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# A THEORETICAL STUDY OF THE STEADY STATE OF A SPACE PLASMA


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An important and outstanding problem in Space Plasma Physics is accounting for the velocity distributions observed in the solar wind and the Earth's magnetosphere. This translates to seeking the steady-state configuration of the plasma, the properties of which are strongly related to the wave activity in the plasma. In many regions, particle distributions are well approximated by a single hump distribution known as the $\kappa$ (Kappa) distribution (see for example, *Lui and Krimigis*, 1981). *Hasegawa et al.* (1985) have derived the $\kappa$ distribution as a steady-state plasma distribution using a Fokker-Planck description of the plasma. However, from the study, it is difficult to actually determine the value for $\kappa$ for comparing to fits of the data. This study restricts itself to the situation of electrostatic wave activity in a plasma with zero magnetic field and where the (positive) ion motion can be neglected. The end result of the study is a simple criterion for determining the shape of the distribution, or, more specifically, $\kappa$.

The fundamental equations for a plasma are the Vlasov equation for the dynamics of the electron distribution function f(t,x,v) and Gauss's law for the electrostatic electric field E(t,x). Written in Gaussian units, they are, respectively,

$$\partial_t f + v\partial_x f - \frac{e}{m} E\partial_v f = 0 \tag{1}$$

$$\partial_x E = 4\pi e\, n_o - 4\pi e \int_{-\infty}^{\infty} dv\, f \tag{2}$$

The independent variables are time (t), space (x) and particle velocity (v) in the x direction. A partial derivative with respect to one of the variables is denoted by subscripting the partial derivative symbol $\partial$ . The electric field vector, and the positive x and v axes all point in the same direction. The electron has mass m and charge $-e$. The ions have charge e and are uniformly distributed over space with number density $n_o$. The normalization for the electron distribution is that the number of electrons between (x,v) and (x+dx,v+dv) is f(t,x,v)dx dv.

Observed particle distributions represent a type of spatial average. For comparison with observations and to get a simpler equation to analyze, the electron distribution is written as the sum of two distributions, $f(t,x,v) = f(t,v) + f'(t,x,v)$. As seen from the arguments of functions, f(t,v) is the spatially uniform part of the distribution and $f'(t,x,v)$ is the part which can vary with x. In practice, f(t,v) would be the observable part of the distribution. It will be assumed that the plasma is infinite in extent. This means that one way to determine f(t,v) is to take the limit of f(t,x,v) as $|x| \to \infty$, and extract the uniform part of the distribution. Another way to determine f(t,v), which will be applied here, is to define the following averaging process

$$\langle A \rangle = \lim_{L \to \infty} \int_{-L}^{L} \frac{dx}{2L} A(x) \tag{3}$$

At infinity $f'$ will at most oscillate with finite amplitude (i.e., wave amplitude is bounded), so that $\langle f' \rangle = 0$. Applying (3) to the total distribution function gives

$$\lim_{L \to \infty} \int_{-L}^{L} \frac{dx}{2L} f(t,x,v) = f(t,v) \tag{4}$$

Applying (3) to (2) gives

$$\lim_{L \to \infty} \frac{E(t,L) - E(t,-L)}{2L} = 4\pi e\, n_o - 4\pi e \int_{-\infty}^{\infty} dv\, f(t,v) \tag{5}$$

It is unrealistic to have a large scale, uniform electric field, therefore, the condition $\langle E \rangle = 0$ is imposed. This amounts to using the boundary condition that E at most oscillates with a finite amplitude at infinite distances. Equation (5) then says that the quasi-neutrality condition is enforced; namely, that

$$\int_{-\infty}^{\infty} dv\, f(t,v) = n_o \tag{6}$$

In place of (2), one has

$$\partial_x E = -4\pi e \int_{-\infty}^{\infty} dv\, f' \tag{7}$$

Applying (3) to (1) gives

$$\partial_t f(t,v) - \frac{e}{m}\langle E\partial_v f'\rangle = 0 \tag{8}$$

As a check of consistency, it will now be verified that (6) is true for all time, along with determining some other important properties of the first and second velocity moments of f(t,v). Integrating (8) over all velocity values gives that the zeroth velocity moment of f(t,v) is constant in time, which is consistent with (6). The first velocity moment of (8) involves the zeroth moment of f′, which is eliminated using (7). This gives that the first velocity moment of f(t,v) is also constant, and its value is taken to be zero by choice of reference frame. Recalling that no large scale electric field is allowed, there can be no uniform current flowing in the plasma. Therefore, the Maxwell-Ampere equation for electrostatic waves simplifies to

$$\partial_t E = 4\pi e \int_{-\infty}^{\infty} dv\, v\, f' \tag{9}$$

Making use of (9), the second moment of (8) is a statement of energy conservation. In summary, the first three moments of (8) give

$$\frac{d}{dt}\int_{-\infty}^{\infty} dv\, f(t,v) = 0\ ;\ \frac{d}{dt}\int_{-\infty}^{\infty} dv\, v\, f(t,v) = 0\ ;\ \frac{d}{dt}\left\{\frac{1}{2}m\int_{-\infty}^{\infty} dv\, v^2 f(t,v) + \frac{1}{8\pi}\langle E^2\rangle\right\} = 0 \tag{10}$$

Equation (8) is just one of the equations required to solve for f(t,v). Since f(t,v) depends on f′ and E, both (1) and (7) are also needed. Substituting (8) into (1) gives

$$\partial_t f' + v\partial_x f' - \frac{e}{m}E\partial_v f(t,v) - \frac{e}{m}\left\{E\partial_v f' - \langle E\partial_v f'\rangle\right\} = 0 \tag{11}$$

The full set of equations for the present study are (7), (8) and (11). Since observations show that the $\kappa$ distribution is a persistent feature in many regions of the space plasma environment, an analysis of the steady-state properties of the plasma is desired. The existing set of equations require that f(t,v) and f′ be specified at some time, call it the "initial" time $t=0$ (E at each time is uniquely specified by (7) subject to the boundary condition that the spatial dependence of E at most oscillates with a finite amplitude at infinity).

Consider the following initial condition

$$f'(0,x,v) = A\, g(v)\sin(kx) \quad \text{and} \quad E(0,x) = E_o\cos(kx) \tag{12}$$

where A, $E_o$ and k are related constants. The averaging process encountered in evaluating the dynamics in (8) and (11) can be performed using the trigonometric identities

$$2\sin(kx)\cos(kx) = \sin(2kx) \quad \text{and} \quad \cos^2(kx) = \frac{1}{2}+\frac{1}{2}\cos(2kx) \tag{13}$$

Application of (3) to (13) gives

$$\langle\sin(kx)\cos(kx)\rangle = 0 \quad \text{and} \quad \langle\cos^2(kx)\rangle = \frac{1}{2} \tag{14}$$

Substitution of (12) into (8) gives that the first partial time derivative of f(t,v) is initially zero. Evaluation of the second partial derivative requires using (9) and (11) to determine the initial values of the first partial time derivatives of E and f′, respectively. The result is that second partial time derivative of f(t,v) is not zero for any f(0,v), which means that the uniform part of the distribution will always change as a result of the initial condition. Therefore, at least for initial conditions of the same type as (12), there can be no steady-state of the plasma, for after the plasma has evolved to a final state, the system can be caused to change again by externally reinitiating a spatial perturbation. In respect to the Vlasov description of the plasma, external means of causing spatial variation would be due to particle effects, such as discreteness and collisions. For later reference, both of the terms in (11) enclosed within $\{\ \cdots\ \}$ give zero contribution to the second derivative of f(t,v).

A second possibility to a steady-state equilibrium is that the plasma eventually resides in a state of minimal change. In other words, as a result of the plasma response to perturbations, it eventually attains a state for which the net change in f(t,v) is a minimum, but not zero. This will be referred to as a quasi-steady state, since the state could still be changed if perturbed. To proceed, the following assumptions are made:

1) Waves are initiated by something external to the Vlasov theory of a plasma.

2) Waves are repeatedly being initiated.

3) The rate at which energy is delivered to the plasma through repeated wave initiations is much slower than the rate at which the quasi-steady state is attained.

It should be emphasized that, in the present context, "external" means external to the Vlasov theory. Ultimately, the source of energy comes from the plasma particles either from kinetic (particle) or potential (electrical) energy, but through a mechanism unaccounted for by the present theory. However, any more encompassing theory would be correspondingly more difficult to handle. Assumptions (1) and (2) provide an operational means of attaining quasi-steady state. Together they imply that it is insufficient to simply solve the initial-value problem one time through, as the final state obtained could readily be change by repeating the perturbation. Assumption (3) is made in light of energy conservation. As seen from the second moment equation in (10), the energy brought into the system from a given perturbation goes into plasma thermal energy. Consistency of Assumptions (1) and (2) together with the fact that plasma temperatures are finite, leads one to the assumption that the energy from the perturbations is slowly supplied to the plasma.

Further work is necessary to specify a precise meaning of the statement that the net change in f(t,v) must be minimized in a quasi-steady state. Clearly, the statement should in some way apply to the whole distribution, rather than some particular velocity or even some particular velocity moment. Consider the $\kappa$ distribution,

$$f_\kappa = \frac{A_\kappa}{(1+\frac{v^2}{2\kappa v_e^2})^\kappa} \tag{15}$$

where $A_\kappa$ is a normalization constant and $v_e$ is the thermal speed. The only parameter that should be affected by minimizing the change in the whole distribution is $\kappa$, since it determines the shape of the function. The other parameters, $A_\kappa$ and $v_e$, are subject to the constraints on the moments of the distribution, the number density and thermal energy of the plasma, respectively.

An analytical expression for the change in f(t,v) is now sought subject to simplifying assumptions. Firstly, as the system approaches quasi-steady state, it is reasonable to assume that $f'$ is small compared to f(t,v). Consistency with (7) implies that the electric field is also small. Consequently, the terms within $\{\ \cdots\ \}$ in (11), which contain products of small quantities, will be neglected. The resulting equation is the linearized Vlasov equation with a time varying uniform distribution. Solving this equation for $f'(t,x,v)$ with $f'(0,x,v)$ as its initial condition gives

$$f'(t,x,v) = \frac{e}{m}\int_0^t dt' E(t',x-v(t-t'))\partial_v f(t',v) + f'(0,x-vt,v) \tag{16}$$

(This expression can be verified by direct substitution into the linearized form of (11).) Substitution of (16) into (8) gives

$$\partial_t f(t,v) = \frac{e^2}{m^2}\int_0^t dt' \langle E(t,x)\, E(t',x-v(t-t'))\rangle + \frac{e}{m}\langle E(t,x)\, f'(0,x-vt,v)\rangle \tag{17}$$

The right-hand side of this equation is quadratic in small quantities, which means that f(t,v) changes slowly. These terms must be included to account for the leading order changes in f(t,v). Even though (17) is an approximation, the main conclusions of the analysis thus far still apply. In particular, (10) still holds as well as the conclusion that f(t,v) does change subject to the perturbation in (12), since the negligible terms in (11) do not contribute to the initial value of the second partial time derivative of f(t,v).

Consistent with quasi-steady state being defined as a state of minimal change in f(t,v) is the assumption that the change in f(t,v) is small for this state. Therefore, in determining the change in f(t,v) at or near quasi-steady state, f(t,v) can be treated as constant in (16) and on the right-hand side of (17), where the function f(0,v) will be used in its place. Note that the starting time $t=0$ here refers to the start time of one of the perturbations that occurs when the system is close to quasi-steady state. Under this assumption, $f'$ and E obey the standard linearized Vlasov and Gauss equations. In particular, the temporal behavior of the electrostatic field can in many situations be approximated by a product of a monotonic and an oscillatory function corresponding to damping with rate $\gamma$ (damping when $\gamma > 0$) and a oscillation of angular frequency $\omega_o$, respectively (see for example, *Nicholson*, 1983). The rates $\gamma$ and $\omega_o$ are determined from the root of the dielectric function, which itself depends on f(0,v). Therefore, the following explicit form for the electrostatic field will be used to calculate the change in f(t,v) near quasi-steady state

$$E(t,x) = E_o e^{-\gamma t} \cos(\omega_o t) \cos(kx) \tag{18}$$

The same initial condition of (12) is in use. Substitution of (12) and (18) into (17) enables one to calculate the net change in f(t,v) due to this perturbation. As an intermediate step, the time integral in (17) is evaluated to give

$$\int_0^t dt' \langle E(t,x)\, E(t', x - v(t-t')) \rangle = \sum_{\sigma=\pm 1} \frac{E_o^2}{2} e^{-\gamma t}\cos(\omega_o t) \left\{ \frac{e^{-\gamma t}\{-\gamma \cos(\omega_o t) + (\omega_o - \sigma kv)\sin(\omega_o t)\} + \gamma\cos(kvt) - \sigma(\omega_o - \sigma kv)\sin(kvt)}{\gamma^2 + (\omega_o - \sigma kv)^2} \right\}$$

The final expression for the net change in f(t,v) is

$$f(\infty,v) - f(0,v) = \int_0^\infty dt\, \partial_t f(t,v) = \sum_{\sigma=\pm 1}\sum_{\mu=\pm 1} \frac{e^2}{4m^2} \frac{E_o^2 \partial_v f(0,v)}{\gamma^2 + (\omega_o - \sigma kv)^2} \left\{ \frac{-1}{4\gamma} + \frac{-\gamma^2 + \omega_o - \sigma kv}{2(\gamma^2 + \omega_o^2)} + \frac{\gamma^2 - \sigma\mu(\omega_o - \sigma kv)(\omega_o - \mu kv)}{\gamma^2 + (\omega_o - \mu kv)^2} \right\}$$

$$+ \sum_{\sigma=\pm 1} \frac{e}{4m} A E_o \partial_v \left\{ g \frac{\omega_o - \sigma kv}{\gamma^2 + (\omega_o - \sigma kv)^2} \right\} \tag{19}$$

From this expression, the damping rate $\gamma$ is the only factor that can be used to minimize the change in f(t,v) without singling out any one of the terms in (19). Therefore, it is concluded that the criterion for attaining quasi-steady-state is that f((t,v) will be such that it maximizes the damping rate, or, in reference to (15), $\kappa$ attains a value such that $\gamma$ is maximized. Physically, greater damping means that the waves dissipate faster and exist for shorter times. That this leads to a smaller change in f(t,v) is reasonable on the basis that it is the waves that change f(t,v) and waves which exist for shorter times with smaller amplitudes will produce less overall effect.

It is necessary to discuss the special case of $\gamma=0$. If such a mode exists, (19) shows that f(t,v) changes infinitely fast at all velocities. This is a clear breakdown of the approximations that lead to (19), chiefly the assumption that f(t,v) changes slowly. In such a situation, one expects that the energy oscillates back and forth between the particles and field without any loss. The criterion of minimal change in f(t,v) in some sense still applies, but now in reference to the time average, with there being no net change on the average. However, in reference to the $\kappa$ distribution, a $\gamma=0$ mode exists only for infinite wavelength waves ($k \rightarrow 0$); namely, a uniform electric field that varies in time. For a uniform field, f(t,v) would have to be changing as fast as the oscillation to produce the required charge separation at infinity. Such a possibility was ruled out from the onset of this study on the grounds that a uniform electric field infinite in extent is physically unlikely to exist.

As a preliminary evaluation of the theory, compare the damping rates $\gamma_1$ and $\gamma_\infty$ for the $\kappa=1$ and $\kappa \rightarrow \infty$ distributions, respectively, the latter being a Maxwellian distribution.

$$\frac{\gamma_1}{\omega_o} = k\lambda_D \;\; ; \;\; \frac{\gamma_\infty}{\omega_o} \approx \left[\frac{\pi}{8}\right]^{\frac{1}{2}} e^{-3/2} \frac{e^{-1/(2k^2\lambda_D^2)}}{k^3\lambda_D^3} \tag{20}$$

The parameter $\lambda_D$ is the Debye length for the plasma. The expression for $\gamma_\infty$ applies only for $k\lambda_D \ll 1$. Of these two values, $\kappa=1$ would be the quasi-steady state distribution, since it gives a larger damping rate, at least for long wavelength perturbations. This is consistent with fits to observed particle distributions, which show that $\kappa$ is not very large. The complete range of allowable $\kappa$ must be analyzed for a better comparison with observations.

In summary, examination of the Vlasov theory of a plasma lead to the hypothesis that a plasma may reside in a state of minimal change of the uniform distribution. This statement was made definite by determining that the change in the whole distribution can be minimized if the damping rate were maximized. A preliminary test of the theory shows that one would expect a plasma well fit by a $\kappa$ distribution to have a low $\kappa$ value.

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