

# HIERARCHIC MODELS FOR LAMINATED PLATES

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## Abstract

This interim report describes the research being conducted in the formulation of hierarchic models for laminated plates. The work, conducted in collaboration with Professors Babuška and Schwab of the University of Maryland, is an extension of the work done for laminated strips [3]. The use of a single parameter  $\beta$ , representing the degree to which the equilibrium equations of three-dimensional elasticity are satisfied, is being investigated. The powers of  $\beta$  identify members in the hierarchic sequence.

Included in this report are numerical examples that were analyzed with the proposed sequence of models. The results obtained for square plates with uniform loading and with homogeneous boundary conditions are very encouraging. Several cross-ply and angle-ply laminates were evaluated and the results compared with those of the fully three-dimensional model, computed using MSC/PROBE, and with previously reported work on laminated strips [3].

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# 1 Models for laminated plates

Consider an infinite flat plate of constant thickness  $h$  composed of thin layers of orthotropic material perfectly bonded together. Each layer (lamina) possesses a plane of elastic symmetry parallel to the  $x-y$  plane. The laminae are symmetrically arranged with respect to the middle surface of the plate (i.e., the  $x-y$  plane). The load  $q(x, y)$  is antisymmetric with respect to the middle plane, and  $q(x, y) = 0$  for  $|x| \geq a$ ,  $|y| \geq b$ , with  $a$  and  $b$  some fixed number. Let  $\alpha = 1/a$  and  $\gamma = 1/b$ , and further let:

$$\beta = \min(\alpha, \gamma) \quad (1)$$

Assume that the displacement field can be written in exponential form:

$$u_x(x, y, z) = \phi(\beta, z) e^{i\beta(x+y)} \quad (2)$$

$$u_y(x, y, z) = \psi(\beta, z) e^{i\beta(x+y)} \quad (3)$$

$$u_z(x, y, z) = \rho(\beta, z) e^{i\beta(x+y)} \quad (4)$$

where

$$\phi(\beta, z) = \phi_a(\beta, z) + i\phi_b(\beta, z) \quad (5)$$

$$\psi(\beta, z) = \psi_a(\beta, z) + i\psi_b(\beta, z) \quad (6)$$

$$\rho(\beta, z) = \rho_a(\beta, z) + i\rho_b(\beta, z) \quad (7)$$

where  $\phi_a$ ,  $\phi_b$ ,  $\psi_a$ ,  $\psi_b$  are antisymmetric real functions, and  $\rho_a$  and  $\rho_b$  are symmetric real functions with respect to the middle surface of the plate (laminate).

The strain components corresponding to the displacement field given by (2), (3) and (4) are:

$$\epsilon_x = \frac{\partial u_x}{\partial x} = i\beta\phi e^{i\beta(x+y)} \quad (8)$$

$$\epsilon_y = \frac{\partial u_y}{\partial y} = i\beta\psi e^{i\beta(x+y)} \quad (9)$$

$$\epsilon_z = \frac{\partial u_z}{\partial z} = \rho' e^{i\beta(x+y)} \quad (10)$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = i\beta(\phi + \psi) e^{i\beta(x+y)} \quad (11)$$

$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = (\phi' + i\beta\rho) e^{i\beta(x+y)} \quad (12)$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = (\psi' + i\beta\rho) e^{i\beta(x+y)} \quad (13)$$

where the primes represent differentiation with respect to  $z$ .

Let  $x'$ ,  $y'$  be the material (lamina) coordinates for any layer rotated an angle  $\theta$  with respect to the global coordinate system about the  $z$  axis. Then the stress-strain relations in the global (laminate) system for any layer can be written as:

$$\{\sigma\} = [T]^{-1} [C] [T] \{\epsilon\} \quad (14)$$

where  $[C]$  is the lamina material stiffness matrix in the lamina coordinate system ( $x'$ ,  $y'$ ,  $z$ ), and  $[T]$  is the transformation matrix. Defining

$$[Q] = [T]^{-1} [C] [T] \quad (15)$$

as the transformed lamina material matrix, equation (14) can be written as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & Q_{16} \\ & Q_{22} & Q_{23} & 0 & 0 & Q_{26} \\ & & Q_{33} & 0 & 0 & Q_{36} \\ & \text{sym.} & & Q_{44} & Q_{45} & 0 \\ & & & & Q_{55} & 0 \\ & & & & & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}. \quad (16)$$

The equilibrium equations with zero body force components are given by:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad (17)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \quad (18)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0. \quad (19)$$

Substituting equations (8) to (13) and (16) into (17) to (19) the following equilibrium equations are obtained:

$$\begin{aligned} & \{-\beta^2[(Q_{11} + 2Q_{16} + Q_{66})\phi + (Q_{12} + Q_{16} + Q_{26} + Q_{66})\psi] + \\ & i\beta[(Q_{13} + Q_{26})\rho' + (Q_{45}\rho)' + (Q_{55}\rho)'] + \\ & (Q_{45}\psi' + Q_{55}\phi')'\} e^{i\beta(x+y)} = 0 \end{aligned} \quad (20)$$

$$\begin{aligned} & \{-\beta^2[(Q_{12} + Q_{16} + Q_{26} + Q_{66})\phi + (Q_{22} + 2Q_{26} + Q_{66})\psi] + \\ & \quad i\beta[(Q_{23} + Q_{36})\rho' + (Q_{44}\rho)' + (Q_{45}\rho)'] + \\ & \quad (Q_{44}\psi' + Q_{45}\phi')\}' e^{i\beta(x+y)} = 0 \end{aligned} \quad (21)$$

$$\begin{aligned} & \{-\beta^2[(Q_{44} + 2Q_{45} + Q_{55})\rho] + \\ & \quad i\beta[(Q_{44} + Q_{45})\psi' + (Q_{45} + Q_{55})\phi' + ((Q_{13} + Q_{36})\phi)' + \\ & \quad ((Q_{23} + Q_{36})\psi)'] + (Q_{33}\rho')\}' e^{i\beta(x+y)} = 0 \end{aligned} \quad (22)$$

Expanding  $\phi(\beta, z)$ ,  $\psi(\beta, z)$  and  $\rho(\beta, z)$  into a Taylor series with respect to  $\beta$ :

$$\begin{aligned} \phi(\beta, z) &= [\phi_{a0}(z) + i\phi_{b0}(z)] + \beta[\phi_{a1}(z) + i\phi_{b1}(z)] + \beta^2[\phi_{a2}(z) + i\phi_{b2}(z)] + \dots \\ \psi(\beta, z) &= [\psi_{a0}(z) + i\psi_{b0}(z)] + \beta[\psi_{a1}(z) + i\psi_{b1}(z)] + \beta^2[\psi_{a2}(z) + i\psi_{b2}(z)] + \dots \\ \rho(\beta, z) &= [\rho_{a0}(z) + i\rho_{b0}(z)] + \beta[\rho_{a1}(z) + i\rho_{b1}(z)] + \beta^2[\rho_{a2}(z) + i\rho_{b2}(z)] + \dots \end{aligned}$$

On substituting into the equilibrium equations (20), (21), (22) and separating into real and imaginary parts (only the real parts are shown here) we have:

The real part of (20):

$$\begin{aligned} & (Q_{45}\psi'_{a0} + Q_{55}\phi'_{a0})' + \beta[(Q_{45}\psi'_{a1} + Q_{55}\phi'_{a1})' - (Q_{13} + Q_{26})\rho'_{b0} - \\ & \quad ((Q_{45} + Q_{55})\rho_{b0})'] + \\ & \quad + \beta^2[(Q_{45}\psi'_{a2} + Q_{55}\phi'_{a2})' - (Q_{13} + Q_{26})\rho'_{b1} - \\ & \quad ((Q_{45} + Q_{55})\rho_{b1})' - (Q_{11} + 2Q_{16} + Q_{66})\phi_{a0} - \\ & \quad (Q_{12} + Q_{16} + Q_{26} + Q_{66})\psi_{a0}] + \dots = 0 \end{aligned} \quad (23)$$

The real part of (21):

$$\begin{aligned} & (Q_{44}\psi'_{a0} + Q_{45}\phi'_{a0})' + \beta[(Q_{44}\psi'_{a1} + Q_{45}\phi'_{a1})' - (Q_{23} + Q_{36})\rho'_{b0} - \\ & \quad ((Q_{44} + Q_{45})\rho_{b0})'] + \\ & \quad + \beta^2[(Q_{44}\psi'_{a2} + Q_{45}\phi'_{a2})' - (Q_{23} + Q_{36})\rho'_{b1} - \\ & \quad ((Q_{44} + Q_{45})\rho_{b1})' - (Q_{12} + Q_{16} + Q_{26} + Q_{66})\phi_{a0} - \\ & \quad (Q_{22} + 2Q_{26} + Q_{66})\psi_{a0}] + \dots = 0 \end{aligned} \quad (24)$$

The real part of (22):

$$\begin{aligned} & (Q_{33}\rho'_{a0})' + \beta[(Q_{33}\rho'_{a1})' - (Q_{44} + Q_{45})\psi'_{b0} - (Q_{45} + Q_{55})\phi'_{b0} - \\ & \quad ((Q_{13} + Q_{36})\phi_{b0})' - ((Q_{23} + Q_{36})\psi_{b0})'] + \\ & \quad + \beta^2[(Q_{33}\rho'_{a2})' - (Q_{44} + Q_{45})\psi'_{b1} - (Q_{45} + Q_{55})\phi'_{b1} - \\ & \quad ((Q_{13} + Q_{36})\phi_{b1})' - ((Q_{23} + Q_{36})\psi_{b1})' - \\ & \quad (Q_{44} + 2Q_{45} + Q_{55})\rho_{a0}] + \dots = 0 \end{aligned} \quad (25)$$

These equations hold for any choice of  $\beta$ . Solving for each power of  $\beta$  we obtain the transverse shape functions as described in next section.

## 1.1 The model characterized by $\beta^0$

Setting  $\beta = 0$  in equations (23) to (25) we have:

$$(Q_{45}\psi'_{a0} + Q_{55}\phi'_{a0})' = 0 \quad (26)$$

$$(Q_{44}\psi'_{a0} + Q_{45}\phi'_{a0})' = 0 \quad (27)$$

$$(Q_{33}\rho'_{a0})' = 0. \quad (28)$$

Knowing that  $\phi_{a0}(z)$ ,  $\psi_{a0}(z)$  are antisymmetric and  $\rho_{a0}(z)$  is symmetric, and integrating, we have:

$$\phi_{a0}(z) = F_0(z) \quad (29)$$

$$\psi_{a0}(z) = G_0(z) \quad (30)$$

$$\rho_{a0}(z) = 1. \quad (31)$$

Similarly, solving the imaginary part of the equilibrium equations we get:

$$\phi_{b0}(z) = F_0(z) \quad (32)$$

$$\psi_{b0}(z) = G_0(z) \quad (33)$$

$$\rho_{b0}(z) = 1. \quad (34)$$

where

$$F_0(z) = \int_0^z \frac{Q_{44} - Q_{45}}{Q_{44}Q_{55} - Q_{45}^2} dz \quad (35)$$

$$G_0(z) = \int_0^z \frac{Q_{55} - Q_{45}}{Q_{44}Q_{55} - Q_{45}^2} dz. \quad (36)$$

To obtain these transverse functions we adopted all integration constants to be either 0 or 1. This accomplishes one important aspect, i.e., there is only 'one' transverse function per field. If the integration constants are arbitrary, the solution is:

$$\phi_{a0}(z) = \int_0^z \frac{a_0 Q_{44} - b_0 Q_{45}}{Q_{44}Q_{55} - Q_{45}^2} dz \quad (37)$$

$$\psi_{a0}(z) = \int_0^z \frac{b_0 Q_{55} - a_0 Q_{45}}{Q_{44}Q_{55} - Q_{45}^2} dz \quad (38)$$

$$\rho_{a0}(z) = c_0. \quad (39)$$

Note that with either choice of the integration constants, the resulting transverse functions (29), (30), (31) or (37), (38), (39) satisfy the equilibrium equations. Selecting the integration constants to be some convenient numbers is acceptable because the equilibrium equations are still satisfied.

The real and imaginary parts are not linearly independent, hence both lead to the same functional form. The mode of deformation corresponding to  $\beta = 0$  can be written in the following form:

$$u_x(x, y, z) = u_1(x, y) F_0(z) \quad (40)$$

$$u_y(x, y, z) = u_2(x, y) G_0(z) \quad (41)$$

$$u_z(x, y, z) = u_3(x, y). \quad (42)$$

When  $Q_{44}$ ,  $Q_{45}$ ,  $Q_{55}$  are constant through the thickness, this model is capable of representing rigid body displacement and rotation. However, this model does not satisfy the condition of converging to the same limit as the problem of elasticity as  $h \rightarrow 0$ , unless some adjustments are introduced to the material properties as discussed later.

## 1.2 The model characterized by $\beta^1$

To find the mode of deformation for the model which satisfies the equilibrium equations up to the first power of  $\beta$ , we differentiate (23) to (25) with respect to  $\beta$  and let  $\beta = 0$ . In this case we have:

$$(Q_{45}\psi'_{a1} + Q_{55}\phi'_{a1})' - (Q_{13} + Q_{26})\rho'_{b0} - ((Q_{45} + Q_{55})\rho_{b0})' = 0 \quad (43)$$

$$(Q_{44}\psi'_{a1} + Q_{45}\phi'_{a1})' - (Q_{23} + Q_{36})\rho'_{b0} - ((Q_{44} + Q_{45})\rho_{b0})' = 0 \quad (44)$$

$$\begin{aligned} (Q_{33}\rho'_{a1})' - (Q_{44} + Q_{45})\psi'_{b0} - (Q_{45} + Q_{55})\phi'_{b0} - \\ ((Q_{13} + Q_{36})\phi_{b0})' - ((Q_{23} + Q_{36})\psi_{b0})' = 0. \end{aligned} \quad (45)$$

Upon integration we have:

$$\phi_{a1}(z) = F_0(z) + z \quad (46)$$

$$\psi_{a1}(z) = G_0(z) + z \quad (47)$$

$$\rho_{a1}(z) = 1 + H_0(z) \quad (48)$$

and solving the imaginary part of the equilibrium equations:

$$\phi_{b1}(z) = F_0(z) - z \quad (49)$$

$$\psi_{b1}(z) = G_0(z) - z \quad (50)$$

$$\rho_{b1}(z) = 1 - H_0(z) \quad (51)$$

where

$$H_0(z) = \int_0^z \left[ \frac{2z + (Q_{13} + Q_{36}) F_0 + (Q_{23} + Q_{36}) G_0}{Q_{33}} \right] dz. \quad (52)$$

The displacement field in this case is:

$$u_x(x, y, z) = u_1(x, y) F_0(z) + u_4(x, y) z \quad (53)$$

$$u_y(x, y, z) = u_2(x, y) G_0(z) + u_5(x, y) z \quad (54)$$

$$u_z(x, y, z) = u_3(x, y) + u_6(x, y) H_0(z). \quad (55)$$

### 1.3 The model characterized by $\beta^2$

To find the mode of deformation for the model which satisfies the equilibrium equations up to the second power of  $\beta$ , we differentiate (23) to (25) twice with respect to  $\beta$  and let  $\beta = 0$ . Upon integration, the following results are obtained:

$$\phi_{a2}(z) = F_0(z) + z + F_1(z) \quad (56)$$

$$\psi_{a2}(z) = G_0(z) + z + G_1(z) \quad (57)$$

$$\rho_{a2}(z) = 1 + H_0(z) - H_1(z) \quad (58)$$

where

$$F_1(z) = \int_0^z \left( \frac{M_0 Q_{44} - N_0 Q_{45}}{Q_{44} Q_{55} - Q_{45}^2} - H_0 \right) dz \quad (59)$$

$$G_1(z) = \int_0^z \left( \frac{N_0 Q_{55} - M_0 Q_{45}}{Q_{44} Q_{55} - Q_{45}^2} - H_0 \right) dz \quad (60)$$

$$H_1(z) = \int_0^z \left[ \frac{(Q_{13} + 2Q_{36} + Q_{23})}{Q_{33}} \right] z dz. \quad (61)$$

and

$$\begin{aligned}
M_0(z) = \int_0^z & \left[ \left( Q_{11} + 2Q_{16} + Q_{66} - (Q_{13} + Q_{26}) \frac{Q_{13} + Q_{36}}{Q_{33}} \right) F_0 \right. \\
& + \left( Q_{12} + Q_{16} + Q_{26} + Q_{66} - (Q_{13} + Q_{26}) \frac{Q_{23} + Q_{36}}{Q_{33}} \right) G_0 \\
& \left. - 2z \frac{Q_{13} + Q_{26}}{Q_{33}} \right] dz. \tag{62}
\end{aligned}$$

$$\begin{aligned}
N_0(z) = \int_0^z & \left[ \left( Q_{12} + Q_{16} + Q_{26} + Q_{66} - (Q_{23} + Q_{36}) \frac{Q_{13} + Q_{36}}{Q_{33}} \right) F_0 \right. \\
& + \left( Q_{22} + 2Q_{26} + Q_{66} - (Q_{23} + Q_{36}) \frac{Q_{23} + Q_{36}}{Q_{33}} \right) G_0 \\
& \left. - 2z \frac{Q_{23} + Q_{36}}{Q_{33}} \right] dz. \tag{63}
\end{aligned}$$

Therefore the displacement field can be written in the form:

$$u_x(x, y, z) = u_1(x, y) F_0(z) + u_4(x, y) z + u_7(x, y) F_1(z) \tag{64}$$

$$u_y(x, y, z) = u_2(x, y) G_0(z) + u_5(x, y) z + u_8(x, y) G_1(z) \tag{65}$$

$$u_z(x, y, z) = u_3(x, y) + u_6(x, y) H_0(z) + u_9(x, y) H_1(z). \tag{66}$$

This mode of deformation satisfies both the real and imaginary parts of the equilibrium equations up to the second power of  $\beta$ . By continuing this process, the equilibrium equations can be satisfied to an arbitrary power of  $\beta$ .

## 2 The limiting case when $\beta \rightarrow 0$

One of the requirements of the hierarchic sequence of models is that each member converges to the same limit as the exact solution of the problem of elasticity as  $h \rightarrow 0$ . We know from the evaluation of the laminated strip that the first two members of the hierarchy do not meet this requirement if some materials properties are not adjusted. We also learned that the adjustments are different for the first and for the second member of the hierarchy. The model which satisfies the equilibrium equations up to the second power of  $\beta$  is the first member of the hierarchic sequence of strip models which converges to the right limit as  $h \rightarrow 0$ .



The exact solution minimizes the potential energy with respect to all functions  $u_i(x, y)$ ,  $i = 1, 2, \dots$  for which the strain energy is finite. The limit for each model is obtained as follows:

1. Start with the expression of the potential energy:

$$\begin{aligned} \Pi = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-h/2}^{+h/2} & \left( \sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} \right. \\ & \left. + \tau_{yz} \gamma_{yz} \right) dx dy dz = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(x, y) u_z(x, y, h/2) dx dy \end{aligned} \quad (67)$$

For a given plate model, compute the strain components and perform the integration through the thickness ( $z$  direction) to obtain the material coefficients. These coefficients form the laminate material stiffness matrix  $E_{ij}$ . Rewrite the potential energy in terms of the  $E_{ij}$ .

$$\Pi = \frac{1}{2} \iint_{\Omega} \{u\}^T [E] \{u\} dx dy - \iint_{\Omega} q u_z(x, y, h/2) dx dy \quad (68)$$

In the case of the  $\beta^0$  model, the potential energy expression is:

$$\begin{aligned} \Pi = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} & \left[ E_1 \left( \frac{\partial u_1}{\partial x} \right)^2 + 2 E_2 \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial x} + E_3 \left( \frac{\partial u_2}{\partial x} \right)^2 + E_4 \left( \frac{\partial u_1}{\partial y} \right)^2 + \right. \\ & 2 E_5 \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial y} + E_6 \left( \frac{\partial u_2}{\partial y} \right)^2 + E_7 \left( \frac{\partial u_3}{\partial x} \right)^2 + 2 E_8 \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial y} + E_9 \left( \frac{\partial u_3}{\partial y} \right)^2 + \\ & E_{10} u_1^2 + 2 E_{11} u_1 u_2 + E_{12} u_2^2 + 2 E_{13} \frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial y} + 2 E_{14} \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} + \\ & 2 E_{15} \frac{\partial u_2}{\partial x} \frac{\partial u_1}{\partial y} + 2 E_{16} \frac{\partial u_2}{\partial x} \frac{\partial u_2}{\partial y} + 2 E_{17} u_1 \frac{\partial u_3}{\partial x} + 2 E_{18} u_2 \frac{\partial u_3}{\partial x} + 2 E_{19} u_1 \frac{\partial u_3}{\partial y} + \\ & \left. 2 E_{20} u_2 \frac{\partial u_3}{\partial y} \right] dx dy - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} q(x, y) u_3(x, y, h/2) dx dy \end{aligned} \quad (69)$$

2. Obtain the Euler equations by taking the variation of the potential energy with respect to each one of the field variables  $u_i(x, y)$ ,  $i = 1, 2, \dots$ . Apply Fourier transform to the Euler equations and construct the system of linear equations in the transformed field variables  $U_i(\xi, \eta)$ :

$$[A] \{U\} = \{R\} \quad (70)$$

The matrix  $[A]$  depends on the material stiffness matrix  $E_{ij}$ , and on the Fourier variables  $\xi$ , and  $\eta$ , and  $\{R\}$  is the load vector obtained from the transformation of the potential of the external forces.

3. Compute values for  $E_i$ ; for different stacking sequences and solve the system of equations for  $U_3$  for each one of them:

$$D(\xi, \eta) U_3 = B(\xi, \eta) Q \quad (71)$$

where

$$D(\xi, \eta) = D_1 \xi^2 + D_2 \xi \eta + D_3 \eta^2 + D_4 \xi^4 + D_5 \xi^3 \eta + D_6 \xi^2 \eta^2 + D_7 \xi \eta^3 + D_8 \eta^4 + \dots \quad (72)$$

is the determinant of  $[A]$ , and

$$B(\xi, \eta) = B_0 + B_1 \xi^2 + B_2 \xi \eta + B_3 \eta^2 + \dots \quad (73)$$

is the determinant of  $[A]$  when the third row was replaced by the load vector  $\{R\}$ .

4. Perform the limit analysis as  $h \rightarrow 0$ . First divide  $D_i$  by  $B_0$ :

$$\alpha_j = \frac{D_i}{B_0} \quad \text{for } i = 1 \rightarrow 3, j = 1 \rightarrow 3 \quad (74)$$

$$\beta_j = \frac{D_i}{B_0} \quad \text{for } i = 4 \rightarrow 8, j = 1 \rightarrow 5 \quad (75)$$

Note that  $B_0$  is the first non-zero term of  $B(\xi, \eta)$ . Neglecting derivatives higher than fourth order we obtain the general equation :

$$(\alpha_1 \xi^2 + \alpha_2 \xi \eta + \alpha_3 \eta^2) U_3 + (\beta_1 \xi^4 + \beta_2 \xi^3 \eta + \beta_3 \xi^2 \eta^2 + \beta_4 \xi \eta^3 + \beta_5 \eta^4) U_3 = Q. \quad (76)$$

This last equation has the following form after inverse Fourier transform:

$$\left( \alpha_1 \frac{\partial^2 u_3}{\partial x^2} + \alpha_2 \frac{\partial^2 u_3}{\partial x \partial y} + \alpha_3 \frac{\partial^2 u_3}{\partial y^2} \right) + \left( \beta_1 \frac{\partial^4 u_3}{\partial x^4} + \beta_2 \frac{\partial^4 u_3}{\partial x^3 \partial y} + \beta_3 \frac{\partial^4 u_3}{\partial x^2 \partial y^2} + \beta_4 \frac{\partial^4 u_3}{\partial x \partial y^3} + \beta_5 \frac{\partial^4 u_3}{\partial y^4} \right) = q. \quad (77)$$

If the hierarchic plate model being evaluated converges to the proper limit, the coefficients  $\alpha_i$  should be zero. This is because all models should converge to the Kirchhoff model as  $h \rightarrow 0$  [3], which requires that  $\alpha_i = 0$ . If the  $\alpha_i$  are not zero, the model needs adjustment of the material properties. The  $\beta_i$  may require adjustment also, so that they have the same values as those of the models that converge to the same limit as the theory of elasticity as  $h \rightarrow 0$ .

Following this procedure, it has been found using Mathematica<sup>1</sup> that the model characterized by  $\beta^0$  is not a member of the hierarchy. However, making the transverse shear moduli constant through the thickness, the  $\alpha_i$  become zero. Further discussion of this model is deferred to next section.

<sup>1</sup>Mathematica: A system for doing mathematics by computer. Wolfrman Research Inc.

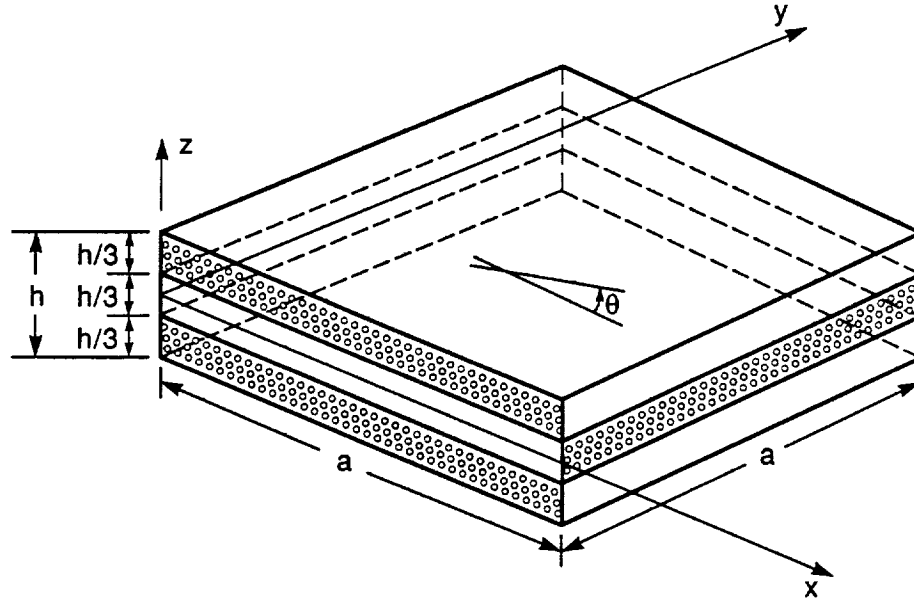


Figure 1: Model problems: Notation.

### 3 Examples

Consider a square plate of uniform thickness  $h$  and planar dimensions  $a$  and  $b = a$ , composed of perfectly bonded orthotropic layers, symmetrically distributed with respect to the middle plane, (Fig. 1). The number of layers, the stacking sequence (orientation of the material axes with respect to the global axes), and the boundary conditions are the variables used in several representative model problems.

Uniform load is applied as a normal traction to the top and bottom surfaces of the plate. All layers in the laminate are of equal thickness, and are of a square symmetric unidirectional fibrous composite material possessing the following stiffness properties, which simulate a high-modulus graphite/epoxy composite:

$$E_L = 25.0 \times 10^6 \text{ psi} \quad E_T = 1.0 \times 10^6 \text{ psi}$$

$$G_{LT} = 0.5 \times 10^6 \text{ psi} \quad G_{TT} = 0.2 \times 10^6 \text{ psi}$$

$$\nu_{LT} = \nu_{TT} = 0.25$$

where  $L$  indicates the direction parallel to the fibers,  $T$  is the transverse direction, and  $\nu_{LT}$  is the Poisson ratio (i.e.,  $\nu_{LT} = -\epsilon_{TT}/\epsilon_{LL}$ , where  $\epsilon_{TT}$ ,  $\epsilon_{LL}$  are, respectively, the normal strains in the directions  $T$  and  $L$ ). These material properties were selected from reference [1].

When the  $L$  direction coincides with the  $x$  direction, we refer to it as the  $\theta = 0^\circ$  orientation. For a three-ply laminate a designation 90/0/90 means that the central lamina is oriented with the  $L$  direction parallel to the global  $x$ -axis, and in the two outer layers  $L$  is at  $90^\circ$  with the global  $x$ -axis.

In order to establish a reference solution which can be regarded as being sufficiently close to the exact solution of the problem of elasticity, the problem was solved using the finite element program MSC/PROBE<sup>2</sup> and an experimental program in which the algorithm described in this report is implemented. In the reference solution obtained with MSC/PROBE each layer was discretized as a three-dimensional element with orthotropic material properties. The solution was obtained for  $p$  ranging from 1 through 8. In energy norm was below 1% at  $p = 8$ . The solution corresponding to  $p = 8$  will be used as the basis for comparison.

The solutions corresponding to the proposed hierarchic models were obtained using only one laminated plate element. The polynomial degree was varied from 1 through 8 and the equilibrium equations were satisfied up to  $\beta^1$ .

The model that satisfies the equilibrium equations up to the zeroeth power of  $\beta$  was modified as indicated in the previous Section. The transverse shear moduli of each layer ( $Q_{44}$  and  $Q_{55}$ ) were made equal to the harmonic averages  $\hat{Q}_{44}$  and  $\hat{Q}_{55}$ , while  $Q_{45}$  was made equal to the average  $\bar{Q}_{45}$ . In the case of three layers for instance, the harmonic average of  $Q_{44}$  is:

$$\hat{Q}_{44} = 3 \left( \frac{1}{Q_{44}^{(1)}} + \frac{2}{Q_{44}^{(2)}} \right)^{-1} \quad (78)$$

The following changes were introduced for each layer:

$$\tilde{Q}_{ij} = Q_{ij} - \frac{Q_{i3} Q_{j3}}{Q_{33}}, \quad i, j = 1, 2, 6. \quad (79)$$

This modification is necessary to account for the plane stress constitutive equations for each layer used in the Kirchhoff and Reissner-Mindling type plate models [2], [6].

We will denote the modified model characterized by  $\beta^0$  with  $\beta_m^0$ . The plate deflection at a given location  $(x_n, y_n, z_n)$  was selected to evaluate the models. The normalized plate deflection is defined as:

$$U_z \stackrel{\text{def}}{=} \frac{100 E_T h^3 u_z(x_n, y_n, 0)}{q a^4} \quad (80)$$

where  $q$  is the applied traction,  $h$  is the thickness of the plate and  $u_z(x_n, y_n, 0)$  is the vertical displacement of the middle plane of the plate at  $x = x_n$ ,  $y = y_n$ .

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<sup>2</sup>MSC/PROBE: User's Manual, The MacNeal-Schwendler Corporation, 1600 S. Brentwood Blvd., Suite 840, St. Louis, Missouri 63144.

## Normalized Central Displacement Three-ply Laminate (90/0/90)

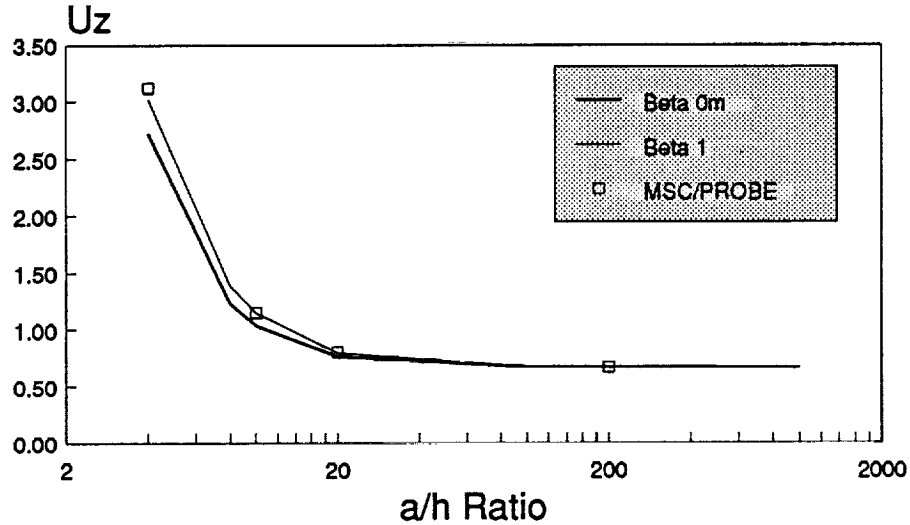


Figure 2: Orthotropic square plate. Four sides simply supported.

### 3.1 Cross-ply laminate

The results for a three-ply orthotropic (or cross-ply) simply supported square plate are shown in Fig. 2. For large  $a/h$  ratios both models yield similar results. As  $a/h$  decreases, the  $\beta^0$  model underestimates the deflection while the  $\beta^1$  model is very close to the MSC/PROBE solution.

Fig. 3 show the results for a three-ply (90/0/90) square plate with two opposite sides simply supported and the other two free. The central deflection of the plate is compared with the results of the laminated strip model described in reference [3]. Fig. 4 correspond to the same problem but with five layers (90/0/90/0/90). The agreement between the beam and plate models is excellent.

The influence of the number of layers in the end deflection of a square plate with one end clamped and the other three free is shown in Fig. 5 for three different  $a/h$  ratios. In all cases the fibers in the outer layers were normal to the clamped edge of the plate. Also included in the figure are the results of the deflection computed using a simplified beam formula which is valid for the case  $a/h \rightarrow \infty$ . According to reference [4] the end deflection of a cantilever beam of length  $a$  and thickness  $h$  with uniform load  $q$  is:

$$u_z = \frac{q a^4}{8 D_{11}} \quad (81)$$

### Normalized Central Displacement Three-ply Laminate (90/0/90)

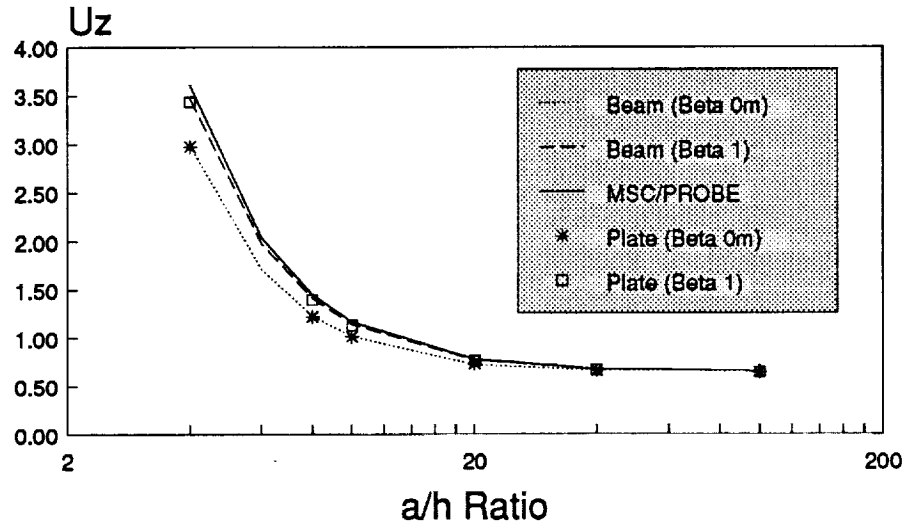


Figure 3: Orthotropic square plate. Two sides simply supported.

### Normalized Central Displacement Five-ply Laminate (90/0/90/0/90)

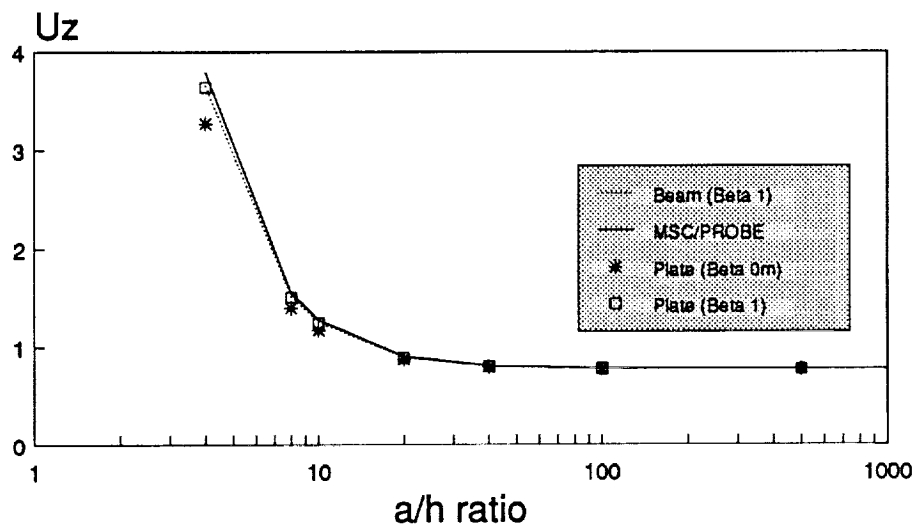
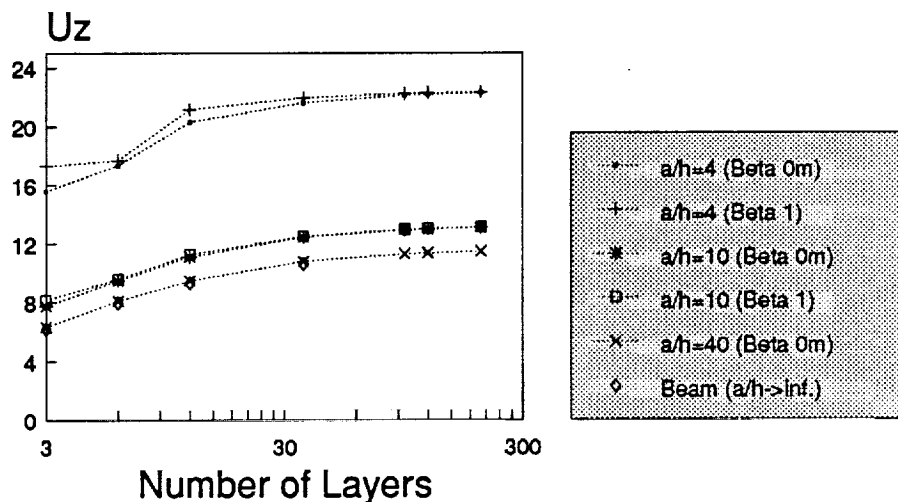


Figure 4: Orthotropic square plate. Two sides simply supported.

## Normalized End Displacement Influence of Number of Layers



Outer fibers normal to clamped edge

Figure 5: Orthotropic square plate. One side clamped.

where

$$D_{11} = \int_{-h/2}^{+h/2} Q_{11} z^2 dz \quad (82)$$

The deflection computed by the use of (81) is identified as 'Beam ( $a/h \rightarrow \infty$ )'. The results indicate that when the number of layers increases the bending stiffness of the plate decreases to an asymptotic value. In the limit when we have an infinite number of alternating plies, the laminate will become quasi-homogeneous. The property of the laminate will be square symmetric but not homogeneous [5].

### 3.2 Angle-ply laminate

The results for a three- and four-ply simply supported square angle-ply laminated plate are shown in Fig. 6 and in Fig. 7 respectively. In this case the stacking sequence is such that all layers are oriented at either  $45^\circ$  or  $-45^\circ$ . For the three-ply plate, the agreement between the MSC/PROBE solution and the hierarchic models is similar to the case of cross-ply laminates. For the four-ply plate no MSC/PROBE results are still available.

Normalized Central Displacement  
Three-ply Laminate (-45/45/-45)

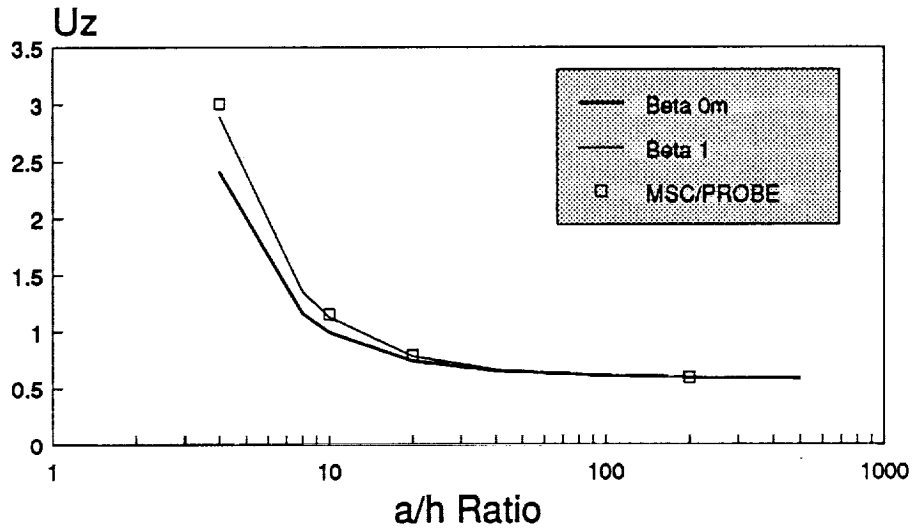


Figure 6: Angle-ply square plate. Four sides simply supported.

Normalized Central Displacement  
Four-ply Laminate (-45/45/45/-45)

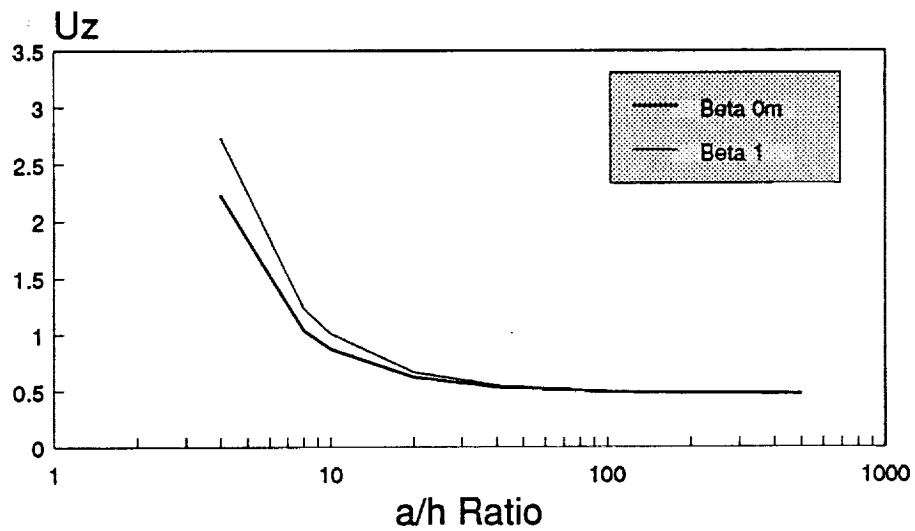


Figure 7: Angle-ply square plate. Four sides simply supported.



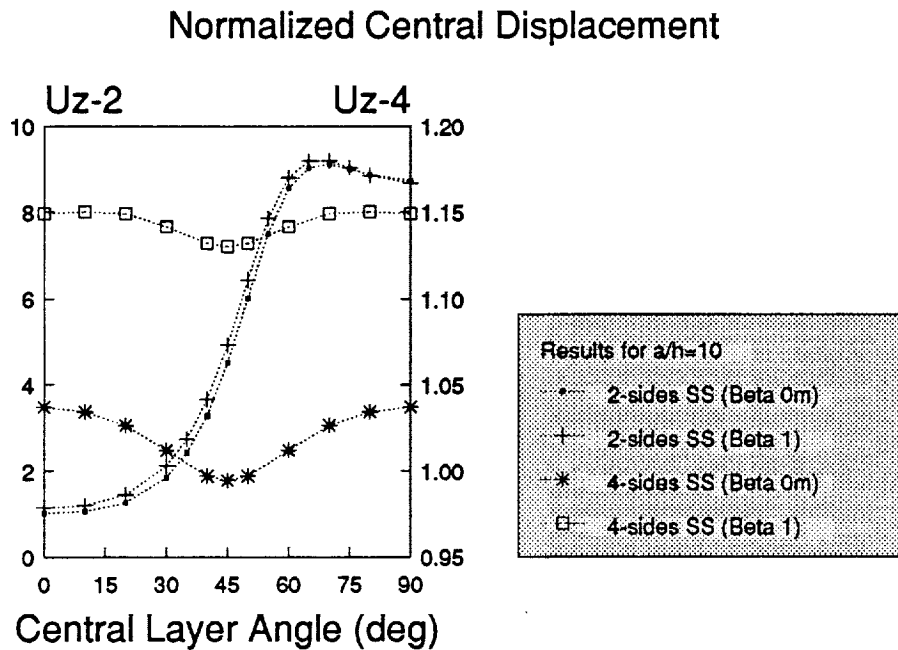


Figure 8: Angle-ply square plate. Influence of fiber orientation.

The influence of fiber orientation in the central deflection of a three-ply square plate with two opposite sides simply supported (2-sides SS) and Four sides simply supported (4-sides SS) is shown in Fig. 8. In this case the  $a/h$  ratio was kept constant at  $a/h = 10$  and the orientation of the fibers in the central layer was varied between 0 and 90°. The fibers in the outer layers were always at 90° with the ones of the central layer. The results for the  $\beta^0$  and  $\beta^1$  models are included for each boundary condition.

## 4 Summary and conclusions

1. Hierarchic models for mid-plane symmetric laminated plates have been developed based on a single parameter  $\beta$ . The powers of the parameter  $\beta$  representing the degree to which the equilibrium equations of three-dimensional elasticity are satisfied, have been used to identify members of the hierarchic sequence.
2. The model characterized by  $\beta^0$  is the Reissner-Mindlin model, generalized for laminated composites, when the modified material properties are used (also

known as *first order shear deformation model*). In the special case, when the shear modulus is independent of  $z$ , the hierarchic model is the Reissner-Mindlin model. The shear correction factor can be assigned arbitrarily since the requirements set for hierarchic models are satisfied independently of the shear correction factor.

3. The advantage of a single parameter sequence of models is that the number of fields added per level is always the same. Three fields are added per increment of power of  $\beta$ , regardless of the number of layers.
4. Good correlation between the proposed hierarchic sequence and a three-dimensional solution has been found for the problems investigated. More work is under way to compare not only displacements but stress distributions as well.

## 5 References

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