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CMU-HEP91-08
FNAL-PUB-91/167-T&A
June 26, 1991

Statistical mechanics of soft-boson phase transitions

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The existence of structure on large (say, 100 Mpc) scales, and limits to anisotropies in the cosmic microwave background radiation (CMBR), have imperiled models of structure formation based solely upon the standard cold dark matter scenario. Novel scenarios, which may be compatible with large-scale structure and small CMBR anisotropies, invoke non-linear fluctuations in the density appearing *after* recombination, accomplished via the use of *late-time phase transitions* involving ultra-low mass scalar bosons. Here, we study the statistical mechanics of such phase transitions in several models involving *naturally* ultra-low mass pseudo-Nambu–Goldstone bosons (pNGB's). These models can exhibit several interesting effects at high temperature, which we argue are the most general possibilities for pNGB's.

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

I. INTRODUCTION

As the Universe is probed on larger scales, evidence for very large scale structures seems to be emerging. Voids, filaments, and walls on scales as large as 100 Mpc have been observed in various redshift surveys.[1] The existence of these structures, constraints on the anisotropies of the microwave background temperature,[2] and the existence of quasars at redshifts larger than 4,[3] makes it extremely difficult to understand the origin of large scale structure within the framework of the standard gravitational instability theory, i.e, cold dark matter with density perturbations coming from inflationary models.[4]

Many of the constraints on structure formation models can be obviated if a mechanism could be found which allows for density inhomogeneities to appear at a redshift z satisfying $z_{\rm rec} > z \gg 1$ with $\delta\rho/\rho \sim 1$ (here $z_{\rm rec} \sim 1000$ is the redshift at recombination). In this case, the perturbations will *not* affect the CMBR directly (but may have significant indirect effects) yet structure on large scales will have ample time to grow so as to satisfy the constraint coming from quasar observations.

Taking our cue from the fact that sources of density fluctuations may arise from the effects of phase transitions, it is interesting to ask whether phase transitions could occur at *late* times (i.e., after decoupling) in such a way as to generate large density fluctuations. This is not a new idea. Wasserman[5] showed that the existence of a first-order phase transition at late times could generate large fluctuations due to bubble collisions. Hill, Schramm, and Fry[6] proposed the idea of domain wall formation in late-time phase transitions in the context of ultra-low-mass pseudo-Nambu–Goldstone bosons that can readily occur in a wide class of models.[7,8,9] The implications of late time phase transitions with soft domain walls has been a subject of considerable activity in recent years.[10]

Press, Ryden, and Spergel considered the possibility of a slow-roll transition in a soft-boson model,[11] which not only drove structure formation, but also implied that the dark

matter is the residual oscillations of the field about the potential minimum. Schramm and Fuller[12] have also considered such scenarios within the context of Majoron models.[13] These different approaches have a common theme: *Ultra-low mass particles, typically spin–0 bosons, are a generic component of all such models.*

The most familiar spin–0 particles occuring in nature are the π mesons. The scale of the masses of the pions in comparison to the scale of nucleon mass is small, $m_\pi^2 \ll m_N^2$. This is well understood: the pions are pseudo–Nambu–Goldstone bosons (pNGB's) associated with the dynamical breaking of fermionic *chiral* symmetries. In the limit of vanishing up and down quark masses, $m_{u,d} \to 0$, the pion masses go to zero, $m_\pi^2 \to 0$, and the pions become exact Nambu–Goldstone bosons (NGB's). Most of our intuition about pNGB's derives from this established system, which is one of the most profound in elementary particle physics. We will exploit and develop the analogy with this system in greater detail in Section IV. The basic lesson is: *Many continuous (perhaps approximate) global symmetries may exist in nature that are spontaneously broken, and have associated NGB's (or pNGB's) with phenomenological implications.*

One such example is the familiar axion,[14] a hypothetical pNGB associated with the Peccei–Quinn (PQ) $U(1)$ symmetry. The PQ–symmetry is broken by QCD instanton effects and the axion thus develops a small mass, $m_{\rm axion} \sim \mathcal{O}(m_\pi f_\pi / f_{\rm axion})$, where one conventionally assumes $f_{\rm axion} \sim 10^{12}$ GeV. Thus, the Compton wavelength of the axion is measured in centimeters, requiring the construction of macroscopic microwave cavities as detectors. Of course, since very low mass particles such as the axion are very difficult to detect, few theorists spend their time trying to invent new ones. However, the axion, which is a respectable, if not desirable, theoretical entity portends an important lesson: the physical world may contain many new phenomena in the far infrared which are not directly accessible, but nevertheless may play an important role in nature.

A generalization of axions to a class of pNGB's with masses of order $m_{\rm fermion}^2/f$ has

been analyzed in some detail.[7] Remarkably, if one associates $m_{\rm fermion} \sim m_{\rm neutrino} \sim 0.01$ eV and $f \sim M_{GUT} \sim 10^{15}$ GeV, one arrives at a cosmologically interesting scale $\lambda \sim f/m_\nu^2 \sim$ Megaparsecs. This can lead naturally to a late-time phase transition.

Embedded into theories that contain NGB's or pNGB's is our only rational guideline in thinking about ultra-low mass particles: the principle of "naturalness." In this regard, the mass scales of such particles must not be fine tuned, and must appear as a consequence of some plausible mechanism. 't Hooft first gave a concrete definition of the principle of naturalness: *a parameter is "naturally" small if when it is set to zero, the symmetry of the Lagrangian is increased.*[15] In this case, the parameter will be *multiplicatively renormalized* and will remain small to all orders of perturbation theory. While the cosmological applications are insensitive to whether or not a given model Lagrangian has been fine tuned, the form of any given low-energy effective Lagrangian, or its finite-temperature corrections, will be strongly influenced by the symmetries of the interactions of the full theory, and therefore we focus on natural models.

More generally, there are two versions of the naturalness principle: (1) "Strong Naturalness," in which the very low mass scales must emerge on the grounds of symmetry and dynamics without the input of any large hierarchy (for example, technicolor theories respect this principle as a means of generating the hierarchy involving the W mass and the Planck mass, $M_W/m_{Pl} \sim 10^{-17}$, although they have difficulty accomodating the observed large quark and lepton masses) (2) "Weak Naturalness," in which one inputs a large hierarchy *ab initio*, which is then protected by a symmetry in the theory from being overturned by radiative corrections (for example, supersymmetry operates in this mode of protecting the hierarchy M_W/m_{Pl}, and "chiral" symmetries protect small ratios like $m_{\rm neutrino}/m_{\rm electron} \leq 10^{-5}$).

The axion falls into the category of strong naturalness, since it would be an identically massless particle by virtue of a symmetry principle if it were not for QCD effects

(instantons) which spoil the symmetry and are operant at energy scales of about 1 GeV (the QCD scale arises naturally from, e.g., any Grand Unified Theory upon specifying α_{QCD} at the GUT scale). Other kinds of pNGB's, having masses given by approximate expressions such as $m_\phi \sim m^2_{\rm fermion}/f$, where the decay constant f can be viewed as large, say 10^{15}GeV to 10^{19}GeV, are technically naturally low mass particles.[7] Here, the boson mass is protected by fermionic chiral symmetries such that if $m_{\rm fermion} = 0$, then m_ϕ vanishes to all orders of perturbation theory.

Theories with naturally low-mass particles should be contrasted with theories where a small mass is unnatural. From a particle theorist's point of view, the model of Press, et al.,[11] suffers from being unnatural. The Lagrangian they considered assumes a mass term for a scalar field multiplet that is fine tuned to be of order $(30\ {\rm kpc})^{-1}$, yet the field is assumed to have normal interactions with other particles. In any quantum field theoretic version of the model this would lead to an additive quadratic divergence in the mass term. Thus, to maintain the small mass term one must fine tune the theory in each order of perturbation theory.

As a general laboratory for the statistical mechanical phenomenology of pNGB's, we will focus on the models developed by Hill and Ross.[7] These models have a light pNGB ϕ which couples to fermions. The effect of these fermions is to induce a potential for ϕ, which can lead to a phase transition. These models are very simple, but we believe that they are sufficiently general to imitate any kind of pNGB dynamics. For example, the $\mathcal{Z}_N$ models for $N > 2$ lead to a phase transition analogous to the axion case. We remark that in this analysis we will not include the potential effects of anomalies, aside from briefly indicating in Section IV how they arise in pNGB physics.

The interesting setting for these theories in a cosmological context is one in which the fermions are the light neutrinos and ϕ is a NGB associated with symmetries of the neutrino masses. We will discuss this below in the context of pNGB's associated purely

with hypothetical Dirac mass terms, as well as those associated with Majorana mass terms.[13,16] The critical temperature of the transition in some cases is either naturally of the order of, or determined by, the small masses of the neutrinos, and as such would be rather small compared to the usual scale of critical temperatures in particle physics models.

The purpose of this paper is to understand the statistical mechanics of pNGB phase transitions. As a "straw man" the first and simplest case we present in Sec. II is an unrealistic one. The model presented in Sec. III is more realistic but technically unnatural; it does however illustrate several features that will be present in more sophisticated models. It is analogous to a Coleman-Weinberg[17] effective potential with thermal or finite density corrections coming from relic neutrinos, and it has some striking features in common with slow-roll inflationary schemes. Sec. IV contains a discussion of the motivation for neutrino pNGB models. In Sec. V, we review the $\mathcal{Z}_2$ model[6] where we give a standard computation of the effective potential in the tadpole formalism (which we use throughtout).[18] We then study the finite temperature effects for this model. The usual trick of using the high temperature expansion in order to determine T_C is unreliable and more delicate methods must be used. We then generalize the $\mathcal{Z}_2$ model to $\mathcal{Z}_N$ models, which softens the fermion loop effects in the UV. We find that these models do not undergo a conventional phase transition. The potential "turns on" at low temperatures in analogy to the axion case. We summarize our results in the final section.

II. SELF-INTERACTING SCALAR MODEL

Let us review a phase transition associated with a simple model consisting of a single real scalar field ϕ with a classical potential of the form

$$V_0(\phi) = -\frac{1}{2}m_0^2\phi^2 + \frac{1}{4}\lambda_0\phi^4. \tag{2.1}$$

The classical potential Eq. (2.1) has minima at $\phi = \pm\sigma$, where $\sigma = \sqrt{m_0^2/\lambda_0}$. The mass of the scalar field is related to the curvature of the potential at the minimum:

$$m_\phi^2 \equiv \left.\frac{d^2V_0(\phi)}{d\phi^2}\right|_{\phi=\sigma} = 2m_0^2 = 2\lambda_0\sigma^2. \tag{2.2}$$

This is a very simple model that illustrates the phenomena of phase transitions and domain wall production. The calculation of the critical temperature of the phase transition in this model is well known and completely straightforward. We review it here to establish some notation and definitions that will be of use in the more complicated models discussed below.

Questions of symmetry breaking, symmetry restoration, finite-temperature effects, etc., are best studied by considering the "effective potential." This will account for the quantum effects of virtual particle emission and absorption, as well as the effect of emission and absorption of particles from the thermal background. Methods of calculating the effective potential are well developed. In one prescription the evaluation of the potential involves shifting the field by an arbitrary amount (say $\phi \to \phi + \tilde{\phi}$), and evaluating the "tadpole" diagram of Fig. 1a in the shifted theory. In this formalism the effective potential to one loop is

$$V(\phi) = V_0(\phi) - \int d\tilde{\phi}\,\Gamma^{(1)}\Big|_{\tilde{\phi}=\phi}, \tag{2.3}$$

where $\Gamma^{(1)}$ is simply a factor of i times the tadpole diagram of Fig. 1a in the shifted theory. In the shifted theory the potential is

$$V_0(\phi) = -\frac{1}{2}m_0^2(\phi - \tilde{\phi})^2 + \frac{1}{4}\lambda(\phi - \tilde{\phi})^4, \tag{2.4}$$

which results in a coupling constant for the cubic term of $\lambda\tilde{\phi}$ and a mass-squared of $-m_0^2 + 3\lambda\tilde{\phi}^2$. Evaluating the tadpole of Fig. 1a, $\Gamma^{(1)}$ is simply

$$\Gamma^{(1)} = \frac{-i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{6\lambda\bar{\phi}}{k^2-(-m_0^2+3\lambda\bar{\phi}^2)}. \tag{2.5}$$

Integrating with respect to $\bar{\phi}$ and rotating to Euclidean momentum k, the one-loop correction to the potential is

$$V_1(\phi) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2 - m_0^2 + 3\lambda\phi^2). \tag{2.6}$$

This expression of course is divergent and it is necessary to cut off the integral at $k^2 = \Lambda^2$, with result

$$\begin{aligned} V(\phi) &= V_0(\phi) + V_1(\phi) \\ &= -\frac{1}{2}m_0^2\phi^2 + \frac{1}{4}\lambda\phi^4 + a\phi^2 + b\phi^4 + \frac{(-m_0^2+3\lambda\phi^2)^2}{64\pi^2}\ln\left(\frac{-m_0^2+3\lambda\phi^2}{\Lambda^2}\right), \end{aligned} \tag{2.7}$$

where a and b include terms proportional to Λ. These constants will be determined by renormalization conditions, e.g., by the definition of a renormalized mass m and a renormalized coupling constant λ. After renormalization, the zero-temperature, one-loop potential may be expressed as

$$V_1(\phi) = \frac{1}{64\pi^2}M^4(\phi)\ln\left(\frac{M^2(\phi)}{\mu^2}\right), \tag{2.8}$$

where μ is an arbitrary mass scale which can be related to the renormalized coupling constants, and $M^2(\phi) = -m_0^2 + 3\lambda\phi^2$ is the mass as a function of ϕ.

The finite-temperature[19] corrections to the potential arise from the interaction of the ϕ field with the ambient background. To calculate the effect of the background, one computes the quantum corrections to the tree-level potential of Eq. (2.1), taking into account the fact that the background influences the ϕ propagator. That the background should have an effect at the one-loop level is easy to see, since evaluation of the effective potential in the one-loop approximation involves evaluation of the tadpole diagram of Fig. 1a, which in turn involves the ϕ propagator. The ϕ propagator is influenced by the distribution of real particles in the background.

If the phase space density of the ϕ's is denoted by $f_\phi(k)$, the ϕ propagator (in the real-time formalism) becomes

$$D_T(k) = i(k^2 - M^2)^{-1} + 2\pi f_\phi(k)\delta(k^2 - M^2). \tag{2.9}$$

Consider the part of the momentum integration of the tadpole diagram for emission and absorption of a particle on shell ($k^2 = M^2$). There is no way to differentiate between the possibility that the absorbed particle is the virtual particle emitted, or the absorbed particle comes from the background. The second term in Eq. (2.9) accounts for the latter possibility.

Clearly the effect of the background particles depends upon their phase-space density. If the ϕ's are in thermal equilibrium, they will be distributed in phase space according to the Bose–Einstein distribution: $f_\phi(k) = [\exp(E/T) - 1]^{-1}$, where $E = \sqrt{|\vec{\mathbf{k}}|^2 + M^2}$. For the moment, we will make the assumption that the phase-space distribution of the ϕ's are described by the equilibrium expression.

In the one-loop approximation, the potential is a sum of the tree-level potential, $V_0(\phi)$ given by given by Eq. (2.1), a zero-temperature one-loop correction, $V_1(\phi)$ of Eq. (2.8), and the temperature-dependent one-loop potential, $\Delta V_T(\phi)$:[20]

$$V(\phi) = V_0(\phi) + V_1(\phi) + \Delta V_T(\phi). \tag{2.10}$$

The temperature-dependent part of the propagator adds to $\Gamma^{(1)}$ a term

$$\Gamma_T^{(1)} = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} 6\lambda\tilde{\phi}\; 2\pi f_\phi(k)\delta[k^2 - (-m_0^2 + 3\lambda\tilde{\phi}^2)]. \tag{2.11}$$

Following through the integration with respect to $\tilde{\phi}$, rotation to Euclidean momentum k, and integration over d^4k, one obtains the (finite) result

$$\Delta V_T(\phi) = \frac{T^4}{2\pi^2}\int_0^\infty dx\, x^2 \ln\left[1 - \exp\left(-\sqrt{x^2 + M^2(\phi)/T^2}\right)\right], \tag{2.12}$$

where again, $M^2(\phi) = -m_0^2 + 3\lambda\phi^2$.

To demonstrate that there is a phase transition and to calculate the critical temperature is straightforward. At zero-temperature the minima of the potential are $\phi = \pm\sigma$, and the curvature at $\phi = 0$ is negative (i.e., $\phi = 0$ is a local maximum). At high temperature $\Delta V_T(\phi)$ can be expanded in ϕ, with the leading-order ϕ-dependent term proportional to $+T^2\phi^2$. Clearly at high temperature the curvature of the potential at $\phi = 0$ is positive, and indeed $\phi = 0$ is the true minimum of the theory at high temperature. We will denote the temperature at which the curvature of the high-temperature minimum vanishes as the critical temperature, T_C. In the above theory, $\partial^2 V/\partial\phi^2$ evaluated at $\phi = 0$ changes sign at a temperature $T_C = 2\sigma$.

This model illustrates the standard scenario for making walls. At temperatures above the critial temperature, the value of the field is pinned at the high-temperature minimum, $\phi = 0$. This is because at high temperatures $\phi = 0$ is the global minimum of the potential, and furthermore, the mass of the field at high temperature is large (of order λT). It is the large mass that pins ϕ to the high-temperature minimum. Now once the temperature drops below the critical temparature, the ϕ field will evolve classically to either of two possible minima. Regions of the Universe in different minima will be separated by a domain wall. This scenario depends upon the fact that as the phase transition starts, ϕ is localized at a low-temperature maximum, which is also a high-temperature minimum.

Note that $T_C/m_\phi = \sqrt{2/\lambda}$. Thus, it appears that by making λ sufficiently small, one might have a late-time transition generating soft-walls. Let us explore this scenario.

Let us assume a generous range for T_C, say $T_0 \lesssim T_C \lesssim T_{\rm rec}$, where T_0 is the present temperature, $T_0 = 2.7$ K $\sim 2.4 \times 10^{-4}$eV, and $T_{\rm rec}$ is the temperature at recombination, $T_{\rm rec} \simeq 0.3$ eV. Let us also assume that the boson has an ultra-low mass, $m_\phi \lesssim 10^{-24}$eV. The combination $T_C \gtrsim T_0$ and $m_\phi \lesssim 10^{-24}$eV leads to the constraint $\lambda \lesssim 10^{-40}$. If this constraint is satisfied, the model as presented will lead to a late-time phase transition,

and contains soft domain walls with thicknesses of order $m_\phi^{-1} \sim$ parsecs.

However, there are two serious problems with the model. The first problem is that it is unreasonable to assume that a fundamental constant, such as λ, has a value of 10^{-40} without some deeper underlying motivation. Such a value is unnatural in the technical sense[15] and is arbitrary. A second difficulty is that with such a small value of λ it is unlikely that the ϕ's were ever in equilibrium and the assumption that they are present in a thermal phase-space density cannot be justified. Unless the potential of Eq. (2.1) is augmented by some additional interaction terms, the only processes leading to thermalization of the ϕ's are ϕ self-interactions. The cross section for this processes in the relativistic limit is $\sigma_{\rm int} = \lambda^2/s \sim \lambda^2/T^2$, where the last approximation assumed that the average energy of the ϕ is characterized by a temperature T. *If* ϕ is in equilibrium and relativistic, $n_\phi \sim T^3$, so the interaction rate of the ϕ's is $\Gamma_{\rm int} \sim n_\phi \sigma_{\rm int} \sim \lambda^2 T$. In the radiation-dominated era, the expansion rate is $H \sim T^2/m_{Pl}$, so $\Gamma_{\rm int}/H \sim \lambda^2 m_{Pl}/T$. If $\lambda = 10^{-40}$, then $\Gamma_{\rm int}/H \ll 1$ for $T \gtrsim 10^{-80} m_{Pl} \ll T_0$. A similar conclusion follows for the expansion rate appropriate to a matter-dominated era.

Clearly the assumption that ϕ's exist as a thermal background cannot be justified. Of course, one might imagine that the background is not established through self-interactions, but rather is the result of some non-standard (but reasonable) process such as primordial black-hole evaporation, quantum effects during inflation, or other such processes.

We will now turn to our attention to developing models where the phase transition is driven not through ϕ self interactions with a background, but rather by ϕ interactions with a background of some other field ψ, typically a fermion. The virtue of this complication is that it is possible to have ϕ–ψ interactions weak enough to provide a late-time, soft-wall transition, but the ψ can have additional interactions that can establish the background by thermal interactions.

III. SCALAR FIELDS WITH YUKAWA INTERACTIONS

To the classical potential of Eq. (2.1) we add a Yukawa coupling of ϕ to a fermion field ψ:[21]

$$V_0(\phi,\psi) = -\frac{1}{2}m_0^2\phi^2 + \frac{\lambda_0}{4}\phi^4 - h\phi\overline{\psi}\psi, \tag{3.1}$$

where the parameters m_0, λ_0, and h are the unrenormalized mass and coupling constants.[22]

Before turning to the temperature-dependent effects, consider the zero-temperature radiative corrections. The one-loop corrections involves calculation of the tadpole diagram of Fig. 1b in addition to the scalar tadpole of Fig. 1a. In the following we will assume that the fermion loops dominate, which will be true if if $h^2 \gg \lambda_0$, and ignore the boson tadpole. The effect of the ϕ–ψ interaction on the effective potential is evaluated by calculating the tadpole as discussed in the previous section. Upon shifting the field $\phi \rightarrow \phi + \bar{\phi}$, the mass of the ψ is $M_\psi = h\bar{\phi}$, and the ϕ–ψ–ψ vertex that appears in the tadpole is proportional to h. Thus $\Gamma^{(1)}$ is obtained from computing the one-loop tadpole diagram of Fig. 1b:

$$\Gamma^{(1)} = i\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\frac{h}{\not{k} - h\bar{\phi}}. \tag{3.2}$$

Following a procedure similar to the one outlined in the previous section, one obtains terms in the one-loop effective potential that are infinite (proportional to a cut off Λ) and terms that are finite. The infinite terms are dealt with by some renormalization prescription, and the renormalized, one-loop effective potential is simply

$$V_1(\phi) = -\frac{1}{16\pi^2}(h\phi)^4 \ln\frac{\phi^2}{\mu^2}, \tag{3.3}$$

where μ again is an arbitrary mass scale related to the values of the coupling constants.

Now the effect of a background of real ψ's on the effective potential is calculated along the line as the previous section. Again the tadpole of Fig. 1b is calculated replacing the fermion propagator by its finite-temperature expression:

$$S_T(k) = i(\not{k} - M_\psi)^{-1} - 2\pi f_\psi(k)(\not{k} + M_\psi)\delta(k^2 - M_\psi^2), \tag{3.4}$$

where here f_ψ is the phase space density for ψ. Again let us assume the phase space density for ψ is that of a thermal distribution (i.e., a Fermi-Dirac distribution with temperature T). This adds to the one-loop effective potential a temperature-dependent term

$$\Delta V_T(\phi) = -4\frac{T^4}{2\pi^2}\int_0^\infty dx\ x^2 \ln\left[1 + \exp\left(-\sqrt{x^2 + M_\psi^2/T^2}\right)\right]. \tag{3.5}$$

In comparison to Eq. (2.12) several differences are obvious. The overall sign is opposite because there is an overall sign difference between fermion and boson loops. The sign difference in the argument of the logarithm arises from the sign difference in the Fermi–Dirac verses Bose–Einstein distribution functions. Finally the overall factor of 4 owes to the trace over γ-matrices involved in the fermion loop.

Now let's consider the phase transition. Let us assume that the zero-temperature one-loop potential has negligible effect and the curvature at $\phi = 0$ remains negative. Expanding Eq. (3.5) for large T, $\Delta V_T(\phi) \simeq +h^2\phi^2T^2/3$. Clearly at high temperature the curvature of the potential at $\phi = 0$ will be positive, and again it will be the global minimum of the potential. We again define the critical temperature for the phase transition to be the temperature where $\partial^2 V/\partial\phi^2$ evaluated at $\phi = 0$ vanishes. This results in a critical temperature of $T_C/m_\phi = h^{-1}\sqrt{3}$. This expression is very similar to the critial temperature in the model of the previous section with the replacement $h \leftrightarrow \sqrt{\lambda}$.

Clearly by making h sufficiently small it is possible to have $T_C \gg m_\phi$ for a late-time, soft-wall phase transition. However the present model is superior to the previous model in one important regard: Although the ψ field driving the transition must be very weakly coupled to ϕ, it may have stronger couplings to other fields. These other (yet unspecified) couplings can be sufficiently strong to establish ψ in thermal equilibrium. Therefore, although ϕ may be completly decoupled from the thermal bath, its interactions

with the thermal background of ψ's can restore the symmetry at high temperature and lead to a phase transition.

The model still suffers from the ugly feature of very small, unnatural dimensionless coupling constants. In the next section we will describe the physical motivation for the origin of such small dimensionless coupling constants. The model will have some of the features of the model of this section. Before proceeding, let us restate the parameters of the present model. The model has a scalar field ϕ with mass $m_\phi \lesssim 10^{-24}$eV. The vacuum expectation value of the scalar field is σ. The Yukawa coupling of ϕ to a fermion field ψ results in a mass $M_\psi = h\sigma$. If h is much larger than the scalar quartic self coupling, the phase transition temperature will be $T_C \sim m_\phi/h$. If we want T_C to be larger than T_0, then $h \lesssim 10^{-20}$. However we are free to choose σ to be as large as desirable, and to have the ψ field coupled to other particles with sufficient strength to establish it in equilibrium.

IV. PSEUDO-NAMBU–GOLDSTONE BOSONS

A. Chiral Lagrangians

The model discussed in Sec. III most simply demonstrates the basic idea of a late-time phase transition, but it suffers from the lack of symmetries that can naturally give a soft boson mass scale for ϕ without fine tuning. Let us now consider models in which these constraints are implemented.[23]

Consider first the low-energy effective Lagrangian which contains a neutrino field ν:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \bar{\nu}_L i\partial\!\!\!/\,\nu_L + \bar{\nu}_R i\partial\!\!\!/\,\nu_R + (m\bar{\nu}_L\nu_R e^{i\phi/f} + \text{h.c.}) \tag{4.1}$$

where ν_L (ν_R) is the left-handed (right-handed) projection: $\nu_L = (1-\gamma_5)\nu/2$ ($\nu_R =$

$(1+\gamma_5)\nu/2)$. The factor of $me^{i\phi/f}$ can be viewed as arising from the vacuum expectation value (VEV) of some $U(1)$ complex scalar field Φ that is coupled as $g\overline{\nu}_L\nu_R\Phi + h.c.$ In a $U(1)$ invariant potential $V(\Phi)$ we assume that Φ develops a VEV of $\langle\Phi\rangle = fe^{i\phi/f}/\sqrt{2}$, and $m = gf/\sqrt{2}$ (the factor $\sqrt{2}$ assures that ϕ has a properly normalized kinetic term of $(\partial\phi)^2/2$ coming from the kinetic term of Φ, $|\partial\Phi|^2$).

Eq. (4.1) is a "chiral Lagrangian," possessing the continuous chiral $U(1)$ symmetry:

$$\nu_L \to e^{i\alpha}\nu_L; \qquad \nu_R \to e^{-i\alpha}\nu_R; \qquad \phi \to \phi + 2\alpha f. \tag{4.2}$$

We emphasize that the symmetry is not broken, and is properly said to be "nonlinearly realized" (this is often a confusing point: spontaneously broken symmetries are in fact equivalent to nonlinearly realized symmetries and are not really broken symmetries). We remark that chiral Lagrangians have several important and well-known properties: (1) as stated above, they can be embedded into a fully renormalizeable theory in which, e.g., a $U(1)$ complex field develops a vacuum expectation value, $\langle\Phi\rangle = f/\sqrt{2}$, and ϕ is then the residual Nambu–Goldstone boson; (2) $\mathcal{L}$ can itself be viewed as renormalizeable for a small cut off $\Lambda \ll f$ up to suppressed counterterms of order Λ/f; (3) ϕ will be identically massless unless terms are introduced which explicitly break the chiral symmetry; (4) ϕ satisfies "Adler decoupling," i.e., we may replace ν everywhere by ν':

$$\nu'_L = \nu_L e^{i\phi/2f}; \qquad \nu'_R = \nu_R e^{-i\phi/2f}; \tag{4.3}$$

and our Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \overline{\nu}'_L i\partial\!\!\!/\,\nu'_L + \overline{\nu}'_R i\partial\!\!\!/\,\nu'_R + m(\overline{\nu}'_L\nu'_R + \text{h.c.}) + \frac{1}{2f}\partial^\mu\phi\,\overline{\nu}'\gamma_5\gamma_\mu\nu' \tag{4.4}$$

and we thus see that ϕ disappears in the mass term but couples derivatively to the neutrino as $\partial^\mu\phi\,\overline{\nu}\gamma_5\gamma_\mu\nu$. Therefore, for small ϕ momentum q_μ, ϕ emission or absorbtion amplitudes will tend to zero as $q_\mu \to 0$ ("Adler zero"). An implication of this is that ϕ will not mediate a long-range $1/r^2$ force as a consequence of this decoupling theorem

(when the symmetry is broken by a nonchirally invariant mass term as discussed below, then the Adler decoupling can be violated and ϕ can mediate a long range force, though this requires CP-violation[7]).

We remark that at this stage on yet another branch of NGB or pNGB physics. If we integrate the last term of Eq. (4.4) by parts we obtain $-(2f)^{-1}\phi\ \partial^\mu\overline{\nu}\gamma_5\gamma_\mu\nu$. Now if we include the effects of generic gauge fields that may be coupled to ν (for example, the ν_L couples to the electroweak gauge fields) then the divergence of the axial current will contain an axial anomaly $\partial^\mu\overline{\nu}\gamma_5\gamma_\mu\nu = cF\tilde{F} + ...$, and therefore we find that ϕ couples to the gauge fields through this anomalous term: $-(2f)^{-1}\phi\ cF\tilde{F}$. This is an *explicit symmetry breaking effect* coming from quantum loops and it generally leads to important consequences. For example, the decay $\pi^0 \to 2\gamma$ involves this term; the gluon field enters the divergence of the PQ current ultimately giving a mixing of the axion to the π^0, η and η' from which the axion mass derives. In principle we should include potential anomaly effects in our effective potential analysis, however we will not do so for a reason: the phase transitions we consider here occur at very low temperature (or finite density) and arise from other explicit symmetry breaking effects. It is hard to see how anything but the electromagnetic anomaly could play a role at these low energy scales. It is conceiveable that an electromagnetic effect, e.g., in a plasma, might trigger a late-time transition through the anomaly, but we will not consider this possibility in the present paper.

Let us now consider the explicit breaking of the symmetry by effects other than anomalies. By this we mean the addition of new terms to Eq. (4.1) which explicitly violate the nonlinearly realized symmetry of Eq. (4.2). For example, to the Lagrangian we may add a small mass term for ϕ of unspecified origin. Usually this comes from some deeper symmetry breaking in the theory which breaks the continuous $U(1)$ down to a discrete subgroup $\mathcal{Z}_N$. For example, let us break $U(1)$ to its trivial center by adding a "soft-breaking" term, which is a cosine potential for ϕ. This implies that $\phi \to \phi + 2n\pi f$

remains an invariance. So we now have:

$$\mathcal{L}' = \mathcal{L} + \widetilde{m}^4 \cos(\phi/f + \theta). \tag{4.5}$$

By expanding the $\cos \phi/f$ term about a local minimum we infer the mass of the ϕ boson:

$$m_\phi^2 = \widetilde{m}^4/f^2 \tag{4.6}$$

and there are also further interaction terms such as a $\lambda\phi^4$ term where

$$\lambda = \widetilde{m}^4/12f^4. \tag{4.7}$$

The physical values of λ and m_ϕ are proportional to the ratio $\widetilde{m}/f$, and can be almost arbitrarily small, while remaining stable under quantum radiative or thermal corrections. This is "natural" in the sense of 't Hooft,[15] and is due to the fact that $\widetilde{m} = 0$ is a symmetry limit of the full theory [in which we recover Eq. (4.2)].

For example, with $f \sim 10^{16}$ GeV and $\widetilde{m} \sim 10^{-2}$ eV we have $m_\phi \sim 10^{-30}$eV, or a Compton wavelength, $\hbar/m_\phi c \sim 10$ Mpc. The incoherent particle interaction rates will be negligible since $\lambda \sim 10^{-109}$! The Adler decoupling theorem still holds with soft breaking since it follows from redefinition of fermion fields. In general, reaction rates involving ϕ coupled *incoherently* to matter will be suppressed, since the cross sections are necessarily proportional to a power of $1/f^2$. Thus it is difficult, if not impossible, to excite ϕ in the laboratory, just as the detection of invisible axions is difficult. Since reaction rates that maintain thermal equilibrium of ϕ are of order T^3/f^2, in a radiation dominated Robertson–Walker phase we see that the condition that ϕ be in equilibrium is $T^3/f^2 \gtrsim T^2/m_{Pl}$ or $T \gtrsim f^2/m_{Pl}$. Hence, a pure pNGB like ϕ decouples very early in the evolution of the Universe.

What kind of deeper structure can give rise to a mass term for ϕ? In the case of QCD the proton and neutron are analogues of the ν field, and the pion is the analogue of ϕ. The deeper structure that breaks the chiral symmetry is the presence of light quark masses,

which are not chirally invariant. This leads to the non-zero pion mass term. However, it is unlikely that the only manifestation of a deeper symmetry breaking term is merely a mass term for ϕ. Indeed, in the case of the nucleon-pion system, the finite quark masses also lead to a small chiral symmetry breaking term in the proton and neutron masses (known as the σ-term). We can make a strict analogy to this situation in the present case by adding an *additional neutrino mass term to the Lagrangian, which explicitly breaks the chiral symmetry*, in analogy to the QCD σ-term. The low-energy Lagrangian then becomes:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \bar{\nu}_L i\partial\!\!\!/\,\nu_L + \bar{\nu}_R i\partial\!\!\!/\,\nu_R \\ &\quad +(m\bar{\nu}_L\nu_R e^{i\phi/f} + \epsilon\bar{\nu}_L\nu_R + \text{h.c.}) + \widetilde{m}^4\cos(\phi/f+\theta),\end{aligned} \tag{4.8}$$

where the term involving ϵ explicitly breaks the symmetry of Eq. (4.2). Now, if $\widetilde{m} \to 0$ we must also set $\epsilon \to 0$ to recover the symmetry limit of Eq. (4.2). However, a nonzero $\widetilde{m}$ will always be induced by the presence of a nonzero ϵ and m. For instance, the diagram of Fig. 2a with a cutoff $\Lambda < f$ gives an induced term in the Lagrangian

$$\mathcal{L}_{\text{induced}} \sim \frac{m\epsilon\Lambda^2}{16\pi^2}\cos(\phi/f). \tag{4.9}$$

In the present case we see that the induced scalar mass will be of order:

$$m_\phi^2 \sim m\epsilon(\Lambda^2/f^2) \sim m\epsilon. \tag{4.10}$$

We can view this as the origin of the scale of $\widetilde{m}^2 \sim \sqrt{m\epsilon}\Lambda$. The mass can be naturally small in the technical sense since we can tune the symmetry breaking parameter ϵ to be arbitrarily tiny for large m so that the observed neutrino mass is, e.g., $m_\nu \sim m_0 \sim 1$ eV, while $m_\phi \sim (100\text{ Mpc})^{-1}$ with $\epsilon \sim 10^{-60}$ eV, and the symmetry will guarantee that we don't have to worry about radiative corrections changing this result. This is arbitrary, however, and this is not the ultra-low mass case we seek for application to a late-time phase transition.

In the Lagrangian of Eq. (4.9) we observed the appearance of a ("large") quadratically divergent contribution to the induced mass of ϕ. Can we somehow reduce the degree of divergence of this induced term? The answer is yes: residual symmetries can readily control this.

Consider the following Lagrangian containing N Dirac neutrino species and invariant under a $\mathcal{Z}_N$ discrete symmetry:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \sum_{j=0}^{N-1}\overline{\nu}_j i\partial\!\!\!/\,\nu_j + \sum_{j=0}^{N-1}\left(m + \epsilon e^{i\phi/f+2ij\pi/N}\right)\overline{\nu}_{jL}\nu_{jR} + \text{h.c.} \tag{4.11}$$

The continuous $U(1)$ chiral symmetry is broken down to a residual $\mathcal{Z}_N$ discrete symmetry:

$$\nu_j \to \nu_{j+1}; \qquad \nu_{N-1} \to \nu_0; \qquad \phi \to \phi + 2\pi f/N. \tag{4.12}$$

If one now computes the induced ϕ mass term, one obtains the ϕ-dependent term

$$\sum_{j=0}^{N-1}\frac{M_j^4}{16\pi^2}\log\left(\Lambda^2/M_j^2\right), \tag{4.13}$$

where $M_j^2 = m^2 + \epsilon^2 + 2m\epsilon\cos(\phi/f + 2j\pi/N)$. Notice that the potential retains the discrete symmetry $\phi \to \phi + 2j\pi f/N$. Now, it is readily seen that $\sum_j M_j^4$ is a constant independent of ϕ for $N > 2$. Therefore the Λ dependence in Eq. (4.13) is illusory; the ϕ-dependent part is Λ independent, and for $N > 2$ we may write

$$V(\phi) = -\sum_{j=0}^{N-1}\frac{M_j^4}{16\pi^2}\log\left(M_j^2\right) + \text{const.} \tag{4.14}$$

Hence, in $\mathcal{Z}_N$ models we can view the symmetry breaking as soft and the potential of ϕ is calculable.

These models can be further generalized. In Ref. 7 the effects of CP violation are also included to contruct models in which the Adler decoupling theorem is violated and the pseudo-Nambu–Goldstone bosons develop CP-violating Yukawa couplings. This is analogous to including a θ-term into QCD (without an axion to kill it!). The net effect is

the possibility of weak, sub–gravitational strength long-range forces in a natural model. This is a further complication of the model which we will not include at present.

B. Majoron Models

In discussing neutrino masses the most sensible framework is that of the "seesaw" mechanism.[16] Here one attempts to explain the comparatively tiny values of neutrino masses relative to their charged lepton counterparts by invoking ultra-large Majorana masses for the right-handed neutrinos. In short, one needs only to assume that *(i) all neutrinos have Dirac mass terms* (perhaps of the same order as their charged lepton partners within a given generation), and *(ii) all right-handed neutrinos have a Majorana mass term.* With both terms present we have a conventional Gell-Mann–Ramond–Slansky–Yanagida[16] see-saw mechanism. The predicted light mass scale of neutrinos will be acceptably small, of order $m^2_{\rm lepton}/M$. The right-handed neutrinos are favored for a large Majorana mass term because they carry no known gauge symmetries. Nonetheless, one can also invoke small left-handed Majorana masses, as in the Gelmini–Roncadelli model.[24] These carry electroweak isospin of $I = 1$ and must be very small, since $I = 1$ effects are suppressed in the Standard Model.

Here we will give only a brief toy model discussion, leaving a more detailed catalogue of schemes to another place.[25] We consider a single Dirac neutrino field, with left- and right-handed components ν_L, ν_R. We now assume the existence of Dirac and both left- and right-handed Majorana mass terms:

$$\frac{1}{2}\overline{\begin{pmatrix} \nu_L \\ \nu_R^C \end{pmatrix}} \begin{pmatrix} \epsilon & m \\ m & Me^{i\phi/f} \end{pmatrix} \begin{pmatrix} \nu_L^C \\ \nu_R \end{pmatrix} + h.c. \tag{4.15}$$

Here superscript-C denotes charge conjugation. We assume $\epsilon \ll m \ll M$. The phase $\exp(i\phi/f)$ is the CMP Majoron,[13] the NGB associated with spontaneous breaking of the

$U(1)_R$ rh-neutrino number symmetry. We take ϵ to be an explicit $U(1)_L$ lh-neutrino number breaking effect. The Dirac masses communicate the explicit breaking in the rh sector into the lh sector, and thus the majoron becomes a pNGB.

If we set either $m = 0$ or $\epsilon = 0$ then the phase $\exp(i\phi/f)$ can be eliminated from the mass matrix by a redefinition of the neutrino fields, and we are left only with the derivative coupling and ϕ remains massless. This is the usual assumption for the majoron. However, we see that the diagram of Fig. 2b implies an induced mass term for ϕ given by

$$\mathcal{L}_{\text{induced}} \sim \frac{\epsilon m^2 M \log(\Lambda^2/m^2)}{16\pi^2} \cos(\phi/f), \tag{4.16}$$

which is highly suppressed owing to the combination of small ϵ and the chiral suppression involving m^2. Thus, we expect that majorons will behave in a mode which is no more divergent than the $\mathcal{Z}_2$ case described above, and in more general schemes will be suppressed as in the $\mathcal{Z}_{N>2}$ case.

In fact, the full behavior of broken majoron models may be very rich. Bjorken (private communication) has suggested considering the full 3-generation standard model to contain a spontaneously broken $SU(3)_R$ (which will be in a sextet mode, corresponding to $\nu_R^i \nu_R^{jC}$) which will produce NGB's. The $SU(3)_R$ is then explicitly broken by the Dirac mass terms, and no lh majorana masses are included. The result is a hierarchy of NGB's, some remaining massless while others acquire a spectrum of induced mass terms. More general schemes such as this will be considered elsewhere.[25]

We turn now to the thermal corrections to the chiral Lagrangians.

V. THERMAL PROPERTIES OF $\mathcal{Z}_N$ MODELS

In this section we will analyze the thermal properties of the $\mathcal{Z}_N$-symmetric chiral Lagrangian models discussed in the previous section. For simplicity we will first consider the $\mathcal{Z}_2$ model, and then generalize our results to the $\mathcal{Z}_N$ case.

A. $\mathcal{Z}_2$-symmetric models

The effective low-energy theory (here, low-energy refers to scales much smaller than f) consists of two fermions (presumably neutrinos) ψ_j, $j = 0, 1$, coupled to the scalar field ϕ by Yukawa couplings of the form:

$$-\mathcal{L}_{\rm YUK} = \sum_{j=0}^{1} \overline{\psi}_j \{m + \epsilon[\cos(\phi/f + j\pi) + i\gamma_5 \sin(\phi/f + j\pi)]\} \psi_j. \tag{5.1}$$

where we have used $\bar{\psi}_L \psi_R e^{i\alpha} + h.c. = \bar{\psi}\psi \cos(\alpha) + i\bar{\psi}\gamma_5\psi \sin(\alpha)$. Here, f can be thought of as the scale at which the continous symmetry, of which ϕ is the Nambu–Goldstone boson, is spontaneously broken. The origin of and motivation for considering such theories was discussed in Sec. IV.

We can rewrite Eq. (5.1) by performing a chiral rotation to eliminate the γ_5 term, with result

$$\begin{aligned} -\mathcal{L}_{\rm YUK} &= -M_+(\phi)\overline{\psi}_0\psi_0 - M_-(\phi)\overline{\psi}_1\psi_1; \\ M_\pm^2(\phi) &= m^2 + \epsilon^2 \pm 2m\epsilon\cos(\phi/f). \end{aligned} \tag{5.2}$$

Note that at this point there is no potential for ϕ. However the effect of the ϕ–ψ coupling will generate a non-trivial potential for ϕ through radiative corrections, rendering it a pseudo-Nambu–Goldstone boson. We first calculate the zero-temperature potential, then consider the finite-temperature potential. We will employ the same methods developed in Sec. III.

We will only consider the fermion contributions.[26] The tadpole method described in Sec. II can be adapted to the present case. Rather than defining tadpoles in ϕ, i.e., $\langle\phi\rangle$, as in the self-interacting scalar model, here we must preserve the full symmetry of the theory and define the tadpole to be expectation value of the mass terms. The one-loop potential now receives contribution from two tadpoles with fermions as in Fig. 1b, from the mass terms $\langle M_+(\phi)\overline{\psi}_0\psi_0\rangle$ and $\langle M_-(\phi)\overline{\psi}_1\psi_1\rangle$. These contribute to the unrenormalized one-loop potential a result given by

$$V_1(\phi) = \sum_{j=+,-}\left[\frac{\Lambda^4}{32\pi^2} - \frac{\Lambda^2}{8\pi^2}M_j^2(\phi) - \frac{1}{16\pi^2}M_j^4(\phi)\left(\ln\frac{M_j^2(\phi)}{\Lambda^2} - \frac{1}{2}\right)\right]. \tag{5.3}$$

Here Λ is an ultraviolet cutoff for the theory; presumably it cannot be larger than f, since at this scale the effective theory with the Nambu–Goldstone boson ϕ must be supplanted by the full theory.

We must now renormalize the potential. In so doing we will introduce an arbitrary energy scale μ; of course no physical effects will depend on μ. To proceed, first introduce the scale μ into the potential:

$$\begin{aligned} V_1(\phi) \;=\; & \sum_{j=+,-}\left[\frac{\Lambda^4}{32\pi^2} - \frac{\Lambda^2}{8\pi^2}M_j^2(\phi)\right. \\ & \left. -\frac{1}{16\pi^2}M_j^4(\phi)\left(\ln\frac{M_j^2(\phi)}{\mu^2} - \frac{1}{2}\right) + \frac{1}{16\pi^2}M_j^4(\phi)\ln\frac{\mu^2}{\Lambda^2}\right]. \end{aligned} \tag{5.4}$$

Before proceeding we make note of the following identities:

$$\begin{aligned} \sum_{j=+,-} M_j^2(\phi) &= 2(m^2+\epsilon^2) \\ \sum_{j=+,-} M_j^4(\phi) &= 2[(m^2+\epsilon^2)^2 + (2m\epsilon)^2\cos^2(\phi/f)]. \end{aligned} \tag{5.5}$$

Because of the residual $\mathcal{Z}_2$ symmetry $M_+^2(\phi)+M_-^2(\phi)$ is independent of ϕ. From Eq. (5.4) we see that we must add counterterms $V_{\rm CT}(\phi) = \mathcal{V}_0 + \mathcal{V}_1\cos^2(\phi/f)$, where $\mathcal{V}_0$ and $\mathcal{V}_1$ are ϕ and μ independent, to the original Lagrangian to cancel the cutoff-dependent terms.

Here $\mathcal{V}_0$ and $\mathcal{V}_1$ are given by

$$\begin{aligned}\mathcal{V}_0 &= \tilde{V}_0(\mu) - \frac{\Lambda^4}{16\pi^2} + \frac{\Lambda^2}{8\pi^2}\sum_{j=+,-} M_j^2(\phi) + \frac{1}{16\pi^2}2(m^2+\epsilon^2)^2 \ln\frac{\mu^2}{\Lambda^2};\\ \mathcal{V}_1 &= \widetilde{M}^4(\mu) + \frac{1}{16\pi^2}2(2m\epsilon)^2\ln\frac{\mu^2}{\Lambda^2},\end{aligned} \tag{5.6}$$

with $\tilde{V}_0(\mu)$, and $\widetilde{M}^4(\mu)$ finite. The final potential is then given by

$$\begin{aligned}V(\phi) &= V_{\rm CT}(\phi) + V_1(\phi)\\ &= \tilde{V}_0(\mu) + \widetilde{M}^4(\mu)\cos^2(\phi/f)\\ &\quad -\frac{1}{16\pi^2}\left[M_+^4(\phi)\left(\ln\frac{M_+^2(\phi)}{\mu^2} - \frac{1}{2}\right) + M_-^4(\phi)\left(\ln\frac{M_-^2(\phi)}{\mu^2} - \frac{1}{2}\right)\right].\end{aligned} \tag{5.7}$$

Now the μ-independent parts of $\tilde{V}_0(\mu)$ and $\widetilde{M}^4(\mu)$ can be fixed by renormalization conditions. In particular, let us choose the renormalization conditions

$$V''(\phi)|_{\phi=0} \equiv m_0^2; \qquad V(\pi/2) = 0. \tag{5.8}$$

At this stage the sign of m_0^2 is not fixed. The final potential becomes

$$\begin{aligned}V(\phi) &= \frac{1}{32\pi^2}(m_+^2+m_-^2)^2\left(\ln\frac{m_+^2+m_-^2}{2\mu^2} - \frac{1}{2}\right)\\ &\quad + \left[-\frac{1}{2}f^2m_0^2 + \frac{1}{32\pi^2}(m_+^2-m_-^2)\left(m_+^2\ln\frac{m_+^2}{\mu^2} - m_-^2\ln\frac{m_-^2}{\mu^2}\right)\right]\cos^2(\phi/f)\\ &\quad -\frac{1}{16\pi^2}\left[M_+^4(\phi)\left(\ln\frac{M_+^2(\phi)}{\mu^2} - \frac{1}{2}\right) + M_-^4(\phi)\left(\ln\frac{M_-^2(\phi)}{\mu^2} - \frac{1}{2}\right)\right],\end{aligned} \tag{5.9}$$

where we have made yet another definition, $m_\pm \equiv (m \pm \epsilon)$. Althought it is not apparent, Eq. (5.9) is *μ-independent*, as can be seen by showing $V(\phi;\mu) - V(\phi;\mu') = 0$. We leave the exercise in algebra to the reader.

Obviously $V(\phi)$ is periodic with period π and that its extrema are at $\phi = 0,\ \pi/2$ (mod π). The location of the *minima* depend on the sign of m_0^2. We show the potential in Fig. 3 for negative m_0^2.

Let us now turn to the finite temperature corrections to the effective potential for ϕ. Given the ϕ-dependent masses $M_+(\phi)$ and $M_-(\phi)$, we can then use the finite-temperature formalism discussed in Sec. III to compute the corrections [cf., Eq. (3.5)]:

$$\Delta V_T(\phi) = -4\frac{T^4}{2\pi^2}\sum_{j=+,-}\int_0^\infty dx\; x^2 \ln\left[1+\exp\left(-\sqrt{x^2+M_j^2(\phi)/T^2}\right)\right]. \tag{5.10}$$

The signal of a second order phase transition is the flattening of the potential at the high-temperature minima, i.e., $V''(\phi=0)|_{T=T_C}=0$. Here, $V(\phi)=V_1(\phi)+\Delta V_T(\phi)$. The temperature-dependent mass squared at $\phi=0$, $m^2(T)$, is given by:

$$m^2(T) = m_0^2 + \frac{4m\epsilon T^2}{\pi^2 f^2}\sum_{j=+,-}(-1)^j\int_0^\infty dx\; x^2\frac{(x^2+m_j^2/T^2)^{-1/2}}{1+\exp(x^2+m_j^2/T^2)^{1/2}}. \tag{5.11}$$

Since $4m\epsilon = m_+^2 - m_-^2 > 0$, it is easy to show that the temperature-dependent term is always *positive*. Thus, if $\phi=0$ is a minimum at zero temperature, it will remain so at *any* finite temperature. This implies that the $T=0$ maximum at $\pi/2$ (when $m_0^2>0$) remains one at finite T. Thus we do not expect any phase transitions when $m_0^2>0$. On the other hand, if m_0^2 is *negative*, so that $\phi=0$ is a maximum at zero temperature, we can balance the negative zero-temperature mass against the positive contribution from the finite temperature piece. Thus, we expect that there will be a phase transition at some critical temperature T_C in this case.

Whether a phase transition occurs depends upon the sign of m_0^2 as can be seen by examining $\Delta V_T(\phi)$. An example of the temperature-dependent part of the potential is shown in Fig. 4. Clearly the curvature at $\phi=0$ becomes more positive as the temperature increases. This does not depend upon the sign of the curvature of the zero-temperature potential as $\Delta V_T(\phi)$ is independent of m_0^2. From Fig. 4 we also see that the finite-temperature corrections will always increase $V(\pi/2)$ more than $V(0)$, so if at zero temperature $\pi/2$ is a maximum of the potential, it will remain so at high temperature. Of course, the actual value of m_0^2 is arbitrary, since it contains a renormalization counterterm. The value of m_0^2 is only technically naturally small, since it is protected by the chiral symmetry (and the residual discrete symmetries).

There is no analytic expression for T_C; however, we can show that T_C must be of order $m_\pm$. First we show that T_C *cannot* be much larger than the fermion masses $m_\pm$

by means of the high temperature expansion.[27] For $m/T \ll 1$ we can expand the finite temperature potential as:

$$\Delta V_T(\phi) = \sum_{j=+,-} \frac{1}{16\pi^2} M_j^4(\phi) \ln \frac{M_j^2(\phi)}{T^2} + \cdots, \tag{5.12}$$

where we neglect terms such as (T-dependent) constants or (T-independent) parts depending on $\cos^2 \phi/f$. These terms are unimportant as far as computing the effective mass. The critical temperature is obtained by setting the second derivative of the full potential at $\phi = 0$ to zero. Doing this within the high temperature approximation yields

$$m_0^2 = \frac{m_-^2 - m_+^2}{8\pi^2 f^2} \left(m_+^2 \ln \frac{m_+^2}{T_C^2} - m_-^2 \ln \frac{m_-^2}{T_C^2} \right). \tag{5.13}$$

We can solve this for T_C:

$$T_C = m_+ \left(\frac{m_-}{m_+} \right)^{m_-^2/(m_-^2 - m_+^2)} \exp \left(\frac{4\pi^2 f^2 m_0^2}{(m_-^2 - m_+^2)^2} \right) \tag{5.14}$$

(recall $m_0^2 < 0$). It is easily seen that this quantity is at most of order $m_\pm$ so that the conditions for the validity of the high temperature expansion do *not* obtain, and the phase transition cannot occur at $T \gg m_\pm$. Now consider the possibility that T_C is much less than $m_\pm$. In the limit $m/T \gg 1$, clearly $\Delta V_T(\phi) \propto \exp(-m_\pm/T)$, so the phase transition cannot take place at $T \ll m_\pm$. It follows that we should expect the phase transition to occur near the scale set by $m_\pm$.

In Fig. 5 we show the total ϕ-dependent potential as a function of temperature. Clearly there is a phase transition somewhere in the range $3m_- < T_C < 5m_-$ when the high-temperature maxima become the low-temperature minima. Just as clearly, the phase transition will be second order. A unique feature of this model is that at the critical temperature the potential is absolutely flat—ϕ becomes a *free* field (not simply massless as in a typical second-order transition). This can be understood by observing that the only extrema of the potential are at $\phi = 0, \pi/2, \pi \cdots$, and when the full potential evaluated at $\pi/2$ becomes equivalent to the potential evaluated at π, there can be no

intervening extrema so the potential must be flat. Thus at the critical temperature the $\mathcal{Z}_2$ symmetry is promoted to a (non-linearly realized) $U(1)$ symmetry.

Above the critical temperature there is still a $\mathcal{Z}_2$ symmetry—there is in no sense a larger symmetry at high temperature. Another interesting feature of this model is that at high temperature, the potential becomes T independent (except for an additive T-dependent, ϕ-independent constant). This in in contrast to usual high-temperature scalar field theory where the mass of the scalar field at high temperature is proportional to T.

Notice that the phase transition can lead to the formation of domain walls. For instance, if ϕ is at the minimum $\phi = \pi$ for $T > T_C$, when the phase transition is complete regions of the Universe with $\phi = \pi/2$ will be separated from regions with $\phi = 3\pi/2$ by a domain wall. However there is one concern with the above scenario: There may be no physical mechanism to set ϕ to its high-temperature minimum. The value of ϕ at high temperature may be free to roam and may not be pinned to any particular value. This is because the $\mathcal{Z}_2$ symmetry of the model implies that the $\phi^2 T^2$ term will not be present, and at high temperatures the leading temperature-dependent, ϕ-dependent term is $\phi^2 \ln(T^2)$, which grows slowly with T. One might well imagine a scenario where ϕ has insufficient time to relax to its high-temperature minimum before the onset of the phase transition. If ϕ has a value away from the low-temperature maximum at the onset of the phase transition, and it is constant throughout the Universe (say set during inflation), then the entire Universe may evolve to the same low-temperature value of ϕ and domain walls would not appear. Therefore, if domain walls are produced, f must be considerably less than the scale of inflation.

Finally, we digress for a moment to make sure we know just exactly *whose* temperature enters into the above expressions. Recall that the light neutrinos decouple from the ambient plasma at $T_D \sim 1$ MeV. Thus after this time the neutrinos are *not* in thermal

equilibrium. However, the neutrino distribution function is *still* that of a particle in thermal equilibrium (so long as T is not too much less than the mass) with an effective temperature given by $a(t_D)T_D/a(t)$ where $a(t)$ is the scale factor, and t_D is the cosmic time at which decoupling occurs. Thus, it is this effective temperature that appears in the finite-temperature effective potential.

B. $\mathcal{Z}_N$-symmetric models

It is simple to generalize the models in the previous section, with its $\mathcal{Z}_2$ symmetry amongst the fermions to one with N fermions and a corresponding $\mathcal{Z}_N$ symmetry. The Yukawa couplings for such a model are:

$$-\mathcal{L}_{\rm Yuk} = \sum_{j=0}^{N-1} \overline{\psi}_j \{m + \epsilon[\cos(\phi/f + 2\pi j/N) + i\gamma_5 \sin(\phi/f + 2\pi j/N)]\} \psi_j. \tag{5.15}$$

This theory has the $\mathcal{Z}_N$ symmetry given by

$$\psi_i \to \psi_{i+1}; \qquad \phi/f \to \phi/f + 2\pi/N, \tag{5.16}$$

where now the index i is taken mod N.

The same methods used in the $\mathcal{Z}_2$ case can be used here to calculate the effective ϕ potential. We find [cf., Eq. 5.3)]

$$V_1(\phi) = \sum_{j=0}^{N-1} \left[\frac{\Lambda^4}{32\pi^2} - \frac{\Lambda^2}{8\pi^2} M_j^2(\phi) - \frac{1}{16\pi^2} M_j^4(\phi) \left(\ln \frac{M_j^2(\phi)}{\Lambda^2} - \frac{1}{2} \right) \right], \tag{5.17}$$

with Λ being the ultraviolet cutoff, as usual, and

$$M_j^2(\phi) = m^2 + \epsilon^2 + 2m\epsilon \cos(\phi/f + 2\pi j/N) \tag{5.18}$$

for $j = 0, \cdots, N-1$. In parallel with the $\mathcal{Z}_2$ case we introduce an arbitrary scale μ, and rewrite the potential as [cf., Eq. (5.4)]

$$V_1(\phi) = \sum_{j=0}^{N-1} \left[\frac{\Lambda^4}{32\pi^2} - \frac{\Lambda^2}{8\pi^2} M_j^2(\phi) \right.$$

$$-\frac{1}{16\pi^2}M_j^4(\phi)\left(\ln\frac{M_j^2(\phi)}{\mu^2}-\frac{1}{2}\right)+\frac{1}{16\pi^2}M_j^4(\phi)\ln\frac{\mu^2}{\Lambda^2}\Bigg]. \tag{5.19}$$

Our next task is to ascertain what types of counterterms must be included to absorb the divergences present in $V_1(\phi)$. Before proceeding, we note the following:

$$\begin{aligned}\sum_{j=0}^{N-1}M_j^2(\phi) &= N(m^2+\epsilon^2) \qquad (\text{for } N>1)\\ \sum_{j=0}^{N-1}M_j^4(\phi) &= N[(m^2+\epsilon^2)^2+2m^2\epsilon^2] \qquad (\text{for } N>2).\end{aligned} \tag{5.20}$$

Whereas in the $\mathcal{Z}_2$ case $\sum_{j=0}^{1}M_j^2(\phi)$ was ϕ independent, if the discrete symmetry is $\mathcal{Z}_{N>2}$, then $\sum_{j=0}^{1}M_j^4(\phi)$ is also ϕ independent. Thus, the only counterterm we need to add is the ϕ-independent term $\mathcal{V}_0$, given by

$$\mathcal{V}_0=\tilde{V}_0(\mu)-\sum_{j=0}^{N-1}\left(\frac{\Lambda^4}{32\pi^2}-\frac{\Lambda^2}{8\pi^2}M_j^2(\phi)+\frac{1}{16\pi^2}M_j^4(\phi)\ln\frac{\mu^2}{\Lambda^2}\right). \tag{5.21}$$

Forming the total potential and dropping irrelevant ϕ-independent terms, we find

$$V(\phi)=\tilde{V}_0(\mu)-\sum_{j=0}^{N-1}\frac{1}{16\pi^2}M_j^4(\phi)\ln\frac{M_j^2(\phi)}{\mu^2}. \tag{5.22}$$

Again $\tilde{V}_0$ can be found by some renormalization condition, and the μ dependence in $\tilde{V}_0(\mu)$ will cancel the μ dependence in the log term rendering the entire potential finite and μ independent. Since we did not need to add any ϕ-dependent counterterms, the ϕ mass is *calculable* in terms of the parameters of the theory, i.e., m and ϵ.

The extrema of $V(\phi)$ are somewhat trickier to find than in the $\mathcal{Z}_2$ case. It can, however, be shown that these are located at $\phi/f=0,\ \pi/N \bmod 2\pi/N$. Whether these are maxima or minima depends on N; for N even (odd) $\phi=0$ is a max (min) while $\phi/f=\pi/N$ is a min (max). The potential again has a simple periodic form. The form of the potential for $N=3$ is shown in Fig. 6.

Now the temperature corrections are easy to calculate—they are given by Eq. (5.10) where now the sum on j runs from 0 to $N-1$. An example of the temperature-dependent

corrections to the potential is shown in Fig. 7. It is clear that the sign of the temperature-dependent part of the potential is opposite to the sign of the zero-temperature potential.

The total potential $V(\phi) + \Delta V_T(\phi)$ is shown in Fig. 8 for several temperatures. The interesting result is that at high temperatures the $\mathcal{Z}_N$ symmetry is promoted to an exact (non-linearly realized) $U(1)$ symmetry. That this should occur is easy to see by examination of the high-temperature expansion of the finite-temperature potential [cf., Eq. (5.12)]:

$$\Delta V_T(\phi) = \sum_{j=0}^{N-1} \frac{1}{16\pi^2} M_j^4(\phi) \ln \frac{M_j^2(\phi)}{T^2} + \cdots, \tag{5.23}$$

which exactly cancels the entire ϕ-dependent part of Eq. (5.22). Thus at high temperature the potential becomes exactly flat.

What cosmology might one expect given the temperature behavior of the potential? Clearly between $T \sim f$ (when the effective potential makes sense) and $T \simeq m$, ϕ is free to take on any value. Below some temperature of order m the potential minima will start to become important and different regions of the Universe will have different values of ϕ with domain walls between them. Thus, effectively there is a phase transition at $T \sim m$ where the order parameter (in this case ϕ) evolves from whatever value it had at high temperatures to a zero-temperature minimum. In this case the transition is similar to the phase transition associated with axions, although we emphasize that the underlying dynamics are quite different in the two cases.

Also in analogy with the axion case, if inflation occurs at a scale less than f, then one might expect ϕ to be set to a single value throughout the Universe. If this happens, when the transition occurs there will be a single initial value of ϕ that will be random, there is nothing to perch the initial value of ϕ on a low-temperature maximum, and the Universe will most likely end up in a single value of ϕ—no domain walls.

VI. CONCLUSIONS

In this paper we have given a general discussion of the thermal physics of pseudo-Nambu–Goldstone bosons. These afford a natural way of generating very soft scales of potential interest to astrophysics, typically of order $m_\phi \sim m_{\rm small}^2/f_{\rm large}$ where we might choose $m_{\rm small} \sim m_{\rm neutrino} \sim 10^{-2}$ eV, and $f_{\rm large} \sim f_{GUT} \sim 10^{15}$GeV. These objects have a precedent in elementary particle physics in the familiar nucleon-meson system, as well as scores of theoretical generalizations, and the models considered here really involve no additional physical components. As in the case of the invisible axion, or familons,[28] arions, etc., we are simply abstracting the scales to those that are of potential interest to astrophysics or cosmology. We have discovered that the thermal behavior of these systems is very simple, controlled largely by the residual symmetries of the low energy potential.

Though we have largely focused on the specific models of Ref. (7), these models capture most of the physics that can generally occur in the context of pNGB's. For example, the $\mathcal{Z}_N$ models for large N have very soft breaking of the continuous $U(1)$ symmetry, due to the cancellation of the fermion loops at high momentum from the discrete symmetry. It is, therefore, not surprising that the thermal behavior of this system imitates that of the axion, since the PQ symmetry of the axion is broken only in the far infrared limit of QCD.

To a good approximation we may summarize the thermal physics as follows. The ϕ-dependent part of the potential has the form:

$$V(\phi) = c(T)m^4 \cos(N\phi/f) \tag{6.1}$$

where $c(T)$ is a smoothly varying function of T with the following possible behaviors:

1. $c(T)$ is slowly varying with no sign change over the full range of temperatures

$(0 \leq T \leq f)$;

2. $c(T)$ is slowly varying with a sign change for $T \sim m$, as in the ($\mathcal{Z}_2$ model);

3. $c(T)$ is slowly varying with asymptotic zero $c(T) \to 0$ as $T \to \infty$ (as in $\mathcal{Z}_N$, for $N > 2$).

Here we have not addressed the issue of cosmological implications (if any) for the formation of structure or other possibile signatures. It seems that the options here are (i) to pursue schemes that lead to soft domain walls, or other topological configurations, with "thicknesses" of order m_ϕ^{-1}, which form after the 3°K microwave background decoupling,[6] or (ii) to try to buld a natural version of the Press, Ryden, Spergel[11] scheme.[29] The latter has an additional potential fine-tuning problem associated with initial conditions that may be remedied in something like the $\mathcal{Z}_2$ scheme with a sign change in $c(T)$. We also mention that other large scale signatures, such as periodic redshifts, might require some bizarre version of schemes as discussed here.[30]

If the symmetries and dynamics of particle physics are a guide, then it seems likely that either ultra-low mass fermions, such as massive neutrinos, or ultra-low mass bosons, such as pNGB's, are the best candidates for potential new cosmological effects. Restricting attention to such classes of particles is a powerful simplification rather than a complication. The existence of such objects implies dramatic new physics at the highest energies, $\mathcal{O}(f)$, that lead to phenomena on the largest distance scales (as large as f/m^2), which are of relevance for cosmology. We thus feel that the general discussion of the thermal behavior of pNGB's given here is an important consideration for future cosmological model building efforts. Cosmologists should learn the physics of pNGB's and think about their potential implications in the early and not-so-early Universe.

ACKNOWLEDGEMENTS

This work was supported in part by the Department of Energy and NASA (grant NAGW-1340) at Fermilab, grants NSF AST-88-22595, NASA NAGW-1321 at the University of Chicago, and DOE contract DE-AC02-76ER3066 at Carnegie Mellon. We would like to thank Josh Frieman for conversations.

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19. In this discussion we refer to the effect of ambient background particles as "finite-temperature" effects, even if the background particles have a large chemical potential, and even if they are not described by any sort of thermal distribution function.

20. Strictly speaking, we should include a source term $-J\phi$ in the Lagrangian, chosen so that $\phi = 0$ is a local minimum. Equivalently, we let $\partial V/\partial J = \phi_c$ define the classical value of ϕ and perform the Legendre transformation to $V(\phi) \to V - J\phi_c$.

21. Since the addition of the $\overline{\psi}\phi\psi$ term breaks the reflection symmetry, we should also include a term proportional to ϕ^3.

22. Notice that h will remain unrenormalized, as we will not consider fermion self-energy corrections.

23. Perhaps one should adopt the following principle in reading the rest of this paper, espoused by Georgi: "a given hypothetical physical phenomenon is worthy of study if it arises in an interesting new way in a wide class of natural models with some potential connection to the world, but one should not take seriously any particular model."

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26. In principle, we should also compute the effect of ϕ loops on the effective potential. The renormalization of such Lagrangians is sensible when the cut off of the theory is significantly less than the decay constant. New (nonrenormalizeable counterterms) are associated with powers of Λ/f and are small, while the unsuppressed terms are renormalizeable by redefinition of f and the coefficient of $\cos\phi/f$. This is not suprising, since the potential of Eq. (5.1) is simply an effective low-energy theory, and it can easily be embedded into a fully renormalizeable Higgs Lagrangian on scales above f. Our procedure is the simplification of not allowing ϕ to appear in loops, that is, keep ϕ a classical background field

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FIGURE CAPTIONS

Figure 1: Tadpole diagrams used in calculating the effective potential. Dashed lines represent bosons and solid lines represent fermions.

Figure 2: Loops that lead to induced pNGB mass terms in (a) a chiral Lagrangian scheme of Eq. (4.8) and (b) the Majorana scheme of Eq. (4.15).

Figure 3: The zero-temperature potential of the $\mathcal{Z}_2$ model.

Figure 4: The temperature-dependent corrections to the $\mathcal{Z}_2$ model.

Figure 5: The total potential of the $\mathcal{Z}_2$ model.

Figure 6: The zero-temperature potential of the $\mathcal{Z}_3$ model.

Figure 7: The temperature-dependent corrections to the $\mathcal{Z}_3$ model.

Figure 8: The total potential of the $\mathcal{Z}_3$ model.

Fig. 1

Fig. 2

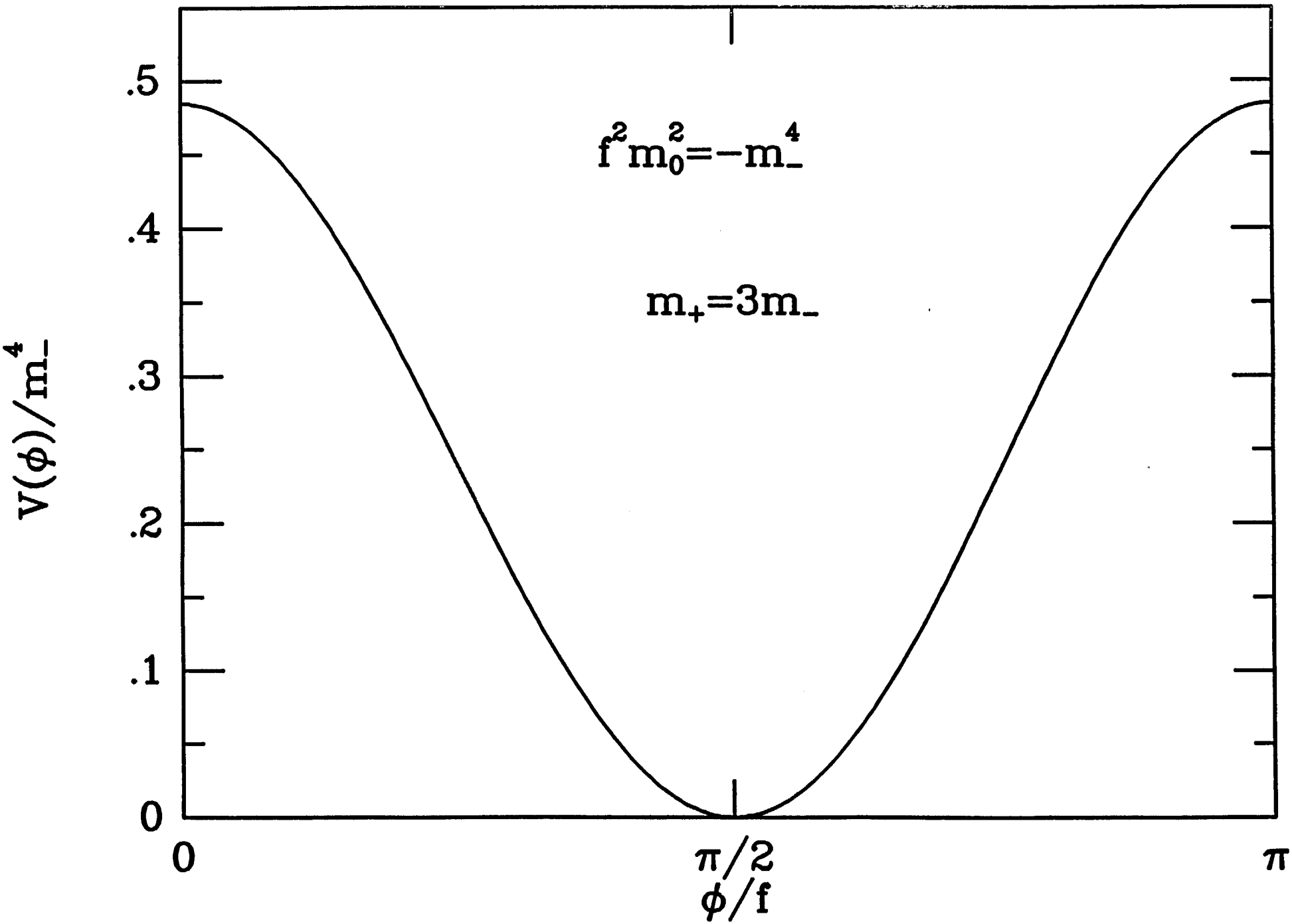

Fig.3

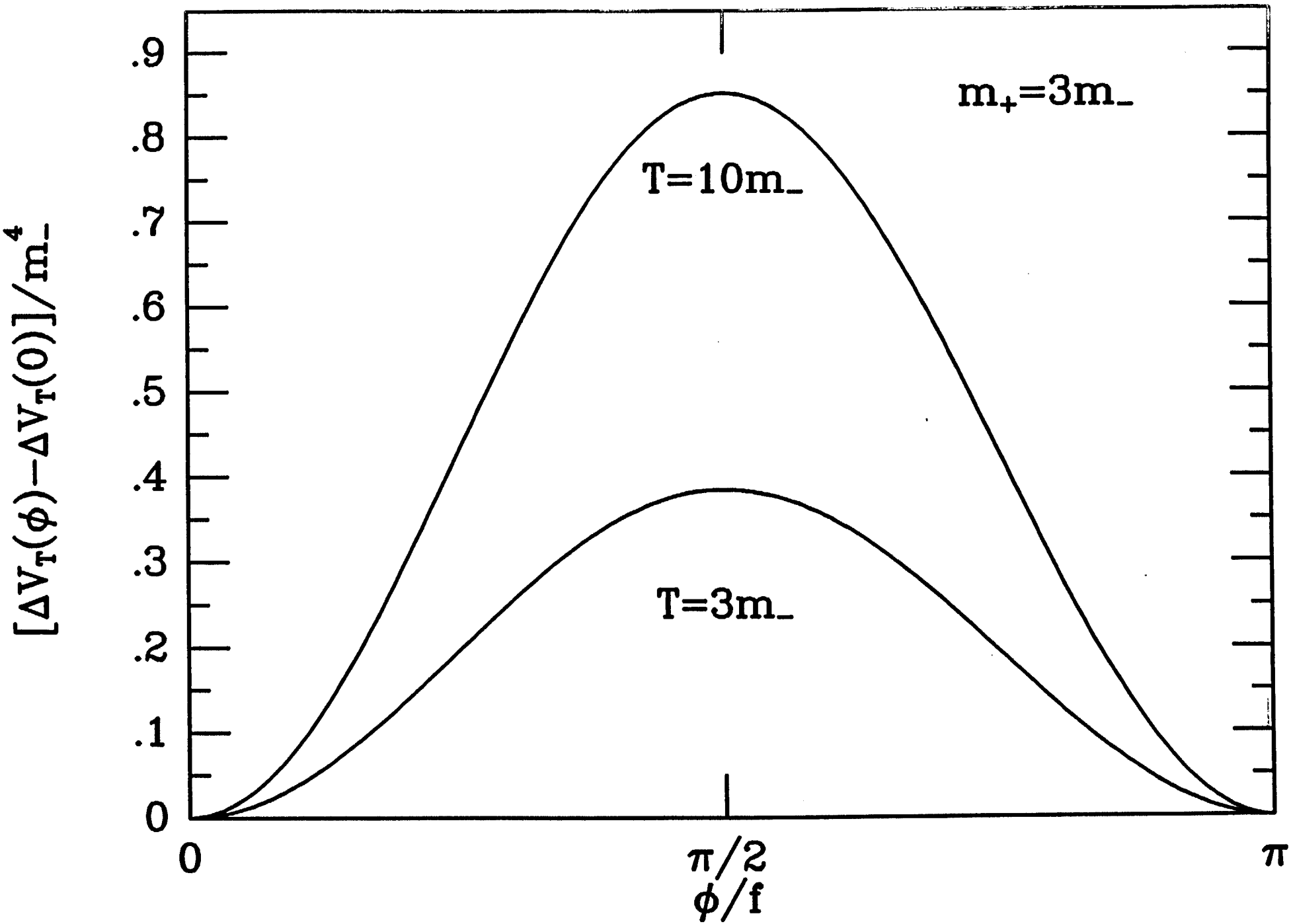
$[\Delta V_T(\phi)-\Delta V_T(0)]/m_-^4$
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
$m_+=3m_-$
$T=10m_-$
$T=3m_-$
0
$\pi/2$
π
ϕ/f

Fig.4

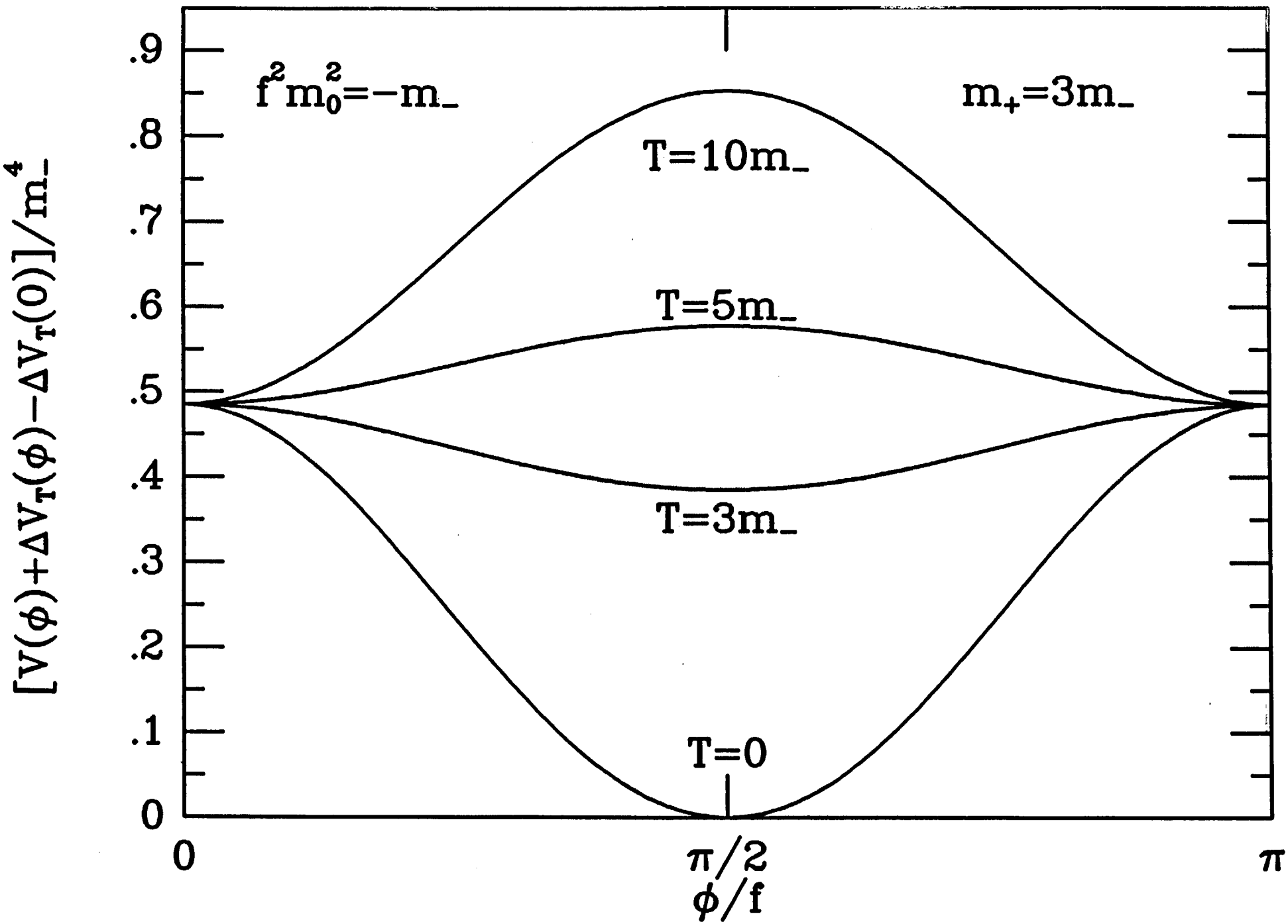
$f^2m_0^2=-m_-$
$m_+=3m_-$
T=10m_
T=5m_
T=3m_
T=0
$[V(\phi)+\Delta V_T(\phi)-\Delta V_T(0)]/m_-^4$
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
$\pi/2$
π
ϕ/f

Fig 5

Fig. 6

$[\Delta V_T(\phi)-\Delta V_T(0)]/m^4$
0
0
$\epsilon=m/3$
$T=10m/3$
$T=2m/3$
$T=m/3$
$\pi/3$
ϕ/f
$2\pi/3$

fig.7

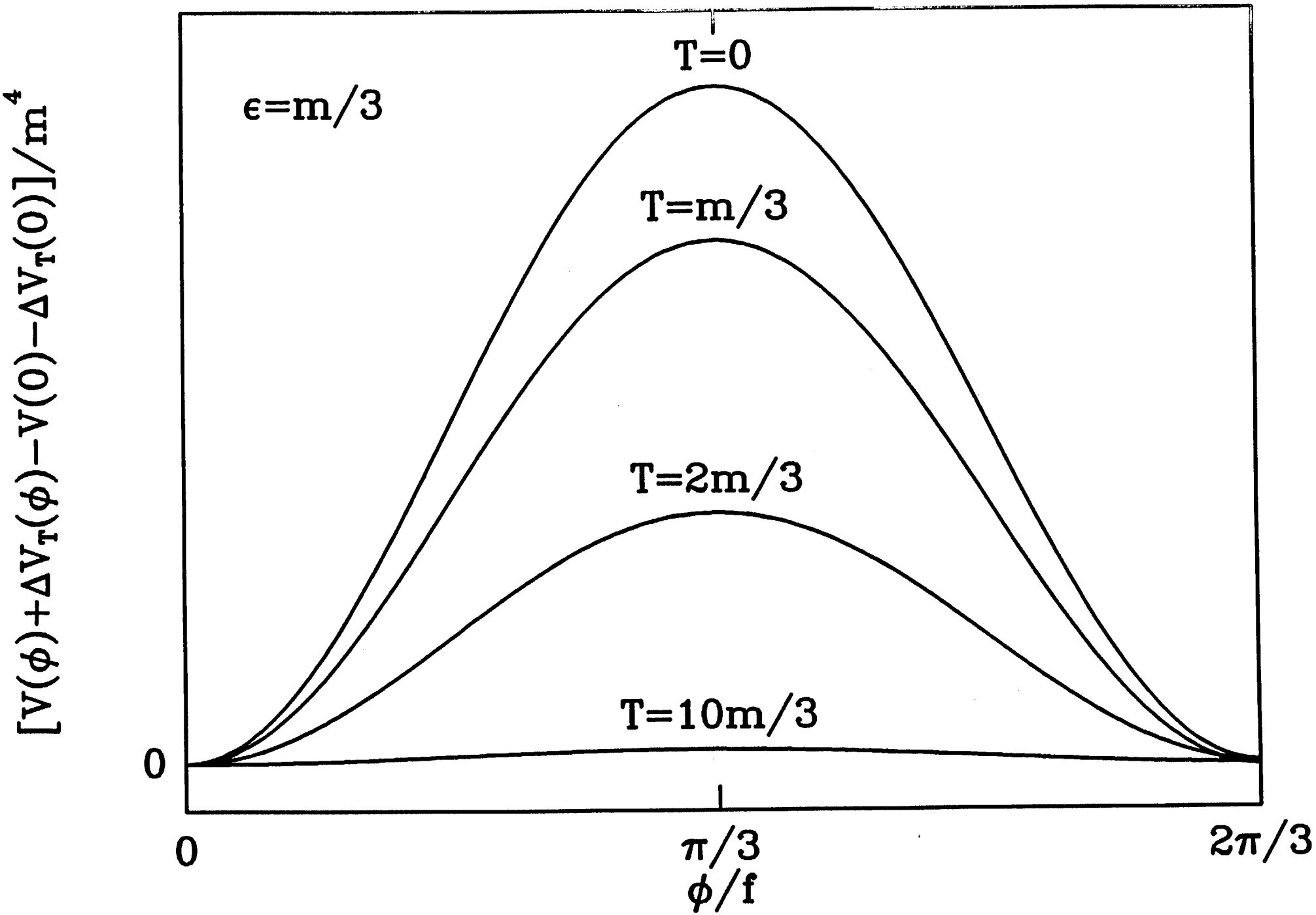

Fig. 8