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NEW BERNSTEIN TYPE INEQUALITIES
FOR POLYNOMIALS ON ELLIPSES

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Abstract. *We derive new and sharp estimates for the growth in the complex plane of polynomials known to have a curved majorant on a given ellipse. These so-called Bernstein type inequalities are closely connected with certain constrained Chebyshev approximation problems on ellipses. We also present some new results for approximation problems of this type.*

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

A classical result due to Bernstein [1] is

Theorem A. *Let $n \in I\!N$. Then*

$$(1.1) \qquad |p(c)| \leq \frac{1}{\sqrt{|1 - c^2|}} \frac{1}{2}\left(R^{n+1} - \frac{1}{R^{n+1}}\right), \quad c = \pm\frac{1}{2}\left(R + \frac{1}{R}\right), \quad R > 1,$$

for any $p \in \Pi_n$ which satisfies

$$(1.2) \qquad |p(z)| \leq \frac{1}{\sqrt{1 - z^2}} \quad for\ all \quad -1 < z < 1.$$

The estimate (1.1) is best possible with equality holding only for $p = e^{i\alpha}U_n$, $\alpha \in I\!R$.

Here and in the sequel,

$$(1.3) \qquad U_n(z) \equiv \frac{v^{n+1} - 1/v^{n+1}}{v - 1/v}, \quad z \equiv \frac{1}{2}\left(v + \frac{1}{v}\right),$$

is the nth Chebyshev polynomial of the second kind. Note that $U_n = T'_{n+1}/(n + 1)$ where

$$(1.4) \qquad T_n(z) \equiv \frac{1}{2}\left(v^n + \frac{1}{v^n}\right), \quad z \equiv \frac{1}{2}\left(v + \frac{1}{v}\right),$$

is the usual nth Chebyshev polynomial. Furthermore, throughout this paper, Π_n denotes the set of all complex polynomials of degree at most n. Bernstein and Markov type inequalities for polynomials with a curved majorant of the form (1.2) were studied by several authors (see [10, p. 90], [11,12,9,8] and the references in the recent paper by Rahman and Schmeisser [13]). Note that often (1.2) is written in the form

$$|q(z)| \leq \sqrt{1 - z^2} \quad for\ all \quad -1 \leq z \leq 1,$$

which, obviously, is equivalent to (1.2) with $q(z) \equiv (1 - z^2)p(z)$.

Interestingly, for the case of complex c, sharp estimates (1.1) for polynomials satisfying (1.2) are known only for special cases. For polynomials $p \in \Pi_n$ with real coefficients, Rahman [11, Theorem 4], [12] has shown that

$$|p(c)| \leq \frac{1}{\sqrt{|1 - c^2|}} \frac{1}{2}\left(R^{n+1} + \frac{1}{R^{n+1}}\right) \quad for\ all \quad c = c(\gamma, R).$$

Here $c \in C \setminus [-1, 1]$ is arbitrary and parametrized in the form

$$(1.5) \qquad c = c(\gamma, R) := \frac{1}{2}\left(R + \frac{1}{R}\right)\cos\gamma + \frac{i}{2}\left(R - \frac{1}{R}\right)\sin\gamma, \quad 0 \leq \gamma < 2\pi, \quad R > 1.$$

For complex polynomials with (1.2) it follows as a special case of Corollary 1 in [5] that

$$(1.6) \qquad |p(c)| \le \frac{1}{\sqrt{|1-c^2|}} \, \frac{1}{2}\Big(R^{n+1} + R^{n-1}\Big), \quad c = \pm\frac{i}{2}\Big(R - \frac{1}{R}\Big), \quad R > 1.$$

Moreover, (1.6) is best possible with equality holding if, and only if,

$$(1.7) \qquad p(z) \equiv \frac{e^{i\alpha}}{R^2+1}\Big(R^2 U_n(z) \pm 2i U_{n-1}(z) - U_{n-2}(z)\Big), \quad \alpha \in \mathbb{R}.$$

It seems that, for complex polynomials satisfying (1.2), the cases (1.1) and (1.6) are the only ones for which sharp bounds are known. In particular, it can be shown that, for $c \in \mathbb{C} \setminus \mathbb{R}$, the polynomials U_n are never extremal for best possible inequalities of the type (1.1).

Since the interval $[-1,1]$ can be viewed as the degenerated case $r = 1$ of the family of ellipses

$$(1.8) \qquad \mathcal{E}_r = \{\, z \in \mathbb{C} \mid |z-1| + |z+1| \le r + \frac{1}{r} \,\}, \quad r \ge 1,$$

with foci at ± 1 and semi-axes $(r \pm 1/r)/2$, it is natural to ask for estimates of the form (1.1) for polynomials

$$(1.9) \qquad p \in \Pi_n(r) := \{\, p \in \Pi_n \mid |\sqrt{1-z^2}\, p(z)| \le 1 \text{ for all } z \in \mathcal{E}_r \,\}$$

which satisfy (1.2) on $\mathcal{E}_r$. In this note, we present several new Bernstein inequalities of this type. In particular, it turns out that, somewhat surprisingly and in contrast to the case $\mathcal{E}_1 = [-1,1]$, the polynomials (1.3) still lead to optimal estimates for the true ellipse case $r > 1$, as long as $c = c(\gamma, R)$ is not "too close" to $\mathcal{E}_r$. Note that, for fixed R, (1.5) is a parametrization of the boundary of the ellipse $\mathcal{E}_R$, and $R - r$ is a measure of the distance of $c(\gamma, R)$ to $\mathcal{E}_r$. More precisely, we will prove the following result.

Theorem 1.1. *Let $n \in \mathbb{N}$ and $r > 1$. There exists a number $R^*(n,r)$ $(> r)$ such that, for all $p \in \Pi_n(r)$,*

$$(1.10) \qquad |p(c)| \le \frac{1}{\sqrt{|1-c^2|}} \, \frac{\sqrt{(R^{n+1} + 1/R^{n+1})^2 - 4\cos^2((n+1)\gamma)}}{r^{n+1} + 1/r^{n+1}}$$

for all $c = c(\gamma, R)$ with $R \ge R^*(n,r)$. The estimate (1.10) is best possible with equality holding if, and only if,

$$(1.11) \qquad p(z) \equiv \frac{2e^{i\alpha}}{r^{n+1} + 1/r^{n+1}} \, U_n(z), \quad \alpha \in \mathbb{R}.$$

Furthermore,

$$(1.12) \qquad R^\star(n,r) \le r\frac{65r^4 - 1}{r^4 - 1}.$$

Remark 1.2. The upper bound (1.12) for $R^\star(n,r)$ is very pessimistic. In particular, numerical tests indicate that $R^\star(n,r) \approx r$ for r or n large. We were not able to prove this.

For special values of γ, the estimate (1.10) is true for $c(\gamma, R)$ with arbitrary $R > r$, as long as r is sufficiently large.

Theorem 1.3. *Let $n \in \mathbb{N}$ and $m \in \{0, 1, \ldots, 2n+1\}$. Then there exists a number $r^\star(n) > 1$ such that, for all $p \in \Pi_n(r)$,*

$$(1.13) \qquad |p(c)| \le \frac{1}{\sqrt{|1 - c^2|}} \frac{R^{n+1} + 1/R^{n+1}}{r^{n+1} + 1/r^{n+1}}, \quad c = c\left(\frac{m + 1/2}{n+1}\pi, R\right), \quad R > r \ge r^\star(n).$$

The estimate (1.13) is best possible with equality holding only for the polynomials (1.11).

Actually, it turns out that the inequalities (1.10) and (1.13) also hold true for polynomials p which, instead of $p \in \Pi_n(r)$, satisfy the weaker condition

$$(1.14) \qquad p \in \Pi_n^{(d)}(r) := \{ \, p \in \Pi_n \mid |\sqrt{1 - z_l^2}\, p(z_l)| \le 1, \; l = 0, 1, \ldots, 2n+1 \, \}.$$

Here and in the sequel,

$$(1.15) \qquad z_l = z_l(r) := \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\varphi_l + \frac{i}{2}\left(r - \frac{1}{r}\right)\sin\varphi_l, \quad \varphi_l := \frac{l + 1/2}{n+1}\pi.$$

Theorem 1.4. *Theorems 1.1 and 1.3 remain true if $p \in \Pi_n(r)$ is replaced by $p \in \Pi_n^{(d)}(r)$.*

By means of this last theorem, we will deduce the following corollary to Theorem 1.1.

Corollary 1.5. *Let $m \in 2\mathbb{N}$ be even and $r > 1$. There exists a number $R^\star(m, r)$ $(> r)$ such that, for all "self-inverse" polynomials*

$$(1.16) \qquad s \in \Sigma_m := \{s \in \Pi_m \mid s(v) \equiv -v^m s(\frac{1}{v})\},$$

the inequality

$$(1.17) \qquad \max_{|v| \le R} |s(v)| \le \frac{R^m + 1}{r^m + 1} \max_{l=1,2,\ldots,m} |s(re^{i(2l-1)\pi/m})|, \quad R \ge R^\star(m, r).$$

holds. The estimate (1.17) is best possible with equality only for $s(v) \equiv \sigma(v^m - 1)$, $\sigma \in C$. Furthermore,

$$R^\star(m, r) \leq r \frac{65r^4 - 1}{r^4 - 1}.$$

Remark 1.6. The inequality (1.17) was motivated by a recent result of Frappier, Rahman, and Ruscheweyh. In [4, Theorem 9], they showed that to each given polynomial $p \in \Pi_m$ there exists a number $\hat{\rho}(p) > 0$ depending on p such that

$$\max_{|v| \leq R} |p(v)| \leq \frac{R^m}{r^m} \max_{j=1,2,\ldots,2m} |p(re^{ij\pi/m})| \quad \text{for all} \quad R \geq r\hat{\rho}(p), \ r > 0.$$

For real c and the true ellipse case $r > 1$, we obtain the following extension of Bernstein's result (1.1).

Theorem 1.7. *Let $n \in I\!\!N$.*

a) Let $r > 1$ and $c \in I\!\!R$ with $|c| \geq (r^4 + r^2 + 1)/(r(r^2 + 1))$. Then, for all $p \in \Pi_n(r)$,

$$(1.18) \qquad |p(c)| \leq \frac{1}{\sqrt{|1 - c^2|}} \frac{R^{n+1} - 1/R^{n+1}}{r^{n+1} + 1/r^{n+1}}, \quad c = \pm\frac{1}{2}\Big(R + \frac{1}{R}\Big).$$

The estimate (1.18) is best possible with equality holding only for the polynomials (1.11).
b) There exists a number $\hat{r}(n) > 1$ such that to any $r \geq \hat{r}(n)$ one can find numbers $R > r$ and polynomials $p \in \Pi_n(r)$ for which (1.18) is not fulfilled.

Remark 1.8. For $r = 1$, the estimate (1.18) reduces to (1.1). Moreover, note that Theorem 1.7 leaves open the problem of finding sharp Bernstein type inequalities for $c \in I\!\!R$ with $1 < (r + 1/r)/2 < |c| < (r^4 + r^2 + 1)/(r(r^2 + 1))$.

The rest of the paper is organized as follows. In Section 2, we collect some auxiliary results. The problem of obtaining sharp Bernstein type inequalities of the type (1.1) can be reformulated via weighted complex Chebyshev approximation. In Section 3, we derive some new results for such approximation problems. Finally, the proofs of the results stated in the introduction and in Section 3 are given in Section 4.

2. PRELIMINARIES

In this section, we introduce some further notation and list some auxiliary results. In the sequel, it is always assumed that $k = 0, 1, \ldots$ and $r > 1$. Moreover, let $n \in I\!\!N$

be fixed and set $\varphi_l := (l + 1/2)\pi/(n + 1)$. Finally, let the branch of the square root in $\omega(z) := \sqrt{1 - z^2}$ be chosen such that

$$\omega(z) \equiv \frac{i}{2}\left(v - \frac{1}{v}\right), \quad z \equiv \frac{1}{2}\left(v + \frac{1}{v}\right).$$

In view of (1.3), this choice guarantees

$$(2.1) \qquad \omega(z)U_k(z) \equiv \frac{i}{2}\left(v^{k+1} - 1/v^{k+1}\right), \quad z \equiv \frac{1}{2}\left(v + \frac{1}{v}\right).$$

Next, let $c = c(\gamma, R)$ be as in (1.5) and set

$$(2.2) \qquad d_k := \omega(c)U_k(c).$$

With (1.5) and (2.1), one readily verifies that

$$(2.3) \qquad \begin{aligned} d_k &= -A_{k+1}\sin((k+1)\gamma) + iB_{k+1}\cos((k+1)\gamma), \quad \text{where} \\ A_{k+1} &:= \frac{1}{2}\left(R^{k+1} + \frac{1}{R^{k+1}}\right) \quad \text{and} \quad B_{k+1} := \frac{1}{2}\left(R^{k+1} - \frac{1}{R^{k+1}}\right). \end{aligned}$$

In particular, (2.3) yields

$$(2.4) \qquad |d_k|^2 = A_{k+1}^2 - \cos^2((k+1)\gamma).$$

Let us introduce the Chebyshev norm $\|f\|_{\mathcal{E}_r} := \max_{z \in \mathcal{E}_r} |f(z)|$ on $\mathcal{E}_r$. Using (1.5) (with R replaced by r) and (2.1), a straightforward computation shows that

$$(2.5) \qquad \|\omega U_n\|_{\mathcal{E}_r} = \frac{1}{2}\left(r^{n+1} + \frac{1}{r^{n+1}}\right)$$

and the maximum is attained precisely for the points $z_l = z_l(r)$, $l = 0, 1, \ldots, 2n+1$, defined in (1.15). Moreover,

$$(2.6) \qquad \begin{aligned} \omega(z_l)U_k(z_l) &= -a_{k+1}\sin((k+1)\varphi_l) + ib_{k+1}\cos((k+1)\varphi_l), \quad \text{where} \\ a_{k+1} &:= \frac{1}{2}\left(r^{k+1} + \frac{1}{r^{k+1}}\right) \quad \text{and} \quad b_{k+1} := \frac{1}{2}\left(r^{k+1} - \frac{1}{r^{k+1}}\right), \end{aligned}$$

and, in particular,

$$(2.7) \qquad \omega(z_l)U_n(z_l) = -(-1)^l a_{n+1}.$$

Next, we state a criterion due to Rogosinski and Szegö [15] for the nonnegativity of cosine polynomials.

Lemma 2.1. *Let* $\lambda_0, \lambda_1, \ldots, \lambda_m$ *be real numbers which satisfy* $\lambda_m \geq 0$, $\lambda_{m-1} - 2\lambda_m \geq 0$, *and* $\lambda_{\nu-1} - 2\lambda_\nu + \lambda_{\nu+1} \geq 0$ *for* $\nu = 1, 2, \ldots, m-1$. *Then*

$$(2.8) \qquad t(\varphi) := \frac{\lambda_0}{2} + \sum_{\nu=1}^{m} \lambda_\nu \cos(\nu\varphi) \geq 0 \quad \text{for all} \quad \varphi \in \mathbb{R}.$$

As a first application of Lemma 2.1, one readily obtains the following

Proposition 2.2. *Let* $n \in \mathbb{N}$ *and* $j \in \{1, 2, \ldots, n\}$. *Then*

$$\frac{1}{2} \frac{(n+1)!}{(n+1-j)!} + \sum_{\nu=1}^{n+1-j} \frac{(n+1-\nu)!}{(n+1-j-\nu)!} \cos(\nu\varphi) \geq 0 \quad \text{for all} \quad \varphi \in \mathbb{R}.$$

Finally, we collect some discrete orthogonality relations which will be used in the next section.

Proposition 2.3.

a)

$$\sum_{l=0}^{2n+1} (-1)^l e^{ij\varphi_l} = \begin{cases} 2i(n+1)(-1)^m & \text{if} \quad j = (n+1)(2m+1), \ m \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

b)

$$\sum_{k=0}^{n} e^{i\frac{2k+1}{n+1}\nu\pi} = 0 \quad \text{for} \quad \nu = 1, 2, \ldots, n.$$

c)

$$\frac{1}{2} + (-1)^k \sum_{\nu=1}^{n} \cos\left(\nu \frac{k}{n+1}\pi\right) = \begin{cases} n + 1/2 & \text{if } k \equiv 0 \pmod{2(n+1)}, \\ -1/2 & \text{if } k \not\equiv 0 \pmod{2(n+1)} \text{ is even}, \\ 1/2 & \text{if } k \text{ is odd.} \end{cases}$$

d)

$$\frac{n+1}{2} + \sum_{\nu=1}^{n} (n+1-\nu) \cos\left(\nu \frac{k}{n+1}\pi\right) = \begin{cases} 0 & \text{if } k \text{ is even}, \\ \dfrac{1}{2\sin^2\left(k\pi/(2(n+1))\right)} & \text{if } k \text{ is odd.} \end{cases}$$

e)

$$\sum_{\nu=1}^{n} \sin(\nu\varphi_l) = (-1)^l\left(-\frac{1}{2} + (-1)^l \frac{\sin\varphi_l}{2(1-\cos\varphi_l)}\right).$$

Proof. The proofs of a) – c) and e) are straightforward. For example, in view of

$$\sum_{l=0}^{2n+1} (-1)^l e^{ij\varphi_l} = e^{\frac{ij\pi}{2(n+1)}} \sum_{l=0}^{2n+1} \left(-e^{\frac{ij\pi}{n+1}}\right)^l,$$

part a) is an immediate consequence of

$$\sum_{l=0}^{2n+1} \left(-e^{\frac{ij\pi}{n+1}}\right)^l = \begin{cases} 2(n+1) & \text{if } j \in (n+1)(2\mathbb{Z}+1), \\ 0 & \text{otherwise.} \end{cases}$$

Parts b), c), and e) follow similarly. For part d), apply the well-known identity (see e.g. [15, p. 75])

$$\frac{n+1}{2} + \sum_{\nu=1}^{n} (n+1-\nu)\cos(\nu\varphi) = \frac{1}{2}\left(\frac{\sin((n+1)\varphi/2)}{\sin(\varphi/2)}\right)^2$$

for $\varphi = k\pi/(n+1)$. ∎

3. A WEIGHTED COMPLEX CHEBYSHEV APPROXIMATION PROBLEM

The problem of determining sharp estimates (1.1) for complex c and polynomials $p \in \Pi_n(r)$ is intimately related to the family of constrained approximation problems

$$(3.1) \qquad \left(E_n(r,c) :=\right) \min_{p \in \Pi_n: \; p(c)=1} \; \max_{z \in \mathcal{E}_r} |\omega_c(z)p(z)|, \quad \omega_c(z) := \frac{\omega(z)}{\omega(c)} = \sqrt{\frac{1-z^2}{1-c^2}}.$$

Here and in the sequel, it is always assumed that $c = c(\gamma, R)$ (cf. (1.5)) and $R > r > 1$. Standard results from approximation theory (see e.g. [7]) then guarantee that there always exists a unique optimal polynomial for (3.1). Clearly, the minimal deviation $E_n(r,c)$ of the approximation problem (3.1) yields the best possible constant in the Bernstein type inequality

$$(3.2) \qquad |p(c)| \le \frac{1}{\sqrt{|1-c^2|}} \frac{1}{E_n(r,c)} \max_{z \in \mathcal{E}_r} |\sqrt{1-z^2}\, p(z)|, \quad p \in \Pi_n.$$

Furthermore, equality in (3.2) holds if, and only if, p is a scalar multiple of the optimal polynomial for (3.1).

The solution of (3.1) is classical for the case $r = 1$, $\mathcal{E}_r = [-1,1]$, and $c \in \mathbb{R} \setminus [-1,1]$. Here Bernstein [1] (cf. Theorem A) proved that the scaled Chebyshev polynomial of the second kind

$$(3.3) \qquad u_n(z;c) \equiv \frac{U_n(z)}{U_n(c)}$$

is optimal for (3.1). For purely imaginary c and, again, $r = 1$, Freund [5] showed that the extremal polynomial for (3.1) is a suitable combination of $u_n(z;c), u_{n-1}(z;c)$, and $u_{n-2}(z;c)$ (compare (1.7) and (1.6)). We are not aware of any other cases for which the solution of (3.1) is explicitly known.

In this section, we will derive conditions for the polynomials (3.3) to be optimal for (3.1) in the general case $r > 1$ and $c \in \mathbf{C} \setminus \mathcal{E}_r$. Our main tool is Rivlin and Shapiro's characterization [14] of the best approximation for general linear approximation problems. Recall (see (2.5) and (2.6)) that $\|\omega_c u_n\|_{\mathcal{E}_r}$ is attained just for the points z_l, $l = 0, \dots, 2n+1$, stated in (1.15). Using (2.7), we then deduce from [14] the following

Criterion 3.1. $u_n(z;c)$ *is the unique optimal polynomial for (3.1) if, and only if, there exist real numbers* $\sigma_0, \sigma_1, \dots, \sigma_{2n+1}$ *(not all zero) such that*

$$(3.4) \qquad \sum_{l=0}^{2n+1} \sigma_l (-1)^l \omega(z_l) q(z_l) = 0 \quad \text{for all} \quad q \in \Pi_n \quad \text{with} \quad q(c) = 0.$$

We remark that (3.4) is a system of linear equations for the unknowns $\sigma_0, \dots, \sigma_{2n+1}$. It turns out that one can derive explicit formulas for all real solutions of this linear system. To this end, note that it suffices to check (3.4) for

$$(3.5) \qquad q(z) = U_k(z) - U_k(c), \quad k = 1, 2, \dots, n,$$

only. Furthermore, we will use the ansatz

$$(3.6) \qquad \sigma_l = \sum_{j=0}^{n+1} (\lambda_j \cos(j\varphi_l) + \mu_j \sin(j\varphi_l)), \quad l = 0, 1, \dots, 2n+1,$$

where $\lambda_j, \mu_j \in \mathbb{R}$, $j = 0, 1, \dots, n+1$, and the φ_l are defined in (1.15), for the unknowns of (3.4). Clearly, every collection of real $\sigma_0, \dots, \sigma_{2n+1}$ admits a representation of the form (3.6). Next, we insert (3.5) and (3.6) into (3.4) and, furthermore, rewrite $\omega(z_l) U_k(z_l)$ by means of (2.6). Using part a) of Proposition 2.3, one readily verifies that the resulting linear system (3.4) decouples as follows:

$$(3.7) \qquad \begin{aligned} \lambda_{n-k} a_{k+1} - i\mu_{n-k} b_{k+1} - (\lambda_n a_1 - i\mu_n b_1) U_k(c) &= 0, \quad k = 1, 2, \dots, n-1, \\ 2\lambda_0 a_{n+1} - (\lambda_n a_1 - i\mu_n b_1) U_n(c) &= 0. \end{aligned}$$

Here a_j, b_j, $j = 1, 2, \dots, n+1$, are defined in (2.6). By determining all real solutions λ_j, μ_j of (3.7) and inserting them into (3.6), we finally obtain all $\sigma_l \in \mathbb{R}$ which satisfy the

linear system (3.4). A straightforward computation shows that this solution space of (3.4) is two dimensional and given by

$$(3.8) \qquad \sigma_l = (-1)^l \mu + \tau \rho_l, \quad l = 0, 1, \ldots, 2n+1, \ \mu, \ \tau \in \mathbb{R},$$

where

$$(3.9) \qquad \rho_l = \frac{1}{2} \frac{|d_n|^2}{a_{n+1}} + (-1)^l \sum_{\nu=1}^{n} \left(\frac{\mathrm{Re}\,(d_n \overline{d_{\nu-1}})}{a_\nu} \sin(\nu \varphi_l) + \frac{\mathrm{Im}\,(d_n \overline{d_{\nu-1}})}{b_\nu} \cos(\nu \varphi_l) \right)$$

with d_ν defined in (2.2).

Note that the sign of σ_l still depends on the choice of the free parameters μ and τ in (3.8). However, it is possible to restate Criterion 3.1 in terms of the ρ_l only.

Theorem 3.2. *Let $n \in \mathbb{N}$, $R > r > 1$, and $c = c(\gamma, R)$. Then, $u_n(z; c)$ is the unique optimal polynomial for (3.1) if, and only if,*

$$(3.10) \qquad t_{jk} := \rho_{2j} + \rho_{2k+1} \geq 0 \quad \text{for all} \quad j, k = 0, 1, \ldots, n.$$

Moreover, if u_n is optimal, then

$$(3.11) \qquad E_n(r, c) = \frac{r^{n+1} + 1/r^{n+1}}{\sqrt{(R^{n+1} + 1/R^{n+1})^2 - 4\cos^2((n+1)\gamma)}} .$$

Proof. In view of Criterion 3.1 and (3.8), it remains to show that there exist $\mu, \ \tau \in \mathbb{R}$, μ and τ not both 0, such that

$$(3.12) \qquad \sigma_l = (-1)^l \mu + \tau \rho_l \geq 0, \quad l = 0, 1, \ldots, 2n+1,$$

iff (3.10) is fulfilled. Clearly, we may assume that $\tau \in \{-1, +1\}$. First let $\tau = 1$. Then (3.12) holds iff there exists a μ such that

$$\min_{0 \leq k \leq n} \rho_{2k+1} \geq \mu \geq - \min_{0 \leq j \leq n} \rho_{2j}.$$

Obviously this condition is equivalent to (3.10). Analogously, for $\tau = -1$ we arrive at

$$- \max_{0 \leq k \leq n} \rho_{2k+1} \geq \mu \geq \max_{0 \leq j \leq n} \rho_{2j}$$

or, equivalently,

$$(3.13) \qquad t_{jk} \leq 0, \quad \text{for all} \quad j, k = 0, 1, \ldots, n.$$

However, the case (3.13) can always be excluded since

$$(3.14) \qquad \sum_{k=0}^{n} t_{kk} = (n+1)\frac{|d_n|^2}{a_{n+1}} > 0$$

holds. It remains to verify (3.14). Using trigonometric formulas, one readily deduces from (3.9) that, for all $j, k = 0, 1, \ldots, n$,

$$(3.15) \qquad t_{jk} = \frac{|d_n|^2}{a_{n+1}} + 2\left(\sum_{\nu=1}^{n}\left(\frac{\text{Re}\,(d_n\overline{d_{\nu-1}})}{a_\nu}\cos\left(\nu\frac{j+k+1}{n+1}\pi\right)\right.\right.$$
$$\left.\left. - \frac{\text{Im}\,(d_n\overline{d_{\nu-1}})}{b_\nu}\sin\left(\nu\frac{j+k+1}{n+1}\pi\right)\right)\right)\sin\left(\nu\frac{j-k-1/2}{n+1}\pi\right).$$

By considering (3.15) for $j = k = 0, \ldots, n$ and applying Proposition 2.3 b), we arrive at (3.14).

Finally, note that the formula (3.11) for $E_n(r, c)$ follows from (2.4) and (2.5). ∎

Since every optimal polynomial is in particular optimal with respect to the set of its extremal points, we have

Corollary 3.3. *Theorem 3.2 remains true if $\mathcal{E}_r$ in (3.1) is replaced by the set $\{z_l \mid l = 0, 1, \ldots, 2n+1\}$ of extremal points of $u_n(z; c)$.*

Every nontrivial solution of (3.4) always leads to a lower bound for the minimal deviation $E_n(r, c)$, which is sharp in a certain sense.

Corollary 3.4. *Let σ_l, $l = 0, 1, \ldots, 2n+1$, be any nontrivial real solution of (3.4), normalized such that $\sum_{l=0}^{2n+1}|\sigma_l| = 1$. Then*

$$(3.16) \qquad \left(L_n(r, c) :=\right) \left|\sum_{l=0}^{2n+1}(-1)^l\sigma_l\omega_c(z_l)\right| \leq E_n(r, c).$$

Proof. Let $q \in \Pi_n$ with $q(c) = 0$. We deduce from (3.4)

$$\left|\sum_{l=0}^{2n+1}(-1)^l\sigma_l\omega_c(z_l)\right| = \left|\sum_{l=0}^{2n+1}(-1)^l\sigma_l\omega_c(z_l)(1 - q(z_l))\right|$$
$$\leq \max_{z\in\mathcal{E}_r}|\omega_c(z)(1 - q(z))|\sum_{l=0}^{2n+1}|\sigma_l|,$$

and the result follows. ∎

The following example illustrates the lower bound (3.16).

Example 3.5. We computed the relative deviation

$$D_n(r,c) := \frac{\|u_n(z;c)\|_{\mathcal{E}_r} - L_n(r,c)}{\|u_n(z;c)\|_{\mathcal{E}_r}}$$

of the lower bound (3.16) from the weighted Chebyshev norm of u_n for various cases. In Figure 3.6 the result for $n = 2$, $r = 3$, and $c \in [-2.5, 2.5] \times i[-2.5, 2.5] \setminus \mathcal{E}_r$ is displayed. For σ_l in (3.16), the numbers (3.8) with $\mu = 0$ and $\tau = 1$ were used. Note that $D_2(r,c) = 0$ if u_2 is optimal for (3.1). Moreover, for points $c \in \mathcal{E}_r$ inside the ellipse, we have set $D_n(r,c) = 0$.

Figure 3.6. Relative deviation of u_2

Figure 3.6 as well as our other numerical experiments suggest that the polynomials $u_n(z;c)$ are optimal for (3.1) as long as c is not "too close" to $\mathcal{E}_r$. Furthermore, for certain fixed values of γ and r sufficiently large, it seems that $u_n(z;c)$ is optimal for all $R > r$. In accordance with these observations, we obtained the following results.

Theorem 3.7. *Let* $n \in \mathbb{N}$, $r > 1$, *and* $c = c(\gamma, R)$. *If* $R \geq r(65r^4 - 1)/(r^4 - 1)$, *then* $u_n(z;c)$ *is the unique optimal polynomial for (3.1).*

Theorem 3.8. *Let* $n \in \mathbb{N}$, *and* $c = c(\gamma, R)$. *There exists a number* $r^*(n) > 1$ *such that, for all* $\gamma = \gamma_m = (m + 1/2)\pi/(n + 1)$, $m = 0, 1, \ldots, 2n + 1$, *and* $R > r \geq r^*(n)$, $u_n(z;c)$ *is the unique optimal polynomial for (3.1).*

Note that for fixed γ_m the points

$$c = c(\gamma_m, R) = \frac{1}{2}\left(R + \frac{1}{R}\right)\cos\gamma_m + \frac{i}{2}\left(R - \frac{1}{R}\right)\sin\gamma_m, \quad R > 1,$$

describe a hyperbola which intersects $\mathcal{E}_r$ just at the extremal point $z_m(r)$ of u_n.

Finally, for the special case of real c, we will prove in the next section the following

Theorem 3.9. *Let* $n \in \mathbb{N}$, $R > r > 1$, *and* $c = \pm(R + 1/R)/2$.

a) If $|c| \geq (r^4 + r^2 + 1)/(r(r^2 + 1))$, *then* $u_n(z;c)$ *is the unique optimal polynomial for (3.1).*

b) There exists a number $\hat{r}(n) > 1$, *such that to any* $r > \hat{r}(n)$ *one can find numbers* $R > r$ *for which* $u_n(z;c)$ *is not optimal for (3.1).*

Remark 3.10. An analogue to Theorem 3.9 for the case of unweighted problems (3.1), i.e. $\omega \equiv 1$, was derived by the authors in [3, Theorem 1b),2b)]. Futhermore, in [2] resp. [6], we obtained a result similar to Theorem 3.7 for approximation problems of type (3.1) with complex c and weight functions $\omega(z) \equiv 1$ resp. $\omega(z) \equiv \sqrt{1 + z}$.

4. PROOFS OF THE MAIN RESULTS

In this section, we give the remaining proofs of the results stated in the introduction and in Section 3. First, recall the connection between the constrained Chebyshev approximation problem (3.1) and the inequality (3.2). Moreover, note that, by (3.11),

$$\frac{1}{E_n(r, c(\gamma, R))} = \frac{\sqrt{(R^{n+1} + 1/R^{n+1})^2 - 4\cos^2((n+1)\gamma)}}{r^{n+1} + 1/r^{n+1}}$$

$$= \begin{cases} \dfrac{R^{n+1} + 1/R^{n+1}}{r^{n+1} + 1/r^{n+1}} & \text{if } \gamma = \frac{(m+1/2)\pi}{n+1}, \ m \in \mathbb{Z}, \\[3mm] \dfrac{R^{n+1} - 1/R^{n+1}}{r^{n+1} + 1/r^{n+1}} & \text{if } \gamma = 0, \pi, \end{cases}$$

and, by (2.5), the polynomials (1.11) are in $\Pi_n(r)$ (see (1.9)). Thus, in view of (3.2), Theorem A is an immediate consequence of Bernstein's results [1], while Theorem 1.1, 1.3, and 1.4 follow directly from Theorem 3.7, 3.8, and Corollary 3.3, respectively.

Corollary 1.5 follows from Theorem 1.1 by rewriting the discrete (cf. Theorem 1.4) version of (3.2) by means of the Joukowsky map

$$z = \frac{1}{2}\left(v + \frac{1}{v}\right)$$

for the disks $|v| \leq R$ and $|v| \leq r$. Let $m \in 2\mathbb{N}$ be even and set $n := m/2 - 1$. Then, using (2.1), one readily verifies that

$$s(v) \equiv v^{n+1}\left(\omega(\frac{1}{2}(v + \frac{1}{v}))p(\frac{1}{2}(v + \frac{1}{v}))\right), \quad p \in \Pi_n, \quad s \in \Sigma_m,$$

defines a one-to-one mapping between Π_n and the class of polynomials (1.16). Therefore, we deduce from (3.2) and (3.11)

$$
\max_{|v| \leq R} |s(v)| \leq \frac{R^{n+1}}{r^{n+1}} \max_{c \in \mathcal{E}_r} \frac{1}{E_n(r,c)} \max_{|v| \leq r} |s(v)|
$$
$$
= \frac{R^m + 1}{r^m + 1} \max_{l=1,2,\ldots,m} |s(re^{i(2l-1)\pi/m})|,
$$

where the last equality holds if $u_n(z;c)$ is optimal for (3.1) for all $c \in \mathcal{E}_r$.

It remains to prove Theorems 3.7 – 3.9. We start with the

Proof of Theorem 3.7. Let $j, k \in \{0, 1, \ldots, n\}$ and t_{jk} be given by (3.15). In view of Theorem 3.2, we need to show that

$$
(4.1) \qquad\qquad R \geq r\frac{65r^4 - 1}{r^4 - 1}, \quad r > 1,
$$

implies $t_{jk} \geq 0$. To this end, note that, by (2.4) and (2.3),

$$
(4.2) \qquad\qquad |d_n|^2 \geq A_{n+1}^2 - 1 \geq \frac{1}{4}(R^{2n+2} - 2)
$$

and

$$
(4.3) \qquad |\mathrm{Re}\,(d_n\overline{d_{\nu-1}})| \leq R^{n+1+\nu}, \quad |\mathrm{Im}\,(d_n\overline{d_{\nu-1}})| \leq R^{n+1+\nu}, \quad \nu = 1, \ldots, n.
$$

Using (3.15), (4.2), (4.3), and $a_{n+1} \leq r^{n+1}$, one obtains

$$
(4.4) \qquad
\begin{aligned}
t_{jk} &\geq \frac{|d_n|^2}{a_{n+1}} - 2\sum_{\nu=1}^{n}\left(\frac{|\mathrm{Re}\,(d_n\overline{d_{\nu-1}})|}{a_\nu} + \frac{|\mathrm{Im}\,(d_n\overline{d_{\nu-1}})|}{b_\nu}\right) \\
&\geq \frac{R^{2n+2} - 2}{4r^{n+1}} - 8R^{n+1}\sum_{\nu=1}^{n}\frac{r^{4\nu}}{r^{4\nu} - 1}\left(\frac{R}{r}\right)^\nu.
\end{aligned}
$$

By means of the estimates

$$
R^{2n+2} - 2 \geq \frac{1}{2}R^{2n+2} \quad \text{and} \quad \frac{r^{4\nu}}{r^{4\nu} - 1} < \frac{r^4}{r^4 - 1}, \quad \nu = 2, 3, \ldots,
$$

which are guaranteed by (4.1), we further deduce from (4.4) the inequality

$$
(4.5) \qquad\qquad t_{jk} \geq \frac{R^{2n+2}}{8r^{n+1}}\left(1 - \frac{64r^5}{(r^4 - 1)(R - r)}\right).
$$

However, by the first condition in (4.1), the lower bound in (4.5) is nonnegative, and this concludes the proof. ∎

Proof of Theorem 3.8. Let $r > 1$, $m \in \{0, 1, \ldots, 2n + 1\}$ be arbitrary, but fixed and let $\gamma = (m + 1/2)\pi/(n + 1)$. For $l = 0, 1, \ldots, 2n + 1$ and $R \geq r$, we consider the numbers ρ_l defined in (3.9). A standard calculation, using (2.3), (2.6), and simple trigonometric identities, yields

$$
\frac{\rho_l}{A_{n+1}} = \frac{1}{2} \frac{R^{n+1} + 1/R^{n+1}}{r^{n+1} + 1/r^{n+1}} +
$$

$$
(4.6) \qquad (-1)^{m-l} \sum_{\nu=1}^{n} \frac{1}{r^{2\nu} - 1/r^{2\nu}} \left[\left((Rr)^\nu - \frac{1}{(Rr)^\nu} \right) \cos\left(\nu \frac{m-l}{n+1} \pi \right) + \right.
$$

$$
\left. \left(\left(\frac{R}{r} \right)^\nu - \left(\frac{r}{R} \right)^\nu \right) \cos\left(\nu \frac{m+l+1}{n+1} \pi \right) \right]
$$

$$
=: f_l(R).
$$

By Proposition 2.3 c), we have

$$
(4.7) \qquad f_l(r) = \frac{1}{2} + (-1)^{m-l} \sum_{\nu=1}^{n} \cos\left(\nu \frac{m-l}{n+1} \pi \right) = \begin{cases} n + 1/2 & \text{if } m - l = 0, \\ -1/2 & \text{if } m - l \neq 0 \text{ is even}, \\ 1/2 & \text{if } m - l \text{ is odd}. \end{cases}
$$

¿From (4.6) one easily deduces that for the derivatives of f_l

$$
(4.8) \qquad f_l^{(j)}(r) = \frac{1}{r^j} c_{m-l}^{(j)} + O\left(\frac{1}{r^{j+1}} \right), \quad j \in \mathbb{N},
$$

holds. Furthermore, $c_{m-l}^{(j)} = 0$ if $j > n$ and

$$
c_{m-l}^{(j)} = \frac{1}{2} \frac{(n+1)!}{(n+1-j)!} + \sum_{\nu=1}^{n+1-j} \frac{(n+1-\nu)!}{(n+1-j-\nu)!} \cos\left(\nu \frac{m-l}{n+1} \pi \right), \quad j = 1, \ldots, n.
$$

Remark that, in view of Proposition 2.2,

$$
(4.9) \qquad c_{m-l}^{(j)} \geq 0 \quad \text{for all} \quad j \in \mathbb{N},
$$

and, by Proposition 2.3 d),

$$
(4.10) \qquad \begin{aligned} c_{m-l}^{(1)} &= \frac{1}{2}(n+1) + \sum_{\nu=1}^{n} (n+1-\nu) \cos\left(\nu \frac{m-l}{n+1} \pi \right) \\ &= \begin{cases} 0 & \text{if } m - l \text{ is even} \\ \dfrac{1}{2\sin^2\left((m-l)\pi/(2(n+1)) \right)} & \text{if } m - l \text{ is odd}. \end{cases} \end{aligned}
$$

Next define

$$(4.11) \qquad p_l(R) := \frac{R^{n+1}}{r^{n+1}} f_l(R).$$

By (4.6), p_l is a polynomial in R of degree not exceeding $2n + 2$. By means of Leibniz's rule, we obtain from (4.11) and (4.8) that

$$(4.12) \quad p_l^{(\nu)}(r) = \begin{cases} \dfrac{1}{r^\nu} \left(f_l(r) \dfrac{(n+1)!}{(n+1-\nu)!} + d_{m-l}^{(\nu)} \right) + O(\dfrac{1}{r^{\nu+1}}) & \text{if } 1 \le \nu \le n+1, \\ \dfrac{1}{r^\nu} d_{m-l}^{(\nu)} + O(\dfrac{1}{r^{\nu+1}}) & \text{if } n+2 \le \nu \le 2n+2, \end{cases}$$

where

$$d_{m-l}^{(\nu)} = \sum_{j=j_\nu}^{\nu} \binom{\nu}{j} \frac{(n+1)!}{(n+1-\nu+j)!} c_{m-l}^{(j)}, \quad j_\nu = \max\{1, \nu - n - 1\}.$$

Note that (4.9) implies

$$(4.13) \qquad d_{m-l}^{(\nu)} \ge 0 \quad \text{for all} \quad \nu \in \mathbb{N}.$$

Next, let $M > 1$ be any fixed constant. Then, by inserting (4.12) into the Taylor series of p_l, we deduce that, for all $1 < r \le R \le Mr$,

$$(4.14)$$

$$p_l(R) = \sum_{\nu=0}^{2n+2} \frac{p_l^{(\nu)}(r)}{\nu!} (R - r)^\nu$$

$$= c_{m-l}^{(1)} \frac{R-r}{r} + \sum_{\nu=0}^{n+1} \binom{n+1}{\nu} f_l(r) \left(\frac{R-r}{r}\right)^\nu + \sum_{\nu=2}^{2n+2} \frac{d_{m-l}^{(\nu)}}{\nu!} \left(\frac{R-r}{r}\right)^\nu + O(\frac{1}{r^2}).$$

Now, let $j, k \in \{0, 1, \dots, n\}$ and t_{jk} be defined by (3.10). From (4.7) resp. (4.10), it follows that

$$(4.15) \qquad f_{2j}(r) + f_{2k+1} \ge 0 \quad \text{resp.} \quad c_{m-2j}^{(1)} + c_{m-2k-1}^{(1)} > 0.$$

Finally, using (4.6), (4.11), (4.13)–(4.15), we conclude that to any fixed $M > 1$ there is a number $r(M) > 1$ such that, for all $r(M) \le r < R \le Mr$,

$$(4.16) \quad t_{jk} = \rho_{2j} + \rho_{2k+1} = A_{n+1} \frac{r^{n+1}}{R^{n+1}} \left(p_{2j}(R) + p_{2k+1}(R) \right) > 0 \quad \text{for all } j, k = 0, 1, \dots, n,$$

and hence, in view of Theorem 3.2, u_n is optimal for (3.1). Furthermore, recall that, by Theorem 3.7, u_n is the extremal polynomial for (3.1) if R satisfies (4.1). With (4.1) and

(4.16), it follows that e.g. $r^*(n) := \max\{2^{1/4}, r(129)\}$ fulfills the requirements of Theorem 3.8. ∎

Proof of Theorem 3.9. Let $r > 1$ be fixed and set $a := a_1$. Since, by (3.3) and (1.3), $u_n(z; c) \equiv (-1)^n u_n(z; -c)$, it suffices to consider only the case $c > 0$, i.e. $\gamma = 0$. Then, the representation (3.9) reduces to

$$(4.17) \qquad \rho_l = B_{n+1}\left(\frac{1}{2}\frac{B_{n+1}}{a_{n+1}} + \sum_{\nu=1}^{n} \frac{B_{n+1-\nu}}{a_{n+1-\nu}} \cos(\nu\varphi_l)\right), \quad l = 0, 1, \ldots, 2n+1.$$

First, we turn to the proof of part a) and assume that

$$(4.18) \qquad c \geq c^* := 2a - \frac{1}{2a} = \frac{r^4 + r^2 + 1}{r(r^2 + 1)}.$$

Note that (4.17) can be rewritten in the form $\rho_l = B_{n+1}t(\varphi_l)$ where t is a trigonometric polynomial of type (2.8) (with $m = n$) and coefficients

$$\lambda_0 = \frac{B_{n+1}}{a_{n+1}}, \quad \lambda_\nu = \frac{B_{n+1-\nu}}{a_{n+1-\nu}}, \quad \nu = 1, 2, \ldots, n.$$

Therefore, Theorem 3.2 in combination with Lemma 2.1 ensures that $u_n(z; c)$ is the optimal polynomial for (3.1) if

$$(4.19) \qquad \frac{B_2}{a_2} - 2\frac{B_1}{a_1} \geq 0$$

and

$$(4.20) \qquad \frac{B_{\nu+2}}{a_{\nu+2}} - 2\frac{B_{\nu+1}}{a_{\nu+1}} + \frac{B_\nu}{a_\nu} \geq 0, \quad \nu = 1, 2, \ldots, n-1.$$

It is readily checked that the condition (4.19) is equivalent to $c \geq 2a - 1/a$ and thus satisfied by (4.18). Furthermore, a lengthy, but routine, calculation shows that (4.20) is fulfilled if

$$F_\nu(c) := 4c^2 a_\nu a_{\nu+1} - 4c a_\nu a_{\nu+2} + a_{\nu+1}(a_{\nu+2} - a_\nu) \geq 0, \quad \nu = 1, 2, \ldots, n-1.$$

One easily verifies that c^* is larger than the zeros of F_ν, and this completes the proof of part a).

Finally, we turn to the proof of part b). Let $a > 1$ be arbitrary, but fixed. Using (1.3), (1.4), (2.3), and (2.6), we rewrite (4.17) in the form

$$(4.21) \qquad \rho_l = B_1 B_{n+1} p_l(c), \quad l = 0, 1, \ldots, 2n+1,$$

where

$$(4.22) \quad \begin{aligned} p_l(c) &:= \frac{1}{2} \frac{T'_{n+1}(c)}{(n+1)T_{n+1}(a)} + (-1)^l \sum_{\nu=1}^{n} \frac{T'_\nu(c)}{\nu T_\nu(a)} \sin(\nu \varphi_l) \\ &= p_l(a) + \sum_{m=1}^{n} \frac{p_l^{(m)}(a)}{m!} (c-a)^m, \end{aligned}$$

is a polynomial in c of degree n. Since T'_ν / T_ν is an odd function, it follows that

$$(4.23) \quad \frac{T'_\nu(a)}{\nu T_\nu(a)} = \frac{1}{a} + O(\frac{1}{a^3}) \quad \text{and} \quad \frac{T_\nu^{(m+1)}(a)}{T_\nu(a)} = O(\frac{1}{a^{m+1}}).$$

With (4.23) and Proposition 2.3 e), we deduce from (4.22) that

$$(4.24) \quad \begin{aligned} p_l(a) &= \frac{1}{a}\Big(\frac{1}{2} + (-1)^l \sum_{\nu=1}^{n} \sin(\nu \varphi_l)\Big) + O(\frac{1}{a^3}) \\ &= \frac{1}{a}\Big((-1)^l \frac{\sin \varphi_l}{2(1 - \cos \varphi_l)}\Big) + O(\frac{1}{a^3}). \end{aligned}$$

Now, let t_{jk}, $j,k \in \{0, 1, \ldots, n\}$, be given by (3.10). Using (4.21)–(4.24), we obtian

$$\begin{aligned} \frac{t_{jk}}{B_1 B_{n+1}} &= p_{2j}(c) + p_{2k+1}(c) \\ &= \frac{1}{a}\Big(\frac{\sin \varphi_{2j}}{2(1 - \cos \varphi_{2j})} - \frac{\sin \varphi_{2k+1}}{2(1 - \cos \varphi_{2k+1})} + O(\frac{1}{a^2}) + O(\frac{c-a}{a})\Big). \end{aligned}$$

Thus, $t_{jk} < 0$ if $j > (n+1/2)/2$ (e.g. $j = n$), $k < (n-1/2)/2$ (e.g. $k = 0$), a sufficiently large, and e.g. $c - a \leq 1$. This concludes the proof. ∎

ACKNOWLEDGEMENT

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FIGURE CAPTION

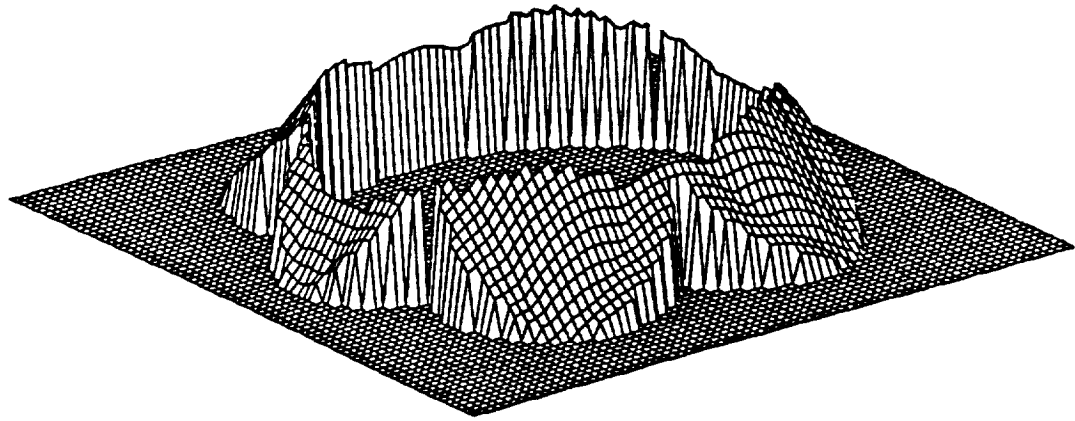**Figure 3.6.** Relative deviation $D_n(r,c)$ of u_n for $c \in \mathbf{C}$ and fixed $n = 2$, $r = 3$.

Figure 3.6