Interface Conditions for Domain Decomposition With Radical Grid Refinement

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Abstract

Interface conditions for coupling the domains in a physically motivated domain decomposition method are discussed. The domain decomposition is based on an asymptotic-induced method for the numerical solution of hyperbolic conservation laws with small viscosity. The method consists of multiple stages. The first stage is to obtain a first approximation using a first-order method, such as the Godunov scheme. Subsequent stages of the method involve solving internal-layer problems via a domain decomposition. The method is derived and justified via singular perturbation techniques.

1 Introduction

This is a report on a preliminary investigation of conditions for the interfaces between subdomains when solving partial differential equations. The analysis for the method is a combination of asymptotics and numerical analysis. The result is a physically motivated domain decomposition method where different partial differential equations may be solved in different domains. Since different modeling equations are in different subdomains for the same problem, we call this *heterogeneous* domain decomposition. The numerical treatment of interface conditions between the subdomains must be addressed. The approach here is to examine the physics reflected in the numerical method used within the subdomains and guarantee that this same physics is reflected in the interface treatment.

The method is best suited to partial differential equations that contain regions of singular behavior. A typical situation is when there are narrow regions where the variation in the solution is large. Such regions are called *boundary layers* or *transition layers* depending on whether they are near a boundary or inside the interior of the domain. Examples of such situations are laminar flow of a slightly viscous fluid or combustion with high activation energy. Classical schemes applied to these types of situations generally fail to correctly describe the behavior inside the layers. This difficulty is overcome by utilizing asymptotic analysis that reflects the physics of the problem. Here we present and motivate the domain decomposition method, but the details of the analysis are presented elsewhere [7].

There have been some intersting results regarding interface conditions for heterogeneous domain decomposition where Euler equations are coupled with Navier-Stokes equations [9],

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and where viscous and inviscid equations where coupled [2, 4]. Many of the basic ideas relating to asymptotic analysis and numerical methods that utilize domain decomposition are found in [10]. These ideas were incorporated into a parallel numerical method in [5]. Specific application to conservation laws have been developed in [1]. There are other important works in these areas–these references are only a small sample of the literature.

The coupling of the problems in the subdomains is based on a balance of the flux across the interface. Each subdomain is treated as a control volume, and the flux into and out-of the control volume is balanced. This is similar to the flux-differencing methods used within the subdomains. The result is a numerical method with no visual artifacts. This numerical treatment of the interface is an extension (to heterogeneous domain decomposition) of the work by Osher and Saunders [11]. We expect extension of this method for the interfaces to work for two dimensional heterogeneous domain decomposition, since it was used for a two-dimensional homogeneous domain decomposition method that utilizes adaptive refinement [8].

2 Problem Setting and Domain Decomposition Motivation

Consider the Cauchy problem

$$\begin{cases} \frac{\partial U}{\partial t} + \frac{\partial}{\partial x} F(U) = \epsilon \frac{\partial}{\partial x} \left(P(U) \cdot \frac{\partial U}{\partial x} \right) & \text{for} \quad (x,t) \in \Omega \\ \\ U(x,0) = V(x) & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Here the solution $U \in \mathbb{R}^n$ is a vector-valued function with n components, the domain is $\Omega = \mathbb{R} \times]0, T[$ and $\epsilon << 1$ is a small parameter.

We assume that V is piecewise smooth. We also assume F and P are regular functions of U. We suppose that P is a *suitable viscosity matrix* [3] for the shocks of the associated inviscid problem

$$\begin{cases} \frac{\partial U^0}{\partial t} + \frac{\partial}{\partial x} F(U^0) = 0 & \text{for} \quad (x,t) \in \Omega \\ \\ U^0(x,0) = V(x) & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{2.2}$$

Namely, a shock-wave solution to (2.2) can be obtained as a limit of progressive wave solutions of (2.1). Problem (2.1) is a parabolic-hyperbolic singular perturbation problem driven by (2.2).

The regions where the solutions to the associated inviscid problem fail to be good approximations to the solution of the full problem are the regions where we use a subdomain to localize the behavior of the solution. Thus, we have two types of domains. The first type of domain is located where the regular expansion

$$U_{\text{as}}^{\text{outer}} = U^0 + \epsilon U^1 + \epsilon^2 U^2 + \ldots \tag{2.3}$$

for U is valid and the solution is smooth. The second type of domain is where the solution exhibits singular behavior and the regular expansion for U is no longer valid.

We substitute $U_{\text{as}}^{\text{outer}}$ in the differential equation of (2.1) and use identification in ϵ to obtain that U^0 must be a solution of (2.2). The inviscid problem (2.2) has many weak solutions; it is possible to uniquely define U^0 by considering the problem that governs U^1 [7].

The failure of the regular expansion is reflected by some of the terms in the PDE governing U^0 being significantly larger than other terms. Typically, the term $RHS(U^0)$ will become unbounded as the small parameter ϵ tends to zero. For finite ϵ, a large $RHS(U^0)$ would indicate that the region should be covered by a subdomain in which we apply techniques designed to capture the singular behavior of the solution. We describe how to use a measure of the numerical approximation of $RHS(U^0)$ to place the subdomain boundaries in a later section of this manuscript.

2.1 Problem in the Singular Region

So that we can handle the regions where solutions to problem (2.1) contain shocks that interact with other singularities we use a brute force approach that will capture all possible behavior of the solution. The approach is to use the coordinate system

$$\xi = \frac{x}{\epsilon}, \quad \tau = \frac{t}{\epsilon}$$

in the regions with shocks. We will present and motivate the domain decomposition method, but the details of the analysis are presented elsewhere [7, 6]. Under this transformation the PDE that governs the solution becomes

$$\frac{\partial \tilde{U}}{\partial \tau} + \frac{\partial}{\partial \xi} F(\tilde{U}) = \frac{\partial}{\partial \xi} \left(P(\tilde{U}) \cdot \frac{\partial \tilde{U}}{\partial \xi} \right), \tag{2.4}$$

where $\tilde{U}(\xi, \tau) = U(x, t)$. This is the equation that is solved in the singular region.

This scaling is most appropriate for regions where shock-layers are interacting with other non-smooth physical phenomena. Because the transformation *a priori* resolves all of the physics. This is reflected by all of the terms in (2.4) having magnitude of order unity or smaller. In general, this method is overkill, similar to using a shotgun to dispatch a housefly. We choose to study only this brute-force approach so that we concentrate on one type of interface. Other treatments that include more of the physics are possible [7]. They can result in more efficient numerical methods than the one discussed here.

The boundary condition at the interface is to impose that the viscous equation from problem (2.1) be the model at the interface between the subdomains. The computational implications of this condition is discussed in §4.

3 Conservative Discretizations

It is important for the discretization techniques to satisfy a discrete conservation relation. One can verify that if the discretizations can be written in the form

$$z_i^{k+1} = z_i^k - \lambda(h_{i+1/2} - h_{i-1/2}),$$

then the method satisfies the appropriate conservation relations. Here we use flux differencing methods based on a finite-volume formulation of the problem.

We will discuss the differencing method for the outer region subdomain where the solution is smooth first. Let W_0 be the discrete numerical approximation to U^0. We use a first-order finite-volume method. This method assumes that the value $W_{0,i}^k$ is an approximation to the average of the desired function U^0 over the spatial interval $]x_{i-1/2}, x_{i+1/2}]$ at time $t = k\Delta t$. The method can also be categorized as a *flux differencing* technique since the general form of the discrete analogue to the original PDE can be written

$$W_{0,i}^{k+1} = W_{0,i}^k - \lambda(F_{i+1/2}^k - F_{i-1/2}^k) \tag{3.5}$$

where

$$F_{i+1/2}^k \approx F(W_{0,i+1/2}^k).$$

Here the fluxes are based on the first-order Godunov scheme; thus, the flux f_j for component w_j of W_0 is approximated as

$$f_{j,i+1/2}^k = \frac{1}{2}\left[f_j(W_{0,i}^k) + f_j(W_{0,i+1}^k) - \alpha_i(W_{j,i+1} - W_{j,i})\right] \tag{3.6}$$

where α_i is an approximation of the upper bound on the local speed of sound.

The discretization that is used for the numerical method in the shock-layer region is a modification of the treatment used for the outer region. We have used a coordinate transformation that creates a smooth problem for this subdomain. Let $\tilde{W}_0$ be the first order numerical approximation to $\tilde{U}$. Let $\tilde{W}_{0,i}^{\tilde{k}}$ be an approximation to the the average of the desired function $\tilde{U}$ over the spatial interval $]\xi_{i-1/2}, \xi_{i+1/2}]$ at time $\tau = \tilde{k}\Delta\tau$. The flux differencing technique is

$$\tilde{W}_i^{\tilde{k}+1} = \tilde{W}_i^{\tilde{k}} - \tilde{\lambda}(\tilde{F}_{i+1/2}^{\tilde{k}} - \tilde{F}_{i-1/2}^{\tilde{k}}) \tag{3.7}$$

where

$$\tilde{F}_{i+1/2}^{\tilde{k}} \approx F(\tilde{W}_{i+1/2}^{\tilde{k}}) - \frac{\partial \tilde{W}_{i+1/2}^{\tilde{k}}}{\partial \xi}$$

The particular discrete form for each component of the flux is obtained using a formula similar to that of Equation (3.6).

We are not restricted to this particular numerical discretization; however, the numerical treatment of the interface will possibly need to be modified for different numerical treatments of the problems within the subdomains.

One can verify that the flux differencing methods given above satisfy the discrete conservation relation. What remains is to formulate the conditions at the interface so that the relation will be satisfied globally.

4 Treatment of the Interface

Using the shock-layer coordinates with $\Delta\xi = C\Delta x$ will result in C/ϵ points in the shock-layer for each point in the outer region. Here, a typical value for ϵ is .01; hence, this results in a *radical* grid refinement for the shock-layer. For the numerical method, since there will be many grid points in the shock-layer for each point in the outer-region, we will refer to the shock-layer grid as the refined grid, and the outer region grid will be called the coarse grid. The temporal coordinate will also be stretched, resulting in the situation outlined in Fig. 4.1.

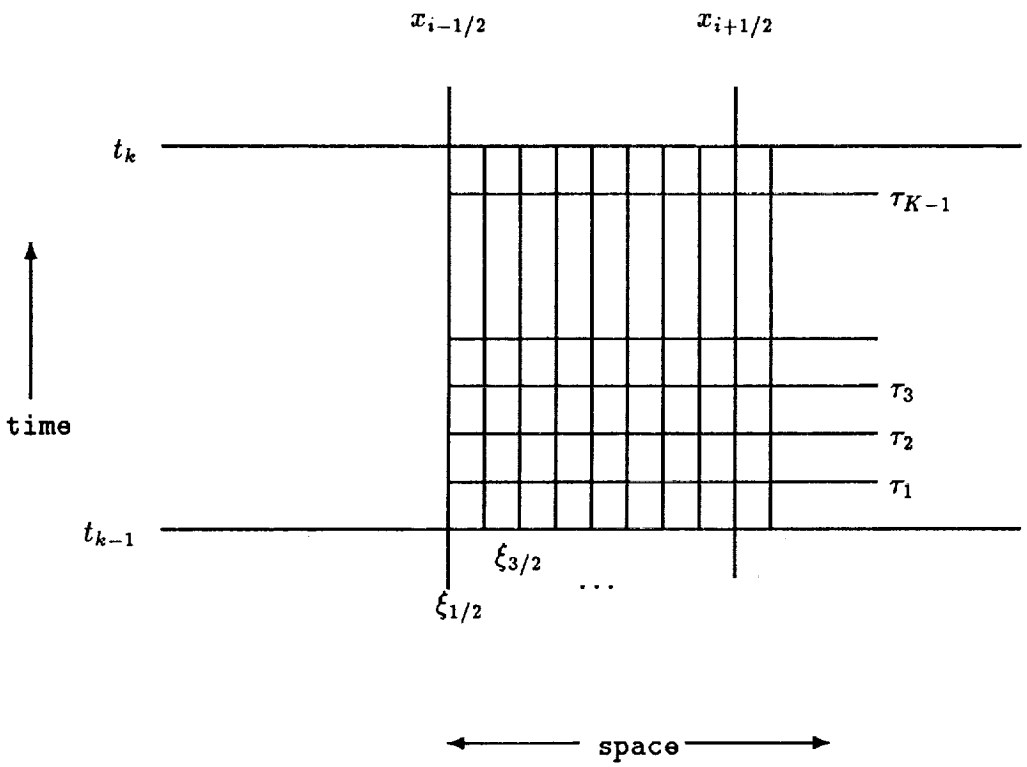

Figure 4.1: Interface at the left boundary

4.1 Flux Treatment of Interface

As in [11], we view the interface treatment as a predictor-corrector method on the coarse mesh. We start at time $t = t^k$. The coarse-grid values are defined everywhere, and are the average of the corresponding fine-grid values when the coarse-grid volume element is within the fine-grid region.

The steps for the first order method are outlined in Algorithm 1 below. At time step k, the shock-layer has $\bar{N}(k)$ points in the interior of the region and a ghost point on each

For $k = 1, \ldots$.

I. March W_0 from t_{k-1} to t_k based on scheme (3.5).

II. Detection.

  A. Compute the residual on the coarse mesh.

  B. Mark regions that should be refined. (Let this be the region between $x_{i_L-1/2}$ and $x_{i_R+1/2}$.

A. Modify shape of refined region.

III. March the shock-layer region from t_k to t_{k+1}. For $\tilde{k} = 1$ to K

  1. Form the initial condition in newly refined regions.

  2. Use linear interpolation to compute the ghost values of $\tilde{W}_{\tilde{O}}^{\tilde{k}}$

  3. March $\tilde{W}_0$ to $\tau_{\tilde{k}+1}$ based on scheme (3.7).

IV. Project $\tilde{W}_0$ onto W_0.

V. Correct values $W_{0,L}^k$ and $W_{0,R}^k$ based on the shock-layer fluxes.

ALGORITHM 1 *Numerical Method.*

side of the refined region. There are a few points that need to be clarified in this algorithm. The interpolation to obtain ghost values (i.e. $\tilde{W}_{0,\tilde{0}}^{\tilde{k}}$) is bi-linear interpolation based on $\tilde{W}_{0,\tilde{1}}^{k-1}$, $W_{0,L-1}^{k-1}$ and $W_{0,L-1}^k$. The initial condition for this problem is derived by imposing mass conservation; thus, the fine-grid values are all initialized to the value of the solution at the cell center. Improvements in the initialization procedure is a subject of further research.

The correction of the coarse-grid values in Step VI is to use the same discretization that was used when the values were originally computed, but to modify the fluxes at the boundary of the domain to reflect what happened on the refined region. That is, to update $W_{0,L}^k$, we would use scheme (3.5) with (3.6) for $F_{L-1/2}^{k-1}$, but we would compute $F_{L+1/2}^{k-1}$ with the formula

$$F_{L+1/2}^{k-1} = \frac{1}{K} \sum_{\tilde{k}=0}^{K-1} \tilde{F}_{\tilde{0}}^{\tilde{k}}.$$

One may verify that this results in a globally conservative method. Also, this treatment of the boundary is consistent with the boundary conditions imposed in §2.1. Namely, this treatment of the interface is consistent with the viscous equation from problem (2.1) being the model at the interface between the subdomains.

4.2 Dirichlet Treatment of Interface

As a comparison to the flux boundary condition, we also implemented the heterogeneous domain decomposition method with dirichlet boundary conditions at the interface. This

is an interesting comparison, since there was little difference in the results when the two different treatments of the interface were used (this is discussed in §6).

5 Detection of Interface

We present the detection of the interface for the sake of completeness. Detection of the interface based on computational data results in a method that can have a different location of the internal-layer subdomain for each time step. The detection for the numerical method is based on obtaining an approximation to

$$\frac{\partial W_0}{\partial t} + \frac{\partial F(W_0)}{\partial x} - \epsilon \frac{\partial}{\partial x}\left(P(W_0)\frac{\partial W_0}{\partial x}\right).$$

This term is the residual from using W_0 as an approximation to the solution of (2.1). The residual is of magnitude $O(\Delta x^{-1})$ in either a shock layer or in a zone where a shock interacts with other singularities.

It is also possible to use an approximation of the viscous term $\frac{\partial^2}{\partial x^2} W_0(\cdot, t_K)$ to localize some of the singularities. For example, this viscous term will be of order $O(\Delta x^{-1})$ in a shock layer or in a zone of interaction. This method is not as reliable as using the residual, however. Other types of behavior can be located and identified using these techniques [7].

6 Application to the Isentropic Gasdynamic Equations

In this section we examine the interface treatments on the viscous isentropic gasdynamic equations

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{u^\gamma}\right) = \epsilon \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right).$$

Here u is the inverse of the density and v is the velocity. These equations are obtained from the conservation of mass and momentum in Lagrangian coordinates assuming that u is equal to the pressure raised to the $-1/\gamma$th power (the perfect gas law). The experiments were run with $\gamma = 2.2$.

The problem is a right-traveling shock interacting with and a left-traveling rarefaction, both of which eminate from the origin. An analytic self-similar solution (a rarefaction eminating from the origin) to the inviscid isentropic gasdynamic equations is given by

$$u(x,t) = \gamma^{1/(\gamma+1)} \left(\frac{x}{t}\right)^{-2/(\gamma+1)} \tag{6.8}$$

$$v(x,t) = \frac{-2\gamma^{1/(\gamma+1)}}{\gamma-1}\left(\frac{x}{t}\right)^{\frac{\gamma-1}{\gamma+1}} + const. \tag{6.9}$$

An initial condition with a shock and rarefaction eminating from the origin is constructed by connecting left values to middle values with a rarefaction. The middle values are connected to the right values with a shock. Thus, the initial condition is given by

$$u(x,0) = \begin{cases} U_L, & \text{for } x < 0 \\ U_R, & \text{for } x \geq 0 \end{cases} \tag{6.10}$$

$$v(x,0) = \begin{cases} V_L, & \text{for } x < 0 \\ V_R, & \text{for } x \geq 0 \end{cases} \tag{6.11}$$

where

$$U_L = 1.4709, \qquad U_R = 2.5000, \qquad V_L = 1.0388, \qquad V_R = 0.8050.$$

The middle value of the solution between the shock and rarefaction is $(U_M, V_M) = (1.973, 1.356)$. We remark that the middle values were was chosen using the Rankine-Hugoniot condition

$$\frac{V_M - V_R}{U_R - U_M} = \frac{1/U_R^\gamma - 1/U_M^\gamma}{V_R - V_M}.$$

We expect the the viscous perturbation to have little or no effect on the speed at which shocks and rarefactions travel; thus, we will compare the viscous solutions to the solutions given above.

The method was run with $\epsilon = .01$. The discretization parameters for numerical solution in the outer region have CFL number $\Delta t/\Delta x = .1$, and $\Delta x = .02$. The discretization on the scaled coordinates inside the shock-layer is based on $\Delta\xi = .1$, with the CFL condition $\Delta\tau/\Delta\xi \leq .025$ and the stability condition $\Delta\tau/\Delta\xi^2 \leq .1$. These values are well within the limits imposed for the stability of the finite difference methods.

Figure 6.2 depicts the evolution of the internal-layer subdomain when the two different boundary conditions are used. The errors generated by using the dirichlet boundary condition when the rarefaction is trying to exit the internal-layer subdomain result in a larger computed second derivative, and the detection scheme kept the rarefaction inside the internal-layer much longer. The solution projected onto the coarse grid at the end of the computations showed little difference between the two methods (Fig. 6.3). The primary difference is the visual artificats at the boundary of the internal-layer subdomain at the point when the rarefaction is exiting the subdomain (Fig. 6.4).

7 Conclusion

Clearly the best interface condition is the flux-based treatment; however, the dirichlet boundary conditions did not induce as many errors as expected. One explaination of the lack of errors may be that the internal-layer subdomain boundary moves fast enough that waves propagating out of the internal-layer subdomain are allowed to pass across the boundary by the oscillations in the boundary. More studies are planned with the goal to identify the precise nature of the errors associated with the interface treatments.

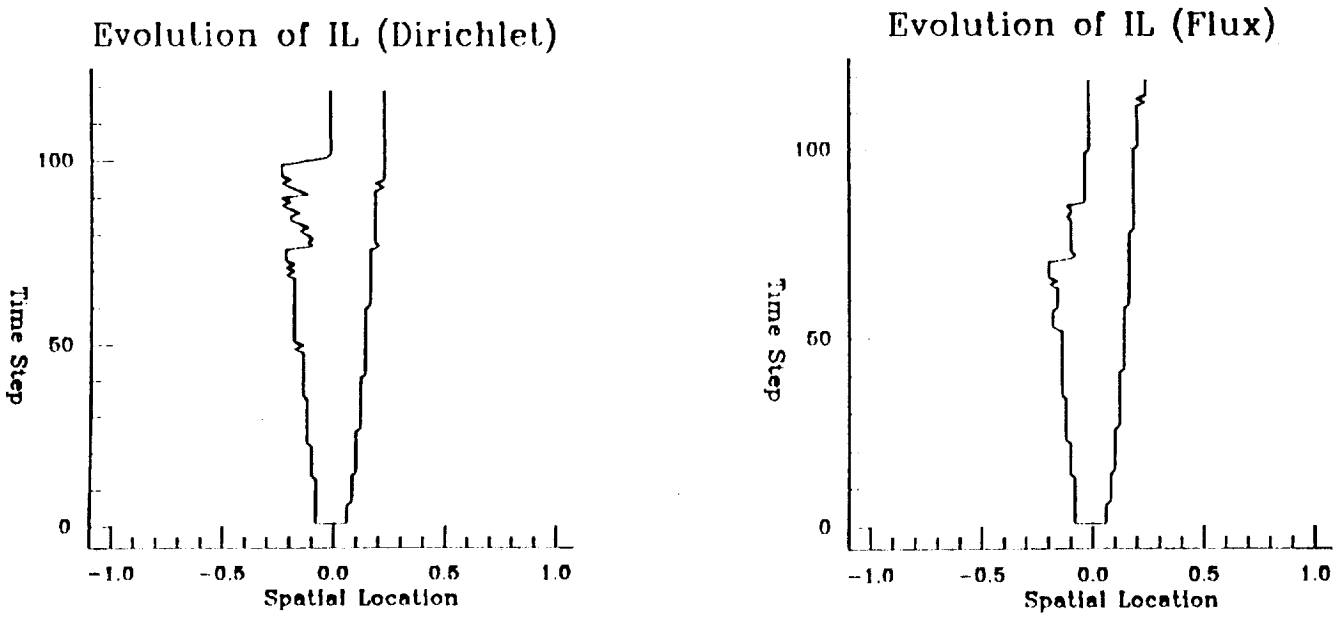

Figure 6.2: Evolution of the Internal-layer Subdomain.

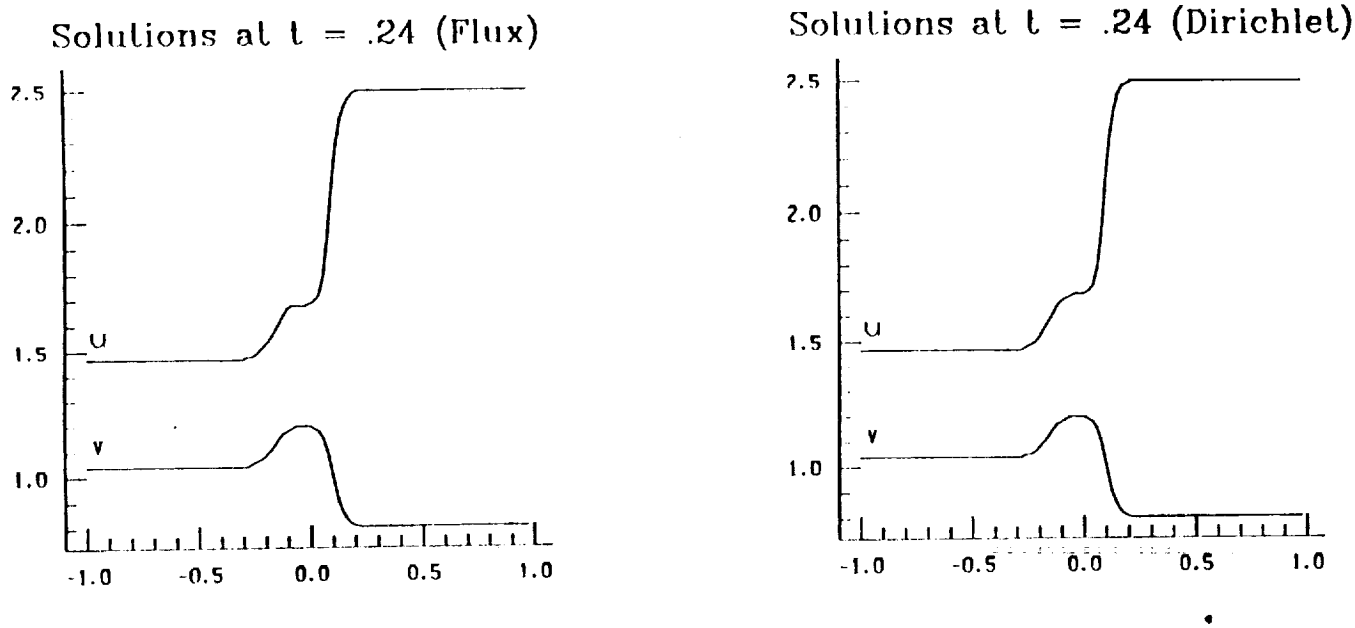

Figure 6.3: Solution on Coarse Mesh at $t = .24$.

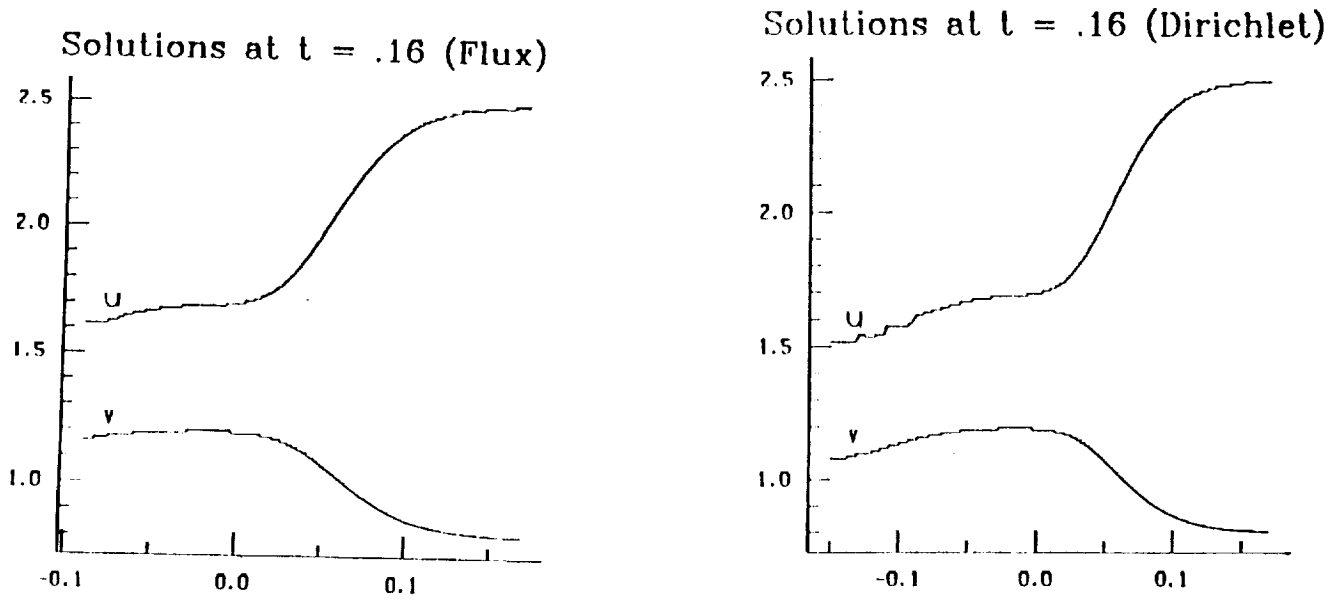

Figure 6.4: Solution on Fine Mesh at $t = .16$.

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