N92-14077

NEW QUESTS FOR BETTER ATTITUDES

Malcolm D. Shuster

The Johns Hopkins University, Applied Physics Laboratory,
Laurel, Maryland 20723-6099

ABSTRACT

During the past few years considerable insight has been gained into the QUEST algorithm both as a maximum-likelihood estimator and as a Kalman filter/smoother for systems devoid of dynamical noise. This conference contribution describes the new algorithms and software and makes analytic comparisons with the more conventional attitude Kalman filter. We also describe how they may be accommodated to noisy dynamical systems.

Introduction: the QUEST Algorithm

The QUEST algorithm is based on a least-square problem first proposed in 1965 by Grace Wahba, then a graduate student in Statistics at George Washington University and working during that summer for IBM in Gaithersberg, Maryland. The problem, which appeared in *SIAM Review* [1], was, in fact, Wahba's first publication. In it she posed the problem of finding the attitude which minimizes the loss function

$$L(A) = \frac{1}{2} \sum_{i=1}^{n} a_1 \, |\hat{\mathbf{W}}_i - A \, \hat{\mathbf{V}}_i|^2 \,, \tag{1}$$

where $\hat{\mathbf{W}}_i$, $i = 1, \ldots, n$, are a set of unit-vector observations in the spacecraft-fixed reference frame, and $\hat{\mathbf{V}}_i$, $i = 1, \ldots, n$, are the representations of the same unit vectors with respect to the primary reference frame (the frame to which the attitude is referred). The a_i are a set of non-negative weights. Provided that at least two of the observation vectors are not parallel (or anti-parallel) and the corresponding weights are positive, a unique minimizing attitude matrix will always exist. Dozens of solutions have been proposed to find this attitude matrix, of which the fastest currently and most frequently used is the QUEST algorithm [2], based on the q-algorithm of Davenport [3].

 PRECEDING PAGE BLANK NOT FILMED

To solve for the optimal attitude we first write equation (1) in the form

$$L(A) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i \hat{\mathbf{W}}_i \cdot A \hat{\mathbf{V}}_i , \tag{2}$$

$$= \sum_{i=1}^{n} a_i - g(A) . \tag{3}$$

The gain function, $g(A)$, may be further manipulated to give

$$g(A) = \mathrm{tr}\big(B^T A\big) , \tag{4}$$

where B, the attitude profile matrix, is given by

$$B = \sum_{i=1}^{n} a_i \hat{\mathbf{W}}_i \hat{\mathbf{V}}_i^T . \tag{5}$$

The minimization of $L(A)$ is equivalent to the maximization of $g(A)$.

We now note that $g(A)$ is linear in A. Nonetheless, the minimization of $g(A)$ is not simple because the 3×3 matrix A is subject to six nonlinear constraints. Thus, the minimization of $g(A)$ over A is not necessarily simple.[1] The attitude matrix, however, can be written as a quadratic function of the quaternion,

$$\bar{q} = [q_1, q_2, q_3, q_4]^T = \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix} , \tag{6}$$

namely,

$$A(\bar{q}) = (q_4^2 - \mathbf{q} \cdot \mathbf{q}) I_{3\times 3} + 2\mathbf{q}\mathbf{q}^T + 2q_4 [[\,\mathbf{q}\,]] , \tag{7}$$

where

$$[[\,\mathbf{q}\,]] \equiv \begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix} . \tag{8}$$

Defining further the quantities

$$S = B + B^T , \quad s = \mathrm{tr}\, B , \quad [[\,\mathbf{Z}\,]] = B - B^T , \tag{9}$$

the gain function may be rewritten in terms of the quaternion as

$$g(\bar{q}) \equiv g(A(\bar{q})) = \bar{q}^T K \bar{q} , \tag{10}$$

where

$$K = \begin{bmatrix} S - s\, I_{3\times 3} & \mathbf{Z} \\ \mathbf{Z}^T & s \end{bmatrix} . \tag{11}$$

[1] Not all students of the Wahba problem will agree, as shown by Markley [4].

The maximization of this gain function, subject to the constraint that the quaternion have unit norm, leads to an eigenvalue equation for $\bar{q}^*$, the optimal quaternion, which is [2, 3]

$$K\,\bar{q}^* = \lambda_{\max}\,\bar{q}^*\,, \tag{12}$$

where $\lambda_{\max}$ is the largest eigenvalue of K. Thus, the optimal quaternion may be found by solving this 4×4 eigenvalue problem and choosing the eigenvector with the largest eigenvalue. This is Davenport's q-method, which was applied in this form to the HEAO mission [5].

The QUEST algorithm, a very fast implementation of Davenport's q-method which avoids the complete solution of the eigenvalue problem, is formulated in terms of the Gibbs vector, $\mathbf{Y}$,

$$\mathbf{Y} \equiv \mathbf{q}/q_4\,. \tag{13}$$

In terms of the Gibbs vector the optimal attitude may be written as

$$\mathbf{Y}^* = \left[(\lambda_{\max} + s)\,I_{3\times 3} - S\right]^{-1}\,\mathbf{Z}\,, \tag{14}$$

and the optimal quaternion then reconstructed as

$$\bar{q}^* = \frac{1}{\sqrt{1+|\mathbf{Y}^*|^2}}\begin{bmatrix}\mathbf{Y}^*\\ 1\end{bmatrix}. \tag{15}$$

Key to the QUEST algorithm is the fact that a very good first approximation of the optimal attitude (accurate to $O(\sigma^4)$, where σ is the standard deviation of a typical sensor error) may be obtained by substituting $\lambda_{\max}^{(o)}$ for $\lambda_{\max}$, with

$$\lambda_{\max}^{(o)} \equiv \sum_{i=1}^{n} a_i\,. \tag{16}$$

It is easy to show that

$$\lambda_{\max} = \lambda_{\max}^{(o)}\,(1 + O(\sigma^2))\,. \tag{17}$$

The further refinement of $\lambda_{\max}$ is described in detail in [2]. This amounts to solving the equation

$$\lambda_{\max} = s + \mathbf{Z}^T\left[(\lambda_{\max} + s)\,I_{3\times 3} - S\right]^{-1}\,\mathbf{Z}\,, \tag{18}$$

by the Newton-Raphson method using $\lambda_{\max}^{(o)}$ as a starting value.

If the measurements are assumed to be corrupted solely by Gaussian random errors of the form,

$$\hat{\mathbf{W}}_i = A\,\hat{\mathbf{V}}_i + \Delta\hat{\mathbf{W}}_i\,, \tag{19}$$

where the sensor error $\Delta\hat{\mathbf{W}}_i$ satisfies

$$E\{\Delta\hat{\mathbf{W}}_i\} = \mathbf{0}\,, \tag{20}$$

$$E\{\Delta\hat{\mathbf{W}}_i\,\Delta\hat{\mathbf{W}}_i^T\} = \sigma_i^2\,[I_{3\times 3} - (A\,\hat{\mathbf{V}}_i)(A\,\hat{\mathbf{V}}_i)^T]\,, \tag{21}$$

and the weights a_i, $i = 1, \ldots, n$, are chosen so that

$$a_i = \frac{c}{\sigma_i^2} \tag{22}$$

for some constant c, then Reference [2] shows that the attitude covariance matrix is given by

$$P_{\theta\theta} = \left[\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\left(I_{3\times 3} - (\hat{\mathbf{W}}_i)_{\text{true}}(\hat{\mathbf{W}}_i)_{\text{true}}^T\right)\right]^{-1}, \tag{23}$$

where

$$(\hat{\mathbf{W}}_i)_{\text{true}} \equiv A_{\text{true}}\,\hat{\mathbf{V}}_i\,. \tag{24}$$

In actual computation we generally substitute $\hat{\mathbf{W}}_i$ for $(\hat{\mathbf{W}}_i)_{\text{true}}$, since the latter value is not known, in general. The attitude covariance matrix is defined here as

$$P_{\theta\theta} = \text{Cov}(\Delta\boldsymbol{\theta}), \tag{25}$$

where $\Delta\boldsymbol{\theta}$, the attitude error, is given by

$$A^*\,A_{\text{true}}^T \approx I_{3\times 3} + [[\Delta\boldsymbol{\theta}]], \tag{26}$$

and Cov denotes the covariance. Thus, the QUEST algorithm gives a fast direct method for constructing the optimal attitude. The algorithm has other valuable properties as well, which are discussed in [2].

The Attitude Kalman Filter for the QUEST Model

QUEST is a batch estimator taking as input a collection of *simultaneously* measured unit vectors. When the data is not simultaneous and we wish to use data at widely different times, the algorithm of choice has been the Kalman filter. In the present section we present the Kalman filter for the measurement model of equations (19)–(21).

Since the QUEST algorithm does not treat dynamical noise (the measurements being all simultaneous, this would hardly be relevant), we examine the Kalman filter for a system without dynamical noise, that is, a system for which the temporal development of the attitude is described by

$$A_k = \Phi_{k-1} A_{k-1}, \tag{27}$$

where the transition matrices, Φ_k, $k = 0, \ldots, N-1$, is known perfectly. In general, the subscript k will indicate the time, and the subscript i will indicate the sensor. For such a system the prediction of the attitude matrix must have the form

$$A_{k|k-1}^* = \Phi_{k-1} A_{k-1|k-1}^*, \tag{28}$$

where $A_{k-1|k-1}^*$ is the estimate of the attitude matrix at time t_{k-1} based on all the measurements up to that time inclusively, and $A_{k|k-1}^*$ is the estimate of the attitude matrix at time t_k based on the same data. Since dynamical noise is absent, the prediction of the attitude covariance matrix is given by

$$P_{k|k-1} = \Phi_{k-1} P_{k-1|k-1} \Phi_{k-1}^T\,. \tag{29}$$

and, since no confusion can result, we have dropped the subscript $\theta\theta$ to make the notation less cumbersome.

For updates the calculation is more involved. Since the attitude has only three free parameters while the attitude matrix has nine, we do not update the attitude matrix directly but compute instead the updated value of $\boldsymbol{\xi}_k$, the incremental rotation vector, which is defined by

$$A_k = e^{[[\boldsymbol{\xi}_k]]} A^*_{k|k-1}\,, \tag{30}$$

$$\approx A^*_{k|k-1} + [[\boldsymbol{\xi}_k]]\, A^*_{k|k-1}\,, \tag{31}$$

so that by definition

$$\boldsymbol{\xi}^*_{k|k-1} = \mathbf{0}\,. \tag{32}$$

Then we can write the linearized measurement as

$$\boldsymbol{\zeta}_k \equiv \hat{\mathbf{W}}_k - \hat{\mathbf{W}}_{k|k-1}\,, \tag{33}$$

where

$$\hat{\mathbf{W}}_{k|k-1} = A^*_{k|k-1}\hat{\mathbf{V}}_k\,. \tag{34}$$

Combining equations (31)–(34) yields

$$\boldsymbol{\zeta}_k = H_k\boldsymbol{\xi}_k + \mathbf{v}_k\,, \tag{35}$$

where

$$H_k = -[[\hat{\mathbf{W}}_{k|k-1}]]\,. \tag{36}$$

The measurement noise of our linearized measurement, $\mathbf{v}_k$, is assumed to be Gaussian and zero-mean. Its covariance matrix can have only rank 2 since unit-vector measurements have only two degrees of freedom. However, it can be shown that the true covariance matrix of $\mathbf{v}_k$ can be replaced by

$$R_k = \sigma_k I_{3\times 3}\,, \tag{37}$$

which is obviously of rank 3. This substitution leads to the same estimates and covariance matrices as the form given by equation (21) [6]. The reason for this is that the additional noise which makes the covariance matrix of rank 3 is along the direction of $\hat{\mathbf{W}}_k$, to which the attitude is not sensitive. It can be seen from equation (36) that H_k annihilates that component from $\hat{\mathbf{W}}_k$.

The Kalman filter update equations now become

$$\mathcal{B}_k = H_k^T P_{k|k-1} H_k + R_k\,, \tag{38}$$

$$K_k = P_{k|k-1} H_k^T \mathcal{B}_k^{-1} \tag{39}$$

$$\boldsymbol{\xi}^*_{k|k} = K_k \hat{\mathbf{W}}_k\,, \tag{40}$$

$$P_{k|k} = (I_{3\times 3} - K_k H_k) P_{k|k-1} \tag{41}$$

$$= (I_{3\times 3} - K_k H_k) P_{k|k-1} (I_{3\times 3} - K_k H_k)^T + K_k R_k H_k\,. \tag{42}$$

The Kalman filter equations, (28) through (42), can treat non-simultaneous data but are considerably more complicated than the QUEST algorithm for simultaneous data. It is natural ask, therefore, whether the QUEST equations can be manipulated to remove the restriction to simultaneous data. The answer is affirmative. In fact, the Wahba problem was applied to non-simultaneous quite some time ago [5] but in a batch framework, not in a sequential framework like the Kalman filter.

The Sequentialization of QUEST: Filter QUEST

Suppose that we have a set of simultaneous measurements at time t_{k-1} which we can denote by $\hat{\mathbf{W}}_{i,k-1}$, $i = 1, \ldots, n$, and let us denote the optimal attitude at time t_{k-1} computed using the QUEST algorithm by $A^*_{k-1|k-1}$. Recalling equation (27), the optimal value of A_k based on the data at time t_{k-1} is obtained by minimizing

$$L_{k-1}(A_k) = \frac{1}{2} \sum_{i=1}^{n_{k-1}} a_{i,k-1} \, |\hat{\mathbf{W}}_{i,k-1} - \Phi_{k-1}^{-1} A_k \, \hat{\mathbf{V}}_{i,k-1}|^2 , \tag{43}$$

where the additional subscript on $L(A)$ indicates the time of the data. Since Φ_k is orthogonal, this is clearly the same as finding the value of A_k which minimizes

$$L_{k-1}(A_k) = \frac{1}{2} \sum_{i=1}^{n_{k-1}} a_{i,k-1} \, |\Phi_{k-1} \hat{\mathbf{W}}_{i,k-1} - A_k \, \hat{\mathbf{V}}_{i,k-1}|^2 , \tag{44}$$

that is, by replacing $\hat{\mathbf{W}}_{i,k-1}$ by $\Phi_{k-1}\hat{\mathbf{W}}_{i,k-1}$, or equivalently, noting equation (5), by replacing

$$B_{k-1|k-1} \equiv \sum_{i=1}^{n_{k-1}} a_{i,k-1} \, \hat{\mathbf{W}}_{i,k-1} \, \hat{\mathbf{V}}_{i,k-1}^T \tag{45}$$

by

$$B_{k|k-1} \equiv \sum_{i=1}^{n_{k-1}} a_{i,k-1} \, \Phi_{k-1} \hat{\mathbf{W}}_{i,k-1} \, \hat{\mathbf{V}}_{i,k-1}^T \tag{46}$$

Thus, for the filter version of QUEST, the prediction step becomes simply [7]

$$B_{k|k-1} = \Phi_{k-1} \, B_{k-1|k-1} \, . \tag{47}$$

We may, in fact, drop the distinction between the indices i and k and treat each unit vector has having a distinct time t_k, reference vector $\hat{\mathbf{V}}_k$ and weight a_k associated with it. If two vector measurements $\hat{\mathbf{W}}_{k+1}$ and $\hat{\mathbf{W}}_k$ are simultaneous, then $t_{k+1} = t_k$ and $\Phi_k = I_{3\times3}$.

For the update step of Filter QUEST, we note that when we increase the number of measurements in the measurement set of equation (5) we simply add a term, $a_k \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T$, to B. Thus, the update step in terms of the attitude profile matrix is

$$B_{k|k} = B_{k|k-1} + a_k \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T \, . \tag{48}$$

The QUEST algorithm requires also that we know the value of $\lambda^{(o)}_{\max k}$ separately. This is given by

$$\lambda^{(o)}_{\max k} = \lambda^{(o)}_{\max k-1} + a_k \, . \tag{49}$$

Equations (47)–(49) are clearly much simpler than the corresponding Kalman filter equations (28)–(42).

The covariance matrix can be computed sequentially also by first computing the attitude information matrix, $F \equiv P^{-1}$. The relevant equations are

$$F_{k|k-1} = \Phi_{k-1} F_{k-1|k-1} \Phi_{k-1}^T \tag{50}$$

$$F_{k|k} = F_{k|k-1} + \frac{1}{\sigma_k^2}\left(I_{3\times 3} - \hat{\mathbf{W}}_k \hat{\mathbf{W}}_k^T\right) . \tag{51}$$

These can be computed likewise [7, 8] from

$$F = \mathrm{tr}\,(A^* B^T) I_{3\times 3} - A^* B^T , \tag{52}$$

without the need to have a separate recursion relation for F. Equation (52), in fact, is very important because it can be solved for B to yield

$$B = \left(\frac{1}{2}\,\mathrm{tr}\,(F)\, I_{3\times 3} - F\right) A^* . \tag{53}$$

Thus, given initial values, $A^*_{o|o}$ and $P_{o|o}$, the initial value of the attitude profile matrix, $B_{o|o}$ can be computed from equation (53). This last fact makes the analogy of Filter QUEST with the Kalman filter complete. In fact, since it can be shown that QUEST is a maximum-likelihood estimator for Gaussian errors [7], the Kalman filter and Filter QUEST will yield identical attitude estimates for the attitude system considered above.

It is well to note that the Filter QUEST is an information filter rather than a covariance filter. This is made clear by the fact that $B_{1|1}$ is a meaningful quantity even though the attitude cannot be calculated from a single measurement.

While the prediction and update equations of Filter QUEST are simple, it is also true that the need to apply the part of QUEST which computes the optimal attitude and covariance matrix from the attitude profile matrix is an additional computational burden. The real advantage of Filter QUEST comes when one does not require an attitude solution at every measurement update. In this case, the efficiency of Filter QUEST relative to the Kalman filter is greatly enhanced.

The Treatment of Noisy Processes

In general, attitude systems are subject to random torques, or the dynamical equations are replaced by the gyro equations [9] so that the gyro measurement noise becomes process noise. In the Kalman filter formulation, this extra complication results in the prediction equation for the attitude being replaced by

$$P_{k|k-1} = \Phi_{k-1} P_{k-1|k-1} \Phi_{k-1}^T + Q_{k-1} \cdot \tag{54}$$

where Q_{k-1} is the covariance of the accumulated process noise from time t_{k-1} to time t_k. Possibly also, the state vector must be augmented to include the state of the Markov process driving the gyro errors [9].

Such an enhancement is not possible in Filter QUEST since there is no simple way of adding process noise to B. One could, of course, compute $P_{k|k-1}$ after every prediction step of Filter QUEST, add the process noise covariance to $P_{k|k-1}$, and then use equation (53) to compute a new $B_{k|k-1}$. This would be extremely burdensome and destroy whatever computational advantage Filter QUEST offered.

An approximate way to simulate the treatment of process noise is to modify Filter QUEST so that it becomes a fading memory filter. Thus, we replace the prediction step in QUEST by

$$B_{k|k-1} = \alpha_k \Phi_{k-1} B_{k-1|k-1} , \tag{55}$$

where α_k is a number between zero and one, and which is also a function of k. Clearly, if α is chosen to be zero, then Filter QUEST will have no memory at all. If the data consists of a sequence of frames each containing several simultaneous vector measurements, then choosing $\alpha_k = 0$ at the end of each frame and $\alpha_k = 1$ otherwise will produce a sequence of single-frame QUEST estimates. Choosing $\alpha_k = 1$ for all k corresponds to infinite memory, which would be appropriate for a genuinely noiseless system.

How should one choose α? Clearly, if the accumulated process noise between measurements is generally much smaller than the measurement noise, then it should be expected that Filter QUEST properly adjusted will average several measurements and obtain a much more accurate result than the single-frame estimate. If α_k is adjusted to be too small, then Filter QUEST will take insufficient advantage of the data and the result will be less accurate. Likewise, if α_k is too large, then the Filter will overweight data which has become less accurate due to the accumulation of process noise, and the solution will be less accurate again. Thus, there is generally an optimal choice for α_k.

Let us consider the case where the process noise is equivalent to

$$A_k = e^{[[\mathbf{w}_{k-1}]]} \Phi_{k-1} A_{k-1} , \tag{56}$$

where $\mathbf{w}_k$ is a white sequence with covariance $q I_{3\times3}$. Such a model is characteristic of an idealized laser gyro and would be appropriate if the dynamical information were coming from laser-gyro measurements. Let us consider also that the spacecraft is equipped with three attitude sensors which at each time t_k sense simultaneously unit vectors along each of the coordinate axes, each with an accuracy of σ. In such a case, clearly, we would choose $\alpha_k = 1$ between the unit-vector measurements in each frame and $\alpha_k = \alpha$ after the last measurement in each frame. In this case, an analytical solution is possible for the covariance matrix of the QUEST filter, which in the limit that an infinite number of measurements have been processed turns out to be

$$P^{\mathrm{QF}}_{k|k} = p^{\mathrm{QF}}_{k|k} I_{3\times3} , \tag{57}$$

with

$$p^{\mathrm{QF}}_{k|k} = \frac{\sigma^2}{2} \left[\frac{1-\alpha}{1+\alpha} + \frac{2}{x} \frac{\alpha^2}{1-\alpha^2} \right] , \tag{58}$$

and

$$x \equiv \sigma^2 / q . \tag{59}$$

This function is a minimum for

$$\alpha_{\mathrm{opt}} = \frac{x + 1 - \sqrt{1+2x}}{x} . \tag{60}$$

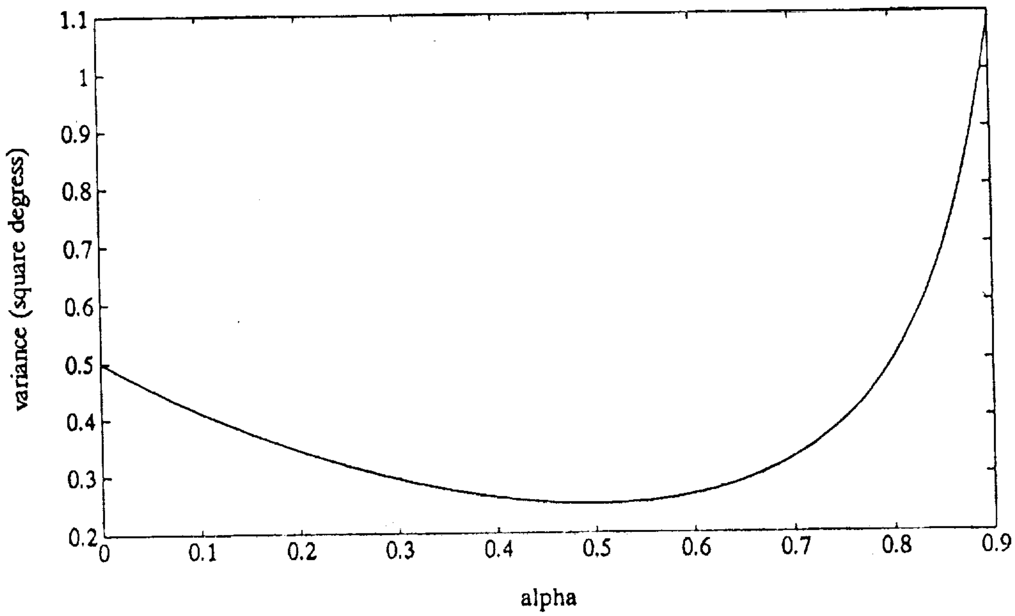

Figure 1. Filter QUEST Covariance as a Function of the Fading Memory Factor for $\sigma^2 \approx q$

If we choose values such as

$$\sigma = 1 \text{ deg}, \quad q = (.5 \text{ deg})^2 , \tag{61}$$

then a plot of $p^{\text{QF}}_{k|k}$ will look like Figure 1, which shows a broad minimum at $\alpha = 0.5$ and a minimum variance of $p^{\text{QF}}_{k|k} = (.5 \text{ deg})^2$. This should be compared with the single-frame result which is

$$p^{\text{single-frame}} = \frac{\sigma^2}{2} = (.707 \text{ deg})^2 . \tag{62}$$

The Filter QUEST solution is not a very large improvement over the single frame solution but not inconsistent with the relatively large gyro noise we have chosen compared to the vector-sensor noise.

The general formula for the minimum Filter QUEST variance for this example as a function of σ and q is

$$p^{\text{QF}}_{\text{min}}(+) = \frac{\sigma^2}{2} \frac{-1 + \sqrt{1 + 2x}}{x} . \tag{63}$$

Thus,

$$p^{\text{QF}}_{\text{min } k|k} \to \frac{\sigma^2}{2} = p^{\text{single-frame}} \quad \text{as} \quad x \to 0 , \tag{64}$$

as expected, and

$$p^{\text{QF}}_{\text{min } k|k} \to \frac{\sigma^2}{2} \sqrt{\frac{2}{x}} \quad \text{as} \quad x \to \infty . \tag{65}$$

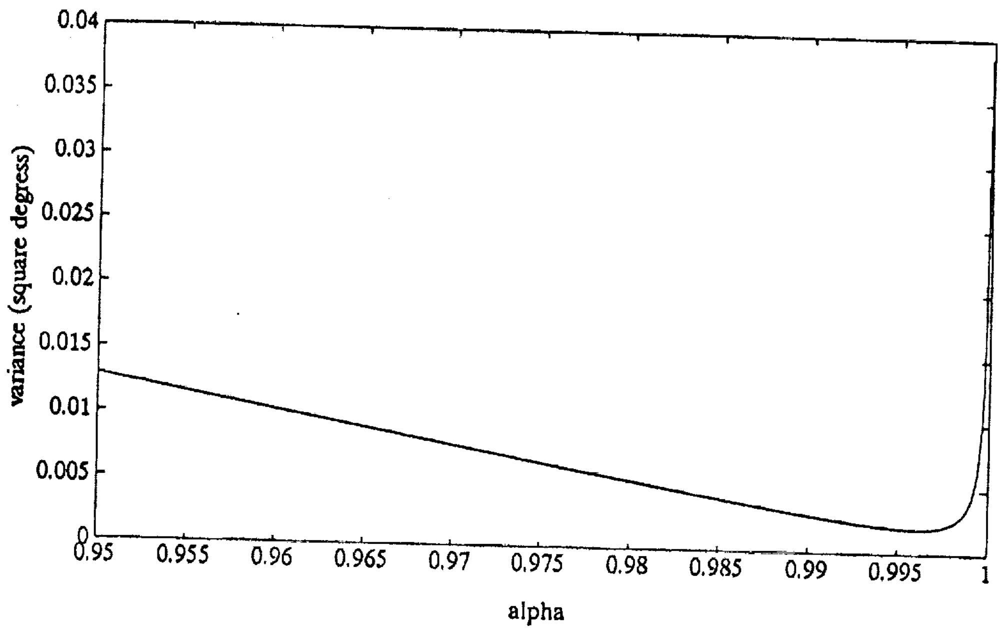

Figure 2. Filter QUEST Covariance as a Function of the Fading Memory Factor for $\sigma^2 >> q$

Equation (65) shows that for x very large, the effective number of previous measurements which Filter QUEST is averaging to reduce the error is

$$N_{\text{eff}} = x/2\,. \tag{66}$$

Note that the dependence of α_{opt} on the measurements is in the limiting cases

$$\alpha_{\text{opt}} \to x \quad \text{as} \quad x \to 0\,, \tag{67}$$

and

$$\alpha_{\text{opt}} \to 1 - \sqrt{\frac{2}{x}} \quad \text{as} \quad x \to \infty\,. \tag{68}$$

Thus, α_{opt} will generally be extremely close to unity for cases where we would generally want to use a filter. For example, if we choose instead of the previous case more physical values such as $\sigma = 1$ deg, $q = (1\text{ arc min})^2$, then $\alpha_{\text{opt}} = .976$ and $p_{k|k}^{\text{QF}} = (.11\text{ deg})^2$, The standard deviation is, thus, almost seven times smaller than the single-frame value. The dependence of the variance on the fading-memory parameter for this case is shown in Figure 2.

Smoother QUEST

The Kalman filter has the disadvantage that only anterior data is used in the estimate. Thus, posterior data, which is equally accurate, is not considered, thereby increasing the covariance by at least a factor of 2 over its achievable value. Also, for early estimates, less data is used leading to a less accurate result than for later estimates.

A Kalman filter/smoother uses both the data which precedes the time of the estimate and the data which follows the time of the estimate. While such an estimator is more accurate, it has the disadvantage that it cannot function in real-time. Thus, a smoother is generally more applicable to background processing on the ground rather than real-time processing on the spacecraft.

The QUEST algorithm also admits a smoother implementation. Suppose we are given measurements $\hat{\mathbf{W}}_k$, $k = 1, \ldots, N$. Then the smoothed attitude profile matrix $B_{k|N}$ at time t_k, $k = 0, \ldots, N$, is given by

$$\begin{aligned} B_{k|N} \quad = \quad & \alpha^k \left(\prod_{i=0}^{k-1} \Phi_i \right) B_{o|o} \\ & + \sum_{i=1}^{k-1} \alpha^{|k-i|} \, \Phi_{k-1} \cdots \Phi_i \, a_i \, \hat{\mathbf{W}}_i \hat{\mathbf{V}}_i^T \\ & + a_k \, \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T \\ & + \sum_{i=k+1}^{N} \alpha^{|k-i|} \, \Phi_k^{-1} \cdots \Phi_{i-1}^{-1} \, a_i \, \hat{\mathbf{W}}_i \hat{\mathbf{V}}_i^T . \end{aligned} \tag{69}$$

The first term in this equation is the contribution of the *a priori* estimate of the attitude. If the smoother were implemented in segments, $B_{o|o}$ would be the attitude profile matrix for the final estimate of the previous segment. The second term gives the predicted contributions of the measurements preceding the current measurement. The third term is the current measurement. The first three terms thus constitute the usual Filter QUEST expression for the attitude profile matrix. The fourth term gives the contribution from the measurements which come after the time of the estimate. The factors of the transition matrices transform the measurements to the body frame at time t_k and the factors of $\alpha^{|k-i|}$ downgrade the data to reflect the ravages of process noise. Equation (70) may be rewritten as

$$B_{k|N} = B_{k|k} + D_k \quad , \quad k = 0, \ldots , N, \tag{70}$$

where $B_{k|k}$ is the "filtered" attitude profile matrix, which satisfies the previous Filter QUEST (forward) recursion relations, and is given by the first three lines of equation (69). D_k is the contribution of the posterior measurements, which is given by the last line of equation (68). By inspection, we see that D_k satisfies a backward recursion relation,

$$D_N = 0 \,, \tag{71}$$

$$D_{k-1} = \alpha \, \Phi_{k-1}^{-1} \left[D_k + a_k \, \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T \right] \,, \tag{72}$$

in complete analogy to the usual Rauch-Tung-Striebel Kalman filter/smoother [10]. The information matrix again is given by equation (52) but with $B_{k|N}$ replacing $B_{k|k}$. Since Φ_k in the present application is orthogonal, the inverse is given by the transpose. Thus, the set of smoothed attitude estimates for an interval of data is obtained with only twice the computational burden of the calculation of the filtered estimates. An obvious drawback, however, is that all of the data, filtered attitude profiles matrices, and attitude transition matrices must be stored. Thus, it is beneficial to process overlapping segments (but whose data length is much greater than $N_{\rm eff}$) in order to keep storage requirements for the processing within reason.

Discussion

Despite its simplicity and obvious power in the above example, Filter QUEST has its drawbacks. First, it only estimates attitude. Thus, in a system in which angular velocity or gyro biases must also be estimated, Filter QUEST will not be sufficient. Also, for systems with poor geometries, say only a single measurement, Filter QUEST's approximation of a single fading-memory factor may be inadequate. Also, Filter QUEST suffers from the short-comings of QUEST, which, if viewed as a maximum-likelihood estimator, effectively assumes the measurement error model given by equations (19)–(21). This is not always the case. However, it is frequently so, and for the most part this model is reasonable, and Filter QUEST offers a useful if limited alternative to the full Kalman filter. In one recent example Filter QUEST has been applied to the COBE mission with encouraging, if not spectacular, results [11].

Acknowledgment

The author is grateful to F. Landis Markley for many interesting discussions and criticisms on this topic and to Orville Filla and Donald Chu for sharing their experience in integrating the algorithm in the COBE software.

References

[1] WAHBA, G., "A Least-Squares Estimate of Satellite Attitude," Problem 65–1, *Siam Review,* Vol. 7, No. 3, July 1965, p. 409.

[2] SHUSTER, M. D., and OH, S. D., "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance, Control and Dynamics,* Vol. 4, No. 1, Jan.–Feb. 1981 pp. 70–77.

[3] J. KEAT, "Analysis of Least Squares Attitude Determination Routine, DOAOP," Computer Sciences Corporation, CSC/TM-77/6034, February 1977.

[4] MARKLEY, F. L., "Attitude Determination Using Vector Measurements and the Singular Value Decomposition," *Journal of the Astronautical Sciences,* Vol. 36, No. 3, January–March 1989, pp. 245–258.

[5] FALLON, L. III, HARROP, I. H., and STURCH, C. R., "Ground Attitude Determination and Gyro Calibration Procedures for the HEAO Mission," *Proceedings, AIAA 17th Aerospace Sciences Meeting,* New Orleans, Louisiana, 1979.

[6] SHUSTER, M. D., "Kalman Filtering of Spacecraft Attitude and the QUEST Model," *Journal of the Astronautical Sciences*, Vol. 38, No. 3, July-September, 1990, pp. 377–393.

[7] SHUSTER, M. D., "Maximum Likelihood Estimation of Spacecraft Attitude," *Journal of the Astronautical Sciences*, Vol. 37, No. 1, January–March 1989, pp. 79–88.

[8] SHUSTER, M. D., "A Simple Kalman Filter and Smoother for Spacecraft Attitude," *Journal of the Astronautical Sciences*, Vol. 37, No. 1, January–March 1989, pp. 89–106.

[9] LEFFERTS, E. J., MARKLEY, F. L., and SHUSTER, M. D., "Kalman Filtering for Spacecraft Attitude Estimation," *Journal of Guidance, Control and Dynamics,* Vol. 5, No. 5, Sept.–Oct. 1982, pp. 417–429.

[10] RAUCH, H. E., TUNG, F., and STRIEBEL, C. T., "Maximum Likelihood Estimation of Linear Dynamic Systems," *AIAA Journal,* Vol. 3, No. 8, pp. 1445–1450, 1965.

[11] FILLA, O., KEAT, J, and CHU, D., "COBE Experience with Filter QUEST," *Proceedings, Flight Mechanics/Estimation Theory Symposium*, NASA Goddard Space Flight Center, Greenbelt, Maryland, May 21–23, 1991.