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COLLEGE OF ENGINEERING & TECHNOLOGY
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NORFOLK, VIRGINIA 23529

INTEGRATED ADAPTIVE FILTERING AND DESIGN FOR CONTROL EXPERIMENTS OF FLEXIBLE STRUCTURES

By

Jen-Kuang Huang, Principal Investigator

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Final Report
For the period ended December 31, 1991

Prepared for
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23665

Under
Research Grant NAG-1-830
Dr. Jer-Nan Juang, Technical Monitor
SDYD-Spacecraft Dynamics Branch

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By

Jen-Kuang Huang*

SUMMARY

Attached is an article entitled, "Identification of Linear Stochastic Systems through Projection Filters" which summarizes the final progress on research grant NAG-1-830. Results of the other research works reported in the previous progress reports include:

1. Large Planar Maneuvers for Articulated Flexible Manipulators and Lyapunov-Based Control Designs for Flexible-Link Manipulators, progress report, 1988. Also published in the Proceedings of AIAA GNC Conference, 1988, pp. 556-570 and AIAA SDM Conference 1989, pp. 497-506, respectively.

2. Rapid Rotational/Transitional Maneuvering Experiments of a Flexible Steel Beam, progress report, 1989. Also published in the Proceedings of American Control Conference, 1989, pp. 1403-1408.

3. Integrated System Identification and State Estimation of Large Flexible Structures, progress report, 1990. Also published in the Proceedings of AIAA GNC Conference, 1990, pp. 1396-1404.

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Identification of Linear Stochastic Systems Through Projection Filters

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Abstract

This paper presents a novel method of identifying a state space model and a state estimator for linear stochastic systems from input and output data. The method is primarily based on the relations between the state space model and the finite difference model for linear stochastic systems derived through projection filters. This paper proves that least-squares identification of a finite difference model converges to the model derived from the projection filters. System pulse response samples are computed from the coefficients of the finite difference model. In estimating the corresponding state estimator gain, a z-domain method is used. First the deterministic component of the output is subtracted out, and then the state estimator gain is obtained by whitening the remaining signal. Experimental example is used to illustrate the feasibility of the method.

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Introduction

System identification, sometimes also called system modelling, deals with the problem of building mathematical models of dynamical systems of interest based on their input/output data. This technique is important in many disciplines such as economics, communication, system dynamics and control[1,2]. The mathematical model allows researchers to understand more about the properties of the systems, so that they can explain, predict or control the behaviors of the systems.

In automatic control of dynamical systems, in order to determine appropriate control force the controller design requires mathematical models of the systems. The quality of the model, therefore, will greatly affect the performance of the controller. Though a great variety of system identification methods have been proposed during the last few decades, still identification of systems of large dimension, or systems with noises in both input and output remains a difficult task.[1,2,3]

Because modern control theories are mostly developed based on state space description of systems, the model in state space format is preferred for control purpose. However, because the relation between input/output data and parameters in state space model is non-linear, if system identification chooses a state space model directly, the parameter estimation of the model becomes a non-linear optimization problem, which is difficult to solve in general. Usually iterative numerical methods should be resorted, but convergence and uniqueness of the solution are not guaranteed. On the other hand, if some special difference model which has linear relation between input/output data and parameters is chosen, the parameter estimation is a linear optimization problem, which has unique solution and can be solved analytically. The least-squares methods provide simple and powerful tools for solving linear optimization problems, either recursively or in batch.

Therefore, in general, difference models are easier to identify than state space models. However, for the demand of state space models in control applications, efforts have been made to convert a difference model to a state space model.[4]

Recently, a method is introduced in Refs. 5 and 8 to identify a state space model from a finite difference model, called the AutoRegressive with eXogeneous input (ARX) model, which is derived through Kalman filter theories. However, the requirement of large order causes intensive computation in the embedded least-squares operation. In Ref. 6 a method is derived to obtain a state space model from input/output data using the notion of state observers. This approach can use an ARX model with the order much smaller than that derived through the Kalman filter, but the derivation is based on the *deterministic* approach. In Ref. 7, it has been proved that as the order of the ARX model increase to infinity, the observer identification converges to the Kalman filter identification. However, for a stochastic system and a small order ARX model, to what the least-squares identification of the ARX model will converge in a stochastic sense is not clear.

This paper addresses the above mentioned problem using the stochastic approach. *The approach is primarily based on the relationship between state space models and finite difference models linked through the projection filter.*[8] First, an ARX model is chosen, and then the ordinary least-squares is used to estimate the coefficient matrices. Based on the relationship between the projection filter and state space parameters the system pulse response samples, i.e., the Markov parameters, can be calculated from the coefficients of the identified ARX model. To decompose the Markov parameters into state space parameters, the Eigensystem Realization Algorithm (ERA)[9] is employed. ERA is effective in realizing state space model from system pulse responses,[10–14] which are the Markov parameters for discrete systems.

To compute the state estimator gain a different method is developed in this paper using a z-domain approach in contrast to the time-domain approaches used in Refs. 5 and 7. After identifying a state space model, the deterministic part of the output is subtracted out. The remaining signal represents the stochastic part and can be modeled by a Moving Average (MA) model of which the coefficients are in terms of state space parameters and the Kalman filter gain. Identifying the MA model is approximated by identifying a corresponding AotoRegressive (AR) model first and then inverting it. From the identified MA model a state estimator gain can be calculated. Finally, the identification of a ten-bay structure is used to illustrate the feasibility of the approach.

Relations between Projection Filter and Finite Difference Model of a Linear System

The projection filter is a linear transformation matrix which projects (transforms) a finite number of input/output data of a system into its current state space. The image of the projection is an optimal estimate of the current state, and the filter is chosen such that the mean square estimation error is minimized.[8] Here "filter" is a generic term referring to a data processing procedure which extracts desired information from data. To explain the relation between the projection filter and the finite difference model of a linear system, we start from a simple case and gradually move to more general ones.

Consider a finite-dimensional, linear, discrete-time, time-invariant, noise-free dynamic system, which can be represented by a state space model as

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$

$$y_k = Cx_k + Du_k, \tag{2}$$

where x is an $n \times 1$ state vector, u an $m \times 1$ input vector, and y a $p \times 1$ measurement or output vector. Matrices A, B, C and D are respectively the state matrix, input matrix, output matrix and direct influence matrix. The integer k is the sample indicator.

From Eqs. (1) and (2) it is easy to follow that

$$\begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_{k-q+1} \end{bmatrix} = \begin{bmatrix} C \\ CA^{-1} \\ \vdots \\ CA^{-q+1} \end{bmatrix} x_k - \begin{bmatrix} -D & 0 & \cdots & 0 \\ 0 & CA^{-1}B - D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & CA^{-q+1}B & \cdots & CA^{-1}B - D \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_{k-q+1} \end{bmatrix}, \tag{3}$$

or in short

$$Y_{q,k} = H_q x_k - G_q U_k, \tag{4}$$

or in normal form

$$H_q x_k = Y_{q,k} + G_q U_k, \tag{5}$$

where q denotes the number of data stacked up to form the equation, and the meanings of other notations are self-evident. If the state vector x_k is the variable to be solved, Eq. (5) contains n unknowns and $p \times q$ equations. However, there are only at most n independent equations. Therefore, for a sufficiently large q ($p \times q \geq n$) which can make H_q full-column-ranked, the unique solution of x_k is

$$\hat{x}_k = F_q(Y_{q,k} + G_q U_k), \tag{6}$$

where

$$F_q = (H_q^T H_q)^{-1} H_q^T \tag{7}$$

is the pseudo-inverse of H_q and also the projection filter in this case. If $p \times q = n$ (i.e., H_q is square), F_q becomes H_q^{-1}. In general the number q can be any integer bigger than an integer q_{min} which is the minimum number required to make H_q full-column-ranked. The solution $\hat{x}_k$ is identical with the true value x_k.

To write a difference model of the system which expresses the current output as a linear transformation of finite previous input/output data, one can use Eqs. (1), (2) and (6)

$$
\begin{aligned}
y_k &= Cx_k + Du_k \\
&= CAx_{k-1} + CBu_{k-1} + Du_k \\
&= CA\left[F_q(Y_{q,k-1} + G_qU_{k-1})\right] + CBu_{k-1} + Du_k \\
&= \sum_{i=1}^{q} CAF_{qi}y_{k-i} + Du_k + (CB - CAF_{q1}D)u_{k-1} + \sum_{i=2}^{q} CAF_qG_{qi}u_{k-i} \\
&= \sum_{i=1}^{q} A_iy_{k-i} + \sum_{i=0}^{q} B_iu_{k-i},
\end{aligned} \tag{8}
$$

where F_{qi} ($\in R^{n\times p}$) and G_{qi} ($\in R^{p\times q\times m}$) are the i-th partitions of F_q and G_q, respectively, defined as

$$F_q = \left[F_{q1} \vdots\ F_{q2} \vdots\ \cdots \vdots F_{qq}\right], \qquad G_q = \left[G_{q1} \vdots\ G_{q2} \vdots\ \cdots \vdots G_{qq}\right], \tag{9}$$

and $A_i = CAF_{qi}$, $(i = 1, \cdots, q)$; $B_0 = D$, $B_1 = CB - CAF_{q1}D$, $B_i = CAF_qG_{qi}$, $(i = 2, \cdots, q)$. The model described by Eq. (8) is an ARX model.

Next, consider a system without process noise but with additive, white, gaussian, and zero-mean measurement noise which is not correlated with the state variable, the output equation becomes

$$y_k = Cx_k + Du_k + v_k,$$

where v_k represents the measurement noise. Similarly, we can derive a matrix equation

$$H_qx_k = Y_{q,k} + G_qU_k - V_{q,k},$$

where $V_{q,k}^T = [v_k^T,\ v_{k-1}^T, \cdots,\ v_{k-q+1}^T]$. The unknown variable x_k is still a deterministic variable in this case. By the theory of parameter estimation for deterministic parameters from a linear equation with independent white noise,[16] one can write the optimal estimate of x_k as shown in Eq. (6) with

$$F_q = (H_q^T \bar{R}^{-1} H_q)^{-1} H_q^T \bar{R}^{-1} \tag{10}$$

which is a weighted pseudo-inverse of H_q and $\bar{R} = R \otimes I_q$, $\otimes$ is the Kronecker product, R the covariance of the measurement noise, and I_q the identity matrix of dimension q. The optimality is defined by the minimum variance of state estimation error.

To derive an ARX model using the relation provided by the projection filter is similar to the previous case. We can form a one-step-ahead output prediction using the last estimated state as

$$\hat{y}_k = CA\hat{x}_{k-1} + CBu_{k-1} + Du_k \tag{11}$$

and define

$$y_k = \hat{y}_k + \eta_k \tag{12}$$

where η_k is the prediction error. Therefore,

$$\begin{aligned} y_k &= CA\hat{x}_{k-1} + CBu_{k-1} + Du_k + \eta_k \\ &= CA\left[F_q(Y_{q,k-1} + G_q U_{k-1})\right] + CBu_{k-1} + Du_k + \eta_k \\ &= \sum_{i=1}^{q} CAF_{qi} y_{k-i} + Du_k + (CB - CAF_{q1}D)u_{k-1} + \sum_{i=2}^{q} CAF_q G_{qi} u_{k-i} + \eta_k \\ &= \sum_{i=1}^{q} A_i y_{k-i} + \sum_{i=0}^{q} B_i u_{k-i} + \eta_k, \end{aligned} \tag{13}$$

where F_{qi} and G_{qi} are defined the same way as in Eq. (9), but F_q is defined by Eq. (10) in this case.

Next, consider a more general case for a system with both process and measurement noises. In state space format the system can be modeled as

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{14}$$

$$y_k = Cx_k + Du_k + v_k, \tag{15}$$

where the sequences $\{w_k\}$ and $\{v_k\}$ are the process (input) noise and the measurement (output) noise, respectively. Both are assumed to be gaussian, zero-mean and white with covariance matrices Q and R, respectively. They are also assumed statistically independent of each other.

Similarly, by writing the previous output in terms of the current state using Eqs. (14) and (15), one can derive

$$Y_{q,k} = H_q x_k - G_q U_k - M_q W_{q,k} + V_{q,k}, \tag{16}$$

where

$$M_q = \begin{bmatrix} 0 & \cdots & 0 \\ CA^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ CA^{-q+1} & \cdots & CA^{-1} \end{bmatrix}, \quad W_{q,k} = \begin{bmatrix} w_{k-1} \\ \vdots \\ w_{k-q+1} \end{bmatrix}$$

Equation (16) can be further simplified to be

$$H_q x_k = Y'_{q,k} + \xi_{q,k} \tag{17}$$

where

$$Y'_{q,k} = Y_{q,k} + G_q U_k, \qquad \xi_{q,k} = M_q W_{q,k} - V_{q,k}.$$

Note that the unknown variable x_k is a random variable in this case. The overall noise vector $\xi_{q,k}$ is still gaussian and zero-mean because $W_{q,k}$ and $V_{q,k}$ are gaussian and zero-mean. It is also correlated with the unknown variable x_k because $W_{q,k}$ is correlated with x_k. Denote the covariance between x_k and $\xi_{q,k}$ by $P_{x\xi}$. For a linear equation like Eq. (17), suppose the mean of the current state $\bar{x}_k$ and its variance P_x is given, by the theory of random parameters estimation[8,16] the optimal estimate of x_k can be obtained by

$$\hat{x}_k = \bar{x}_k + F_q(Y'_{q,k} - \bar{Y}'_{q,k}), \tag{18}$$

where the overbar "–" denotes the expectation value,

$$\bar{Y}'_{q,k} = H_q \bar{x}_k,$$

and

$$F_q = (P_x H_q^T + P_{x\xi})(H_q P_x H_q^T + H_q P_{x\xi} + P_{x\xi}^T H_q^T + R_\xi)^{-1} \tag{19}$$

is the projection filter in this case. Matrix R_ξ denotes the covariance of $\xi_{q,k}$. The optimality is under the minimum variance of state estimation error.

Similarly, to derive an ARX model, we can use one-step-ahead output prediction as Eq. (11) and have

$$\begin{aligned}
\hat{y}_k &= CA\hat{x}_{k-1} + CBu_{k-1} + Du_k \\
&= CA\left[\bar{x}_{k-1} + F_q(Y'_{q,k-1} - \bar{Y}'_{q,k-1})\right] + CBu_{k-1} + Du_k \\
&= CAF_q Y_{q,k-1} + CBu_{k-1} + CAF_q G_q U_{k-1} + CA(I_n - F_q H_q)\bar{x}_{k-1} + Du_k \\
&= \sum_{i=1}^{q} CAF_{qi} y_{k-i} + Du_k + (CB - CAF_{q1}D)u_{k-1} \\
&\quad + \sum_{i=2}^{q} CAF_q G_{qi} u_{k-i} + CAL\bar{x}_{k-1},
\end{aligned} \tag{20}$$

where

$$L = I_n - F_q H_q,$$

F_{qi} and G_{qi} are again defined the same way as in Eq. (9) but F_q is defined by Eq. (19) instead.

Equation (20) represents the best prediction of y_k one can make using q previous input/output data. If the prediction is made once and for all, namely, no prediction of previous state is made, the best value assigned to $\bar{x}_k$ is zero. However, if previous state estimation has been carried out, the best choice for $\bar{x}_k$ is the a priori Kalman filter estimation. Note that for the Kalman filter

$$\begin{aligned}
\hat{x}_{k-1}^- &= A\hat{x}_{k-2}^- + AK(y_{k-2} - C\hat{x}_{k-2}^- - Du_{k-2}) + Bu_{k-2} \\
&= \cdots \\
&= \sum_{i=1}^{q-1} \bar{A}^{i-1} AKy_{k-1-i} + \sum_{i=1}^{q-1} \bar{A}^{i-1}(B - AKD)u_{k-1-i} + \bar{A}^q \hat{x}_{k-q},
\end{aligned} \tag{21}$$

where

$$\bar{A} = A(I_n - KC),$$

and K is the optimal steady state Kalman filter gain. Based on the argument above, we can replace $\bar{x}_{k-1}$ in Eq. (20) by Eq. (21) and obtain

$$\begin{aligned}
y_k &= \hat{y}_k + \eta_k \\
&= CAF_{q1}y_{k-1} + \sum_{i=2}^{q} CA\left(F_{qi} + L\bar{A}^{i-2}AK\right)y_{k-i} + Du_k + (CB - CAF_{q1}D)u_{k-1} \\
&\quad + \sum_{i=2}^{q} CA\left(F_qG_{qi} + L\bar{A}^{i-2}(B - AKD)\right)u_{k-i} + \eta'_k \\
&= \sum_{i=1}^{q} A_iy_{k-i} + \sum_{i=0}^{q} B_iu_{k-i} + \eta'_k, \qquad (22)
\end{aligned}$$

where $\eta'_k = \eta_k + C\bar{A}^q\hat{x}_{k-q}$. Note that if q is not large, $\{\eta'_k\}$ is not white. Equations (8), (13) and (22) represent the AutoRegressive with eXogeneous input (ARX) models of linear systems in various different noise situations. The equation in each case provides a best prediction of the output measurement at time k in the sense of minimum state error at time $k-1$ using q previous input and output data.

Least-Squares Identification of ARX Model

A general ARX model of a linear system can be written as

$$y_k = \sum_{i=1}^{q1} A_iy_{k-i} + \sum_{i=0}^{q2} B_iu_{k-i} + \epsilon_k, \qquad (23)$$

where $(q1, q2)$ is the order of the model. Given a set of input and output data $\{y_k, \cdots, y_0, u_k, \cdots, u_0\}$ of the system, we can use the least-squares method to find a set of matrix coefficients $\{\hat{A}_1, \cdots, \hat{A}_{q1}, \hat{B}_0, \cdots, \hat{B}_{q2}\}$ which fits the equation and the data "best" under least-squares error of output prediction sense. The least-squares method for single-input single-output ARX model (a scalar equation) can be found in many text books.[17,18] The extension to multi-input multi-output model is straightforwards.[8]

The ARX models derived in the last section have an order $(q,\ q)$, or just q in short, which are special cases of the general ARX model. It can be proved (see appendix) that if we choose an ARX model of order q and use the least-squares method to identify the parameters of the model, the parameters will converge to that derived from the projection filter.

Obtaining System Markov Parameters from ARX Model

There are some special relations between the system pulse response samples, i.e., the system Markov parameters, and the coefficient matrices of the ARX models derived in the previous section. Based on these relations we can obtain the system Markov parameters from the ARX models.

For noise-free systems, from Eq. (8) if we denote the coefficient matrices of y_{k-j} and u_{k-j} by A_j and B_j, respectively, we can have

$$CA^jB = B_{j+1} + \sum_{i=1}^{j} A_i CA^{j-i}B + A_{j+1}D. \tag{24}$$

Note that this equation can calculate the system Markov parameters CA^jB $(j = 1, \cdots, q-1)$ iteratively from the coefficient matrices of a ARX model of order q (note $B_0 = D$ and $CB = B_1 + A_1D$).

Proof:

By definition

$$G_{q1} = [-D^T,\ 0^T, \cdots,\ 0^T]^T$$

and for $j \geq 2$

$$G_{qj} = \begin{bmatrix} 0 \\ \vdots \\ CA^{-1}B - D \\ \vdots \\ CA^{-q+j-1}B \end{bmatrix} = \begin{bmatrix} C \\ \vdots \\ CA^{-j+2} \\ CA^{-j+1} \\ \vdots \\ CA^{-q+1} \end{bmatrix} A^{j-2}B - \begin{bmatrix} CA^{j-2} \\ \vdots \\ C \\ 0 \\ \vdots \\ 0 \end{bmatrix} B - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ D \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= H_q A^{j-2}B - E^{(j-2)}B - D^{(j)} \tag{25}$$

where

$$E^{(j-2)} = \left[(CA^{j-2})^T, \; \cdots, \; C^T, \; 0^T, \; \cdots 0^T\right]^T,$$

$$D^{(j)} = [0^T, \cdots, 0^T, \; D^T, \; 0^T, \cdots, 0^T]^T,$$

$D^{(j)}$ has D in the j-th block and is zero elsewhere. Therefore, with $j \geq 1$

$$\begin{aligned} CAF_q G_{q(j+1)} &= CAF_q H_q A^{j-1}B - CAF_q E^{(j-1)}B - CAF_{q(j+1)}D \\ &= CAF_q H_q A^{j-1}B - \sum_{i=1}^{j} CAF_{qi} CA^{j-i}B - CAF_{q(j+1)}D \end{aligned} \tag{26}$$

It is noted that Eq. (26) also holds for the system with both process and measurement noises. Now, because

$$F_q H_q = I_n, \tag{27}$$

and from Eqs. (8) and (26), we have

$$B_{j+1} = CAF_q G_{q(j+1)} = CA^j B - \sum_{i=1}^{j} A_i CA^{j-1}B - A_{j+1}D.$$

So, Eq. (24) follows.

Q.E.D.

We can also iteratively calculate CA^jB ($j = q, \; q+1, \; ...$) by

$$\begin{aligned} CA^jB &= CA(F_q H_q)A^{j-1}B \\ &= \sum_{i=1}^{q} A_i CA^{j-i}B. \end{aligned} \tag{28}$$

Though derived from noise-free systems, the above equations (Eqs. (24) to (28)) also hold for systems with additive white measurement noise. Because for systems with white measurement noise the projection filter F_q is nothing but a weighted pseudo-inverse of H_q, hence Eq. (27) also holds.

It is interesting to see that Eq. (24) also holds for systems with both process and measurement noise even though Eq. (27) does not hold in this case. This can be proved as follows.

Proof:

Be definition (see Eq. (22))

$$
\begin{aligned}
&B_{j+1} + \sum_{i=1}^{j} A_i C A^{j-i} B + A_{j+1} D \\
&= CA(F_q G_{q(j+1)} + L\bar{A}^{j-1}(B - AKD)) + CAF_{q1}CA^{j-1}B + CA(F_{q2} + LAK)CA^{j-2}B \\
&\quad + \cdots + CA(F_{qj} + L\bar{A}^{j-2}AK)CB + CA(F_{q(j+1)} + L\bar{A}^{j-1}AK)D \\
&= CAF_q G_{q(j+1)} + CAL\bar{A}^{j-1}(B - AKD) + \sum_{i=1}^{j} CAF_{qi}CA^{j-i}B \\
&\quad + \sum_{i=1}^{j-1} CAL\bar{A}^{i-1}AKCA^{j-i-1}B + CA(F_{q(j+1)} + L\bar{A}^{j-1}AK)D \\
&= CAF_q H_q A^{j-1}B - \sum_{i=1}^{j} CAF_{qi}CA^{j-i}B - CAF_{q(j+1)}D + CAL\bar{A}^{j-1}B \\
&\quad - CAL\bar{A}^{j-1}AKD + \sum_{i=1}^{j} CAF_{qi}CA^{j-i}B + \sum_{i=1}^{j-1} CAL\bar{A}^{i-1}AKCA^{j-i-1}B \\
&\quad + CAF_{q(j+1)}D + CAL\bar{A}^{j-1}AKD \\
&= CAF_q H_q A^{j-1}B + CAL\bar{A}^{j-1}B + CAL\bar{A}^{j-2}AKCB \\
&\quad + \sum_{i=1}^{j-2} CAL\bar{A}^{i-1}AKCA^{j-i-1}B \\
&= CAF_q H_q A^{j-1}B + CAL\bar{A}^{j-2}(\bar{A} + AKC)B + \sum_{i=1}^{j-2} CAL\bar{A}^{i-1}AKCA^{j-i-1}B
\end{aligned}
$$

$$
\begin{aligned}
&= CAF_qH_qA^{j-1}B + CAL\bar{A}^{j-2}AB + \sum_{i=1}^{j-2} CAL\bar{A}^{i-1}AKCA^{j-i-1}B \\
&= \cdots \\
&= CAF_qH_qA^{j-1}B + CAL\bar{A}A^{j-2}B + CALAKCA^{j-2}B \\
&= CAF_qH_qA^{j-1}B + CALA^{j-1}B \\
&= CA(F_qH_q + L)A^{j-1}B \\
&= CA^jB
\end{aligned}
\tag{29}
$$

where the relations $\bar{A} + AKC = A$, $F_qH_q + L = I_n$ and Eq. (26) are used.

Q.E.D.

However, Eq. (28) does not hold for systems with process noise. Hence, for an ARX model of order q, only q terms of the system Markov parameters can be obtained. A system of order n has only n independent Markov parameters; all the rest are the linear combination of these n independent ones. Therefore, the order chosen for the ARX model q should be greater than or equal to n. In general, more Markov parameters can improve the accuracy of the identified state space model in the later procedure; however, a trade-off is that a larger q will increase computational load.

To decompose the identified Markov parameters into state space parameters $[A,\ B,\ C]$, one can use the Eigensystem Realization Algorithm (ERA). ERA is a simple and accurate algorithm for identification of linear systems from pulse response samples. It has been proved valuable for modal state parameter identification from test data.[13,14] The algorithm uses the pulse response samples (i.e., the Markov parameters for a discrete-time system) to form a large block data matrix which is referred to as the general Hankel matrix. Then the technique of singular value decomposition is used to decompose the Hankel matrix. The system order is determined by counting the number of singular values retained. The small singular values are attributed to noises and are truncated. The state space model can be computed from the decomposed matrices. The realized model is not unique, but the Markov parameters are unique. For further details, the readers are referred to Ref. 9.

Identification of a State Estimator Gain

After obtaining a set of state space parameters via the ERA, a corresponding state estimator gain can be estimated. This method is enlightened by the Kalman filter theory.[15] From the Kalman filter formulations, a filter model in innovation form is[19]

$$\hat{x}_{k+1}^{-} = A\hat{x}_k^{-} + Bu_k + AK_k\varepsilon_k \tag{30}$$

$$y_k = C\hat{x}_k^{-} + Du_k + \varepsilon_k \tag{31}$$

where $\hat{x}_k^{-}$ is the optimal prediction made by the Kalman filter based on all the data prior to the moment k, and K_k the Kalman filter gain; the quantity ε_k, called residual, is the difference between true output y_k and predicted output $\hat{y}_k$ $(= C\hat{x}_k^{-})$. In steady state the filter gain is constant and the subscript of it can be omitted. Equations (30) and (31) are called "innovation model", because the quantity ε_k is also called "innovation" because in a sense it contains new information which can not be obtained from previous data. *For an optimal Kalman filter, the sequence* $\{\varepsilon_k\}$ *is white,*[19] which is a useful property.

From the innovation model, the Kalman filter can be viewed as driven by deterministic input u_k through B and by stochastic input ε_k through AK. Hence, the filter state and output can be decomposed into two parts — one caused by the deterministic input and the other caused by the stochastic input. Accordingly, the innovation model can be divided into two models:

$$\hat{x}_{k+1,1}^{-} = A\hat{x}_{k,1}^{-} + Bu_k \tag{32}$$

$$y_{k,1} = C\hat{x}_{k,1}^{-} + Du_k \tag{33}$$

and

$$\hat{x}^-_{k+1,2} = A\hat{x}^-_{k,2} + AK_k\varepsilon_k \tag{34}$$

$$y_{k,2} = C\hat{x}^-_{k,2} + \varepsilon_k \tag{35}$$

where $x^-_k = x^-_{k,1} + x^-_{k,2}$ and $y_k = y_{k,1} + y_{k,2}$. Expanding Eqs. (33) and (35) based on Eqs. (32) and (34), respectively, one can derive

$$y_{k,1} = Du_k + \sum_{i=1}^{k} CA^{i-1}Bu_{k-i}, \tag{36}$$

$$y_{k,2} = \varepsilon_k + \sum_{i=1}^{k} CA^iK\varepsilon_{k-i}. \tag{37}$$

Combining the above two equations, one obtains

$$y_k = Du_k + \sum_{i=1}^{k} CA^{i-1}Bu_{k-i} + \varepsilon_k + \sum_{i=1}^{k} CA^iK\varepsilon_{k-i}. \tag{38}$$

Equation (38) clearly shows the two parts of which the output is composed. If the accurate state space parameters $[A,\ B,\ C,\ D]$ are known, one can subtract the deterministic component $y_{k,1}$ out from the output y_k; that is, by defining $s_k \triangleq y_{k,2} = y_k - y_{k,1}$, then

$$s_k = \varepsilon_k + \sum_{i=1}^{k} CA^iK\varepsilon_{k-i} = \sum_{i=0}^{k} C_i\varepsilon_{k-i}, \tag{39}$$

where $C_0 = I_p,\ C_i = CA^iK$. The remaining signal s_k represents the stochastic component and is driven by the sequence $\{\varepsilon_k\}$. For a stable system all the terms CA^iK, $i > q$, are negligibly small when q is sufficiently large; therefore, when k is large the upper limit of the summation on the right side of Eq. (39) can be replaced by q. Equation (39) describes s_k as a linear transformation of a white sequence $\{\varepsilon_k\}$; therefore, it is called a Moving Average (MA) model.[18,19] The matrices $C_1, \cdots, C_q$ are constants, called the MA parameters. The term moving average arose because s_k can be regarded as a weighted average of $\varepsilon_k, \cdots, \varepsilon_{k-q}$. Note that the MA parameters are expressed in terms of the state space parameters A, C and steady state Kalman filter gain K. If the MA parameters are known one can compute the filter gain from them.

The problem of estimating the MA model in Eq. (39) is that the white sequence $\{\varepsilon_k\}$ is not readily available; therefore, ordinary least-squares methods frequently used estimating the coefficients of linear equations can not be used directly. However, we can estimate a corresponding autoregressive (AR) model first, and then invert it to approximate the original MA model. To highlight this point, taking z-transform of both sides of Eq. (39) to become

$$S = \sum_{i=0}^{q} C_i z^{-i} E = M(z^{-1})E, \tag{40}$$

where $M(z^{-1})$ is a polynomial matrix in z^{-1} (a matrix whose entries are polynomials in z^{-1}). Matrix $M(z^{-1})$ can be regarded as a filter which receives ε_k and its delayed versions as the input and yields s_k as the output. If we can find the inverse filter $N(z^{-1})$ of $M(z^{-1})$ such that $N(z^{-1})M(z^{-1}) = I_p$, by pre-multiplying Eq. (40) with $N(z^{-1})$ we have

$$N(z^{-1})S = E. \tag{41}$$

Matrix $N(z^{-1})$ in general is an infinite-ordered polynomial matrix in z^{-1}. In Eq. (41), $N(z^{-1})$ can be viewed as a whitening filter which receives s_k and its delayed versions as the input and yields white sequence $\{\varepsilon_k\}$ as the output.

To obtain a whitening filter for the signal s_k, we can write a AutoRegressive model of s_k with order r in time domain as

$$\sum_{i=0}^{r} N_i s_{k-i} = \varepsilon_k, \tag{42}$$

where $N_0 = I_p$, and estimate the AR parameters $N_i, \cdots, N_r$. [18] Comparing Eq. (41) with Eq. (42) it can be seen that the infinite-ordered polynomial matrix $N(z^{-1})$ is approximated by a finite-ordered polynomial matrix $\sum_{i=0}^{r} N_i z^{-i}$. The parameter estimation of the AR model can be accomplished using the ordinary least-squares method.

After obtaining an identified $N(z^{-1})$, we can invert it to approximate $M(z^{-1})$. Inverting a square polynomial matrix is similar to inverting an ordinary square matrix (a matrix with scalar entries), and the result is the adjoint matrix divided by the determinant of the matrix. In the operation, multiplying two polynomials is equivalent to convoluting the coefficient sequences of the two polynomial, dividing two polynomials is equivalent to deconvoluting the coefficient sequence of the numerator polynomial over that of the denominator, expanding to the number of terms desired.

After obtaining the estimated MA model and collecting $q1$ coefficients, one can form a matrix

$$M = \begin{bmatrix} \widehat{CAK} \\ \widehat{CA^2K} \\ \vdots \\ \widehat{CA^{q1}K} \end{bmatrix}, \tag{43}$$

and the least-squares solution of K is

$$\hat{K} = (H^T H)^{-1} H^T M = H^{\dagger} M \tag{44}$$

where

$$H = \left[(CA)^T,\ (CA^2)^T,\ \cdots,\ (CA^{q1})^T\right]^T \tag{45}$$

is an observability-like matrix, which is full-column-ranked for an observable system; $H^{\dagger}$ is the pseudo-inverse of H.

Because of the approximation used in the process, $\hat{K}$ is not a real optimal Kalman filter gain; however, it represents an identified state estimator gain (or suboptimal Kalman filter gain). The quality of the identified gain relies on the accuracy of the identified state space parameters and the order of the whitening filter r. If the identified state space model is accurate and the order r is chosen large enough, the identified gain will converge to the optimal steady state Kalman filter gain.

Experimental Example

An experimental example is used to demonstrate the feasibility of the integrated system identification and state estimation method developed above. A ten-bay structure as shown in Fig. 1 is considered. The truss is one of the structures built in NASA Langley Research Center for experiments in studies of control and structure interaction (CSI). It is 100 inches long, with a square cross section of 10 in $\times$ 10 in. All the tubing (longerons, battens, and diagonals) and ball joints are made of aluminum. The structure is in a vertical configuration attached from the top using an L-shaped fixture to a backstop. Two cold air thrusters acting in the same direction are placed at the tip. The thrusters which are used for excitation and control have a maximum thrust of 2.2 lb each. A mass of approximately 20 lb is attached at the beam tip to lower the fundamental frequency of the truss. Two servo accelerometers located at a corner of the square cross section provide the in-plan tip acceleration.

The structure was excited using random inputs to both thrusters for 30 seconds. The input signals were filtered to concentrate the energy in the low frequency range. A total of 7499 data points at sampling rate 250 Hz is taken. The two output acceleration signals were filtered using a three-pole Bessel filter with a break frequency of 20 Hz.

From the output we can tell the dominant mode is about 5 to 6 Hz. To avoid using too large order in the least-squares filter the sampling rate is reduced to 1/2 of the original one by choosing one out of every two samples. Hence, the sampling rate becomes 125 Hz and totals 3750 data. The order of the ARX model is set to 100. Figure 2 shows the identified system Markov parameters $\widehat{CA^{i-1}B}$ $(i = 1, \cdots, 100)$. By the ERA three modes are identified and the identified modal frequencies and dampings are listed as follows:

Mode	Frequency (rad/sec)	Damping (%)
1	37.0988	0.27
2	46.1175	2.87

3	304.4817	0.40

The corresponding state space parameters in normalized modal format are

$$\hat{A} = diag\left\{\begin{bmatrix} 0.9555 & 0.2922 \\ -0.2922 & 0.9555 \end{bmatrix} \begin{bmatrix} 0.9229 & 0.3568 \\ -0.3568 & 0.9229 \end{bmatrix} \begin{bmatrix} -0.7538 & 0.6423 \\ -0.6423 & -0.7538 \end{bmatrix}\right\}$$

$$\hat{B} = \begin{bmatrix} 0.1725 & -0.1117 & 0.1122 & -0.0321 & 0.3241 & -0.1871 \\ -0.1789 & 0.1267 & -0.1522 & 0.0556 & 0.3309 & -0.2725 \end{bmatrix}^T$$

$$\hat{C} = \begin{bmatrix} 1.7754 & 0.0000 & 1.0023 & 0.0000 & 1.3946 & 0.0000 \\ 0.9201 & 0.0362 & -1.6909 & -0.3692 & -1.4287 & 0.1185 \end{bmatrix}$$

$$\hat{D} = \begin{bmatrix} -0.0042 & 0.0165 \\ 0.0215 & 0.0009 \end{bmatrix}.$$

The identified state space parameters are used to estimate the corresponding Kalman filter gain. The identified stochastic Markov parameters $\widehat{CA^iK}$ $(i = 1, \cdots, 100)$ are shown in Fig. 3, and the estimated state estimator gains is

$$\hat{K} = \begin{bmatrix} 0.2787 & 0.1807 & 0.3437 & 0.1072 & 0.0065 & 0.0357 \\ 0.2277 & -0.0685 & 0.1236 & -0.0884 & -0.0874 & -0.0182 \end{bmatrix}^T.$$

To show the results of state estimation, the first state of each mode is shown in Fig. 4. Since the modal model has been normalized, the amplitude of each modal state indicates the energy allocated in that mode. To evaluate the quality of the system identification and state estimation, the estimated outputs calculated based on the estimated state are compared to the true outputs. Because the true state is not available, output comparison is the only way to validate the results. The comparison of the first output is shown in Fig. 5, where we can see the estimated and the true outputs are in good agreement. The covariance of the error is less than 1.5 % of the covariance of the output.

Concluding Remarks

In contrast to most existing system identification methods of which the great majority use deterministic approach, the method developed in this paper is derived under the stochastic framework, taking into account the effects of process noise as well as measurement noise. The use of projection filter to derive a state space model provides stochastic insight into the model. The accuracy of the identified state estimator gain relies on the accuracy of the identified state space model and the order of the whitening filter. The order of the whitening filter is not necessary to be equal to the order of the system. The larger the filter order is the whiter the residual will be. If the identified model is accurate and the order of the whitening filter is sufficiently large, the identified gain converges to the optimal steady state Kalman filter gain. An experimental example shows the feasibility of the method.

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Appendix

In the appendix, we want to show that for a linear time-invariant system described by a known state space model, the system output predicted by an optimal state (in the least-mean-square sense) based on the system model is also optimal (in the least-mean-square sense) if the number of the stationary data sample is sufficiently large. In other words, though the projection filter is derived based on the criterion of resulting least-mean-square state error, it also provides least-squares output error.

An ARX model of order q of a linear time-invariant system can be written as

$$y_k = \sum_{i=1}^{q} A_i y_{k-i} + \sum_{i=0}^{q} B_i u_{k-i} + \varepsilon_k = \hat{y}_k + \varepsilon_k \qquad (A1)$$

where ε_k is the output error. Given a set of input/output data $\{y_k, \cdots, y_0, u_k, \cdots, u_0\}$ of the system, we can use the least-squares method to find a set of matrix coefficients $\{\hat{A}_1, \cdots, \hat{A}_q, \hat{B}_0, \cdots, \hat{B}_q\}$ which fits the equation optimally in the least-squares error of output prediction sense. It minimizes a scalar cost function J_1, defined by

$$J_1 = \sum_{i=q}^{k}[(y_i - \hat{y}_i)^T(y_i - \hat{y}_i)] = \sum_{i=q}^{k}[\varepsilon_i^T \varepsilon_i] \tag{A2}$$

However, assume ε_i is stationary and k is sufficiently large,

$$J_1 = (k - q + 1)\left(\frac{1}{k - q + 1}\sum_{i=q}^{k}[\varepsilon_i^T \varepsilon_i]\right) \approx (k - q + 1)E[\varepsilon_i^T \varepsilon_i], \tag{A3}$$

for a stationary random process the sample average can represent the ensemble average (expectation) due to its ergodic property. Therefore, minimizing J_1 is equivalent to minimizing mean-square output error.

On the other hand, the projection filter provides a least-mean-square state estimate $\hat{x}_{k-1}$ of x_{k-1} from a set of previous input/output data $Y^T = [y_{k-1}^T, \cdots, y_0^T, u_{k-1}^T, \cdots, u_0^T]$ of the system. Similar to $\hat{y}_k$, the estimated state $\hat{x}_{k-1}$ is also a linear combination of previous input/output data and can be shown as (see Eq. (20))

$$\hat{x}_{k-1} = FY$$

where F is the projection filter. The optimal estimated state minimizes a scalar cost function J_2, defined by

$$\begin{aligned} J_2 &= E[(x_{k-1} - \hat{x}_{k-1})^T(x_{k-1} - \hat{x}_{k-1})] \\ &= E[(x_{k-1} - FY)^T(x_{k-1} - FY)] \\ &= trace\ E[(x_{k-1} - FY)(x_{k-1} - FY)^T] \\ &= trace\ E[x_{k-1}x_{k-1}^T - x_{k-1}Y^TF^T - FYx_{k-1}^T + FYY^TF^T] \end{aligned} \tag{A4}$$

with

$$\frac{dJ_2}{d\hat{x}_{k-1}} = 0 \quad \text{or} \quad \frac{dJ_2}{dF} = -2E[x_{k-1}Y^T] + 2F(E[YY^T]) = 0$$

$$F(E[YY^T]) - E[x_{k-1}Y^T] = 0 \tag{A5}$$

The optimal predicted output $\hat{y}_k$ can be derived by using the one-step-ahead prediction from the optimal state $\hat{x}_{k-1}$ as shown in Eq. (11). From Eqs. (11) and (15) one can obtain

$$\epsilon_k = y_k - \hat{y}_k = CA(x_{k-1} - FY) + Cw_{k-1} + v_k$$

and

$$\begin{aligned} J_1 &= E[\epsilon_k^T \epsilon_k] \\ &= trace\ E[\epsilon_k \epsilon_k^T] \\ &= trace\ E[CA(x_{k-1} - FY)(x_{k-1} - FY)^T A^T C^T] + trace[CQC^T + R] \end{aligned}$$

Because the process noise covariance Q and the measurement noise covariance R are constant, we have

$$\begin{aligned} \frac{dJ_1}{dF} &= \frac{d}{dF}\{trace\ E[CA(x_{k-1} - FY)(x_{k-1} - FY)^T A^T C^T]\} \\ &= -2A^T C^T CA(E[x_{k-1} Y^T]) + 2A^T C^T CAF(E[YY^T]) \\ &= 2A^T C^T CA\{F(E[YY^T]) - E[x_{k-1} Y^T]\} \end{aligned}$$

From Eq. (A5), we get $dJ_1/dF = 0$. This proves that the predicted output $\hat{y}_k$ shown in Eq. (11) from the projection filter is also the lest-squares output. The coefficients in Eq. (22) should be equivalent to those in ARX model (A1) derived from the least squares method if the number of data is sufficiently large.

Figure Captions

Fig. 1 Ten-bay truss structure test configuration.

Fig. 2 Identified system Markov parameters $\widehat{CA^{i-1}B}$ (the (1,1) element).

Fig. 3 Identified stochastic Markov parameters $\widehat{CA^{i}K}$ (the (1,1) element).

Fig. 4 Estimated Modal states.

Fig. 5 Comparison of true and estimated outputs.

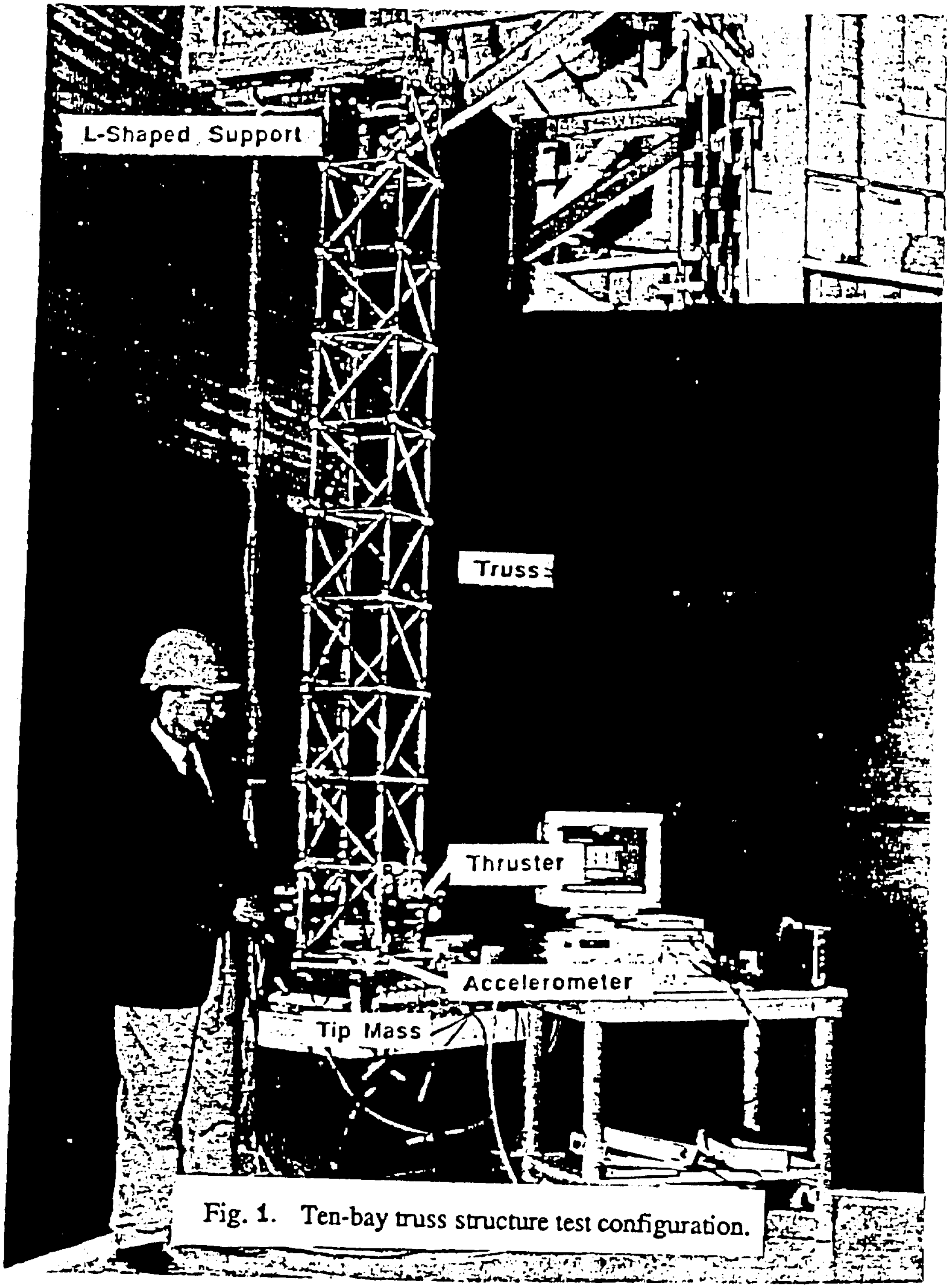


Fig. 1. Ten-bay truss structure test configuration.

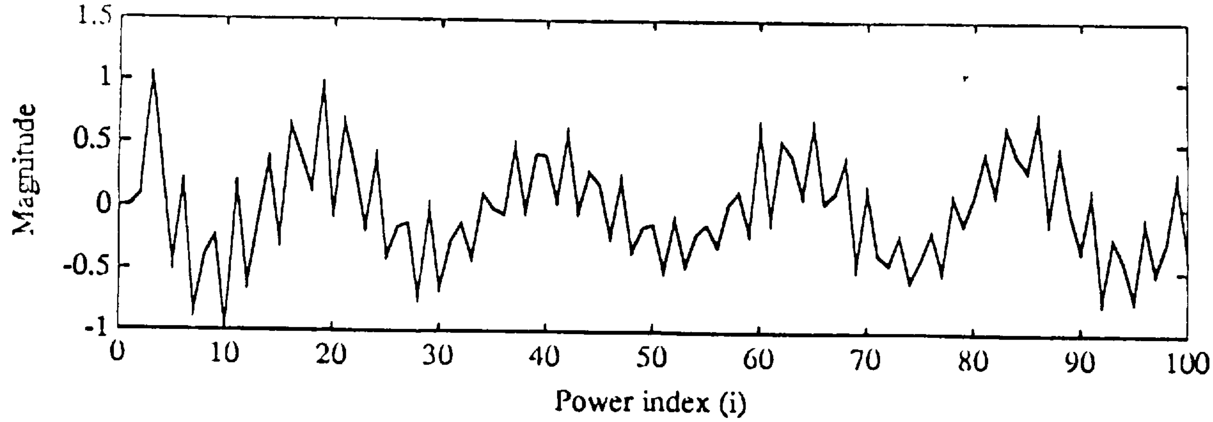

1.5
1
0.5
0
-0.5
-1
Magnitude
0
10
20
30
40
50
60
70
80
90
100
Power index (i)

Fig. 2

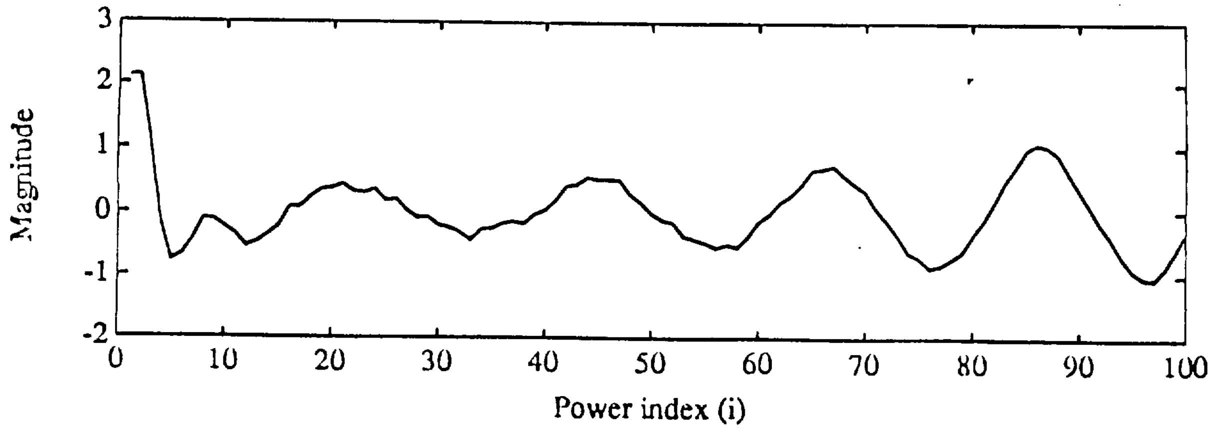


Fig 3

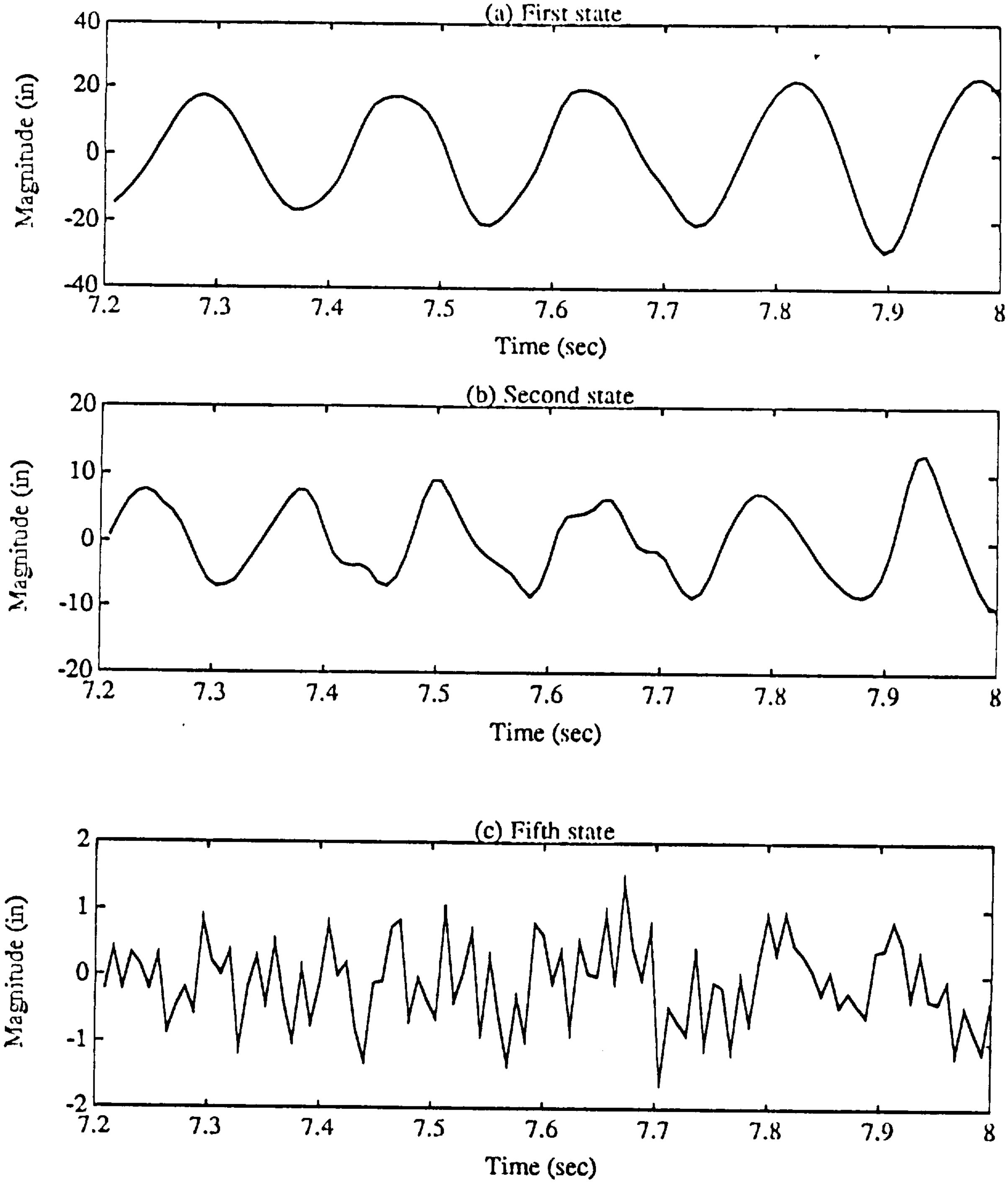
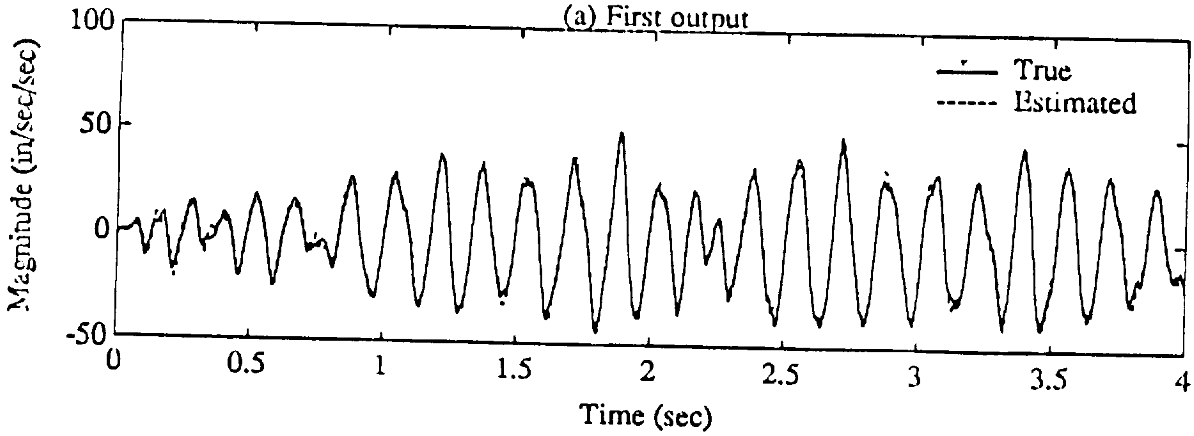

(a) First state
Magnitude (in)
40
20
0
-20
-40
7.2
7.3
7.4
7.5
7.6
7.7
7.8
7.9
8
Time (sec)
(b) Second state
Magnitude (in)
20
10
0
-10
-20
7.2
7.3
7.4
7.5
7.6
7.7
7.8
7.9
8
Time (sec)
(c) Fifth state
Magnitude (in)
2
1
0
-1
-2
7.2
7.3
7.4
7.5
7.6
7.7
7.8
7.9
8
Time (sec)

Fig 4


(a) First output
True
Estimated
Magnitude (in/sec/sec)
Time (sec)

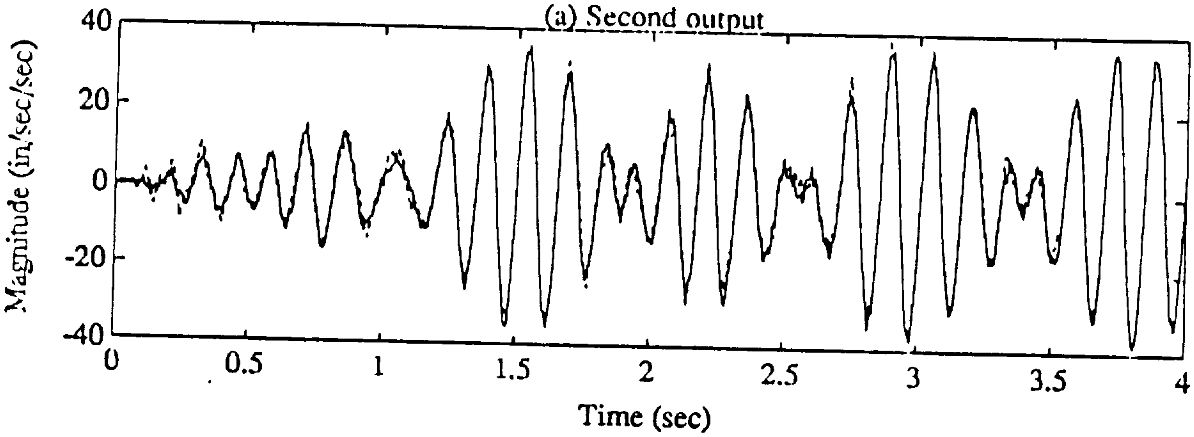

(a) Second output
Magnitude (in/sec/sec)
Time (sec)

Fig 5.