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# The effects of suction on the nonlinear stability of the three-dimensional boundary layer above a rotating disc<sup>†</sup>

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## Abstract

There exist two types of stationary instability of the flow over a rotating disc corresponding to the upper-branch, inviscid mode and the lower-branch mode, which has a triple deck structure, of the neutral stability curve. A theoretical investigation of the linear problem and an account of the weakly nonlinear properties of the lower branch-modes have been undertaken by Hall (1986) and MacKerrell (1987) respectively. Motivated by recent reports of experimental sightings of the lower-branch mode and an examination of the role of suction on the linear stability properties of the flow here we investigate the effects of suction on the nonlinear disturbance described by MacKerrell (1987). The additional analysis required in order to incorporate suction is relatively straightforward and enables us to derive an amplitude equation which describes the evolution of the mode. For each value of the suction a threshold value of the disturbance amplitude is obtained; modes of size greater than this threshold grow without limit as they develop away from the point of neutral stability.

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# 1. Introduction

The stability of rotating disc flows has been the subject of many studies, one reason for this being that the basic boundary layer flow given by Von Kármán's (1921) exact solution is fully three-dimensional. Interest also stems from the observed similarities in the type of instability occurring in rotating disc flow work (Gregory, Stuart & Walker 1955) and near the leading edge of a swept back wing, Poll (1978). This instability, known as cross-flow instability, is due to the inflectional character of the basic velocity profile and was first studied extensively by Gregory *et al.* (1955). They showed with china clay techniques that this instability took the form of a regularly spaced pattern of equi-angular spiral vortices which was stationary relative to the disc. Stuart (in Gregory *et al.*), using inviscid theory, suggested that the instability could be associated with a particular inflectional profile in which the inflection point coincided with a point of zero velocity somewhere in the flow. His calculation gave the predicted number of vortices to be approximately four times greater than the observed value of about 30 but the angle of  $13^\circ$  between the axes of the vortices and the radius vector was in excellent agreement with their experiments.

Several attempts have been made to explain the observed spiral patterns by means of a linear stability theory (see, for example, Cebeci & Stewartson (1980), Malik *et al.* (1981), Mack (1985), Malik (1986)). Malik *et al.* (1981) showed that Coriolis force and streamline curvature cannot be neglected— a point first made by Faller & Kaylor (1966) and Lilly (1966) in the context of the Ekman boundary layer. The theoretical results of Mack (1985) strongly suggest that the spiral streaks observed in flow visualisation experiments are the constant phase lines of the merged wave patterns produced by several random sources on the disc. This was first found by the experimental investigations of Wilkinson & Malik (1985) who revealed that the wave patterns from each point source spread out circumferentially downstream of the source and that the wave patterns from a number of sources eventually merge and cover the entire circumference of the disc.

Experiments by Fedorov *et al.* (1976) showed that for various rotation rates of the disc an instability occurred which also appeared as a pattern of spiral vortices which numbered in the region 14 to 16 and which had their axes inclined at angles of about  $20^\circ$  to the radius vector. These discrepancies between these observations and those of Gregory *et al.* (1955) imply that the vortices seen by Fedorov *et al.* were not those of Gregory *et al.*

Malik (1986) calculated the neutral stability curve for stationary disturbances and demonstrated that two types of mode could exist. The first, the upper branch of the neutral curve, corresponds to the inviscid mode described by Stuart in Gregory *et al.* The second, the lower branch mode, is an essentially viscous disturbance which corresponds to zero wall shear stress of the crossflow velocity profile.

A linear asymptotic investigation of the inviscid mode at high Reynolds number was conducted by Hall (1986); work that has been extended to include nonlinear effects by Gajjar (personal communication). Further, Hall (1986) demonstrated that the lower branch viscous mode is governed by a triple deck structure (which is derived from the classical structure as reviewed by, for example, Smith 1982) and it is this mode which is thought to have been observed by Fedorov *et al.* (1976). MacKerrell (1987), hereafter referred to as M, extended the study of Hall (1986) in order to give an account of the weakly nonlinear properties of the lower branch mode. Following the framework set up in Hall (1986) M derived solutions which enabled the finite amplitude growth of a disturbance close to neutral to be described. Importantly, she found that nonlinearity has a destabilising effect on the vortices and discovered the existence of a threshold amplitude (in the sense of Stuart 1960, 1971 and Watson 1960) such that disturbances smaller than the threshold value decay as they move away from the neutral position whereas those above threshold grow explosively. This led M to conclude that for small disturbances the inviscid mode of Gregory *et al.* (1955) is probably dominant whereas for larger amplitude perturbations the short wavelength mode observed by Fedorov *et al.* (1976) is the more important.

Until very recently there have been, to the best of the authors' knowledge, few, if any, other observations of the short wavelength disturbance. At the 1991 ICASE/LaRC workshop on transition and turbulence Corke reported that in his experiments on rotating disc flows a subcritical stationary instability had been observed—the mode which is almost certainly that seen by Fedorov *et al.* (1976) and analysed in M. At the same meeting we learned of an investigation by Dhanak (1991) who has extended the calculations of Malik (1986) to include the effects of suction. The role of suction in practical flows is an important one for it is believed that imposition of suitable suction on a boundary layer can often stabilise the flow and thence delay the onset of transition. Dhanak (1991) found that if suction is applied to the rotating disc flow then the overall effect, at least according to linear theory, is that the flow is indeed stabilised: conversely blowing destabilises the flow. This finding invites the question as to the effect of suction on the nature of the short wavelength mode described by Hall (1986) and M and it is this that we address here.

In §2 we follow the nonlinear analysis of M and indicate the modifications required in order to incorporate suction into the flow. We then execute some numerical work to obtain a quantitative description of the flow properties and conclude with some discussion.

## 2. Formulation of the problem

We consider the case in which the disc  $z = 0$  rotates about the  $z$ -axis with angular velocity  $\Omega$ . Relative to cylindrical polar co-ordinates  $(r, \theta, z)$  which rotate with the disc, the continuity and Navier–Stokes equations, when suitably non-dimensionalised, become

$$\nabla \cdot \mathbf{u} = 0, \quad (2.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2(\mathbf{k} \wedge \mathbf{u}) - r \hat{\mathbf{r}} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad (2.2)$$

where  $\mathbf{u} = (u, v, w)$  are the velocity components,  $p$  is the fluid pressure,  $\hat{\mathbf{r}}$  and  $\mathbf{k}$  are unit vectors in the  $r$ - and  $z$ - co-ordinate directions and the Reynolds number  $Re = \Omega L^2 / \nu$  with  $\nu$  the kinematic viscosity of the fluid and  $L$  a reference lengthscale. The Reynolds number is taken to be large throughout.

For the ensuing analysis it is convenient to define  $\varepsilon = Re^{-\frac{1}{16}}$  and with  $z = \varepsilon^8 \eta$  the basic steady flow is given by the solution

$$\mathbf{u} = \mathbf{u}_B = (r\bar{u}(\eta), r\bar{v}(\eta), \varepsilon^8 \bar{w}(\eta)), \quad p = p_B = \bar{p}(\eta),$$

where  $\bar{u}$ ,  $\bar{v}$ ,  $\bar{w}$  and  $\bar{p}$  satisfy

$$\bar{u}^2 - (1 + \bar{v})^2 + \bar{u}'\bar{w} = \bar{u}'', \quad 2\bar{u}(1 + \bar{v}) + \bar{v}'\bar{w} = \bar{v}'', \quad (2.3a, b)$$

$$\bar{w}' + 2\bar{u} = 0, \quad \bar{p}' + \bar{w}\bar{w}' - \bar{w}'' = 0. \quad (2.3c, d)$$

The appropriate far field conditions take the forms

$$\bar{u} \longrightarrow 0, \quad \bar{v} \longrightarrow -1 \quad \text{as} \quad \eta \longrightarrow \infty, \quad (2.4a)$$

whereas on the disc surface we impose

$$\bar{u} = \bar{v} = 0, \quad (2.4b)$$

together with

$$\bar{w} = -\bar{S}, \quad (2.4c)$$

where  $\bar{S}$  is a measure of the suction ( $\bar{S} > 0$ ) or blowing ( $\bar{S} < 0$ ) applied to the flow.

The linear stability of the basic flow (2.3), (2.4) in the absence of suction or blowing was addressed by Hall (1986). He demonstrated that the flow is susceptible to a stationary disturbance which is governed by an appropriate triple deck structure whose lower, main and upper decks are of thicknesses  $O(\varepsilon^9)$ ,  $O(\varepsilon^8)$  and  $O(\varepsilon^4)$  respectively. Hall showed that

the disturbance wavenumbers in the  $r$  and  $\theta$  directions are  $O(\varepsilon^{-4})$  and solutions were sought proportional to

$$E \equiv \exp \left[ \frac{i}{\varepsilon^4} \left( \int^r \alpha(r, \varepsilon) dr + \theta \beta(\varepsilon) \right) \right], \quad (2.5)$$

where the wavenumbers  $\alpha$  and  $\beta$  expand as

$$\alpha = \alpha_0 + \varepsilon^2 \alpha_1 + \varepsilon^3 \alpha_2 + \dots, \quad \beta = \beta_0 + \varepsilon^2 \beta_1 + \varepsilon^3 \beta_2 + \dots \quad (2.6)$$

It was found by Hall (1986) that, as shown by Gregory *et al.* (1955), the 'effective' velocity profile for a three-dimensional disturbance with wavenumbers  $\alpha$  and  $\beta$  in the  $r$  and  $\theta$  directions is  $r\alpha\bar{u} + \beta\bar{v}$ . However, if the effective wall shear  $r\alpha\bar{u}' + \beta\bar{v}'$  does not vanish the modes are necessarily time dependent for  $\alpha$  and  $\beta$  real. Therefore, in order to seek stationary modes which are neutrally stable at the location  $r$  the values  $\alpha_j$  and  $\beta_j$  must be chosen so as to reduce the effective wall shear to zero at leading order.

MacKerrell (1987) extended the work of Hall (1986) to a weakly nonlinear setting. She demonstrated that in order to obtain a classical evolution equation for the disturbance it is necessary to examine disturbances of amplitude  $O(\varepsilon^{\frac{5}{2}})$  within the lower deck. Further, the mode then evolves in an  $O(\varepsilon)$  neighbourhood of the neutral position  $r = \bar{r}$  and, as in usual nonlinear triple deck work, it is the lower tier of the triple deck where nonlinearity manifests itself. Our objective here is to generalise the work of M to investigate the role of the suction parameter  $\bar{S}$  and in the following we deliberately give the briefest of details. The analysis differs only slightly from that of M and so we highlight the main differences between the present work and that contained in M. The omitted details may be found in M and the reader is referred to that paper where necessary.

Following the argument of M we define a new variable  $r_1$  in the neighbourhood of the neutral point  $\bar{r}$  by

$$r = \bar{r} + \varepsilon r_1, \quad (2.7a)$$

and seek disturbances whose radial dependence may be written in the form

$$\frac{\partial}{\partial r} \longrightarrow \frac{i\alpha_0}{\varepsilon^4} + \frac{i\alpha_1}{\varepsilon^2} + \frac{1}{\varepsilon} \frac{\partial}{\partial r_1}. \quad (2.7b)$$

Within the lower deck, where we define the  $O(1)$  co-ordinate  $\xi$  by  $z = \varepsilon^9 \xi$ , we suppose that the basic flow velocity components (2.3) expand according to

$$\begin{aligned} \bar{u} &= \varepsilon \bar{u}_0 \xi + \varepsilon^2 \bar{u}_1 \xi^2 + \varepsilon^3 \bar{u}_2 \xi^3 + \dots, \\ \bar{v} &= \varepsilon \bar{v}_0 \xi + \varepsilon^2 \bar{v}_1 \xi^2 + \varepsilon^3 \bar{v}_2 \xi^3 + \dots, \\ \bar{w} &= -\bar{S} + \varepsilon^2 \bar{w}_1 \xi^2 + \varepsilon^3 \bar{w}_2 \xi^3 + \dots \end{aligned} \quad (2.8)$$



If we perturb the basic flow by writing

$$(u, v, w, p) = (\bar{u} + U, \bar{v} + V, \varepsilon^8 \bar{w} + W, \bar{p} + P), \quad (2.9)$$

then in the lower deck the disturbance quantities develop according to

$$U = \varepsilon^{\frac{7}{2}} \left[ \frac{\bar{r}\bar{\gamma}_0 C_1}{\varepsilon \beta_0^2} (\bar{u}_0 + 2\varepsilon \bar{u}_1 \xi + \dots) + \frac{\bar{U}_{-1}}{\varepsilon} + \bar{U}_0 + \dots \right] E \\ + \text{harmonic terms} + \text{complex conjugate}, \quad (2.10a)$$

together with a similar expansion for  $V$ . In addition

$$W = \varepsilon^{\frac{7}{2}} \left[ -\frac{i\bar{\gamma}_0 \varepsilon^5 C_1}{\beta_0^2} [(\alpha_0 \bar{r} \bar{u}_1 + \beta_0 \bar{v}_1) \xi^2 + \dots] + \varepsilon^6 \bar{W}_1 + \varepsilon^7 \bar{W}_2 + \dots \right] E \\ + \text{harmonic terms} + \text{complex conjugate}, \quad (2.10b)$$

and

$$P = \varepsilon^{\frac{7}{2}} [(\varepsilon^3 \bar{P}_1 + \varepsilon^4 \bar{P}_2 + \dots)] E \\ + \text{harmonic terms} + \text{complex conjugate}. \quad (2.10c)$$

Here  $C_1(r_1)$  is a scaled amplitude of the disturbance and we have made the definition

$$\bar{\gamma}_0^2 \equiv \alpha_0^2 + \frac{\beta_0^2}{\bar{r}^2}. \quad (2.11)$$

These forms of the disturbance, in which all the unknowns are functions of  $\xi$  and  $r_1$ , were first derived by M who also elucidated the forms of the harmonic terms. However, we find that the crucial harmonic and mean flow terms which interact to drive the fundamental and hence lead to the desired amplitude equation are unchanged by the imposition of suction or blowing. Hence, for brevity we do not repeat these results from M and now merely state how her overall findings need to be modified.

Firstly, following the procedure adopted in M, we find that  $\bar{W}_2$  satisfies

$$\frac{\partial^3 \bar{W}_2}{\partial s^3} - s^2 \frac{\partial^2 \bar{W}_2}{\partial s^2} + 2s \bar{W}_2 = \Delta^{-\frac{3}{4}} \bar{\gamma}_0^2 D_1 - 2r_1 \bar{\gamma}_0^2 C_1 \frac{\Delta^{-\frac{3}{4}}}{\bar{r}} + i\bar{\gamma}_0 \bar{v}_0 \beta_1 \Delta^{-\frac{1}{4}} \frac{s^2 r_1 C_1}{\beta_0^2 \bar{r}} \\ + ik_1 (\bar{r} \alpha_1 \bar{u}_0 + \beta_1 \bar{v}_0) \Delta^{-\frac{5}{4}} s^2 - i\beta_2 \bar{v}_0 \Delta^{-\frac{1}{4}} \bar{\gamma}_0 \frac{C_1 s^2}{\beta_0^2} \\ - \bar{\gamma}_0^2 C_2 C_1^* \frac{\Delta^{\frac{3}{4}} s^2}{\beta_0^4} - 2i\bar{\gamma}_0 r_1 C_1 \frac{\bar{u}_0 \Delta^{-\frac{3}{4}}}{\beta_0 \bar{r}} \left( 1 + \frac{\bar{v}_0^2}{\bar{u}_0^2} \right) \frac{U(0, \sqrt{2}s)}{U(0, 0)} \\ - \bar{r} \bar{u}_0 \Delta^{-\frac{1}{4}} \frac{\bar{\gamma}_0 s^2}{\beta_0^2} \frac{dC_1}{dr_1} + 2\bar{S} \Delta^{\frac{1}{4}} \frac{\bar{\gamma}_0 D_1}{\beta_0^2} + RHS, \quad (2.12a)$$

where  $\Delta \equiv i(\alpha_0 \bar{r} \bar{u}_1 + \beta_0 \bar{v}_1)$ ,  $D_1(r_1)$  and  $C_2(r_1)$  are functions of  $C_1$  given in M,  $k_1$  is a real constant whose precise value will turn out to be unimportant,  $s \equiv \Delta^{\frac{1}{4}} \xi$  and an asterisk on a quantity denotes the complex conjugate of that quantity. Further,  $U(a, x)$  denotes the usual parabolic cylinder function (see, for example, Abramowitz & Stegun (1964)) and

$$\begin{aligned}
RHS \equiv & -2i\bar{r} \frac{\bar{\gamma}_0^2}{\beta_0} \bar{U}_0 \Delta^{-\frac{3}{4}} - ik_1 \Delta^{-\frac{7}{4}} s^4 (\bar{r} \alpha_0 \bar{u}_2 + \beta_0 \bar{v}_2) + 2i\bar{\gamma}_0 C_1 \frac{\Delta^{-1}}{\beta_0} s \bar{u}_0 \bar{v}_0 \frac{U(0, \sqrt{2}s)}{U(0, 0)} \\
& - \left[ i(\bar{r} \alpha_1 \bar{u}_0 + \beta_1 \bar{v}_0) \Delta^{-\frac{3}{2}} + 3i \Delta^{-2} s^2 (\bar{r} \alpha_0 \bar{u}_2 + \beta_0 \bar{v}_2) \right] \\
& \quad \times \left[ \bar{\gamma}_0^2 C_1 F_1(s) + \frac{2i\bar{\gamma}_0}{\beta_0} C_1 \left( 1 + \frac{\bar{v}_0^2}{\bar{u}_0^2} \right) \frac{\bar{u}_0 F_2(s)}{U(0, 0)} \right] \\
& + \left[ i(\bar{r} \alpha_0 \bar{u}_2 + \beta_0 \bar{v}_2) \Delta^{-2} s^3 + i(\bar{r} \alpha_1 \bar{u}_0 + \beta_1 \bar{v}_0) \Delta^{-\frac{3}{2}} s \right] \\
& \quad \times \left[ \bar{\gamma}_0^2 C_1 F_1'(s) + \frac{2i\bar{\gamma}_0}{\beta_0} C_1 \left( 1 + \frac{\bar{v}_0^2}{\bar{u}_0^2} \right) \frac{\bar{u}_0 F_2'(s)}{U(0, 0)} \right] \\
& - \bar{S} \Delta^{-1} \left[ \bar{\gamma}_0^2 C_1 F_1''(s) + \frac{2i\bar{\gamma}_0}{\beta_0} C_1 \left( 1 + \frac{\bar{v}_0^2}{\bar{u}_0^2} \right) \frac{\bar{u}_0 F_2''(s)}{U(0, 0)} \right], \tag{2.12b}
\end{aligned}$$

where  $F_1(s)$  and  $F_2(s)$  satisfy

$$F_1'''' - s^2 F_1' + 2s F_1 = 1, \quad F_1(0) = F_1(\infty) = 0,$$

and

$$F_2'''' - s^2 F_2' + 2s F_2 = U(0, s\sqrt{2}), \quad F_2(0) = F_2(\infty) = 0.$$

The solution for  $\bar{U}_0$  when  $\bar{S} = 0$ , say  $\bar{U}_{00}$ , is given in Hall (1986). It is then straightforward to obtain

$$\bar{U}_0 = \bar{U}_{00} + \bar{S} \Delta^{-\frac{1}{4}} C_1 \bar{r} \frac{\bar{\gamma}_0^2 \bar{u}_0}{\sqrt{2} \beta_0^2} \frac{s U(0, s\sqrt{2})}{U(0, 0)}.$$

In order for the disturbance to vanish on the disc surface we find from the continuity equation that  $\partial \bar{W}_2 / \partial s = 0$  on  $s = 0$ . By solving (2.12) subject to this condition we obtain the desired amplitude equation in the form

$$\frac{dC_1}{dr_1} = (a + ic)r_1 C_1 + (b + id)C_1 |C_1|^2 + (e + if)C_1, \tag{2.13}$$

where  $a, b, c, d, e$  and  $f$  are real valued constants, the more important of which are detailed presently. Multiplying (2.13) through by  $C_1^*$ , adding the complex conjugate and replacing  $r_1$  by  $r_1' = r_1 + (e/a)$  yields the amplitude evolution equation

$$\frac{d}{dr_1'} (|C_1|^2) = 2ar_1' |C_1|^2 + 2b|C_1|^4, \tag{2.14}$$

and the form of the solution of (2.14) depends on the signs of  $a$  and  $b$ . From (2.3) and (2.4) we find that

$$\bar{u}_1 = -\frac{1}{2} - \frac{\bar{S}\bar{u}_0}{2} \quad \text{and} \quad \bar{v}_1 = -\frac{\bar{S}\bar{v}_0}{2}$$

and recalling the imposition that the leading order wall shear ( $= \alpha_0 \bar{r} \bar{u}_0 + \beta_0 \bar{v}_0$ ) must be necessarily zero in order to obtain stationary vortices we have  $\Delta \equiv -i\bar{r}\alpha_0/2$ . Following the manipulations of M we finally obtain

$$a = -\frac{\beta_0^2 \bar{\gamma}_0 I_1}{\bar{r}^2 \bar{u}_0 (\alpha_0 \bar{r})^{\frac{1}{2}}}, \quad (2.15a)$$

and

$$b = \frac{3(\alpha_0 \bar{r})^{\frac{5}{2}} \bar{\gamma}_0}{2\sqrt{2}\bar{r}\beta_0^3} \left(1 + \frac{\bar{v}_0^2}{\bar{u}_0^2}\right) \frac{I_2}{\lambda^2}, \quad (2.15b)$$

where

$$\lambda^2 = \left[4\bar{\gamma}_0\beta_0^2 I_1 - \left(\frac{\alpha_0 \bar{r}}{2}\right)^{\frac{1}{2}} (\bar{r}\alpha_1 \bar{u}_0 + \beta_1 \bar{v}_0)\right]^2 + \left[\bar{\gamma}_0\beta_0^2 I_1 - \left(\frac{\alpha_0 \bar{r}}{2}\right)^{\frac{1}{2}} (\bar{r}\alpha_1 \bar{u}_0 + \beta_1 \bar{v}_0)\right]^2, \quad (2.15c)$$

$$I_1 \equiv F_1'(0) = \int_0^\infty \frac{\theta U(0, \theta)}{2U(0, 0)} d\theta = 0.59907, \quad (2.15d)$$

and

$$I_2 \equiv \frac{2F_2'(0)}{U(0, 0)} = \int_0^\infty \frac{\theta U^2(0, \theta)}{U^2(0, 0)} d\theta = 0.45695. \quad (2.15e)$$

Consequently, we now have the outcome that the formulae (2.15) for the coefficients of the amplitude equation appear to be independent of the suction parameter  $\bar{S}$  and hence are identical to those of M. However, the constants appearing in the expansions (2.8) for the basic flow quantities close to  $\xi = 0$  are functions of  $\bar{S}$  and thence so are  $a$  and  $b$ . In the following section we conduct a few calculations for various values of  $\bar{S}$  and draw some conclusions.

### 3. Results and discussion

In order to calculate the coefficients  $a$  and  $b$  in (2.14) we merely need to solve the basic equations (2.3) subject to boundary conditions (2.4) and note that the function  $(\bar{r}\alpha_1 \bar{u}_0 + \beta_1 \bar{v}_0)$  appearing in definition (2.15c) is given by

$$\bar{\gamma}_0^2 I_1 + \frac{i\bar{\gamma}_0 \bar{u}_0}{\beta_0} \left(1 + \frac{\bar{v}_0^2}{\bar{u}_0^2}\right) I_2 = i\Delta^{\frac{1}{2}} (\alpha_1 \bar{r} \bar{u}_0 + \beta_1 \bar{v}_0) \frac{\bar{\gamma}_0}{\beta_0^2},$$

a relation which was deduced in M. This relation also serves to determine the value of  $\bar{\gamma}_0$  and this, in addition to the requirement that the leading order wall shear  $\alpha\bar{r}\bar{u}_0 + \beta_0\bar{v}_0$  vanishes, furnishes the leading order radial and azimuthal wavenumbers  $\alpha_0, \beta_0$ . The equations for the basic flow were solved using a suitable NAG routine and then we found it convenient to write

$$a = A\bar{r}^{-\frac{7}{4}}, \quad b = B\bar{r}^{-\frac{13}{4}}, \quad (3.1)$$

where the functions  $A(\bar{S})$  and  $B(\bar{S})$  are given in Figure (1). We note that for all values of suction or blowing,  $-\infty < \bar{S} < \infty$ , we have  $A < 0$  and  $B > 0$  and the former condition implies that the amplitude of the solution increases or decreases depending on whether  $r$  is less than or greater than the neutral value. It is also straightforward to demonstrate that if we write

$$|C_1|^2 = \left(\sqrt{-a/2b}\right) y \quad \text{and} \quad r'_1 = x/\sqrt{-a}$$

then (2.14) becomes

$$\frac{dy}{dx} = -2xy + y^2,$$

whose solution is

$$y(x) = \frac{e^{-x^2}}{\left(\frac{1}{y_0} - \frac{\sqrt{\pi}}{2}\text{erf}(x)\right)},$$

where  $y_0 = y(0)$  and  $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ . Since  $y(x)$  must be greater than or equal to zero and  $0 \leq \text{erf}(x) < 1$ , the solution is bounded and valid for all  $x$  if  $y_0 < 2/\sqrt{\pi}$ . However, for  $y_0 \geq 2/\sqrt{\pi}$ ,  $y(x)$  becomes infinite at some point  $x_0$ ,  $0 < x_0 \leq \infty$ , and the solution is only valid for  $x \leq x_0$ . Thus the value  $y_0 = 2/\sqrt{\pi}$  represents a threshold between solutions which grow indefinitely and those that decay to zero. The corresponding disturbance threshold value

$$|C_1|_e = \frac{2}{\sqrt{\pi}} \sqrt{-\frac{a}{2b}} = \frac{2}{\sqrt{\pi}} \sqrt{-\frac{A}{2B}} \bar{r}^{\frac{3}{4}}, \quad (3.2)$$

is shown in Figure (2). We note that for large suction ( $\bar{S} \gg 1$ ) the threshold amplitude becomes small whereas for large blowing this amplitude grows. We now verify these qualitative results by considering the appropriate limits.

### 3.1. The large suction limit, $\bar{S} \gg 1$ .

For  $\bar{S} \gg 1$ , Stuart (1954) has demonstrated that the basic flow profile is confined within a zone of thickness  $O(1/\bar{S})$  and if we write  $\zeta = \bar{S}\eta$  where  $\zeta = O(1)$  then, to leading order,

$$\bar{u} = \frac{1}{2\bar{S}^2} (e^{-\zeta} - e^{-2\zeta}), \quad \bar{v} = 1 - e^{-\zeta} \quad \text{and} \quad \bar{w} = -\bar{S}.$$

Consequently the constants defined in the basic flow expansions (2.8) assume the forms

$$\bar{u}_0 = \frac{1}{2\bar{S}}, \quad \bar{v}_0 = -\bar{S}$$

and these lead to the asymptotic expressions

$$A \sim -2^{\frac{3}{2}} I_2^{\frac{5}{4}} I_1^{-\frac{1}{4}} \bar{S}^{\frac{13}{4}} \approx -1.2078 \bar{S}^{\frac{13}{4}}, \quad (3.3a)$$

$$B \sim \frac{24 I_1^{\frac{1}{4}}}{(21 - 10\sqrt{2}) I_2^{\frac{5}{4}}} \bar{S}^{\frac{27}{4}} \approx 8.1952 \bar{S}^{\frac{27}{4}}, \quad (3.3b)$$

where  $I_1$  and  $I_2$  are given by (2.13b, c) and  $A$  and  $B$  are the coefficients in the amplitude equations (2.14) and which were defined in (3.1). In turn, the threshold amplitude

$$|C_1|_e \approx 0.3063 \bar{S}^{-\frac{7}{4}} \bar{r}^{\frac{3}{4}}. \quad (3.3c)$$

### 3.2. The large blowing limit, $\bar{S} \ll -1$ .

In his consideration of boundary layer flows Pretsch (1944) demonstrated that the influence of viscosity becomes almost negligible for sufficiently large blowing and this work, together with the results emanating from studies of fluid injection in supersonic boundary layers by Smith & Stewartson (1973a, b), enable us to derive the flow structure for the present problem when  $\bar{S} \ll -1$ . We find it convenient to define the positive quantity  $\bar{\bar{S}} = -\bar{S}$ . It is now the case that the extent of the boundary layer increases and formally becomes  $O(\bar{\bar{S}})$ . Therefore, if we define the  $O(1)$  co-ordinate  $z_1$  by  $z_1 = \eta/\bar{\bar{S}}$  we find that in the region  $0 \leq z_1 \leq \pi/2$  the basic flow solution develops according to

$$\begin{aligned} \bar{u} &= F_0(z_1) + \frac{1}{\bar{\bar{S}}^2} F_1(z_1) + \dots, & \bar{v} &= G_0(z_1) + \frac{1}{\bar{\bar{S}}^2} G_1(z_1) + \dots, \\ \bar{w} &= \bar{\bar{S}} H_0(z_1) + \frac{1}{\bar{\bar{S}}} H_1(z_1) + \dots, \end{aligned} \quad (3.4)$$

with

$$H_0 = G_0 = \cos^2 z_1 \quad \text{and} \quad F_0 = \cos z_1 \sin z_1. \quad (3.5)$$

The profile decays algebraically as  $z_1 \rightarrow \pi/2$  and a thin viscous layer centred on this point ensures that the solutions for  $\bar{u}$  and  $\bar{v}$  are brought to zero at infinity. Details of this viscous structure are not needed for present purposes as we are solely concerned with the values of  $\bar{u}'(0)$  and  $\bar{v}'(0)$ . Consideration of the problem for  $G_1(z_1)$  enables us to conclude that the flow constants  $\bar{u}_0$  and  $\bar{v}_0$  in (2.8a, b) are given, at leading orders, by

$$\bar{u}_0 = \frac{1}{\bar{\bar{S}}} \quad \text{and} \quad \bar{v}_0 = -\frac{2}{\bar{\bar{S}}^3}.$$

In turn this leads to the asymptotic results

$$A \sim -2^{\frac{1}{2}} I_2^{\frac{5}{4}} I_1^{-\frac{1}{4}} \bar{S}^{\frac{3}{4}} \approx -0.3020 (-\bar{S})^{\frac{3}{4}}, \quad (3.6a)$$

$$B \sim \frac{6I_1^{\frac{1}{4}}}{(21 - 10\sqrt{2})I_2^{\frac{5}{4}}} \bar{S}^{-\frac{11}{4}} \approx 2.0488 (-\bar{S})^{-\frac{11}{4}}, \quad (3.6b)$$

with a threshold amplitude given by

$$|C_1|_e \approx 0.3063 (-\bar{S})^{\frac{7}{4}} \bar{r}^{\frac{3}{4}}. \quad (3.6c)$$

## 4. Conclusions

On Figures (1) & (2) we have indicated the asymptotic behaviours (3.3) & (3.6) both for  $\bar{S} < -1$  and for  $\bar{S} > 1$ . We observe that these one term asymptotic predictions are very accurate for a surprisingly large range of values of  $\bar{S}$ ; indeed they are graphically indistinguishable from the respective curves for  $|\bar{S}|$  greater than about five.

The most important deduction to be drawn from Figure (1b) is that the nonlinear coefficient in evolution equation (2.14) is positive for all values of blowing and suction. Thus in all cases nonlinearity has a destabilising influence and we have found the existence of a threshold amplitude  $|C_1|_e$  as discussed by Stuart (1960, 1971) and Watson (1960) such that all disturbances of amplitude less than threshold decay whereas those greater than threshold become infinite as the distance from the neutral stability position increases. This in turn leads to turbulence.

Our results described above seem to suggest, perhaps unexpectedly, that the effect of including suction in the rotating disc problem tends to lower the threshold amplitude for the disturbance, see Figure (2) and result (3.3c). Moreover, Figure (2) indicates that only very moderate suction or blowing is needed in order to have a significant effect on the critical disturbance amplitude. In the work of Dhanak (1991) it was found that suction stabilises the flow in as much that inclusion of suction reduces the region of wavenumber/Reynolds number parameter space in which the basic profile is unstable to stationary, infinitesimal modes. However, since the threshold amplitude decreases with increasing suction we speculate that an experimental configuration would need to be less strongly forced in order for the subcritical instability to occur than would be the case for the zero suction problem:— conversely, imposition of blowing destabilises the flow according to linear analysis but the much larger threshold amplitude suggests that exciting the subcritical disturbance experimentally might be extremely difficult.

We now say a few words about the extension of our present problem into the compressible regime. Seddougui (1990) considered the effects of compressibility on the subcritical mode of  $M$  and found that for both an adiabatic wall and an isothermal wall the stationary mode is only possible over a finite range of Mach numbers. The analysis conducted by Seddougui (1990) is naturally more involved than that of  $M$  but we have seen here that, at least for the incompressible flow problem, the inclusion of suction requires only relatively minor changes to the nonlinear analysis. With suction the variations in the coefficients of amplitude equation (2.14) are predominantly due to the changes in the wall shears; these are in turn due to the changing boundary condition (2.4c) for the basic flow. Preliminary consideration of the compressible version of the present problem also suggests that the modifications required in order to account for suction in Seddougui's (1990) work will also be relatively straightforward. Further studies on this aspect are in progress together with an investigation into the properties of the time dependent version of the mode considered here.

Finally there is the question of the extension of the present work into the fully nonlinear regime. Examples of fully nonlinear calculations for suction/blowing problems are typified by those of Smith & Stewartson (1973*a, b*) for plate and slot injections into supersonic boundary layers. Extension of our work along the lines suggested in these papers would be valuable indeed.

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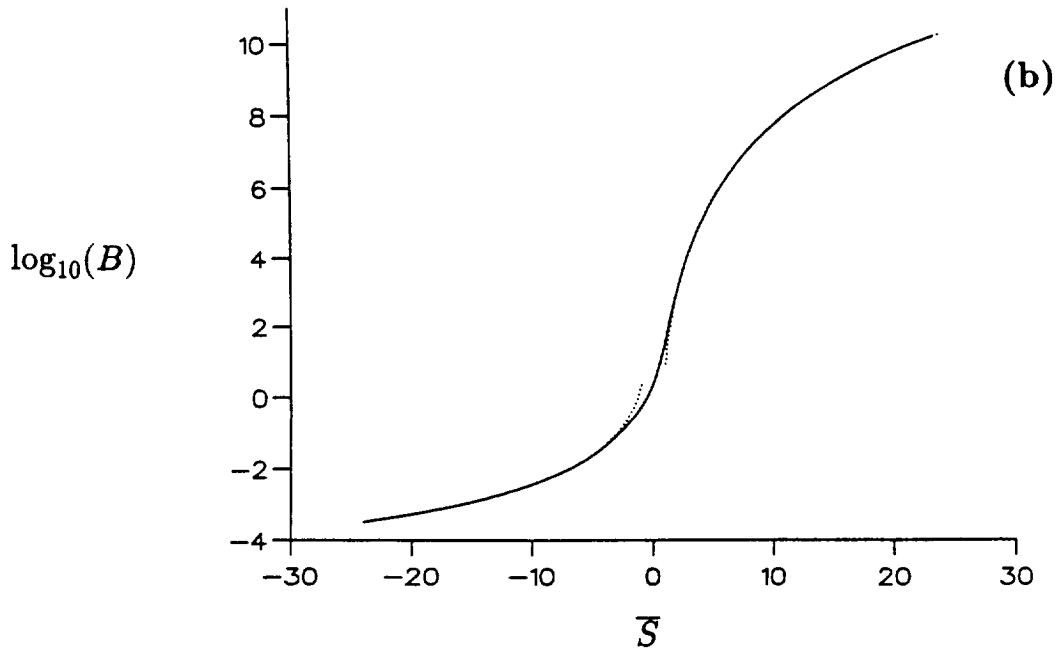
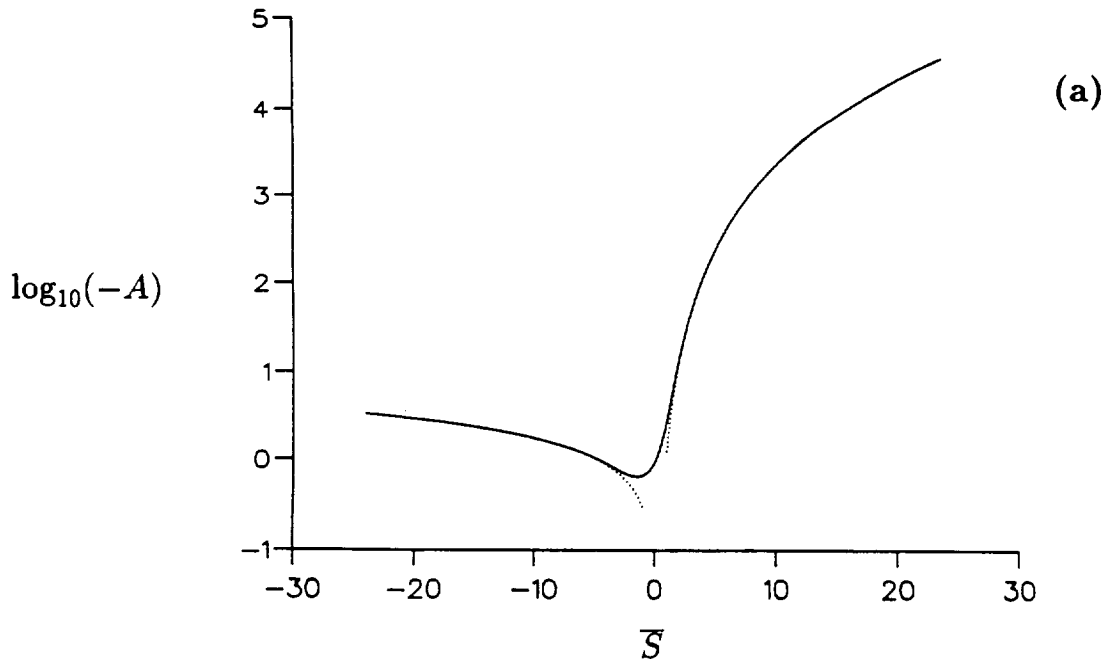


Figure (1). The dependences of amplitude equation coefficients  $A$  and  $B$  defined by (2.14), (2.15) and (3.1) on the suction parameter  $\bar{S}$ . *a*)  $\log_{10}(-A)$ : dotted lines indicate the asymptotic values (3.3a) and (3.6a), *b*)  $\log_{10}B$ : dotted lines indicate the asymptotic values (3.3b) and (3.6b).

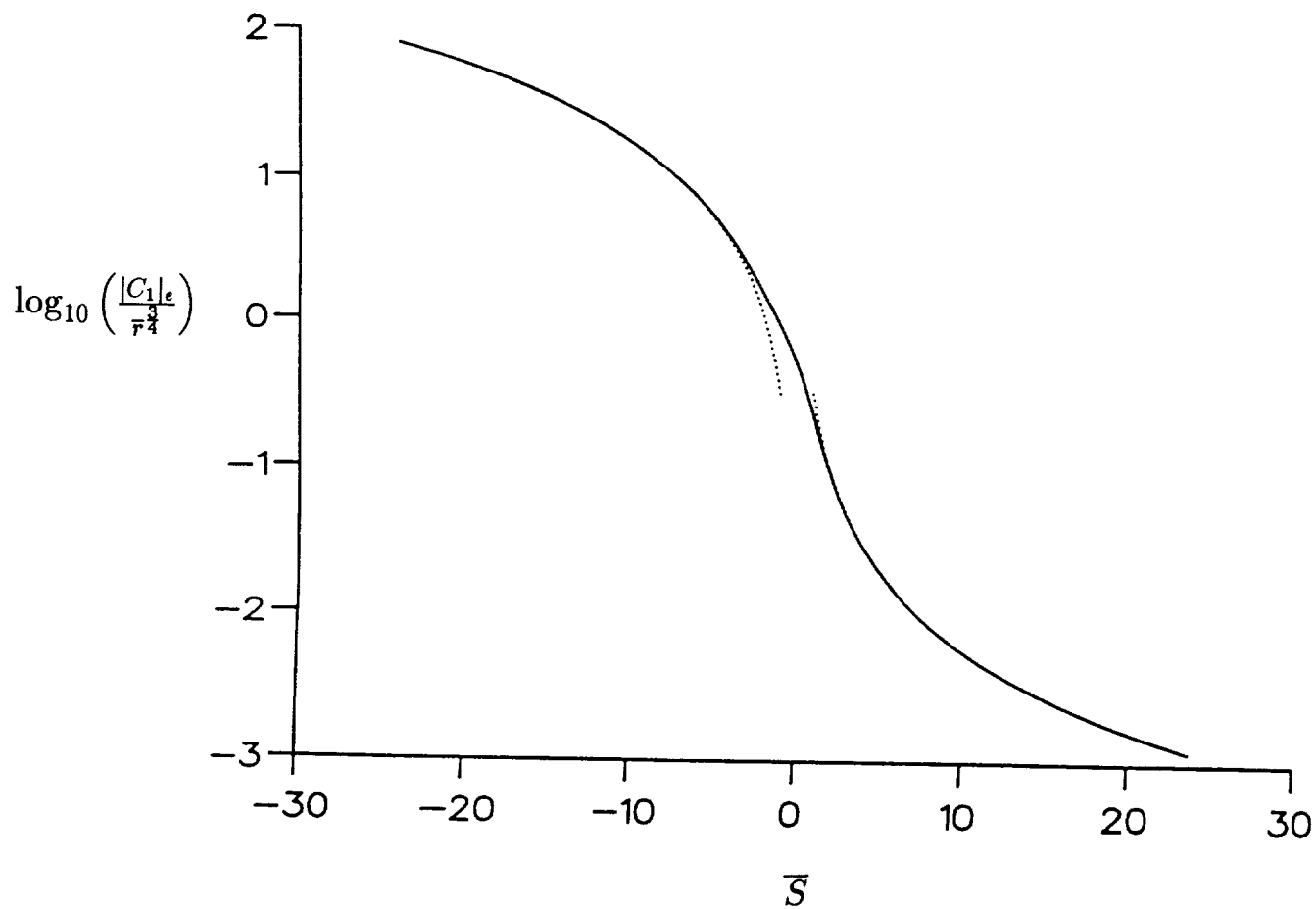


Figure (2). Threshold amplitude parameter  $|C_1|_e \bar{r}^{-3/4}$  as a function of suction parameter  $\bar{S}$  together with asymptotic values (3.3c) and (3.6c) shown dotted.







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