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VARIATIONAL FORMULATION OF HIGH PERFORMANCE FINITE ELEMENTS: PARAMETRIZED VARIATIONAL PRINCIPLES

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SUMMARY

High performance elements are simple finite elements constructed to deliver engineering accuracy with coarse arbitrary grids. This paper is part of a series on the variational basis of high-performance elements, with emphasis on those constructed with the free formulation (FF) and assumed natural strain (ANS) methods. The present paper studies parametrized variational principles that provide a foundation for the FF and ANS methods, as well as for a combination of both.

1. INTRODUCTION

For 25 years researchers have tried to construct "best" finite element models for problems in structural mechanics. The quest appeared to be nearly over in the late 1960s when higher order displacement elements dominated the headlines. But these elements did not dominate the marketplace. The overwhelming preference of finite element code users has been for simple elements that deliver engineering accuracy with coarse meshes. The search for these "high-performance" (HP) elements began in the early 1970s and by now it represents an important area of finite element research in solid and structural mechanics. Many ingenious schemes have been tried: reduced and selective integration, incompatible modes, mixed and hybrid formulations, stress and strain projections, the free formulation (FF), and the assumed natural strain (ANS) method.

The present paper is part of a series [8-12] that studies how several high performance element construction methods can be embedded within an *extended* variational framework that uses parametrized hybrid functionals. The general plan of attack is sketched in Figure 1. Heavy line boxes are those emphasized in the present paper. The extensions, shown on

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the left, involve parametrization of the conventional elasticity functionals and treatment of element interfaces through generalizations of the hybrid approach of Pian [14-16].

The effective construction of HP elements relies on devices, sometimes derisively called "tricks" or "variational crimes," that do not fit *a priori* in the classical variational framework. The range of tricks range from innocuous collocation and finite difference constraints to more drastic remedies such as selective integration. Despite their unconventional nature, tricks are an essential part of the construction of high-performance elements. They collectively represent a fun-and-games ingredient that keeps the derivation of HP finite elements as a surprisingly enjoyable task.

The present treatment "decriminalizes" kinematic constraint tricks by adjoining Lagrange multipliers, hence placing the ensemble in a proper variational setting. Placing formulations within a variational framework has the great advantage of supplying the *general structure* of the matrices and forcing vectors of high performance elements, and of allowing a systematic derivation of classes of elements by an array of powerful techniques.

Note the reliance of the program of Figure 1 on hybrid functionals. The original 1964 vision of Pian [14] is thus seen to acquire a momentous significance. It is perhaps appropriate to quote here the prediction of another great contributor to finite elements:

> *T. H. H. Pian responded to the problem of plate bending by inventing the "hybrid formulation", which avoids the problem of slope continuity. He assumed that the element responds not according to shape functions but according to element stress fields. These communicate with the outside world via the boundaries .... Hybrid elements can be the most competitive and we believe that the future lie in that direction. However, the formulation is more complicated. Therefore we advocate that researchers should try to cajole their formulation into shape function form, so that users do not have to struggle. In the form, hybrid elements are no more difficult to use than the Iso-P elements ... Unfortunately at the time of writing we have no uniform technique to achieve this.*
>
> *B. Irons and S. Ahmad, Techniques of Finite Elements (1980), p. 159*

Fulfillment of the prophecy appears to be near.

2. THE ELASTICITY PROBLEM

Consider a *linearly elastic body* under static loading that occupies the volume V. The body is bounded by the surface S, which is decomposed into $S : S_d \cup S_t$. Displacements are prescribed on S_d whereas surface tractions are prescribed on S_t. The outward unit normal on S is denoted by $n \equiv n_i$.

The three unknown volume fields are displacements $u \equiv u_i$, infinitesimal strains $e \equiv e_{ij}$, and stresses $\sigma \equiv \sigma_{ij}$. The problem data include: the body force field $b \equiv b_i$ in V, prescribed displacements $\hat{d}$ on S_d, and prescribed surface tractions $\hat{t} \equiv \hat{t}_i$ on S_t.
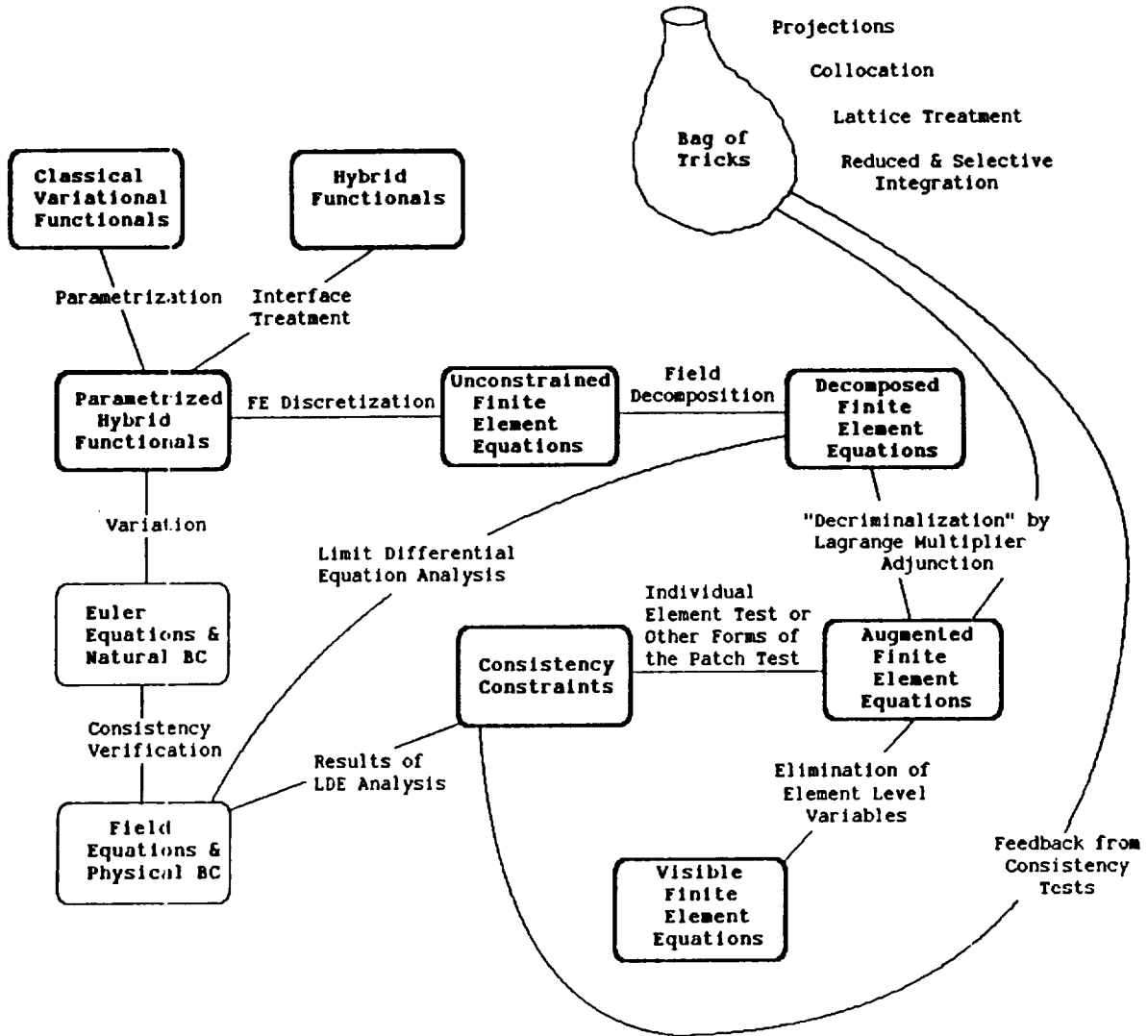
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Figure 1   Program of attack for variational formulation of HP elements

The relations between the volume fields are the strain-displacement equations

$$\mathbf{e} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u}) = \mathbf{Du} \qquad \text{or} \qquad e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } V, \tag{1}$$

the constitutive equations

$$\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e} \qquad \text{or} \qquad \sigma_{ij} = E_{ijkl}e_{kl} \quad \text{in } V, \tag{2}$$

and the equilibrium (balance) equations

$$-\text{div}\,\boldsymbol{\sigma} = \mathbf{D}^{*}\boldsymbol{\sigma} = \mathbf{b} \qquad \text{or} \qquad \sigma_{ij,j} + b_i = 0 \quad \text{in } V, \tag{3}$$

in which $\mathbf{D}^* = -\mathrm{div}$ denotes the adjoint operator of $\mathbf{D} = \frac{1}{2}(\nabla + \nabla^T)$.

The stress vector with respect to a direction defined by the unit vector $\mathbf{v}$ is denoted as $\sigma_v = \sigma.\mathbf{v}$, or $\sigma_{vi} = \sigma_{ij}v_j$. On S the surface-traction stress vector is defined as

$$\sigma_n = \sigma.\mathbf{n}, \qquad \text{or} \qquad \sigma_{ni} = \sigma_{ij}n_j. \tag{4}$$

With this definition the traction boundary conditions may be stated as

$$\sigma_n = \hat{\mathbf{t}} \qquad \text{or} \qquad \sigma_{ij}n_j = \hat{t}_i \qquad \text{on } S_t, \tag{5}$$

and the displacement boundary conditions as

$$\mathbf{u} = \hat{\mathbf{d}} \qquad \text{or} \qquad u_i = \hat{d}_i \qquad \text{on } S_d. \tag{6}$$

3. NOTATION

3.1 Field Dependency

In variational methods of approximation we do not work of course with the exact fields that satisfy the governing equations (1-3,5-6), but with *independent* (primary) fields, which are subject to variations, and *dependent* (secondary, associated, derived) fields, which are not. The approximation is determined by taking variations with respect to the independent fields.

An *independently varied* field will be identified by a superposed tilde, for example $\tilde{\mathbf{u}}$. A dependent field is identified by writing the independent field symbol as superscript. For example, if the displacements are independently varied, the derived strain and stress fields are

$$\mathbf{e}^u = \frac{1}{2}(\nabla + \nabla^T)\tilde{\mathbf{u}} = \mathbf{D}\tilde{\mathbf{u}}, \qquad \sigma^u = \mathbf{E}\mathbf{e}^u = \mathbf{E}\mathbf{D}\tilde{\mathbf{u}}. \tag{7}$$

An advantage of this convention is that $\mathbf{u}$, $\mathbf{e}$ and σ may be reserved for the *exact* fields.

3.2 Integral Abbreviations

Volume and surface integrals will be abbreviated by placing domain-subscripted parentheses and square brackets, respectively, around the integrand. For example:

$$(f)_V \stackrel{\text{def}}{=} \int_V f\, dV, \quad [f]_S \stackrel{\text{def}}{=} \int_S f\, dS, \quad [f]_{S_d} \stackrel{\text{def}}{=} \int_{S_d} f\, dS, \quad [f]_{S_t} \stackrel{\text{def}}{=} \int_{S_t} f\, dS. \tag{8}$$

If $\mathbf{f}$ and $\mathbf{g}$ are vector functions, and $\mathbf{p}$ and $\mathbf{q}$ tensor functions, their inner product over V is denoted in the usual manner

$$(\mathbf{f},\mathbf{g})_V \stackrel{\text{def}}{=} \int_V \mathbf{f}.\mathbf{g}\, dV = \int_V f_i g_i\, dV, \qquad (\mathbf{p},\mathbf{q})_V \stackrel{\text{def}}{=} \int_V \mathbf{p}.\mathbf{q}\, dV = \int_V p_{ij}q_{ij}\, dV, \tag{9}$$

and similarly for surface integrals, in which case square brackets are used.
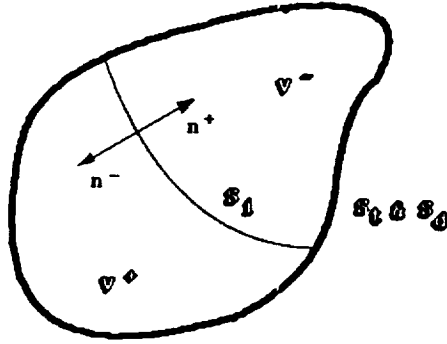
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Figure 2. Internal interface example.

3.3 Domain Assertions

The notation

$$(a = b)_V, \qquad [a = b]_S, \qquad [a = b]_{S_d}, \qquad [a = b]_{S_t}, \tag{10}$$

is used to assert that the relation $a = b$ is valid at each point of V, S, S_d and S_t, respectively.

3.4 Internal Interfaces

In the following subsections we construct *hybrid variational principles* in which boundary displacements d can be varied independently from the internal displacements u. These displacements play the role of Lagrange multipliers that relax internal displacement continuity. Variational principles containing P^d will be called *displacement-generalized*, or d-generalized for short.

The choice of d as independent field is *not* variationally admissible on S_d or S_t. We must therefore extend the definition of boundary to include *internal interfaces* collectively designated as S_i. Thus

$$S : S_d \cup S_t \cup S_i. \tag{11}$$

On S_i neither displacements nor tractions are prescribed. A simple case is illustrated in Figure 2, in which the interface S_i divides V into two subvolumes: V^+ and V^-. An interface such as S_i on Figure 2 has two "sides" called S_i^+ and S_i^-, which identify S_i viewed as boundary of V^+ and V^-, respectively. At smooth points of S_i the unit normals n^+ and n^- point in opposite directions.

The integral abbreviations (8)-(9) generalize as follows, using Figure 2 for definiteness. A volume integral is the sum of integrals over the subvolumes:

$$(f)_V \stackrel{\text{def}}{=} \int_{V^+} f \, dV + \int_{V^-} f \, dV. \tag{12}$$

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An integral over S_i includes two contributions:

$$[g]_{S_i} \stackrel{\text{def}}{=} \int_{S_i^+} g^+ \, dS + \int_{S_i^-} g^- \, dS, \tag{13}$$

where g^+ and g^- denotes the value of the integrand g on S_i^+ and S_i^-, respectively. These two values may be different if g is discontinuous or involves a projection on the normals.

Following a finite element discretization, the union of interelement boundaries becomes S_i.

4. THE ELASTICITY FUNCTIONALS

The variational principles of linear elasticity are based on functionals of the form

$$\Pi = U - P, \tag{14}$$

where U characterizes the internal energy stored in the body volume and P includes other contributions such as work of applied loads and energy stored on internal interfaces. We shall call U the *generalized strain energy* and P the *forcing potential*.

It must be pointed out that all functionals considered here include *independently varied displacements*. Thus, the class of dual functionals such as the complementary energy are *not* included in the following study.

4.1 Volume Integrals

The generalized strain energy has the following structure:

$$U = \tfrac{1}{2}j_{11}(\tilde{\sigma}, e^\sigma)_V + j_{12}(\tilde{\sigma}, \tilde{e})_V + j_{13}(\tilde{\sigma}, e^u)_V + \tfrac{1}{2}j_{22}(\sigma^e, \tilde{e})_V + j_{23}(\sigma^e, e^u)_V + \tfrac{1}{2}j_{33}(\sigma^u, e^u)_V \tag{15}$$

where j_{11} through j_{33} are numerical coefficients. For example, the Hu-Washizu principle is obtained by setting $j_{12} = -1$, $j_{13} = 1$, $j_{22} = 1$, all others being zero. The matrix representation of the general functional (15) and the relations that must exist between the coefficients are studied in §5.1.

4.2 Hybrid Forcing Potentials

Variational principles of linear elasticity are constructed by combining the volume integral (15) with the forcing potential P. Two forms of the forcing potential, called P^d and P^t in the sequel, are of interest in the hybrid treatment of interface discontinuities. The d-generalized (displacement-generalized) forcing potential introduces an independent boundary displacement field $\tilde{d}$ over S_i:

$$P^d(\tilde{u}, \tilde{\sigma}, \tilde{d}) = (b, \tilde{u})_V + [\tilde{\sigma}_n, \tilde{u} - \hat{d}]_{S_d} + [\hat{t}, \tilde{u}]_{S_t} + [\tilde{\sigma}_n, \tilde{u} - \tilde{d}]_{S_i}. \tag{16}$$

The t-generalized (traction generalized) forcing potential introduces an independently varied traction displacement field $\tilde{t}$ over S_i:

$$P^t(\tilde{u}, \tilde{\sigma}, \tilde{t}) = (b, \tilde{u})_V + [\tilde{t}, \tilde{u} - \hat{d}]_{S_d} + [\hat{t}, \tilde{u}]_{S_t} + [\tilde{t}, \tilde{u}]_{S_i}. \tag{17}$$

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The "conventional" form P^c of the forcing potential is obtained if the interface integral vanishes and one sets $[t = \sigma_n]_S$. If so P^t and P^d coalesce into P^c, which retains only two independent fields:

$$P^c(\tilde{u}, \tilde{\sigma}) = (b, \tilde{u})_V + [\tilde{\sigma}_n, \tilde{u} - \hat{d}]_{S_d} + [\hat{t}, \tilde{u}]_{S_t}. \tag{18}$$

4.3 Modified Forcing Potentials

Through various manipulations and assumptions detailed in [10] the forcing potential P^d may be transformed to

$$P^d(\tilde{u}, \tilde{\sigma}, \tilde{d}) = (b, \tilde{u})_V + [\hat{t}, \tilde{d}]_{S_t} + [\tilde{\sigma}_n, \tilde{u} - \tilde{d}]_S. \tag{19}$$

where the all-important surface dislocation integral is taken over S rather than S_i. One of the assumptions is that displacement boundary conditions (6) are exactly satisfied. This expression of P^d is used in the sequel. A similar technique can be used to modify P^t, but that expression will not be required in what follows.

4.4 Complete Functionals

Complete elasticity functionals are obtained by combining the generalized strain energy with one of the forcing potentials. For example, the d and t generalized versions of the Hu-Washizu functional are

$$\Pi_W^d = U_W - P^d, \qquad \Pi_W^t = U_W - P^t. \tag{20}$$

where U_W is obtained by setting $j_{22} = j_{13} = 1$, $j_{12} = -1$, others zero, in (15).

5. MATRIX REPRESENTATION OF ELASTICITY FUNCTIONALS

The generalized strain energy (15) can be presented in matrix form as*

$$U = \tfrac{1}{2} \int_V \langle \tilde{\sigma} \quad \sigma^e \quad \sigma^u \rangle \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ & j_{22} & j_{23} \\ symm & & j_{33} \end{bmatrix} \begin{Bmatrix} e^\sigma \\ \tilde{e} \\ e^u \end{Bmatrix} dV. \tag{21}$$

The symmetric matrix

$$\mathbf{J} = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ & j_{22} & j_{23} \\ symm & & j_{33} \end{bmatrix} \tag{22}$$

characterizes the volume portion of the variational principle. Using the relations $\sigma^e = Ee$, $\sigma^u = ED\tilde{u}$, $e^e = E^{-1}e$, and $e^u = D\tilde{u}$, the above integral may be rewritten in terms of the independent fields as

$$U = \tfrac{1}{2} \int_V \langle \tilde{\sigma} \quad \tilde{e} \quad \tilde{u} \rangle \begin{bmatrix} j_{11}E^{-1} & j_{12}I & j_{13}D \\ j_{12}I & j_{22}E & j_{23}ED \\ j_{13}D^T & j_{23}D^TE & j_{33}D^TED \end{bmatrix} \begin{Bmatrix} \tilde{\sigma} \\ \tilde{e} \\ \tilde{u} \end{Bmatrix} dV. \tag{23}$$

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* To justify the symmetry of $\mathbf{J}$ note, for example, that $j_{13}(\tilde{\sigma}, e^u)_V = \tfrac{1}{2} j_{13}(\tilde{\sigma}, e^u)_V + \tfrac{1}{2} j_{13}(e^\sigma, \sigma^u)_V$, and so on.

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5.1 First Variation of Generalized Strain Energy

The first variation of the volume term (15) may be presented as

$$\delta U = (\Delta e, \delta\tilde{\sigma})_V + (\Delta\sigma, \delta\tilde{e})_V - (\text{div}\,\sigma', \delta\tilde{u})_V + [\sigma'_n, \delta\tilde{u}]_S. \tag{24}$$

where

$$\begin{aligned}
\Delta e &= \dot{j}_{11}e^\sigma + \dot{j}_{12}\tilde{e} + \dot{j}_{13}e^u, \\
\Delta\sigma &= \dot{j}_{12}\tilde{\sigma} + \dot{j}_{22}\sigma^e + \dot{j}_{23}\sigma^u, \\
\sigma' &= \dot{j}_{13}\tilde{\sigma} + \dot{j}_{23}\sigma^e + \dot{j}_{33}\sigma^u.
\end{aligned} \tag{25}$$

The last two terms combine with contributions from the variation of P. For example, if $P = P^c$ the complete variation of $\Pi^c = U - P^c$ is

$$\delta\Pi^c = (\Delta e, \delta\tilde{\sigma})_V + (\Delta\sigma, \delta\tilde{e})_V - (\text{div}\,\sigma' + b, \delta\tilde{u})_V + [\sigma'_n - \hat{t}, \delta\tilde{u}]_{S_t} + -[\tilde{u} - \hat{d}, \delta\tilde{\sigma}_n]_{S_d}. \tag{26}$$

Using P^d or P^t does not change the volume terms. The Euler equations corresponding to P^d and P^t are studied in [10,11] for a more restrictive form of functionals U.

Since the Euler equations associated with the first two terms are $\Delta\sigma = 0$ and $\Delta e = 0$, these quantities may be regarded as deviations from stress-balance and strain-compatibility, respectively. For consistency of the Euler equations with the field equations of §2 we must have $\Delta e = 0$, $\Delta\sigma = 0$ and $\sigma' = \sigma$ if the assumed stress and strain fields reduce to the exact ones. Consequently

$$\begin{aligned}
\dot{j}_{11} + \dot{j}_{12} + \dot{j}_{13} &= 0, \\
\dot{j}_{12} + \dot{j}_{22} + \dot{j}_{23} &= 0, \\
\dot{j}_{13} + \dot{j}_{23} + \dot{j}_{33} &= 1.
\end{aligned} \tag{27}$$

Because of these constraints, the maximum number of independent parameters that define the entries of J is three.

5.2 Specific Functionals

Expressions of J for some classical and parametrized variational principles of elasticity are tabulated below. The subscript of J is used the identify the functionals, which are listed roughly in order of ascending complexity. The fields included in parentheses after the functional name are those subject to independent variations.

Potential energy $(\tilde{u})$:

$$J_P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{28}$$

Stress-displacement Reissner, also called Hellinger-Reissner, $(\tilde{\sigma}, \tilde{u})$:

$$J_R = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{29}$$

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Unnamed stress-displacement functional listed in Oden and Reddy [13] $(\tilde{\sigma}, \tilde{u})$:

$$\mathbf{J}_U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}. \tag{30}$$

Strain-displacement Reissner-type [13] $(\tilde{e}, \tilde{u})$:

$$\mathbf{J}_S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{31}$$

Hu-Washizu $(\tilde{\sigma}, \tilde{e}, \tilde{u})$:

$$\mathbf{J}_W = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{32}$$

One-parameter stress-displacement family $(\tilde{\sigma}, \tilde{u})$ that includes U_P, U_R and U_U as special cases [9,10,11]:

$$\mathbf{J}_\gamma = \begin{bmatrix} -\gamma & 0 & \gamma \\ 0 & 0 & 0 \\ \gamma & 0 & 1-\gamma \end{bmatrix}. \tag{33}$$

One-parameter strain-displacement family $(\tilde{e}, \tilde{u})$ that includes U_P and U_S as special cases [9]:

$$\mathbf{J}_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\beta & \beta \\ 0 & \beta & 1-\beta \end{bmatrix}. \tag{34}$$

Two-parameter strain-displacement family $(\tilde{\sigma}, \tilde{e}, \tilde{u})$ that includes U_β and U_γ as special cases [9]:

$$\mathbf{J}_{\beta\gamma} = (1-\beta)\mathbf{J}_\gamma + (1-\gamma)\mathbf{J}_\beta - (1-\beta-\gamma)\mathbf{J}_P$$
$$= \begin{bmatrix} -\gamma(1-\beta) & 0 & \gamma(1-\beta) \\ 0 & -\beta(1-\gamma) & \beta(1-\gamma) \\ \gamma(1-\beta) & \beta(1-\gamma) & 1-\beta-\gamma+2\beta\gamma \end{bmatrix}. \tag{35}$$

Three-parameter (α, β, γ) family $(\tilde{\sigma}, \tilde{e}, \tilde{u})$ that includes U_W and $U_{\beta\gamma}$ as special cases [9]:

$$\mathbf{J}_{\alpha\beta\gamma} = \alpha\mathbf{J}_W + (1-\alpha)\mathbf{J}_{\beta\sigma}$$
$$= \begin{bmatrix} -\gamma(1-\beta)(1-\alpha) & -\alpha & \alpha+\gamma(1-\beta)(1-\alpha) \\ -\alpha & \alpha-\beta(1-\gamma)(1-\alpha) & \beta(1-\gamma)(1-\alpha) \\ \alpha+\gamma(1-\beta)(1-\alpha) & \beta(1-\gamma)(1-\alpha) & (1-\beta-\gamma+2\beta\gamma)(1-\alpha) \end{bmatrix}. \tag{36}$$

The last form, which contains three independent parameters, supplies all matrices $\mathbf{J}$ that satisfy the constraints (21). It yields stress-displacement functionals for $\alpha = \beta = 0$, strain displacement functionals for $\alpha = \gamma = 0$, and 3-field functionals otherwise. A graphic representation of $\mathbf{J}_{\alpha\beta\gamma}$ in (α, β, γ) space is given in Figure 3.
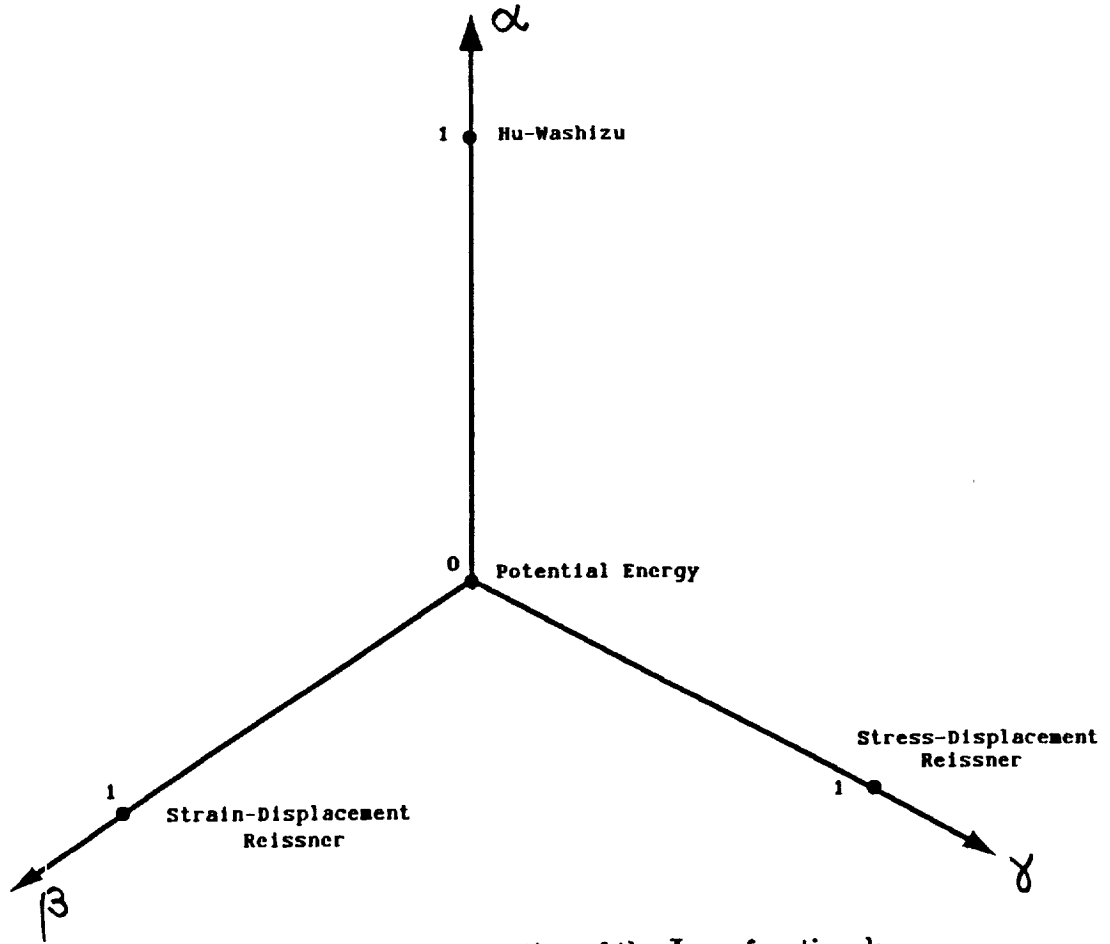
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Figure 3   Graphical representation of the $\mathbf{J}_{\alpha\beta\gamma}$ functionals

5.3 Energy Balancing

A prime motivation for introducing the j coefficients as free parameters is optimization of finite element performance. The determination of "best" parameters for specific elements relies on the concept of energy balance. Let $\mathcal{U}(\epsilon) = \frac{1}{2}(\mathbf{E}\epsilon, \epsilon)_V$ denote the strain energy associated with the strain field ϵ. If $\mathbf{E}$ is positive definite, $\mathcal{U}(\epsilon)$ is nonnegative. We may decompose the generalized strain energy into the following sum of strain energies:

$$U = j_{33}\mathcal{U}(\mathbf{e}^u) + c_1\mathcal{U}(\mathbf{e}^\sigma - \tilde{\mathbf{e}}) + c_2\mathcal{U}(\tilde{\mathbf{e}} - \mathbf{e}^u) + c_3\mathcal{U}(\mathbf{e}^u - \mathbf{e}^\sigma), \tag{37}$$

where $\mathcal{U}_P(\mathbf{e}^u) = U_P$ is the usual strain energy, $c_1 = \frac{1}{2}(j_{11} + j_{22} - j_{33} + 1)$, $c_2 = \frac{1}{2}(-j_{11} + j_{22} + j_{33} - 1)$, and $c_3 = \frac{1}{2}(j_{11} - j_{22} + j_{33} - 1)$. Equation (37) is equivalent to decomposing $\mathbf{J}$ into the sum of four rank-one matrices:

$$\mathbf{J} = j_{33}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + c_1\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \tag{38}$$

Decompositions of this nature can be used to derive energy balanced finite elements by considering element "patches" under simple load systems. This technique is discussed for the one-parameter functionals generated by (34) in [5,7,8].

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6. FINITE ELEMENT DISCRETIZATION

In this section assumptions invoked in the finite element discretization of the functional Π^d for arbitrary J are stated. Following usual practice in finite element work, the components of stresses and strains are arranged as one-dimensional arrays whereas the elastic moduli in E are arranged as a square symmetric matrix. In the sequel we shall consider an *individual element* of volume V and surface $S : S_t \cup S_d \cup S_i$, where S_i is the portion of the boundary in common with other elements.

6.1 Boundary Displacement Assumption

The boundary displacement assumption is

$$[\tilde{d} = N_d v]_S. \tag{39}$$

Here matrix N_d collects the boundary shape functions for the boundary displacement d whereas vector v collects the degrees of freedom of the element, also called the *connectors*. These boundary displacements must be unique on common element boundaries. This condition is verified if the displacement of the common boundary portion is uniquely specified by degrees of freedom located on that boundary. There are no derived fields associated with $\tilde{d}$.

6.2 Internal Displacement Assumption

The displacement assumption in the interior of the element is

$$(\tilde{u} = N_u q)_V, \tag{40}$$

where matrix N_u collects the internal displacement shape functions and vector q collects generalized coordinates for the internal displacements. The assumed $\tilde{u}$ need not be continuous across interelement boundaries.

The displacement derived fields are

$$(e^u = DNq = Bq)_V, \quad (\sigma^u = EBq)_V. \tag{41}$$

To link up with the FF and ANS formulations, we proceed to break up the internal displacement field as follows. The assumed $\tilde{u}$ is decomposed into rigid body, constant strain, and higher order displacements:

$$\tilde{u} = N_r q_r + N_c q_c + N_h q_h. \tag{42}$$

Applying the strain operator $D = \frac{1}{2}(\nabla + \nabla^T)$ to $\tilde{u}$ we get the associated strain field:

$$e^u = DN_r q_r + DN_c q_c + DN_h q_h = B_r q_r + B_c q_c + B_h q_h. \tag{43}$$

But $B_r = DN_r$ vanishes because N_r contains only rigid-body modes. We are also free to select $B_c = DN_c$ to be the identity matrix I if the generalized coordinates q_c are identified with the mean (volume-averaged) strain values $\bar{e}^u$. Consequently (44) simplifies to

$$e^u = \bar{e}^u + e_h^u = \bar{e}^u + B_h q_h, \tag{44}$$

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in which

$$q_c \equiv \bar{e}^u = (e^u)_V / v, \qquad (\mathbf{B}_h)_V = 0. \tag{45}$$

where $v = (1)_V$ is the element volume measure. The second relation is obtained by integrating (44) over V and noting that q_h is arbitrary. It says that the mean value of the higher-order displacement-derived strains is zero over the element.

6.3 Stress Assumption

The stress field will be assumed to be *constant* over the element:

$$(\tilde{\sigma} = \bar{\sigma})_V. \tag{46}$$

This assumption is sufficient to construct high-performance elements based on the free formulation [1-10]. Higher order stress variations are computationally effective if they are divergence free [10] but such a requirement makes extension to geometrically nonlinear problems difficult. The only derived field is

$$(\bar{e}^\sigma = \mathbf{E}^{-1}\bar{\sigma})_V \tag{47}$$

6.4 Strain Assumptions

The assumed strain field $\tilde{e}$ is decomposed into a mean constant strain $\bar{e}$ and a higher order variation:

$$(\tilde{e} = \bar{e} + \mathbf{A}\mathbf{a})_V. \tag{48}$$

where $\bar{e} = (\tilde{e})_V / v$, $\mathbf{A}$ collects higher order strain modes with mean zero value over the element:

$$(\mathbf{A})_V = 0, \tag{49}$$

and $\mathbf{a}$ collects the corresponding strain parameters. The only derived field is

$$(\sigma^\epsilon = \mathbf{E}\tilde{e} = \mathbf{E}\bar{e} + \mathbf{E}\mathbf{A}\mathbf{a})_V. \tag{50}$$

7. UNCONSTRAINED FINITE ELEMENT EQUATIONS

For simplicity we shall assume that all elastic moduli in $\mathbf{E}$ are *constant* over the element. Inserting the above assumptions into Π^d with the forcing potential (19), we obtain a quadratic algebraic form, which is fairly sparse on account of the conditions (45) and (49). Making this form stationary yields the finite element equations

$$\begin{bmatrix} j_{11}v\mathbf{E}^{-1} & j_{12}v\mathbf{I} & 0 & -\mathbf{P}_r^T & j_{13}v\mathbf{I}-\mathbf{P}_u^T & -\mathbf{P}_h^T & \mathbf{L}^T \\ j_{12}v\mathbf{I} & j_{22}v\mathbf{E} & 0 & 0 & j_{23}v\mathbf{I} & 0 & 0 \\ 0 & 0 & j_{22}\mathbf{C}_h & 0 & 0 & j_{23}\mathbf{R}^T & 0 \\ -\mathbf{P}_r & 0 & 0 & 0 & 0 & 0 & 0 \\ j_{13}v\mathbf{I}-\mathbf{P}_u & 0 & 0 & 0 & j_{33}v\mathbf{E} & 0 & 0 \\ -\mathbf{P}_h & 0 & j_{13}\mathbf{R} & 0 & 0 & j_{33}\mathbf{K}_{qh} & 0 \\ \mathbf{L} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\sigma} \\ \bar{e} \\ \mathbf{a} \\ q_r \\ \bar{e}^u \\ q_h \\ \mathbf{v} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f_{qr} \\ f_{qu} \\ f_{qh} \\ f_v \end{Bmatrix}. \tag{51}$$

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where

$$\mathbf{K}_{qh} = (\mathbf{B}_h^T \mathbf{E} \mathbf{B}_h)_V = \mathbf{K}_{qh}^T, \quad \mathbf{C}_h = (\mathbf{A}^T \mathbf{E} \mathbf{A})_V = \mathbf{C}^T, \quad \mathbf{R} = (\mathbf{B}_h^T \mathbf{E} \mathbf{A})_V,$$

$$\mathbf{L} = [\mathbf{N}_{dn}^T]_S, \quad \mathbf{P}_r = [\mathbf{N}_{rn}^T]_S, \quad \mathbf{P}_c = [\mathbf{N}_{cn}^T]_S, \quad \mathbf{P}_h = [\mathbf{N}_{hn}^T]_S, \tag{52}$$

$$\mathbf{f}_r = (\mathbf{N}_r^T \mathbf{b})_V, \quad \mathbf{f}_q = (\mathbf{N}_q^T \mathbf{b})_V, \quad \mathbf{f}_h = (\mathbf{N}_h^T \mathbf{b})_V, \quad \mathbf{f}_v = [\mathbf{N}_d^T \hat{\mathbf{t}}]_{S_t}.$$

in which $\mathbf{N}_{dn}$ denotes the projection of shape functions $\mathbf{N}_d$ on the exterior normal $\mathbf{n}$, and similarly for $\mathbf{N}_r$, $\mathbf{N}_c$ and $\mathbf{N}_h$. Coefficient matrix entries that do not depend on the j's come from the last boundary term in (19).

7.1 The P matrices

Application of the divergence theorem to the work of the mean stress on $\mathbf{e}^u$ yields

$$(\bar{\sigma}, \mathbf{e}^u)_V = (\bar{\sigma}, \bar{\mathbf{e}}^u + \mathbf{B}_h \mathbf{q}_h)_V = v\bar{\sigma}^T \bar{\mathbf{e}}^u + \bar{\sigma}^T (\mathbf{B}_h)_V \mathbf{q}_h = v\bar{\sigma}^T \bar{\mathbf{e}}^u$$
$$= [\bar{\sigma}_n, \hat{\mathbf{u}}]_S = [\bar{\sigma}_n, \mathbf{N}_r \mathbf{q}_r + \mathbf{N}_c \bar{\mathbf{e}}^u + \mathbf{N}_h \mathbf{q}_h]_S = \bar{\sigma}^T (\mathbf{P}_r \mathbf{q}_r + \mathbf{P}_c \bar{\mathbf{e}}^u + \mathbf{P}_h \mathbf{q}_h). \tag{53}$$

Hence $\mathbf{P}_r = 0$, $\mathbf{P}_c = v\mathbf{I}$, $\mathbf{P}_h = 0$, and the element equations simplify to

$$
\begin{bmatrix}
j_{11}v\mathbf{E}^{-1} & j_{12}v\mathbf{I} & 0 & 0 & (j_{13}-1)v\mathbf{I} & 0 & \mathbf{L}^T \\
j_{12}v\mathbf{I} & j_{22}v\mathbf{E} & 0 & 0 & j_{23}v\mathbf{I} & 0 & 0 \\
0 & 0 & j_{22}\mathbf{C}_h & 0 & 0 & j_{23}\mathbf{R}^T & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(j_{13}-1)v\mathbf{I} & j_{23}v\mathbf{I} & 0 & 0 & j_{33}v\mathbf{E} & 0 & 0 \\
0 & 0 & j_{23}\mathbf{R} & 0 & 0 & j_{33}\mathbf{K}_{qh} & 0 \\
\mathbf{L} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\bar{\sigma} \\ \bar{\mathbf{e}} \\ \mathbf{a} \\ \mathbf{q}_r \\ \bar{\mathbf{e}}^u \\ \mathbf{q}_h \\ \mathbf{v}
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\ 0 \\ 0 \\ \mathbf{f}_{qr} \\ \mathbf{f}_{qu} \\ \mathbf{f}_{qh} \\ \mathbf{f}_v
\end{Bmatrix}. \tag{54}
$$

The simplicity of the $\mathbf{P}$ matrices comes from the mean-plus-deviator expression (44) for $\mathbf{e}^u$. If this decomposition is not enforced, $\mathbf{P}_r = 0$ but $\mathbf{P}_c = (\mathbf{B}_c)_V$ and $\mathbf{P}_h = (\mathbf{B}_h)_V$.

8. KINEMATIC CONSTRAINTS

The "tricks" we shall consider here are *kinematic constraints* that play a key role in the development of high-performance FF and ANS elements. These are matrix relations between kinematic quantities that are established *independently* of the variational equations. Two types of relations will be studied.

8.1 Constraints Between Internal and Boundary Displacements

Relations linking the generalized coordinates $\mathbf{q}$ and the nodal connectors $\mathbf{v}$ were introduced by Bergan and coworkers in conjunction with the free formulation (FF) of finite elements [2-3]. For simplicity we shall assume that the number of freedoms in $\mathbf{v}$ and $\mathbf{q}$ is the same; removal of this restriction is discussed in [10]. By collocation of $\mathbf{u}$ at the element node points one easily establishes the relation

$$\mathbf{v} = \mathbf{G}_r \mathbf{q}_r + \mathbf{G}_c \mathbf{q}_c + \mathbf{G}_h \mathbf{q}_h = \mathbf{G}\mathbf{q}, \tag{55}$$

where $\mathbf{G}$ is a square transformation matrix that will be assumed to be nonsingular. On inverting this relation we obtain

$$\mathbf{q} = \mathbf{G}^{-1} = \mathbf{H}\mathbf{v}, \qquad \text{or} \qquad \mathbf{q} = \left\{ \begin{array}{c} \mathbf{q}_r \\ \bar{\mathbf{e}}^u \\ \mathbf{q}_h \end{array} \right\} = \left[\begin{array}{c} \mathbf{H}_r \\ \mathbf{H}_c \\ \mathbf{H}_h \end{array} \right] \mathbf{v}. \tag{56}$$

The following relations between $\mathbf{L}$ and the above submatrices hold as a consequence of the individual element test performed in §9.3:

$$\mathbf{L}^T \mathbf{G}_r = 0, \qquad \mathbf{L}^T \mathbf{G}_c = v\mathbf{I}, \qquad v\mathbf{H}_c = \mathbf{L}^T. \tag{57}$$

If the decomposition (44) is not enforced, the last two should read $\mathbf{L}^T \mathbf{G}_c = v\mathbf{B}_c$, a relation first stated in [3], and $\mathbf{P}_c \mathbf{H}_c + \mathbf{P}_h \mathbf{H}_h = \mathbf{L}^T$.

8.2 Constraints Between Assumed Higher Order Strains and Boundary Displacements

Constraints linking $\tilde{\mathbf{e}}_h$ to $\mathbf{v}$ are of fundamental importance in the assumed natural strain (ANS) formulation. The effect of these constraints in a variational framework is analyzed in some detail in [11-12]. Here we shall simply postulate the following relation between higher order strains and nodal displacements:

$$\mathbf{a} = \mathbf{Q}\mathbf{v}. \tag{58}$$

where $\mathbf{Q}$ is generally a rectangular matrix determined by collocation and/or interpolation. The individual element test in §9.3 requires that $\mathbf{Q}$ be orthogonal to $\mathbf{G}_r$ and $\mathbf{G}_c$:

$$\mathbf{Q}\mathbf{G}_r = 0, \qquad \mathbf{Q}\mathbf{G}_c = 0. \tag{59}$$

The constraint (58) still leaves the independently varied mean strain $\bar{\mathbf{e}}$ to be determined variationally.

9. VISIBLE STIFFNESS EQUATIONS

Enforcing the constraints $\mathbf{a} = \mathbf{Q}\mathbf{v}$, $\mathbf{q}_r = \mathbf{H}_r\mathbf{v}$, $\mathbf{q}_c = \mathbf{H}_c\mathbf{v} = v^{-1}\mathbf{L}^T\mathbf{v}$, $\mathbf{q}_h = \mathbf{H}_h\mathbf{v}$, through Lagrange multiplier vectors λ_a, λ_r, λ_c, and λ_h, respectively, we get the augmented finite element equations

$$\left[\begin{array}{ccccccccccc} j_{11}v\mathbf{E}^{-1} & j_{12}v\mathbf{I} & 0 & 0 & (j_{13}-1)v\mathbf{I} & 0 & 0 & 0 & 0 & 0 & \mathbf{L}^T \\ j_{12}v\mathbf{I} & j_{22}v\mathbf{E} & 0 & 0 & j_{23}v\mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j_{22}\mathbf{C}_h & 0 & 0 & j_{23}\mathbf{R}^T & -\mathbf{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{I} & 0 & 0 & 0 \\ (j_{13}-1)v\mathbf{I} & j_{23}v\mathbf{I} & 0 & 0 & j_{33}v\mathbf{E} & 0 & 0 & 0 & -\mathbf{I} & 0 & 0 \\ 0 & 0 & j_{23}\mathbf{R} & 0 & 0 & j_{33}\mathbf{K}_{qh} & 0 & 0 & 0 & -\mathbf{I} & 0 \\ 0 & 0 & -\mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{Q} \\ 0 & 0 & 0 & -\mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{H}_r \\ 0 & 0 & 0 & 0 & -\mathbf{I} & 0 & 0 & 0 & 0 & 0 & v^{-1}\mathbf{L}^T \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{I} & 0 & 0 & 0 & 0 & \mathbf{H}_h \\ \mathbf{L} & 0 & 0 & 0 & 0 & 0 & \mathbf{Q}^T & \mathbf{H}_r^T & v^{-1}\mathbf{L} & \mathbf{H}_h^T & 0 \end{array} \right] \left\{ \begin{array}{c} \bar{\sigma} \\ \bar{\mathbf{e}} \\ \mathbf{a} \\ \mathbf{q}_r \\ \bar{\mathbf{e}}^u \\ \mathbf{q}_h \\ \lambda_a \\ \lambda_r \\ \lambda_c \\ \lambda_h \\ \mathbf{v} \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ \mathbf{f}_{qr} \\ \mathbf{f}_{qu} \\ \mathbf{f}_{qh} \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathbf{f}_v \end{array} \right\}. \tag{60}$$

Condensation of all degrees of freedom except v yields the visible * element stiffness equations

$$\mathbf{Kv} = (\mathbf{K}_b + \mathbf{K}_h)\mathbf{v} = \mathbf{f} \qquad (61)$$

where

$$\mathbf{K}_b = v^{-1}\mathbf{LEL}^T, \qquad (62)$$

$$\mathbf{K}_h = j_{33}\mathbf{H}_h^T\mathbf{K}_{qh}\mathbf{H}_h + j_{23}(\mathbf{H}_h^T\mathbf{RQ} + \mathbf{Q}^T\mathbf{R}^T\mathbf{H}_h) + j_{22}\mathbf{Q}^T\mathbf{C}_h\mathbf{Q}, \qquad (63)$$

$$\mathbf{f} = \mathbf{f}_v + \mathbf{H}_r^T\mathbf{f}_{qr} + v^{-1}\mathbf{L}^T\mathbf{f}_{qc} + \mathbf{H}_h^T\mathbf{f}_{qh}. \qquad (64)$$

Adopting the nomenclature of the free formulation [3], we shall call $\mathbf{K}_b$ the *basic stiffness matrix* and $\mathbf{K}_h$ the *higher order stiffness matrix*.

9.1 Relation to Previous HP Element Formulations

If $\mathbf{J} = \mathbf{J}_\gamma$ of (33), $j_{33} = 1 - \gamma$, $j_{22} = j_{23} = 0$, and we recover the scaled *free formulation* stiffness equations studied in [5,7,9,10]:

$$\mathbf{K}_h = (1 - \gamma)\,\mathbf{H}_h^T\mathbf{K}_{qh}\mathbf{H}_h. \qquad (65)$$

If we take $\mathbf{J} = \mathbf{J}_W$ of (32), $j_{22} = 1$, $j_{33} = j_{23} = 0$ and we obtain

$$\mathbf{K}_h = \mathbf{Q}^T\mathbf{C}_h\mathbf{Q}. \qquad (66)$$

This is similar to the stiffness produced by the ANS hybrid variational formulation studied in [11-12], in which the potential P^t was used instead of P^d.

But the term with coefficient j_{23} in (63) is new. It may be viewed as coupling the FF and ANS formulations. It is not known at this time whether (61-64) represents the most general structure of the visible stiffness equations of HP elements.

9.2 Recovery of Element Fields

For simplicity suppose that the *body forces vanish* and so do $\mathbf{f}_{qr}$, $\mathbf{f}_{qc}$ and $\mathbf{f}_{qh}$. If $\mathbf{v}$ is known following a finite element solution of the assembled system, solving the equations (60) for the internal degrees of freedom yields

$$\bar{\mathbf{e}} = v^{-1}\mathbf{L}^T\mathbf{v}, \quad \bar{\sigma} = \mathbf{E}\bar{\mathbf{e}}, \quad \mathbf{a} = \mathbf{Qv}, \quad \mathbf{q}_r = \mathbf{H}_r\mathbf{v}, \quad \bar{\mathbf{e}}^u = \bar{\mathbf{e}}, \quad \mathbf{q}_h = \mathbf{H}_h\mathbf{v},$$
$$\lambda_a = (j_{22}\mathbf{C}_h\mathbf{Q} + j_{33}\mathbf{R}^T\mathbf{H}_h)\mathbf{v}, \quad \lambda_r = 0, \quad \lambda_c = 0, \quad \lambda_h = (j_{23}\mathbf{RQ} + j_{33}\mathbf{K}_{qh}\mathbf{H}_h)\mathbf{v}. \qquad (67)$$

It is seen that the *mean* strains $\bar{\mathbf{e}}$, $\bar{\mathbf{e}}^u$ and $\bar{\mathbf{e}}^\sigma = \mathbf{E}^{-1}\bar{\sigma}$ agree, and so would the mean stresses. This is not the case, however, if the body forces are not zero. It is also worthwhile to mention that a nonzero Lagrange multiplier vector flags a deviation of the associated fields from the *variationally consistent* fields that would result on using the unconstrained FE equations (54) without "tricks".

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* The qualifier *visible* emphasizes that these are the stiffness equations other elements "see", and consequently are the only ones that matter insofar as computer implementation on a displacement-based finite element program.

9.9 The Individual Element Test

To conclude the paper, we investigate the conditions under which HP elements based on the foregoing general formulation pass the individual element test of Bergan and Hanssen [1-3]. To carry out the test, assume that the "free floating" element* under zero body forces is in a *constant stress state* σ_0, which of course is also the mean stress. Insert the following data in the left-hand side vector of (60):

$$\bar{\sigma} = \sigma_0 = \bar{\sigma}^u, \quad \bar{e} = E^{-1}\sigma_0, \quad a_h = 0, \quad q_r = arbitrary, \quad e^u = \bar{e}^u = E^{-1}\bar{\sigma}_0, \quad q_h = 0,$$

$$\lambda_a = 0, \quad \lambda_r = 0, \quad \lambda_c = 0, \quad \lambda_h = 0, \quad v = G_r q_r + G_c \bar{e}^u = G_r q_r + G_c E^{-1}\sigma_0.$$

(68)

Premultiply by the coefficient matrix, and demand that all terms on the right-hand side vanish but for $f_v = L\sigma_0$. Then the orthogonality conditions in (57) and (59) emerge. This form of the patch test is very strong, and it may well be that relaxing circumstances can be found for specific problems such as shells.

10. CONCLUSIONS

The results of the present paper may be summarized as follows.

1. The classical variational principles of linear elasticity may be embedded in a parametrized matrix form.

2. The elasticity principles with assumed displacements are members of a three-parameter family.

3. Finite element assumptions for constructing high-performance elements may be conveniently investigated on this family.

4. Kinematic constraints established outside the realm of the variational principle may be incorporated through Lagrange multiplier adjunction.

5. The FF and ANS methods for constructing HP finite elements may be presented within this variational setting. In addition, combined forms emerge naturally from the general parametrized principle.

6. The satisfaction of the individual element test yields various orthogonality conditions that the kinematic constraints should satisfy *a priori*.

The construction of high performance elements based on a weighted mix of FF and ANS "ingredients" will be examined in sequel papers, and specific examples given to convey the power and flexibility of the present methods.

Acknowledgements

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* Mathematically, the entire element boundary is traction-specified, *i.e.*, $S \equiv S_t$.

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13. ABSTRACT (Maximum 200 words)

High performance elements are simple finite elements constructed to deliver engineering accuracy with coarse arbitrary grids. This paper is part of a series on the variational basis of high-performance elements, with emphasis on those constructed with the free formulation (FF) and assumed natural strain (ANS) methods. The present paper studies parametrized variational principles that provide a foundation for the FF and ANS methods, as well as for a combination of both.

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