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Abstract

The design of iterative schemes for sparse matrix computations often leads to constrained polynomial approximation problems on sets in the complex plane. For the case of ellipses, we introduce a new class of complex polynomials which are in general very good approximations to the best polynomials and even optimal in most cases.

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§1. Introduction

We consider complex Chebyshev approximation problems of the type

$$E_n(r,c) = \min_{p\in\Pi_n:\, p(c)=1} \|p\|_{\mathcal{E}_r}, \quad \|p\|_{\mathcal{E}_r} := \max_{z\in\mathcal{E}_r} |p(z)|. \tag{1}$$

Here Π_n denotes the space of all complex polynomials of degree at most n, $\mathcal{E}_r$ is any ellipse with foci ± 1 and semi-axes $(r \pm r^{-1})/2$, $r > 1$, and $c \in \mathbf{C}\setminus\mathcal{E}_r$. It will be convenient to express c as a point on the boundary $\partial\mathcal{E}_R$ of the ellipse $\mathcal{E}_R$, $R > r$, i.e. $c = c(R,\gamma) = ((R+R^{-1})\cos(\gamma) + \mathrm{i}(R-R^{-1})\sin(\gamma))/2$, $\gamma \in [0, 2\pi)$. Since Haar's condition is satisfied, there always exists a unique optimal polynomial $p_n(z;r,c)$ of (1).

Problems (1), in general with $\mathcal{E} \subset \mathbf{C}$ any compact set instead of $\mathcal{E}_r$, arise in numerical linear algebra. E.g. the design of iterative methods for the solution of large sparse non-Hermitian linear systems $Ax = b$ with best possible convergence rates [2], the computation of optimal polynomial preconditioners for conjugate gradient type algorithms for $Ax = b$ [7], or the acceleration of eigenvalue methods for A [6] all lead to problems of this type. However, for arbitrary sets $\mathcal{E}$ the optimal polynomials are in general not known explicitly and therefore the methods are usually based on polynomials which are only asymptotically optimal. A popular choice for the set $\mathcal{E}$ are ellipses, and then the scaled Chebyshev polynomials $t_n(z;c) := T_n(z)/T_n(c)$ are used as approximations to the optimal polynomials of (1) [4, 6]. Clayton [1] showed that even $t_n(z;c) \equiv p_n(z;r,c)$ if c is real, and in general t_n is nearly optimal for (1) as long as n is large. However, in some of the applications we mentioned, polynomials with small degree are used and typically the distance between c and $\mathcal{E}_r$ is small. Depending on the position of c on $\partial\mathcal{E}_R$, $\|t_n(z;c)\|_{\mathcal{E}_r} > 1$ can occur, and then t_n yields no useful approximation (cf. Example 1 given below).

In this note, we introduce a new class of asymptotically optimal polynomials q_n for Problem (1) which always satisfy $\|q_n(z;c)\|_{\mathcal{E}_r} \le \|t_n(z;c)\|_{\mathcal{E}_r}$ and $\|q_n(z;c)\|_{\mathcal{E}_r} < 1$. Moreover, they are even optimal in most cases.

§2. Results

The q_n are defined by

$$q_n(z;c) = \frac{T_n(z) + \alpha_n}{T_n(c) + \alpha_n}, \quad \alpha_n = 2\mathrm{i}\frac{\sin(n\gamma)}{(R^n - R^{-n})}. \tag{2}$$

Here α_n is the solution of the extremal problem

$$M_n(r,c) = \min_{\alpha\in\mathbf{C}} \max_{z\in\mathcal{E}_r} \left|\frac{T_n(z)+\alpha}{T_n(c)+\alpha}\right|. \tag{3}$$

We summarize the important properties of $q_n(z;c)$ in the following

Theorem 1. [3]

(a) $q_n(z;c)$ *has precisely* $2n$ *extremal points* z_j, $j = 1, 2, \ldots, 2n$, *on* $\partial\mathcal{E}_r$ *with* $\|q_n(z;c)\|_{\mathcal{E}_r} = M_n(r,c) = (r^n + r^{-n})/(R^n + R^{-n})$.

(b) There exists a number $R_0(n,r)$ *such that* $q_n(z;c) \equiv p_n(z;r,c)$ *for all* $c \in \partial\mathcal{E}_R$ *with* $R \geq R_0(n,r)$.

(c) Let $c \in \partial\mathcal{E}_R$ *be such that* $R > r(9r^4 - 1)/(r^4 - 1)$. *Then, there exists an integer* $n_0(r,R)$ *such that* $q_n(z;c) \equiv p_n(z;r,c)$ *for all* $n \geq n_0(r,R)$.

Discussion: Supported by numerical tests (c.f. Example 2), we conjecture that (c) is true for arbitrary $R > r > 1$. $\|q_n(z;c)\|_{\mathcal{E}_r}$ does not depend on the position of c on $\partial\mathcal{E}_R$ and $\|q_n(z;c)\|_{\mathcal{E}_r} \leq \|t_n(z;c)\|_{\mathcal{E}_r}$, where equality holds iff $\sin(n\gamma) = 0$, e.g. for $c \in \mathbb{R}$ (cf. Example 1). The proof of Theorem 1 is based on the following characterization [5]: $q_n(z;c) \equiv p_n(z;r,c)$ iff the linear system

$$\sum_{j=1}^{2n} \sigma_j \overline{q_n(z_j;c)}(z_j - c)p(z_j) = 0 \quad \text{for all } p \in \Pi_{n-1} \tag{4}$$

has a nontrivial and nonnegative solution. See [3] for the explicit solution of (4).

Example 1. *We compare* $\|q_n(z;c)\|_{\mathcal{E}_r}$ *(continous curve) and* $\|t_n(z;c)\|_{\mathcal{E}_r}$ *(dashed curve) where* $r = 1.1$, $R = 1.2$ *for* $\gamma \in [0, \pi]$ *and* $n = 3, 4$ *(cf. Figure 1).*

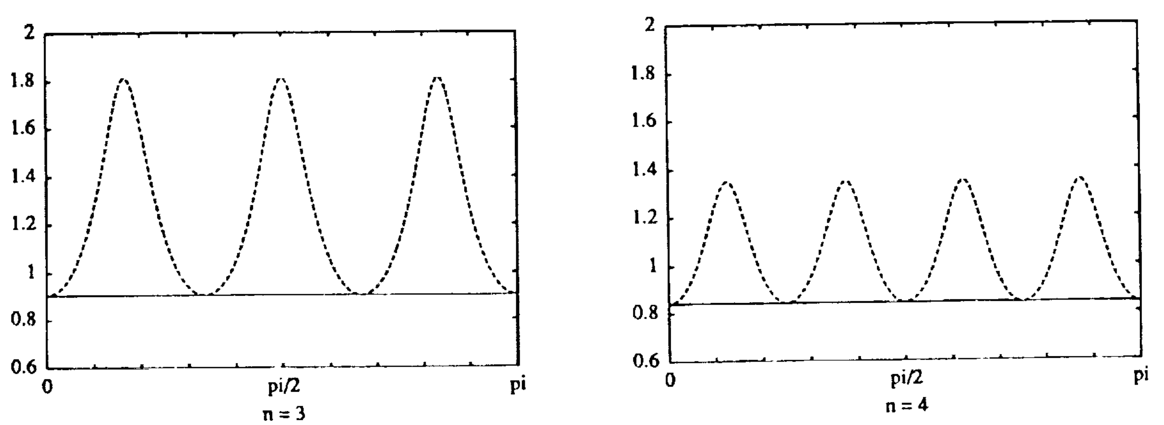


Figure 1. Maximum norm of q_n and t_n

The nontrivial solutions of (4) always lead to a lower bound for the minimal deviation of problem (1), which is sharp in a certain sense:

Theorem 2. *Let* σ_j, $j = 1, 2, \ldots, 2n$, *be any nontrivial real solution of (4), normalized such that* $\sum_{j=1}^{2n} |\sigma_j| = 1$, *then*

$$L_n(r,c) = \frac{1}{M_n(r,c)} \left| \sum_{j=1}^{2n} \sigma_j q_n(z_j;c) \right| \leq E_n(r,c),$$

where equality holds iff $q_n(z;c) \equiv p_n(z;r,c)$.

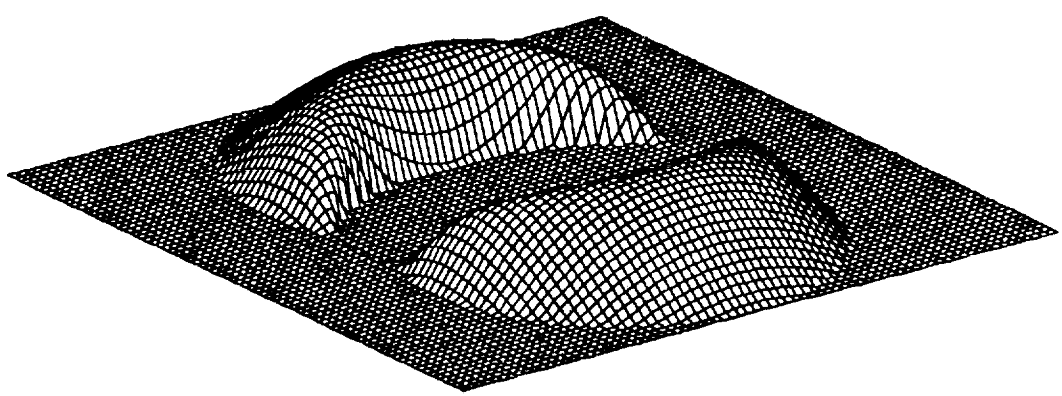
Proof: Let $p \in \Pi_{n-1}$. From (4) we obtain

$$\left|\sum_{j=1}^{2n} \sigma_j q_n(z_j;c)\right| = \left|\sum_{j=1}^{2n} \sigma_j \overline{q_n(z_j;c)}\right| = \left|\sum_{j=1}^{2n} \sigma_j \overline{q_n(z_j;c)}(1-(z_j-c)p(z_j))\right|$$
$$\leq \left(M_n(r,c)\sum_{j=1}^{2n}|\sigma_j|\right) \quad \|1-(z-c)p(z)\|_{\mathcal{E}_r},$$

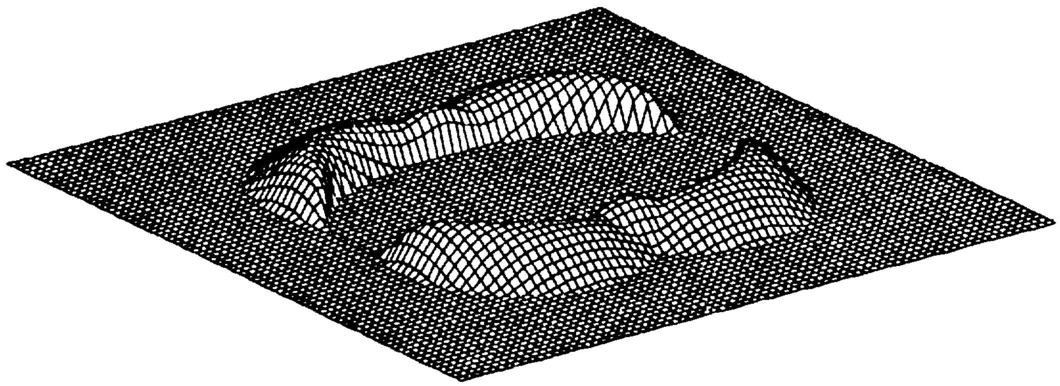
and the result follows. ■

We illustrate that q_n is in general nearly optimal in the following

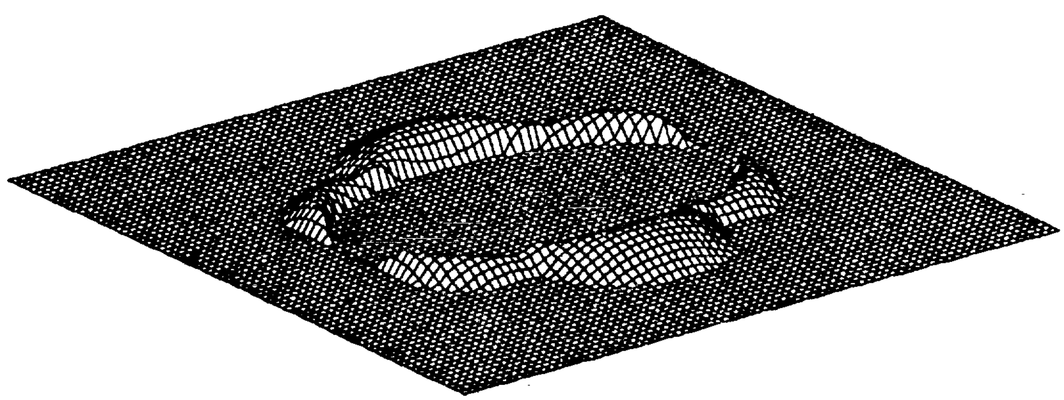
Example 2. *In this example we compute the relative deviation* $D_n(r,c) = (M_n(r,c) - L_n(r,c))/M_n(r,c)$ *where* $r = 2$ *for* $c \in [-2.1, 2.1] \times [-i2.1, i2.1]$ *(here* $D_n(r,c) := 0$ *if* $c \in \mathcal{E}_r$*) and* n = 2, 3, 4, 5. *Note that* $D_n(r,c) = 0$ *if* $q_n(r,c)$ *is optimal (cf. Figure 2). We obtain* $\max_{c\in C} D_n(2,c) <$ *0.1024, 0.0498, 0.0336, 0.0210 for* n = 2, 3, 4, 5 *resp.*



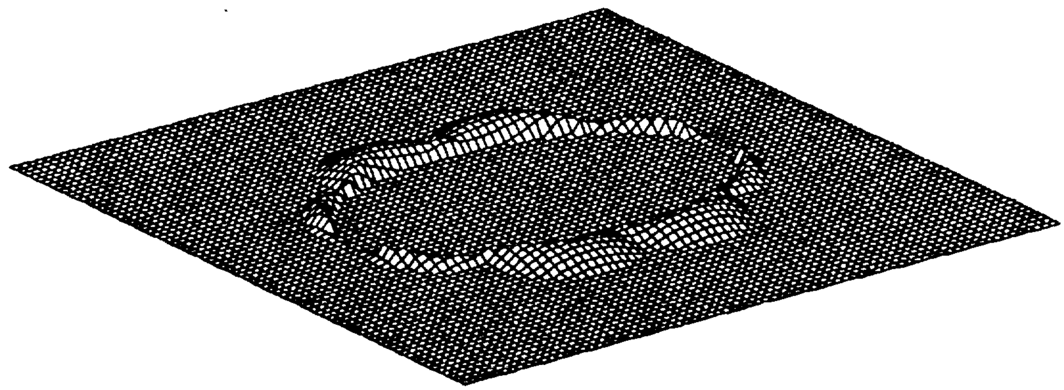
r = 2, n = 2



r = 2, n = 3



r = 2, n = 4



r = 2, n = 5

Figure 2. *Relative deviation of* q_n

References

1. Clayton, A. J., Further results on polynomials having least maximum modulus over an ellipse in the complex plane, U.K.A.E.A. Memorandum - AEEW - M 348, 1963.
2. Eiermann, M., W. Niethammer, and R. S. Varga, A study of semiiterative methods for nonsymmetric systems of linear equations, Numer. Math. **47** (1985), 505-533.
3. Fischer, B. and R. Freund, On the constrained Chebyshev approximation Problem on ellipses, J. Approx. Th. , to appear.
4. Manteuffel, T. A., The Tchebychev iteration for nonsymmetric linear systems, Numer. Math. **28** (1977), 307-327.
5. Rivlin, T. J. and H. S. Shapiro, A unified approach to certain problems of approximation and minimization, J. Soc. Indust. Appl. Math. **9** (1961), 670-699.
6. Saad, Y., Chebyshev acceleration techniques for solving nonsymmetric eigenvalue problems, Math. Comp. **42** (1984), 567-588.
7. Saad, Y., Krylov subspace methods on supercomputers, SIAM J. Sci. Stat. Comp., to appear.