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# First-Passage Problems: A Probabilistic Dynamic Analysis for Degraded Structures

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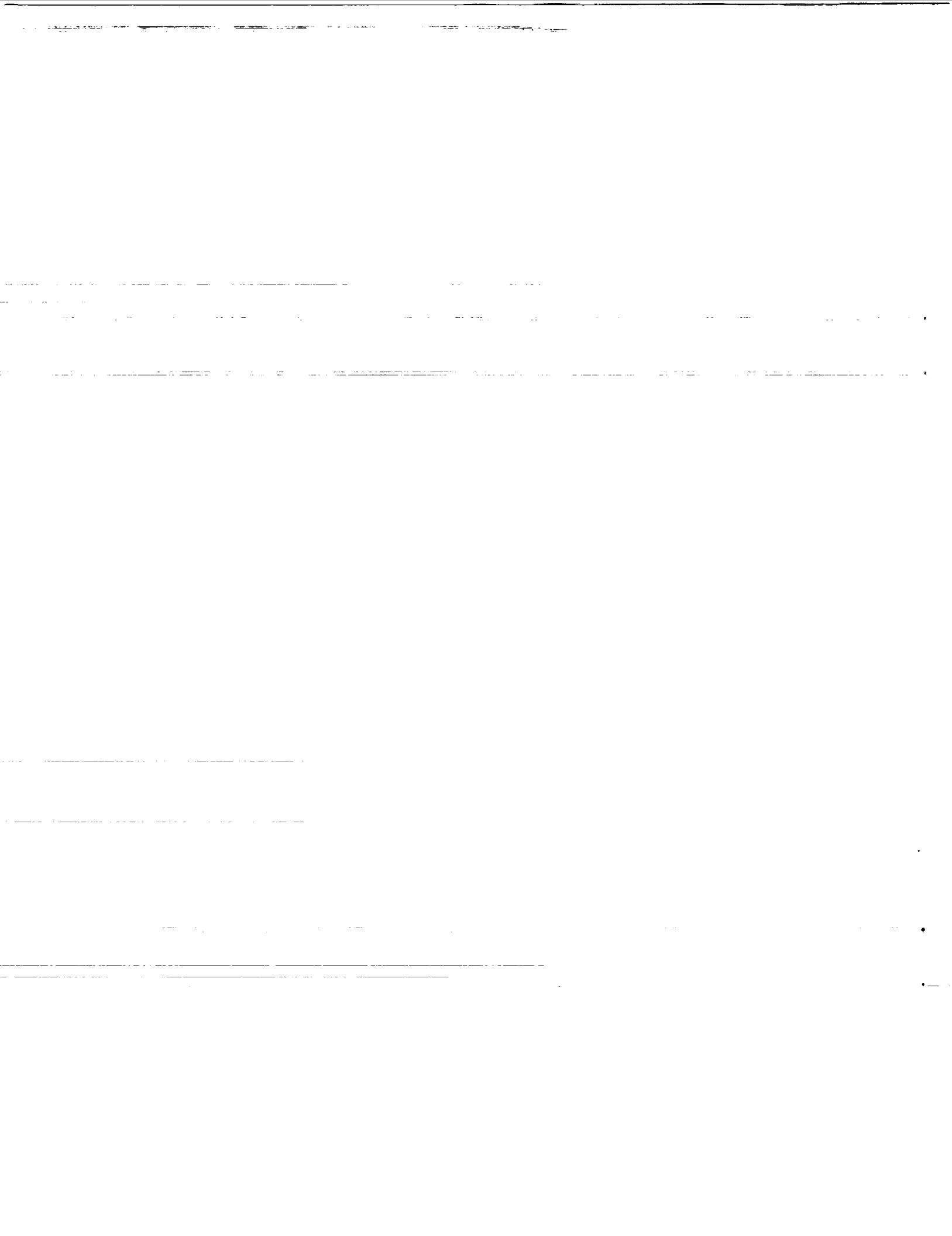


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# FIRST-PASSAGE PROBLEMS: A PROBABILISTIC DYNAMIC ANALYSIS FOR DEGRADED STRUCTURES

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## Summary

Structures subjected to random excitations with uncertain system parameters degraded by surrounding environments (a random time history) are studied. Methods are developed to determine the statistics of dynamic responses, such as the time-varying mean, the standard deviation, the autocorrelation functions, and the joint probability density function of any response and its derivative. Moreover, the first-passage problems with deterministic and stationary/evolutionary random barriers are evaluated. The time-varying (joint) mean crossing rate and the probability density function of the first-passage time for various random barriers are derived.

## Introduction

Many structures that require high performance or high reliability and durability often operate under complex environments including random excitations and random temperatures. These excitation and temperature variations not only degrade the material but also cause an additional randomness in the uncertain material behavior. In addition, a small variation in the structural shape may have an important effect on the structural responses. Therefore, the deterministic

structural analysis is not sophisticated enough to quantify the uncertain structural responses, and the design requirements cannot be satisfied. For this reason a methodology has been developed for a probabilistic dynamic analysis of stochastic structures degraded by surrounding service environments. This methodology evolves from the concept of NESSUS (Numerical Evaluation of Stochastic Structures Under Stress), which is a probabilistic structural analysis code developed under NASA's Probabilistic Structural Analysis Methods Project (Chamis, 1986). The methodology consists of five parts: (1) random process decomposition, (2) probabilistic material behavior model, (3) perturbed dynamic analysis of uncertain systems, (4) the first-order-second-moment method, and (5) a set of reliability algorithms. The random process decomposition procedure expresses the random phenomenon by a set of independent random variables and their respective characteristic functions. The probabilistic material behavior model defines the relationship between material properties (Young's modulus, damping ratio, thermal expansion coefficient, strength, etc.) and the random phenomena (temperature, fatigue, etc.). The perturbed dynamic analysis of uncertain systems produces perturbed time histories (dynamic responses), which are used to determine the response functions numerically in terms of independent random variables at each time. The first-order-second-moment method redefines the limit state function in the normalized

probabilistic space. A set of reliability algorithms are then used to determine the important response statistics and to solve the first-passage problems.

This newly developed method is demonstrated by performing a transient analysis of a stochastic structure subjected to random excitations under severe random temperature conditions. The structure is modeled by a single-degree-of-freedom oscillator, or only the fundamental mode is considered in the dynamics analysis. The natural frequency of this oscillator, which is a function of the uncertain system parameters (geometry, material properties, etc.), is randomly degraded by the temperature process. The excitations can be either stationary or evolutionary random processes. Second-order statistics of any response, such as the time-varying mean, the standard deviation, and the autocorrelation functions, are determined. Also, the first-passage problems with constant or stationary evolutionary barriers are evaluated. A random barrier, such as the uncertain material strength or clearance for maximum displacement, is composed of its original randomness and of a variation due to the effect of random temperature and fatigue. The probability events in which crossing does or does not occur at any time are defined in terms of a reliability index and sensitivity factors from the first-order-second-moment method. The time-varying mean crossing rate and the joint mean crossing rate are derived. The probability density functions of the first-passage time for various random barriers are determined by two methods. The details of the methodology are explained in this paper. The application of this methodology to a multiple-degree-of-freedom system in conjunction with NESSUS is also addressed.

## Random Process Decomposition

For a random process in time, such as load and temperature, a Karhunen-Loeve expansion is used (Spanos and Ghanem, 1989). This expansion is a representation of the random process in terms of independent random variables  $U_n$  and their associated time-varying characteristic functions  $f_n(t)$ , as shown in equation (1).

$$S(t) = \bar{S}(t) + \sum_{n=1}^r U_n f_n(t) \quad (1)$$

where  $\bar{S}(t)$  is the mean value of the random process and  $r$  is the number of terms included. The smoother the random process is, the smaller the number  $r$  is.

## Probabilistic Material Behavior Model

A generic material behavior model (multifactor interaction model) (Boyce and Chamis, 1988) is used to synthesize the variations of material properties for structures operating under

hostile environments, such as high or cryogenic random temperature conditions. The fundamental assumption for this model is that the material behaviors can be synthesized by independent random variables  $V$ . The general form of this model is shown in equation (2),

$$M_p = M_{P0} \prod_{j=1}^N \left( \frac{V_F - V}{V_F - V_0} \right)^a \quad (2)$$

where  $N$  is the number of environmental effects (temperature, fatigue, etc.);  $M_{P0}$ ,  $V_F$ ,  $V_0$  are independent random variables;  $M_{P0}$  is the reference material properties;  $V_F$ ,  $V_0$ , and  $V$  are final, reference, and current environmental variables, respectively; and  $M_p$  is the current material property. The exponent  $a$  is determined from available experimental data or can be estimated from anticipated material behavior due to a particular environmental effect.

## Perturbed Dynamic Analysis of Uncertain Systems

The uncertain structural parameters, such as geometry or material properties, are random processes in space. The uncertain service environments, such as random excitations and temperature, are random processes in time. Furthermore, the material properties are affected and degraded by temperature. Because a probabilistic structural analysis by the first-order-second-moment method requires the input to be independent random variables, a decomposition procedure is applied to expand the random process in terms of independent random variables and their respective time-varying characteristic functions. Some other variables, such as initial boundary conditions, damping ratio, and random amplitude for harmonic loads, can also be included in the analysis.

A dynamic analysis is performed repeatedly to generate a set of parallel time histories (dynamic responses) corresponding to a small change in each of the independent random variables. These perturbed time histories provide information on how the uncertainties in the problem parameters propagate through time and affect the transient responses. The structural responses  $R(t)$ , in terms of independent random variables  $U_i$ , can be expressed as

$$R(t) = a_0 + \sum_{i=1}^N a_i U_i + \sum_{i=1}^N b_i U_i^2 \quad (3)$$

where  $a_i$  and  $b_i$  are the coefficients computed by using perturbed time histories in a least-squares procedure. In the following, a first-order-second-moment method is used, together with the explicit response function determined by

equation (3), to derive the probabilistic information of structural responses. Also derived are the time-varying (joint) crossing rate and the probability density function of the first-passage time.

## Statistics of Dynamic Responses

The dynamic responses of uncertain structures subjected to random excitation are also random processes in time. Although the first- and second-order statistical properties do not characterize a random process completely, they still contain the most important information of that process. Subsequent structural reliability assessments will be based on this information. In this paper the mean, the standard deviation, and the autocorrelation function of any random response process are derived in appendix A together with the joint probability density function (pdf) of any response and its derivative. The results are summarized in the following paragraphs.

The cumulative distribution function (cdf)  $F$ , the mean  $E[R(t)]$ , and the standard deviation  $\sigma$  for any structural response  $R(t)$  (displacement, velocity, acceleration, stresses, etc., at any time) are

$$F_{R(t)}(z_i) = \Phi(-\beta_i^t) \quad i = 1, \dots, k \quad (4)$$

$$E[R(t)] = \bar{R}(t) = \sum_{i=1}^k z_i f_R(z_i) \Delta z \quad (5)$$

and

$$\sigma_{R(t)} = \sqrt{\sum_{i=1}^k [z_i - \bar{R}(t)]^2 f_R(z_i) \Delta z} \quad (6)$$

where  $\Phi$  is the standard normal cumulative distribution function;  $\beta$  is the reliability index from first-order-second-moment (FORM) analysis (Lind et al., 1985);  $k$  is the number of discretizations; and  $f_{R(t)}$  is the probability density function (pdf) of  $R(t)$ .

Defining  $X(t)$  as a random process (stress, displacement, etc.) and letting  $R_1$  and  $R_2$  be any two responses from  $X(t_1)$ ,  $\dot{X}(t_1)$ ,  $\ddot{X}(t_1)$ ,  $X(t_2)$ ,  $\dot{X}(t_2)$ ,  $\ddot{X}(t_2)$ , the joint cumulative distribution function (cdf)  $F_{R_1, R_2}$  derived in appendix A is

$$F_{R_1, R_2}(z_i, y_j) = \Phi(-\beta_i) \Phi(-\beta_j) + \int_0^{\rho_{ij}} \phi(-\beta_i, -\beta_j; u) du \quad (7)$$

$i = 1, \dots, k \quad j = 1, \dots, l$

where  $(z_i, y_j)$  are any pair of realizations;  $k$  and  $l$  are the number of discretizations for  $R_1$  and  $R_2$ , respectively;  $\rho_{ij}$  is determined by equation (A10); and  $\phi$  is the standard normal pdf. Letting  $R_1 = X(t_1)$  and  $R_2 = X(t_2)$ , the autocorrelation function of the random process  $X(t)$  is

$$\rho(t_1, t_2) = \frac{\sum \sum [z_i - \bar{X}(t_1)] [y_j - \bar{X}(t_2)] f_{X(t_1)X(t_2)}(z_i, y_j) \Delta z \Delta y}{\sigma_{X(t_1)} \sigma_{X(t_2)}} \quad (8)$$

where  $f_{X(t_1)X(t_2)}$  is the joint pdf of  $X(t_1)$  and  $X(t_2)$ .

Letting  $R_1 = X(t)$  and  $R_2 = \dot{X}(t)$ , equation (7) becomes the joint cdf of  $X(t)$  and  $\dot{X}(t)$ . The mean rate at which a random process  $X(t)$  crosses a random barrier  $\xi(t)$  can be calculated by

$$v_{\xi}(t) = \int_0^{\infty} f_{X(t)\dot{X}(t)}[\xi(t), \dot{x}(t)] \dot{x} dx \quad (9)$$

where  $f_{X(t)\dot{X}(t)}$  is the joint pdf of  $X(t)$  and  $\dot{X}(t)$ .

## Probability Density Function of First-Passage Time

The objective of a probabilistic dynamic analysis is to assess the reliability of structures subjected to random excitations. One of the failure modes concerning the structural reliability is so-called first-excursion failure. It states that the structure fails at the first excursion of a response to a given threshold. The response can be displacement, stress, strain, or eigenvalue. The threshold can be thought of as the maximum clearance for displacement at any time, the material strength to resist crack initiation caused by high stress, or the upper and lower bounds for eigenvalues within which responses can be minimized. The reliability for such problems is often represented by the probability density function of the time when the response first crosses the threshold (pdf of the first-passage time). In this analysis the threshold is modeled as a deterministic time function and an evolutionary random process. The probability density function of the first-passage time is calculated by two methods, (1) a crossing-rate-based method and (2) an equivalent-system-based method, as described here.

### Crossing-Rate-Based Method

The probability density function  $f_1(t)$  of the first-passage time can be obtained by solving the integral equation

$$f_1(t) + \int_0^t K(t|\tau) f_1(\tau) d\tau = v_{\xi}(t) \quad (10)$$

where  $\xi(t)$  is a random barrier process;  $v_{\xi}(t)$  is the mean crossing rate;  $K(t|\tau)$  is the crossing rate at  $t$  conditional on the first crossing having occurred at  $\tau$ . It is an integral identity expressing the fact that either a barrier crossing at time  $t$  must be the first or the first crossing must have occurred at some previous time  $\tau$ . Thus, the crossing rate can be given as the sum of two contributions from the left-hand side of equation (10). The simplest approximation of the pdf of the

first-passage time is from the approximation  $K(t|\tau) = v_{\xi}(t)$ ; that is, the crossing rate at  $t$  is assumed to be independent of a prior first crossing at  $\tau$ . In this case

$$f_1(t) = v_{\xi}(t) - v_{\xi}(t) \int_0^t f_1(\tau) d\tau \quad (11)$$

In the case where the Poisson hypothesis about independent barrier crossings is invalid, an improved approximation for the kernel  $K(t|\tau)$  is necessary. If the requirement that a barrier crossing at  $\tau$  should be first is dropped, then

$$K(t|\tau) = \frac{v_{\xi\xi}(t, \tau)}{v_{\xi}(\tau)} \quad (12)$$

The pdf of the first-passage time is then calculated by the equation

$$f_1(t) = v_{\xi}(t) - \int_0^t \frac{v_{\xi\xi}(t, \tau)}{v_{\xi}(\tau)} f_1(\tau) d\tau \quad (13)$$

where  $v_{\xi\xi}(t, \tau)$  is the joint mean crossing rate. Letting  $t = n \Delta t$  and  $\tau = i \Delta t$ , equation (13) can be discretized as

$$f_1(n \Delta t) = v_{\xi}(n \Delta t) - \sum_{i=0}^{n-1} \frac{v_{\xi\xi}(n \Delta t, i \Delta t)}{v_{\xi}(i \Delta t)} f_1(i \Delta t) \Delta t \quad (14)$$

The mean crossing rate and the joint mean crossing rate are derived in appendixes B and C, respectively.

#### Equivalent-System-Based Method

The probability density function  $f_1(n \Delta t)$  of the first-passage time is defined by the equation

$$f_1(n \Delta t) \Delta t = \text{Prob} \left\{ \begin{array}{l} \text{Positive crossing in time } [(n-1)\Delta t, n\Delta t] \\ \bigcap_{i=1}^{n-1} \text{No crossing in time } [(i-1)\Delta t, i\Delta t] \end{array} \right\} \quad (15)$$

Assume  $\Delta t$  is small enough to allow at most one crossing occurrence in this time interval. Defining the limit state function  $g_i$  and its corresponding safety margin  $M_i$  by the equations

$$g_i = \xi(i \Delta t) - X(i \Delta t) \quad M_i = \beta_i - \sum_{r=1}^N \alpha_{ir} U_r \quad i = 1, \dots, n \quad (16)$$

where  $\xi$  is a random barrier process and  $X$  is a random response process;  $\beta$  and  $\alpha$  are the reliability index and sensitivity factor, respectively;  $U_r$  is a standardized normally distributed random variable; and  $N$  is the number of independent random variables. Substituting equation (16) into equation (15) gives

$$\begin{aligned} f_1(n \Delta t) \Delta t &= \text{Prob} \left[ (g_n \leq 0 \text{ and } g_{n-1} > 0) \bigcap_{i=1}^{n-1} (g_i > 0 \text{ and } g_{i-1} > 0) \right] \\ &= \text{Prob} \left[ (g_n \leq 0) \bigcap_{i=1}^{n-1} g_i > 0 \right] \\ &= \text{Prob} \left[ (M_n \leq 0) \bigcap_{i=1}^{n-1} M_i > 0 \right] \end{aligned} \quad (17)$$

The probability event  $\bigcap_{i=1}^{n-1} M_i > 0$  is approximated by an equivalent event  $M_e(k-1) > 0$ , which is determined by a recursive equation as follows:

$$M_e(k-1) > 0 = M_{(k-1)} > 0 \bigcap M_e(k-2) > 0 \quad k = 2, \dots, n \quad (18)$$

Therefore,

$$f_1(n \Delta t) \Delta t = \text{Prob} [M_n \leq 0 \text{ and } M_e(n-1) > 0] \quad (19)$$

The probability on the right-hand side of equation (19) will be calculated by equation (B8).

## Numerical Examples and Discussion

The algorithms derived previously were applied to a single-degree-of-freedom oscillator shown in figure 1. The initial system parameter  $K_0$  was affected by a stationary random temperature process  $T(t)$ , and the current system parameter  $K(t)$  was characterized by the following material behavior model:

$$K(t) = K_0 \left[ \frac{T_F - T(t)}{T_F - T_0} \right]^a \quad (20)$$

where  $K_0$ ,  $T_F$ , and  $T_0$  are independent random variables and  $T(t)$  is a stationary random process whose covariance kernel is defined by the equation

$$\text{Cov} [T(t), T(\tau)] = \sigma_T^2 e^{-C_T |t-\tau|} \quad (21)$$

where  $\sigma_T$  is the standard deviation of  $T(t)$  and the constant  $C_T$  is the parameter that controls the smoothness of the random process. If  $C_T = 0$ , the random process reduces to a single random variable. When  $C_T$  approaches infinity, the random process becomes a white noise.

The natural frequency of the oscillator is obtained by the equation

$$\omega_n(t) = \sqrt{\frac{K(t)}{M}} \quad (22)$$

where  $M$  is the lump mass. Substituting equation (20) into equation (22) gives

$$\omega_n(t) = \sqrt{\left(\frac{K_0}{M}\right) \left[\frac{T_F - T(t)}{T_F - T_0}\right]^a} \quad (23)$$

Therefore,  $\omega_n(t)$  becomes a random process in time. The equation of the motion is then defined by the equation

$$\ddot{x} + 2\beta\omega_n(t)\dot{x} + \omega_n^2(t)x = F(t) \quad (24)$$

where  $\beta$  is the damping ratio. In the following numerical examples, these values are assumed:  $M = 300\,000$ ;  $E[K_0] = 30\,000\,000$ ;  $\sigma(K_0) = 600$ ;  $E[T_F] = 2100$ ;  $\sigma(T_F) = 42$ ;  $T_0 = 260$ ;  $a = 0.25$ ;  $E[T(t)] = 1800$ ;  $\sigma[T(t)] = 36$ ;  $C_T = 0.25$ ;  $\beta = 0.10$ .

The oscillator is subjected to an evolutionary random excitation  $F(t)$ , which is defined by

$$F(t) = H(t)S(t) \quad (25)$$

where  $H(t)$  is a deterministic function defined by

$$H(t) = \begin{cases} \frac{t}{t_c} & t \leq t_c \\ 1 & t > t_c \end{cases} \quad (26)$$

as shown in figure 2. Here  $S(t)$  is a stationary random process with mean  $E[S(t)] = \bar{S}(t) = 20$  and standard deviation  $\sigma_S = 2$ . The covariance kernel of  $S(t)$  is defined by the equation

$$\text{Cov}[S(t), S(\tau)] = \sigma_S^2 e^{-c_s|t-\tau|} \quad (27)$$

Again, the constant  $C_S$  will determine the smoothness of the random excitation.

A procedure is used next to decompose the random excitation  $S(t)$  and the random temperature  $T(t)$  into a set of independent random variables and their respective time-varying characteristic functions. In the time interval  $[-l_i, l_i]$  the process  $S(t)$  can be expanded by the equation

$$S(t) = \bar{S}(t) + \sum_{i=0}^r b_i f_i(t) + \sum_{j=0}^{r'} b_j f_j(t) \quad (28)$$

where  $b_i$  and  $b_j$  are the zero-mean independent random variables and

$$f_i(t) = \sqrt{\frac{2C_S\sigma_S^2}{\omega_i^2 + C_S^2}} \frac{\cos(\omega_i t)}{\sqrt{t_i + \frac{\sin(2\omega_i t_i)}{2\omega_i}}} \quad (29)$$

and

$$f_j(t) = \sqrt{\frac{2C_S\sigma_S^2}{\omega_j^2 + C_S^2}} \frac{\cos(\omega_j t)}{\sqrt{t_j + \frac{\sin(2\omega_j t_j)}{2\omega_j}}} \quad (30)$$

in which  $\omega_i$  and  $\omega_j$  are the solution to the transcendental equations

$$C_S - \omega_i \tan(\omega_i t_i) = 0 \quad (31)$$

and

$$\omega_j + C_S \tan(\omega_j t_j) = 0 \quad (32)$$

The random temperature  $T(t)$  will be decomposed in a similar way.

### Statistics of Dynamic Responses

The covariance of the random excitations is defined by equation (27) with  $C_S = 5$ . This value will produce a nonsmooth random process with a correlation function similar to that for earthquake records. Thirty-six independent random variables are used to represent this process. The covariance of the random temperature is defined by equation (21) with  $C_T = 0.25$  because temperature oscillates slowly in time. Ten independent random variables are used to define this process. The mean, the standard deviation, and the autocorrelation function of the dynamic displacement were determined from equations (4) to (6). The results were compared with Monte Carlo simulation solutions and are shown in figure 3. Good agreement is observed. It is also noticed that in the time interval (0,2), the displacement process  $X(t)$  is nonstationary with varying mean, standard deviation, and autocorrelation function. However, the process becomes stationary after 2 seconds.

### Probability Density Function of First-Passage Time

The first-passage problems have been studied with different barriers and loading conditions. Two types of random barriers are used; they are defined by equations (33) and (34).

Type A:

$$\xi(t) = \xi_0 \left(1 - \frac{t}{t_F}\right) \quad (33)$$

Type B:

$$\xi(t) = \xi_0 \left(1 - \frac{t}{t_F}\right) \left[ \frac{T_F - T(t)}{T_F - T_0} \right]^a \quad (34)$$

where  $\xi_0$  is the reference barrier,  $t_F$  is the final time, and  $1/t_F$  is the degradation slope, which reflects the environmental degradation on the barriers. The type B random barrier includes the temperature effect more specifically. Two types of random loading are considered. The first type is the one with  $C_S$  in equation (27) equal to 5. This type of loading is similar to earthquake records, which are nonsmooth random processes. The second type of loading has  $C_S = 1$ , corresponding to a smooth random process. Two different degradation slopes are used, 0.02 and 0.15, to represent slow and fast environmental degradation. The mean reference barriers are chosen such that they correspond to  $\sigma$  and  $3\sigma$  levels of the response process at  $t = 0$ . The statistics of random loads and barriers for each problem are listed in table I.

The probability density functions of the first-passage time were calculated by both the crossing-rate-based method and the equivalent-system-based method. The results were compared with Monte Carlo simulation, as shown in figures 4 to 7. With type A deterministic barriers, a nonsmooth random excitation ( $C_S = 5$ ), and a small barrier degradation slope (0.02), the crossing-rate-based method performed well, as shown in figures 4(a) and (b), but the equivalent-system-based method did not. However, the equivalent-system-based method gave satisfactory results for problems with a large barrier degradation slope (0.15), as shown in figures 4(c) and (d), but the crossing-rate-based method did not. With the same barrier conditions as for problems in figures 4(a) and (b) and a smooth random excitation ( $C_S = 1$ ), the results from the equivalent-system-based method improved compared with figures 4(a) and (b) as shown in figures 5(a) and (b). The pdf of the first-passage time predicted by the crossing-rate-based method for a large degradation slope also improved compared with figures 4(c) and (d), as shown in figures 5(c) and (d). Figure 6 shows the pdf of the first-passage time with nonsmooth random excitation ( $C_S = 5$ ) and type B random barriers. As expected, the equivalent-system-based method performed well for large degradation slopes, as shown in figures 6(c) and (d). It also gave a satisfactory answer for a small degradation slope, as shown in figures 6(a) and (b). For problems with type B random barriers and subjected to smooth random excitations ( $C_S = 1$ ), some improvements in the results predicted by the equivalent-system-based method were also observed for a small degradation slope (0.02), as shown in figures 7(a) and (b). As usual, it was good for a large degradation slope, as shown in figures 7(c) and (d).

In general, the crossing-rate-based method is good for problems with a small degradation slope (0.02) subjected to nonwhite ( $C_S \leq 5$ ) excitations and for either low or high thresholds. The equivalent-system-based method is good for problems with a large degradation slope (0.15) subjected to nonwhite ( $C_S \leq 5$ ) excitations and for either low or high

thresholds. However, when any of the following conditions are present, the results of the equivalent-system-based method will converge to the solutions by Monte Carlo simulation. The conditions are (1) smoother random excitations, (2) more uncertainties in the barrier, and (3) larger degradation slope. The reason is stated as follows: In order to have the equivalent-system-based method perform well, either the reliability index  $\beta_i$  defined in equation (16) decreases rapidly or the correlations between the safety margins are large. Otherwise, a quadratic equivalent system should be developed to improve the accuracy. The degradation slope essentially controls the decreasing speed of  $\beta_i$ . The degree of uncertainty in the barriers and the smoothness of the random excitations, on the other hand, determine the correlations of the safety margins.

Although the proposed method was applied to a single-degree-of-freedom oscillator, it can be directly applied to a multiple-degree-of-freedom system as addressed in the next section. The computing time required by Monte Carlo simulation is highly dependent on the degree of freedom of the finite element model. However, the computing time required for the proposed methods is dependent on the number of random variables and is not very sensitive to the degree of freedom of the finite element model. Therefore, they are more attractive for problems with complex structures or for nonlinear problems where analytical solutions do not exist. It is necessary to point out that the proposed method is not suitable for problems with white noise excitations because the infinite number of independent random variables required to represent this special random process makes the proposed method inefficient.

## Application to Multiple-Degree-of-Freedom System in Conjunction With NESSUS

The algorithms derived in this paper were verified by applying them to a single-degree-of-freedom oscillator. However, the application to multiple-degree-of-freedom structures is straightforward and is discussed here. The equation of motion for such a system is defined by

$$[M]\ddot{X}(t) + [C]\dot{X}(t) + [K]X(t) = F(t) \quad (35)$$

where the mass matrix  $[M]$ , the damping matrix  $[C]$ , and the stiffness matrix  $[K]$  can be random processes in space. Because the explicit response functions in terms of independent random variables are required for the methods developed previously, a random process decomposition procedure is needed to represent the correlated random process in space by a set of independent random variables. This can be achieved by using NESSUS/PRE. NESSUS/PRE is a preprocessor used for the preparation of the statistical data needed to perform the probabilistic structural analysis.



It allows the user to describe the uncertainties in the structural parameters (primitive random variables) at nodal points of a finite element mesh. The uncertainties in these parameters are specified over this mesh by defining the mean value and the standard deviation of the random variable at each point, together with an appropriate form of correlation. Correlated random variables are then decomposed into a set of independent vectors by a modal analysis.

The perturbed time histories of any structural response corresponding to a small variation of each independent random variable can be generated by any finite element code with transient dynamic capability. The Finite Element Methods in NESSUS, called NESSUS/FEM, is one of the codes that can be used for this purpose. The explicit response functions can then be computed accordingly.

NESSUS/FPI (Fast Probability Integrator) (Wu, 1985) provides the first-order-second-moment method necessary for determining the probabilistic responses and solving the first-passage problems. It is obvious that the newly developed methods not only utilize NESSUS but also enhance its capability to perform various dynamic problems. It is worth pointing out that when the mass, damping, and stiffness matrices are independent of time, the transient dynamic responses can be obtained in closed forms by a modal analysis. The perturbed eigenvalues and corresponding eigenvectors are computed by using an efficient algorithm in NESSUS/FEM.

## Conclusions

Methods have been developed for the probabilistic dynamic analysis of uncertain structures subjected to random excitations with random environmental effects on the uncertain structural parameters. The statistics of dynamic analysis such as time-varying mean, standard deviation, and autocorrelation function were determined. The first-passage problems, with barriers being deterministic functions and stationary/evolutionary random processes, were studied. The mean crossing rate and the joint mean crossing rate were derived. Two methods were proposed to determine the pdf of the first-passage time. The crossing-rate-based method is for problems of nonsmooth random excitations with small uncertainties in the barrier and small environmental effect on the barrier. The equivalent-system-based method is for problems of either large barrier degradation rate, smooth random excitations, or large uncertainties in the barrier.

## APPENDIX A STATISTICS OF DYNAMIC RESPONSES

In order to calculate the response statistics, a limit state function  $g$  is first defined by

$$g^t = R(t) - z \quad (A1)$$

where  $R(t)$  is the structural response ( $X(t)$ ,  $\dot{X}(t)$ ,  $\ddot{X}(t)$ ) at time  $t$ , which is a function of independent random variables, and  $z$  is any realization. This limit state is subsequently transformed into a standardized normally distributed probability space and is replaced by a linear safety margin  $M$  through a first-order-second-moment (FORM) analysis, where  $M$  is defined by

$$M^t = \beta^t - \sum_{r=1}^N \alpha_r^t U_r \quad (A2)$$

and where  $\beta^t$  and  $\alpha_r^t$  are the reliability index and the sensitivity factor, respectively;  $U_r$  is a standardized normally distributed random variable; and  $N$  is the number of independent random variables. The probability that  $g^t \leq 0$  or  $R(t) \leq z$  is determined by

$$\text{Prob}(g^t \leq 0) = \text{Prob}[R(t) \leq z] = \text{Prob}(M^t \leq 0) = \Phi(-\beta^t) \quad (A3)$$

where  $\Phi$  is the standard normal cumulative distribution function. The cumulative distribution function (cdf)  $F_R$  for any structural response  $R(t)$  at any time can be generated from equation (A3) by selecting a number of realization  $z$  from an appropriate response range, as shown in equation (A4).

$$F_{R(t)}(z_i) = \Phi(-\beta_i^t) \quad i = 1, \dots, k \quad (A4)$$

If  $f_{R(t)}$  is denoted as the probability density function (pdf) of  $R(t)$ , the mean and the standard deviation of  $R(t)$  can be calculated by the equations

$$E[R(t)] = \bar{R}(t) = \sum_{i=1}^k z_i f_{R(t)}(z_i) \Delta z \quad (A5)$$

and

$$\sigma_{R(t)} = \sqrt{\sum_{i=1}^k [z_i - \bar{R}(t)]^2 f_{R(t)}(z_i) \Delta z} \quad (A6)$$

The joint cdf  $F_{R_1 R_2}$  is derived as follows: Let  $R_1$  and  $R_2$  be any two of  $X(t)$ ,  $X(\tau)$ ,  $\dot{X}(t)$ ,  $\dot{X}(\tau)$ ,  $\ddot{X}(t)$ ,  $\ddot{X}(\tau)$ , etc., where  $X(t)$  is a random response process. Define the limit state functions  $g_i$  and  $g_j$  by

$$g_i = R_1 - z_i, \quad g_j = R_2 - y_j \quad (A7)$$

where  $z_i$  and  $y_j$  are different realizations. Also define their corresponding linear safety margins  $M_i$  and  $M_j$  by

$$M_i = \beta_i - \sum_{r=1}^N \alpha_{r_i} U_r, \quad M_j = \beta_j - \sum_{r=1}^N \alpha_{r_j} U_r \quad (A8)$$

The joint cdf  $F_{R_1, R_2}$  will be determined by the equation

$$\begin{aligned} F_{R_1, R_2}(z_i, y_j) &= \text{Prob}(R_1 \leq z_i \text{ and } R_2 \leq y_j) \\ &= \text{Prob}(M_i \leq 0 \text{ and } M_j \leq 0) \\ &= \Phi(-\beta_i) \Phi(-\beta_j) + \int_0^{\rho_{ij}} \phi(-\beta_i - \beta_j; u) du \quad (\text{A9}) \\ & \quad i = 1, \dots, k, \quad j = 1, \dots, l \end{aligned}$$

where

$$\rho_{ij} = \sum_{r=1}^N \alpha_{r1} \alpha_{rj} \quad (\text{A10})$$

Let  $R_1 = X(t)$  and  $R_2 = X(\tau)$  and calculate the autocorrelation function of  $X(t)$  by the equation

$$\rho(t, \tau) = \frac{\sum \sum [z_i - \bar{X}(t)][y_j - \bar{X}(\tau)] f_{X(t)X(\tau)}(z_i, y_j) \Delta z \Delta y}{\sigma_{X(t)} \sigma_{X(\tau)}} \quad (\text{A11})$$

where  $f_{X(t)X(\tau)}$  is the joint pdf of  $X(t)$  and  $X(\tau)$ .

## APPENDIX B MEAN CROSSING RATE $v_{\xi}(t)$

The mean rate at which a random process  $X(t)$  crosses a random barrier  $\xi(t)$  is determined by the equation

$$\begin{aligned} &[\text{Average number of positive crossings in time } (t, t + \Delta t)] \\ &= \text{Prob}[\text{Positive slope crossings in time } (t, t + \Delta t)] \\ &= \text{Prob}[X(t) \leq \xi(t) \text{ and } X(t + \Delta t) > \xi(t + \Delta t)] \\ &= P'_c \quad (\text{B1}) \end{aligned}$$

where  $\xi(t)$  is the barrier. Therefore,

$$v_{\xi}(t) \Delta t = P'_c \quad (\text{B2})$$

Let  $t = n \Delta t$ ; the crossing rate then becomes

$$v_{\xi}(n \Delta t) = \frac{P'_c}{\Delta t} \quad (\text{B3})$$

where  $P'_c$  is determined as follows: The limit state functions  $g^t$  and  $g^{t+\Delta t}$  are defined by the equations

$$g^t = \xi(t) - X(t) \quad g^{t+\Delta t} = \xi(t + \Delta t) - X(t + \Delta t) \quad (\text{B4})$$

The corresponding linear safety margins  $M^t$  and  $M^{t+\Delta t}$  in the standardized normally distributed probability space are

$$M^t = \beta^t - \sum_{r=1}^N \alpha'_r U_r, \quad M^{t+\Delta t} = \beta^{t+\Delta t} - \sum_{r=1}^N \alpha'^{t+\Delta t}_r U_r \quad (\text{B5})$$

Substitute equations (B4) and (B5) into equation (B1) to get

$$P'_c = \text{Prob}(g^t > 0 \text{ and } g^{t+\Delta t} \leq 0) = \text{Prob}(M^t > 0 \text{ and } M^{t+\Delta t} \leq 0) \quad (\text{B6})$$

Define the probability events  $A$ ,  $B$ ,  $\bar{A}$ , and  $\bar{B}$  by

$$\begin{aligned} A &= M^t \leq 0 & \bar{A} &= M^t > 0 \\ B &= M^{t+\Delta t} \leq 0 & \bar{B} &= M^{t+\Delta t} > 0 \end{aligned} \quad (\text{B7})$$

Then substitute equation (B7) into equation (B6) to get

$$\begin{aligned} P'_c &= \text{Prob}(\bar{A} B) \\ &= \text{Prob}(B) - \text{Prob}(A B) \\ &= \text{Prob}(M^{t+\Delta t} \leq 0) - \text{Prob}(M^{t+\Delta t} \leq 0 \text{ and } M^t \leq 0) \\ &= \Phi(-\beta^{t+\Delta t}) - \left[ \Phi(-\beta^{t+\Delta t}) \Phi(-\beta^t) + \int_0^{\rho} \phi(-\beta^{t+\Delta t} - \beta^t; u) du \right] \end{aligned} \quad (\text{B8})$$

where  $\phi$  is the standard normal pdf and

$$\rho = \sum_{r=1}^N \alpha'_r \alpha'^{t+\Delta t}_r \quad (\text{B9})$$

## APPENDIX C JOINT MEAN CROSSING RATE $v_{\xi\xi}(t, \tau)$

The joint mean rate at which a random process  $X$  crosses a random barrier  $\xi$  at times  $t$  and  $\tau$  is derived by the following equations:

$$\begin{aligned} &[\text{Average number of positive crossings in time } (t, t + \Delta t) \\ & \quad \text{and positive crossings in time } (\tau, \tau + \Delta t)] \\ &= \text{Prob}[\text{Positive crossings in time } (t, t + \Delta t) \text{ and} \\ & \quad \text{positive crossings in time } (\tau, \tau + \Delta t)] \\ &= \text{Prob}[X(t) \leq \xi(t) \text{ and } X(t + \Delta t) > \xi(t + \Delta t) \text{ and} \\ & \quad X(\tau) \leq \xi(\tau) \text{ and } X(\tau + \Delta t) > \xi(\tau + \Delta t)] \\ &= P''_c \quad (\text{C1}) \end{aligned}$$

where  $t > \tau$ . Therefore, the joint mean crossing rate is determined by the following equation with  $t = n \Delta t$  and  $\tau = i \Delta t$ :

$$v_{\xi\xi}(n \Delta t, i \Delta t) = \frac{P_c''}{(\Delta t)^2} \quad (C2)$$

where  $P_c''$  is determined by first defining the limit state functions  $g_1, g_2, g_3$ , and  $g_4$ .

$$\begin{aligned} g_1 &= \xi(t) - X(t) & g_2 &= \xi(t + \Delta t) - X(t + \Delta t) \\ g_3 &= \xi(\tau) - X(\tau) & g_4 &= \xi(\tau + \Delta t) - X(\tau + \Delta t) \end{aligned} \quad (C3)$$

The corresponding linear safety margins  $M_1, M_2, M_3$ , and  $M_4$  in the standardized normally distributed probability space are

$$M_i = \beta_i - \sum_{r=1}^N \alpha_{ri} U_r \quad i = 1, 2, 3, 4 \quad (C4)$$

Then substituting equations (C3) and (C4) into equation (C1) gives

$$\begin{aligned} P_c'' &= \text{Prob} \left[ \left( g^t > 0 \text{ and } g^{t+\Delta t} \leq 0 \right) \text{ and } \left( g^\tau > 0 \text{ and } g^{\tau+\Delta t} \leq 0 \right) \right] \\ &= \text{Prob} \left[ \left( M^t > 0 \text{ and } M^{t+\Delta t} \leq 0 \right) \text{ and } \left( M^\tau > 0 \text{ and } M^{\tau+\Delta t} \leq 0 \right) \right] \end{aligned} \quad (C5)$$

Also the probability events are defined as

$$\begin{aligned} A &= M_1 \leq 0 & \bar{A} &= M_1 > 0 \\ B &= M_2 \leq 0 & \bar{B} &= M_2 > 0 \\ C &= M_3 \leq 0 & \bar{C} &= M_3 > 0 \\ D &= M_4 \leq 0 & \bar{D} &= M_4 > 0 \end{aligned} \quad (C6)$$

Substituting equation (C6) into equation (C5) gives

$$\begin{aligned} P_c'' &= \text{Prob}(\bar{A}\bar{B}\bar{C}D) \\ &= \text{Prob}(BD) + \text{Prob}(ABCD) - \text{Prob}(ABD) - \text{Prob}(BCD) \end{aligned} \quad (C7)$$

To alleviate the computational difficulty in equation (C7), two equivalent probability events  $E_{e1}$  and  $E_{e2}$  were derived to replace probability events  $AB$  and  $CD$ , respectively. Therefore, equation (C7) becomes

$$P_c'' = \text{Prob}(BD) + \text{Prob}(E_{e1}E_{e2}) - \text{Prob}(E_{e1}D) - \text{Prob}(BE_{e2}) \quad (C8)$$

Each term on the right-hand side of the equation can be easily determined from equation (A9). The equivalent systems  $E_{e1}$  and  $E_{e2}$  are determined as follows: Because the safety margins  $M_1$  and  $M_2$  are highly correlated, the probability event  $AB$  can be replaced by an equivalent event  $E_{e1}$  (Gollwitzer and Rackwitz, 1983) in such a way that

$$\begin{aligned} \text{Prob}[AB] &= \text{Prob}[M_1 \leq 0 \text{ and } M_2 \leq 0] \\ &= \text{Prob}[M_{e1} \leq 0] = \text{Prob}[E_{e1}] \end{aligned} \quad (C9)$$

where  $M_{e1}$  is the equivalent linear safety margin defined by

$$M_{e1} = \beta_{e1} - \sum_{r=1}^N \alpha_r^{e1} U_r \quad (C10)$$

where

$$\beta_{e1} = -\Phi[\text{Prob}(AB)] \quad (C11)$$

and

$$\alpha_r^{e1} = \frac{\frac{\partial \beta_{e1}}{\partial U_r}}{\sqrt{\sum_{j=1}^N \left( \frac{\partial \beta_{e1}}{\partial U_j} \right)^2}} \quad r = 1, \dots, N \quad (C12)$$

The probability event  $CD$  is replaced by equivalent event  $E_{e2}$  in a similar way.

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TABLE I.—STATISTICS FOR RANDOM LOADS AND BARRIERS

Figure	Random barriers				Loads, $C_s$ in eq.(27)
	Barrier type	Reference barrier		Degradation slope	
		Mean	Covariance		
4(a)	A	0.37	0	0.02	5
4(b)	↓	.47	↓	.02	↓
4(c)		.37		.15	
4(d)		.47		.15	
5(a)		.37		.02	
5(b)	↓	.47	↓	.02	↓
5(c)		.37		.15	
5(d)		.47		.15	
6(a)		B		.58	
6(b)	↓	.74	↓	.02	↓
6(c)		.58		.15	
6(d)		.74		.15	
7(a)		.58		.02	
7(b)	↓	.74	↓	.02	↓
7(c)		.58		.15	
7(d)		.74		.15	

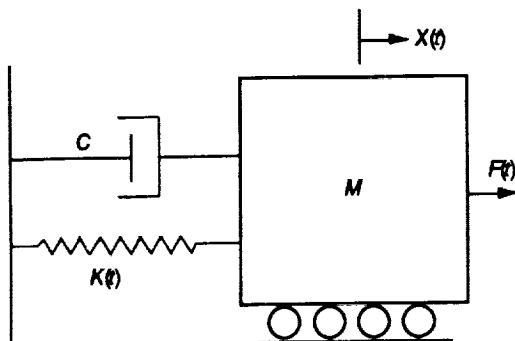


Figure 1.—Single-degree-of-freedom oscillator subjected to random excitation.

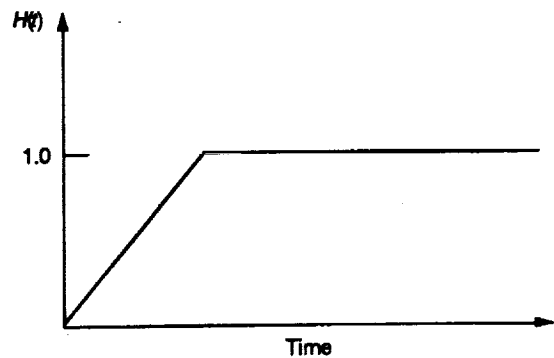
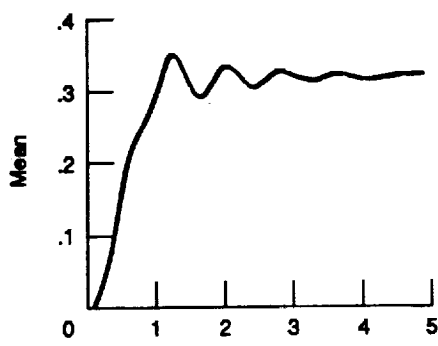
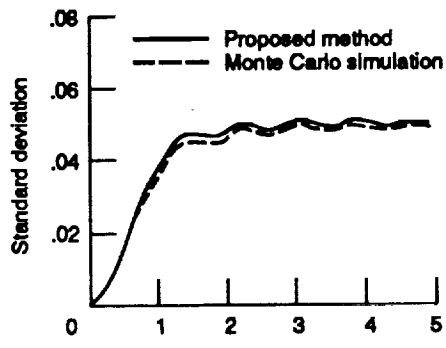


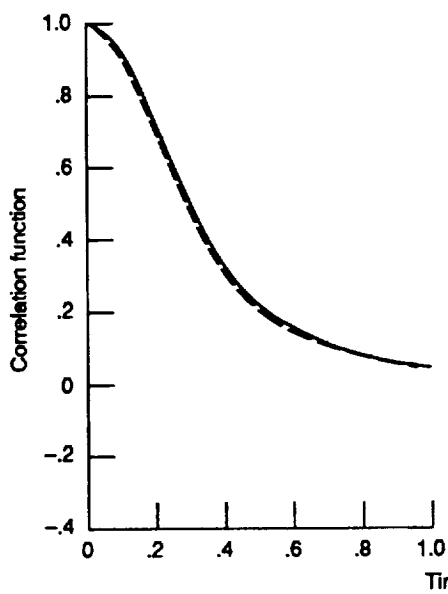
Figure 2.—Modulating function  $H(t)$ .



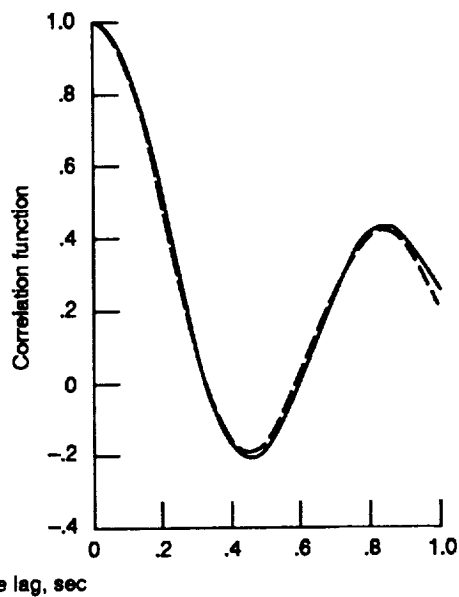
(a) Mean.



(b) Standard deviation.



(c) Autocorrelation function at 1 sec.



(d) Autocorrelation function at 5 sec.

Figure 3.—Statistics of displacement in equation (24) ( $C_S$  in eq. (27), 5).

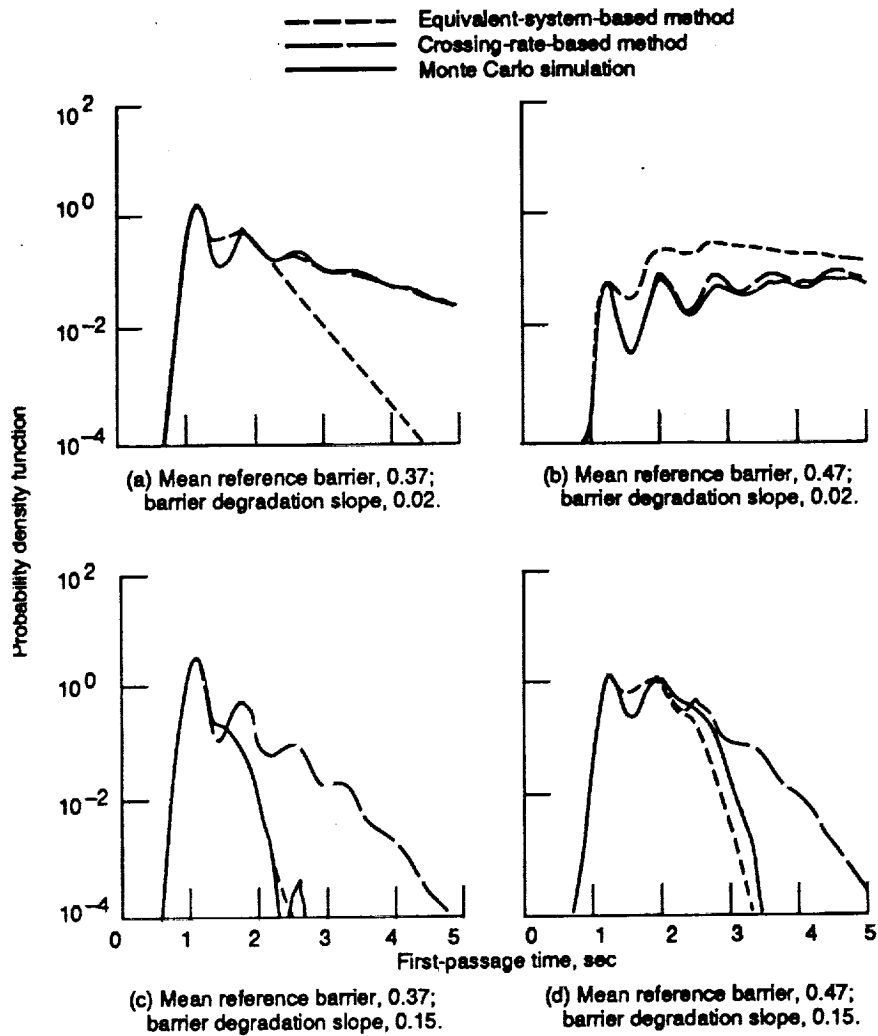


Figure 4.—Probability density function of first-passage time ( $C_S$  in eq. (27), 5; coefficient of variation of reference barrier, 0; type A deterministic barrier model).

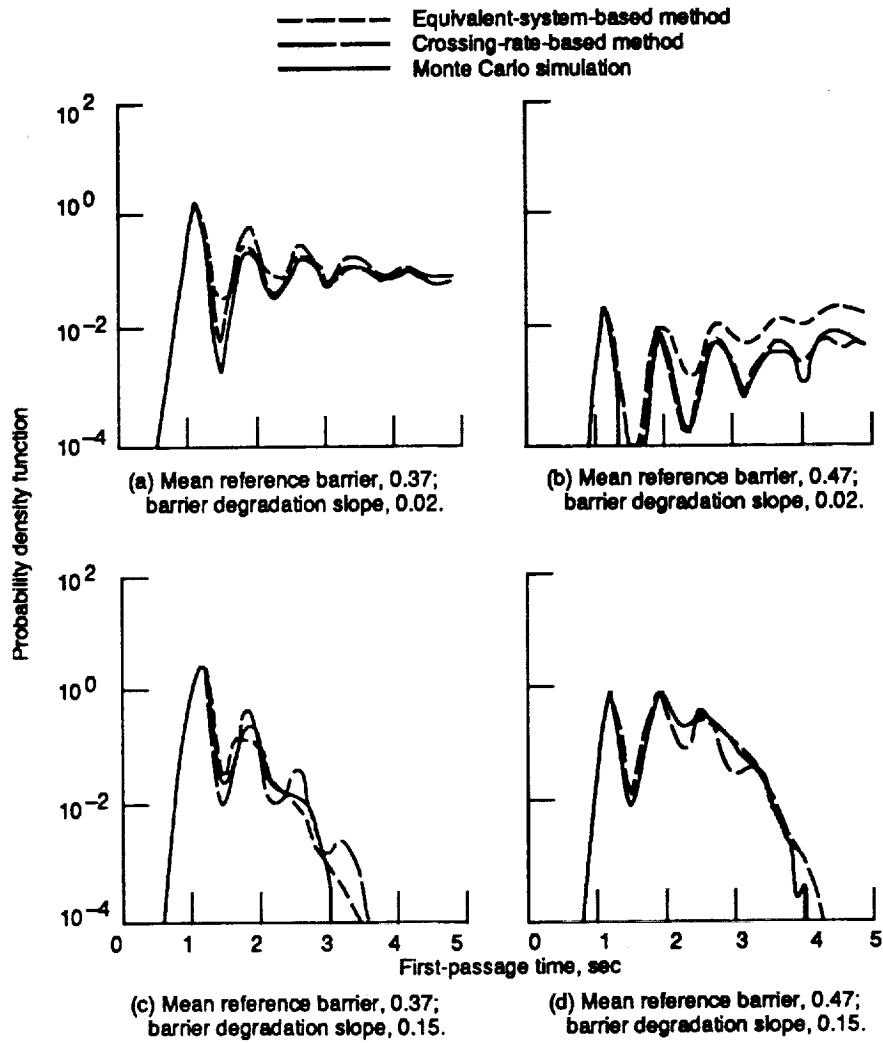


Figure 5.—Probability density function of first-passage time ( $C_S$  in eq. (27), 1; coefficient of variation of reference barrier, 0; type A deterministic barrier model).

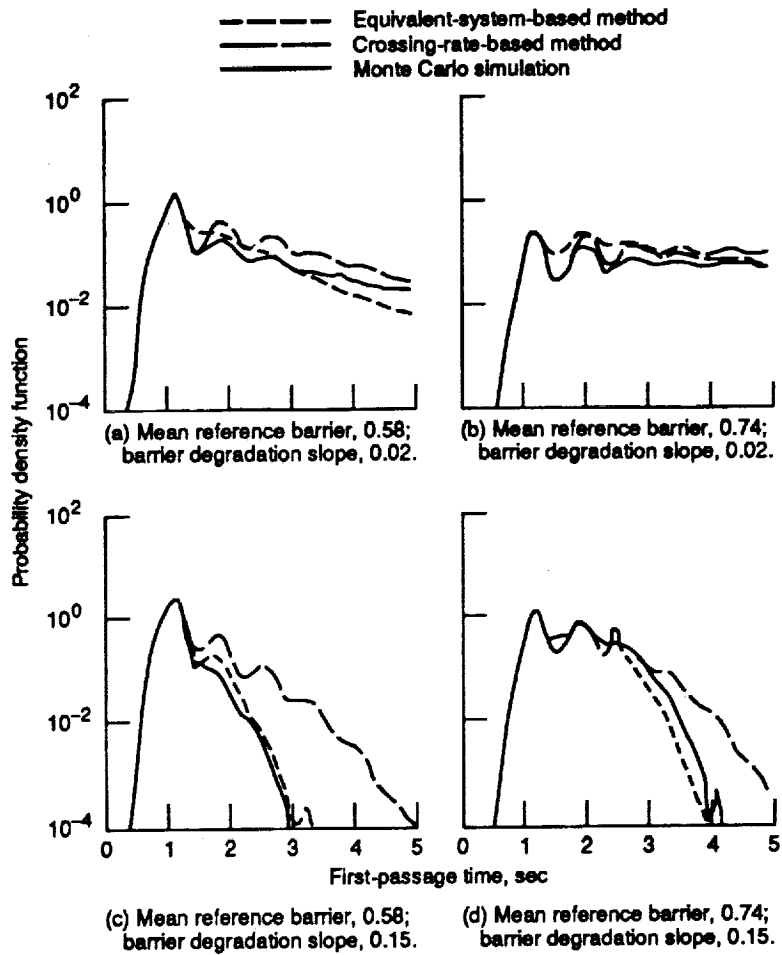


Figure 6.—Probability density function of first-passage time ( $C_S$  in eq. (27), 5; coefficient of variation of reference barrier, 0.1; type B random barrier model).



- - - - - Equivalent-system-based method  
 - - - - - Crossing-rate-based method  
 ——— Monte Carlo simulation

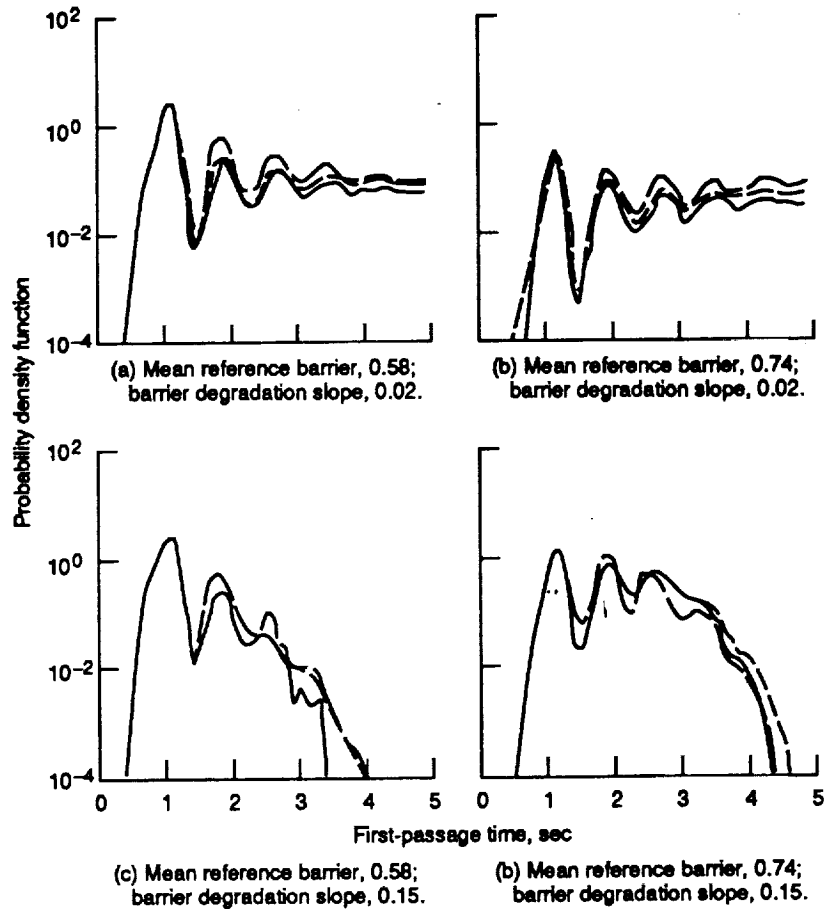


Figure 7.—Probability density function of first-passage time ( $C_S$  in eq. (27), 1; coefficient of variation of reference barrier, 0.1; type B random barrier model).

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16. Abstract Structures subjected to random excitations with uncertain system parameters degraded by surrounding environments (a random time history) are studied. Methods are developed to determine the statistics of dynamic responses, such as the time-varying mean, the standard deviation, the autocorrelation functions, and the joint probability density function of any response and its derivative. Moreover, the first-passage problems with deterministic and stationary/evolutionary random barriers are evaluated. The time-varying (joint) mean crossing rate and the probability density function of the first-passage time for various random barriers are derived.					
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