbf{H}}{dt} = \boldsymbol{\omega}_p \times \mathbf{H} = \mathbf{I}_1 \, (\boldsymbol{\omega}_p \times \boldsymbol{\omega}_0) \tag{4.138}$$

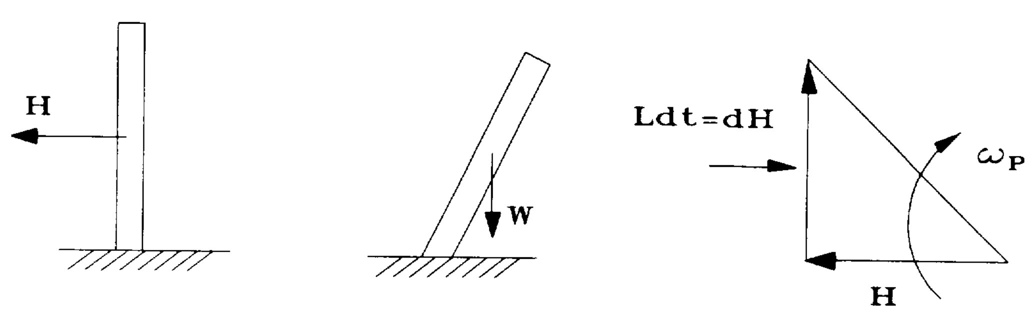
Another formulation which aids the understanding of the gyroscopic behavior is:

$$d\mathbf{H} = \mathbf{L}_0 \, dt \tag{4.139}$$

This means that the change of angular momentum vector is in the direction of the moment applied.

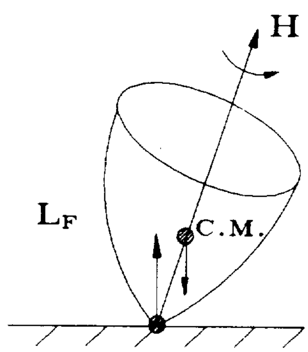
A few examples are presented to illustrate the applications of these considerations.

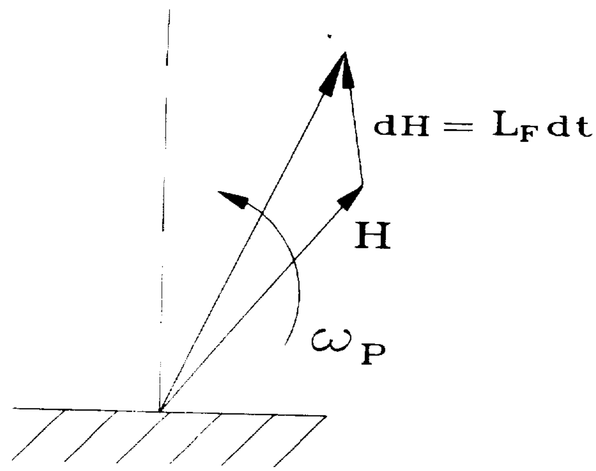
Example 1 Bicycle wheel.
Consider the front wheel of a bicycle. The angular moment of the wheel is to the left.



If the rider tilts to the right, the external moment **L** is positive along the forward speed direction and causes a precession of the wheel to the right. The ensuing centrifugal force restores the bicycle to its upright position allowing a free-hand ride.

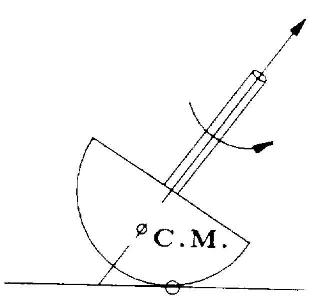
Example 2 Heavy Top.

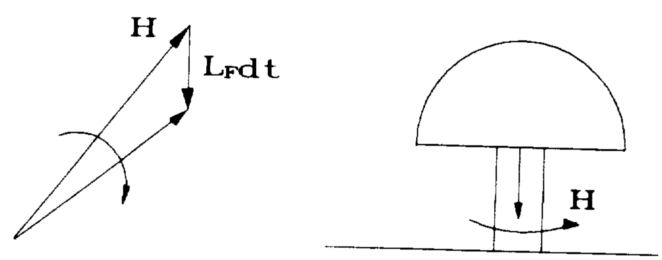




The friction on the surface produces a positive torque about the C.M. in the upward direction. It has a tendency to bring the heavy top in the upright position.

Example 3 Tippy Top (Class Ring)





The friction force being directed out of the paper plane produces a downward torque which precesses the top such as to invert its orientation.

4.7 Rheonomic Systems

Dynamic systems containing internal mass elements whose motions relative to the main body are explicit time functions are known as rheonomic systems. It is apparent when inspecting the translational Equation 3.4 and the rotational Equation 3.12 of section III that such internal moving parts affect the overall dynamics of the main body. If these effects are undesirable they induce disturbance forces and torques (e.g. crew motion, running machinery, etc.). If these motions are used to produce desirable effects they result in control forces and torques. In this case, the motion is usually restricted to rotational motion. We will therefore refer to those as gyroscopic systems.

For simplicity, the following discussion will deal only with a single internal gyroscopic system. Generalization to a situation involving multiple internal moving elements should be straightforward.

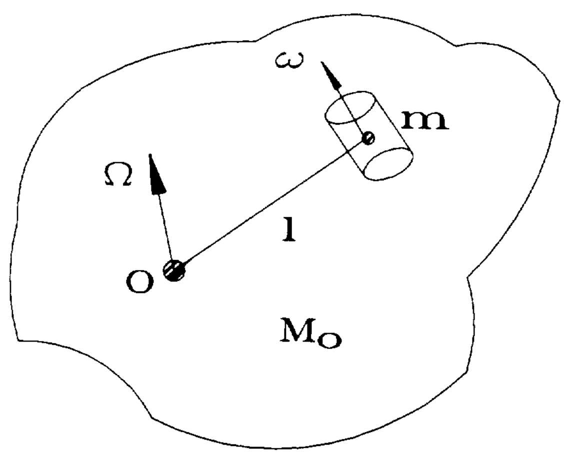


Figure 4.6: Internal Gyroscopic System

Let Ω be the angular velocity of the main body M_0 relative to inertial space and $\boldsymbol{\omega}$ be the angular velocity of the rotating mass m relative to the main body. If the origin of the main body reference frame coincides with the total mass center then the rotational equations are:

$$\mathcal{I} \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathcal{I} \cdot \boldsymbol{\Omega} + \int \mathbf{R} \times [2(\boldsymbol{\Omega} \times \mathbf{v}) + \mathbf{a}]\, dm = \mathbf{L} \tag{4.140}$$

where $\mathcal{I}$ is the moment of inertia of the total system including the moving mass. In most applications it can be assumed that the change in orientation of the rotating mass caused by its relative motion has a negligible effect on the overall moment of inertia, which is therefore, usually treated as a constant. This must, of course, not necessarily be true of all systems.

For the rotational motion of the internal mass, we have the relative velocity and acceleration:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$
$$\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{4.141}$$

where $\mathbf{r}$ is the position of the moving mass with respect to its own center of mass. Therefore:

$$\int \mathbf{r}\, dm = 0 \tag{4.142}$$

The position vector $\mathbf{R}$ of a moving mass element can be written as:

$$\mathbf{R} = \boldsymbol{\ell} + \mathbf{r} \tag{4.143}$$

where $\boldsymbol{\ell}$ is the position of the mass center of the moving mass relative to the mass center of the total system.

We can write then the Coriolis term in Equation 4.140 in the form:

$$2 \int \mathbf{R} \times (\boldsymbol{\Omega} \times \mathbf{v})\, dm = 2 \int (\boldsymbol{\ell} + \mathbf{r}) \times [\boldsymbol{\Omega} \times (\boldsymbol{\omega} \times \mathbf{r})]\; dm$$
$$= 2 \int \mathbf{r} \times [\boldsymbol{\Omega} \times (\boldsymbol{\omega} \times \mathbf{r})]\; d\,m = 2 \int (\boldsymbol{\Omega} \cdot \mathbf{r})(\mathbf{r} \times \boldsymbol{\omega})\, dm$$

Introducing the Coriolis dyadic

$$\mathcal{C}_m = \int (\mathbf{r}\;\mathbf{r})\; d\,m = (\frac{1}{2}\; tr\; \mathcal{I}_m)\mathcal{E} - \mathcal{I}_m \tag{4.144}$$

where $\mathcal{I}_m$ is the moment of inertia dyadic of the moving mass. The Coriolis term is then:

$$2 \int \mathbf{R} \times (\boldsymbol{\Omega} \times \mathbf{v})\; d\,m = 2\boldsymbol{\Omega} \cdot \mathcal{C}_m \times \boldsymbol{\omega} \tag{4.145}$$

The relative acceleration term in Equation 4.140 can also be written in a more convenient form by using Equations 4.141 and 4.142:

$$\int \mathbf{R} \times \mathbf{a}\, d\, m = \int \mathbf{r} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + (\dot{\boldsymbol{\omega}} \times \mathbf{r})]\ d\, m$$
$$= \mathcal{I}_m \cdot\ \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathcal{I}_m \cdot \boldsymbol{\omega} \tag{4.146}$$

The final form of the rotational equation for the rheonomic systems is:

$$\mathcal{I} \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathcal{I} \cdot \boldsymbol{\Omega} + 2\boldsymbol{\Omega} \cdot \mathcal{C}_m\ \times \boldsymbol{\omega}$$
$$+ \mathcal{I}_m\ \cdot\ \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathcal{I}_m \cdot \boldsymbol{\omega} = \mathbf{L} \tag{4.147}$$

a) Gyroscopic Attitude Control

There exist in practice essentially two different techniques of controlling the orientation of a spacecraft in space using internal gyroscopic devices. These devices are generically called (angular) momentum exchange controllers. They are both using the two relative acceleration terms of Equation 4.147 to affect the desired attitude control. For the purpose of attitude control analysis and simulation, these terms are brought over to the right hand side of Equation 4.147 and then looked upon as effective control torques about the mass center of the main body.

1) Momentum Wheel

This device is also known as reaction wheel, inertia wheel or flywheel. In this technique, the spin axis of the controller is kept in a fixed direction relative to the main body and only the magnitude of the rotational speed of the spinning wheel is changed. Since the wheel is spinning about a principal axis the term $\boldsymbol{\omega} \times \mathcal{I}_m \cdot \boldsymbol{\omega}$ is zero. The so called control torque of the device is then:

$$\mathbf{L}_c = -\mathcal{I}_m \cdot \dot{\boldsymbol{\omega}}_c \tag{4.148}$$

where $\dot{\boldsymbol{\omega}}_c$ is the controlled change of the spin rate which is determined by the desired control torque $\mathbf{L}_c$.

2) Control Moment Gyro (C. M. G.)

In this technique the rotational speed of the device is kept constant at a very high level. The desired control torque is generated by a proper change of the direction of the spin axis. This change can be affected by a single and two degree of freedom (single and double gimbal) C. M. G. system.

Since the spin rate of the C. M. G. is very high and practically constant, the term $I_m \cdot \dot{\omega}$ in Equation 4.147 can be safely neglected. Furthermore, the angular momentum vector can be assumed to coincide with the spin axis of the C. M. G. rotor. Therefore:

$$\mathbf{H}_m = \mathcal{I}_m \cdot \boldsymbol{\omega} = \mathcal{I}_m \cdot \boldsymbol{\omega}_0 \tag{4.149}$$

where $\boldsymbol{\omega}_0$ = spin rate vector.

The control torque of the C. M. G. can then be approximated with a very high degree of precision by:

$$\mathbf{L}_c = -\boldsymbol{\omega}_c \times \mathbf{H}_m \tag{4.150}$$

This particular form of the C. M. G. control torque allows a highly useful treatment of the C. M. G. behavior in terms of angular momentum considerations especially when, as in most practical applications, multiple C. M. G. systems are employed for attitude control.

A major task in the proper use of C. M. G.'s is to find optimum "steering" laws for the directional changes of the C. M. G. spin axes and to avoid "saturation" of the C. M. G.'s, a condition in which the controllers loose their control effectiveness along certain directions. Some of the difficulties associated with this task stem from the fact that the solution of Equation 4.150 in terms of the unknown angular velocities $\boldsymbol{\omega}_c$ is not unique, because it involves, in general, more unknowns than equations.

The literature in this area known as angular momentum management is quite extensive. Any further discussion would go beyond the scope of the present exposition.

NOTE:

No viable implementation is known which combines both techniques in one controller.

b) Gyroscopic Reaction Torques

In many practical applications it is required to determine the reaction torques which a gyroscopic system exerts on its surroundings (e. g. bearings) when its motion is known. The solution of this problem is relatively simple. Knowing the motion of the gyroscope one can calculate the left-hand side of Euler's equation to obtain the external torque which must be applied to generate the assumed motion. The reaction torque is then represented by an equal torque of opposite sign.

If there exists, however, a considerable interaction effect between the gyroscopic system and the main body it would be necessary to work with the rotational equation (4.146). Application of this equation requires the evaluation of the Coriolis term. Transferring this term to the right-hand side yields the reaction torque arising from it as:

$$\mathbf{L}_R = -2\boldsymbol{\Omega} \cdot \mathcal{C}_m \times \boldsymbol{\omega} \tag{4.151}$$

If the main body angular velocity is resolved along the principal body axis system of the gyroscope corresponding to:

$$\Omega = \Omega_1 \boldsymbol{e}_1 + \Omega_2 \boldsymbol{e}_2 + \Omega_3 \boldsymbol{e}_3$$

the Coriolis reaction torque becomes in component form:

$$\begin{aligned} L_1 &= \Omega_3\, \omega_2\, (I_1 + I_2 - I_3) - \;\Omega_2\, \omega_3\, (I_1 - I_2 + I_3) \\ L_2 &= \Omega_1\, \omega_3\, (I_2 + I_3 - I_1) - \;\Omega_3\, \omega_1\, (I_1 + I_2 - I_3) \\ L_3 &= \Omega_2\, \omega_1\, (I_1 - I_2 + I_3) - \;\Omega_1\, \omega_2\, (I_2 + I_3 - I_1) \end{aligned}$$

Reaction torques can, of course, also arise from the other relative motion terms.

Example 1: Airplane propeller.

Determine the gyroscopic reaction torque of a two-blade airplane propeller when an airplane makes a horizontal turn with constant yaw rate $\dot{\psi}$. Rotational speed of the propeller is ω_0.

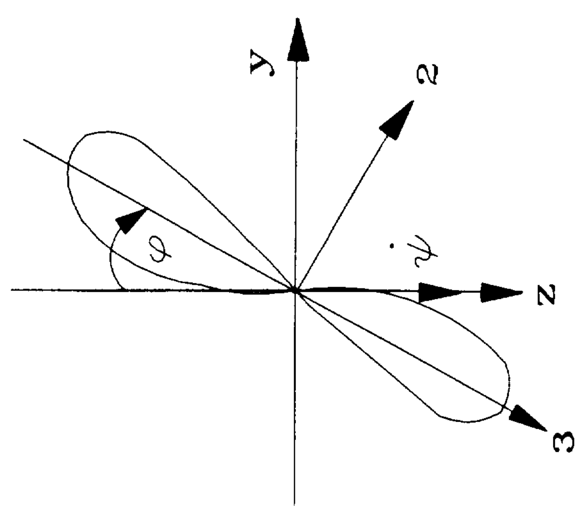


Figure 4.7: Airplane propeller

The moment of inertia of the propeller is approximated by a uniform slender rod of length ℓ:

$$I = I_2 = I_1 = \frac{m\,\ell^2}{12} \qquad I_3 = 0$$

The pertinent angular velocities are:

$$\Omega_2 = \dot{\psi}\sin\phi \qquad \Omega_3 = \dot{\psi}\cos\phi \qquad \Omega_1 = 0$$
$$\omega_1 = \omega_0 \qquad \omega_2 = \omega_3 = 0$$

The Coriolis reaction torque becomes then:

$$L_1 = 0 \quad L_2 = -2I\,\psi\,\omega_0\,\cos\,\phi \quad L_3 = 0$$

This torque can be referred to the airplane coordinate system (X, Y, Z) as:

$$L_Y = -2\,I\,\dot{\psi}\,\omega_0\,\cos^2\,\phi$$
$$L_Z = -2\,I\,\dot{\psi}\,\omega_0\,\sin\,\phi\,\cos\,\phi$$

where $\phi = \omega_0 t$
Notice that the frequency of this torque is twice the frequency of the propeller angular speed. By the way a three-blade propeller ($\mathcal{I}_2 = \mathcal{I}_3$) eliminates this cyclic propeller torque. Since the propeller is spinning about a principal moment of inertia at a constant speed no reaction torques arise from the last two relative motion terms of Equation 4.147 However, there is an interesting torque coming from the second (centrifugal) torque term. Remember that the moment of inertia in this term refers to the total moment of inertia of the main body and the moving mass. Looking only at the torque coming from the propeller we have ($\mathcal{I}_p$ = propeller moment of inertia).

$$\begin{aligned} \mathbf{L}_R &= -\boldsymbol{\Omega} \times \mathcal{I}_p \cdot \boldsymbol{\Omega} \\ &= +\Omega_2\,\Omega_3\,I_2\,\mathbf{e}_1 = I_2\dot{\psi}^2\,\sin\,\phi\,\cos\,\phi\,\mathbf{e}_1 \end{aligned}$$

This torque is along the x_1-axis which coincides with the inertial x-axis. The corresponding component is:

$$L_x = I_2\,\dot{\psi}^2\,\sin\,\phi\,\cos\,\phi$$

It is not a gyroscopic reaction torque because it really depends only on the orientation of the propeller relative to the airplane and exists also when the propeller is not spinning. In fact, it is related to the internal shear stresses which are even present in a rotating system. The torque disappears for $\phi = 0$ and $\phi = 90^o$ when the airplane rotates the propeller about a principal axis and therefore does not induce a centrifugal torque.

NOTE:

Euler's equations could be used directly to determine the reaction torques. The present method shows more clearly how an internal gyroscope affects the total systems dynamics. It is seen that the gyroscopic reaction torque of a symmetric gyroscopic system is exactly given by the Coriolis torque of Equation 4.151. Assuming a steady spin about the I_1-axis such that:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 = \boldsymbol{\omega}_0 \mathbf{e}_1 \qquad \boldsymbol{\omega}_0 = \text{CONSTANT}$$

and axial symmetry, which means $I_2 = I_3$, we obtain the gyroscopic reaction torque of a symmetric gyroscope from the components of Equation 4.151 which reads then in vector form:

$$\mathbf{L}_R = -\Omega_3\, \omega_0\, I_1 \mathbf{e}_2 + \Omega_2\, \omega_0\, I_1\, \mathbf{e}_3$$

or in another form:

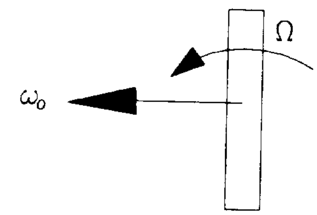
$$\mathbf{L}_R = -\Omega \times (I_1\, \boldsymbol{\omega}_0) = -\boldsymbol{\Omega} \times \mathbf{H} \qquad (4.152)$$

where $\mathbf{H} = I_1\, \boldsymbol{\omega}_0$ is the angular momentum of the gyroscope relative to the rotating total system. The negative sign indicates that the gyroscopic reaction torque always tends to bring the two vectors $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}_0$ into coincidence. This is the principle of homologous parallelism as enunciated by Leon Foucault (1819-1868).

It is very important to notice that the relationship established in Equation 4.152 is an exact one and holds true regardless of whether $\boldsymbol{\omega}_0$ is small or large relative to $\boldsymbol{\Omega}$. This is in distinct contrast to the approximate relationship given in Equation 4.138 for the precessional motion of a symmetric gyroscope under the influence of an externally applied torque. Here we have the situation of an externally applied (forced) precession.

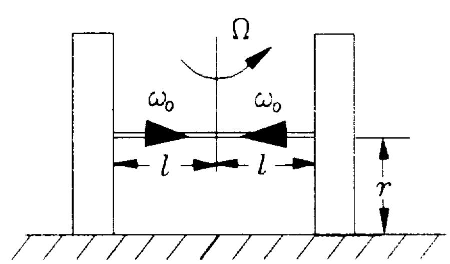
Example 2: Bicycle Wheel

Consider the top view of the front wheel of a bicycle



If the rider forces a precession on the front wheel by turning the handlebar to the left, the corresponding reaction torque will tilt the bicycle to the right according to the principle of homologous parallelism. (Foucault's Principle). This maneuver can be used to initiate a right-hand turn.

Example 3: Ore Crusher



The angular velocity Ω about the vertical axis causes each roller to have a relative angular velocity:

$$\omega_0 = \frac{\Omega\, \ell}{r}$$

The gyroscopic torque is obtained from Equation 4.152 as

$$| L_R | = I_1 \frac{\Omega^2 \ell}{r}$$

By Foucault's principle it will increase the normal force between the rollers and the surface.

Chapter 5

Lagrangian Dynamics

Newtonian or vectorial mechanics bases everything on two fundamental vectors: force and acceleration. Each component of a dynamical system is isolated and treated individually. Free body diagrams are introduced by which all forces are represented which contribute to the acceleration of the individual components. These forces include both the externally applied forces (known) and the internal reaction forces (unknown) acting on the isolated component. The latter are also called constraint forces because they act as to maintain the kinematical constraint conditions existing between the individual components of a dynamical system. To distinguish between the two forces one can call the applied forces also "forces of physical origin" and the reaction forces also "forces of geometric origin" because they emerge from the geometric configuration of the system. Applied forces reveal their origin by the fact that their mathematical expression contain quantities which can only be determined by an experiment.

Example:

Static friction is a constraint (reaction) force whereas sliding (kinetic) friction is an applied force involving the experimentally determined coefficient of friction μ.

Lagrangian or analytical mechanics bases everything on two fundamental scalars: kinetic energy and work. From a philosophical standpoint it is indeed surprising that two scalar quantities contain all the information regarding the motion of the most complicated system because motion is by its very nature a vectorial quantity.

A component is no longer considered an isolated unit but a part of an overall system of interacting components. The great superiority of the Lagrangian dynamics over the vectorial dynamics stems mainly from the fact that knowledge of the unknown constraint forces is not required, but only knowledge of the kinematical conditions. Working with scalars rather than with vectors is another contributing factor to the simplicity of the analytical treatment.

Before deriving the Lagrangian equations of motion it is necessary to introduce and define some distinctive concepts of analytical mechanics.

5.1 Constraint Equations

The kinematical constraints existing between systems components can be mathematically expressed as equations connecting conditions between their positions or between their velocities.

Example:

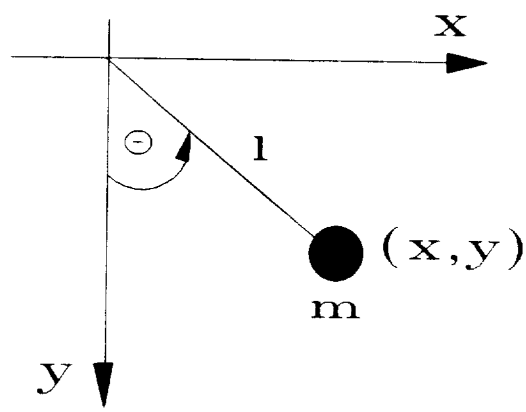


Figure 5.1: Simple Pendulum

Consider the simple pendulum constrained to move in a vertical plane.
The position of the mass m can be defined as:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \tag{5.1}$$

and its velocity as:

$$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \tag{5.2}$$

One way of expressing the kinematical constraint is to state that the velocity of the mass particle must always be perpendicular to its position vector i.e.

$$\mathbf{v} \cdot \mathbf{r} = 0 \tag{5.3}$$

In component form Equation 5.3 reads:

$$x\,\dot{x} + y\,\dot{y} = 0 \tag{5.4}$$

This is a kinematical condition on the velocity of the mass particle. It can be integrated and yields:

$$x^2 + y^2 = \ell^2 \tag{5.5}$$

This is a condition on the position of the mass particle and could have been obtained directly by observing that the pendulum mass is constrained to move on a circle.

NOTE:

It is often easier to determine the kinematical condition on the velocity of system components than on their position.

If the conditions on the velocities can be integrated to yield conditions on the positions of the system components the system (or the constraint) is called holonomic ("holonomic" is the Greek word for the Latin word "integrable"). If this integration cannot be performed the system (or the constraint) is called nonholonomic.

Example:

A characteristic and often quoted example is that of a ball which rolls without slipping on a horizontal plane. The kinematical constraint of "rolling" (and pivoting) requires that the instantaneous axis of rotation goes through the point of contact O.

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_1 \mathbf{e}_1$$
$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3$$
$$\mathbf{R}_0 = R_0 \mathbf{e}_2$$

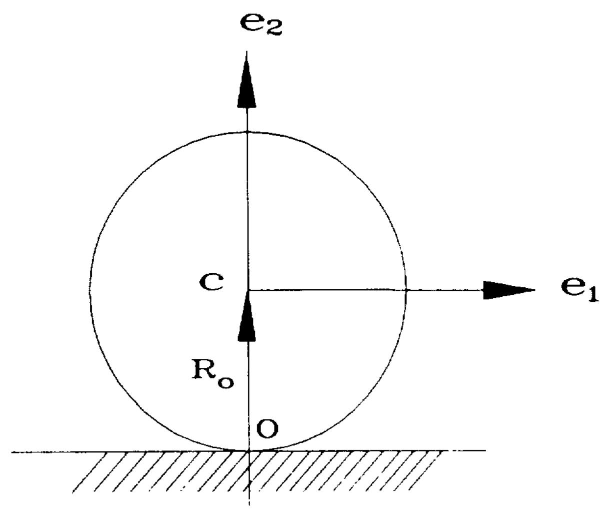


Figure 5.2: Rolling Ball

The velocity of the center C of the ball is then given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R}_0 \qquad \mathbf{R}_0 = R_0\,\mathbf{e}_2 \tag{5.6}$$

where $\mathbf{R}_0$ defines the position of C relative to the contact point 0. Using a body-fixed reference frame with unit vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$, Equation 5.6 can be written in component form ("ROLL" CONDITION):

$$v_1 = -\omega_3\, R_0 \qquad v_2 = 0 \qquad v_3 = \omega_1\, R_0 \tag{5.7}$$

These are three constraint conditions on the velocity of the center of the ball, but only one, namely $v_2 = 0$, can be expressed as a condition on the position. In fact it states simply that the ball has to move in a horizontal plane such that its center has a constant distance from the surface. The other two conditions cannot be integrated without solving the entire dynamical problem. This is due to the fact that the angular velocity $\boldsymbol{\omega}$ of the ball is expressed in a moving reference frame whose orientation is given by Euler's kinematical differential equations of the preceding section IV and this instantaneous orientation changes with time. It is, however, possible to establish differential equations of constraint from Equation 5.7 either in terms of Euler angles or by considering the angular velocity components ω_1, ω_2 and ω_3 to be time derivatives of angular displacements α_1, α_2, and α_3 (so-called quasicoordinates) about the instantaneous body axes such that:

$$\omega_1 = \frac{d\alpha_1}{dt} \quad , \quad \omega_2 = \frac{d\alpha_2}{dt} \quad , \quad \omega_3 = \frac{d\alpha_3}{dt} \tag{5.8}$$

With these defined, the roll condition 5.7 can be written as:

$$d\,x_1 + R_0 d\,\alpha_3 = 0; \tag{5.9}$$

$$d\,x_3 + R_0 d\,\alpha_1 = 0 \tag{5.10}$$

But these differential relations are not integrable.

A system for which the kinematical conditions change with time is called rheonomic otherwise it is called scleronomic.

Examples of a rheonomic system is a mass particle which moves on a surface which itself is moving according to a prescribed time function. Another example is a pendulum whose length is a given time-function.

The essential difference between rheonomic and scleronomic constraints is that rheonomic constraints do work. As a consequence rheonomic systems are not conservative. This is the reason why rheonomic systems can become unstable in a very unsuspecting way.

Illustration:

Consider a tennis racket. If it is held fixed, the ball is reflected without change in energy. If the racket yields energy is taken out of the ball, if it moves against the ball, it transfers energy to the ball.

Generalized Coordinates

Lagrangian dynamics extensively uses coordinates other than Cartesian coordinates, which are then called generalized coordinates. They are any set of parameters which can be used to define the configuration of a dynamical system. Some of the generalized coordinates may not have geometrical significance and are therefore also called hybrid coordinates. For example, the amplitudes in a Fourier expansion of the position vector **R** may be used as generalized coordinates.

Using generalized coordinates $q_1, q_2, \ldots, q_n$ the holonomic kinematical constraints can be mathematically expressed as:

$$\Phi_j\,(q_1, q_2, \ldots, q_n, t) = 0 \quad (j = 1, 2, \ldots, m) \tag{5.11}$$

Nonholonomic constraints are conventionally written in the form:

$$\sum_{k=1}^{n} c_{jk}\,(q_1, q_2, \ldots, q_n, t)\; d\, q_k + c_j(t) = 0 \quad (j = 1, 2, \ldots, m) \tag{5.12}$$

Degrees of Freedom (D. O. F.)

The number of degrees of freedom of a system is equal to the number of independent generalized coordinates necessary to define the configuration of a dynamical system. This number is characteristic of a given dynamical system. If N parameters are necessary and sufficient to define the system configuration we say it has "N degrees of freedom." Each independent kinematical constraint condition reduces the number of degrees of freedom by one.

Examples:

One D. O. F.: A mass particle moving along a given curve.

Two D. O. F.: A mass particle moving on a given surface.

Three D. O. F.: A mass particle moving freely in space.

Five D. O. F.: Two mass particles connected by a massless rod (Dumbbell).

Six D. O. F.: A rigid body moving freely in space.

NOTE 1:

A ball (coin) rolling on a horizontal plane is sometimes said to have five finite D. O. F. 's and three infinitesimal D. O. F. 's.

NOTE 2:

Linear and angular velocity components of a rotating reference frame cannot be integrated to furnish position and orientation. They are called nonholonomic velocity components.

NOTE 3:

Some authors only refer to independent coordinates as generalized coordinates. In this case, the number of generalized coordinates is equal to the number of D. O. F.'s.

Virtual Displacement/Virtual Work

A virtual displacement is an infinitesimal change of a generalized coordinate which is compatible with the kinematical constraints existing at that instant of time. Any moving constraints are temporarily stopped. This process is a kind of mathematical thought experiment. To emphasize its virtual character Lagrange introduced the special symbol δ. It has the usual properties of the ordinary differential d; for example $\delta(\sin\theta) = \cos\theta\ \delta\ \theta$. A virtual displacement can, of course, be applied to several coordinates simultaneously, but they are, in general, not independent because of the kinematical constraints.

The work done by a force $\mathbf{F}$ during a virtual displacement $\delta\mathbf{r}$ is called virtual work defined by the scalar product:

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} \tag{5.13}$$

To illustrate the difference between actual work and virtual work consider the work done by the Coriolis force acting on a mass particle.

$$dW = -2m(\boldsymbol{\omega} \times \mathbf{v}) \cdot d\mathbf{r} \tag{5.14}$$

where $d\mathbf{r}$ is an actual displacement and is therefore related to the velocity as $d\mathbf{r} = \mathbf{v}dt$. Equation 5.14 becomes then:

$$dW = -2m(\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{v}dt = 0 \tag{5.15}$$

The actual work of the Coriolis force during an actual infinitesimal displacement is zero. The virtual work, however, is:

$$\delta W = -2m(\boldsymbol{\omega} \times \mathbf{v}) \cdot \delta\mathbf{r} \neq 0 \tag{5.16}$$

5.2 Principle of Virtual Work (Bernoulli 1717)

Consider a mechanical system in motion. According to d'Alembert's Principle, the vector sum of all forces acting on each particle is zero. These forces include the external forces $\mathbf{F}_i$, the inertial forces $\mathbf{I}_i$ due to the acceleration of each particle and the constraint (reaction) forces $\mathbf{R}_i$ which maintains the given kinematical constraints. Thus, the equilibrium of the force system on each particle is:

$$\mathbf{F}_i + \mathbf{I}_i + \mathbf{R}_i = 0 \text{ where } \mathbf{I}_i = -m_i\ddot{\mathbf{R}}_i \tag{5.17}$$

The virtual work of all these forces during the virtual displacements must likewise be zero:

$$\delta W = \sum(\mathbf{F}_i + \mathbf{I}_i + \mathbf{R}_i) \cdot \delta\mathbf{r}_i = 0 \tag{5.18}$$

The principle of virtual work establishes now the following postulate:

"For any mechanical system, the virtual work of the constraint forces is zero."

Mathematically expressed it is:

$$\delta W = \sum \mathbf{R}_i \cdot \delta\mathbf{r}_i = 0 \tag{5.19}$$

This postulate is the corner stone of analytical mechanics. Many scientists Ref. (2,3) consider it to be an additional axiom of mechanics which cannot be derived from Newton's laws. Adopting this viewpoint, analytical mechanics is more than just a different mathematical formulation of Newtonian mechanics.

5.3 Generalized Forces

Another fundamental concept of analytical dynamics is that of the generalized force. The generalized forces acting on a dynamical system are determined by calculating the virtual work done by the external forces during the virtual displacements δq_i of the coordinates q_i. Each virtual displacement δq_i will produce the virtual work

$$\delta W_i = Q_i\, \delta q_i \tag{5.20}$$

where Q_i is a quantity containing the external forces acting on the system. This quantity is called the generalized force Q_i associated with the generalized coordinate q_i.

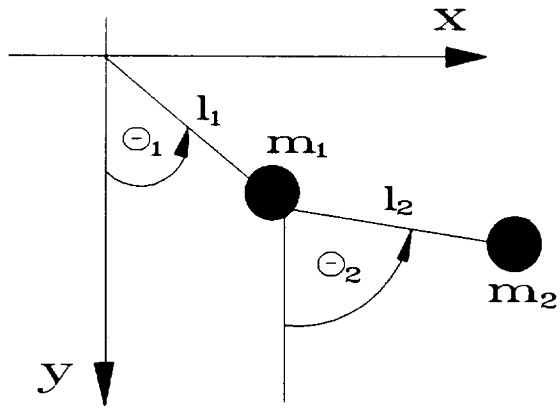


Figure 5.3: Double Pendulum

To illustrate the situation, we consider a double pendulum which can be defined by the two independent generalized coordinates θ_1 and θ_2.

To calculate the virtual work done by the external gravity force, we first define the position of the two mass particles m_1 and m_2 in Cartesian coordinates as follows:

$$\begin{aligned} \text{mass } m_1: \quad & x_1 = \ell_1 \sin \theta_1 \\ & y_1 = \ell_1 \cos \theta_1 \\ \text{mass } m_2: \quad & x_2 = \ell_1 \sin \theta_1 + \ell_2 \sin \theta_2 \\ & y_2 = \ell_1 \cos \theta_1 + \ell_2 \cos \theta_2 \end{aligned}$$

Next we perform the virtual displacements of the two masses:

$$\delta x_1 = \ell_1 \cos \theta_1 \, \delta \theta_1$$

$$\delta y_1 = -\ell_1 \sin \theta_1 \, \delta \theta_1$$

$$\delta x_2 = \ell_1 \cos \theta_1 \, \delta \theta_1 + \ell_2 \cos \theta_2 \, \delta \theta_2$$

$$\delta y_2 = -\ell_1 \sin \theta_1 \, \delta \theta_1 - \ell_2 \sin \theta_2 \, \delta \theta_2$$

The virtual work associated with these virtual displacements is:

$$\delta W = F_{x1} \, \delta x_1 + F_{x2} \delta x_2 + F_{y1} \delta y_1 + F_{y2} \delta y_2$$

Because the external forces are only due to gravity we have

$$F_{x1} = F_{x2} = 0 \text{ and } F_{y1} = m_1 g \ , \ F_{y2} = m_2 g$$

The virtual work is then:

$$\delta W = m_1 g(-\ell_1 \sin \theta_1 \, \delta \theta_1) + m_2 g(-\ell_1 \sin \theta_1 \, \delta\theta_1 - \ell_2 \sin \theta_2 \, \delta \theta_2)$$

Collecting terms with $\delta \theta_1$ and $\delta \theta_2$ yields:

$$\begin{aligned} \delta W &= -(m_1 + m_2) g \, \ell_1 \sin \theta_1 \, \delta \theta_1 \\ &\quad - m_2 \, g \ell_2 \sin \theta_2 \, \delta \theta_2 \end{aligned}$$

According to Equation 5.20 the generalized forces associated with the generalized coordinates are then:

$$\begin{aligned} \text{for } \theta_1 : &\quad Q_1 = -(m_1 + m_2) g \, \ell_1 \sin \theta_1 \\ \text{for } \theta_2 : &\quad Q_2 = -m_2 g \, \ell_2 \sin \theta_2 \end{aligned}$$

One could, of course, derive these two generalized forces more directly by finding the work done by the gravity force on the masses during an independent virtual displacement of their corresponding generalized coordinates θ_1 and θ_2.

For the general definition of the generalized forces, we consider a system of N particles whose positions are defined by the vector $\mathbf{r}_i$ and acted upon by the forces $\mathbf{F}_i$.

The virtual work is then:

$$\delta W = \sum \mathbf{F}_i \cdot \delta \mathbf{r}_i \qquad i = 1, 2 \ldots N \tag{5.21}$$

The positions of the particles are related to the generalized coordinates by:

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2 \ldots q_m, t) \tag{5.22}$$

The virtual displacements of the generalized coordinates are obtained by differentiating Equation 5.22 keeping in mind that a virtual displacement requires that $\delta t = 0$:

$$\delta \mathbf{r}_i = \sum_{1}^{N} \frac{\partial \mathbf{r}_i}{\partial q_k} \, \delta \, q_k \quad \cdot \quad k = 1, 2 \ldots, n \tag{5.23}$$

Substituting Equation 5.23 into Equation 5.21 yields:

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot (\sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k} \, \delta \, q_k) \tag{5.24}$$

Changing the order of summation yields:

$$\delta W = \sum_{k=1}^{n} (\sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}) \, \delta q_k = \sum_{1}^{n} Q_k \, \delta q_k \tag{5.25}$$

The generalized forces are therefore:

$$Q_k = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \quad k = 1, 2, \ldots, n \tag{5.26}$$

The forces $\mathbf{F}_i$ can be separated into external forces and constraint forces:

$$\mathbf{F}_i = \mathbf{F}_i^{(E)} + \mathbf{R}_i \tag{5.27}$$

Consequently, the generalized forces can be separated into generalized external forces and generalized constraint (reaction) forces:

$$Q_k^{(E)} = \sum_{i=1}^{N} \mathbf{F}_i^{(E)} \cdot \frac{\partial \, \mathbf{r}_i}{\partial q_k} \tag{5.28}$$

$$Q_k^{(R)} = \sum_{i=1}^{N} \mathbf{R}_i \cdot \frac{\partial \, \mathbf{r}_i}{\partial q_k} \tag{5.29}$$

5.4 Classical Lagrange Equations

Consider a mechanical system of N particles of constant mass m_i with position vectors $\mathbf{r}_i$ in an inertial reference frame.

The kinetic energy is:

$$T = \frac{1}{2} \sum m_i \, \dot{\mathbf{r}}_i^2 \qquad i = 1, 2, \ldots, N \tag{5.30}$$

The kinetic energy can be expressed in terms of generalized coordinates using the transformation equations:

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n, t) \tag{5.31}$$

Differentiating Equation 5.31 with respect to time gives:

$$\dot{\mathbf{r}}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\,\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \tag{5.32}$$

The kinetic energy is by substituting Equation 5.32 in Equation 5.30:

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{k=1}^{N} \frac{\partial \mathbf{r}_i}{\partial q_k}\,\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}\right)^2 \tag{5.33}$$

Define the generalized momentum p_k associated with the generalized coordinate q_k:

$$p_k = \frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \tag{5.34}$$

The time rate of change of p_k is:

$$\frac{d\,p_k}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_{1}^{N}\left(m_i\,\ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} + m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k}\right) \tag{5.35}$$

Next using Equation 5.30 we calculate

$$\frac{\partial T}{\partial q_k} = \sum m_i\,\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k} \tag{5.36}$$

This equation is seen to be identical with the last term of Equation 5.35.

Therefore we obtain:

$$\frac{d}{dx}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \sum_{i=1}^{N} m_i\,\ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \tag{5.37}$$

Applying Newton's law to the right hand side and introducing the generalized forces of 5.28 and 5.29

$$\sum m_i\,\ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum(\mathbf{F}_i^{(E)} + \mathbf{R}_i) \cdot \frac{\delta \mathbf{r}_i}{\delta q_k} = Q_k^{(E)} + Q_k^{(R)} \tag{5.38}$$

We arrive at Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k^{(E)} + Q_k^{(R)} \qquad k = 1, 2, \ldots, n \tag{5.39}$$

The generalized reaction forces $Q_k^{(R)}$ can be eliminated if the given mechanical system configuration can be defined in terms of independent generalized coordinates q_j^*. This is only possible if the kinematical constraint equations can be expressed in holonomic form as in Equation 5.11. Using the principle of virtual work of Equation 5.19 we have:

$$\delta W = \sum_{i=1}^{N} \mathbf{R}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^{n-m} Q_j^{(R)} \, \delta \, q_j^* = 0 \tag{5.40}$$

where m is the number of holonomic constraint equations. Now the δq_j^* can be chosen independently and the generalized reaction force associated with each generalized coordinate must be zero. This is so because we can now let all $\delta \, q_j^*$ be zero except for any one $\delta \, q_j^*$ which will be chosen not to be zero.

For holonomic systems, the Lagrange equations assume the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \qquad k = 1, 2, \ldots, n \tag{5.41}$$

In this case, the number of generalized coordinates is equal to the number of degrees of freedom of the system. The superscript E has been dropped for simplicity of notation.

Usually some or all external forces can be derived from a potential energy V such that

$$\mathbf{F} = -\text{grad} V \tag{5.42}$$

where $V = V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t) = V(q_1, q_2, \ldots, q_n, t)$.

We calculate the generalized forces arising from such an irrotational force field from the associated virtual work:

$$\begin{aligned}
\delta W &= \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = -\sum_{i=1}^{N} \text{grad } V_i \cdot \delta \mathbf{r}_i \\
&= -\sum_{i=1}^{N} \text{grad } V_i \cdot \left(\sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k} \, \delta \, q_k\right) \\
&= -\sum_{k=1}^{N} \left(\sum_{i=1}^{N} \frac{\text{grad } V_i \cdot \partial \mathbf{r}_i}{\partial q_k}\right) \delta q_k = -\sum_{k=1}^{n} \frac{\partial V}{\partial q_k} \, \delta q_k
\end{aligned}$$

The generalized force is then:

$$Q_k = -\frac{\partial V}{\partial q_k} \tag{5.43}$$

NOTE:

If the potential energy V is independent of time, the force field is called conservative. This does not imply that the mechanical system is conservative. In this case, the Lagrange equations are written in the form:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_k}) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_k \qquad k = 1, 2, \ldots, n \tag{5.44}$$

In theoretical mechanics, it is often convenient to introduce the Lagrangian function L as follows:

$$L = T - V \tag{5.45}$$

If furthermore all forces are derivable from a potential energy, Lagrange equations become

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k}) - \frac{\partial L}{\partial q_k} = 0 \qquad k = 1, 2, \ldots, n \tag{5.46}$$

This is the standard form of Lagrange's equations for holonomic systems.

NOTE:

For practical applications of the form of Equation 5.44 is more useful.

In some electromechanical systems, the potential function V can also be dependent on the velocities of the generalized coordinates, i.e., $V = V(q_j, \dot{q}_j, t)$. The generalized forces are then obtained by the prescription:

$$Q_k = -\frac{\partial V}{\partial q_k} + \frac{d}{dt}(\frac{\partial V}{\partial \dot{q}_k}) \tag{5.47}$$

It is apparent that in such a case, Equation 5.46 is still applicable.

Examples

a) Scleronomic System

Consider the double pendulum swinging in the vertical plane where the two masses m_1 and m_2 are connected by massless bars of length ℓ_1 and ℓ_2 as shown in the figure.

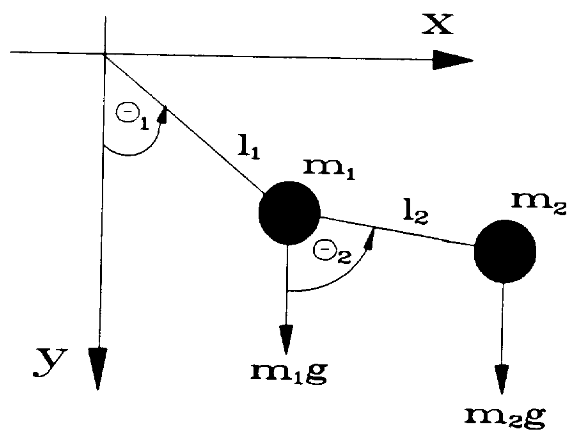


Figure 5.4: Double Pendulum

The configuration of this system can be defined by the Cartesian coordinates (x_1, y_1) of mass m_1 and (x_2, y_2) of mass m_2. These four coordinates must satisfy the kinematical constraint conditions.

(a)
$$x_1^2 + y_1^2 = \ell_1^2$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = \ell_2^2$$

The system has therefore two degrees of freedom. As a consequence, it can be defined by two independent generalized coordinates for which we choose the two angles θ_1 and θ_2 which the two bars make with the local vertical. These two angles are related to the four rectangular coordinates as follows:

(b)
$$x_1 = \ell_1 \sin\theta_1 \qquad y_1 = \ell_1 \cos\theta_1$$
$$x_2 = \ell_2 \sin\theta_1 \; + \ell_2 \sin\theta_2$$
$$y_2 = \ell_2 \cos\theta_1 \; + \ell_2 \cos\theta_2$$

The kinetic energy of the system is:

(c)
$$T = \tfrac{1}{2}\, m_1(\dot{x}_1^2 + \dot{y}_1^2) \quad + \tfrac{1}{2}\, m_2\,(\dot{x}_2 + \dot{y}_2^2)$$

This can be expressed in terms of the generalized coordinates by differentiating Eqs. (b) with respect to time which yields after some algebraic manipulation:

(d)
$$T = \tfrac{1}{2}\, m_1(\ell_1\dot{\theta}_1)^2 + \tfrac{1}{2}\, m_2 \left[(\ell_1\dot{\theta}_1)^2 + (\ell_2\,\dot{\theta}_2)^2 \right.$$
$$\left. +2\,\ell_1\,\ell_2\,\dot{\theta}_1\,\dot{\theta}_2\cos(\theta_2 - \theta_1)\right]$$

The potential energy of the system is:

(e) $$V = -m_1 g y_1 - m_2 g y_2$$

Expressed in terms of generalized coordinates it is given by:

(f) $$V = -m_1 g \ell_1 \cos\theta_1 - m_2 g(\ell_1 \cos\theta_1 + \ell_2 \cos\theta_2)$$

NOTE:

The potential energy was referenced to the level $y = 0$ at which $V = 0$.

The equations of motion are now obtained by performing the differentiations called for in Equation 5.44. The right hand side of Equation 5.44 is zero in this case because the external force was expressed by its potential energy.

θ_1-equation:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} &= (m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\, \ell_1\, \ell_2\, \ddot{\theta}_2 \cos(\theta_2 - \theta_1) \\ &\quad - m_2\, \ell_1\, \ell_2\, \dot{\theta}_2^{\,2} \sin(\theta_2 - \theta_1) + (m_1 + m_2)\, \ell_1\, g \sin\theta_1 \\ &= 0\end{aligned}$$

θ_2-equation:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} &= m_2\, \ell_2^2\ddot{\theta}_2 + m_2\ell_1\, \ell_2\ddot{\theta}_1 \cos(\theta_2 - \theta_1) \\ &\quad + m_2\, \ell_1\, \ell_2\, \dot{\theta}_1^{\,2} \sin(\theta_2 - \theta_1) + m_2\, \ell_2\, g \sin\theta_2 \\ &= 0\end{aligned}$$

b) Rheonomic System

As an example of the rheonomic constraint, i.e., a constraint which contains time explicitly, consider a simple pendulum of mass m attached to a string whose free length ℓ can be varied by pulling the end A.

The position of the mass is defined by the length ℓ and the angle θ. But since ℓ is a function of time depending on the motion of A, the system has only one degree of freedom with θ being the only generalized coordinate.

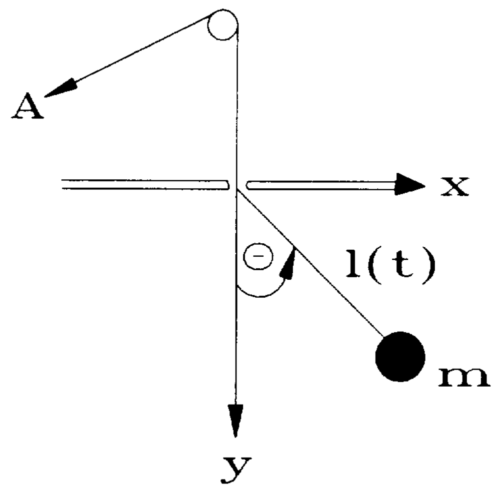


Figure 5.5: Rheonomic Pendulum

The constraint equation is

$$x^2 + y^2 = \ell^2(t) \tag{5.48}$$

The kinetic energy of the system is:

$$T = \frac{1}{2}\, m\, (\dot{x}^2 + \dot{y}^2) = \frac{1}{2}\, m\, (\dot{\ell}^2 + \ell^2\, \dot{\theta}^2) \tag{5.49}$$

The potential energy of the system is:

$$V = -mg\, y = -mg\, \ell(t)\, \cos\, \theta \tag{5.50}$$

Substituting these terms in the Lagrange Equation 5.44 we obtain

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta}}) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = m\, \ell^2 \ddot{\theta} + 2m\ell\, \dot{\ell}\dot{\theta} + mg\, \ell \sin\, \theta = 0 \tag{5.51}$$

It is interesting to introduce a small angle approximation such that $\sin\theta \approx \theta$ and write the equation of motion as

$$\ddot{\theta} + \frac{2\dot{\ell}}{\ell}\dot{\theta} + \frac{g}{\ell}\, \theta = 0 \tag{5.52}$$

Comparing this result with the well-known equation of a damped linear system, it is seen that the second term containing the rate of change of the pendulum length acts as an effective damping. In fact, the damping will be positive for $\dot{\ell} > 0$ and negative for $\dot{\ell} < 0$. It is obvious that the mechanical energy of this system is no longer conserved. The agent producing this change in energy is the centrifugal force in the string.

Another example of a rheonomic system is a launch vehicle whose attitude is controlled by proper thrust vectoring. We consider the motion of the vehicle in an inertial reference plane X-Y, called the yaw plane. The vehicle is assumed to be rigid and only subject to the thrust force of a single gimballed engine.

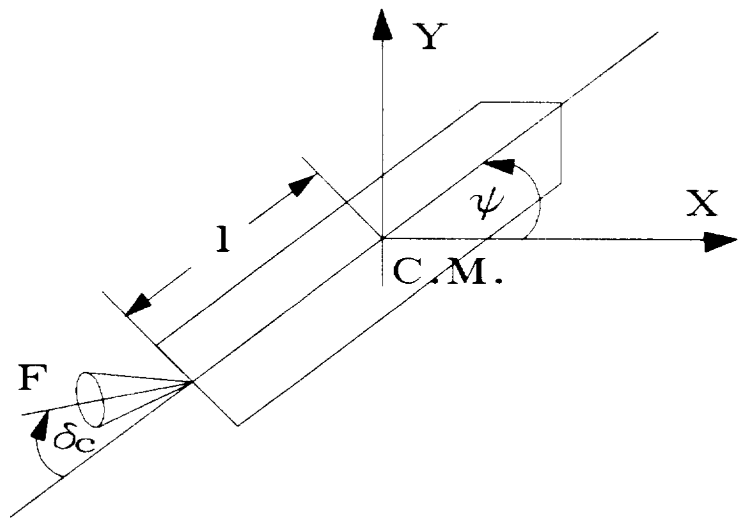


Figure 5.6: Attitude Control

The system is defined by the position of its mass center relative to the inertial X-Y plane, its yaw angle ψ and the engine deflection δ_c. However, the engine deflection is not a generalized coordinate because it is controlled by the autopilot which operates from signals generated by position and rate gyros located on the vehicle. It is therefore a rheonomic constraint and referred to as "control variable." Consequently, the system has three degrees of freedom. Its kinetic energy is given by:

$$T = \frac{1}{2}\, m\, (\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\, I\, \dot{\psi}^2 \tag{5.53}$$

where I is the moment of inertia of the vehicle about its mass center.

The generalized forces associated with the three generalized coordinates are:

$$Q_x = F\ \cos\,(\psi - \delta_c)\ ,\ Q_y = F\ \sin(\psi - \delta_c)\ ,\ Q_\psi = F\ \ell\ \sin\ \delta_c \tag{5.54}$$

where ℓ is the distance of the mass center from the engine survival point.

From the Lagrange Equations 5.44 we obtain the equations of motion as (consider m to be constant):

$$m\,\ddot{x} = F\ \cos\left(\psi - \delta_c\right)$$
$$m\,\ddot{y} = F\ \sin\left(\psi - \delta_c\right)$$
$$I\,\ddot{\psi} = F\,\ell\ \sin\delta_c$$

For preliminary control dynamics analysis, it is useful to assume small angle deflections. A very simple and effective altitude control can be obtained by a so-called attitude/attitude rate control law expressed as:

$$\delta_c = -(k_\psi\ \psi + k_{\dot{\psi}}\ \dot{\psi})$$

where

$$k_\psi = \text{position gyro gain}$$
$$k_{\dot{\psi}} = \text{rate gyro gain}$$

Assuming small engine deflection angle δ_c the rotational (attitude) equation becomes by introducing the above control laws:

$$I\,\ddot{\psi} + (F\,\ell\,k_{\dot{\psi}})\,\dot{\psi} + (F\,\ell\,k_\psi)\,\psi = 0$$

This is seen to be the differential equation for a damped oscillation. It should, of course, be realized that this is the simplest mathematical model and its sole purpose was to expose the rheonomic nature of a feedback control system.

5.5 Lagrange Equations With Reaction Forces

If the system is nonholonomic, a reduction of the generalized coordinates to independent generalized coordinates is not possible and one has to use the Lagrange equations in the form given by Equation 5.39. This form of the Lagrange equations allows furthermore to calculate internal dynamic stresses of critical system elements. Sometimes it is also possible to introduce surplus coordinates in order to simplify the equations of motion. This procedure provides a means to trade off simplicity versus number of equations to be solved.

The Lagrange multiplier method provides an elegant and efficient way to solve for these unknown reaction forces caused by the kinematical constraints. The method is equally applicable to holonomic and nonholonomic systems. However, if applied to holonomic constraints as given in Equation 5.11 they have to be written in differential form as virtual constraints:

$$\delta\phi_j = \sum_{k=1}^{n} \frac{\partial \phi_j}{\partial q_k}\, \delta\, q_k = 0 \quad j = 1, 2, \ldots, m \tag{5.55}$$

Recall that virtual displacements are taken irrespective of time such that $\delta t = 0$.

According to the principle of virtual work the mathematical problem is that of determining the stationary value of a function namely the virtual work, of the reaction forces subject to the kinematical constraints of Equation 5.55

$$\delta W_R = \sum_{k=1}^{n} Q_k^{(R)} \delta q_k = 0 \tag{5.56}$$

In the subsequent analysis, we can treat holonomic and nonholonomic systems alike. The only difference between the two systems is that for holonomic constraints, the constraint coefficients c_{jk} take the form of partial derivatives such that:

$$c_{jk} = \frac{\partial \phi_j}{\partial q_k}$$

Applying the Lagrange multiplier rule, the stationary value of the virtual work can be determined by multiplying the m virtual constraint equations

$$\sum_{k=1}^{n} c_{jk}(q_1, q_2, \ldots, q_n, t)\delta q_k = 0 \quad j = 1, 2, \ldots, m \tag{5.57}$$

by the m Lagrange multipliers λ_j and <u>add</u> them to Equation 5.56 to obtain:

$$\sum_{k=1}^{n} (Q_k^{(R)} + \lambda_1\, c_{1k} + \lambda_2 c_{2k} + \ldots \lambda_m c_{mk})\delta q_k = 0 \tag{5.58}$$

In this sum of n terms, we can now select the m Lagrange multipliers in such a way that the last m terms vanish. The remaining (n-m) terms contain then only (n-m) variations δq_k which are independent. Therefore, each associated coefficient has to be zero.

Thus, we obtain the set of equations for the reaction forces:

$$Q_k^{(R)} = -\sum_{j=1}^{m} \lambda_j \, c_{jk} \qquad k = 1, 2, \ldots, n \tag{5.59}$$

NOTE:

Some authors multiply the constraint equations (5.57) by the Langrange multipliers and subtract them from the virtual work equation (5.56). This only changes the sign of the right-hand side of Equation 5.59.

Despite the fact that these n equations all have the same form, it is well to remember, that they have different origins. The last m equations hold because we selected the m Lagrange multipliers to make them true. The first (n-m) equations are true because the Lagrange multipliers are selected to make the associated virtual displacements independent. (See Appendix for details on Lagrange Multiplier Rule.)

We have now (n+m) unknowns, namely the n generalized coordinates and the m Lagrange multipliers. But we have also the same number of equations, namely the n equations of motion from Equation 5.39 and m equations of constraint from Equation 5.57.

For holonomic systems, it is useful to express the constraint equations in terms of velocities as:

$$\sum_{k=1}^{n} \left(\frac{\partial \phi_j}{\partial q_k}\right)\dot{q}_k + \frac{\partial \phi_j}{\partial t} = 0 \qquad j = 1, 2, \ldots, m \tag{5.60}$$

In special cases the set of equations can be solved by ad hoc elimination and substitution methods. However, a systematic algorithm can be setup taking the following steps:

1) The Lagrange equations are a set of n second order ordinary differential equations which can be written in matrix form as:

$$M\ddot{\mathbf{x}} = \mathbf{Q}_A + \mathbf{Q}_I + \mathbf{Q}_R \tag{5.61}$$

where M is the $(n \times n)$ generalized mass matrix which is nonsingular for a proper mechanical system. The vector $\mathbf{x}$ is the $(n \times 1)$ column matrix of the generalized coordinates. The three terms $\mathbf{Q}_A, \mathbf{Q}_I$ and $\mathbf{Q}_R$ represent the generalized applied force, the generalized inertia force and the generalized reaction force, respectively. All these forces depend only on $\mathbf{x}, \dot{\mathbf{x}}$ and time.

2) Introduce the constraint equations (holonomic or nonholonomic) in matrix form as:

$$C\,\dot{\mathbf{x}} + \mathbf{b}(t) = 0 \tag{5.62}$$

where C is the $(m \times n)$ constraint matrix. The generalized reaction force of Equation 5.59 is likewise expressed in matrix rotation as:

$$\mathbf{Q}_R = -C^T \boldsymbol{\lambda} \tag{5.63}$$

where $\boldsymbol{\lambda}$ is the $(m \times 1)$ column matrix consisting of the Lagrange multipliers.

3) Differentiate the constraint Equation 5.62 to obtain:

$$C\,\ddot{\mathbf{x}} + \dot{C}\dot{\mathbf{x}} + \dot{\mathbf{b}}(t) = 0 \tag{5.64}$$

4) Premultiply the equations of motion (5.61) by the inverse generalized mass matrix M^{-1} and substitute in Equation 5.64. This yields:

$$CM^{-1}(\mathbf{Q}_A + \mathbf{Q}_I + \mathbf{Q}_R) = -\dot{C}\dot{\mathbf{x}} - \dot{\mathbf{b}}(t) \tag{5.65}$$

5) Using Equation 5.63 we solve the preceding equation for the Lagrange multipliers:

$$\boldsymbol{\lambda} = (CM^{-1}C^T)^{-1}\left\{\dot{C}\dot{\mathbf{x}} + \dot{\mathbf{b}}(t) + CM^{-1}(\mathbf{Q}_A + \mathbf{Q}_I)\right\} \tag{5.66}$$

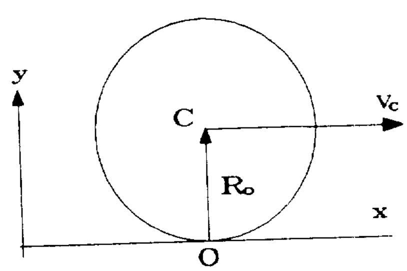
The reaction can now be readily obtained by going back again to Equation 5.63:

$$\mathbf{Q}_R = -C^T(CM^{-1})^{-1}\left\{CM^{-1}(\mathbf{Q}_A + \mathbf{Q}_I) + \dot{C}\dot{\mathbf{x}} + \dot{\mathbf{b}}(t)\right\} \tag{5.67}$$

Examples

a) Nonholonomic System

A uniform sphere of mass m and radius a rolls without slipping on a plane horizontal surface.



The kinetic energy of the Sphere is given by:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}\, I\omega^2 \tag{5.68}$$

The motion of the center C of the sphere is constrained by the roll condition

$$\mathbf{v}_c = \boldsymbol{\omega} \times \mathbf{R}_0 \text{ where } \mathbf{R}_0 = a\mathbf{j} \tag{5.69}$$

The rotational velocity $\boldsymbol{\omega}$ of the sphere has to be expressed in components of the inertial reference frame x, y, z.

We can relate the unit vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ of a body-fixed system to the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of an inertial frame and obtain the following result:

$$\begin{aligned}\boldsymbol{\omega} &= \omega_1\mathbf{e}_1 + \omega_2\,\mathbf{e}_2 + \omega_3\mathbf{e}_3 \\ &= (\omega_1 A_{11} + \omega_2 A_{21} + \omega_3 A_{31})\mathbf{i} + (\omega_1 A_{12} + \omega_2 A_{22} + \omega_3 A_{32})\mathbf{j} \\ &\quad +(\omega_1 A_{13} + \omega_2 A_{23} + \omega_3 A_{33})\mathbf{k}\end{aligned}$$

Likewise, the velocity $\mathbf{v}_c$ of the sphere's center is expressed in inertial components as:

$$\mathbf{v}_c = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{5.70}$$

Substituting the above two equations, Equation 5.69 yields:

$$\begin{aligned}\dot{x} &= -a(\omega_1 A_{13} + \omega_2 A_{23} + \omega_3 A_{33}) \\ \dot{y} &= 0 \\ \dot{z} &= a(\omega_1 A_{11} + \omega_2 A_{21} + \omega_3 A_{31})\end{aligned} \tag{5.71}$$

The middle equation represents a holonomic constraint which states geometrically that the center of the sphere maintains a constant distance above the horizontal surface. It can be easily taken into account by just setting $\dot{y} = 0$ in the kinetic energy expression of Equation 5.68.

The other two constraint conditions can be expressed in terms of generalized velocities by introducing the classical or modern Euler angle system. Using the modern Euler angles we obtain:

$$\dot{x} + a\ A_{13}\ \dot{\phi} + (\sin\theta A_{13} - \cos\theta\sin\phi A_{23} - \cos\theta\cos\phi A_{33})$$

$$+(\sin\phi A_{33} - \cos\phi A_{23})\dot{\theta} = 0$$

$$\dot{z} + (-A_{11})\dot{\phi} + (\sin\theta\ A_{11} - \cos\phi A_{21} - \cos\theta\cos\phi A_{31})\dot{\psi}$$

$$+(\sin\phi A_{31} - \cos\phi\ A_{21})\dot{\theta} = 0$$

This set of constraint equations can be expressed in matrix form as:

$$C\,\dot{\mathbf{q}} = 0 \qquad C = (2\times 5)\text{matrix} \tag{5.72}$$

where $\mathbf{q}^T = [x\ z\ \phi\ \psi\ \theta]$.

The equations of motion are then obtained by expressing the kinetic energy in terms of the generalized coordinates

$$T = \frac{1}{2}\ m\ (\dot{x}^2 + \dot{z}^2) + \frac{1}{2}I(\dot{\theta}^2 + \dot{\psi}^2 + \dot{\phi}^2 - 2\dot{\phi}\dot{\psi}\ \sin\theta) \tag{5.73}$$

and performing the differentiations required by the Lagrange equations.

$$\begin{aligned}
m\ \ddot{x} &= -(c_{11}\ \lambda_1 + c_{21}\ \lambda_2)\\
m\ \ddot{y} &= -(c_{12}\ \lambda_1 + c_{22}\ \lambda_2)\\
I(\ddot{\theta} + \dot{\phi}\dot{\psi}\cos\theta) &= -(c_{13}\ \lambda_1 + c_{23}\ \lambda_2)\\
I(\ddot{\phi} - \ddot{\psi}\sin\theta) - \dot{\psi}\dot{\theta}\cos\theta) &= -(c_{14}\ \lambda_1 + c_{24}\ \lambda_2)\\
I(\ddot{\psi} - \ddot{\phi}\sin\theta - \dot{\phi}\dot{\theta}\cos\theta) &= -(c_{15}\ \lambda_1 + c_{25}\ \lambda_2)
\end{aligned}$$

The solution of this set of equations could formally proceed along the steps outlined above. The initial conditions would have to be chosen to be compatible with the requirement of pure rolling without slipping.

b) Holonomic System (Scleronomic)

Two particles m_1 and m_2 are connected by a massless rod. They move in a vertical plane under a frictionless constraint which keeps m_1 on the horizontal x-axis and m_2 on the vertical y-axis. Calculate the stress in the rod.

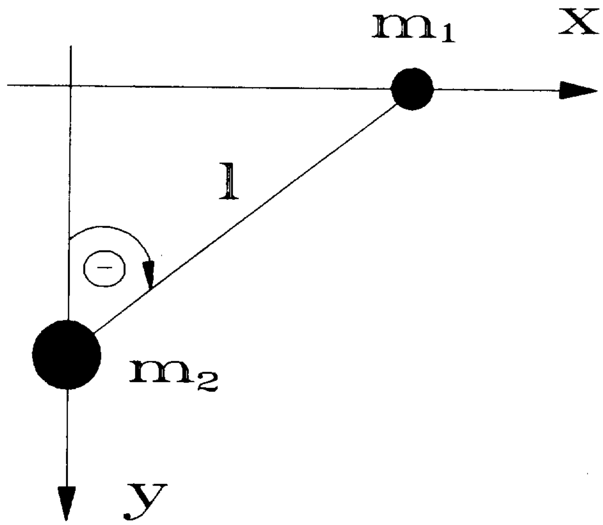


Figure 5.7: Constrained Dumbbell

The kinetic energy is:

$$T = \frac{1}{2}\, m(\ell^2 \dot{\theta}^2 + \dot{\ell}^2) \tag{5.74}$$

and the potential energy:

$$V = -mg\ell \cos\theta \tag{5.75}$$

The kinematical constraint condition expressed in terms of the velocity:

$$\dot{\ell} = 0 \tag{5.76}$$

The Lagrangian equations yield the following two equations of motion:

$$\begin{gathered} \ell\, \ddot{\theta} + g\ \sin\ \theta = 0 \\ m\, \ddot{\ell} - m\, \ell\dot{\theta}^2 - mg\cos\theta + \lambda = 0 \end{gathered}$$

The first of these equations can be solved by itself and furnishes the time histories of the angle θ and the angular velocity $\dot{\theta}$. With these known, one can solve for the reaction force associated with the constraint $\dot{\ell} = \ddot{\ell} = 0$:

$$R_\ell = -\lambda = -m\,\ell\,\dot{\theta}^2 - mg\,\cos\,\theta$$

The reaction force is negative, indicating that the rod is in tension. Physically interpreted it is observed that the reaction force is caused by the gravity component along the rod and the centrifugal face arising from its rotation.

c) Holonomic System (Rheonomic)

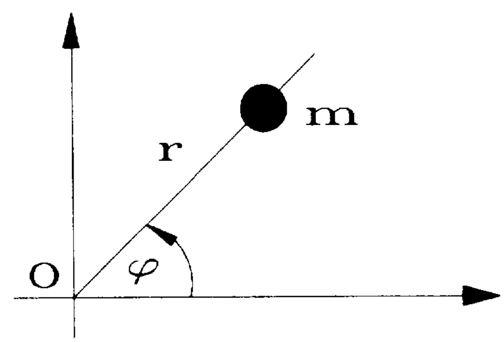


Figure 5.8: Sliding Mass

A rigid wire is pivoted at one end so that it can be rotated in a horizontal plane with prescribed angular velocity $\omega = f(t)$. A particle of mass m can slide without friction along the wire. Calculate the reaction force of the wire on the mass.

The kinetic energy is:

$$T = \frac{1}{2}\,m\,(\dot{r}^2 + r\,\dot{\phi}^2) \tag{5.77}$$

The rheonomic constraint condition:

$$\omega = \dot{\phi} = f(t) \;\text{ or }\; \dot{\phi} - f(t) = 0 \tag{5.78}$$

The Lagrange equations are:

$$m\,\ddot{r} - m\,r\,\dot{\phi}^2 = 0$$

$$m\,r^2\,\ddot{\phi} + 2mr\,\dot{r}\,\dot{\phi} + \lambda = 0$$

The generalized reaction force associated with the generalized coordinate ϕ is the reaction torque about the pivot point 0 and given by:

$$Q_\phi^{(R)} = -\lambda = m\, r^2\, \ddot{\phi} + 2\, m\, r\, \dot{r}\dot{\phi} \qquad (5.79)$$

The second term is the Coriolis torque exerted by the wire on the sliding mass.

Chapter 6

Modal Synthesis Technique

6.1 Boltzmann-Hamel Equations

When applied to complex dynamic configurations, the classical Lagrange equations become formidably lengthy and their computer coding leads to low computational efficiency. This is due to the fact that they contain complete information concerning the dynamics and kinematics of the system. It is possible to separate these two aspects of a dynamical system and obtain a substantial reduction in complexity of the equations of motion by transforming the classical Lagrange equations from an inertial reference frame to a moving reference frame. The accompanying loss of information concerning position and orientation can be supplied by the appropriate kinematical differential equations governing the relation between inertial and non-inertial (nonholonomic) velocities. To achieve the desired transformation we write the classical Lagrange equations in matrix form as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} = \mathbf{Q} - C^T \boldsymbol{\lambda} = \mathbf{Q} + \mathbf{Q}_R \tag{6.1}$$

where the partial derivatives represent the $(n \times 1)$ column matrices containing as elements the partial derivatives with respect to the n generalized coordinates $q_i (i = 1, 2, \ldots, n)$.

The transformation from inertial velocity components to non-inertial velocity components is effected by the kinematical differential equations.

$$\boldsymbol{\Omega} = A(\mathbf{q})\dot{\mathbf{q}} \tag{6.2}$$

The nonholonomic velocity vector $\boldsymbol{\Omega}$ is, in general, composed of both linear and angular velocity components. The kinematical conditions for the angular velocities are given by Euler's kinematical differential equations of section IV. The transformation from translational inertial velocities to non-inertial velocities is accomplished by the direction cosine matrix A such that:

$$\mathbf{v} = A\,\dot{\mathbf{x}} \tag{6.3}$$

where $\dot{\mathbf{x}}^T = (\dot{x}, \dot{y}, \dot{z})$ and $\mathbf{v}^T = (v_1\ v_2\ v_3)$. The kinetic energy of the system when expressed in nonholonomic velocity components is denoted by

$$T^*(\boldsymbol{\Omega}, \mathbf{q}) = T(\dot{\mathbf{q}}, \mathbf{q}) \tag{6.4}$$

The superscript star should indicate that the mathematical form of the transformed kinetic energy differs from the original one although its scalar value is, of course, the same.

The differentiations required by the Lagrange equations are first given in scalar form as:

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{\ell} \frac{\partial T^*}{\partial \Omega_\ell} \cdot \frac{\partial \Omega_\ell}{\partial \dot{q}_k} = \sum_{\ell} \frac{\partial T^*}{\partial \Omega_\ell} A_{\ell\, k} \tag{6.5}$$

where $A_{\ell\, k}$ are the components of the kinematical transformation matrix $A(\mathbf{q})$ of Equation 6.2. Also:

$$\frac{\partial T}{\partial q_k} = \sum_{\ell} \frac{\partial T^*}{\partial \Omega_\ell} \cdot \frac{\partial \Omega_\ell}{\partial q_k} + \frac{\partial T^*}{\partial q_k} \tag{6.6}$$

Equation 6.5 can be directly written in matrix form as:

$$\frac{\partial T}{\partial \dot{\mathbf{q}}} = A^T \frac{\partial T^*}{\partial \boldsymbol{\Omega}} \tag{6.7}$$

To be able to express Equation 6.6 in matrix form we introduce the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial \Omega_1}{\partial q_1} & \frac{\partial \Omega_1}{\partial q_2} & \cdots & \frac{\partial \Omega_1}{\partial q_\ell} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial \Omega_\ell}{\partial q_1} & \frac{\partial \Omega_\ell}{\partial q_2} & \cdots & \frac{\partial \Omega_\ell}{\partial q_\ell} \end{bmatrix} \equiv \frac{\partial \boldsymbol{\Omega}}{\partial \mathbf{q}}$$

Equation 6.6 can then be written in matrix form:

$$\frac{\partial T}{\partial \mathbf{q}} = J^T \frac{\partial T^*}{\partial \boldsymbol{\Omega}} + \frac{\partial T^*}{\partial \mathbf{q}} \tag{6.8}$$

Substituting Equation 6.7 and 6.9 into Equation 6.1 yields:

$$A^T \frac{d}{dt}(\frac{\partial T^*}{\partial \boldsymbol{\Omega}}) + \dot{A}^* \frac{\partial T^*}{\partial \boldsymbol{\Omega}} - J^T \frac{\partial T^*}{\partial \boldsymbol{\Omega}} - \frac{\partial T^*}{\partial \mathbf{q}} = \mathbf{Q} - C^T \boldsymbol{\lambda} = \mathbf{Q} + \mathbf{Q}_R \qquad (6.9)$$

Next we premultiply this equation by $(A^{-1})^T$ and obtain:

$$\frac{d}{dt}(\frac{\partial T^*}{\partial \boldsymbol{\Omega}}) + [(\dot{A} - J)A^{-1}]^T \frac{\partial T^*}{\partial \boldsymbol{\Omega}} - (A^{-1})^T \frac{\partial T^*}{\partial \mathbf{q}} = (A^{-1})^T \mathbf{Q} - (CA^{-1})^T \boldsymbol{\lambda} \qquad (6.10)$$

This is the desired transformation of the classical Lagrange equations from an inertial frame to a moving frame. These equations are known as Boltzmann-Hamel equations or Lagrange equations for quasi-coordinates.

It remains to transform the generalized forces from the inertial reference frame to the moving reference frame. This is accomplished by the so called quasi-coordinates. These serve only as a means to an end and play only a brief cameo role after which they disappear from the scene.

The quasi-coordinates are defined such that their time derivatives are equal to the nonholonomic velocities, i.e.,

$$\frac{d\xi}{dt} = \boldsymbol{\Omega} \qquad \boldsymbol{\xi} = \text{quasi-coordinate vector} \qquad (6.11)$$

The virtual displacements of the generalized coordinates q_i can then be related to the virtual displacements of the quasi-coordinates ξ_i by Equation 6.2 as:

$$\delta \boldsymbol{\xi} = A(q)\, \delta \mathbf{q} \qquad (6.12)$$

Physically interpreted the virtual displacements of the quasi-coordinates are infinitesimal translations along or rotations about the instantaneous axes of the moving reference frame. Calculating now the virtual work of the external forces $\mathbf{Q}$ through a virtual displacement we obtain:

$$\delta W = (\delta \mathbf{q})^T\, \mathbf{Q} = (A^{-1}\delta\boldsymbol{\xi})^T\, \mathbf{Q} = (\delta\boldsymbol{\xi})^T (A^{-1})^T \mathbf{Q} = (\delta\boldsymbol{\xi}^T)\mathbf{k} \qquad (6.13)$$

where $\mathbf{k}$ is the generalized force associated with the quasi-coordinate $\boldsymbol{\xi}$. Mathematically expressed:

$$\mathbf{k} = (A^{-1})^T \mathbf{Q} = (A^T)^{-1} \mathbf{Q} \qquad (6.14)$$

The generalized reaction force in the moving reference frame can be obtained by transformation of the constraint condition.

$$C\dot{\mathbf{q}} + \mathbf{b}(t) = 0 \tag{6.15}$$

Using the kinematical differential equation (6.2) we obtain:

$$C(A^{-1}\boldsymbol{\Omega}) + \mathbf{b}(t) = B\boldsymbol{\Omega} + \mathbf{b}(t) = 0 \tag{6.16}$$

where $B = CA^{-1}$ is simply the constraint matrix relating the nonholonomic velocities of the system elements as required by the kinematical constraints.

The Boltzmann-Hamel equations are therefore finally:

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \boldsymbol{\Omega}}\right) + [(\dot{A} - J)A^{-1}]^T \frac{\partial T^*}{\partial \boldsymbol{\Omega}} - (A^{-1})^T \frac{\partial T^*}{\partial \mathbf{q}} = \mathbf{k} - B^T\boldsymbol{\lambda} = \mathbf{k} + \mathbf{k}_R \tag{6.17}$$

Judged by their outward appearance the transformed equations (6.17) seem to be much more complicated than their classical counterpart given in Equation 6.1. However, their hidden simplicity will be brought to light when the detailed steps of introducing the above-mentioned moving reference frame are carried out. To this end we introduce the linear and angular velocity of the moving reference frame as:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

The column matrix Ω can be written in partitioned form as:

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\omega} \\ -- \\ \mathbf{v} \end{bmatrix}$$

It can be shown by some rather lengthy but straight-forward process that the following relationship holds:

$$\left[(\dot{A} - J)A^{-1}\right]^T = \begin{bmatrix} \tilde{\omega} & | & \tilde{v} \\ -- & -- & -- \\ 0 & | & \tilde{\omega} \end{bmatrix}$$

where

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \qquad \tilde{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

Next we assume without any practical loss of generality that the kinetic energy of the system can be expressed as:

$$T = T(\boldsymbol{\Omega}, \dot{\boldsymbol{\eta}}, \boldsymbol{\eta}) \tag{6.18}$$

where $\boldsymbol{\eta}$ are generalized coordinates defining the system configuration relative to the moving frame. The superscript star has been dropped for simplicity. With this assumption the term $\partial T^*/\partial \mathbf{q}$ in Equation 6.17 is equal to zero. The equations of motion can now be written as two vector equations:

A) Translational Equations:

$$\frac{d}{dt}(\frac{\partial T}{\partial \mathbf{v}}) + \boldsymbol{\omega} \times \frac{\partial T}{\partial \mathbf{v}} = \mathbf{F} + \mathbf{F}_R \tag{6.19}$$

B) Rotational Equations:

$$\frac{d}{dt}(\frac{\partial T}{\partial \boldsymbol{\omega}}) + \boldsymbol{\omega} \times \frac{\partial T}{\partial \boldsymbol{\omega}} + \mathbf{v} \times \frac{\partial T}{\partial \mathbf{v}} = \mathbf{L} + \mathbf{L}_R \tag{6.20}$$

The right-hand side of Equation 6.19 are the forces acting on the system in the direction of the instantaneous moving frame axes, that of Equation 6.20 are the moments of these forces about these instantaneous axes. The latter can be derived easily by calculating the virtual work done by the forces through the virtual rotations of the moving frame. Using the quasi-coordinate definition of Equation 6.11 the virtual displacement of a point located at $\mathbf{R}$ in the moving frame is given by:

$$\delta \mathbf{x}_R = \delta \boldsymbol{\xi}_R \times \mathbf{R} \qquad \text{Remember: } \mathbf{v} = \boldsymbol{\omega} \times \mathbf{R} \tag{6.21}$$

The virtual work is therefore:

$$\delta W = \mathbf{F} \cdot \delta \mathbf{x}_R = \mathbf{F} \cdot (d\boldsymbol{\xi}_R \times \mathbf{R}) = (\mathbf{R} \times \mathbf{F}) \cdot d\boldsymbol{\xi}_R = \mathbf{L} \cdot d\boldsymbol{\xi}_R \tag{6.22}$$

NOTE:

The moving reference frame does not have to be a body-fixed frame but can be any conveniently chosen frame relative to which the system motion is to be defined ("Floating" frame).

The Lagrange equations for the relative motion retain their classical form. Since the relative motion is often associated with flexible system components we call them flexibility equations:

C) Flexibility Equations:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\boldsymbol{\eta}}}) - \frac{\partial T}{\partial \boldsymbol{\eta}} = \mathbf{Q} + \mathbf{Q}^R \tag{6.23}$$

The real physical meaning of the equations of motion given by Equation 6.19, 6.20, and 6.23 can be revealed by introducing the explicit expression of the kinetic energy of the system in terms of the linear velocity $\mathbf{v}$ of the origin of the reference frame, its angular velocity $\boldsymbol{\omega}$ against inertial space and the relative motion $\dot{\mathbf{R}}$ of the system masses as viewed from the moving frame. With these terms we obtain:

$$T = \frac{1}{2} \int \left[\mathbf{v} + (\boldsymbol{\omega} \times \mathbf{R}) + \dot{\mathbf{R}}\right]^2 d\,m \tag{6.24}$$

In the subsequent mathematical manipulations the following relations and identities are used:

(α) If $y = (\mathbf{a} \times \mathbf{b})^2$ then $\frac{\partial y}{\partial \mathbf{a}} = 2\mathbf{b} \times (\mathbf{a} \times \mathbf{b})$.

(β) If $y = (\mathbf{a} \cdot \mathbf{b})$ then $\frac{\partial y}{\partial \mathbf{a}} = \mathbf{b}$

(γ) $\boldsymbol{\omega} \times (\mathbf{R} \times \mathbf{v}) + \mathbf{v} \times (\boldsymbol{\omega} \times \mathbf{R}) = \mathbf{R} \times (\boldsymbol{\omega} \times \mathbf{v})$

(∂) $\boldsymbol{\omega} \times [\mathbf{R} \times (\boldsymbol{\omega} \times \mathbf{R})] = \mathbf{R} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})]$

(ϵ) $\boldsymbol{\omega} \times (\mathbf{R} \times \dot{\mathbf{R}}) + \dot{\mathbf{R}} \times (\boldsymbol{\omega} \times \mathbf{R}) = \mathbf{R} \times (\boldsymbol{\omega} \times \dot{\mathbf{R}})$

It is also important to realize that all time derivatives are taken relative to the moving reference frame.

We obtain:

$$\frac{\partial T}{\partial \boldsymbol{\omega}} = \int \mathbf{R} \times [\mathbf{v} + (\boldsymbol{\omega} \times \mathbf{R}) + \dot{\mathbf{R}}]dm \tag{6.25}$$

$$\boldsymbol{\omega} \times \frac{\partial T}{\partial \boldsymbol{\omega}} = \int \boldsymbol{\omega} \times \left\{\mathbf{R} \times [\mathbf{v} + (\boldsymbol{\omega} \times \mathbf{R}) + \dot{\mathbf{R}}]\right\} dm \tag{6.26}$$

$$\mathbf{v} \times \frac{\partial T}{\partial \mathbf{v}} = \int \mathbf{v} \times [\mathbf{v} + (\boldsymbol{\omega} \times \mathbf{R}) + \dot{\mathbf{R}}]dm \tag{6.27}$$

$$\begin{aligned} \frac{d}{dt}(\frac{\partial T}{\partial \boldsymbol{\omega}}) &= \int \dot{\mathbf{R}} \times [\mathbf{v} + (\boldsymbol{\omega} \times \mathbf{R}) + \dot{\mathbf{R}}]dm \\ &+ \int \mathbf{R} \times [\dot{\mathbf{v}} + (\dot{\boldsymbol{\omega}} \times \mathbf{R}) + (\boldsymbol{\omega} \times \dot{\mathbf{R}}) + \ddot{\mathbf{R}}]dm \end{aligned} \tag{6.28}$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \mathbf{v}}) = \int [\dot{\mathbf{v}} + (\dot{\boldsymbol{\omega}} \times \mathbf{R}) + (\boldsymbol{\omega} \times \dot{\mathbf{R}}) + \ddot{\mathbf{R}}]\mathbf{dm} \tag{6.29}$$

$$\boldsymbol{\omega} \times \frac{\partial T}{\partial \mathbf{v}} = \int \boldsymbol{\omega} \times [\mathbf{v} + (\boldsymbol{\omega} \times \mathbf{R}) + \dot{\mathbf{R}}]dm \tag{6.30}$$

Collecting all terms associated with the translational equation we obtain:

$$\int [\dot{\mathbf{v}} + (\boldsymbol{\omega} \times \mathbf{v}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) + \dot{\boldsymbol{\omega}} \times \mathbf{R} + 2(\boldsymbol{\omega} \times \dot{\mathbf{R}}) + \ddot{\mathbf{R}}]dm = \mathbf{F} + \mathbf{F}_R \tag{6.31}$$

For the rotational equation:

$$\int \mathbf{R} \times [\dot{\mathbf{v}} + (\boldsymbol{\omega} \times \mathbf{v}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) + \dot{\boldsymbol{\omega}} \times \mathbf{R} + 2(\boldsymbol{\omega} \times \dot{\mathbf{R}}) + \ddot{\mathbf{R}}]dm = \mathbf{L} + \mathbf{L}_R \tag{6.32}$$

It is observed that the terms within the square bracket taken together determine the absolute acceleration of a mass point relative to inertial space. The first two terms represent the acceleration of the origin of the moving frame expressed in components of the moving frame. The bracketed term is thus seen to be identical with Equation 1.5 of section I. The translational and rotational Equations 6.19 and 6.20 could have been obtained also by a Newton-Euler approach. The main difference occurs in the physical interpretation of the right-hand side. Here they are seen to be generalized forces associated with the virtual displacements of the instantaneous reference frame. Furthermore the elimination of constraint forces can be very efficient by accomplished by the Lagrange multiplier method. Constraint forces will arise when several moving coordinate frames are introduced for various system elements and then conjoined for dynamic simulation.

6.2 Component Modes

In many situations the motion of flexible components can be described by the superposition of appropriately chosen mode (shape) functions. The success of this method often referred to as component mode synthesis depends largely on the proper choice of these assumed mode functions. These are often selected from the natural modes (eigen functions) of the isolated structural component (substructure) using boundary conditions which are geometrically and dynamically resembling the actual ones. Other mode functions can be obtained by the static deflections of the substructure due to unit displacements or unit forces imposed upon suitable coordinates.

Many other types of component modes have been advocated to describe the flexural motion of the substructure (Ref: Roy R. Craig Jr. "Structural Dynamics" 1981).

In all these cases the position of a mass element in the moving reference frame can be defined as:

$$\mathbf{R} = \mathbf{r} + \sum_i \boldsymbol{\psi}_i(\mathbf{r})\ \eta_i(t) \tag{6.33}$$

where $\boldsymbol{\psi}_i(\mathbf{r})$ is the vector mode function and $\eta_i(t)$ its associated generalized coordinate. The vector mode function is often specified in terms of a translational mode of the center of mass of a mass element and a rotational mode about its mass center such that

$$\boldsymbol{\psi}_i(\mathbf{r}) = \boldsymbol{\phi}_i(\mathbf{r}) + \boldsymbol{\phi}_i' \times \boldsymbol{\rho} \tag{6.34}$$

where $\boldsymbol{\rho}$ is the position of a mass particle relative to the center of mass of the mass element. The position vector $\mathbf{r}$ denotes the location of the undeformed mass element.

The flexibility Equations 6.23 can then be further manipulated using the following partial differentiations:

$$\frac{\partial T}{\partial \eta_i} = \frac{\partial T}{\partial \mathbf{R}} \cdot \frac{\partial \mathbf{R}}{\partial \eta_i} = -\int \boldsymbol{\psi}_i \cdot \left\{\boldsymbol{\omega} \times [\mathbf{v} + (\boldsymbol{\omega} \times \mathbf{R}) + \dot{\mathbf{R}}]\right\} dm \tag{6.35}$$

$$\frac{\partial T}{\partial \dot{\eta}_i} = \frac{\partial T}{\partial \dot{\mathbf{R}}} \cdot \frac{\partial \dot{\mathbf{R}}}{\partial \dot{\eta}_i} = \int \boldsymbol{\psi}_i \cdot [\mathbf{v} + (\boldsymbol{\omega} \times \mathbf{R}) + \dot{\mathbf{R}}] dm \tag{6.36}$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\eta}_i}) = \int \boldsymbol{\psi}_i \cdot [\dot{\mathbf{v}} + (\dot{\boldsymbol{\omega}} \times \mathbf{R}) + (\boldsymbol{\omega} \times \dot{\mathbf{R}}) + \ddot{\mathbf{R}}] dm \tag{6.37}$$

Collecting all terms we obtain Equation 6.23 in the form

$$\int \boldsymbol{\psi}_i \cdot [\dot{\mathbf{v}} + (\boldsymbol{\omega} \times \mathbf{v}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) + (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + 2(\boldsymbol{\omega} \times \dot{\mathbf{R}}) + \ddot{\mathbf{R}}] dm = \mathbf{Q}_i + \mathbf{Q}_i^{(R)} \tag{6.38}$$

The bracketed term is again seen to be the total acceleration of the mass particle relative to inertial space. Therefore the generalized inertial forces are given by the scalar product of the vector mode function with the total acceleration and summing over all mass elements. The generalized forces are obtained via the virtual work through a virtual displacement of the generalized flexural coordinate η_i. Therefore:

$$Q_i = \mathbf{F} \cdot \boldsymbol{\psi}_i(\mathbf{r}) \ \text{ and } \ Q_i^{(R)} = \mathbf{F}_R \cdot \boldsymbol{\psi}_i(\mathbf{r}) \tag{6.39}$$

Another contribution to the right-hand side arises from the strain energy of the flexible component. For a beam, for instance, undergoing a transverse deflection $\boldsymbol{\phi}_i(\mathbf{r})\eta_i(t)$ the strain energy is given by

$$V = \frac{1}{2}\int EI(\boldsymbol{\phi}_i''\,\eta_i)^2 dx \tag{6.40}$$

Its generalized force is obtained by partial differentiation

$$\frac{\partial V}{\partial \eta_i} = Q_i^{(B)} = \int EI\boldsymbol{\phi}_i'' dx\,\eta_i \tag{6.41}$$

Similar expressions can be obtained for the strain energy of other system elements which undergo a deflection as dictated by the assumed mode function.

It is often advantageous to select mode functions which are orthogonal with respect to the mass distribution of the component. This eliminates or reduces the number of dynamic and static coupling terms.

As an example, consider the following integral which is encountered when Equation 6.38 is expanded into its various components:

$$I = \int(\boldsymbol{\phi}_i + \boldsymbol{\phi}_i' \times \boldsymbol{\rho}) \cdot \sum_j(\boldsymbol{\phi}_j + \boldsymbol{\phi}_j' \times \boldsymbol{\rho})dm \tag{6.42}$$

Orthogonality of the mode functions implies that

$$\int(\boldsymbol{\phi}_i + \boldsymbol{\phi}_i' \times \boldsymbol{\rho}) \cdot (\boldsymbol{\phi}_j + \boldsymbol{\phi}_j' \times \boldsymbol{\rho})dm = \delta_{ij} \tag{6.43}$$

where δ_{ij} is the Kronecker symbol.

Therefore, the integral of Equation 6.42 reduces to

$$I = \int(\boldsymbol{\phi}_i + \boldsymbol{\phi}_i' \times \boldsymbol{\rho})^2 dm \tag{6.44}$$

$$= \int(\boldsymbol{\phi}_i^2 + \boldsymbol{\phi}_i' \cdot \mathcal{I} \cdot \boldsymbol{\phi}_i')dm = 1 \tag{6.45}$$

where $\mathcal{I} = \rho^2\mathcal{E} - (\boldsymbol{\rho}\boldsymbol{\rho})$.

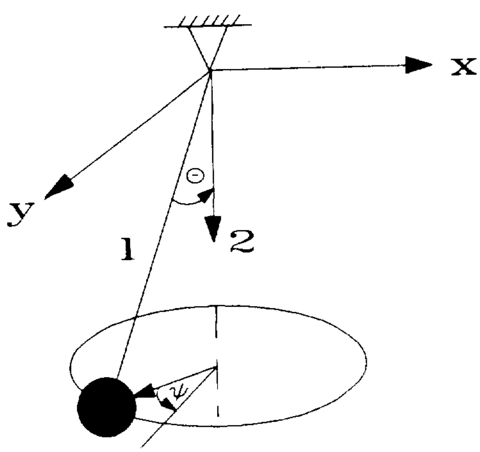
The integral I is the generalized mass of the mode function normalized to unity.

If the systems contains continuous and discrete elements simultaneously, the above mathematical formulations remain unaltered when all integrals are defined as Stieltjes integrals.

6.3 Applications to Aerospace Systems

Example 1:

Spherical Pendulum



The direction of the pendulum is defined by the two Euler angles θ and ψ

$I_1 = m\ell^2 + I = I_2$

$I_3 = I$

The rotating reference frame is defined such that the $\mathbf{e}_3$ axis is along the tether. The rotational motion is governed by the rotational Equation 6.20 which becomes

$$\mathcal{I} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathcal{I} \cdot \boldsymbol{\omega} = \mathbf{L} + \mathbf{L}_R = \mathbf{L} - \mathcal{B}^T \boldsymbol{\lambda}$$

The position of the pendulum in inertial coordinates is given by:

$$x = \ell \sin\theta \cos\psi$$
$$y = \ell \sin\theta \sin\psi$$
$$z = \ell \cos\theta$$

Its angular velocity is given by:

$$\omega_1 = -\dot{\psi}\sin\theta$$
$$\omega_2 = \dot{\theta}$$
$$\omega_3 = \dot{\psi}\cos\theta$$

Because $\dot{\phi} = 0$ then exists a kinematical constraint between the angular velocities. It is:

$$\omega_1 \cos\theta + \omega_3 \sin\theta = 0$$

which leads to the constraint matrix:

$$B = [\cos\theta \quad 0 \quad \sin\theta]$$

The torque acting on the pendulum comes from the gravity force and is:

$$\mathbf{L} = \mathbf{r} \times \mathbf{F} = \ell\mathbf{e}_3 \times mg\mathbf{k} = -mg\ell\sin\theta\mathbf{e}_2$$

The equations of motion can now be obtained in scalar form:

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = \lambda\cos\theta$$
$$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = -mg\ell\sin\theta$$
$$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = \lambda\sin\theta$$

To eliminate the Lagrange multiplier we multiply the first equation by $\sin\theta$ and the third equation by $-\cos\theta$ and add. We obtain the following two differential equations:

$$(I + m\ell^2\sin^2\theta)\ddot{\psi} + 2m\ell^2\dot{\psi}\,\dot{\theta}\,\sin\theta\cos\theta = 0$$
$$I_2\ddot{\theta} - m\ell^2\dot{\psi}^2\sin\theta\cos\theta + mg\ell\sin\theta = 0$$

Those equations could have been obtained by using the classical Lagrange equations. It can be seen that for $I << m\ell^2$ these equations become almost singular for small θ and present serious computational problems.

To avoid this singularity it is necessary to express the direction of the pendulum in terms of the modern Euler angles $\underline{\theta \text{ and } \phi}$ ($\psi = 0$).

The angular velocity of the pendulum is then

$$\begin{aligned} \omega_1 &= \dot{\phi} \\ \omega_2 &= \dot{\theta}\cos\phi \\ \omega_3 &= -\dot{\theta}\sin\phi \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= \ell\sin\theta\cos\phi \\ y &= -\ell\sin\phi \\ z &= \ell\cos\theta\cos\phi \end{aligned}$$

The constraint imposed upon the angular velocity because of $\dot{\psi} = 0$ is:

$$\omega_2 \sin\phi + \omega_3 \cos\phi = 0$$

which leads to the constraint equation:

$$B = [0 \quad \sin\phi \quad \cos\phi]$$

The torque acting on the pendulum is given by

$$\mathbf{L} = \ell\mathbf{e}_3 \times mg\mathbf{k} = -mg\ell\sin\theta\mathbf{e}_2 - mg\ell\sin\phi\cos\theta\mathbf{e}_1$$

The rotational motions are therefore:

$$\begin{aligned} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) &= -mg\ell\sin\phi\cos\theta \\ I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) &= -mg\ell\sin\theta + \lambda\sin\phi \\ I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) &= \lambda\cos\phi \end{aligned}$$

To eliminate the Lagrange multiplier we multiply the second equation by $\cos\phi$ and the third equation by $\sin\phi$ and obtain again two differential equations:

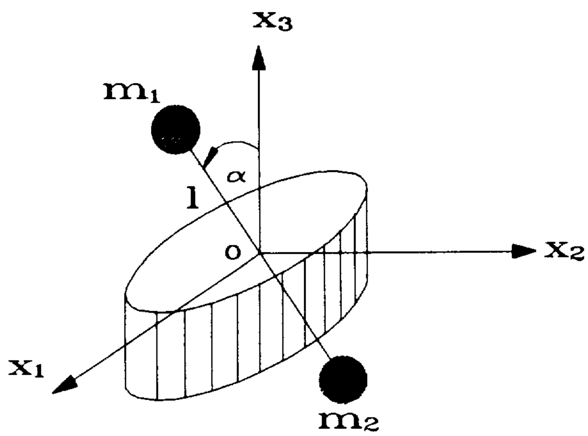
$$\begin{aligned} (m\ell^2 + I)\ddot{\phi} + m\ell^2\dot{\theta}^2 \ \sin\phi\cos\phi &= -mg\ell\sin\phi\cos\theta \\ (m\ell^2\cos\phi + I)\ddot{\theta} - 2m\ell^2\dot{\theta}\dot{\phi}\sin\phi\cos\phi &= -mg\ell\cos\phi\sin\theta \end{aligned}$$

It is seen that these equations are well-behaved for small angles θ and ϕ and in fact, reduce to the simple pendulum equations:

$$\begin{aligned} (I + m\ell^2)\ddot{\phi} + mg\ell\phi &= 0 \\ (I + m\ell^2)\ddot{\theta} + mg\ell\theta &= 0 \end{aligned}$$

<u>Example 2:</u>

A dumbbell is elastically coupled to a flat disk at its mass center and can rotate in a plane through an angle α as shown in the figure.



x_1, x_2, x_3 body-fixed reference frame.
$m_1 = m_2 = m$
The origin is at the center of mass
$I_0 = 2m\ell^2$

Moment of Inertia of Disk:

$$\mathcal{I} = A\mathbf{e}_1\mathbf{e}_1 + B\mathbf{e}_2\mathbf{e}_2 + C\mathbf{e}_3\mathbf{e}_3$$

Moment of Inertia of dumbbell:

$$\mathcal{I}_0 = I_0 \cos^2 \alpha \mathbf{e}_1\mathbf{e}_1 + I_0\mathbf{e}_2\mathbf{e}_2 + I_0 \sin^2 \alpha \mathbf{e}_3\mathbf{e}_3 - I_0 \sin\alpha\cos\alpha \mathbf{e}_1\mathbf{e}_3$$

Location of dumbbell masses m_1 and m_2:

$$\mathbf{R}_1 = \ell \cos\alpha \mathbf{e}_3 + \ell \sin\alpha \mathbf{e}_1$$
$$\mathbf{R}_2 = -(\ell \cos\alpha \mathbf{e}_3 + \ell \sin\alpha \mathbf{e}_1)$$

Angular Velocity of Disk:

$$\boldsymbol{\Omega} = \Omega_1 \mathbf{e}_1 + \Omega_2 \mathbf{e}_2 + \Omega_3 \mathbf{e}_3$$

Kinetic and Potential Energy of Dumbbell:

$$T_0 = \frac{1}{2} I_0 (\dot{\alpha} + \Omega_2)^2 + \frac{1}{2} I_0 (\Omega_3 \sin\alpha - \Omega_1 \cos\alpha)^2$$
$$V_0 = \frac{1}{2} k \alpha^2 \qquad k = \text{spring constant}$$

<u>Rotational Equation:</u>

$$\mathcal{I} \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathcal{I} \cdot \boldsymbol{\Omega} + \mathcal{I}_0 \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathcal{F}_0 \cdot \boldsymbol{\Omega} + 2\mathbf{R}_1 \times (\boldsymbol{\Omega} \times \mathbf{v}_1) m_1$$
$$+ 2\mathbf{R}_2 \times (\boldsymbol{\Omega} \times \mathbf{v}_2) m_2 + (\mathbf{R}_1 \times \mathbf{a}_1) m_1 + (\mathbf{R}_2 \times \mathbf{a}_2) m_2 = 0$$

where **v** and **a** are the time derivatives of the position vector **R** of the dumbbell masses relative to the body-fixed frame.

<u>Flexibility Equation:</u>

$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\alpha}}\right) - \frac{\partial T_0}{\partial \alpha} + \frac{\partial V_0}{\partial \alpha} = 0$$

$$\frac{\partial T_0}{\partial \dot{\alpha}} = I_0 (\dot{\alpha} + \Omega_2)$$
$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\alpha}}\right) = I_0 (\ddot{\alpha} + \dot{\Omega}_2)$$

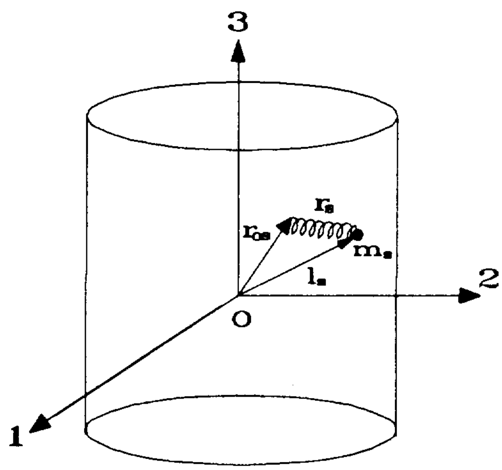
$$\frac{\partial T_0}{\partial \alpha} = I_0 (\Omega_3 \sin\alpha - \Omega_1 \cos\alpha)(\Omega_3 \cos\alpha + \Omega_1 \sin\alpha)$$

$$\frac{\partial V_0}{\partial \alpha} = k\,\alpha$$

Information about the attitude of the main body can be supplied by the Euler kinematical equations. For setting up perturbation (linearized) equations about a nominal spin condition it is, of course, necessary to use the modern Euler angle system to avoid the gimbal look singularity.

Example 3: (Slosh Model)

A satellite contains an internal linear oscillator system located at $\mathbf{r}_{0s}$ having a spring constant k and a damping constant c. The motion of the oscillator is given by its displacement vector $\mathbf{r}_s$.



m_s = mass of oscillator
M = total mass of satellite including m_s

$\boldsymbol{\ell}_s = \mathbf{r}_{os} + \mathbf{r}_s$

The body-fixed reference frame origin coincides with the center of mass of the satellite for $\mathbf{r}_s = 0$. The location of the center of mass of the satellite for a displaced oscillator mass is:

$$M\,\boldsymbol{\ell}_c = \int \mathbf{R}dm = m_s\mathbf{r}_s$$

Translation:

The translational Equation 6.31 contains the following terms:

$$\int [\dot{\mathbf{v}}_0 + (\boldsymbol{\omega} \times \mathbf{v}_0)]dm = M(\dot{\mathbf{v}}_0 + \boldsymbol{\omega} \times \mathbf{v}_0)$$

$$\int \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})dm = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int \mathbf{R}dm) = M\ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\ell}_c)$$

$$\int (\dot{\boldsymbol{\omega}} \times \mathbf{R})dm = \dot{\boldsymbol{\omega}} \times \int \mathbf{R}dm = M(\dot{\boldsymbol{\omega}} \times \boldsymbol{\ell}_c)$$

$$2\int (\boldsymbol{\omega} \times \dot{\mathbf{r}}_s)dm = 2m_s(\boldsymbol{\omega} \times \dot{\mathbf{r}}_s)$$

$$\int \ddot{\mathbf{r}}_s dm = m_s \ddot{\mathbf{r}}_s$$

Combining these terms yields:

$$\begin{aligned} M(\dot{\mathbf{v}}_0 + \boldsymbol{\omega} \times \mathbf{v}_0) + M\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\ell}_c) + M(\dot{\boldsymbol{\omega}} \times \boldsymbol{\ell}_c) \\ +2\ m_s\ (\boldsymbol{\omega} \times \dot{\mathbf{r}}_s) + m_s\ \ddot{\mathbf{r}}_s = \mathbf{F} \end{aligned}$$

Rotation:

The rotational Equation 6.32 contains the following terms:

$$\int \mathbf{R} \times (\dot{\mathbf{v}}_0 + \boldsymbol{\omega} \times \mathbf{v}_0)dm = M[\boldsymbol{\ell}_c \times (\dot{\mathbf{v}}_0 + \boldsymbol{\omega} \times \mathbf{v}_0)]$$

$$\int \mathbf{R} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})]dm = \boldsymbol{\omega} \times \mathcal{F} \cdot \boldsymbol{\omega}$$

$$\int \mathbf{R} \times (\dot{\boldsymbol{\omega}} \times \mathbf{R})dm = \mathcal{F} \cdot \dot{\boldsymbol{\omega}}$$

$$2\int \mathbf{R} \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_s)dm = 2\boldsymbol{\ell}_s \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_s)m_s$$

$$\int \mathbf{R} \times \ddot{\mathbf{r}}_s dm = (\boldsymbol{\ell}_s \times \ddot{\mathbf{r}}_s) m_s$$

Combining these terms yields:

$$\begin{aligned} &\mathcal{I} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathcal{I} \cdot \boldsymbol{\omega} + M[\boldsymbol{\ell}_c \times (\dot{\mathbf{v}}_0 + \boldsymbol{\omega} \times \mathbf{v}_0)] \\ &+2m_s \boldsymbol{\ell}_s \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_s) + m_s(\boldsymbol{\ell}_s \times \ddot{\mathbf{r}}_s) = \mathbf{R} \times \mathbf{F} \end{aligned}$$

The effect of the acceleration of the origin of the body fixed reference frame can be eliminated from the above equation by substituting the equation before it into it:

$$\begin{aligned} &\mathcal{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathcal{I} \cdot \boldsymbol{\omega} + \boldsymbol{\ell}_c \times [\mathbf{F} - M\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\ell}_c) - M(\dot{\boldsymbol{\omega}} \times \boldsymbol{\ell}_c) - 2m_s \\ &(\boldsymbol{\omega} \times \dot{\mathbf{r}}_s) - m_s \ddot{\mathbf{r}}_s] + 2m_s \boldsymbol{\ell}_s \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_s) + m_s(\boldsymbol{\ell}_s \times \ddot{\mathbf{r}}_s) = \mathbf{R} \times \mathbf{F} \end{aligned}$$

If we define the dyadic:

$$\mathcal{I}_2 = M[\boldsymbol{\ell}_c^2 \mathcal{F} - (\boldsymbol{\ell}_c \, \boldsymbol{\ell}_c)]$$

the moment of inertia about the center of mass of the satellite can be written as:

$$\mathcal{I}_c = \mathcal{I} - \mathcal{I}_2$$

The rotational motion equation is then

$$\begin{aligned} &\mathcal{I}_c \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathcal{I}_c \cdot \boldsymbol{\omega} + 2m_s(\boldsymbol{\ell}_s - \boldsymbol{\ell}_c) \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_s) \\ &+m_s(\boldsymbol{\ell}_s - \boldsymbol{\ell}_c) \times \ddot{\mathbf{r}}_s = (\mathbf{R} - \boldsymbol{\ell}_c) \times \mathbf{F} \end{aligned}$$

Flexibility:

The motion of the linear oscillator can be written in the form as:

$$\boldsymbol{\ell}_s = \mathbf{r}_{0s} + \mathbf{e}_s \, r_s \qquad \mathbf{e}_s = \text{unit vector}$$

It is seen that the unit vector $\mathbf{e}_s$ can be looked upon as the vector mode function of the linear oscillator and its displacement r_s as the associated generalized coordinate.

The flexibility equation (6.39) contains, therefore, the following terms:

$$\mathbf{e}_s \cdot [\dot{\mathbf{v}}_0 + (\boldsymbol{\omega} \times \mathbf{v}_0)]m_s$$

$$\mathbf{e}_s \cdot [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\ell}_s)]m_s$$

$$\mathbf{e}_s \cdot (\dot{\boldsymbol{\omega}} \times \boldsymbol{\ell}_s)m_s$$

$$2\mathbf{e}_s \cdot (\boldsymbol{\omega} \times \dot{\mathbf{r}}_s)m_s$$

$$\mathbf{e}_s \cdot \ddot{\mathbf{r}}_s m_s$$

Collecting these terms yields:

$$\mathbf{e}_s \cdot \{[\dot{\mathbf{v}}_0 + (\boldsymbol{\omega} \times \mathbf{v}_0)] + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\ell}_s) + (\dot{\boldsymbol{\omega}} \times \boldsymbol{\ell}_s)$$
$$+2(\boldsymbol{\omega} \times \dot{\mathbf{r}}_s) + \ddot{\mathbf{r}}_s\}\, m_s = -k\, r_s - c\, \dot{r}_s$$

Substituting the translational equation eliminates the effect of the acceleration of the origin:

$$\mathbf{e}_s \cdot \left\{ [\frac{\mathbf{F}M}{-} \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\ell}_c) - (\dot{\boldsymbol{\omega}} \times \boldsymbol{\ell}_c) - \frac{2m_s}{M}(\boldsymbol{\omega} \times \dot{\mathbf{r}}_s) \right.$$
$$\left. -\frac{m_s}{M}\ddot{\mathbf{r}}_s] + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\ell}_s) + (\dot{\boldsymbol{\omega}} \times \boldsymbol{\ell}_s) + 2(\boldsymbol{\omega} \times \dot{\mathbf{r}}_s) + \ddot{\mathbf{r}}_s \right\} m_s = -k\, r_s - c\dot{\mathbf{r}}_s$$

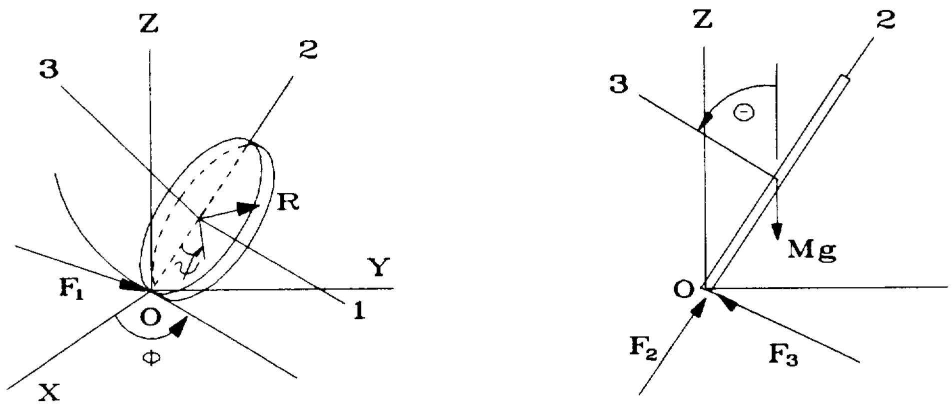
Observing furthermore that $\mathbf{e}_s \cdot (\boldsymbol{\omega} \times \dot{\mathbf{r}}_s) = 0$ we can rewrite this equation in the final form:

$$\mathbf{e}_s \cdot \left\{ (1 - \frac{m_s}{M})\ddot{\mathbf{r}}_s + \dot{\boldsymbol{\omega}} \times (\boldsymbol{\ell}_s - \boldsymbol{\ell}_c) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\boldsymbol{\ell}_s - \boldsymbol{\ell}_c)] \right\}$$
$$= -\frac{m_s}{M}\mathbf{e}_s \cdot \mathbf{F} - \frac{k}{m_s} r_s - \frac{c}{m_s}\dot{r}_s$$

The above equations of motion representing three differential equations of first order and one second-order differential equation together with the appropriate kinematical equations determine the dynamics of the system.

Example 4: Rolling Coin

A classical example of a nonholonomic system is the coin rolling on a rough horizontal plane.



We choose a coordinate system 1, 2, 3 with the origin at the mass center of the coin. The 3-axis is the axis of symmetry; the 1-axis is in the plane of the coin and remains horizontal. It is important to notice that this coordinate system is not a body-fixed system but is "floating" relative to the body. Therefore, the angular velocity of the coin is different from that of the floating coordinate system.

Angular Velocity of Coin:

$$\boldsymbol{\Omega} = \boldsymbol{\omega} + \boldsymbol{\Omega}_0$$

$$\text{where } \boldsymbol{\Omega}_0 = \dot{\psi}\mathbf{e}_3 \qquad (6.46)$$

and $\boldsymbol{\omega}$ the angular velocity of the coordinate system.

Roll Condition:

$$\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{R}_0 \quad \text{where } \mathbf{R}_0 = R\mathbf{e}_2 \tag{6.47}$$

$\mathbf{R}_0$ is the position vector drawn from contact point O to the mass center of the coin, and $\mathbf{v}$ is the velocity of the mass center.

The velocity components for the roll condition are therefore:

$$v_1 = -\boldsymbol{\Omega}_3 R, \quad v_2 = 0, \quad v_3 = \boldsymbol{\Omega}_1 R \tag{6.48}$$

For calculating the constraint forces we need the instantaneous constraints of the system. They are obtained by performing virtual displacements of the coordinate system in conformity with the kinematical constraints which apply at that instant of time.

Constraint Condition:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R}_0 \tag{6.49}$$

or in component form:

$$v_1 = -\omega_3 R, \quad v_2 = 0, \quad v_3 = \omega_1 R \tag{6.50}$$

Notice that the constraint condition (6.50) is different from the roll condition (6.48) for the coin.

In matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & R \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -R & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = 0 \tag{6.51}$$

Constraint Forces:

$$F^R = B^T \boldsymbol{\lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -R \\ 0 & 0 & 0 \\ R & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ -R\lambda_3 \\ 0 \\ R\lambda_3 \end{bmatrix}$$

Kinetic Energy:

$$T = \frac{1}{2}\int (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{R})^2 dm \qquad \text{where} \quad \int \mathbf{R} dm = 0$$

Equations of Motion:

$$\frac{d}{dt}(\frac{\partial T}{\partial \boldsymbol{\omega}}) + \boldsymbol{\omega} \times \frac{\partial T}{\partial \boldsymbol{\omega}} + \mathbf{v} \times \frac{\partial T}{\partial \mathbf{v}} = \mathbf{L} + \mathbf{L}_R$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \mathbf{v}}) + \boldsymbol{\omega} \times \frac{\partial T}{\partial \mathbf{v}} = \mathbf{F} + \mathbf{F}_R$$

Because of $\int \mathbf{R} dm = 0$ $\qquad \mathbf{v} \times \frac{\partial T}{\partial \mathbf{v}} = 0$

The generalized forces $\mathbf{L}$ and $\mathbf{F}$ in the above equations are obtained by the virtual work done by the external gravity force when virtual translations and rotations about the mass center of the coin are performed.

We find that $\mathbf{L} = 0$ and $\mathbf{F}^T = [0, -mg\sin\theta, -mg\cos\theta]$.

The equations of motion in component form are then:

Translation:

$$m\ \dot{v}_1 + m(\omega_2 v_3 - \omega_3 v_2) = 0 + \lambda_1$$

$$m\ \dot{v}_2 + m(\omega_3 v_1 - \omega_1 v_3) = -mg\sin\theta + \lambda_2$$

$$m\ \dot{v}_3 + m(\omega_1 v_2 - \omega_2 v_1) = -mg\cos\theta + \lambda_3$$

Rotation: $(I_1 = I_2 = A, I_3 = C)$

$$A\ \dot{\Omega}_1 + C\omega_2\Omega_3 - A\omega_3\Omega_2 = -R\lambda_3$$

$$A\,\dot{\Omega}_2 + A\omega_3\Omega_1 - C\omega_1\Omega_3 = 0 \tag{6.52}$$

$$C\,\dot{\Omega}_3 + A\omega_1\Omega_2 - A\omega_2\Omega_1 = -R\lambda_3$$

Combining Equations 6.51 and 6.52 we obtain:

$$(A + mR^2)\dot{\Omega}_1 + (C + mR^2)\omega_2\Omega_3 - A\omega_3\Omega_2 = -mgR\cos\theta$$

$$A\dot{\Omega}_2 + A\omega_3\Omega_1 - C\omega_1\Omega_3 = 0 \tag{6.53}$$

$$(c + mR^2)\dot{\Omega}_3 + A\omega_1\Omega_2 - (A + mR^2)\omega_2\Omega_1 = 0$$

This equation is the moment equation about the contact point O.

To complete the formulation of the problem we write the angular velocity ω of the floating coordinate system in terms of Euler angles as

$$\omega_1 = \dot{\theta}, \quad \omega_2 = \dot{\phi}\sin\theta, \quad \omega_3 = \dot{\phi}\cos\theta$$

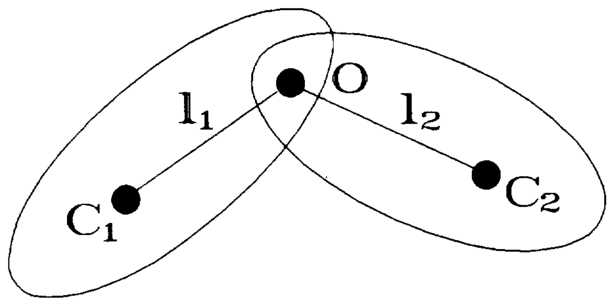
Likewise the angular velocity $\boldsymbol{\Omega}$ of the body as:

$$\Omega_1 = \dot{\theta}, \quad \Omega_2 = \dot{\phi}\sin\theta, \quad \Omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

A closed-form solution is only possible for certain special cases.

Example 5:

Consider two rigid bodies connected by a frictionless hinge in free space.



We select 2 principal axes systems each located at the mass centers of each body. The angular velocities of each reference frame against inertial space are $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$ respectively, their linear velocities are $\mathbf{v}_1$ and $\mathbf{v}_2$. The distance from C_1 to hinge point O is $\boldsymbol{\ell}_1$ and that from O to C_2 is $\boldsymbol{\ell}_2$.

The equations of translation and rotation for each body are then

$$M_1(\dot{\mathbf{v}}_1 + \boldsymbol{\Omega}_1 \times \mathbf{v}_1) = \mathbf{F}_1 + \mathbf{F}_1^{(R)} \tag{6.54}$$

$$\mathcal{I}_1 \cdot \dot{\boldsymbol{\Omega}}_1 + \boldsymbol{\Omega}_1 \times \mathcal{I}_1 \cdot \boldsymbol{\Omega}_1 = \mathbf{L}_1 + \mathbf{L}_1^{(R)} \tag{6.55}$$

$$M_2(\dot{\mathbf{v}}_2 + \boldsymbol{\Omega}_2 \times \mathbf{v}_2) = \mathbf{F}_2 + \mathbf{F}_2^{(R)} \tag{6.56}$$

$$\mathcal{I}_2 \cdot \dot{\boldsymbol{\Omega}}_2 + \boldsymbol{\Omega}_2 \times \mathcal{I}_2 \cdot \boldsymbol{\Omega}_2 = \mathbf{L}_2 + \mathbf{L}_2^{(R)} \tag{6.57}$$

The constraint equation imposed upon the two bodies by the hinge is:

$$\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\Omega}_1 \times \boldsymbol{\ell}_1 + \boldsymbol{\Omega}_2 \times \boldsymbol{\ell}_2 \tag{6.58}$$

The constraint equation has to be written in matrix form to find the constraint matrix B. It is:

$$\mathbf{v}_2 - A\mathbf{v}_1 - A\tilde{\boldsymbol{\ell}}_1\boldsymbol{\Omega}_1 - \tilde{\boldsymbol{\ell}}_2 - \boldsymbol{\Omega}_2 = 0$$

where A is the direction cosine matrix of body 2 relative to body 1 and $\tilde{\ell}$ the familiar skew-symmetric matrix. Introducing the vector:

$$\boldsymbol{\Omega}^T = [\mathbf{v}_1|\boldsymbol{\Omega}_1|\mathbf{v}_2|\boldsymbol{\Omega}_2]$$

we can write the constraint equation in the desired matrix form as:

$$B\boldsymbol{\Omega} = [-A| - A\tilde{\ell}_1|E_3|\tilde{\ell}_2]\boldsymbol{\Omega} = 0$$

The equations of motion are in matrix form:

$$M\dot{\boldsymbol{\Omega}} = \mathbf{Q}^{(I)} + \mathbf{Q}^{(A)} + \mathbf{Q}^{(R)}$$

where M is the diagonal matrix:

$$M = \begin{bmatrix} M_1 & & & 0 \\ & I_1 & & \\ & & M_2 & \\ 0 & & & I_2 \end{bmatrix}$$

and $\mathbf{Q}^{(I)}, \mathbf{Q}^{(A)}, \mathbf{Q}^{(R)}$ are the associated generalized inertial, applied and reaction forces. The latter can be calculated by the same method outlined before. We obtain:

$$\mathbf{Q}^{(R)} = -B^T(BM^{-1}B^T)^{-1}\left\{BM^{-1}(\mathbf{Q}^{(I)} + \mathbf{Q}^{(A)}) + \dot{B}\boldsymbol{\Omega}\right\}$$

The time derivative of the constraint matrix B can be obtained by noting that:

$$\dot{A} = -\tilde{\omega}A$$

where $\tilde{\omega} = \tilde{\boldsymbol{\Omega}}_2 - A\tilde{\Omega}_1 A^T$. It is therefore given by:

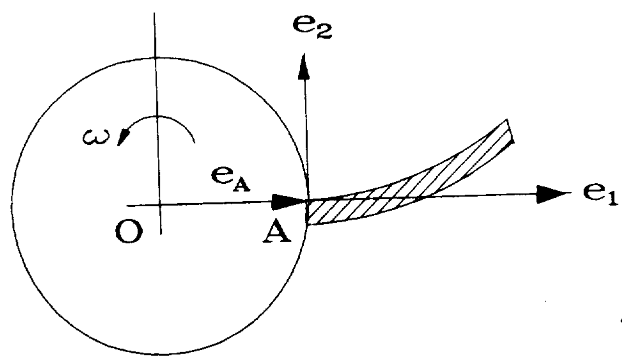
$$\dot{B} = [\tilde{\omega}A|\tilde{\omega}A\tilde{\ell}|0|0]$$

N.B.

It is important to notice that the translational Equation 6.56 is only needed to calculate the constraint forces $\mathbf{Q}^{(R)}$ and does not have to be integrated, because the translational velocity v_2 can be directly obtained from the constraint Equation 6.58. Therefore it is only necessary to integrate as many equations of motion as there are degrees of freedom, namely nine.

Example 6:

A spinning satellite considered to be a rigid body has flexible antennas attached to it.



The antennas are treated as uniform cantilever (clamped-free) beams. The differential equation of a uniform beam is:

$$EI\frac{\partial^4 w}{\partial x^4} + m\,\frac{\partial^2 w}{\partial t^2} = 0$$

where EI is the flexural stiffness and m the mass per unit length of the beam. At the clamped end $x = 0$ we have the geometric boundary conditions:

$$w(0) = w'(0) = 0$$

At the free end ℓ, we have the dynamical (natural) boundary conditions of vanishing moment and shear force. This means that:

$$w''(\ell) = w'''(\ell) = 0$$

The differential equation and its associated boundary conditions furnish the normal modes $\phi_m(x)$ to be used in describing the total deflection of the antennas as a superposition of these mode shapes each multiplied by a generalized coordinate. The transverse deflections of each antenna are expressed relative to a body fixed frame

($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) whose origin coincides with the attachment point A of the antenna. The position of a mass element of the antenna is defined as:

$$\mathbf{r}_F = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

According to the model synthesis method, the "in-plane" deflection y and the "out-of-plane" deflection z are given in the series form:

$$y = \sum Y_i(x)\eta_i(t)$$
$$z = \sum Z_k(x)\xi_k(t)$$

To define the location of the antenna mass element relative to the rigid body satellite we select a satellite-fixed reference frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ with origin at the mass center of the undeformed body. For the subsequent discussion it is assumed that the shift of mass center location caused by the antenna deflections is neglible. As a consequence the translational and rotational motion become dynamically uncoupled. To evaluate the integrals of the equations of motion we need the following quantities:

$$\mathbf{R}_F = \boldsymbol{\ell}_A + \mathbf{r}_F \qquad \text{location of antenna mass element}$$

$$\mathbf{v}_F = \dot{y}\mathbf{e}_2 + \dot{z}\mathbf{e}_3 = \sum Y_i(x)\dot{\eta}_i(t)\mathbf{e}_2 + \sum Z_k(x)\dot{\xi}_k(t)\mathbf{e}_3$$

$$\mathbf{a}_F = \ddot{y}\mathbf{e}_2 + \ddot{z}\mathbf{e}_3 = \sum Y_i(x)\ddot{\eta}_i(t)\mathbf{e}_2 + \sum Z_k(x)\ddot{\xi}_k(t)\mathbf{e}_3$$

$$\ddot{\mathbf{R}}_A = \ddot{R}_0 + (\dot{\boldsymbol{\Omega}} \times \boldsymbol{\ell}_A) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\ell}_A)$$

where $\ddot{\mathbf{R}}_A$ = acceleration of attachment point A. $\ddot{\mathbf{R}}_0$ = acceleration of satellite frame origin.

The following equations of motion can then be established:

Rotation:

$$\int \mathbf{R} \times [\dot{\boldsymbol{\Omega}} \times \mathbf{R} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}) + 2(\boldsymbol{\Omega} \times \mathbf{v}_F) + \mathbf{a}_F]d\mathbf{m} = \mathbf{L}$$

or

$$\mathcal{I} \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathcal{I} \cdot \boldsymbol{\Omega} + 2\int \mathbf{R}_F \times (\boldsymbol{\Omega} \times \mathbf{v}_F) dm_F + \int (\mathbf{R}_F \times \mathbf{a}_F) dm_F = \mathbf{L}$$

where dm_F is a mass element of the flexible antenna.

Flexibility:

$$\mathbf{e}_2 \cdot \int Y_i(x)[\ddot{\mathbf{R}}_A + (\dot{\boldsymbol{\Omega}} \times \mathbf{R}_F) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}_F) + 2(\boldsymbol{\Omega} \times \mathbf{v}_F) + \mathbf{a}_F] dm_F = Q_i$$

$$\mathbf{e}_3 \cdot \int Z_k(x)[\ddot{\mathbf{R}}_A + (\dot{\boldsymbol{\Omega}} \times \mathbf{R}_F) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}_F) + 2(\boldsymbol{\Omega} \times \mathbf{v}_F) + \mathbf{a}_F] dm_F = Q_k$$

The evaluation of the last two integrals is greatly simplified by observing that:

$$Y_i(x) = Z_i(x)$$

$$\int Y_i(x) Y_j(x) dm_F = \delta_{ij} \quad \text{orthogonality}$$

$$\int Z_i(x) Z_j(x) dm_F = \delta_{ij}$$

The acceleration $\ddot{\mathbf{R}}_0$ of the satellite origin is usually caused by thruster firings performed for attitude control or spin-up/spin down maneuvers.

Appendix A

The Lagrange Multiplier Method

The Lagrange Multiplier Rule finds application in determining extremes of a function of several variables subject to constraints.

a) Special Case:

$$f = f(x, y, z) \tag{A.1}$$

Constraint: $\phi(x, y, z) = 0$

Necessary Condition for Extremum:

$$df = f_x dx + f_y dy + f_z dz = 0 \tag{A.2}$$

From the constraint equation, we obtain:

$$d\phi = \phi_x dx + \phi_y dy + \phi_z dz = 0 \tag{A.3}$$

Elimination of dz yields:

$$dz = -(\frac{\phi_x}{\phi_z})dx - (\frac{\phi_y}{\phi_z})dy \tag{A.4}$$

Inserting Equation A.4 into Equation A.2 leads to:

$$df = f_x dx + f_y dy - \frac{f_z}{\phi_z}(\phi_x dx + \phi_y dy) = (f_x - \frac{f_z}{\phi_z})dx + (f_y - \frac{f_z}{\phi_z})dy = 0 \quad \text{(A.5)}$$

Since dx and dy are independent:

$$f_x - (\frac{f_z}{\phi_z})\phi_x = 0 \quad f_y - (\frac{f_z}{\phi_z})\phi_j = 0 \quad \text{(A.6)}$$

Define the Lagrange Multiplier λ as:

$$\lambda \equiv -f_z/\phi_z \qquad (\phi_z \neq 0) \quad \text{(A.7)}$$

Then we obtain as necessary conditions for extrema:

$$f_x + \lambda\phi_x = 0 \quad f_y + \lambda\phi_y = 0 \quad f_z + \lambda\phi_z = 0 \quad \text{(A.8)}$$

Despite their identical outward appearance, it is important to realize that their origin is quite different. The last equation holds because we have selected λ to make it true, whereas the first two equations hold because of the independence of the associated variables x and y.

The Lagrange Multiplier Rule arrives at the same result by introducing an augmented function f^* such that

$$f^*(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

The necessary conditions for the extrema of a function subject to constraints can then be formulated as:

$$f_x^* = 0, \quad f_y^* = 0, \quad f_z^* = 0$$

b) General Case:

The foregoing technique can be readily extended to the general case of n variables.

$$f = f(x_1, x_2, \ldots, x_n) \qquad m[n$$
$$\phi_m = \phi_m(x_1, x_2, \ldots, x_n)$$

Define the augmented function

$$f^* = f + \sum_{i=1}^{m} \lambda_i \phi_i$$

Necessary Conditions:

$$\frac{\partial f^*}{\partial x_k} = \frac{\partial f}{\partial x_k} + \sum_{i=1}^{m} \lambda_i \frac{\partial \phi_i}{\partial x_k} = 0 \qquad k = 1, 2, \ldots, n$$

Using Matrix Notation:

Define: $\boldsymbol{\lambda}^T = [\lambda_1, \lambda_2, \ldots, \lambda_m]$

Jacobian:

$$J \equiv \begin{bmatrix} \partial\phi_1/\partial x_1 & \partial\phi_1/\partial x_2 \ldots & \partial\phi_1/\partial x_n \\ \partial\phi_2/\partial x_1 & \partial\phi_2/\partial x_2 \ldots & \partial\phi_2/\partial x_n \\ \ldots & \ldots & \ldots \\ \partial\phi_m/\partial x_1 & \partial\phi_m/\partial x_2 \ldots & \partial\phi_m/\partial x_m \end{bmatrix}$$

$$\frac{\partial f^*}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{x}} + J^T \boldsymbol{\lambda} = 0 \qquad \text{Rank } J = m$$

There are $n + m$ unknowns:

$$x_1, x_2, \ldots, x_n$$
$$\lambda_1, \lambda_2, \ldots, \lambda_m$$

There are $n + m$ equations:

n necessary conditions

m constraint conditions

The advantage of the Lagrange Multiplier Rule is the symmetry of the formulation and avoiding the awkward elimination process of the variables.

Example 1:

Find the dimensions of a cylindrical can of maximum volume for a given surface area:

a) Elimination Method:

$$V = \pi\, r^2\, h \qquad S = 2\pi(r^2 + rh) = \text{CONSTANT}$$

$$dV = \pi(2rhdr + r^2dh) = 0 \tag{A.9}$$

$$dS = 2\pi(2rdr + hdr + rdh) = 0 = 2\pi[(2r + h)dr + rdh] = 0 \tag{A.10}$$

Eliminate dh: $dh = -\frac{2h}{r}dr$

Inserting in Equation A.10: $(2r + h)dr - 2hdr = 0 \quad \underline{h = 2r}$

b) Lagrange Multiplier Rule:

$$V^* = r^2h + \lambda(r^2 + rh)$$

$$\frac{\partial V^*}{\partial r} = 2rh + \lambda(2r + h) = 0 \tag{A.11}$$

$$\frac{\partial V^*}{\partial h} = r^2 + \lambda r = 0 \rightarrow \lambda = -r \tag{A.12}$$

Inserting λ in Equation A.11:

$$2rh - r(2r + h) = 0 \rightarrow rh = 2r^2 \quad \underline{h = 2r}$$

Example 2:

Find the dimensions of a rectangular box without a top, of maximum capacity, whose surface is s = 108 m^2.

$$f(x, y, z) = xyz$$

$$\phi(x, y, z) = xy + 2xz + 2yz - 108 = 0$$

$$f^* = xyz + \lambda(xy + 2xz + 2yz - 108)$$

$$\frac{\partial f^*}{\partial x} = yz + \lambda(y + 2z) = 0 \tag{A.13}$$

$$\frac{\partial f^*}{\partial y} = xz + \lambda(x + 2z) = 0 \tag{A.14}$$

$$\frac{\partial f^*}{\partial z} - xy + \lambda(2x + 2y) = 0 \tag{A.15}$$

Multiply Equation A.13 by x, Equation A.14 by y and Equation A.15 by z and add:

$$x(xy + 2xz + 2yz) + \frac{3}{2}xyz = 0$$

Using the constraint equation we find:

$$108\lambda + \frac{3}{2}xyz = 0 \rightarrow \lambda = -\frac{xyz}{72}$$

Inserting the Lagrange Multiplier λ in (A.13), (A.14), and (A.15)

$$1 - \frac{x}{72}(y + 2z) = 0 \quad \rightarrow x = y$$

$$1 - \frac{y}{72}(x + 2z) = 0$$

$$1 - \frac{z}{72}(2x + 2y) = 0 \quad \rightarrow z = \frac{18}{y}$$

$x = 6 \quad y = 6 \quad z = 3$

Example 3:

Find the minimum distance of a point in a plane from the origin.

$$F(x, y, z) = x^2 + y^2 + z^2 \quad \text{Distance} \tag{A.16}$$

$$G(x, y, z) = Ax + By + Cz + D = 0 \quad \text{Plane} \tag{A.17}$$

Augmented Function: $F^* = F + 2\lambda\theta$

$$\frac{\partial F^*}{\partial x} = 2x + 2\lambda A = 0 \tag{A.18}$$

$$\frac{\partial F^*}{\partial y} = 2y + 2\lambda B = 0 \tag{A.19}$$

$$\frac{\partial F^*}{\partial z} = 2z + 2\lambda C = 0 \tag{A.20}$$

Multiply Equations A.18, A.19 and A.20 by A, B, C respectively, and add:

$$\lambda = -D/(A^2 + B^2 + C^2) \tag{A.21}$$

Insert in Equations A.18, A.19 and A.20:

$$x_M = -AD/(A^2 + B^2 + C^2) \tag{A.22}$$

$$y_M = -BD/(A^2 + B^2 + C^2) \tag{A.23}$$

$$z_M = -CD/(A^2 + B^2 + C^2) \tag{A.24}$$

For minimum distance insert Equations A.22, A.23 and A.24 in Equation A.16:

$$d_M = D/\sqrt{(A^2 + B^2 + C^2)}$$

Example 4:

Find the minimum and maximum distance from the origin to the ellipse.

$$\phi(x, y) = 5x^2 + 6xy + 5y^2 - 8 = 0 \tag{A.25}$$

$$f(x, y) = x^2 + y^2 \tag{A.26}$$

Lagrange Multiplier Rule:

$$2x + 2\lambda(5x + 3y) = 0 \tag{A.27}$$

$$2y + 2\lambda(3x + 5y) = 0 \tag{A.28}$$

For convenience replace λ by $-\frac{1}{\lambda}$ and divide by 2:

$$(5 - \lambda)x + 3y = 0 \tag{A.29}$$

$$3x + (5 - \lambda)y = 0 \quad \text{Eigenvalue problem} \tag{A.30}$$

Solve for λ: $\lambda_1 = 2$ $\lambda_2 = 8$

Inserting in Equation A.29 or A.30 yields:

$y_1 = x_1$ and $y_2 = -x_2 \leftarrow$ Insert in Equation A.25

From Equation A.26 one obtains then the minimum and maximum distances:

$$d_1 = \sqrt{x_1^2 + y_1^2} = \sqrt{1/2 + 1/2} = 1$$
$$d_2 = \sqrt{x_2^2 + y_2^2} = \sqrt{2 + 2} = 2$$

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16. Abstract

The monograph was prepared to give the practicing engineer a clear understanding of dynamics with special consideration given to the dynamic analysis of aerospace systems. It is conceived to be both a desk-top reference and a refresher for aerospace engineers in government and industry. It could also be used as a supplement to standard texts for in-house training courses on the subject.

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