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Turbulent Fluid Motion III—

Basic Continuum Equations

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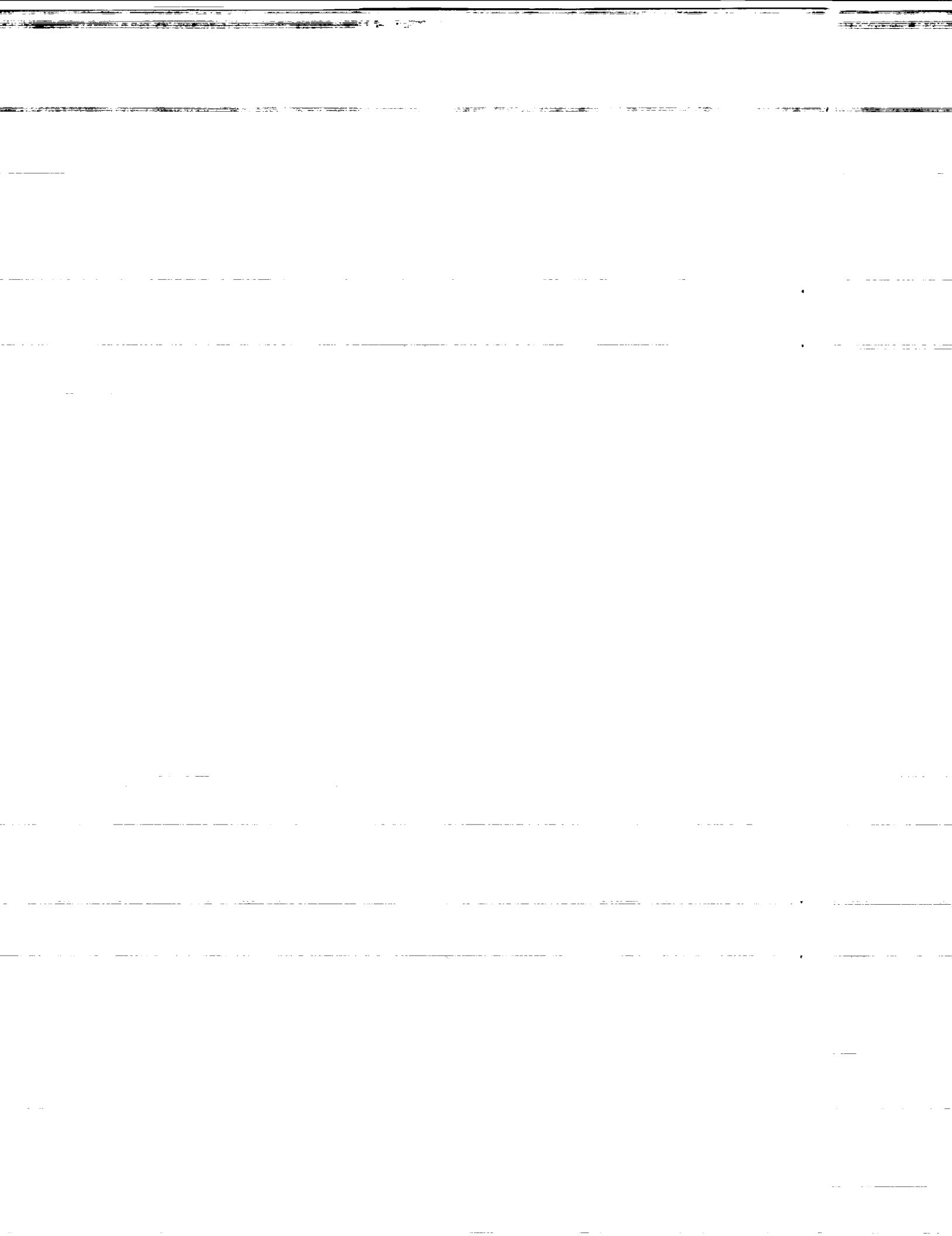
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TURBULENT FLUID MOTION III -

Basic Continuum Equations

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SUMMARY

A derivation of the continuum equations used for the analysis of turbulence is given. These equations include the continuity equation, the Navier-Stokes equations, and the heat transfer or energy equation. An experimental justification for using a continuum approach for the study of turbulence is also given.

INTRODUCTION

The Navier-Stokes equations, with the other continuum equations for fluids, form the basis for the analysis of turbulence in this series. One might, in fact, refer to that analysis as a Navier-Stokes theory of turbulence. Therefore a derivation of the Navier-Stokes equations and of the other fluids equations seems appropriate, if the series is to be reasonably self-contained. Moreover, the derivations given here, at least those for the Navier-Stokes equations, proceed from fewer assumptions than do those generally given.

3.1 JUSTIFICATION OF THE USE OF A CONTINUUM APPROACH FOR TURBULENCE

Most workers in fluid dynamics, since the early days of the science, have ignored the molecular structure of fluids. They consider fluids as continua for the purpose of mathematical analysis. This point of view is generally justified by pointing out that the macroscopic lengths in most fluid-dynamic problems are many times as large as the corresponding molecular lengths. This may be true even if the flow is turbulent, the characteristic lengths of the turbulence being much larger than the molecular lengths.

In the case of turbulent flow, however, there may be some question about the size of the smallest important lengths. According to the spectral theory of turbulence, to be discussed later, there is no definite lower limit on eddy size. But the presence of viscosity might be expected to limit the efficacy of the smallest eddies. For that reason most workers feel intuitively that the molecular structure of the fluid is unimportant for turbulence analysis. (An exception would be highly rarefied gases.)

However there is a solid experimental basis for a continuum theory of turbulence. This lies in the excellent macroscopic correlation of data for fluids of widely different molecular structure. For instance gases and liquids have completely different molecular structures. The molecules in a gas are generally so far apart that each molecule can interact with only one other molecule at a time. In a liquid, on the other hand, each closely spaced molecule interacts

simultaneously with many others.¹ In spite of this considerable difference in structure, experimental data for turbulent flow of liquids correlate well macroscopically with those for gases.

Figure 3-1 compares experimental fully developed turbulent friction factors at various Reynolds numbers for water and for air flowing at low speeds in smooth pipes. Fully developed means that the time-averaged velocity does not vary with time or axial position. The dimensionless friction factor f is the ratio of time-averaged wall-shear stress $\bar{\tau}_w$ to dynamic pressure $(1/2)\rho U_a^2$, U_a is the velocity averaged with respect to time and cross-sectional area, D is the pipe diameter, ρ is the density, and ν is the kinematic viscosity. It can be shown from the equations for continuum fluid flow (to be considered later in this chapter) that f is a function only of Reynolds number for a fully developed low-speed continuum flow (see section 3.3.1). If the molecular structure were important, at least one additional dimensionless group would be required (e.g., the ratio of intermolecular distance to the tube diameter). In spite of the great differences between the molecular structures of water and air, the macroscopic or continuum correlation of the experimental data in figure 3-1 is excellent. General experience in correlating data for turbulent flow of liquids and gases indicates results similar to those in figure 3-1. It thus appears that a continuum theory of turbulence which ignores differences between the molecular structures of various fluids is realistically based.

3.2 EQUATION OF CONTINUITY

(Conservation of Mass)

The rate of mass flow per unit area in the x_1 -direction is

$$m_1 = \rho u_1, \quad (3-1)$$

where ρ is the density and u_1 is the velocity component in the x_1 -direction. In the x_i -direction this becomes

$$m_i = \rho u_i. \quad (3-2)$$

If m_i is measured at x_i , then at $x_i + \Delta x_i$, m_i is replaced by

$$m_i + \frac{\partial m_i}{\partial x_{(i)}} \Delta x_{(i)} + \dots (\text{no sum on } i),$$

where m_i has been expanded in a Taylor series. For small Δx_i only the first two terms need be retained. Then the change in m_i in going from x_i to $x_i + \Delta x_i$ is $(\partial m_i / \partial x_{(i)}) \Delta x_{(i)}$.

¹There is an interesting analogy between the theory of liquids and the theory of strong turbulence. In both theories interactions among many modes or degrees of freedom must be considered.

Consider a small *stationary* volume element $\Delta x_1 \Delta x_2 \Delta x_3 = \Delta \mathbf{x}$.² The net flow of mass into the element through all the faces in a short time Δt is $-(\partial m_i / \partial x_i) \Delta \mathbf{x} \Delta t$ (sum on i). Equating this to the change of mass within the element in time Δt gives

$$\frac{\partial \rho}{\partial t} \Delta t \Delta \mathbf{x} = -\frac{\partial m_i}{\partial x_i} \Delta t \Delta \mathbf{x}$$

or, using equation (3-2),

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u_i)}{\partial x_i} \quad (3-3)$$

which is the continuity equation for a compressible fluid. If the fluid is incompressible, equation (3-3) becomes

$$\frac{\partial u_i}{\partial x_i} = 0. \quad (3-4)$$

That is, for an incompressible flow, the net rate of flow of fluid through the faces of a small volume element $\Delta \mathbf{x}$ is zero.

3.3 NAVIER-STOKES EQUATIONS

(Conservation of Momentum)

3.3.1 The Stress Tensor

3.3.1.1 *Experimental basis and the fundamental assumption.* - We begin with the basic experimental fact concerning the motion of a viscous Newtonian fluid: In a pure shearing motion the shear stress is proportional to the velocity gradient.³ If, for instance, the velocity is u_1 , in the x_1 -direction, and the velocity gradient is in the x_2 -direction, the shear stress σ_{21} is

²A volume of arbitrary shape together with Gauss's theorem (divergence theorem), is often used in the derivation of the conservation equations. However the use of a small rectangular volume element is more direct and gives results as general as does the method which uses Gauss's theorem. Moreover the derivation of Gauss's theorem generally makes use of small volume elements.

³In most derivations of the Navier-Stokes equations the normal components as well as the shear components of the stress are assumed proportional to a velocity gradient. However, the relations for the normal components are difficult to verify experimentally and are not specified here. Rather they come naturally out of the derivation.

$$\sigma_{21} = \mu \frac{du_1}{dx_2}, \quad (3-5)$$

where the scalar μ is called the viscosity and is a property of the fluid but not of its motion. Equation (3-5) is found experimentally to be an excellent assumption for a great many liquids and gases. It gives the shear stress on a nonrotating face 2 (normal to x_2) of a small cube of fluid which is deforming in shear. We can also write equation (3-5) as

$$\sigma_{21} = -\mu \frac{d\theta_{21}}{dt}, \quad (3-6)$$

where $d\theta_{21}/dt$ is the time rate of change of the angle between the intersecting faces 1 and 2 (respectively normal to x_1 and x_2) of the deforming cube; the angle θ_{21} is measured from face 2 to face 1. If we change the rate of rotation of the cube as a whole but keep $d\theta_{21}/dt$ constant, σ_{21} will not change; it is the rate of deformation of the cube, not its rotation rate, that determines its state of stress. But both faces 1 and 2 now rotate, whereas before, (eq. (3-5)) only face 1 rotated. In general, for two intersecting faces i and j of a cube deforming in shear, those faces being initially normal to x_i and x_j respectively, we can write for the shear stress

$$\sigma_{ji} = -\mu \frac{d\theta_{ji}}{dt}, \text{ for } i \neq j \quad (3-7)$$

in place of equation (3-6), where $d\theta_{ji}/dt$ is the time rate of change of the angle between faces i and j , the angle θ_{ji} is originally $\pi/2$, and σ_{ji} is the shear stress on the face initially normal to x_j . For two intersecting faces of a cube deforming in shear, but which are initially normal to x_i^* and x_j^* in a rotated (nonrotating) coordinate system, we write, instead of equation (3-7),

$$\sigma_{ji}^* = -\mu \frac{d\theta_{ji}^*}{dt}, \text{ for } i \neq j, \quad (3-7a)$$

where $d\theta_{ji}^*/dt$ is the time rate of change of the angle between the two faces which are initially normal to x_i^* and x_j^* , and σ_{ji}^* is the shear stress on the face initially normal to x_j^* . Equation (3-7) or (3-7a) is taken to apply in a general viscous flow, in which all faces of the deforming cube of fluid may rotate and move normally, and for all orientations of the cube. The viscosity μ is a scalar, so that it is invariant with rotations. Our fundamental assumption can be stated somewhat formally as follows: There exists at each point in the fluid a scalar μ called the viscosity, such that for all orientations of the initially normal face pair (i, j), equation (3-7) or (3-7a) holds.

Since equation (3-7) applies in a general viscous flow in which all faces of a fluid cube may rotate, $-d\theta_{ji}/dt$ is given in the x_i coordinate system by

$$-\frac{d\theta_{ji}}{dt} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad (3-8)$$

or, in the x_i^* system,

$$-\frac{d\theta_{ji}^*}{dt} = \frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \quad (3-8a)$$

Using equations (3-7) and (3-8), the expression for the shear stress is

$$\sigma_{ji} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \text{ for } i \neq j. \quad (3-9)$$

In the rotated coordinate system x_i^* , equations (3-7a) and (3-8a) give

$$\sigma_{ji}^* = \mu \left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \right), \text{ for } i \neq j. \quad (3-9a)$$

3.3.1.2 *The expression for the stress tensor.* - As shown in section 2.4.2 (ref. 5), a stress is a second-order tensor, so that the general expression for σ_{ji} must be such. As it stands, σ_{ji} in equation (3-9) (or σ_{ji}^* in eq. (3-9a)) is not a tensor because of the qualifying statement $i \neq j$. If we remove that statement, then $\mu (\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is a second-order tensor; as shown in section 2.4.1 (ref. 5), the spatial derivative of a vector is a second-order tensor, as is the sum of second-order tensors and the product of a scalar and a second-order tensor. Since the sum of tensors of the same order is also a tensor, the general expression for σ_{ji} may contain a term (which may in turn be a sum of terms) in addition to $\mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, call it B_{ji} , so that

$$\sigma_{ji} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + B_{ji} \quad (3-10)$$

where, in contrast to equation (3-9), $i, j = 1, 2, \text{ or } 3$. Using equations (2-11) and (2-14) (ref. 5), we get, in a rotated coordinate system x_i^* ,

$$\sigma_{ji}^* = \mu \left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \right) + B_{ji}^*. \quad (3-10a)$$

But to ensure that the shear components are still given by equation (3-9) or ((3-9a)), the additional term B_{ji} or B_{ji}^* must be zero for $i \neq j$. However, one cannot know that a particular second-order tensor which is zero for $i \neq j$, say $\psi\delta_{ji}$, is necessarily the complete B_{ji} , unless it can be shown that it is the most general second-order tensor which is zero for $i \neq j$.⁴

Thus we need to determine the form of B_{ji} , that being the most general second-order tensor for which

⁴For example, one might imagine that a second-order tensor which is zero for $i \neq j$ could depend on the motion of the fluid, as well as on pressure forces that are not necessarily accompanied by motion.

$$B_{ji} = B_{ji}^* = 0 \text{ for } i \neq j. \quad (3-11)$$

Note that we have not said anything about B_{ji} for $i = j$; that will be considered in what follows.

Since B_{ji} is a second-order tensor, its transformation law is (eq. (2-11), ref. 5)

$$B_{ji}^* = a_{jk} a_{i\ell} B_{k\ell}$$

Multiplying this equation by a_{im} and using equation (2-5) give

$$a_{im} B_{ji}^* = a_{jk} a_{i\ell} a_{im} B_{k\ell} = a_{jk} \delta_{im} B_{k\ell} = a_{jk} B_{km}.$$

Rewriting the first and last members of this equation,

$$a_{im} B_{ji}^* = a_{jk} B_{km}. \quad (3-12)$$

The unrepeated (assignable) subscripts in equation (3-12) are m and j . We set $m = j = 1$, carry out the summations on the repeated subscripts i and k , and, with q and r as general subscripts, let $B_{qr} = B_{qr}^* = 0$ for $q \neq r$ (for shear) (eq. (3-11)). The quantities B_{qr} and B_{qr}^* are both set equal to zero for $q \neq r$ to ensure that for shear, equations (3-10) and (3-10a) reduce respectively to (3-9) and (3-9a), as they should. Then equation (3-12) becomes

$$a_{11} (B_{11}^* - B_{11}) = 0,$$

or

$$B_{11}^* = B_{11}.$$

Similarly, by setting $m = 2, i = 1$, equation (3-12) becomes

$$B_{11}^* = B_{22},$$

and by setting $m = 3, i = 1$,

$$B_{11}^* = B_{33}.$$

From the last three equations,

$$B_{11} = B_{22} = B_{33}.$$

Thus, starting from equation (3-11), we have shown that $B_{11} = B_{22} = B_{33}$ if $B_{ij} = B_{ij}^* = 0$ for $i \neq j$. It follows from the definition of the Kronecker delta (eq. (2-3), ref. 5) that

$$B_{ij} = \psi \delta_{ij},$$

where ψ is a scalar. We have placed no restrictions on B_{ij} other than that it be a second-order tensor with the property that it and B_{ij}^* are zero for $i \neq j$. Therefore, $B_{ij} = \psi \delta_{ij}$ is the most general second-order tensor for which $B_{ij} = B_{ij}^* = 0$ for $i \neq j$. Note that B_{ij} turns out to be an isotropic tensor, although we have not explicitly made that assumption. It is of interest that $B_{ij} = \psi \delta_{ij}$ is the most general second-order isotropic tensor (section 2.4.3, ref. 5), as well as the most general second-order tensor for which $B_{ij} = B_{ij}^* = 0$ for $i \neq j$.

Thus the most general expression for the stress tensor which satisfies equation (3-9) or (3-9a) for the shear components is $\sigma_{ji} = \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i) + B_{ji}$ (eq. (3-10)), where $B_{ji} = \psi\delta_{ji}$. So

$$\sigma_{ji} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \psi\delta_{ji}.$$

Note that the above equation, besides reducing to (3-9) for $i \neq j$, reduces to the isotropic form for $u_i = 0$, as it should for a motionless fluid. By contracting that equation, we get

$$\psi = \frac{1}{3}\sigma_{kk} - \frac{2}{3}\mu \frac{\partial u_k}{\partial x_k},$$

so that

$$\sigma_{ji} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ji} \frac{\partial u_k}{\partial x_k} \right) - \sigma\delta_{ji} = \sigma'_{ji} - \sigma\delta_{ji}, \quad (3-13)$$

where σ'_{ji} is the part of the stress produced by viscous action (friction), and

$$\sigma \equiv -\sigma_{kk}/3. \quad (3-14)$$

The quantity σ is, by definition, minus the average of the three normal stresses at a point in the fluid and is sometimes called the mechanical pressure. Equation (3-13) gives the stress tensor which results from our fundamental assumption, equation (3-7) (or (3-7a)). The first term on the right side of equation (3-13) gives the stress produced by viscous or frictional (irreversible) processes, the second term gives the stress produced by pressure forces (nonviscous or reversible processes). For $i = j$, equation (3-13) gives the normal stresses acting on the faces of a small fluid element; for $i \neq j$ it gives the shear stresses. Inspection of the form of equation (3-13) shows that $\sigma_{ij} = \sigma_{ji}$.

3.3.2 The Equations of Motion

To obtain the equations of motion for a viscous fluid, consider the force ΔF_i^* acting on a small fluid element which moves with the fluid and whose volume is $\Delta \mathbf{x} = \Delta x_1 \Delta x_2 \Delta x_3$. Newton's law, applied to the small element, is

$$\Delta F_i^* = \Delta m \frac{du_i}{dt}, \quad (3-15)$$

where t is the time. The quantity Δm is the mass of the element and does not change, although the volume $\Delta \mathbf{x}$ can undergo a change of shape in such a way as to continue to enclose the same fluid.

Next obtain the surface force acting on the volume element $\Delta \mathbf{x}$. We note that if the stress is σ_{ji} at x_j , then at $x_j + \Delta x_j$, σ_{ji} is replaced by

$$\sigma_{ji} + \frac{\partial \sigma_{ji}}{\partial x_{(j)}} \Delta x_{(j)} + \dots (\text{no sum on } j)$$

where, as was done for m_i in the last section, σ_{ji} has been expanded in a Taylor series. For small Δx_j only the first two terms need be retained. Then the change in σ_{ji} as one moves from x_j to $x_j + \Delta x_j$ is $(\partial \sigma_{ji} / \partial x_{(j)}) \Delta x_{(j)}$. The sum of all the forces in the i -direction acting on the faces of Δx is $\Delta x \partial \sigma_{ji} / \partial x_j$ (sum on j).

In terms of σ_{ji} and an external force, equation (3-15) becomes

$$\frac{\partial \sigma_{ji}}{\partial x_j} \Delta x + \rho \Delta x g_i = \rho \Delta x \frac{du_i}{dt} \quad (3-16)$$

or

$$\rho \frac{du_i}{dt} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho g_i, \quad (3-17)$$

where ρ is the density of the fluid and g_i is an external force per unit mass. Introducing the expression for σ_{ji} from equation (3-13) and the Eulerian derivatives of u_i into equation (3-17) ($d/dt = \partial/\partial t + u_k \partial/\partial x_k$), we get

$$\rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial \sigma}{\partial x_i} + \frac{\partial}{\partial x_k} \left[\mu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \right] - \frac{2}{3} \frac{\partial}{\partial x_i} \left(\mu \frac{\partial u_k}{\partial x_k} \right) + \rho g_i. \quad (3-18)$$

Equation (3-18) is the Navier-Stokes equation for a viscous, compressible fluid.

For incompressible flow and constant viscosity, equation (3-18) simplifies to

$$\frac{\partial u_i}{\partial t} = -u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial \sigma}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + g_i \quad (3-19)$$

where the incompressible continuity relation

$$\frac{\partial u_k}{\partial x_k} = 0 \quad (3-20)$$

was used, and ν is the kinematic viscosity μ/ρ . Most turbulence studies have been carried out for constant properties for simplicity. The flow is realistic if the turbulence velocities are reasonably small compared with the velocity of sound, and if temperature gradients are not large (small external heat transfer). Although equation (3-19) is based on a linear stress-strain relationship, the equation is essentially nonlinear because of the presence of the nonlinear convective term $-u_k \partial u_i / \partial x_k$. In fact it is that term which causes most of the difficulties in the turbulence problem; in particular it gives rise to the closure problem, a problem that will be considered in later chapters.

In order to interpret the terms in equation (3-19), it is convenient to multiply it through by ρ and by the *stationary* volume element $dx_1 dx_2 dx_3$. Then the term on the left side of the

equation is the time rate of change of the i^{th} component of momentum $\rho u_i dx_1 dx_2 dx_3$ in the element. This rate of change is contributed to by the terms on the right side of the equation. The first term on the right side, the nonlinear convective or inertia term, is the net rate of flow of the i^{th} component of momentum into the element through its faces. The next term is a (mechanical) pressure-force term and gives the net force acting on the element by virtue of the pressure gradient in the x_i -direction. The term containing ν , a linear viscous-force term, gives the net force acting on the element in the x_i -direction by virtue of viscous or frictional action. Finally the last term is the external force acting on the element in the x_i -direction.

Equations (3-19) and (3-20) constitute a set of four equations in the four unknowns u_i ($i = 1,2,3$) and σ . If we take the divergence of equation (3-19) (differentiate it with respect to x_i) and apply the continuity equation (3-4), we get

$$\frac{1}{\rho} \frac{\partial^2 \sigma}{\partial x_i \partial x_i} = - \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} + \frac{\partial g_i}{\partial x_i} \quad (3-21)$$

Equation (3-21) is a Poisson equation which describes how the scalar σ varies with position under the influence of the source terms on the right side of the equation. Note that it is analogous to the steady-state heat-conduction equation with heat sources, where σ would be the temperature, $1/\rho$ would be a constant thermal conductivity, and the terms on the right side would be heat sources.

As mentioned earlier, the quantity σ , which is the average of the three normal stresses at a point, is sometimes called the mechanical pressure and is often replaced by the symbol p . Here the symbol σ will be retained to distinguish the average of the three normal stresses from the thermodynamic pressure p .

The set of equations (3-19) and (3-21) is often more convenient to use than (3-19) and (3-20). This set, which follows from equation (3-7) (or (3-7a)) with no further assumptions, is complete for incompressible flow with constant viscosity. We will be mainly concerned with incompressible flow in this series.

A fundamental turbulence problem for a constant-property fluid is this initial-value problem: Given initial values for the u_i as functions of position, a value for ν , and suitable boundary conditions; to calculate u_i and σ/ρ as functions of time and position. Equations (3-19) and (3-21) should be sufficient for doing that. But, because of sensitive dependence on initial conditions, the initial values of u_i would have to be given with infinite precision. However, even if they are not, as in the real world, the equations should, in principle, be sufficient for calculating the evolution of *averaged* values.

Equation (3-18), for variable-property flow, also follows from equation (3-7) (or (3-7a)) with no further assumptions. However, in order to form a complete set for variable-property flow we need, in addition to the compressible continuity relation (eq. (3-3)), relations giving the variations of properties with temperature and pressure, an expression of the conservation of energy, and finally an assumption which relates the average normal stress σ to the thermodynamic pressure p . The appropriate assumption for σ is in general, controversial (see, e.g., ref. 6). However, a sufficiently accurate expression for most fluid-dynamics work is Stokes hypothesis, obtained by equating σ to the thermodynamic pressure:

$$\sigma \equiv -\sigma_{kk}/3 = p. \quad (3-22)$$

For cases where buoyancy is important, the velocities and temperature differences are often small, and a good approximation can be obtained by considering properties to be variable only in the buoyancy term (Boussinesq approximation). For doing that, we first write equation (3-18) for the equilibrium case ($u_i = 0$):

$$0 = -\frac{\partial \sigma_e}{\partial x_i} + \rho_e g_i,$$

where the subscript e designates equilibrium values. Subtracting this equation from equation (3-18) and assuming that variable properties are important only in the buoyancy term result in

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho} \frac{\partial(\sigma - \sigma_e)}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + \frac{(\rho - \rho_e)}{\rho} g_i.$$

If $(\rho - \rho_e)/\rho \ll 1$ and is produced mainly by temperature differences, with pressure differences having a negligible effect, we can write this equation as

$$\frac{\partial u_i}{\partial t} = -u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial(\sigma - \sigma_e)}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} - \beta(T - T_e)g_i, \quad (3-18a)$$

where

$$\beta = -(1/\rho) (\partial\rho/\partial T)_\sigma$$

is the thermal expansion coefficient. Note that the equilibrium temperature T_e is uniform, whereas the equilibrium pressure σ_e is not. Taking the divergence of equation (3-18a) and using the incompressible continuity equation give the Poisson equation for the (mechanical) pressure difference as

$$\frac{1}{\rho} \frac{\partial^2(\sigma - \sigma_e)}{\partial x_i \partial x_i} = -\frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} - \beta g_i \frac{\partial T}{\partial x_i}, \quad (3-21a)$$

where g_i is constant.

3.3.1 Dimensionless Form of Constant-Property Fluid-Flow Equations, and Dimensionless Correlation of Friction-Factor Data

Equations (3-19) and (3-21) can be rescaled or written in dimensionless form for $g_i = 0$ as

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial \sigma}{\partial x_i} + \frac{1}{\text{Re}} \frac{\partial^2 u_i}{\partial x_k \partial x_k} \quad (3-23)$$

and

$$\frac{\partial^2 \sigma}{\partial x_i \partial x_i} = -\frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} \quad (3-24)$$

where

$$\begin{aligned}\frac{u_i}{U_a} &\rightarrow u_i, & \frac{x_k}{L} &\rightarrow x_k, \\ \frac{U_a t}{L} &\rightarrow t, & \frac{\sigma}{\rho U_a^2} &\rightarrow \sigma, \\ \frac{U_a L}{\nu} &\rightarrow \text{Re}\end{aligned}$$

and the arrow \rightarrow means "has been replaced by." The quantity U_a is a characteristic average velocity independent of position, L is a characteristic length for the flow, and Re is a Reynolds number. All of the quantities appearing in equations (3-23) and (3-24) are dimensionless.

Equations (3-23) and (3-24) can be used to justify the friction-factor correlation for fully developed turbulent flow through a pipe in figure 3-1. From equation (3-9), we get, in *dimensional* form,

$$\left(\sigma_{21}\right)_w \equiv \tau_w = \mu \left(\frac{\partial u_1}{\partial x_2}\right)_w \quad (3-25)$$

where the subscript w designates values at the wall (τ_w is the shear stress at the wall), u_1 is in the direction along the pipe and x_2 is normal to the wall. Equation (3-25) can be written in dimensionless (rescaled) form, by using the transformations following equation (3-24), as

$$\frac{\tau_w}{(1/2)\rho U_a^2} = \frac{2}{\text{Re}} \left(\frac{\partial u_1}{\partial x_2}\right)_w, \quad (3-26)$$

where the characteristic average velocity U_a is considered to be the axial velocity averaged with respect to time and area over a cross section of the pipe, and the characteristic length L is taken as the pipe diameter D . (Note that the left side of eq. (3-26) is dimensionless, although the individual quantities there are dimensional, whereas the individual quantities on the right side are dimensionless.)

Consider a long time interval and an axial position for which the flow is fully developed. That is, initial transients have died out and time-averaged velocities \bar{u}_1 do not change with increasing time or axial distance. Averaging equation (3-26) over the long time interval gives, in dimensionless form,

$$\frac{\bar{\tau}_w}{(1/2)\rho U_a^2} \equiv f = \frac{2}{\text{Re}} \left(\overline{\frac{\partial u_1}{\partial x_2}}\right)_w \quad (3-27)$$

where the overbars indicate time-averaged values, and f is the friction factor. The time-averaged velocity gradient in equation (3-27) can be obtained by first computing the velocity field $u_i(x_k, t)$ from equations (3-23) and (3-24), starting from a fully developed instantaneous turbulent velocity distribution, and applying appropriate boundary conditions (e.g., $u_i = 0$ at

the wall).⁵ Then the calculated velocity gradient at the wall is averaged over the time interval at the axial position for which equation (3-27) was obtained. Thus, for a given value of Reynolds number $Re = u_a D / \nu$ (in eqs. (3-23) and (3-27)) the friction factor $f = 2 \bar{\tau}_w / \rho U_a^2$ is, in principle, known from equations (3-23), (3-24), and (3-27), together with appropriate boundary conditions. That is,

$$f = f(Re) \quad (3-28)$$

for a constant-property fully developed turbulent continuum flow, as was assumed in figure 3-1.⁶

3.4 HEAT TRANSFER OR ENERGY EQUATION

(Conservation of Energy)

The heat-transfer rate per unit area by molecular conduction in the x_1 -direction is

$$q_1 = -k \frac{\partial T}{\partial x_1},$$

where k is the thermal conductivity, and T is the temperature. In any direction x_i ,

$$q_i = -k \frac{\partial T}{\partial x_i} \quad (3-29)$$

where q_i is the molecular heat transfer vector.

To obtain the energy equation for a viscous fluid, consider the energy added in time Δt to a small element which moves with the fluid and whose volume is $\Delta x = \Delta x_1 \Delta x_2 \Delta x_3$. Then the law of conservation of energy states that the change of energy in the element equals the heat added plus the work done on the element. Herein we consider the case where the only heat added to the element is that transferred into it by molecular conduction. Then an energy balance on the element gives

$$\rho \Delta x \Delta t \frac{d}{dt} (e + \frac{1}{2} u_i u_i) = - \Delta x \Delta t \frac{\partial q_i}{\partial x_i} + \Delta x \Delta t \frac{\partial}{\partial x_j} (u_i \sigma_{ji}) + \rho \Delta x \Delta t u_i g_i \quad (3-30)$$

where the energy per unit mass is made up of the internal energy e and the kinetic energy $(1/2) u_i u_i$. The term on the left side of equation (3-30) gives the change of energy in the element in time Δt ; the first term on the right side gives the heat added by molecular

⁵The difficulties involved in actually doing this need not concern us here; this is a thought experiment.

⁶Equation (3-28) can also be obtained by using the method of dimensional analysis, from which equation (3-28) follows if we assume that $\bar{\tau}_w / \rho = f(U_a, L, \nu)$. That relation may be obtained intuitively, but in order to be sure that it holds it would seem safer to use the equations of motion for a fluid and an analysis similar to that given here. But consider the argument near the end of section 3.5.

conduction. The last two terms are respectively the work done on the element by surface forces and by the external body force.

If we substitute for q_i from equation (3-29) and for $u_i du_i/dt$ and σ_{ji} from equations (3-17) and (3-13) respectively, we get, after some manipulation,

$$\rho \frac{de}{dt} = \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{\partial x_j} \right) + \sigma'_{ji} \frac{\partial u_i}{\partial x_j} - \sigma \frac{\partial u_i}{\partial x_i}, \quad (3-31)$$

where, as in equation (3-13), σ'_{ji} is the part of the stress produced by viscous action, and σ is the mechanical pressure (eq. (3-14)).

If the fluid is incompressible, $de = cdT$, where c is the specific heat at constant volume, and $\partial u_i/\partial x_i = 0$ (eq. (3-4)). Equation (3-31) then becomes

$$\rho c \left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{\partial x_j} \right) + \sigma'_{ji} \frac{\partial u_i}{\partial x_j}, \quad (3-32)$$

where the relation between total and Eulerian derivatives ($d/dt = \partial/\partial t + u_k \partial/\partial x_k$) was used. Equation (3-32) works well for a liquid.

For a perfect gas the term $\sigma \partial u_i/\partial x_i$ in equation (3-31) cannot be neglected because, even at velocities small compared with the velocity of sound, and for small temperature differences (small external heat transfer), that term is not small compared with the term $\rho de/dt$. (As $\sigma \partial u_i/\partial x_i$ becomes small, so also does $\rho de/dt$.)

The relation $de = cdT$ applies to a perfect gas as well as to an incompressible fluid. On using that relation and the perfect-gas relation $c_p dT = cdT + d(p/\rho)$ where c_p is the specific heat at constant pressure, and equating the mechanical pressure σ to the thermodynamic pressure p (eq. 3-22)), equation (3-31) becomes

$$\rho c_p \frac{dT}{dt} = \frac{dp}{dt} + \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) + \sigma'_{ji} \frac{\partial u_i}{\partial x_j}. \quad (3-33)$$

For many flows the velocities are small enough that we can neglect the terms dp/dt and $\sigma'_{ji} \partial u_i/\partial x_j$ in equations (3-33) and (3-32) in comparison with the heat-conduction term. Moreover, the rate of variation of thermal conductivity with temperature is often small enough that we can consider k to be constant, except for very large temperature differences. Then to a good approximation, equations (3-32) and (3-33) can be written as

$$\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \alpha \frac{\partial^2 T}{\partial x_i \partial x_i}, \quad (3-34)$$

where α is the thermal diffusivity and is given by

$$\alpha = \begin{cases} k/(\rho c) & \text{for a liquid,} \\ k/(\rho c_p) & \text{for a perfect gas.} \end{cases} \quad (3-35)$$

Equation (3-34) is the form of the energy equation which will be used in this series. It can also apply to diffusion processes other than for energy. It is only necessary that the accumulation of some quantity in an element of fluid be balanced by the diffusion of that quantity into the element by a law of the form of equation (3-29), where q_i is a flux vector for that quantity, k is a constant, and T is a scalar. For instance the scalar T might be a dilute concentration of a foreign substance in the fluid and q_i the flux vector of that substance.

3.4.1 Dimensionless Form of Constant-Property Energy Equation, and Dimensionless Correlation of Heat-Transfer Data

Equation (3-34) can be rescaled or written in dimensionless form as

$$\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \frac{1}{Pe} \frac{\partial^2 T}{\partial x_i \partial x_i} \quad (3-36)$$

where

$$\frac{T_w - T}{T_w - T_a} \rightarrow T, \quad \frac{u_i}{U_a} \rightarrow u_i,$$

$$\frac{x_k}{L} \rightarrow x_k, \quad \frac{U_a}{L} t \rightarrow t,$$

$$\frac{U_a L}{\alpha} \rightarrow Pe$$

and, as for equations (3-23) and (3-24), the arrow means "has been replaced by." In this section we consider the wall temperature T_w to be spatially uniform and constant in time. The quantity T_a is a characteristic average fluid temperature independent of position, U_a is a characteristic average fluid velocity, also independent of position, L is a characteristic length for the flow, Pe is a Peclet number, and α is the thermal diffusivity given by equation (3-35). All of the quantities appearing in equation (3-36) are dimensionless. Note that t , x_i , and u_i have been nondimensionalized in the same way as in equations (3-23) and (3-24).

From equation (3-29) we get, in *dimensional* form,

$$(q_2)_w \equiv q_w = -k \left(\frac{\partial T}{\partial x_2} \right)_w \quad (3-37)$$

where the subscript w designates values at the wall, q_w is the heat flux at the wall and x_2 is normal to the wall. Equation (3-37) can be written in dimensionless (rescaled) form, by using the transformations following equation (3-36), as

$$\frac{q_w D}{(T_w - T_a)k} = \left(\frac{\partial T}{\partial x_2} \right)_w \quad (3-38)$$

where the characteristic average temperature T_a is considered to be the temperature averaged with respect to time and area over a *particular* cross section of the pipe, and the characteristic length L is taken as the pipe diameter D . (Note that, as for eq. (3-26), the left side of eq. (3-38) is dimensionless, although the individual quantities there are dimensional, whereas the individual quantities on the right side are dimensionless.)

Consider a long time interval and an axial position for which the flow and heat transfer are fully developed. That is, initial transients have died out and the shapes of the time-averaged velocity and temperature distributions remain similar with increasing time and axial distance. Averaging equation (3-38) over the long time interval gives, in dimensionless form,

$$\frac{\bar{q}_w D}{(T_w - T_a)k} = \frac{hD}{k} \equiv Nu = \left(\overline{\frac{\partial T}{\partial x_2}} \right)_w \quad (3-39)$$

where the overbars indicate time-averaged values, $h \equiv \bar{q}_w / (T_w - T_a)$ is the heat-transfer coefficient, and Nu is the Nusselt number. The time-averaged temperature gradient in equation (3-39) can be obtained by first computing the velocity field $u_i(x_k, t)$ for a given Reynolds number Re from equations (3-23) and (3-24). As in section 3.3.1, one starts from a fully developed instantaneous turbulent velocity distribution and applies appropriate boundary conditions (e.g., $u_i = 0$ at the wall). With the velocity field known for a given Reynolds number Re , the temperature field $T(x_k, t)$ for a given Peclet number Pe can be calculated from equation (3-36), starting from a fully developed instantaneous turbulent temperature distribution, and applying appropriate boundary conditions (e.g., $T = 0$ at the wall (see transformation following eq. (3-36))).⁵ Then the calculated temperature gradient at the wall is averaged over the time interval at the axial position for which equation (3-39) was obtained. Thus, for a given value of Reynolds number $Re = U_a D / \nu$ (in eq. (3-23)) and of Peclet number $Pe = U_a D / \alpha$ (in eq. (3-36)), the Nusselt number $Nu = hD/k$ is, in principle, known from equations (3-23), (3-24), (3-36) and (3-39), together with appropriate boundary conditions. That is,

$$Nu = Nu(Re, Pe) \quad (3-40)$$

for a constant property fully-developed turbulent continuum flow and heat transfer. In terms of the Prandtl number $Pr = \nu / \alpha$ and Reynolds number Re , equation (3-40) can be written in the perhaps more familiar form (since $Pe = RePr$), as

$$Nu = Nu(Re, RePr) = Nu(Re, Pr). \quad (3-40a)$$

3.5 A RULE FOR OBTAINING ADDITIONAL DIMENSIONLESS PARAMETERS AS A SYSTEM BECOMES MORE COMPLEX

The method which was used in sections 3.3.1 and 3.4.1 to obtain correlating dimensionless parameters from the basic turbulence equations can be extended to more complex systems.

However, once a functional relation such as equation (3-28) or (3-40a) has been obtained for a given system, the functional relation for a system which depends on an additional physical quantity can be written down by the following simple rule: If a system becomes dependent on one additional physical quantity, one additional dimensionless parameter (containing the additional physical quantity) is added to the functional relation.

For example, if the flow or heat transfer in a pipe becomes dependent on the distance from the entrance x_1 , as it will if the pipe becomes short, then one dimensionless parameter containing x_1 , say x_1/D , where D is the pipe diameter (already included), will be added to the functional relation. Thus equations (3-28) and (3-40a) become respectively

$$f = f(\text{Re}, x_1/D) \quad (3-41)$$

and

$$\text{Nu} = \text{Nu}(\text{Re}, \text{Pr}, x_1/D). \quad (3-42)$$

If, in addition, the system becomes dependent on the velocity of sound c , as it will for high fluid velocities, then an additional dimensionless parameter, say U_a/c , where U_a is a characteristic velocity (already included in the functional relation), becomes operative, and equations (3-41) and (3-42) become, respectively,

$$f = f(\text{Re}, x_1/D, U_a/c) \quad (3-43)$$

and

$$\text{Nu} = \text{Nu}(\text{Re}, \text{Pr}, x_1/D, U_a/c). \quad (3-44)$$

If the system is also dependent on an external force g_i , then a dimensionless parameter such as $g_i D/U_a^2$ will be added in the functional relationship. Finally, although we will not deal with that effect in this series, if intermolecular distances d_m become important, as in a highly rarefied gas, then the functional relationship will contain a dimensionless parameter such as d_m/D . If the system is dependent on all the physical quantities mentioned above, then equations (3-28) and (3-40a) become, respectively,

$$f = f(\text{Re}, x_1/D, U_a/c, g_i D/U_a^2, d_m/D) \quad (3-45)$$

and

$$\text{Nu} = \text{Nu}(\text{Re}, \text{Pr}, x_1/D, U_a/c, g_i D/U_a^2, d_m/D). \quad (3-46)$$

Next we ask whether our rule can be applied when we have only one dimensionless parameter to start with. In the dimensional analysis of a system it is often convenient to group the physical variables in such a way as to eliminate as many dimensions as possible. For our case of flow and heat transfer in a pipe we can easily eliminate all dimensions but length and time. Then, assuming that the dimensionless friction factor f is not a function of any variables, that is, starting with only one dimensionless parameter, we have

$$f = \frac{2(\bar{r}_w/\rho)}{U_a^2} = f(\text{const}), \quad (3-47)$$

where the dimensions of $(\bar{\tau}_w/\rho)$ are $(\text{length})^2/(\text{time})^2$, and of U_a are length/time. In order to obtain a more realistic relation for f , one might expect that the kinematic viscosity ν and/or the pipe diameter D should be included in the functional relationship. But neither of those quantities will combine by itself with quantities already present ($(\bar{\tau}_w/\rho)$ or U_a) to form a new dimensionless parameter; the relationship resulting by adding ν or D separately would not be dimensionally correct. However the quantity (ν/D) will combine with U_a to form $U_a/(\nu/D) = \text{Re}$, so that (3-47) becomes

$$f = f(\text{Re}),$$

which agrees with equation (3-28). Thus adding one physical quantity (ν/D) to the functional relationship in equation (3-47) adds one dimensionless parameter Re to the relationship, in accordance with our rule.

Similarly, considering the heat transfer and assuming that the dimensionless Nusselt number Nu is not a function of any variables, that is, starting with one dimensionless parameter, we have

$$\text{Nu} = \frac{hD}{k} = \frac{[h/(\rho c)]D}{\alpha} = \text{Nu}(\text{const}) \quad (3-48)$$

where c is the specific heat and α is the thermal diffusivity. As in equation (3-47), we have grouped the physical variables in such a way as to eliminate all the dimensions but length and time; the dimensions of $[h/(\rho c)]$ are length/time, and of α are $(\text{length})^2/\text{time}$. In order to obtain a more realistic relation for Nu , one might expect that the kinematic viscosity ν and/or the mean velocity U_a should be included in the functional relationship. It turns out that each of those can be combined with quantities already present in the functional relationship to form a new dimensionless parameter. Thus ν can be combined with α to form the Prandtl number $\text{Pr} = \nu/\alpha$, after which U_a can be combined with ν and D to form the Reynolds number $\text{Re} = U_a D/\nu$. Equation (3-48) for the Nusselt number then becomes

$$\text{Nu} = \text{Nu}(\text{Pr}, \text{Re}),$$

in agreement with equation (3-40a). Thus, once again a dimensionless parameter is added to the functional relationship each time a physical variable is added.

3.6 REMARKS

The analyses given in sections 3.3.1, 3.4.1, and 3.5, which are based on dimensional considerations, whether they proceed in conjunction with the basic fluids equations as in sections 3.3.1 and 3.4.1, or in conjunction with experience with flow situations, as in section 3.5, deal mainly with functional relations between dimensionless variables, and are primarily an aid to the correlation of experimental data. Thus they do not by themselves of course, constitute solutions of the fundamental equations.

Theoretically it should be possible to apply the continuum equations obtained in this chapter directly to turbulent flow. In that case the velocities, temperatures, etc. in the equations are local instantaneous values in the turbulent field. When appropriate initial and boundary conditions are given, the temporal evolution of the turbulent field can, in principle, be calculated from the equations. However, the implementation of such calculations, which will be discussed in succeeding chapters, is not easy.

Before considering solutions of the unaveraged equations of this chapter, we will consider averaged equations, as well as some equations that contain both averaged and unaveraged quantities.

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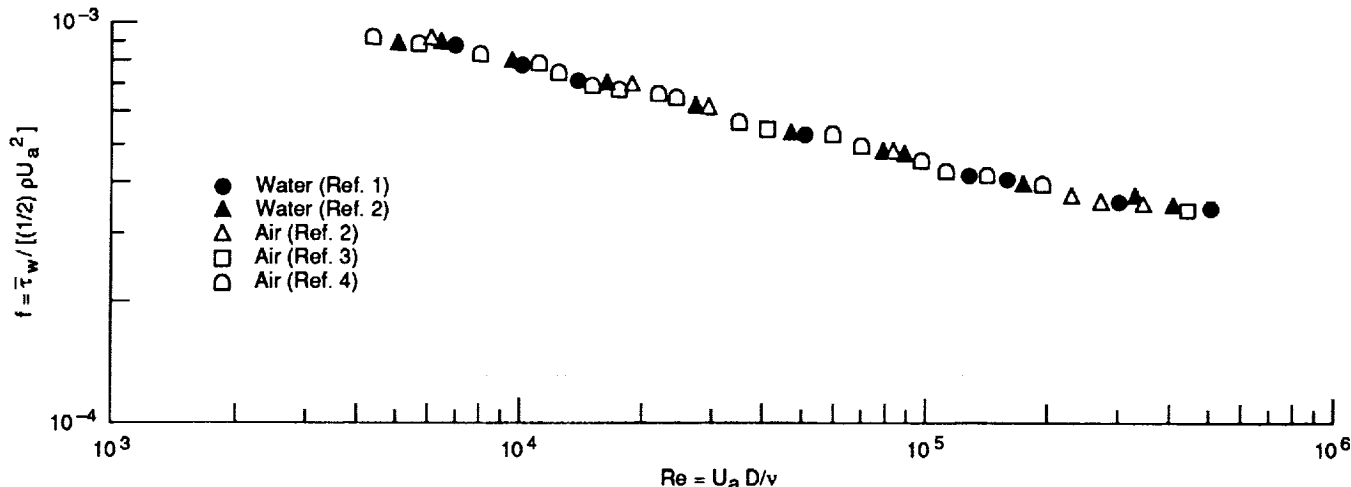


Figure 3-1.—Experimental fully developed turbulent friction factors vs. Reynolds number for water and for air. Agreement between the results for the two fluids shows that the differences in molecular structure do not affect the turbulent flow.



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