Chapter 10

INCOHERENT SCATTER RADAR OBSERVATIONS OF THE IONOSPHERE

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1. Introduction

Incoherent scatter radar (ISR) has become the most powerful means of studying the ionosphere from the ground. Many of the ideas and methods underlying the troposphere and stratosphere (ST) radars have been taken over from ISR. Whereas the theory of refractive index fluctuations in the lower atmosphere, depending as it does on turbulence, is poorly understood, the theory of the refractivity fluctuations in the ionosphere, which depend on thermal fluctuations, is known in great detail. The underlying theory is one of the most successful theories in plasma physics, and allows for many detailed investigations of a number of parameters such as electron density n_o, electron temperature T_e, ion temperature T_i, electron mean velocity v_e, ion mean velocity V_i as well as parameters pertaining to composition, neutral density and others.

Here we shall review the fundamental processes involved in the scattering from a plasma undergoing thermal or near thermal fluctuations in density. We shall relate the fundamental scattering properties of the plasma to the physical parameters characterizing it from first principles. We shall not discuss the observation process itself, as the observational principles are quite similar whether they are applied to a neutral gas or a fluctuating plasma. These observational principles are dealt with in other ISAR presentations.

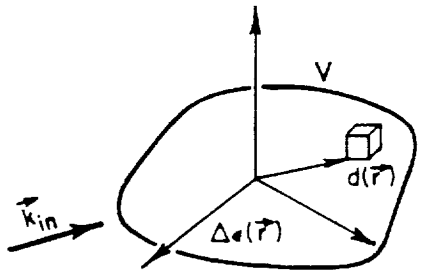

Figure 1. Volume element $d(\bar{r})$ in volume V illuminated by plane wave.

2. Volume Scattering from Random Irregularities. Continuum

In the neutral atmosphere the scattering is derived from the following consideration:

With an incoming wave: $\vec{E}_o e^{-i\bar{k}_{in}\cdot\bar{r}}$ [note that exp $(i\omega_o t)$ is understood] and a dielectric constant in medium:

$$\varepsilon = \varepsilon_o + \Delta\varepsilon(\bar{r}),$$

the polarization of the medium becomes:

$$\vec{P}(\vec{r}) = \Delta\varepsilon(\vec{r}) \cdot \vec{E}_o \cdot e^{-i\vec{k}_{in}\cdot\vec{r}} \tag{2}$$

This oscillating polarization acts as an equivalent current:

$$\vec{j}(\vec{r}) = i\omega_o \vec{P}(\vec{r}) = i\omega_o \, \Delta\varepsilon \vec{E}_o \cdot e^{-i\vec{k}_{in}\cdot\vec{r}}, \tag{3}$$

and the vector potential at a point far from volume V is:

$$\vec{A}(\vec{r}) \simeq \frac{\mu_0}{4\pi} \cdot \frac{i\omega_o \vec{E}_o}{|r|} \int \Delta\varepsilon(\vec{r}')e^{-i\vec{k}_{in}\cdot\vec{r}'} \cdot e^{-ik|\vec{r}-\vec{r}'|} d(\vec{r}') \tag{4}$$

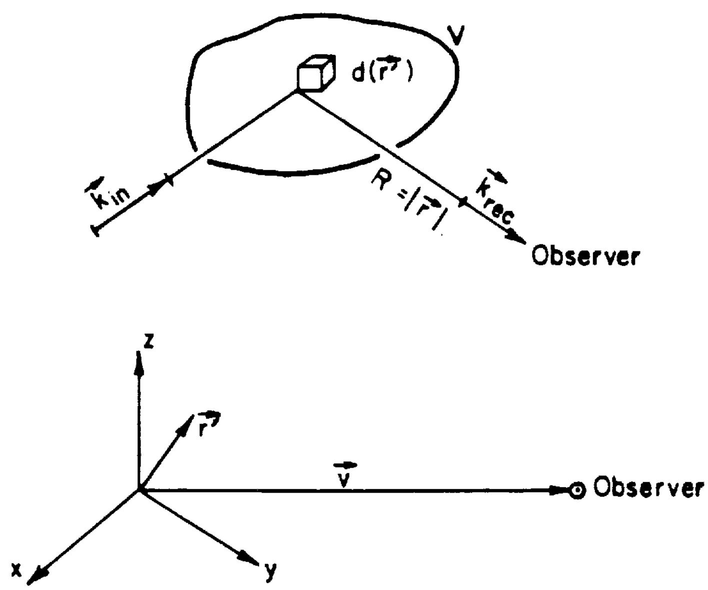

Figure 2. Scattering geometry to calculate vector potential at the observer.

with $|\vec{r} - \vec{r}'| \approx |r| - \vec{n}_1 \cdot \vec{r}'$, where $\vec{n}_1 \approx \frac{\vec{k}_{rec}}{k}$

We obtain:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \cdot \frac{i\omega_o \vec{E}_o}{|\vec{r}|} \cdot e^{-ik\cdot|r|} \int_V \underbrace{\Delta\varepsilon(\vec{r}') \cdot e^{-i\vec{r}'\cdot(\vec{k}_{in}-\vec{k}_{rec})} d(\vec{r}')}_{\Delta\varepsilon(\vec{k}_{in}-\vec{k}_{rec})} \tag{5}$$

which means that the received field depends on the spatial Fourier component with wave vector $\vec{k} = \vec{k}_{in} - \vec{k}_{rec}$.

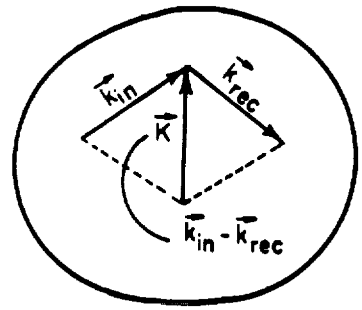

Figure 3. Relation between the wave vectors of the transmitted and revised waves and the spatial Fourier component of the dielectric fluctuation.

It is easy to show from this that for a plane incoming wave of flux (Poynting vector magnitude) S_{in}, the flux at the receiver is:

$$S_{rec} = \frac{\omega_o^4 \mu_0^2}{(4\pi)^2 R_1^2} sin^2\chi \, |\Delta\varepsilon(\vec{k}_{in} - \vec{k}_{rec})|^2 \, S_{in} \qquad (6)$$

$\mu_0 = 4\pi \cdot 10^{-7}$ Henry/m

$sin\chi$ = polarization factor, see next section.

R_1 = distance between scattering volume and observer

In a plasma

$$\Delta\varepsilon(\vec{r}) = -\frac{\Delta n(\vec{r}) \cdot e^2}{m \cdot \omega_o^2} \qquad (7)$$

Where:

$\Delta n(\vec{r})$ = electron density fluctuation
$e = 1.602 \cdot 10^{-19}$ Coulomb = elementary charge
$m = 9.110 \cdot 10^{-31} kg$ = electronic mass

and one obtains by substitution:

$$S_{rec} = \left(\frac{\omega_o}{R_1}\right)^2 |\Delta n(\vec{k}_{in} - \vec{k}_{rec})|^2 \cdot sin^2\chi \cdot S_{in} \qquad (8)$$

where r_o is the classical radius of the electron defined by

$$r_o = \frac{e^2}{4\pi mc^2 \cdot \varepsilon_o}$$

$$\varepsilon_o = 8.854 \cdot 10^{-12}\, F/m$$

$$c = 2.998 \cdot 10^8\, m/s$$

3. Scattering from Individual Electrons, Discrete Particles

Let us recapitulate the derivation of the scattering by a free electron of electromagnetic waves impinging on it. It was originally thought that the scattering from a plasma could be considered a super-position of scattering from individual free electrons and that the strength and spectral broadening can be used to determine density and electron temperature. For the ionospheric plasma and for the frequency used in such scatter experiments, it turned out that the scattering could not usually be considered that simply and that the actual situation was more favorable from an experimental point of view. Nevertheless, even with the more complex theory of interacting electrons the scattering from an individual free electron forms an important and essential ingredient.

Assume as in Section 2 that the electric field set up at the position occupied by the electron is:

$$\vec{E}(t) = \vec{E}_o \cdot e^{+i\omega_o t} \tag{9}$$

where E_o is a complex electric field amplitude which allows for an arbitrary polarization. When we assume that $\omega_o >> \Omega_e$, where Ω_e is the angular gyrofrequency of an electron, the equation of motion of the electron becomes:

$$-\vec{E}_o\, e^{+i\omega_o t} = m \cdot \dot{\vec{v}}/e \tag{10}$$

Solving for $\vec{v}(t)$ with the substitution $\vec{v}(t) = \vec{v}_o\, e^{+i\omega_o t}$ one obtains:

$$\vec{v}_o = +i\, \frac{e}{m\omega_o}\, \vec{E}_o \tag{11}$$

Note that we have neglected the spatial variation of the external electric field and the fact that the electron moves in this field. We have also ignored the force caused by the motion of the electron in the magnetic field of the incoming wave. Both of these effects could contribute to a ponderomotive force which we ignore. We also note that the motion of the electron is considered undamped. This cannot be strictly true since the electron, even without collisions, is re-radiating because of the oscillation and hence, must experience damping.

The current density associated with the motion of this electron becomes:

$$\vec{j}(\vec{r},t) = -e\vec{v}\,(t)\,\delta[\vec{r} - \vec{r}_e(t)] \tag{12}$$

where $\vec{r}_e(t)$ is the position of the electron and where $\delta(\vec{r})$ is a spatial deltafunction. With such an oscillating current at the origin we obtain as in the previous section:

$$\vec{A}(\vec{r},t) = -i\, \frac{\mu_o e^2}{4\pi m\omega_o}\, \vec{E}_o e^{+i(\omega_o t - \vec{k}_{rec}\cdot\vec{r})}\, \frac{1}{R_1} \tag{13}$$

The position of the observer $\vec{r}$ may vary with time due to motion of the electron and we must substitute when necessary:

$$\vec{r} = \vec{r}\,'(t) = \vec{r}\left(t - \frac{|\vec{r}\,'|}{c}\right)$$

where $\vec{r}\,'(t)$ is the radius vector describing the position of the electron relative to the position of the observer at time t.

The Poynting flux vector at the receiver due to the radiation from this electron becomes:

$$S_{rec} = \frac{1}{2}\eta\,|\vec{H}|^2 = \frac{1}{2}\eta\left(\frac{e^2}{4\pi m\omega_o}\right)^2 \frac{|\vec{k}_{rec} \times \vec{E}_o|^2}{R_1^2} \tag{14}$$

where $\eta = \sqrt{\mu_o/\varepsilon_o} = 376.7$ ohms. Introducing the polarization angle χ through

$$sin\chi = \frac{|\vec{k}_{rec} \times \vec{E}_o|}{|\vec{k}_{rec}| \cdot |\vec{E}_o|} \tag{15}$$

and the power density incident on the electron, S_{in} by:

$$S_{in} = |E_o|^2/\eta \cdot \frac{1}{2} \tag{16}$$

we obtain:

$$S_{rec} = (r_o/R_1)^2 \cdot sin^2\chi \cdot S_{in} = 10^{-28}\, sin^2\chi \cdot S_{in} \tag{17}$$

The usual radar cross section σ_s is defined by:

$$\sigma_s = 4\pi R_1^2 \cdot \frac{S_{rec}}{S_{in}} = 4\pi r_o^2\, sin^2\chi \simeq 10^{-28}\, sin^2\chi(m^2) \tag{18}$$

In calculating the single electron scattering we have implicitly assumed that the driving electric field is linearly polarized and that the angle between the field and the direction of the receiver is χ. In actual fact the electron is often oscillating in two linearly polarized fields of arbitrary relative phase and amplitude. When this is the case the interpretation of $sin^2\chi$ is more complex and the amount of scattered energy available may only be received provided the receiver is properly "matched" to the scattered wave.

Consider a plane wave propagating along a positive z-axis. The complex amplitude $\vec{E}_o$ may then be represented as;

$$\vec{E}_o = \xi_o(cos\beta \cdot \vec{e}_x + e^{i\delta}\, sin\beta \cdot \vec{e}_y) = \xi_o \begin{Bmatrix} cos\beta \\ sin\beta \cdot e^{i\delta} \end{Bmatrix} = \xi_o \vec{p} \tag{19}$$

A linear polarization along the x-axis corresponds to:

$$\vec{p} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

and along the y-axis:

$$\vec{p} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Circular polarization is represented by:

$$\vec{p} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -i \end{Bmatrix} \text{ right circular}$$

$$\vec{p} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ +i \end{Bmatrix} \text{ left circular}$$

It follows that the scattered field at the receiver can be expressed as follows:

$$\vec{E}_r(\vec{r},t) = \frac{r_o}{R_1}\vec{n}_1 \times (\vec{n}_1 \times \vec{E}_o) e^{-i(\omega_o t - \vec{k}_{rec} \cdot \vec{r})} \tag{20}$$

where $\vec{n}_1$ is a unit vector along the direction of the scattered wave. In order to specify the polarization both of the transmitted and scattered wave we introduce the following two coordinate systems:

A. For the transmitted wave:

$\vec{e}_z = \vec{n}_o = \vec{k}_{in}/|\vec{k}_{in}|$

$\vec{e}_y$ = unit vector normal to plane defined by $\vec{k}_{in}; \vec{k}_{rec}$

$\vec{e}_x$ = normal to both $\vec{e}_y$ and $\vec{e}_z$ in a right handed coordinate system.

B. For the scattered wave:

$\vec{e}_z = \vec{n}_1 = \vec{k}_{rec}/|\vec{k}_{rec}|$

$\vec{e}'_y = \vec{e}_y$

$\vec{e}'_x$ = normal to $\vec{e}'_y$ and $\vec{e}'_z$ in right handed coordinate system.

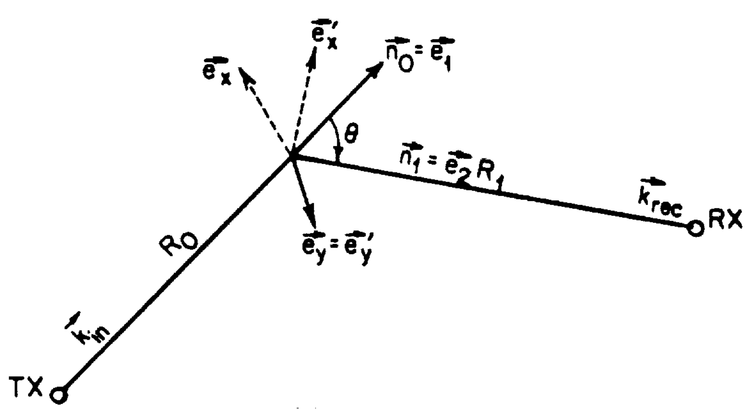

Figure 4. Relationship between polarization of incoming and scattered waves.

In terms of these two coordinate systems we may write:

$$\begin{Bmatrix} E'_{rx} \\ E'_{ry} \end{Bmatrix} = -\frac{r_0}{R_1}\xi_0 \begin{Bmatrix} cos\theta & 0 \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} cos\beta \\ sin\beta \cdot e^{i\delta} \end{Bmatrix} e^{i(\omega_o t - \vec{k}_{rec}\cdot\vec{r})}$$

or:

$$\xi_{r0} \begin{Bmatrix} cos\beta' \\ sin\beta' e^{i\delta'} \end{Bmatrix} = -\frac{r_0}{R_1} \begin{Bmatrix} cos\theta & 0 \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} cos\beta \\ sin\beta e^{i\delta} \end{Bmatrix} \xi_0 \qquad (21a)$$

This determines the relationship of the two fields. In particular we note that

$$sin^2\chi = sin^2\beta + cos^2\beta \cdot cos^2\theta \qquad (22)$$

where θ is the angle between $\vec{n}_o$ and $\vec{n}_1$.

It is now clear that one may express the relationship between scattered and incident electric field amplitudes as:

$$\xi_{r0}\vec{p}\,' = -\frac{r_0}{R_1}\xi_0\Psi\vec{p} \qquad (21b)$$

where:

$$\Psi = \begin{Bmatrix} cos\theta & 0 \\ 0 & 1 \end{Bmatrix} \text{(geometry)}$$

$$\vec{p} = \begin{Bmatrix} cos\beta \\ sin\beta e^{i\delta} \end{Bmatrix} \text{(transmitter)}$$

$$\vec{p}\,' = \begin{Bmatrix} cos\beta' \\ sin\beta' e^{i\delta'} \end{Bmatrix} \text{(receiver)}$$

4. The Scattering from a Collection of Electrons

When many scattering electrons are present inside a volume V rather than a single one the observed field is given by the sum

$$\vec{E}_r = -\frac{r_o}{R_1}\xi_o \Psi\vec{p} \sum_{p=1}^{N} e^{i\vec{k}\cdot\vec{r}_p(t)} \qquad (23)$$

where N is the total number of electrons within the volume V, where the polarization and geometry of all the electrons are the same so that we do not have to sum over different $\Psi\vec{p}\,'s$. The previous distance between a scattering electron and the observer, R_1, now represents the distance from an origin within the scattering volume and $\vec{r}_p(t)$ represents the position of electron number p within this volume. The vector $\vec{k} = \vec{k}_{rec} - \vec{k}_{in}$ comes about as follows:

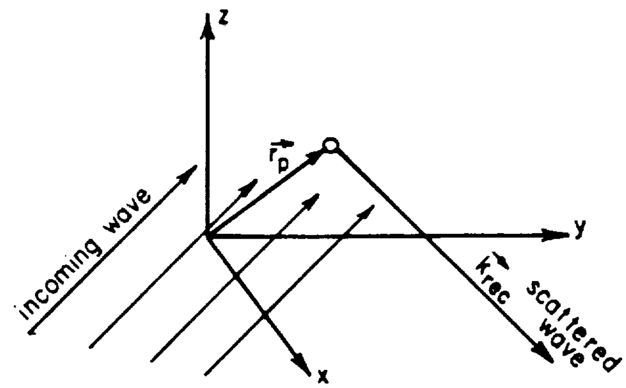

Figure 5. Scattering from electron at $\vec{r}_p$.

In the scattering volume, apart from a phase factor, the incoming wave has the form:

$$\vec{E}_{in}(\vec{r},t) = \xi_o \vec{p} \cdot e^{i(\omega_o t - \vec{r}\cdot\vec{k}_{in})}$$

so that the field at electron number p is:

$$\vec{E}_{in}(\vec{r}_p,t) = \xi_o \vec{p} \cdot e^{i(\omega_o t - \vec{k}_{in}\cdot\vec{r}_p)} \tag{24}$$

From the previous section the field at the receiver due to electron p becomes

$$\vec{E}_r = -\frac{r_o}{R_1} \xi_o \Psi \vec{p} e^{i(\omega_o t - \vec{k}_{in}\cdot\vec{r}_p)} \cdot e^{-ikR_{1p}} \tag{25}$$

If the distance from the receiver to the origin in the scattering volume is R_1 we have:

$$R_{1p} \approx R_1 - \vec{k}_{rec}\cdot\vec{r}_p$$

Substituting this into (25), ignoring an irrelevant phase factor (e^{ikR_1}) and summing over all electrons gives eqn. (23) provided

$$\vec{k} = \vec{k}_{rec} - \vec{k}_{in}, \tag{26}$$

We next imagine the particle density to be expanded in a spatial Fourier series through:

$$n(\vec{r},t) = \frac{1}{V}\sum_{\vec{k}} n(\vec{k},t)e^{-i\vec{k}\cdot\vec{r}} \tag{27}$$

where

$$n(\vec{k},t) = \int_V d(\vec{r}) n(\vec{r},t) e^{+i\vec{k}\cdot\vec{r}} \tag{28}$$

Since the scattering electrons must be considered point particles at $\vec{r}_p(t)$, $p = 1,..N$ the density is:

$$n(\vec{r},t) = \sum_{p=1}^{N} \delta[\vec{r} - \vec{r}_p(t)]$$

with a spatial spectrum

$$n(\vec{k},t) = \sum_{p=1}^{N} e^{+i\vec{k}\cdot\vec{r}_p(t)} = \sum_{p=1}^{N} n_p(\vec{k},t) \tag{29}$$

A comparison with (23) shows that the observed scattered field from the electrons may be expressed as:

$$\vec{E}_r = -\frac{r_o}{R_1}\xi_o \Psi \vec{p} \cdot e^{i\omega_o(t - R_1/c)} \cdot n(\vec{k},t) \tag{30}$$

The complex amplitude of the received signal is

$$\begin{aligned} A_r(t) &= |\frac{r_o}{R_1}\xi_o \Psi \vec{p}| \cdot n(\vec{k},t) \\ &= A_o \cdot n(\vec{k},t) \end{aligned} \tag{31}$$

$$\begin{aligned} < A_r^*(t) A_r(t+\tau) > &= A_o^2 < n^*(\vec{k},t) n(\vec{k},t+\tau) > \\ &= A_o^2 n_o \cdot V < n_p^*(\vec{k},t) n_p(\vec{k},t+\tau) > \end{aligned} \tag{32}$$

where < ... > denotes ensemble average, where individual particles are assumed to be independent, where n_o is the mean electron density and where:

$$< n_p^*(\vec{k},t) n_p(\vec{k},t+\tau) > = < e^{i\vec{k}\cdot[\vec{r}_p(t+\tau) - \vec{r}_p(t)]} > = \rho_p(\vec{k},\tau) \tag{33}$$

is the autocorrelation of density fluctuations associated with a single electron.

Similarly, the power spectrum received is determined from the Wiener-Khinchine's theorem, and

$$P_{rec}(\omega) = n_o \cdot V \cdot P_{oe} \Phi_p(\vec{k},\omega) \tag{34}$$

where P_{oe} is the power scattered by an individual electron under the same geometrical and external conditions and where:

$$\begin{aligned} \Phi_p(\vec{k},\omega) &= \int_{-\infty}^{+\infty} < n_p^*(\vec{k},t) n_p(\vec{k},t+\tau) > e^{-i\omega\tau} d\tau = \\ &= \int_{-\infty}^{+\infty} < e^{i\vec{k}[\vec{r}_p(t+\tau) - \vec{r}_p(t)]} > \cdot e^{-i\omega\tau} d\tau \\ &= \int_{-\infty}^{+\infty} \rho_p(\vec{k},\tau) e^{-i\omega\tau} d\tau \end{aligned} \tag{35}$$

We often need the integral:

$$G(\vec{k},\omega) = \int_{o}^{\infty} \rho_p(\vec{k},\tau) e^{-i\omega\tau} d\tau \tag{36}$$

With this function the spectrum of a particle becomes

$$\Phi_p(\vec{k},\omega) = 2 \cdot Re[G(\vec{k},\omega)]. \tag{37}$$

It will become apparent in the next two sections that both the plasma response to an electric field as well as the thermal driving force can be described on the basis of the motion of non- interacting particles. It is, therefore, useful to study their motion in some detail.

We assume that the static magnetic field $\vec{B}_o$ is directed along the z-axis. It will be convenient to introduce "polarized" coordinates to describe the position of a particle:

Cartesian: Polarized:

$$\vec{r} = \{x, y, z\} \qquad \vec{r} = \{r_{+1}, r_{-1}, r_o\} \tag{38}$$

The relationship between the two descriptions is

$$\begin{aligned} r_1 &= \tfrac{1}{\sqrt{2}}(x+iy) & x &= \tfrac{1}{\sqrt{2}}(r_1 + r_{-2}) \\ r_{-1} &= \tfrac{1}{\sqrt{2}}(x-iy) & y &= \tfrac{i}{\sqrt{2}}(r_{-1} - r_2) \\ r_o &= z & z &= r_o \end{aligned} \tag{39}$$

The advantage of these polarized coordinates becomes evident when we state the equation of free motion of a particle:

$$\begin{aligned} \frac{dv_\alpha}{dt} &= -i\alpha\Omega v_\alpha \qquad (\alpha = \pm 1, 0) \\ \Omega &= \frac{qB_o}{m} \end{aligned} \tag{40}$$

where q and m are the charge and the mass of the particle (electron or ion) respectively. From this we determine the relationship between the (past) velocity at time $t' = t - \tau$ in terms of the present velocity (at t)

$$\vec{v}(t-\tau) = \dot{\Gamma}(\tau)\,\vec{v}(t) \tag{41}$$

where $\dot{\Gamma}(\tau)$ is a diagonal matrix with elements.

$$[\dot{\Gamma}(\tau)]_{\alpha\alpha} = e^{i\alpha\Omega\tau} = \dot{g}_\alpha(\tau) \tag{42}$$

The past particle position can be determined similarly in terms of present position and present velocity through

$$\vec{r}(t-\tau) = \vec{r}(t) - \Gamma(\tau)\,\vec{v}(t) \tag{43}$$

where $\Gamma(\tau)$ is a diagonal matrix which determines the particle helical motion. The elements are given by

$$[\Gamma(\tau)]_{\alpha\alpha} = \frac{e^{i\alpha\Omega\tau} - 1}{i\alpha\Omega} = g_\alpha(\tau) \qquad (\alpha = \pm 1, 0) \tag{44}$$

The single particle autocorrelation, eqn. (33) now becomes:

$$\rho_p(\vec{k},\tau) = < e^{i\vec{a}\cdot\vec{v}} > \tag{45}$$

where

$$a_\alpha = k_\alpha \cdot g_{-\alpha} \tag{46}$$

The actual form of $\rho_p(k,\tau)$ depends on the statistical distribution of velocities, the magnetic field strength and the angle γ between $\vec{k}$ and the magnetic field direction.

For a Maxwellian velocity distribution:

$$f_o(\vec{v}) = (2\pi)^{-3/2}\, v_{th}^{-3}\, e^{-v^2/2v_{th}^2} \tag{47}$$

where

$$v_{th}^2 = T/m$$
T = kinetic temperature (in energy units)

we obtain:

$$\rho_p(\vec{k},\tau) = e^{-\frac{1}{2}v_{th}^2|\vec{a}|^2}$$
$$= e^{-(kR)^2\left\{\left(\frac{\Omega\tau}{2}\right)^2 cos^2\gamma + sin^2\left(\frac{\Omega\tau}{2}\right)sin^2\gamma\right\}} \tag{48}$$

The gyration radius R is defined as:

$$R = \frac{\sqrt{2}v_{th}}{\Omega} \tag{49}$$

It equals the ratio of the r.m.s. orbital velocity and the angular gyration frequency.

We note that for weak magnetic fields and arbitrary γ, or for $\vec{k} \parallel \vec{B}_o$ (i.e. $\gamma = 0$) and arbitary magnetic field strength the autocorrelation becomes

$$\rho_p(\vec{k},\tau) \rightarrow e^{-\frac{1}{2}(kv_{th}\tau)^2} \tag{50}$$

When $\vec{k}$ is exactly perpendicular to the magnetic field, the autocorrelation for the plasma-density becomes periodic with period $T = 2\pi/\Omega$. The depth of modulation increases with R and decreases with the scale of the density fluctuation $\Lambda = 2\pi/k$, see Figure 6.

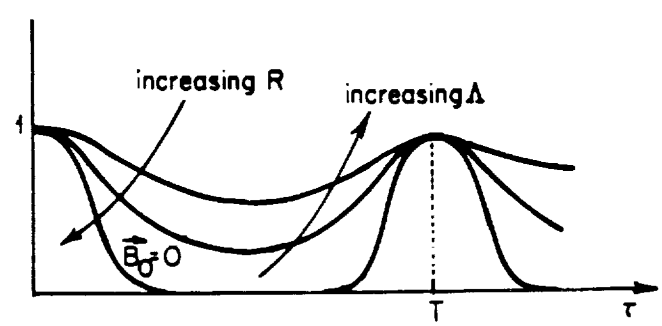

Figure 6. Autocorrelation of density fluctuation when $\vec{k} \perp \vec{B}_o$.

For intermediate angles between k and B_o the autocorrelation of the density fluctuation can be regarded a product of two factors, see (48).

A) $$e^{-(kR)^2\left(\frac{\Omega\tau}{2}\right)^2 cos^2\gamma}$$

B) $$e^{-(kR)^2 sin^2\left(\frac{\Omega\tau}{2}\right)\cdot sin^2\gamma}$$

which are sketched in Figure 7.

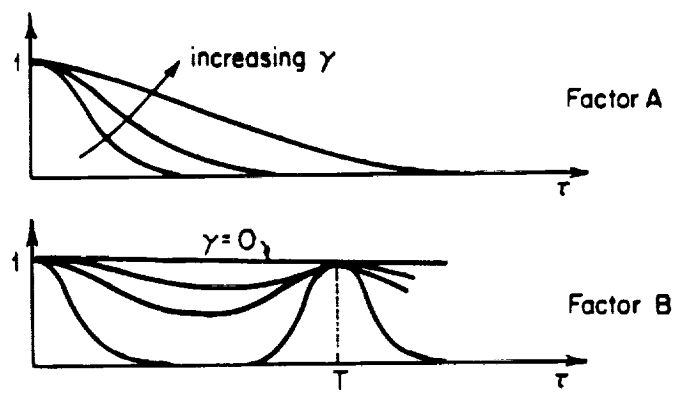

Figure 7. The two factors which in the autocorrelation of in the density fluctuation of non-interacting particles.

The spectral function $\phi(\vec{k},\omega)$ becomes (see 37):

$$\phi_p(\vec{k},\omega) = 2Re\left\{G_p(\vec{k},\omega)\right\} \tag{51}$$

where

$$G_p(\vec{k},\omega) = \int_o^\infty e^{-\frac{1}{2}v_{th}^2|\vec{a}_p|^2 - i\omega\tau} d\tau \tag{52}$$

The next section introduces particle interaction through a self-consistent electric field.

5. Response to a Field Particle. Electrostatic Interactions.

We now introduce particle correlations in the plasma components (electrons, various types of ions) by considering the particles smeared out into a continuum and by assessing the response of this continuum to each discrete particle separately.

The plasma response to an electric field $\vec{E}(\vec{r},t)$can be obtained by solving the Vlasov equation to first order in $\delta n(\vec{r},\vec{v},t)$ and $\vec{E}(\vec{r},t)$. The perturbation solution for any of the species is given by (see Appendix A)

$$\delta n(\vec{r},\vec{v},t) = -\frac{n_o q}{m}\int_{-\infty}^{t} \vec{E}(\vec{r}\,',t')\,\frac{\partial f_o'}{\partial \vec{v}\,'}\,dt' \tag{53}$$

Here $\vec{E}(\vec{r},t)$ is a small electric field which we shall later assume set up by the fluctuating charges, often referred to as the self-consistent field. The integration is carried out along the unperturbed particle orbits which were studied previously. This is indicated by the primed quantities in the integrand.

We make the following substitutions:

$$\begin{aligned} t' &= t-\tau \\ \vec{E}(\vec{r}\,',t') &= \frac{1}{V}\sum_{\vec{k}} \vec{E}(\vec{k},t-\tau)e^{-i\vec{k}\cdot\vec{r}}e^{+i\vec{k}\cdot\Gamma(\tau)\vec{v}} \\ \frac{\partial f_o'}{\partial \vec{v}\,'} &= \dot{\Gamma}(\tau)\frac{\partial f_o(v')}{\partial \vec{v}} \end{aligned} \tag{54}$$

Integrating over all possible $\vec{v}$ we obtain the induced density fluctuation:

$$n(\vec{k},t) = \frac{n_o q}{m}\int_o^{\infty} d\tau\,\vec{E}(\vec{k},t-\tau)\int d(\vec{v})\dot{\Gamma}(\tau)\,\frac{\partial f_o(\vec{v}\,')}{\partial \vec{v}}\,e^{i\vec{k}\cdot\Gamma(\tau)\vec{v}} \tag{55}$$

We now take the Fourier transform with respect to time, and relate the electric field to the <u>total</u> electric charge fluctuation $Q(\vec{k},\omega)$ by:

$$\vec{E}(\vec{k},\omega) = \frac{i\vec{k}}{\varepsilon_o k^2}\,Q(\vec{k},\omega) \tag{56}$$

We, therefore, only include longitudinal electrostatic interactions. This is exact when there is no external magnetic field. When such a field is present the longitudinal and transverse modes are coupled and the electrostatic approximation breaks down particularly at very long wavelengths. In diagnostic experiments on the ionosphere only short wavelengths are used and the electrostatic approximation is adequate.

Substitution of (56) into (55) gives

$$n(\vec{k},\omega) = -\frac{in_o q}{\varepsilon_o m\cdot k^2}\,Q(\vec{k},\omega)\int_o^{\infty} d\tau\cdot e^{-i\omega\tau}\int d(\vec{v})\dot{\vec{a}}(\tau)\frac{\partial f_o}{\partial \vec{v}}\,e^{i\vec{a}\cdot\vec{v}} \tag{57}$$

For each of the species we now introduce

$$\frac{n_o q_\sigma^2}{\varepsilon_o m_\sigma} = \omega_\sigma^2$$

and,

$$\chi_\sigma = +i\,\frac{\omega_\sigma^2}{k^2}\int_o^{\infty} d\tau e^{-i\omega\tau}\int d(\vec{v})\dot{\vec{a}}_\sigma(\tau)\frac{\partial f_o}{\partial \vec{v}}\,e^{+i\vec{a}_\sigma\cdot\vec{v}} \tag{58}$$

Hence, we obtain for the fluctuation induced in species σ:

$$n_\sigma(\vec{k},\omega) = -\frac{1}{q_\sigma}\chi_\sigma(\vec{k},\omega)\cdot Q(\vec{k},\omega) \tag{59}$$

We now have all the ingredients necessary to compute the density fluctuations in the plasma.

6. Calculation of the Fluctuation Spectrum

Let us now assume that we have a plasma of electrons and one type of singly charged ions.

For the electrons we substitute:

$\omega_\sigma^2 \to \omega_e^2$
$q_\sigma \to -e$
$m_\sigma \to m$
$\vec{a}_\sigma \to \vec{a}_e$
$f_{o\sigma} \to f_o$
$\chi_\sigma \to \chi_e$
$n_\sigma \to n$

For the ions we substitute:

$\omega_\sigma^2 \to \omega_i^2$
$q_\sigma \to +e$
$m_\sigma \to M$
$\vec{a}_\sigma \to \vec{a}_i$
$f_{o\sigma} \to F_o$
$\chi_\sigma \to \chi_i$
$n_\sigma \to N$

Consider first the fluctuation associated with a particular electron.

A. Intrinsic fluctuation $n_p(\vec{k},\omega)$, See Section 4.

B. Induced electron fluctuation $n_e^1(\vec{k},\omega) = +\frac{\chi_e}{e} Q_e(\vec{k},\omega)$
(electron dressing on electron)

C. Induced ion fluctuation $N_e^1(\vec{k},\omega) = -\frac{\chi_i}{e} Q_e(\vec{k},\omega)$
(ion-dressing on electron)

The total charge fluctuation associated with this single electron is

$$Q_e(\vec{k},\omega) = -e(n_p + \frac{\chi_e}{e} Q_e) + e(-\frac{\chi_i}{e} Q_e) \tag{60}$$

from which:

$$Q_e(\vec{k},\omega) = \frac{-e \cdot n_p(\vec{k},\omega)}{1+\chi_e+\chi_i} \tag{61}$$

Since we are interested in the total electron-density fluctuation induced by electrons, not the charge fluctuation, we have:

$$\begin{aligned} n_e(\vec{k},\omega) &= n_p(\vec{k},\omega) + n_e^1(\vec{k},\omega) \\ &= n_p(\vec{k},\omega) + \frac{-\chi_e n_p(\vec{k},\omega)}{1+\chi_e+\chi_i} = \frac{n_p(\vec{k},\omega)(1+\chi_i)}{1+\chi_e+\chi_i} \end{aligned} \tag{62}$$

The mean power spectrum associated with the thermal excitation by electrons is, therefore, found by averaging over the electron velocity distribution. If the velocity distribution is Maxwellian (47), then the independent electron spectrum is given by (51) and the result is

$$< |n_e(\vec{k},\omega)|^2 > = \frac{< |n_p(\vec{k},\omega)|^2 > |1+\chi_i|^2}{|1+\chi_e+\chi_i|^2} = \frac{\Phi_e(\vec{k},\omega)| \, |1+\chi_i|^2}{|1+\chi_e+\chi_i|^2} \tag{63}$$

where $\Phi_e(\vec{k},\omega)$ is the independent single electron mean power spectrum discussed in Section 5.

Next consider the more important fluctuation arising from the thermal motion of an ion:

A. Intrinsic fluctuation $N_p(\vec{k},\omega)$, see Section 4.

B. Induced electron fluctuation $n_i^1(\vec{k},\omega) = +\frac{\chi_e}{e}\, Q_i(\vec{k},\omega)$
(electron dressing on ion)

C. Induced ion fluctuation $N_i^1(\vec{k},\omega) = -\frac{\chi_i}{e}\, Q_i(\vec{k},\omega)$

Solving for the charge fluctuation $Q_i(\vec{k},\omega)$ we obtain:

$$Q_i(\vec{k},\omega) = \frac{eN_p(\vec{k},\omega)}{1+\chi_e+\chi_i} \tag{64}$$

and the fluctuation induced in the electron density is given by:

$$n_i^1(\vec{k},\omega) = \frac{\chi_e N_p(\vec{k},\omega)}{1+\chi_e+\chi_i} = n_i(\vec{k},\omega) \tag{65}$$

The total electron density fluctuation in the plasma, therefore, can be expressed in terms of the independent particle spectra for electrons and ions and the response functions of the plasma as follows:

$$\begin{aligned} < |n(\vec{k},\omega)|^2 > &= [< |n_e(\vec{k},\omega)|^2 > + < |n_i(\vec{k},\omega)|^2 >]\, n_o V \\ &= \frac{|1+\chi_i|^2\Phi_e(\vec{k},\omega) + |\chi_e|^2\Phi_i(\vec{k},\omega)}{|1+\chi_e+\chi_i|^2}\, n_o V \end{aligned} \tag{66}$$

This is the form of the spectrum which has been widely used in the analysis of the incoherent scatter data, and which has remained valid for nearly 30 years!

Fluctuations in ion density, charge density, electric field, currents, etc. can all be obtained by an analogous procedure. An extension to a multi-ion plasma is relatively trivial. Different temperatures for electrons and ions are allowed.

Collisions have not been considered but can, in some cases, be taken into account by regarding the particle motion as a stochastic rather than a deterministic process, see Section 8.

7. Discussion of Results

The spectrum function contains the χ_e and χ_i together with Φ_e and Φ_i. All of these functions are related to the autocorrelation of independent particle fluctuations as follows:

Introduce for electrons (see 48):

$$\rho_o(\vec{k},\tau) = \rho_e(\vec{k},\tau) = e^{-\frac{1}{2}v_{th}^2|\vec{a}|^2}$$

$$v_{th}^2 = \frac{T_e}{m}$$

$$a_{-\alpha} = k_{-\alpha}\cdot g_\alpha \qquad (g\alpha \text{ computed with } \Omega = \Omega_e = -\frac{eB_o}{m})$$

$$D_e^2 = \frac{\varepsilon_o T_e}{n_o\cdot e^2} \qquad \text{(Debye length)}$$

And for the ions:

$$\rho_o(\vec{k},\tau) = \rho_i(\vec{k},\tau) = e^{-\frac{1}{2}v_{th}^2|\vec{A}|^2}$$

$$V_{th}^2 = \frac{T_i}{M}$$

$$M = \text{ionic mass}$$

$$A_{-\alpha} = k_{-\alpha} \cdot g_\alpha \qquad (g_\alpha \text{ computed with } \Omega = \Omega_i = +\frac{eB_o}{M})$$

$$D_i^2 = \frac{\varepsilon_o T_i}{n_o \cdot e^2}$$

With these definitions we obtain:

$$\chi_e(\vec{k},\omega) = \left(\frac{1}{k\,D}\right)^2 [1 + \omega \cdot Im\left\{G_e(\vec{k},\omega)\right\} - i\omega \cdot Re\left\{G_e(\vec{k},\omega)\right\}]$$

$$\chi_i(\vec{k},\omega) = \left(\frac{1}{k\,D_i}\right)^2 [1 + \omega \cdot Im\left\{G_i(\vec{k},\omega)\right\} - i\omega \cdot Re\left\{G_i(\vec{k},\omega)\right\}] \tag{67}$$

and:

$$\Phi_e(\vec{k},\omega) = 2 \cdot Re\left\{G_e(\vec{k},\omega)\right\}$$

$$\Phi_i(\vec{k},\omega) = 2 \cdot Re\left\{G_i(\vec{k},\omega)\right\}$$

Here, from Section 4:

$$Re\left\{G_e(\vec{k},\omega)\right\} = \int_o^\infty e^{-\frac{1}{2}v_{th}^2|\vec{a}|^2} \cos\omega\tau\, d\tau$$

$$Im\left\{G_e(\vec{k},\omega)\right\} = -\int_o^\infty e^{-\frac{1}{2}v_{th}^2|\vec{a}|^2} \sin\omega\tau\, d\tau$$

with similar definitions for the ions.

When there is no magnetic field or when $\vec{k}||\vec{B}_o$ one obtains:

$$\omega Re\left\{G(\vec{k},\omega)\right\} = \sqrt{\frac{\pi}{2}}\, Z\, e^{-\frac{1}{2}Z^2} \tag{68}$$

$$1 + \omega Im\left\{G(\vec{k},\omega)\right\} = \begin{cases} 1 - Z^2 + \frac{1}{3}Z^4 - \frac{1}{15}Z^6 ..... Z << 1 \\ -\frac{1}{Z^2} - \frac{3}{Z^4} - ..... Z >> 1 \end{cases} \tag{69}$$

where we have put

$$Z = \frac{\omega}{kv_{th}} = \frac{v_\phi}{v_{th}}$$

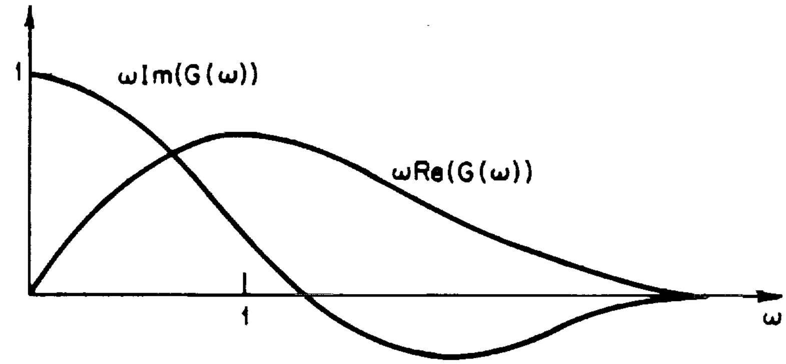

Figure 8. Plasma dispersion functions.

The only difference in the electronic and the ionic functions comes from the difference in thermal velocities.

For values of ω such that $\omega < kV_{th}$ χ_i and χ_e are of the same order of magnitude.

However, $\Phi_e/\Phi_i \sim Re\, G_e/Re\, G_i \sim V_{th}/v_{th} \sim \sqrt{m/M}$

It follows that as long as kD_e and $kD_i << 1$ the dominant excitation of density waves at low frequencies must stem from the ion excitation, see the numerator of (66). In the denominator $\chi_e \simeq (\frac{1}{kD_e})^2$ whenever $\omega < kV_{th}$.

It follows that the low frequency part of the spectrum simplifies to:

$$< |n(\vec{k},\omega)|^2 > \approx 2n_o V \frac{Re\left\{G_i(\omega,\vec{k})\right\}}{|1+\frac{T_e}{T_i}[1-i\omega G_i(\omega,\vec{k})]|^2} \tag{70}$$

For $T_e = T_i$ the factor multiplying $Re\left\{G_i(\omega,\vec{k})\right\}$ starts at $\frac{1}{4}$ at $\omega = 0$. As T_e/T_i increases above unity the depression near $\omega = 0$ increases and a near line spectrum develops, see Figure 9.

When kD_e and kD_i become much larger than unity, then the electronic part of the spectrum will dominate and we obtain a Gaussian spectrum with a width corresponding to the thermal motion of electrons.

A resonance occurs near the electron plasma frequency ω_e. This can be established by looking for a zero in the denominator of (66) near the plasma frequency. Since χ_i has vanished near ω_e we have:

$$1+\left(\frac{1}{kD_e}\right)^2 \{1+\omega Im(G_e) - i\omega Re(G_e)\} \equiv 0 \tag{71}$$

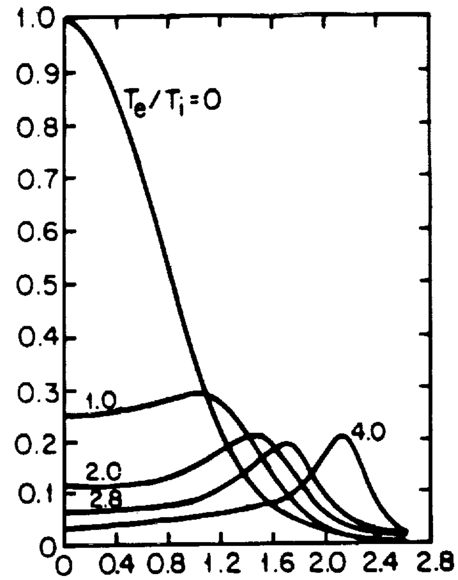

Figure 9. The ionic part of the fluctuation spectrum for various values of T_e/T_i.

As ω is close to ω_e the expansion of the plasma dispersion functions for large arguments can be used and we obtain

$$1+\left(\frac{1}{kD_e}\right)^2\left\{-\frac{(kv_{th})^2}{\omega^2}-3\frac{(kv_{th})^4}{\omega^4}\right\}=0 \tag{72}$$

Where we have neglected the imaginary part. The solution is the familiar expression:

$$\omega^2=\omega_e^2[1+3(kD_e)^2] \tag{73}$$

the spectral peak associated with this oscillation is apparent in Figure 10. The peak can be strongly enhanced by the presence of photoelectrons or other suprathermal charged particles. The actual enhancement level involves the short-range Coulomb collisions as well as the angular distribution of the suprathermal electrons.

Let us now briefly turn to the effect of the magnetic field. As far as the ions are concerned the gyrofrequency Ω_i satifies the relation:

$$kV_{th}>>\Omega_i$$

This means that the correlation function

$$e^{-(kR)^2(\frac{\Omega\tau}{2})^2\cdot\cos^2\gamma}$$

becomes modulated with a periodicity $T_i=2\pi/\Omega_i$.

In practice, however, these modulations are blurred out because of the diffusion of the ions away from their deterministic orbits. Very close to perpendicularity with the magnetic field (i.e. $\gamma=90^o$) the spectrum may become very narrow if the radius of gyration of the

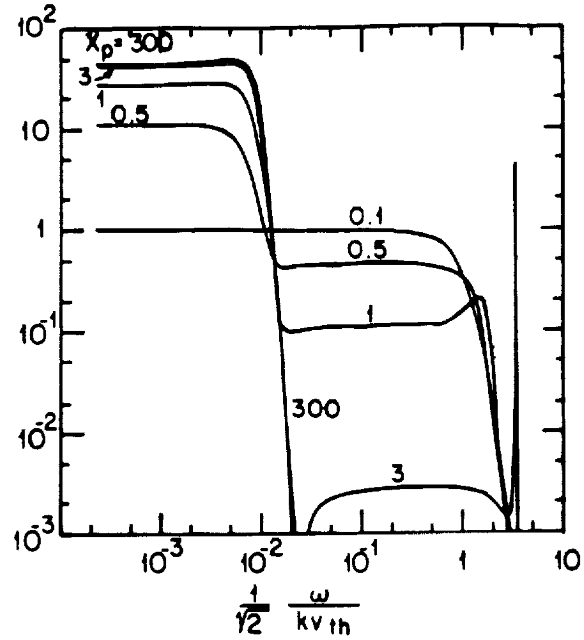

Figure 10. The fluctuation spectrum for $T_e/T_i = 1$ for varying Debye length D_e, $X_p = \frac{1}{\sqrt{2}} \cdot \frac{\omega_e}{kv_{th}}$.

electrons is so small that the electrons are prevented from participating in the fluctuations of the ions.

With a magnetic field the resonance associated with the plasma frequency becomes modified. When $\omega_e >> \Omega_e$, when $kR < 1$ and when γ is not too close to 90° one obtains:

$$\omega^2 = \omega_p^2[1 + 3(kD_e)^2] + \Omega_e^2 \cdot sin^2\gamma \tag{74}$$

An additional resonance, which can also be observed, arises because of the presence of the magnetic field. As $\gamma \to 90°$:

$$\omega_h^2 = \Omega_e^2(\omega_{pi}^2 + \Omega_i^2)/(\omega_e^2 + \Omega_e^2) \tag{75}$$

When $\omega_p^2 > \Omega_e^2$ one obtains:

$$\omega_h^2 = \Omega_e\Omega_i \tag{76}$$

which is sometimes referred to as the lower hybrid frequency. The strength of these lines depend on the relative magnitudes of plasma frequency and gyrofrequency, on the presence of suprathermal electrons (or ions) etc.

The total power residing in the electron plasma oscillation at thermal equilibrium is determined by integrating the spectrum through the electron lines with the result that

$$\int_{electron\ line} \Phi d\omega = \frac{(kD_e)^2}{1 + (kD_e)^2} \tag{77}$$

whereas the power in the ion line becomes:

$$\int_{ion} \Phi \, d\omega = \frac{1}{[1 + (kD_e)^2][1 + \frac{T_e}{T_i} + (kD_e)^2]} \tag{78}$$

Hence, whenever kD_e is small the ion contribution dominates.

Note that (78) $\to \frac{1}{2}$ when $(kD)^2 \to 0$ or more generally $\to \frac{1}{1+T_e/T_i}$ when $T_e \neq T_i$. Hence, the "ion-scattering" is half of the free electron scattering, which is obtained from (77). When $(kD)^2 \to \infty$ (77) $\to 1$.

8. The Effects of Collisions

Let us return to the solution of the inhomogeneous Vlasov first order equation:

$$\delta n(\vec{r}, \vec{v}, t) = -\frac{q n_o}{m} \int_o^\infty \vec{E}(\vec{r}', t - \tau) \frac{\partial f_o(\vec{v}')}{\partial \vec{v}'} d\tau \tag{79}$$

Remember that the past $\vec{R}, \vec{V}$ at time $t - \tau$ approach $\vec{r}, \vec{v}$ at time t, and that the position/velocity travel back in time exactly as in deterministic orbit in accordance with the equations of motion of free particles.

Suppose that the particles suffer collisions. In this case, it is no longer possible to say for certain where they came from, because the previous history of the particle arriving at $\vec{r}, \vec{v}$ at time t is a random process. If the collisions occur with like particles the process becomes difficult because pairs of particles have unrelated motions in this theory. However it is often the case that the collisions occur with particles of another kind with little dynamical mutual coupling. Important examples are:

ions in low ionosphere collide with neutral gas molecules which are much more numerous than the ions.

electrons deviating from their deterministic orbits because they have to move in the random field of near-stationary ions.

In cases such as the two quoted we introduce:

$W_+(\vec{r}, \vec{v}, t | \vec{r} - \vec{R}, \vec{V}, t - \tau)$ = conditional probability density of finding a particle at $\vec{r}, \vec{v}$ at time t given that at time $t - \tau$ it was at $\vec{r} - \vec{R}, \vec{V}$

The joint probability density of the two events $(\vec{r}, \vec{v}, t)$ and $(\vec{r} - \vec{R}, \vec{V}, t - \tau)$ is

$$W_+(\vec{r}, \vec{v}, t | \vec{r} - \vec{R}, \vec{V}, t - \tau) f_o(\vec{V})$$

The individual probability of $\vec{r} - \vec{R}, \vec{v}, t - \tau$ given that the present coordinates are $\vec{r}, \vec{v}, t$ is:

$$W_-(\vec{r} - \vec{R}, \vec{V}, t - \tau | \vec{r}, \vec{v}, t)$$

and the joint probability density of $\vec{r}, \vec{v}, t$ and $\vec{r} - \vec{R}, \vec{V}, t - \tau$ is:

$$W_-(\vec{r}-\vec{R},\vec{V},t-\tau|\vec{r},\vec{v},t)f_o(\vec{v}) =$$

$$= W_+(\vec{r},\vec{v},t|\vec{r}-\vec{R},\vec{V},t-\tau)f_o(\vec{V}) \tag{80}$$

but,

$$W_-(-\vec{R},\vec{V},-\tau|o,\vec{v},o)$$

clearly must equal

$$W_+(\vec{R},\vec{V},\tau|o,\vec{v},o)$$

from the symmetry of the equations of motion.

Taking the spatial Fourier transform of the perturbation solution one obtains:

$$n(\vec{k},\vec{v},t) = -\frac{qn_o}{m}\int_o^\infty \vec{E}(\vec{k},t-\tau) < \frac{\partial f_o(\vec{v}\,')}{\partial \vec{v}\,'}\cdot e^{+i\vec{k}\cdot\vec{R}} > d\tau \tag{81}$$

where the average is taken over all the different particle orbits which lead to $(\vec{r},\vec{v},t)$.

Explicitly with $\frac{\partial f_o'}{\partial v'} = \frac{\partial f_o(\vec{V})}{\partial \vec{V}} = -\frac{\vec{V}\cdot m}{T}\cdot f_0$:

$$n(\vec{k},\vec{v},t) = +\frac{\varepsilon_o}{qD^2}\int_o^\infty \vec{E}(\vec{k},t-\tau)\iint W_+(\vec{R},\vec{V},\tau|o,\vec{v},o)\vec{V}\cdot e^{+i\vec{k}\cdot\vec{R}}d(\vec{R})d(\vec{V})\cdot f_o(\vec{v}) \tag{82}$$

Hence, if we introduce:

$$f_o(\vec{v})\iint d(\vec{R})d(\vec{V})\,W_+(\tau)\vec{V}\cdot e^{+i\vec{k}\cdot\vec{R}} = \vec{A}(\vec{k},\vec{v},\tau) \tag{83}$$

we obtain:

$$n(\vec{k},\vec{v},t) = \frac{\varepsilon_o}{qD^2}\int_o^\infty \vec{E}(\vec{k},t-\tau)\cdot\vec{A}(\vec{k},\vec{v},\tau)d\tau \tag{84}$$

Let us formulate the modified scatter theory in terms of transition probability averaging.

Introducing as before:

$$\vec{E}(\vec{k},\omega) = \frac{i\vec{k}}{\varepsilon_o k^2}\,Q(\vec{k},\omega) \tag{85}$$

and taking Fourier transforms, one obtains:

$$n(\vec{k},\vec{v},\omega) = i\,\frac{Q(\vec{k},\omega)}{q(kD)^2}\int_0^\infty d\tau\, e^{-i\omega\tau}\,\vec{k}\cdot\vec{A}(\vec{k},\vec{v},\tau) \tag{86}$$

Integrating over all arrival velocities one finally obtains:

$$n(\vec{k},\omega) = -\frac{1}{q}\,\chi(\vec{k},\omega)\cdot Q(\vec{k},\omega) \tag{87}$$

where now

$$\chi(\vec{k},\omega) = -i\,\frac{1}{(kD)^2}\int_o^\infty d\tau\, e^{-i\omega\tau}\int d(\vec{v})\cdot f_o(\vec{v})\cdot$$

$$\iint d(\vec{R})d(\vec{v})\; W_+(\vec{R},\vec{v},\tau|o,\vec{v},o)(\vec{k}\vec{V})e^{-i\vec{k}\cdot\vec{R}} \tag{88}$$

Similarly - studying independent particles - diffusing along as a result of collisions with another gas - but <u>not</u> exposed to an external field - one obtains the expression:

$$|n_p(\vec{k},\omega|^2 = 2\int_o^\infty d\tau\cdot e^{-i\omega\tau}\int d(\vec{v})\, f_o(\vec{v})\cdot$$

$$\iint d(\vec{R})d(\vec{v})\; W_+(\vec{R},\vec{V},\tau|o,\vec{v},o)\, e^{-i\vec{k}\vec{R}} \tag{89}$$

So that what we previously referred to as G now takes the form:

$$G(\vec{k},\omega) = \int_o^\infty d\tau\, e^{-i\omega\tau}\int d(\vec{v})\, f_o(\vec{v})\iint d(\vec{R})d(\vec{V})\cdot$$

$$\cdot W_+(\vec{R},\vec{V},\tau|o,\vec{v},o)\, e^{-i\vec{k}\cdot\vec{R}} \tag{90}$$

By properly manipulating the expression for $\chi(\vec{k},\omega)$ we obtain

$$\chi(\vec{k},\omega) = \frac{1}{(kD)^2}(1+\omega\, Im\, G\, - i\omega\, Re\, G) \tag{91}$$

which is of the same form as before. Consider a model for $W(\vec{R},v,\tau|o,\vec{v},o)$:

Suppose the particle is moving as if in Brownian motion with an equation

$$\frac{d\vec{v}}{dt} = -\beta\vec{v} + \vec{A}(t) \qquad \text{(Langevin's equation)} \tag{92}$$

(do not confuse with β in Section 2!)

Then, from Chandrasekhar's work:

$$W_+(\vec{R},\vec{V},\tau|\vec{v}) = \frac{1}{8\pi^3(FG-H^2)^{3/2}}\, e^{-\frac{G\cdot R_o^2 - 2H R_o S + F\cdot S^2}{2(FG-H^2)}} \tag{93}$$

$$\vec{R}_o = \vec{R} - \tfrac{\vec{v}}{\beta}(1-e^{-\beta\tau})$$

$$\vec{S} = \vec{V} - \vec{v}\cdot e^{-\beta\tau}$$

$$F = \tfrac{T}{m\beta^2}\,[2\beta\tau - 3 + 4\cdot e^{-\beta\tau} - e^{-2\beta\tau}]$$

$$G = \tfrac{T}{m}[1-e^{-2\beta\tau}]$$

$$H = \tfrac{T}{m\beta}(1-e^{-\beta\tau})^2$$

The correspondence for $\beta \to 0$

$$W_+(\vec{R},\vec{V},\tau|\vec{v}) = \delta(\vec{R}-\vec{v}\cdot\tau)\delta(\vec{V}-\vec{v}) \tag{94}$$

This is a formulation which preserves particles, momentum and energy (with the background). We shall now use it for a study of plasma line enhancement due to photoelectrons, or other energetic "tail" electrons, neglecting the magnetic field.

By substitution one obtains:

$$G(\vec{k},\omega) = \int_o^\infty e^{-\frac{1}{2}\frac{k^2 v_{th}^2}{\beta^2}[\beta\tau - 1 + e^{-\beta\tau}]} \cdot e^{-i\omega\tau} d\tau \tag{95}$$

If we assume β to be small we obtain:

$$G(\vec{k},\omega) \approx \underbrace{\int_o^\infty e^{-\alpha\tau^2 - i\omega\tau} d\tau}_{G_0(\vec{k},\omega)} + \frac{\alpha\beta}{3} \int_o^\infty \tau^3 \, e^{-\alpha\tau^2 - i\omega\tau} d\tau \tag{96}$$

with $\alpha = \frac{k^2 v_{th}^2}{2}$

Hence: ↙friction term collision frequency

$$G(\vec{k},\omega) = G_o(\vec{k},\omega) - i\,\frac{\alpha\beta}{3}\,\frac{\partial^2}{\partial\alpha\partial\omega}\,G_o(\vec{k},\omega) \tag{97}$$

The asymptotic expansion for $Im\,G$ for <u>large</u> $\frac{\omega}{kv_{th}} = Z$ is

$$-\frac{1}{\omega}\left\{1 + Z^{-2} + 3\,Z^{-4}...\right\}$$

$$= -\left\{\frac{1}{\omega} + \frac{(kv_{th}^2)}{2\omega^3}\cdot 2 + ...\right\} = -\left\{\frac{1}{\omega} + 2\frac{\alpha}{\omega^3}...\right\} \tag{98}$$

Hence, we obtain approximately:

$$\begin{aligned} G(\vec{k},\omega) &\approx G_o(\vec{k},\omega) + \frac{\alpha\beta}{3}\cdot 2\cdot 3\cdot\frac{1}{\omega^4}... \\ &= G_o(\vec{k},\omega) + \beta\cdot\frac{k^2 v_{th}^2}{\omega^4} + ... \\ &\simeq G_o(\vec{k},\omega) + \beta\cdot\frac{(kD)^2}{\omega^2} \end{aligned} \tag{99}$$

Turning to the fluctuation spectrum for electron plasma oscillations one obtains

$$< |n(\vec{k},\omega)|^2 >= \frac{\Phi_e}{|1+\chi_e|^2} \tag{100}$$

Denominator:

$$1 + \chi_e = 1 + \left(\frac{1}{kD}\right)^2 [1 + \omega\,Im\,G - \omega Im\,G - i\omega Re\,G]$$

$$\approx 1 - \frac{\omega_p^2}{\omega^2} - 3\,\frac{(kv_{th})^2}{\omega^4}\,w_e^2 - \frac{i}{(kD)^2}\,\omega\cdot Re\,G$$

$$\approx 1 - \frac{\omega_p^2}{\omega^2} - 3(kD)^2 \cdot \frac{\omega_e^2}{\omega^2} \tag{101}$$

Expand to first order in ω about $\omega = \omega_r$

$$1 + \chi_e \approx +\frac{2}{\omega_r}(\omega - \omega_r) - \frac{i}{(kD)^2}\omega \, Re\, G$$

$$= \frac{2}{\omega_r}(\omega - \omega_r) + \frac{i}{(kD)^2} \cdot \sqrt{\frac{\pi}{2}} \frac{d}{dz}(e^{-\frac{1}{2}z^2}) + \ldots \tag{102}$$

The power spectral shape in this approximation becomes:

$$< |n(\vec{k},\omega)|^2 > \simeq \frac{2n_o \, Re\, G}{\frac{4}{\omega_r^2}(\omega - \omega_r)^2 + (\frac{1}{kD})^4(\frac{d}{dz}\ldots)^2} \tag{103}$$

This is a Lorentzian which can be integrated. If this is done one obtains for the intensity:

$$I \sim \frac{1}{2(kD)^2} \frac{f_m(v_\phi) + f_p(v_\phi) + \beta}{f_m(v_\phi) - T\frac{d}{dE_\phi} f_p(v_\phi) + \beta} \tag{104}$$

When electron-ion collisions dominate, β can be expressed approximation by as $\omega_p \frac{ln\Lambda}{\Lambda}$ ($\Lambda \sim$ number of electrons in Debye sphere.)

$v_\phi \sim \frac{\omega_e}{k}$

$f_m \sim n_m(\frac{m}{2\pi T})^{1/2} a, e^{-mv_\phi^2/2T}$ background plasma

$f_p \sim$ same for <u>hot</u> plasma.

Collisions between ions and the neutrals cause the mobility of the ions to decrease. The collisions, therefore, effectively damps the ion-acoustic waves, which causes the frequency spectrum to narrow. Studies of the narrowing of the frequency spectrum with decreasing height can be used for studies of the neutral density at D and the E-region heights.

9. Summary and some final remarks

A summary of the application of the ion line of the incoherent scatter radar technique to ionospheric measurements can best be given in terms of the sketches shown in Figure 11.

The plasma line, in addition to providing information on suprathermal flux, see equation (104), also provides the possibility to determine the electron temerature and the electron density. Ignoring the geomagnetic field for simplicity we have, for the effective k-vectors $\vec{k}_1$ and $\vec{k}_2 (= \vec{k}_{in} - \vec{k}_{rec})$ the following plasma line frequency offsets apply, see equation (73):

$$f_{1R}^2 = f_e^2[1 + 3(k_1 D)^2]$$

$$f_{2R}^2 = f_e^2[1 + 3(k_2 D)^2]$$

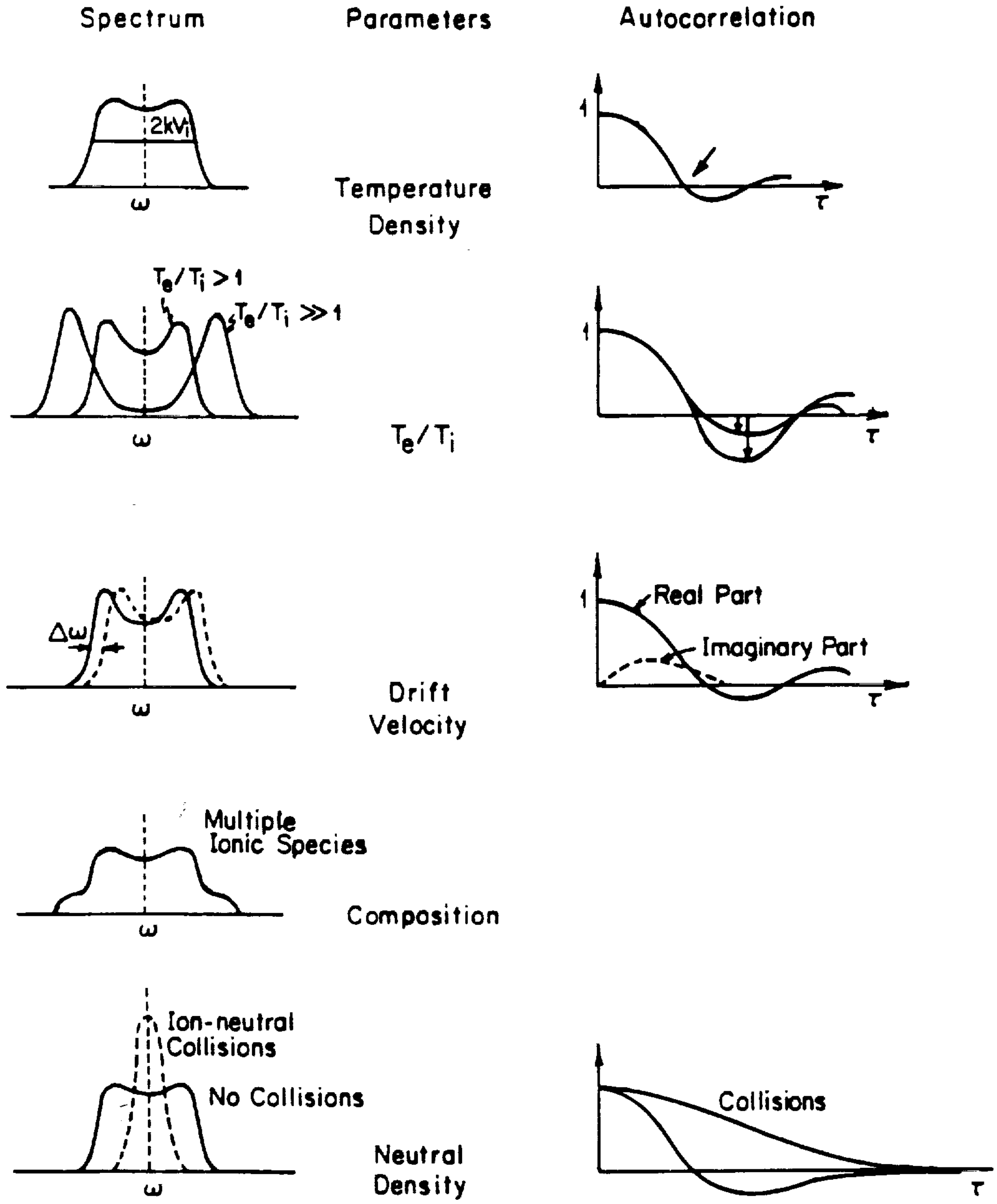

Figure 11. Sketches showing various observations leading to several plasma parameters.

If the two different k-vectors are generated by shifting the radar center frequency, the plasma line spectra at the two radar frequencies f_{01} and f_{02} will appear as shown in Figure 12. Different k-values can also be obtained by changing the geometry in a bistatic setup. In the case shown in Figure 12 we have:

$$(f_{2R} - f_{1R})\underbrace{(f_{2R} + f_{2R})}_{\approx 2 f_e} = 3\,D^2 \cdot f_e^2(k_2 - k_1)(k_2 + k_1) \tag{107}$$

$$\delta f = f_{2R} - f_{1R} = 3\,D^2 \cdot \frac{f_e}{2} \cdot \frac{4\pi(f_{02} - f_{01})4\pi(f_{02} + f_{01})}{e^2} =$$

$$= 24\pi^2 \cdot \frac{v_{th}^2}{\omega_R^2} \cdot f_R \frac{f_{02}^2 - f_{01}^2}{e^2} = \frac{6}{f_R}\frac{T_e}{m}\frac{f_{02}^2 - f_{01}^2}{e^2} \tag{108}$$

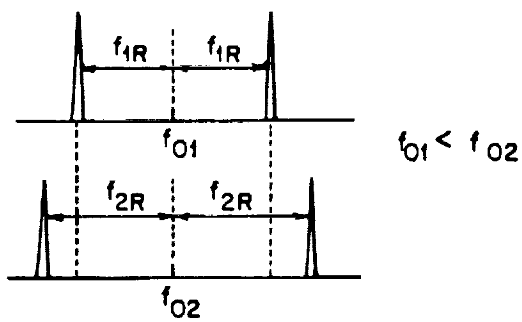

Figure 12. Plasma line spectra at radar frequencies f_{01} and f_{02}.

All of the quantities involved are well determined except for T_e which can be found this way.

Numerical example (EISCAT parameters)

$$T_e \sim 2000°K$$
$$f_{02} = 933MHz$$
$$f_{01} = 224MHz \qquad \delta f \approx 335kHz$$
$$f_e = 5MHz$$

Hence, from a two-frequency plasma line experiment one can deduce the electron temperature accurately and independently.

In Section 2 of these lecture notes we started out considering the scattering from random irregularities in the dielectric function. Throughout the dielectric considered in this paper was a nearly lossless plasma. However, a neutral gas also exhibits random density fluctuations, and the curious readar might wonder whether they are detectable at radio wavelengths. In order to answer this question, consider a gas dense enough to support sound waves. The density fluctuation may, therefore, be considered as a superposition of thermally excited sound-works of varying wavelength and direction. As in Section 4 we expand the parameters (velocity, density, temperature) associated with the acoustic wave-field in a spatial Fourier series:

$$n(\vec{r},t) = \frac{1}{V}\sum_{\vec{k}} n(\vec{k},t)\cdot e^{-i\vec{k}\cdot\vec{r}}$$

where n can be density, pressure, velocity etc. The wave-field amplitudes $n(\vec{k},t)$ all must satisfy the wave-equation

$$\ddot{n}(\vec{k},t) + k^2 c_s^2\, n(\vec{k},t) = 0 \tag{109}$$

where

$$c_s^2 = \frac{T}{M}\cdot\gamma \tag{110}$$

where T is the mean gas temperature in energy units, M the molecular mass and γ (1.4 for air) the ratio of specific heat at constant pressure and constant volume. From equipartition arguments (assuming minute losses to insure equipartition) as used in deriving specific heat of solids we find that

$$< |n(k\omega)|^2 >= \frac{n_0\cdot V}{2\gamma}\left[\delta(\omega - kc_s) + \delta(\omega + kc_s)\right] \tag{111}$$

From Toru Sato's lecture, his equation 2, we see that for dry, nonionized air:

$$\begin{aligned}\Delta\varepsilon(k) &= \varepsilon_0\cdot\frac{1.55\cdot 10^{-9}\, v(k)(mb)}{T(^\circ K)} = \\ &= \varepsilon_0\cdot\frac{1.55\cdot 10^{-7}\, v(k)(N/m^2)}{T(^\circ K)} = \\ &= \varepsilon_0\cdot 1.55\cdot 10^{-7}\,\delta\cdot\kappa\cdot n(k)\end{aligned} \tag{112}$$

where

$$\kappa = \text{Boltzmann's constant} = 1.38\cdot 10^{-23}\, J/^\circ K$$

combining 111 and 112, one obtains:

$$< |\Delta\varepsilon(k)|^2 >= \varepsilon_0^2\,\gamma\cdot\kappa^2\cdot n_0\cdot V\cdot 2.4\cdot 10^{-14} \tag{113}$$

The radar cross section per unit volume of the gas is formed by combining equations (6) and (18) to give:

$$\sigma = \frac{k_0^4}{4\pi}\, 3.73\cdot 10^{-60}\, n_0\, (m^2/m^3)$$

As a specific numerical example, take the Arecibo S-band wavelength and the atmospheric number density at sea level:

$$\lambda_0 = 0.125\, m$$

$$n_0 = 2.688\cdot 10^{28}\, m^{-3}$$

This gives for the troposphere:

$$\sigma_{trop} \approx 5\cdot 10^{-26}\, m^2/m^3$$

As a comparison the cross section per unit volume in the F-region is typically:

$$\sigma_F \approx 10^{-18}\, m^2/m^3$$

If the Arecibo beam could be focused at lower altitude, if full advantage could be taken of the reduced distance and the reduced bandwidth of scattering one might be able to make up for the nearly seven orders of magnitude discrepancy in specific radar cross section. It is probably a much more practical approach to excite low frequency sound waves as done with the Shigaraki MV radar system, and scatter from them.