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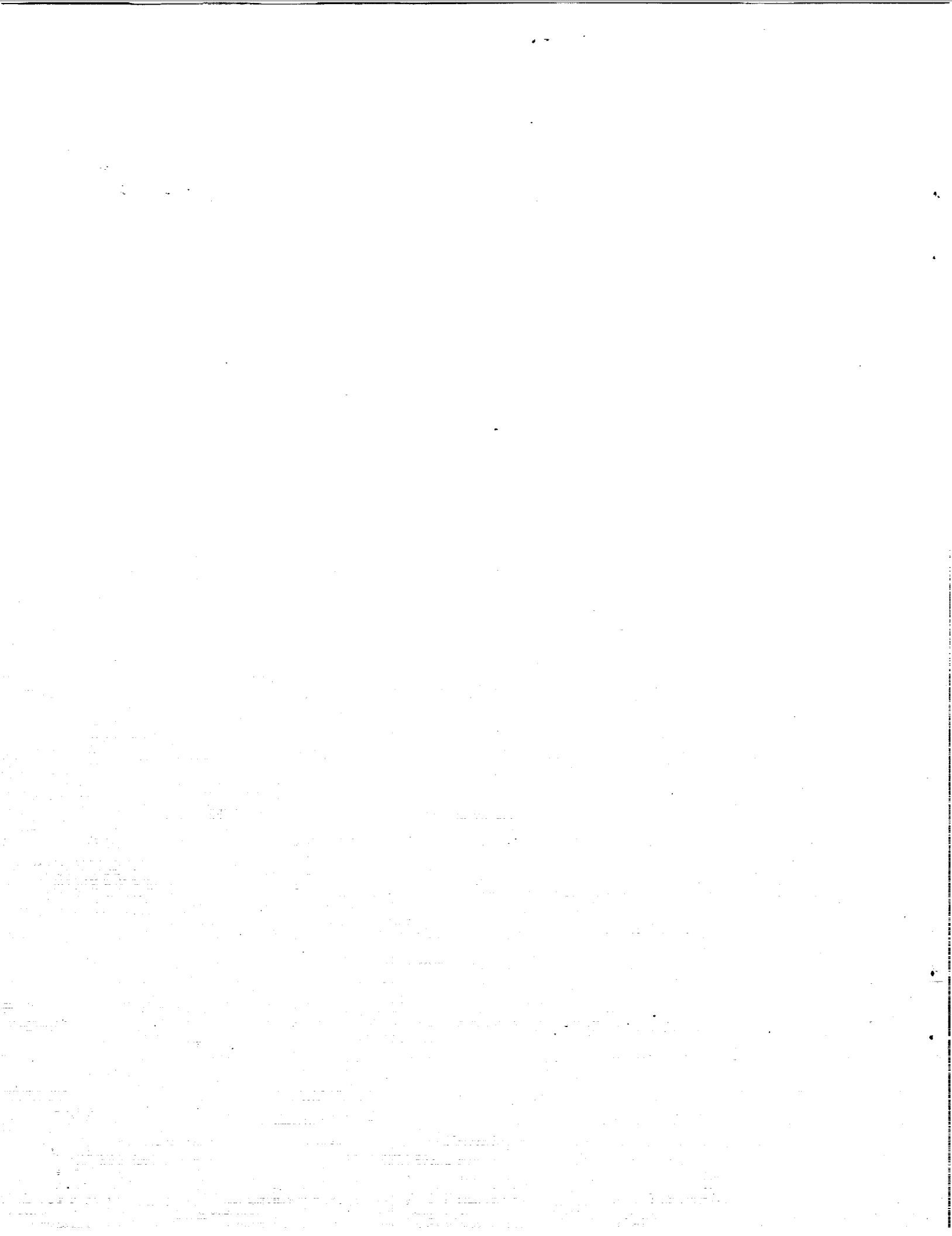
Mixed Variational Formulations of Finite Element Analysis of Elastoacoustic/Slosh Fluid-Structure Interaction

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MIXED VARIATIONAL FORMULATION OF FINITE ELEMENT ANALYSIS OF ACOUSTOELASTIC/SLOSH FLUID-STRUCTURE INTERACTION

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ABSTRACT

A general three-field variational principle is obtained for the motion of an acoustic fluid enclosed in a rigid or flexible container by the method of canonical decomposition applied to a modified form of the wave equation in the displacement potential. The general principle is specialized to a mixed two-field principle that contains the fluid displacement potential and pressure as independent fields. This principle contains a free parameter α. Semidiscrete finite-element equations of motion based on this principle are displayed and applied to the transient response and free-vibrations of the coupled fluid-structure problem. It is shown that a particular setting of α yields a rich set of formulations that can be customized to fit physical and computational requirements. The variational principle is then extended to handle slosh motions in a uniform gravity field, and used to derived semidiscrete equations of motion that account for such effects.

1. INTRODUCTION

An elastic container (the *structure*) is totally or partly filled with a compressible liquid or gas (the *fluid*). The fluid structure system is initially in static equilibrium in a steady body force field such as gravity or centrifugal forces. We consider *small departures from equilibrium* that result in forced or free vibratory motions. To analyze these motions the fluid is treated as a linear acoustic fluid, *i.e.*, compressible but irrotational and inviscid. The purpose of the present work is

1. To derive variational equations of motion based on a mixed variational principle for the fluid subsystem.

2. To obtain semidiscrete equations of motion following spatial discretization of the coupled problem by the finite element method.

The derivation of the mixed variational principle for the fluid is based on the method of *canonical equations* advocated by Oden and Reddy [13] for mechanical applications. The most general dynamical principle derived in this paper contains three primary variables: pressure-momentum vector, dilatation-velocity vector, and displacement potential.

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The general principle is specialized to a two-field functional of Reissner type that has pressure and displacement potential as primary variables, as well as a free coefficient α that parametrizes the application of the divergence theorem. The coupled variational equations are then discretized by the finite element method, and semidiscrete equations for a rigid container established. Linkage with the structure is then made to establish coupled semidiscrete equations of motion for a flexible container. By appropriate selection of the coefficient α a continuum of finite element formulations results. One particular setting yields a rich set of symmetric and unsymmetric formulations for the transient and free-vibrations elastoacoustic problems. From this set selections can be made to satisfy various physical and computational criteria. The implications of these selections as regards efficiency and numerical stability are discussed.

The variational formulation is then extented to cover slosh motions in a uniform gravity field. It is shown that the surface slosh equations may be incorporated as Galerkin terms in several forms, and that one of these forms merges naturally with the mixed variational principle to form an augmented functional. Semidiscretization of this functional produces finite element equations of motions that may be used for a rigid or flexible container.

2. GOVERNING EQUATIONS

The three-dimensional volume domain occupied by the fluid is denoted by V. This volume is assumed to be simply connected. The fluid boundary S consists generally of two portions

$$S : S_d \cup S_p. \tag{1}$$

S_d is the interface with the container at which the normal displacement d_n is prescribed (or found as part of the coupled fluid-structure problem) whereas S_p is the "free surface" at which the pressure p is prescribed (or found as part of the "fluid slosh" problem). If the fluid is fully enclosed by the container, as is necessarily the case for a gas, then S_p is missing and $S \equiv S_d$. The domain is referred to a Cartesian coordinate system (x_1, x_2, x_3) grouped in vector $\mathbf{x}$.

The fluid is under a body force field $\mathbf{b}$ which is assumed to be the gradient of a *time independent* potential $\beta(\mathbf{x})$, *i.e.* $\mathbf{b} = \nabla\beta$. All displacements are taken to be infinitesimal and thus the fluid density ρ may be taken as invariant.

We consider three states or configurations: *original*, from which displacements, pressures and forces are measured, *current*, where the fluid is in dynamic equilibrium at time t, and *reference*, which is obtained in the static equilibrium limit of slow motions. *Transient* motions are the difference between current and reference states. It should be noted that in many situations the original configuration is not physically attainable. Table 1 summarizes the notation used in relation to these states.

2.1 Field Equations

The governing equations of the acoustic fluid are the momentum, state and continuity equations. They are stated below for the current configuration, and specialized to the reference configuration

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Table 1   Notation for Fluid States

Quantities	Domain	Original	Reference	Current	Transient
Displacements	V	0	$\mathbf{d}^0$	$\mathbf{d}^t$	$\mathbf{d} = \mathbf{d}^t - \mathbf{d}^0$
Velocities	V	0	$\dot{\mathbf{d}}^0$	$\dot{\mathbf{d}}^t$	$\dot{\mathbf{d}} = \dot{\mathbf{d}}^t - \dot{\mathbf{d}}^0$
Boundary displacements*	S	0	d_n^0	d_n^t	$d_n = d_n^t - d_n^0$
Displacement potential	V	0	ψ^0	ψ^t	$\psi = \psi^t - \psi^0$
Pressures (+ if compressive)	V	0	p^0	p^t	$p = p^t - p^0$
Body forces	V	0	$\mathbf{b} = \nabla\beta$	$\mathbf{b} = \nabla\beta$	
Density	V	ρ	ρ	ρ	

* Positive along outward normal

later. The momentum (balance) equation expresses Newton's second law for a fluid particle:

$$\rho\ddot{\mathbf{d}}^t = -\nabla p^t + \mathbf{b} = -\nabla p^t + \nabla\beta. \tag{2}$$

The continuity equation may be combined with the linearized equation of state to produce the *constitutive equation* that expresses the small compressibility of a liquid:

$$p^t = -K\nabla\mathbf{d}^t = -\rho c^2 \nabla\mathbf{d}^t, \tag{3}$$

where K is the bulk modulus and $c = \sqrt{K/\rho}$ the fluid sound speed. If the fluid is incompressible, $K, c \to \infty$. This relation is also applicable to nonlinear elastic fluids such as gases undergoing small excursions from the reference state, if the constitutive equation is linearized there so that $K = \rho_0 (dp/d\rho)_0$.

The boundary conditions are

$$d_n^t = \bar{d}_n^t \quad \text{on} \quad S_d, \qquad p^t = \bar{p}^t \quad \text{on} \quad S_p, \tag{4}$$

where $\bar{d}_n^t$ is either prescribed or comes from the solution of an auxiliary problem as in fluid-structure interaction, and $\bar{p}$ may be either prescribed or a function of d_n and $\mathbf{b}$, as in the surface-wave ("slosh") problem.

2.2 Integral Abbreviations

In the sequel the following abbreviations for the volume and surface integrals are used:

$$(f)_V \overset{\text{def}}{=} \int_V f\,dV, \qquad [g]_S \overset{\text{def}}{=} \int_S g\,dS, \qquad [g]_{S_d} \overset{\text{def}}{=} \int_{S_d} g\,dS, \quad \text{etc.} \tag{5}$$

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That is, domain-subscripted parentheses (square brackets) are used to abbreviate volume (surface) integrals. Abbreviations for function innerproducts are illustrated by

$$(f,g)_V \stackrel{\text{def}}{=} \int_V fg\, dV, \qquad (f,g)_{V \times t} \stackrel{\text{def}}{=} \int_{t_0}^{t_1} \int_V fg\, dV\, dt, \qquad [f,g]_{S_4 \times t} \stackrel{\text{def}}{=} \int_{t_0}^{t_1} \int_{S_4} fg\, dS\, dt, \quad \text{etc.}$$

(6)

3. THE DISPLACEMENT POTENTIAL

3.1 The Reference State

Taking the curl of both sides of (2) yields

$$\text{curl}\, \bar{\mathbf{d}}^t = 0.$$

(7)

The general integral of this equation for a simply connected domain is

$$\mathbf{d}^t = \nabla \psi^t + \mathbf{a} + \mathbf{b}t,$$

(8)

where $\psi^t = \psi^t(\mathbf{x}, t)$ is the displacement potential, $\mathbf{a} = \mathbf{a}(\mathbf{x})$ and $\mathbf{b} = \mathbf{b}(\mathbf{x})$ are time-independent vector functions, and t denotes the time. If accelerationless motions (for example, rigid body motions) are precluded by the boundary conditions, $\mathbf{a}$ and $\mathbf{b}$ vanish. Replacing $\bar{\mathbf{d}}^t = \nabla \bar{\psi}^t$ into the momentum equation (2) we get

$$\nabla p^t = -\rho \nabla \bar{\psi}^t + \nabla \beta,$$

(9)

which spatially integrated gives

$$p^t = -\rho \bar{\psi}^t + \beta + C(t),$$

(10)

where the scalar $C(t)$ is not spatially dependent. Next integrate the constitutive equation (3) over V and apply the divergence theorem to $\nabla \mathbf{d}$:

$$(p^t)_V + (\rho c^2 \nabla \mathbf{d}^t)_V = (p^t)_V + [\rho c^2 d_n^t]_S = 0.$$

(11)

Inserting p^t from (10) into the above equation furnishes a condition on $C(t)$, which gives

$$C(t) = -\frac{\rho c^2}{v}[d_n^t]_S + \frac{\rho}{v}(\bar{\psi}^t)_V - \frac{1}{v}(\beta)_V = -\frac{\rho c^2}{v}[d_n^t]_S + \rho \overline{\bar{\psi}^t} - \overline{\beta},$$

(12)

where $v = (1)_V$ is the fluid volume and $\overline{f} = (f)_V / v$ denotes the volume average of a function f defined over V. Substituting $C(t)$ into (10) we get

$$p^t = -\rho(\bar{\psi}^t - \overline{\bar{\psi}^t}) + (\beta - \overline{\beta}) - \frac{\rho c^2}{v}[d_n^t]_S.$$

(13)

In the static limit of very slow motions, the inertia terms may be neglected and we recover the reference solution

$$p^0 = (\beta - \overline{\beta}) - \frac{\rho c^2}{v}[d_n^0]_S.$$

(14)

For an incompressible fluid $[d_n]_S = 0$ but $c \to \infty$; thus it would be incorrect to conclude that $p^0 = \beta - \overline{\beta}$. A counterexample to this effect is provided in [14].

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3.2 Transient Motions

Subtracting the constitutive relations at the current and reference states we get

$$p = -\rho c^2 \nabla^2 \psi = \rho c^2 s, \tag{15}$$

where $s = -\nabla^2 \psi$ is called, following Lamb [10], the condensation. Subtracting (14) from (13) yields

$$p = -\rho(\ddot{\psi} - \overline{\ddot{\psi}}) - \frac{\rho c^2}{v}[d_n]s. \tag{16}$$

On equating (15) and (16) we get modified forms of the wave equation that account for mean boundary surface motions:

$$s = \nabla^2 \psi = \frac{\ddot{\psi} - \overline{\ddot{\psi}}}{c^2} + \frac{1}{v}[d_n]s, \qquad \text{or} \qquad c^2(\nabla^2 \psi - \overline{\nabla^2 \psi}) = \ddot{\psi} - \overline{\ddot{\psi}}. \tag{17}$$

The second form follows from $-v\bar{s} = [d_n]s$, which is a consequence of the divergence theorem. For an incompressible fluid, $c \to \infty$ and $[d_n]s = 0$, and from the first of (17) we recover the Laplace equation $\nabla^2 \psi = 0$.

3.3 Adjusting the Displacement Potential

If the transient displacement potential is modified by a function of time:

$$\psi = \hat{\psi} + P(t), \tag{18}$$

where $\hat{\psi}$ is the potential of (8)-(17), we may chose $P(t)$ so that $c^2 \overline{\ddot{\psi}} = \overline{\nabla^2 \psi} = -\bar{s}$ for any t. Then we obtain the classical wave equation

$$c^2 \nabla^2 \psi = \ddot{\psi}, \qquad \text{or} \qquad \left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2 \right) \psi = 0. \tag{19}$$

In the sequel it is assumed that this adjustment has been made. If so, $C(t)$ vanishes and (16) reduces to

$$p = -\rho \ddot{\psi}. \tag{20}$$

4. MIXED VARIATIONAL PRINCIPLES

4.1 Canonical Decomposition

In this section we derive multifield variational principles for the fluid domain following the canonical decomposition method advocated by Oden and Reddy [13]. This method is applicable to self-adjoint boundary value problems (BVP) of the form

$$Au = f \qquad \text{in } D \tag{21}$$

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where u is the unknown function, f the data, A a symmetric linear operator, and D the domain of existence of the solution. For time-dependent problems D is the tensor product of the time domain (typically 0 to t) and the volume V. To apply this method, the operator A is factored as

$$Au = W^* E W u = f, \tag{22}$$

where T and E are linear operators in V and W^* is the adjoint of T. This is called a *canonical decomposition*. This decomposition may be represented as the operator composition sequence

$$Wu = e, \qquad Ee = \sigma, \qquad W^* \sigma = f, \tag{23}$$

where e and σ denote intermediate field variables in D. The three equations (23) are called the kinematic, constitutive and balance equations, respectively, in mechanical applications. The canonical representation of boundary conditions on the surface $S = S_u \cup S_\sigma$ is

$$B_S u_S = g \quad \text{on} \quad S_u, \qquad B_S^* \sigma_S = h \quad \text{on} \quad S_\sigma. \tag{24}$$

where B_S and B_S^* are surface operators, g and h denote boundary data, and $u_S = \gamma_S u$ and $\sigma_S = \Gamma_S \sigma$ are extensions of u and σ to the boundary S. The extension operators γ_S and δ_S often involve normal derivatives.

4.2 The Wave Equation

The classical wave equation (19) is *not* a good basis for the canonical decomposition (22). Its principal drawback is that the pressure field does not appear naturally as an intermediate variable in (23). A more convenient form for our purposes is obtained by taking the Laplacian of both sides of (19), and multiplying through by the density ρ:

$$\rho \nabla^2 (\ddot{\psi} - c^2 \nabla^2 \psi) = 0, \qquad \text{whence} \quad A = \rho \nabla^2 \left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2 \right), \quad f = 0. \tag{25}$$

A suitable canonical decomposition is $A = \mathbf{W}^* \mathbf{E} \mathbf{W}$, where

$$\mathbf{W} = \begin{bmatrix} i\nabla \frac{\partial}{\partial t} \\ -\nabla^2 \end{bmatrix}, \qquad \mathbf{E} = \rho \begin{bmatrix} \mathbf{I} & 0 \\ 0 & c^2 \end{bmatrix}, \qquad \mathbf{W}^* = \begin{bmatrix} -i\nabla \frac{\partial}{\partial t} & \nabla^2 \end{bmatrix} = -\mathbf{W}^T, \tag{26}$$

in which $i = \sqrt{-1}$. Boldface symbols are used for $\mathbf{W}$ and $\mathbf{E}$ because these are 4×1 and 4×4 matrices, respectively. The operator product sequence (23) becomes

$$e = \mathbf{W}\psi = \begin{bmatrix} i\nabla\dot{\psi} \\ -\nabla^2\psi \end{bmatrix} = \begin{bmatrix} i\mathbf{v} \\ s \end{bmatrix}, \qquad \sigma = \mathbf{E}e = \begin{bmatrix} i\rho\nabla\dot{\psi} \\ -\rho c^2 \nabla^2\psi \end{bmatrix} = \begin{bmatrix} i\mathbf{m} \\ p \end{bmatrix}, \qquad \mathbf{W}'\sigma = \rho\nabla^2\ddot{\psi} - \rho c^2 \nabla^4 \psi = 0. \tag{27}$$

The intermediate fields e and σ are 4×1 column vectors. These vectors are partitioned into their temporal and spatial derivative subvectors for convenience in subsequent manipulations. Note that

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the transient pressure p appears naturally as the spatial component of σ. The temporal components of e and σ are the complex velocity iv and complex specific momentum im, respectively.

The boundary portions S_u and S_σ of (24) are relabeled S_d and S_p, respectively, to match the notation (1). Boundary and initial conditions may be stated as

$$\mathbf{B}\psi(\mathbf{x},t) = \mathbf{g}(\mathbf{x},t) \text{ on } S_d, \qquad \mathbf{B}^*\sigma(\mathbf{x},t) = h(\mathbf{x},t) \text{ on } S_p, \qquad (28)$$
$$\mathbf{d}(\mathbf{x},t_0) = \mathbf{d}_0(\mathbf{x}) \quad \text{or} \quad \mathbf{m}(\mathbf{x},t_0) = \mathbf{m}_0(\mathbf{x}), \qquad \mathbf{d}(\mathbf{x},t_1) = \mathbf{d}_1(\mathbf{x}) \quad \text{or} \quad \mathbf{m}(\mathbf{x},t_1) = \mathbf{m}_1(\mathbf{x}).$$

Here $\mathbf{B}$ and $\mathbf{B}^*$ are time-independent 4×1 and 1×4 vectors, respectively, related to the canonical $\mathbf{B}_S$ and $\mathbf{B}_S^*$ operators of (24) by $\mathbf{B} = \mathbf{B}_S\gamma_S$ and $\mathbf{B}^* = \mathbf{B}_S^*\Gamma_S$, where γ_S (a scalar) and Γ_S (a 4×4 matrix) are boundary extension operators for ψ and σ, respectively. Comparison with (4) and the use of Green's function reveals that

$$\mathbf{B}_S^T = -\mathbf{B}_S^* = [0 \quad 0 \quad 0 \quad 1], \quad \mathbf{g}^T = [0 \quad 0 \quad 0 \quad \bar{d}_n], \quad \gamma_S = \frac{\partial}{\partial n}, \quad \Gamma_S = \mathbf{I}, \quad h = -\bar{p}. \quad (29)$$

4.3 Three Field Principle

The most general variational principle for the canonical decomposition (26) allows the three fields: ψ, e, and σ, to be varied independently. The principle may be stated as $\delta L(\psi, e, \sigma) = 0$, where the functional L is [13]

$$L(u, e, \sigma) = L_V + L_S = \tfrac{1}{2}(\mathbf{E}e, e)_{V \times t} + (\sigma, \mathbf{W}\psi - v)_{V \times t} - (f, \psi)_{V \times t} + (\sigma_S, \mathbf{B}\psi - \mathbf{g})_{S_d \times t} - (h, \psi_S)_{S_p \times t}, \quad (30)$$

where L_V and L_S collect volume and surface terms, respectively. On inserting (27-29) into (30) we get

$$L_V = \tfrac{1}{2}(\mathbf{E}e, e)_{V \times t} + (\sigma, \mathbf{W}\psi - e)_{V \times t} = \int_{t_0}^{t_1} \int_V \left[\tfrac{1}{2}\rho(-v^T v + c^2 s^2) - m^T(\nabla\psi - v) - p(\nabla^2\psi + s) \right] dV\, dt,$$

$$L_S = (\sigma_S, \mathbf{B}\psi - \mathbf{g})_{S_d \times t} - (h, \psi_S)_{S_p \times t} = \int_{t_0}^{t_1} \left[\int_{S_d} p\left(\frac{\partial\psi}{\partial n} - \bar{d}_n\right) dS + \int_{S_p} \bar{p}\frac{\partial\psi}{\partial n}\, dS \right] dt. \quad (31)$$

The term $(f, \psi)_{V \times t}$ vanishes and does not contribute to L_V.

4.4 Two Field Principles

A two field principle of Reissner type can be derived from the functional L by enforcing the inverse constitutive equations $e = \mathbf{E}^{-1}\sigma$ a priori. The resulting principle, which allows ψ and σ to be varied simultaneously, is $\delta R(\psi, \sigma) = 0$, where

$$R(\psi, \sigma) = R_V + R_S = -\tfrac{1}{2}(\mathbf{E}^{-1}\sigma, \sigma)_{V \times t} + (\sigma, \mathbf{W}\psi)_{V \times t} - (f, \psi)_{V \times t} + (\sigma_S, \mathbf{B}\psi - \mathbf{g})_{S_d \times t} - (h, \psi_S)_{S_p \times t}. \quad (32)$$

where $R_S = L_S$ and

$$R_V(\psi, \sigma) = -\tfrac{1}{2}(\mathbf{E}^{-1}\sigma, \sigma)_{V \times t} + (\sigma, \mathbf{W}\psi)_{V \times t} = \int_{t_0}^{t_1} \int_V \left(\frac{1}{2\rho}\mathbf{m}^T\mathbf{m} - \frac{p^2}{2\rho c^2} - \mathbf{m}^T \nabla\dot\psi - p\nabla^2\psi \right) dV \, dt.$$
(33)

The specific momentum disappears as an independent field if we enforce $\mathbf{m} = \rho\nabla\dot\psi$ a priori, whereupon the functional R becomes a function of ψ and p only and the volume term contracts to

$$R_V(\psi, p) = \int_{t_0}^{t_1} \int_V \left(-\tfrac{1}{2}\rho(\nabla\dot\psi)^T\nabla\dot\psi - \tfrac{1}{2}\frac{p^2}{\rho c^2} - p\nabla^2\psi \right) dV \, dt.$$
(34)

To check $R = R_V(\psi, p) + R_S$ we form its first variation[*]

$$\delta R = -\left(\rho\nabla^2\ddot\psi + \nabla^2 p, \delta\psi\right)_{V \times t} - \left(\frac{1}{\rho c^2}p + \nabla^2\psi, \delta p\right)_{V \times t} + \left[\rho\frac{\partial\ddot\psi}{\partial n} + \frac{\partial p}{\partial n}, \delta\psi\right]_{S \times t}$$

$$- \left[p - \bar p, \delta\frac{\partial\psi}{\partial n}\right]_{S_p \times t} + \left[\frac{\partial\psi}{\partial n} - \bar d_n, \delta p\right]_{S_d \times t} - \left(\rho\nabla\dot\psi, \delta\nabla\psi\right)_V \Big|_{t_0}^{t_1}.$$
(35)

Setting $\delta R = 0$ provides the field equations, boundary and initial conditions.

4.5 Parametrization

A one parameter family of variational principles can be obtained by transforming all or part of the last term in (34), viz. $p\nabla^2\psi$, by the divergence theorem (Green's first formula for the Laplace operator)

$$\int_V p\nabla^2\psi \, dV + \int_V (\nabla\psi)^T\nabla p \, dV = \int_S p\frac{\partial\psi}{\partial n} \, dS = \int_{S_d} p\frac{\partial\psi}{\partial n} \, dS + \int_{S_p} p\frac{\partial\psi}{\partial n} \, dS.$$
(36)

Let $0 \le \alpha \le 1$ be the portion of that term to be transformed. Insert $p\nabla^2\psi = \alpha p\nabla^2\psi + (1-\alpha)p\nabla^2\psi$ in (35) and apply (36) to $\alpha p\nabla^2\psi$ to get

$$R_{\alpha V} = \int_{t_0}^{t_1} \left[\int_V \left(\tfrac{1}{2}\rho(\nabla\dot\psi)^T\nabla\dot\psi - \tfrac{1}{2}\frac{p^2}{\rho c^2} + \alpha(\nabla\psi)^T\nabla p - (1-\alpha)p\nabla^2\psi \right) dV - \alpha\int_{S_d} p\frac{\partial\psi}{\partial n} \, dS - \alpha\int_{S_p} p\frac{\partial\psi}{\partial n} \, dS \right] dt.$$
(37)

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[*] The variation of the kinetic energy integral term may be expressed in two different ways,

$$\delta\left(\rho\nabla\dot\psi^T, \nabla\dot\psi\right)_{V \times t} = \left(\rho\nabla^2\ddot\psi, \delta\psi\right)_{V \times t} - \left[\rho\frac{\partial\ddot\psi}{\partial n}, \delta\psi\right]_{S \times t} + \left(\rho\nabla\dot\psi, \delta\nabla\psi\right)_V \Big|_{t_0}^{t_1},$$

$$\delta\left(\rho\nabla\dot\psi^T, \nabla\dot\psi\right)_{V \times t} = \left(\rho\nabla^2\ddot\psi, \delta\psi\right)_{V \times t} + \left[\rho\frac{\partial\dot\psi}{\partial n}, \delta\dot\psi\right]_{S \times t} - \left(\rho\nabla^2\dot\psi, \delta\psi\right)_V \Big|_{t_0}^{t_1}.$$

depending on whether integration by parts is performed first in time or space, respectively. The first form, which provides physically significant initial conditions, is used in constructing (35).

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Finally, replace the Laplacian $\nabla^2\psi$ left over in (37) by $c^{-2}\ddot{\psi}$ to arrive at the parametrized two-field functional†

$$R_\alpha(\psi,p) \doteq R_{\alpha V} + R_S = \int_{t_0}^{t_1}\left[\int_V \left(-\tfrac{1}{2}\rho(\nabla\dot\psi)^T\nabla\dot\psi - \tfrac{1}{2}\frac{p^2}{\rho c^2} + \alpha(\nabla\psi)^T\nabla p - (1-\alpha)\frac{p\ddot\psi}{c^2}\right)dV\right.$$

$$\left. + \int_{S_d} p\left[(1-\alpha)\frac{\partial\psi}{\partial n} - \bar d_n\right]dS + \int_{S_p}(\bar p - \alpha p)\frac{\partial\psi}{\partial n}\,dS\right]dt.$$

(38)

The highest spatial derivative index for both primary variables ψ and p is 1, except if $\alpha = 0$, in which case it is only 0 for p. The two interesting limit cases are of course $\alpha = 0$ and $\alpha = 1$, for which

$$R_0(\psi,p) = \int_{t_0}^{t_1}\left[\int_V\left(-\tfrac{1}{2}\rho(\nabla\dot\psi)^T\nabla\dot\psi - \tfrac{1}{2}\frac{p^2}{\rho c^2} - \frac{p\ddot\psi}{c^2}\right)dV + \int_{S_d} p\left(\frac{\partial\psi}{\partial n} - \bar d_n\right)dS + \int_{S_p}\bar p\frac{\partial\psi}{\partial n}\,dS\right]dt, \quad (39)$$

$$R_1(\psi,p) = \int_{t_0}^{t_1}\left[\int_V\left(-\tfrac{1}{2}\rho(\nabla\dot\psi)^T\nabla\dot\psi - \tfrac{1}{2}\frac{p^2}{\rho c^2} + (\nabla\psi)^T\nabla p\right)dV - \int_{S_d} p\bar d_n\,dS - \int_{S_p}(p-\bar p)\frac{\partial\psi}{\partial n}\,dS\right]dt.$$

(40)

5. FINITE ELEMENT DISCRETIZATION

5.1 Discretization of R_α

In the following we derive semidiscrete finite-element equations of motion based on the R_α functional (38). The volume V is subdivided into fluid finite elements. Over each fluid element the state is represented by the primary variables ψ and p, which are defined as functions of position in the usual shape-function interpolation procedure. The finite element interpolation in V may be expressed as

$$\psi(\mathbf{x},t) = \mathbf{N}_\psi(\mathbf{x})\,\boldsymbol{\Psi}(t), \qquad p(\mathbf{x},t) = \mathbf{N}_p(\mathbf{x})\,\mathbf{p}(t), \tag{41}$$

where $\boldsymbol\Psi$ and $\mathbf p$ are computational column vectors that contain nodal values of ψ and p, respectively, and $\mathbf N_\psi$ and $\mathbf N_p$ are corresponding row-vector arrays of dimensionless shape functions. The specified displacement over S_d is interpolated by

$$\bar d_n(\mathbf{x},t) = \mathbf{n}^T d(\mathbf{x},t) = \mathbf{n}^T\mathbf{N}_d(\mathbf{x})\bar{\mathbf{d}}, = \mathbf{N}_{dn}^T(\mathbf{x})\bar{\mathbf{d}}, \tag{42}$$

where $\mathbf n$ is the external-normal unit vector on S_d, $\mathbf N_d$ contains the displacement shape functions of the enclosing container, $\mathbf N_{dn}$ are these shape functions projected on the outward normal $\mathbf n$ on S_d,

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† If $\alpha \neq 1$, $\delta R_\alpha = 0$ is a *restricted* variational principle because the substitution $\nabla^2\psi = c^{-2}\ddot\psi$ holds only at the exact solution.

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and $\tilde{\mathbf{d}}$ contains nodal displacement values. For now the container displacements will be assumed to be prescribed, hence the superposed tilde.

In the following three Sections (5-8) we shall assume that *the prescribed-pressure boundary conditions are exactly satisfied* by the finite element interpolation, *i.e.* $p \equiv \bar{p}$ on S_p. If so the S_p integral of R_α simplifies to

$$\int_{S_p} (1-\alpha)\bar{p}\frac{\partial\psi}{\partial n}\,dS,\tag{43}$$

which vanishes for $\alpha = 1$. Inserting (41)-(42) into the functional (38) with the simplified S_p integral (43) yields the semidiscrete quadratic form

$$R_\alpha(\boldsymbol{\Psi},\mathbf{p}) = -\tfrac{1}{2}\rho\dot{\boldsymbol{\Psi}}^T\mathbf{H}\dot{\boldsymbol{\Psi}} - \frac{1}{2\rho}\mathbf{p}^T\mathbf{G}\mathbf{p} + \alpha\boldsymbol{\Psi}^T\mathbf{F}\mathbf{p} + (1-\alpha)\left[\boldsymbol{\Psi}^T\mathbf{V}\mathbf{p} - \tilde{\boldsymbol{\Psi}}^T\mathbf{D}\mathbf{p} + \boldsymbol{\Psi}^T\mathbf{f}_\psi\right] - \mathbf{p}^T\tilde{\mathbf{T}}^T\tilde{\mathbf{d}}, \tag{44}$$

where

$$\mathbf{H} = \int_V \nabla\mathbf{N}_\psi^T\nabla\mathbf{N}_\psi\,dV = \mathbf{H}^T, \qquad \mathbf{F} = \int_V \nabla\mathbf{N}_p^T\nabla\mathbf{N}_\psi\,dV, \qquad \mathbf{G} = \int_V c^{-2}\mathbf{N}_p^T\mathbf{N}_p\,dV = \mathbf{G}^T,$$

$$\mathbf{D} = \int_V c^{-2}\mathbf{N}_\psi^T\mathbf{N}_p\,dV, \quad \mathbf{V} = \int_{S_d} (\nabla_n\mathbf{N}_\psi)^T\mathbf{N}_p\,dS, \quad \tilde{\mathbf{T}}^T = \int_{S_d} \mathbf{N}_p^T\mathbf{N}_{dn}\,dS, \quad \mathbf{f}_\psi = \int_{S_p} \bar{p}\,\nabla_n\mathbf{N}_\psi\,dS.\tag{45}$$

The integration with respect to time is dropped as it has no effect on the variation process described below.

5.2 Continuity Requirements

The interelement continuity requirements of the shape functions of ψ and p depend on the index of the highest spatial derivatives that appears in R_α. If $\alpha \neq 0$, this index is 1 for both ψ and p and consequently C^0 continuity is required. It is then natural to take the *same* shape functions for both variables:

$$\mathbf{N}_\psi \equiv \mathbf{N}_p \tag{46}$$

with both vectors $\boldsymbol{\Psi}$ and $\mathbf{p}$ of equal dimension and evaluated at the same nodes. Then some of the matrices in (45) coalesce as

$$\mathbf{H} = \mathbf{F}, \qquad \mathbf{G} = \mathbf{D} = \mathbf{D}^T. \tag{47}$$

The case $\alpha = 0$ is exceptional in that no spatial derivatives of p appear. One can then chose C^{-1} (discontinuous) pressure shape functions; for example, constant over each fluid element. If this is done, obviously

$$\mathbf{N}_\psi \neq \mathbf{N}_p \tag{48}$$

because ψ must be C^0 continuous. Furthermore the dimensions of $\mathbf{p}$ and $\boldsymbol{\Psi}$ will not be generally the same.

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5.3 Singularity of H

For later use, we note that matrix H (as well as F if different from H) before the application of any essential boundary conditions at fluid nodes, is singular because

$$He = 0 \tag{49}$$

where e denotes the vector of all ones. This follows from (45) and expresses the fact that a constant potential generates no pressures or displacements.

6. TRANSIENT RESPONSE EQUATIONS

6.1 The Rigid-Container Equations of Motion

Since R_α contains time derivatives of of order up to 2 in Ψ, the appropriate Euler-Lagrange variational equation is

$$\delta R_\alpha = \left(\frac{\partial R_\alpha}{\partial \Psi} - \frac{\partial}{\partial t} \frac{\partial R_\alpha}{\partial \dot{\Psi}} + \frac{\partial^2}{\partial t^2} \frac{\partial R_\alpha}{\partial \ddot{\Psi}} \right) \delta \Psi + \frac{\partial R_\alpha}{\partial p} \delta p = 0, \tag{50}$$

which applied to (44) yields

$$\left[\rho H \ddot{\Psi} + \alpha F p - (1-\alpha) D \ddot{p} + (1-\alpha) V p + (1-\alpha) f_\psi \right] \delta \Psi = 0,$$
$$\left[-\rho^{-1} G p + \alpha F^T \Psi - (1-\alpha) D^T \ddot{\Psi} + (1-\alpha) V^T \Psi - \widetilde{T}^T \tilde{d} \right] \delta p = 0. \tag{51}$$

These equations can be presented in partitioned matrix form as

$$\begin{bmatrix} \rho H & -(1-\alpha)D \\ -(1-\alpha)D^T & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\Psi} \\ \ddot{p} \end{Bmatrix} + \begin{bmatrix} 0 & J \\ J^T & -\rho^{-1}G \end{bmatrix} \begin{Bmatrix} \Psi \\ p \end{Bmatrix} = \begin{Bmatrix} -(1-\alpha)f_\psi \\ \widetilde{T}^T \tilde{d} \end{Bmatrix}, \tag{52}$$

where $J = (1-\alpha)V + \alpha F$.

6.2 The Flexible-Container Equations of Motion

If the fluid is enclosed in a flexible container, the boundary displacements $\tilde{d}$ are no longer prescribed on S_d but must be incorporated in the problem by including them on the left hand side of the equations of motion. In the sequel, vector d collects *all structural node displacements*, of which $\tilde{d}$ is a subset on S_d. Matrix $\widetilde{T}$, suitably expanded with zeros to make it conform to d, becomes T. We shall only consider here the case in which the container is modelled as a *linear undamped* structure for which the standard mass/stiffness semidiscrete equation of motion is

$$M\ddot{d} + Kd = f_d + Tp, \tag{53}$$

where M is the mass matrix, K the tangent stiffness matrix at the reference state, Tp is the pressure force on the structure, and f_d is the externally applied force on the structure. Note that K

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in general must account for container prestress effects through the geometric stiffness. Combining (52) and (53) we get the coupled system

$$
\begin{bmatrix} \mathbf{M} & 0 & 0 \\ 0 & \rho\mathbf{H} & -(1-\alpha)\mathbf{D} \\ 0 & -(1-\alpha)\mathbf{D}^T & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & 0 & -\mathbf{T} \\ 0 & 0 & \mathbf{J} \\ -\mathbf{T}^T & \mathbf{J}^T & -\rho^{-1}\mathbf{G} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ -(1-\alpha)\mathbf{f}_\psi \\ 0 \end{Bmatrix}.
$$
(54)

If $\alpha = 0$,

$$
\begin{bmatrix} \mathbf{M} & 0 & 0 \\ 0 & \rho\mathbf{H} & \mathbf{D} \\ 0 & \mathbf{D}^T & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & 0 & -\mathbf{T} \\ 0 & 0 & \mathbf{V} \\ -\mathbf{T}^T & \mathbf{V}^T & -\rho^{-1}\mathbf{G} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ -\mathbf{f}_\psi \\ 0 \end{Bmatrix}.
$$
(55)

There is little than can be done beyond this point, as the shape functions for p and ψ will be generally different. Although the pressure may be constant over each element, no condensation of p is possible in the dynamic case.

If $\alpha = 1$,

$$
\begin{bmatrix} \mathbf{M} & 0 & 0 \\ 0 & \rho\mathbf{H} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & 0 & -\mathbf{T} \\ 0 & 0 & \mathbf{F} \\ -\mathbf{T}^T & \mathbf{F}^T & -\rho^{-1}\mathbf{G} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ 0 \\ 0 \end{Bmatrix}.
$$
(56)

Note that all these systems, (54) through (56), are *symmetric*.

6.3 Identical Shape Functions

Further progress in the case $\alpha = 1$ can be made if we assume, as discussed in §5.2, that the shape functions for p and ψ coincide. Taking then (47) into account, (54) simplifies to

$$
\begin{bmatrix} \mathbf{M} & 0 & 0 \\ 0 & \rho\mathbf{H} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & 0 & -\mathbf{T} \\ 0 & 0 & \mathbf{H} \\ -\mathbf{T}^T & \mathbf{H} & -\rho^{-1}\mathbf{G} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ 0 \\ 0 \end{Bmatrix}.
$$
(57)

The second matrix equation gives $\rho\mathbf{H}\ddot{\boldsymbol{\Psi}} + \mathbf{H}\mathbf{p} = 0$. Since $\mathbf{H}$ is nonnegative definite we must have

$$
\mathbf{p} = -\rho\ddot{\boldsymbol{\Psi}}.
$$
(58)

This is the discrete analog of the continuous relation (20) for the dynamic overpressure. For future use let us note that if the container is rigid, (57) reduces to

$$
-\rho^{-1}\mathbf{G}\mathbf{p} + \mathbf{H}\boldsymbol{\Psi} = \mathbf{G}\ddot{\boldsymbol{\Psi}} + \mathbf{H}\boldsymbol{\Psi} = \mathbf{T}^T\ddot{\mathbf{d}}.
$$
(59)

6.4 Unsymmetric Elimination

If (58) is used to eliminate the pressure vector from (57) we obtain

$$
\begin{bmatrix} \mathbf{M} & \rho\mathbf{T} \\ 0 & \mathbf{G} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & 0 \\ -\mathbf{T}^T & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ 0 \end{Bmatrix}.
$$
(60)

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Conversely, eliminating the displacement potential vector gives

$$
\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \rho\mathbf{T}^T & \mathbf{G} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & -\mathbf{T} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ \mathbf{0} \end{Bmatrix}.
\tag{61}
$$

Unlike previous systems, both (60) and (61) are *unsymmetric*. Thus the straightforward elimination of a field variable, be it p or ψ, causes symmetry to be lost. These forms will be called unsymmetric two-field forms, or U2 for short. System (60) reduces to (59) if the container is rigid.

7. REFORMULATIONS OF THE TRANSIENT RESPONSE EQUATIONS

7.1 S3 Forms

Starting from (57) and (58) it is possible to derive three more symmetric forms that are formally equivalent. One is obtained by differentiating the last matrix equation twice in time, transforming the first equation via (57), and finally including (57) premultiplied by $\rho^{-1}\mathbf{G}$ as third matrix equation:

$$
\begin{bmatrix} \mathbf{M} & \rho\mathbf{T} & \mathbf{0} \\ \rho\mathbf{T}^T & -\rho\mathbf{H} & \mathbf{G} \\ \mathbf{0} & \mathbf{G} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \rho^{-1}\mathbf{G} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}
\tag{62}
$$

Another one is obtained by integrating the first matrix equation of (57) twice in time, using (58) to eliminate the pressure, and including $\mathbf{Kd} - \mathbf{Kd} = 0$ as trivial equation:

$$
\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \\ \ddot{\mathbf{d}} \end{Bmatrix} + \begin{bmatrix} -\mathbf{M} & -\rho\mathbf{T} & -\mathbf{K} \\ -\rho\mathbf{T}^T & \rho\mathbf{H} & \mathbf{0} \\ -\mathbf{K} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \\ \ddot{\mathbf{d}} \end{Bmatrix} = \begin{Bmatrix} -\ddot{\mathbf{f}}_d \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}
\tag{63}
$$

where superposed stars denote integration with respect to t. Finally, differentiating the first matrix equation of (63) twice in time, moving $\rho\mathbf{T}^T\mathbf{d}$ to the left, and including $\mathbf{Md} - \mathbf{Md} = 0$ as trivial equation, we get

$$
\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{M} \\ \mathbf{0} & \rho\mathbf{G} & -\rho\mathbf{T}^T \\ -\mathbf{M} & -\rho\mathbf{T} & -\mathbf{K} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \\ \ddot{\mathbf{d}} \end{Bmatrix} + \begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \rho\mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \\ \ddot{\mathbf{d}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\mathbf{f}_d \end{Bmatrix}
\tag{64}
$$

The four symmetric forms, (57), (62), (63) and (64), will be called *symmetric three field* forms, or S3 forms for short. It should be noted that there is no symmetric S3 form with a state vector consisting of $\mathbf{d}$, $\mathbf{p}$ and $\ddot{\mathbf{d}}$.

7.2 S2 Forms

Each of the S3 forms has a statically condensable matrix equation that allows one field to be eliminated. For example, the last matrix equation of (57) is $-\mathbf{T}^T\mathbf{d} + \mathbf{H}\boldsymbol{\Psi} - \rho^{-1}\mathbf{Gp} = 0$ which can be solved for the pressure vector $\mathbf{p}$ if $\mathbf{G}$ is nonsingular. Assuming that all matrix inverses

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indicated below exist (more will be said about this later), the condensation process yields four two-field symmetric forms:

$$\begin{bmatrix} \mathbf{M} & 0 \\ 0 & \rho\mathbf{H} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} + \rho\mathbf{TG}^{-1}\mathbf{T}^T & \rho\mathbf{TG}^{-1}\mathbf{H} \\ \rho\mathbf{HG}^{-1}\mathbf{T}^T & \rho\mathbf{HG}^{-1}\mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ 0 \end{Bmatrix}, \tag{65}$$

$$\begin{bmatrix} \mathbf{M} + \rho\mathbf{TH}^{-1}\mathbf{T}^T & \mathbf{TH}^{-1}\mathbf{G} \\ \mathbf{GH}^{-1}\mathbf{T}^T & \rho^{-1}\mathbf{GH}^{-1}\mathbf{G} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & 0 \\ 0 & \rho^{-1}\mathbf{G} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ 0 \end{Bmatrix}, \tag{66}$$

$$\begin{bmatrix} \rho\mathbf{G} & 0 \\ 0 & \mathbf{K} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{\Psi}} \\ \ddot{\mathbf{d}} \end{Bmatrix} + \begin{bmatrix} \rho\mathbf{H} + \rho^2\mathbf{T}^T\mathbf{M}^{-1}\mathbf{T} & \rho\mathbf{T}^T\mathbf{M}^{-1}\mathbf{K} \\ \rho\mathbf{KM}^{-1}\mathbf{T} & \mathbf{KM}^{-1}\mathbf{K} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Psi} \\ \mathbf{d} \end{Bmatrix} = - \begin{bmatrix} \rho\mathbf{T}^T \\ \mathbf{K} \end{bmatrix} \mathbf{M}^{-1} \ddot{\mathbf{f}}_d, \tag{67}$$

$$\begin{bmatrix} \mathbf{MK}^{-1}\mathbf{M} & \rho\mathbf{MK}^{-1}\mathbf{T} \\ \rho\mathbf{T}^T\mathbf{K}^{-1}\mathbf{M} & \rho\mathbf{G} + \rho^2\mathbf{T}^T\mathbf{K}^{-1}\mathbf{T} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \end{Bmatrix} + \begin{bmatrix} \mathbf{M} & 0 \\ 0 & \rho\mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \end{Bmatrix} = - \begin{bmatrix} \mathbf{M} \\ \rho\mathbf{T}^T \end{bmatrix} \mathbf{K}^{-1}\mathbf{f}_d. \tag{68}$$

These will be called *symmetric two-field forms*, or S2 forms for brevity. The condensation process reduces the number of degrees of freedom but is detrimental to matrix sparsity. The last property may be recovered to some extent by taking advantage of *factored* forms of the matrices affected by the inverses; for example

$$\begin{bmatrix} \mathbf{K} + \rho\mathbf{TG}^{-1}\mathbf{T}^T & \rho\mathbf{TG}^{-1}\mathbf{H} \\ \rho\mathbf{HG}^{-1}\mathbf{T}^T & \rho\mathbf{HG}^{-1}\mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{T} \\ 0 & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0 & \rho\mathbf{G}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{T}^T & \mathbf{H} \end{bmatrix}. \tag{69}$$

Expressions for the matrices in (66)-(68) are given in [3].

7.3 Advantages and Restrictions

The eight symmetric forms (S3 and S2), plus the two unsymmetric forms (U2), represent ten formulations of the R_1-based fluid-structure interaction problem for the identical-shape-function case. Although formally equivalent, they may have different behavior in terms of numerical stability and computational efficiency. The following items may affect the choice among the various forms.

Matrix sparseness retention. Matrices G and M are often diagonal. The S2 forms that involve $\mathbf{G}^{-1}$ and $\mathbf{K}^{-1}$, whether in direct or factored form, are (other things being equal) preferable to the others.

Existence of inverses. If the fluid does not have a free surface, H is singular on account of (49), and consequently (65) does not exist. If the container has some unsuppressed rigid body modes, K is singular and consequently (68) does not exist.

Applied force processing. Forms (63) and (67) require that the applied structural forces, $\mathbf{f}_d$, be integrated twice in time before being used. Both S2 forms (67) and (68) require additional matrix-vector operations on the force vectors. These disadvantages, however, disappear in the free-vibrations case discussed in §8.

Explicit versus implicit time integration. If M and G are diagonal, both unsymmetric forms (60) and (61) are attractive for explicit time integration because the leftmost coefficient matrices are

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Table 2 Limit Conditions

Limit condition	Matrix expression	Recommended form(s)
Incompressible fluid $(c \to \infty)$	$G \to 0$	(60), (61), (62), (66)
Cavitating fluid $(c \to 0)$	$G \to \infty$	(57), (65)
Stiff container	$K \to \infty$	(64), (68)
Hyperlight container	$M \to 0$	(64), (68)

upper and lower triangular, respectively. Therefore equations may be solved directly in a forward or backward direction without prior factorization. No symmetric form exhibits a similar property.

Physical limit conditions. Those collected in Table 2 are of interest in the applications. Recommended forms, if applicable (restrictions are analyzed in §7.3), are preferable because of numerical stability or suitability for perturbation analysis. Of all conditions listed in Table 2 the incompressible fluid case is of central importance. There must be a free surface S_f, else the contained fluid would behave as a rigid body. Consequently H is nonsingular. Setting $G = 0$ in (66) we obtain the so-called added mass equations

$$M_a \ddot{d} + Kd = f_d,\tag{70}$$

where M_a is the *added mass* of the coupled system:

$$M_a = M + \rho T H^{-1} T^T.\tag{71}$$

Preservation of structural rigid body motions. This is discussed in more detail in §8.5 in conjunction with the free-vibration eigenproblem. Suffices to say that forms (63)-(64) and (67)-(68) do not generally preserve such motions and are inappropiate for treating unsupported structures (for example, liquid tanks in orbit).

Presence of constant potential mode (CPM). This is covered in detail in §8.6. If the fluid is totally enclosed by the container so that there is no free surface, forms (57) and (65) should not be used.

8. FREE VIBRATIONS

To obtain the elastoacoustic free-vibrations problem, we make the standard substitutions

$$d = u e^{j\omega t}, \quad \Psi = q e^{j\omega t}, \quad p = r e^{j\omega t}, \quad f_d = 0,\tag{72}$$

where $j = \sqrt{-1}$ and ω is the circular frequency, into the transient response equations. Thus we obtain ten algebraic eigenproblems, eight symmetric and two unsymmetric, which are displayed below. General properties of these eigensystems are summarized in the Appendix. In the following eigenproblem statements, subscript m is a mode index. The following eigenvector relations should be noted:

$$r_m = -\rho \omega_m^2 q_m, \quad \ddot{u}_m = \omega_m^{-2} u_m \quad (\omega_m \neq 0).\tag{73}$$

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For the unsymmetric forms given in §8.3 one must distinguish between left and right eigenvectors. Supercript R is applied to right eigenvectors wherever necessary; otherwise left eigenvectors are assumed.

8.1 S3 Forms

$$\omega_m^2 \begin{bmatrix} M & 0 & 0 \\ 0 & \rho H & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \\ r_m \end{Bmatrix} = \begin{bmatrix} K & 0 & -T \\ 0 & 0 & H \\ -T^T & H & -\rho^{-1}G \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \\ r_m \end{Bmatrix}, \tag{74}$$

$$\omega_m^2 \begin{bmatrix} M & \rho T & 0 \\ \rho T^T & -\rho H & G \\ 0 & G & 0 \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \\ r_m \end{Bmatrix} = \begin{bmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho^{-1}G \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \\ r_m \end{Bmatrix}, \tag{75}$$

$$\omega_m^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & K \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \\ \ddot{u}_m \end{Bmatrix} = \begin{bmatrix} -M & -\rho T & -K \\ -\rho T^T & \rho H & 0 \\ -K & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \\ \ddot{u}_m \end{Bmatrix}, \tag{76}$$

$$\omega_m^2 \begin{bmatrix} 0 & 0 & -M \\ 0 & \rho G & -\rho T^T \\ -M & -\rho T & -K \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \\ \ddot{u}_m \end{Bmatrix} = \begin{bmatrix} M & 0 & 0 \\ 0 & \rho H & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \\ \ddot{u}_m \end{Bmatrix}. \tag{77}$$

8.2 S2 Forms

$$\omega_m^2 \begin{bmatrix} M & 0 \\ 0 & \rho H \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \end{Bmatrix} = \begin{bmatrix} K + \rho TG^{-1}T^T & \rho TG^{-1}H \\ \rho HG^{-1}T^T & \rho HG^{-1}H \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \end{Bmatrix}, \tag{78}$$

$$\omega_m^2 \begin{bmatrix} M + \rho TH^{-1}T^T & TH^{-1}G \\ GH^{-1}T^T & \rho^{-1}GH^{-1}G \end{bmatrix} \begin{Bmatrix} u_m \\ r_m \end{Bmatrix} = \begin{bmatrix} K & 0 \\ 0 & \rho^{-1}G \end{bmatrix} \begin{Bmatrix} u_m \\ r_m \end{Bmatrix}, \tag{79}$$

$$\omega_m^2 \begin{bmatrix} \rho G & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} q_m \\ \ddot{u}_m \end{Bmatrix} = \begin{bmatrix} \rho H + \rho^2 T^T M^{-1}T & \rho T^T M^{-1}K \\ \rho KM^{-1}T & KM^{-1}K \end{bmatrix} \begin{Bmatrix} q_m \\ \ddot{u}_m \end{Bmatrix}, \tag{80}$$

$$\omega_m^2 \begin{bmatrix} MK^{-1}M & \rho MK^{-1}T \\ \rho T^T K^{-1}M & \rho G + \rho^2 T^T K^{-1}T \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \end{Bmatrix} = \begin{bmatrix} M & 0 \\ 0 & \rho H \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \end{Bmatrix}. \tag{81}$$

8.3 U2 Forms

$$\omega_m^2 \begin{bmatrix} M & \rho T \\ 0 & G \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \end{Bmatrix} = \begin{bmatrix} K & 0 \\ -T^T & H \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \end{Bmatrix}, \tag{82}$$

$$\omega_m^2 \begin{bmatrix} M & 0 \\ \rho T^T & G \end{bmatrix} \begin{Bmatrix} u_m \\ r_m \end{Bmatrix} = \begin{bmatrix} K & -T \\ 0 & H \end{bmatrix} \begin{Bmatrix} u_m \\ r_m \end{Bmatrix}. \tag{83}$$

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8.4 Computational Considerations

The considerations of §7.3 apply for the most part to these ten eigensystems. However, matrix symmetry is more important in free vibrations than in the transient response problem. This is because eigensolution extraction methods that take advantage of sparsity are more highly developed for the symmetric eigenproblem than for its unsymmetric counterpart. For an up-to-date exposition of those methods see Parlett [17].

The presence of zero eigenfrequencies ($\omega_m = 0$ roots) may cause serious numerical difficulties in some eigensystem formulations. Two sources of such roots may be distinguished: rigid body structural modes, and the constant-potential mode.

8.5 Rigid-Body Structural Modes

If the container is not fully supported, $\mathbf{K}\mathbf{u}_r = 0$ for structural rigid body eigenmodes $\mathbf{u}_r$. If H is nonsingular eigensystems (74)-(75), their condensed versions (78)-(79), as well as the two U2 eigensystems, preserve such modes. To verify this assertion, substitute

$$\mathbf{u}_m = \mathbf{u}_r, \quad \mathbf{q}_m = -\mathbf{H}^{-1}\mathbf{T}^T\mathbf{u}_r, \quad \mathbf{r}_m = 0, \tag{84}$$

into the Rayleigh quotients (A.12) or (A.15) of the eigensystems. If H is singular, form (79), which contains $\mathbf{H}^{-1}$, does not exist, whereas (74) preserves the modes if there exist $\mathbf{q}_r$ modes such that $\mathbf{H}\mathbf{q}_r + \mathbf{T}\mathbf{u}_r = 0$. Eigensystems (76)-(77) and (80) do not generally preserve rigid-body modes, whereas (81), which contains $\mathbf{K}^{-1}$, does not exist.

8.6 Constant Potential Mode and Spectrum Contamination

Suppose the container is supported so K is nonsingular but the enclosed fluid has no pressure-specified surface S_p. If so H is singular because of (49). Both U2 eigensystems then possess an $\omega = 0$ root which conventionally will be assigned modal index 0. This root is associated with the following left/right eigenvectors

$$\text{Eigensystem (82):} \quad \mathbf{u}_0 = 0, \quad \mathbf{q}_0 = \mathbf{e}, \quad \mathbf{u}_0^R = \mathbf{K}^{-1}\mathbf{T}\mathbf{e}, \quad \mathbf{q}_0^R = \mathbf{e}, \tag{85}$$

$$\text{Eigensystem (83):} \quad \mathbf{u}_0 = \mathbf{K}^{-1}\mathbf{T}\mathbf{e}, \quad \mathbf{r}_0 = \mathbf{e}, \quad \mathbf{u}_0^R = 0, \quad \mathbf{r}_0^R = \mathbf{e}, \tag{86}$$

This statement is readily verified by taking the Rayleigh quotients (A.12). The eigenpairs (85-86) are collectively called *constant potential mode* or CPM. The existence and computational implications of this mode have been discussed by Geradin *et. al.* [7]. The mathematical interpretation of (85) is "dual" to that of a structural rigid-body mode. Under a rigid-body motion the displacements are nonzero but the strains vanish. Under the CPM the potential is nonzero but all fluid displacements and dynamic pressures vanish. But unlike rigid-body modes, the CPM *has no physical significance*: it is spurious.

According to the eigenfunction theory summarized in the Appendix, all non-CPM modes ($\mathbf{u}_m$, $\mathbf{q}_m$, $\mathbf{r}_m$) of (82) and (83) for $m \neq 0$, $\omega_m \neq 0$ satisfy the bi-orthogonality conditions

$$\langle \mathbf{0} \quad \mathbf{e}^T \rangle \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \rho\mathbf{T}^T & \mathbf{G} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_m \\ \mathbf{r}_m \end{Bmatrix} = \mathbf{e}^T(\rho\mathbf{T}^T\mathbf{u}_m + \mathbf{G}\mathbf{r}_m) = 0, \tag{87}$$

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$$\langle e^T T^T K^{-1} \quad e^T \rangle \begin{bmatrix} M & \rho T^T \\ 0 & G \end{bmatrix} \begin{Bmatrix} u_m \\ q_m \end{Bmatrix} = e^T (T^T K^{-1} M u_m + \rho T^T K^{-1} T q_m + G q_m) = 0. \quad (88)$$

As regards the symmetric forms, eigensystems (74) and (78) are adversely affected by the singularity of H and should not be used. This is because substituting the CPM left eigenvector (85) into either one, with $r_m = 0$ for (74), produces a Rayleigh quotient for ω of the form 0/0. This means that both coefficient matrices have a common null space (the CPM) and every ω is an eigenvalue. Such an eigenproblem is called *defective* (see Appendix). If one attempts to numerically solve "untreated" defective eigenproblems, nonsensical results can be expected because the whole spectrum is likely to be contaminated.

9. SLOSH MOTIONS IN A GRAVITY FIELD

A liquid with a free surface in equilibrium in a time-independent acceleration field may exhibit *surface waves*, informally called "slosh" motions. From an applications standpoint the most important acceleration fields are gravity and rotational motion, the latter being of interest in rotating tanks. In this section we shall be content with formulating slosh effects in a uniform gravity field. More general fields, including time-dependent body forces, may be variationally treated by the method of canonical decomposition of the non-homogeneous wave equation, but that general method will not be followed here as it is not necessary for the gravity case.

The fluid volume V is in equilibrium in the reference state discussed in §3.1 under the time-invariant body force *per unit of volume* $b = \nabla\beta$, where β is a potential field. As noted above we restrict developments here to a gravity field of strength g uniform in space and time. The boundary S_p is then the *equilibrium free surface* normal to the gravity field. The axes (x_1, x_2, x_3) are selected so that g acts along the $-x_3 \equiv -z$ axis. Hence $\beta = -\rho g z + B$, where B is an arbitrary constant. If we chose B so that β vanishes at the free surface $z = z_0$, then

$$\beta = -\rho g(z - z_0). \quad (89)$$

In the so-called *hydrostatic approximation* for small-amplitude gravity waves [9] sloshing is considered equivalent to a free surface pressure

$$p = \bar{p} + \rho g d_n = \bar{p} + \rho g \eta, \quad \text{where} \quad \eta = d_n = \frac{\partial \psi}{\partial n} \quad \text{on} \quad S_p. \quad (90)$$

Here $\bar{p}$ as before denotes the prescribed part of the pressure (for example, atmospheric pressure) and η is called the *elevation* of the liquid with respect to the equilibrium free surface. This approximation assumes that the displacements are infinitesimal and that the z-acceleration of the slosh motion is negligible.

9.1 Variational Principle

For the variational derivation of "slosh equations" it is advantageous to chose the elevation η as an independently varied field. This choice simplifies the reduction to surface unknowns as well as the treatment of more complex interface conditions such as capillary effects.

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To incorporate slosh effects into the mixed variational principles based on the functionals studied in §4, it is convenient to follow a Galerkin technique by adding weighted forms of (88) to their first variation. The following combinations may be considered:

$$\pm\left(p-\bar{p}-\rho g\eta,\delta\frac{\partial\psi}{\partial n}\right)_{S_p} \pm\left(\frac{\partial\psi}{\partial n}-\eta,\delta\eta\right)_{S_p}, \quad \pm\left(p-\bar{p}-\rho g\eta,\delta\frac{\partial\psi}{\partial n}\right)_{S_p} \pm\left(\frac{\partial\psi}{\partial n}-\eta,\delta p\right)_{S_p},$$

$$\pm\left(p-\bar{p}-\rho g\eta,\delta p\right)_{S_p} \pm\left(\frac{\partial\psi}{\partial n}-\eta,\delta\frac{\partial\psi}{\partial n}\right)_{S_p}, \quad \pm\left(p-\bar{p}-\rho g\eta,\delta p\right)_{S_p} \pm\left(\frac{\partial\psi}{\partial n}-\eta,\delta\eta\right)_{S_p}, \quad (91)$$

$$\pm\left(p-\bar{p}-\rho g\eta,\delta\eta\right)_{S_p} \pm\left(\frac{\partial\psi}{\partial n}-\eta,\delta p\right)_{S_p}, \quad \pm\left(p-\bar{p}-\rho g\eta,\delta\eta\right)_{S_p} \pm\left(\frac{\partial\psi}{\partial n}-\eta,\delta\frac{\partial\psi}{\partial n}\right)_{S_p}.$$

Of these the first expression, with signs $-$ and $+$, offers two advantages: (1) it is derivable from a functional, and (2) it combines naturally with the S_p integral in the first variation (35). Of the "base" parametrized functional R_α the most computationally advantageous choice is again $\alpha = 1$. The expanded functional (40), denoted as $R_{1\eta}$ in the sequel, is

$$R_{1\eta}(p,\psi,\eta) = R_{1V} - \int_{t_0}^{t_1}\left[\int_{S_d} p\bar{d}_n\, dS + \int_{S_p}(p-\bar{p}-\rho g\eta)\frac{\partial\psi}{\partial n} + \tfrac{1}{2}\rho g\eta^2\, dS\right]dt. \quad (92)$$

where R_{1V} is the volume integral of (40). Note that setting $\eta = 0$ restores R_1.

9.2 Finite Element Discretization

In addition to the assumptions (41), (42) and (46) we interpolate η as

$$\eta = N_\eta\eta \quad \text{on} \quad S_p, \quad (93)$$

where column vector η contains n_η fluid elevations at nodes on S_p, and row vector N_η contains the corresponding elevation shape functions. The semidiscrete quadratic form for (92), again excluding the time integral, is

$$R_{1\eta}(\Psi,p,\eta) = -\tfrac{1}{2}\rho\dot{\Psi}^T H\dot{\Psi} - \frac{1}{2\rho}p^T Gp + p^T(H-Q_{p+})\Psi - p^T\tilde{T}^T\tilde{d} + \rho g\eta^T(Q_{\eta+}\Psi - \tfrac{1}{2}S\eta) - \Psi^T f_\psi, \quad (94)$$

where

$$Q_{\eta+} = \int_{S_p} N_\eta^T\nabla N_\psi\, dS, \quad Q_{p+} = \int_{S_p} N_p^T\nabla N_\psi\, dS, \quad S = \int_{S_p} N_\eta^T N_\eta\, dS = S^T, \quad f_\psi = \int_{S_p}\nabla N_\psi^T\bar{p}. \quad (95)$$

The $+$ subscripts in $Q_{\eta+}$ and Q_{p+} convey that the nonzero, "surface" portion of these matrices is augmented with zeros to conform to vectors Ψ and p. To display this structure, Ψ, p and related matrices are partitioned as

$$\Psi = \left\{\begin{array}{c}\Psi_s\\\Psi_v\end{array}\right\}, \quad p = \left\{\begin{array}{c}p_s\\p_v\end{array}\right\}, \quad Q_{\eta+} = [Q_\eta\quad 0], \quad Q_{p+} = \begin{bmatrix}Q_p & 0\\0 & 0\end{bmatrix}, \quad H = \begin{bmatrix}H_{ss} & H_{sv}\\H_{vs} & H_{vv}\end{bmatrix}, \quad (96)$$

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where Ψ_s contains potentials at $n_{\eta\psi}$ nodes of *elements* connected to S_p and p_s contains n_η pressures on S_p. The dimensions of Q_η and Q_p are $n_\eta \times n_{\eta\psi}$. In general $n_\eta < n_{\eta\psi}$ (in fact, about one half). Also typically $n_\eta \ll n_\psi = n_p$ as the latter pertain to a volume mesh. If η is interpolated by the same surface functions as p, *i.e.* $N_\eta \equiv N_p$ on S_p, then

$$Q_\eta = Q_p = Q, \qquad Q_{\eta+} = [Q \quad 0], \qquad Q_{p+} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}. \qquad (97)$$

9.3 The Rigid Container

The following equations of motion for the rigid but mobile container are obtained on rendering (94) stationary:

$$\begin{bmatrix} \rho H & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\Psi} \\ \ddot{p} \\ \rho g \ddot{\eta} \end{Bmatrix} + \begin{bmatrix} 0 & H - Q_{p+}^T & Q_{\eta+}^T \\ H - Q_{p+} & -\rho^{-1}G & 0 \\ Q_{\eta+} & 0 & -S \end{bmatrix} \begin{Bmatrix} \Psi \\ p \\ \rho g \eta \end{Bmatrix} = \begin{Bmatrix} f_\psi \\ \tilde{T}^T \tilde{d} \\ 0 \end{Bmatrix}. \qquad (98)$$

Assuming G and S to be nonsingular and identical p and η shape functions so that (97) hols, the nodal pressures and elevations may be statically condensed from (98) thus producing the single matrix equation

$$\rho H \ddot{\Psi} + (P + R_+)\Psi = f_\psi + \rho(H - Q_{p+})G^{-1}\tilde{T}^T \tilde{d}, \qquad (99)$$

where

$$R_+ = Q_{\eta+}^T S^{-1} Q_{\eta+} = \begin{bmatrix} Q^T S^{-1} Q & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} = R_+^T, \quad P = \rho(H - Q_{p+}^T)G^{-1}(H - Q_{p+}) = P^T. \qquad (100)$$

The rank of R_+ and R is the same as that of S, that is, n_η. For most real liquids, acoustic and slosh motions take place in very different time scales. This is the basis for the common assumption in slosh analysis that the fluid is *incompressible*, *i.e.* $c \to \infty$, $G \to 0$ and $R \to \infty$. If $G \to 0$ the response of the above system tends is forced to occur in the displacement-potential subspace defined by the second matrix equation of (98):

$$(H - Q_{p+})\Psi = \tilde{T}\tilde{d}. \qquad (101)$$

For simplicity let us assume that the container is not only rigid but *motionless*, that is, $\tilde{d} = 0$. The incompressible-fluid equations become

$$\rho \begin{bmatrix} H_{ss} & H_{sv} \\ H_{sv}^T & H_{vv} \end{bmatrix} \begin{Bmatrix} \ddot{\Psi}_s \\ \ddot{\Psi}_v \end{Bmatrix} + \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \Psi_s \\ \Psi_v \end{Bmatrix} = \begin{Bmatrix} f_\psi \\ 0 \end{Bmatrix}, \qquad (102)$$

subject to the constraint $(H - Q_{p+})\Psi = 0$. Subvector Ψ_v may be statically condensed from these two relations, which may be combined as the system

$$\begin{bmatrix} \rho H_s & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\Psi}_s \\ \ddot{\lambda}_\psi \end{Bmatrix} + \begin{bmatrix} R & H_s - Q_s^T \\ H_s - Q_s & 0 \end{bmatrix} \begin{Bmatrix} \Psi_s \\ \lambda_\psi \end{Bmatrix} = \begin{Bmatrix} f_\psi \\ 0 \end{Bmatrix}. \qquad (103)$$

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where λ_ψ are Lagrangian multipliers (in fact, the pressures at nodes of Ψ_s), and

$$H_s = H_{ss} - H_{sv}H_{vv}^{-1}H_{vs}, \qquad Q_s = \begin{bmatrix} Q \\ 0 \end{bmatrix}. \tag{104}$$

If $\bar{d} \neq 0$ the force term in (103) must be appropriately modified.

9.4 The Flexible Container

For a flexible container the equations of motion accounting for fluid compressibility are

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & \rho H & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{d} \\ \ddot{\Psi} \\ \ddot{p} \\ \rho g \ddot{\eta} \end{Bmatrix} + \begin{bmatrix} K & 0 & -T & 0 \\ 0 & 0 & H - Q_{p+}^T & Q_{\eta+}^T \\ -T^T & H - Q_{p+} & -\rho^{-1}G & 0 \\ 0 & Q_{\eta+} & 0 & -S \end{bmatrix} \begin{Bmatrix} d \\ \Psi \\ p \\ \rho g \eta \end{Bmatrix} = \begin{Bmatrix} f_d \\ f_\psi \\ 0 \\ 0 \end{Bmatrix}. \tag{105}$$

Eliminating η and p by static condensation yields

$$\begin{bmatrix} M & 0 \\ 0 & \rho H \end{bmatrix} \begin{Bmatrix} \ddot{d} \\ \ddot{\Psi} \end{Bmatrix} = \begin{bmatrix} K_a & -Y \\ -Y^T & P+R \end{bmatrix} \begin{Bmatrix} d \\ \Psi \end{Bmatrix} = \begin{Bmatrix} f_d \\ f_\psi \end{Bmatrix}, \tag{106}$$

where

$$K_a = K + \rho T G^{-1} T^T, \qquad Y = \rho T G^{-1}(H - Q_{p+}). \tag{107}$$

System (106) is the counterpart of (65). If the fluid is treated as incompressible, a subspace reduction procedure similar to that used in §9.3 can be invoked.

9.5 Slosh Vibrations

Algebraic eigenproblems to investigate slosh vibrations may be constructed following essentially the same techniques as in §8, and reduced to S_p node elevations and pressures. We illustrate the reduction technique for the incompressible fluid held in a motionless rigid container. The eigenproblem associated with (103), suppressing the modal index m for simplicity, may be written as

$$\omega^2 \begin{bmatrix} \rho H_s & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} q_s \\ r_\psi \end{Bmatrix} = \begin{bmatrix} R & H_s - Q^T \\ H - Q & 0 \end{bmatrix} \begin{Bmatrix} q_s \\ r_\psi \end{Bmatrix} \tag{108}$$

where q_s and r_ψ are the modal amplitudes of Ψ_s and λ_ψ, respectively. The last matrix equation in (98) provides $Q\Psi_s = S\eta$, or $Qq_s = Sz$, where z is the vector of modal amplitudes of η, i.e. $\eta = ze^{j\omega t}$. Using these relations we can transform the eigenproblem (108) to

$$\omega^2 \begin{bmatrix} \rho g S & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} z \\ r_s \end{Bmatrix} = \begin{bmatrix} C & Q^T - C \\ Q - C & 0 \end{bmatrix} \begin{Bmatrix} z \\ r_s \end{Bmatrix} \tag{109}$$

in which

$$C = QH_s^{-1}Q^T \tag{110}$$

and r_s are Lagrange-multiplier modal amplitudes at nodes of η. This generalized symmetric eigensystem of order $2n_\eta$ provides n_η solutions to the slosh eigenproblem. A similar technique may be followed for the flexible container case. This finite element reduction-to-surface technique provides an alternative to boundary integral methods [1,8].

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10. CONCLUDING REMARKS

Displacement-potential formulations are of practical interest in fluid-structure transient-response and vibration analysis as they provide the basis for effective numerical computations. For some recent applications see [2,6,7,9,12,14] and references therein. The preceding treatment unifies a number of previous continuum-based and algebraic statements [3,4,5,9,11,12,14-16] of the coupled problem. It may be further extended in the following directions:

(1) The inhomogeneous wave equation $c^2 \nabla^2 \psi - \ddot{\psi} = f$, $f \neq 0$, when the body force field $b(x,t)$ is time-dependent and $\nabla^2 b \neq 0$. Additional forcing terms appear in the equations of motion. These are of interest for slosh of fluids in rotating containers.

(2) Retaining the specific momentum m as independent field in functional (33).

(3) Inclusion of additional physical effects: capillarity, cavitation and viscosity.

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Appendix A: THE GENERALIZED ALGEBRAIC EIGENPROBLEM

Some facts about the algebraic eigenproblem are collected here for convenient reference. These facts are relevant to the study of the free vibrations of the coupled fluid-structure system.

A.1 The Standard Unsymmetric Eigenproblem

The standard eigenproblem for a *real unsymmetric* square matrix A may be stated as

$$A x_i = \lambda_i x_i, \tag{A.1}$$

where λ_i the eigenvalues (which may be complex), and x_i the corresponding *right* eigenvectors *normalized to unit length*. The eigenproblem for the transposed matrix is

$$A^T y_i = \lambda_i y_i. \tag{A.2}$$

This problem has the same eigenvalues but in general the eigenvectors y_i will be different. The y_i are called *left* eigenvectors of A because they satisfy the problem $y_i^T A = \lambda_i y_i$; this in turn explains the qualifier 'right' applied to x_i. The system of left and right eigenvectors of A satisfies *bi-orthogonality* relations:

$$y_i^T x_j = \begin{cases} 0 & \text{if } i \neq j, \\ \mu_i & \text{if } i = j. \end{cases} \tag{A.3}$$

This μ_i is called the condition number of λ_i with respect to the eigenproblem (A.1); it is always less or equal than 1 in absolute value, and may be zero in pathological cases. (The closer to 1, the better conditioned λ_i is.)

Premultiplying (A.1) by y_i and assuming that $\mu_i \neq 0$ yields

$$\lambda_i = y_i^T A x_i / \mu_i = x_i^T A^T y_i / \mu_i, \tag{A.4}$$

which is the Rayleigh quotient for unsymmetric matrices. If $\mu_i = 0$ and $y_i^T A x = 0$, (A.5) takes the undetermined form $0/0$ so every λ_i is an eigenvalue. In such a case the eigenproblem (A.1) is said to be *defective*.

A.2 The Standard Symmetric Problem

If A is symmetric then $x_i = y_i$, $\mu_i = 1$ and (A.3) reduce to the usual orthogonality conditions

$$x_i^T x_j = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{A.5}$$

whereas (A.4) becomes the usual Rayleigh quotient for a unit length vector:

$$\lambda_i = x_i^T A x_i. \tag{A.6}$$

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A.3 The Generalized Unsymmetric Eigenproblem

The generalized unsymmetric eigenproblem is

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{B}\mathbf{x}_i, \tag{A.7}$$

where $\mathbf{A}$ and $\mathbf{B}$ are unsymmetric real matrices. Assuming that $\mathbf{B}^{-1}$ exists, this problem can be reduced to the standard problem

$$\mathbf{C}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \tag{A.8}$$

in which $\mathbf{C} = \mathbf{B}^{-1}\mathbf{A}$. The transposed problem is

$$\mathbf{C}^T \mathbf{z}_i = \mathbf{A}^T \mathbf{B}^{-T} \mathbf{z}_i = \lambda_i \mathbf{z}_i. \tag{A.9}$$

Defining $\mathbf{B}^T \mathbf{y}_i = \mathbf{z}_i$, (A.9) can be transformed to

$$\mathbf{A}^T \mathbf{y}_i = \lambda_i \mathbf{B}^T \mathbf{y}_i. \tag{A.10}$$

The bi-orthogonality conditions (A.3) become

$$\mathbf{z}_i^T \mathbf{x}_j = \mathbf{y}_i^T \mathbf{B}\mathbf{x}_j = \mathbf{x}_i^T \mathbf{B}^T \mathbf{y}_j = \begin{cases} 0 & \text{if } i \neq j, \\ \mu_i & \text{if } i = j. \end{cases} \tag{A.11}$$

The Rayleigh quotient (A.4) generalizes to

$$\lambda_i = \frac{\mathbf{y}_i^T \mathbf{A}\mathbf{x}_i}{\mathbf{y}_i^T \mathbf{B}\mathbf{x}_i} = \frac{\mathbf{y}_i^T \mathbf{A}\mathbf{x}_i}{\mu_i}. \tag{A.12}$$

As in §A.1, if (A.12) takes on the form $0/0$ for some i, every λ_i is an eigenvalue and the eigenproblem (A.7) is said to be *defective*; mathematically, $\mathbf{A}$ and $\mathbf{B}$ share a common null space. A defective eigenproblem cannot be solved numerically by conventional root-extraction methods because the $0/0$ roots contaminate the entire spectrum.

A.4 The Generalized Symmetric Eigenproblem

If both $\mathbf{A}$ and $\mathbf{B}$ are symmetric,

$$\mathbf{x}_i = \mathbf{y}_i, \qquad \mathbf{z}_i = \mathbf{B}^{-1}\mathbf{y}_i. \tag{A.13}$$

and we recover the usual orthonormality conditions

$$\mathbf{x}_i^T \mathbf{B}\mathbf{x}_j = \begin{cases} 0 & \text{if } i \neq j, \\ \mu_i & \text{if } i = j. \end{cases} \tag{A.14}$$

In mechanical vibration problems for which $\mathbf{B}$ is the mass matrix, μ_i is called the *generalized mass*. Finally, (A.12) reduces to the usual Rayleigh quotient

$$\lambda_i = \frac{\mathbf{x}_i^T \mathbf{A}\mathbf{x}_i}{\mathbf{x}_i^T \mathbf{B}\mathbf{x}_i}. \tag{A.15}$$

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A general three-field variational principle is obtained for the motion of an acoustic fluid enclosed in a rigid or flexible container by the method of canonical decomposition applied to a modified form of the wave equation in the displacement potential. The general principle is specialized to a mixed two-field principle that contains the fluid displacement potential and pressure as independent fields. This principle contains a free parameter α. Semidiscrete finite-element equations of motion based on this principle are displayed and applied to the transient response and free-vibrations of the coupled fluid-structure problem. It is shown that a particular setting of α yields a rich set of formulations that can be customized to fit physical and computational requirements. The variational principle is then extended to handle slosh motions in a uniform gravity field, and used to derived semidiscrete equations of motion that account for such effects.

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