Houston, Texas

1

69295
P—10

Final Technical Report on NASA Grant NAGW–1781,
"A Numerical Code for a Three-Dimensional Magnetospheric MHD Equilibrium Model"
(prepared on February 04, 1992 by G.-H. Voigt)

Report Summary

Title: A Numerical Code for a Three-Dimensional Magnetospheric MHD Equilibrium Model
Principal Investigator: G.-H. Voigt;
Institution: Rice University.

Hann Voigt   02/04/92

This is a summary report of a two-year grant. We started to develop two- and three-dimensional MHD equilibrium models for Earth's magnetosphere. Our original proposal was motivated by realizing that global, purely data-based models of Earth's magnetosphere are inadequate for investigating the underlying plasma-physical principles according to which the magnetosphere evolves on the quasi-static convection time scale (see Section I).

During the two years of grant NAGW–1781, we established complex numerical grid-generation schemes for a three-dimensional Poisson solver, and we succeeded in coding a rather robust Grad-Shafranov solver for high-beta MHD equilibria. Thus, we were able to calculate the effects of both the magnetopause geometry and boundary conditions on the magnetotail current distribution. That work is summarized in Section II.

Table of Contents

I. Objectives:

Magnetospheric MHD Equilibrium Configurations:

The problem of plasma equilibrium for Earth's magnetosphere is of fundamental importance. Any attempt to model processes on the quasi-static time scale, such as magnetospheric convection, or the interaction between the ionosphere and the magnetosphere in terms of field aligned Birkeland currents, requires that the magnetic field model satisfy the condition of quasi-static MHD equilibrium.

Conventional magnetospheric B-field models which only satisfy Maxwell's equations merely *reproduce* the observed magnetic field structure in terms of empirically determined current distributions (magnetopause currents, tail currents, ring current). Those models do not include the underlying plasma physics that would *explain* any particular magnetic field configuration.

In the slow-flow (quasi-static) MHD approximation, the convection flow field is decoupled from the the force balance equation; thus one has to solve the equilibrium equations

$$\mathbf{j} \times \mathbf{B} = \nabla \cdot \mathbf{P} \tag{1}$$

$$\nabla \times \mathbf{B} = \mu_0 \, \mathbf{j} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

with appropriate magnetopause boundary conditions for the magnetic field, and plasma sheet boundary conditions for the thermal plasma pressure.

Ideally, one would like to have a three-dimensional magnetospheric equilibrium model that would allow us to "play around" with various magnetopause boundary conditions, boundary shapes, and thermodynamic conditions for both isotropic and anisotropic thermal plasma. Such an ideal model would then allow us to see whether the system reaches a steady state or conserves one or the other thermodynamic quantity in the course of its time evolution. Also, MHD equilibrium configurations can be tested for stability, a problem which is intimately related to the problem of magnetospheric substorms!

Thus we began two years ago with a rather ambitious (NASA supported) research program that eventually will provide us with an "ideal" MHD equilibrium model explained above. Numerical equilibrium codes adopted from fusion plasma physics cannot simply be used in a high plasma beta ($\beta \gg 1$) environment such as Earth's magnetotail. We therefore started from the very beginning from building codes that are based on sophisticated numerical grid generation schemes (see Section II.1). Following that, we were able to develop a three-dimensional Poisson solver for Neumann boundary value problems (Section II.2), and a Grad-Shafranov solver for two-dimensional equilibria (Section II.3).

Those new codes have very recently allowed us to address interesting magnetospheric physics questions regarding the influence of the magnetopause shape and boundary conditions on the self-consistent magnetotail currents (Section II.4).

II. Accomplishments from Previous Grant NAGW–1781:

We spent most of the our budgeted time during the last two years with the development of rather sophisticated numerical codes we describe in this section. Since the material discussed below has not been published yet, we explain our main accomplishments in same detail.

1. Numerical Grid Generation:

During the first funding cycle, Frank Toffoletto has implemented a computational grid- gener- ation scheme [Thompson et al., 1985] that allows us to formulate numerical difference equations such that derivatives can be calculated with high accuracy. Numerical grid methods have the ad- vantage that the computational domain becomes a uniformly spaced rectangular grid that greatly simplifies the implementation of the the boundary conditions: the grid boundary exactly corre- sponds to the physical boundary of the system.

The proper choice of an optimal non-orthogonal, non-cartesian grid is important for the con- vergence behavior of an MHD equilibrium scheme. The optimal choice of the grid often depends on the problem at hand. However, maximum orthogonality in the numerical grid pattern is desir- able in order to minimize truncation errors in the difference equations.

We decided to use a grid generation algorithm of maximum flexibility: Our magnetospheric grids were computed by the algebraic grid generation program TBGG. TBGG was developed at the Los Alamos National Laboratory; it allows interactive user control of the grid structure via modification of such features as grid spacing and orthogonality. The program generates compu- tational grids of rectangular structure. Such grids are convenient in "finite difference" or "finite volume" solutions of boundary and initial value problems. Computational grids of rectangular structure are topologically equivalent to a square: for each grid point in the physical space (x, y, z coordinates) there is a corresponding point in rectangular computational space (ξ^i, ξ^j, ξ^k).

2. The Three-Dimensional Poisson Solver:

Based on the grid generation method described above, we were able to establish a code that solves Poisson's equation (or Laplace's equation as a special case) in three dimensions. The conservative version of the Laplacian operator is given by

$$\nabla^2 \varphi = \frac{1}{g^{1/2}} \sum_{i=1}^{3} \sum_{j=1}^{3} [a^i \cdot (a^j \varphi)_{\xi^j}]_{\xi^i} \tag{4}$$

where $()_{\xi^i}$ is the differentiation with respect to the coordinate ξ^i. By using the metric identity

$$\frac{1}{g^{1/2}} \sum_{i=1}^{3} (g^{1/2} \, a^i)_{\xi^i} = 0 \qquad (5)$$

the Laplacian (4) reduces to the form

$$\nabla^2 \varphi = \frac{1}{g^{1/2}} \sum_{i=1}^{3} \sum_{j=1}^{3} (g^{1/2} \, g^{ij} \, \varphi_{\xi^j})_{\xi^i} \qquad (6)$$

which is conservative in the ξ^i direction. The term $g^{1/2}$ is the Jacobian defined from the volume element $dV = g^{1/2} \, d\xi^i \, d\xi^j \, d\xi^k$, and g^{ij} is defined as the dot product

$$g^{ij} = \nabla \xi^i \cdot \nabla \xi^j \qquad (7)$$

The expression $\nabla \xi^i$ is determined from

$$\nabla \xi^i = \frac{1}{g^{1/2}} \frac{\partial r}{\partial \xi^j} \times \frac{\partial r}{\partial \xi^k} \quad (i, j, k \text{ cyclic}) \qquad (8)$$

where r is the position vector of the grid in physical space at the location (ξ^i, ξ^j, ξ^k). The magnetic field components are determined from the conservative expression

$$B \equiv - \nabla \varphi = \frac{1}{g^{1/2}} \sum_{i=1}^{3} (g^{1/2} \, a_i \, \varphi)_{\xi^i} \qquad (9)$$

Grid Discretization:

To solve the Poisson equation on a numerical grid, we tried two methods of discretization, namely, the "finite difference", and the "finite volume" approach.

In the "finite difference" approach, the scalar potential ϕ and its derivatives are calculated at the grid locations. In two dimensions, we have used both an average of a forward-backward scheme and a second order scheme [Thompson et al., 1985] for solving Laplace's equation. A three-dimensional generalization of those two schemes is straightforward.

Both schemes are symmetric with respect to the grid points which ensures hat the second derivatives satisfy the relation

$$\frac{\partial^2 \varphi}{\partial \xi^i \, \partial \xi^j} = \frac{\partial^2 \varphi}{\partial \xi^j \, \partial \xi^i} \qquad (10)$$

In the "finite volume" approach, the scalar potential ϕ and its derivatives are calculated at the center of a cell which is defined as a square surrounding the grid point (a cube in three dimensions); the derivatives of ϕ are calculated at the boundaries between the cells. The "finite

volume" scheme also satisfies (10) and reduces to the well known second order scheme for a purely rectangular grid ($\lambda = 1$), namely,

$$\nabla^2 \varphi = \varphi_{i+\lambda,j} + \varphi_{i-\lambda,j} + \varphi_{i,j+\lambda} + \varphi_{i,j-\lambda} - 4\,\varphi_{i,j} \tag{11}$$

Magnetic Boundary Conditions:

Magnetospheric physics problems typically require the solution of Neumann boundary value problems at the magnetopause [e.g., Voigt, 1981]. For Neumann boundary conditions, the normal component of the magnetic field is given by

$$B_n = \frac{\nabla \xi^j}{|\nabla \xi^j|} \cdot B \tag{12a}$$

By using (5) and (9), equation (12a) reduces to

$$B_n = -\frac{1}{\sqrt{g^{jj}}} \sum_{i=1}^{3} g^{ij} A_{\xi_i} \tag{12b}$$

A "finite volume" approach is more convenient for the solution of Neumann boundary value problems. The implementation, however, is a little more complex, because it requires the calculation of the metric coefficients at the cell faces as well as the cell centers. For a (m × n) grid there are (n-1) × (m-1) cells and (n+m-1) 'ghost' cells that are needed to impose the Neumann boundary conditions. These 'ghost' cells lie outside the range of the computational box, so that the normal component at the magnetopause, for example, is given by

$$B_n = -\frac{1}{g^{1/2}_{i,j-1/2}} \left(\frac{1}{4} G^{12}_{i,jmax-1/2}(\varphi_{i+1,jmax} + \right.$$

$$\varphi_{i+1,jmax-1} - \varphi_{i-1,jmax} - \varphi_{i-1,jmax-1}) + G^{22}_{i,jmax-1/2}(\varphi_{i,jmax} - \varphi_{i,jmax-1})) \tag{13}$$

which provides a second-order accurate, discretization of the magnetic normal component at the magnetopause.

Solution of Poisson's Equation:

We briefly mention that the algorithms sketched above lead to the solution of the matrix equation

$$M A = S \tag{14a}$$

$$A = M^{-1} S \tag{14b}$$

For a two-dimensional problem on a $(n \times m)$ grid, M is a sparse $(n \times m)^2$ matrix, A is a vector of order $(n \times m)$ corresponding to the potential in equation (4), and S is a vector denoting the source term for (4) as well as the magnetic boundary conditions.

In three dimensions, owing to the large sizes of the matrices, an iterative scheme must be employed to evaluate (14b). We have utilized the software package PCGPAK available on the CRAY-YMP at NCSA for that purpose. The package uses a conjugate gradient method adopted for non-symmetric matrices. The successful inversion of the matrix M depends on the choice of grid which effects the size and number of the off-diagonal terms (i.e., terms involving g^{ij}, where i≠j). A second-order difference scheme in three dimensions typically involves over 30 off-diagonal terms. The ultimate solution is usually obtained after several trial grids.

The two-dimensional version of our codes produce sufficiently small matrices that direct matrix inversion techniques can be employed. for this we use the matrix package SMPAK which is based on the Yale Sparse Package and is also available on the CRAY-YMP at NCSA. Again, care must be taken in choosing the appropriate grid.

3. The Two-dimensional Grad-Shafranov Solver:

Two-dimensional magnetospheric MHD equilibria for isotropic plasma pressure can be calculated by solving the Grad-Shafranov equation,

$$\nabla^2 \alpha + \mu_0 \frac{dP}{d\alpha} = -\mu_0 j_{dipole} \qquad (15)$$

This equation is equivalent with the equilibrium equations (1) to (3) for $\nabla \cdot \mathbf{P} \rightarrow \nabla P$. The function $\alpha(x, z)$ is a (scalar) magnetic flux function and satisfies the conditions $\mathbf{B} \cdot \nabla \alpha = 0$ and $\nabla \cdot \mathbf{B} = 0$. The variables x and z are used in a (GSM) coordinate system where the x axis points toward the sun, and the x - z plane contains the planetary dipole. In the isotropic limit, the plasma pressure $P(\alpha)$ is constant on magnetic field lines. The plasma currents are given by

$$j_y (\alpha) = \frac{dP}{d\alpha} \qquad (16)$$

and the magnetization current which describes the planetary magnetic dipole depends on the dipole tilt angle ψ according to

$$\mu_0 j_{dipole} = - M_D [\frac{\partial}{\partial x} \delta(x) \delta(z) \cos \psi - \delta(x) \frac{\partial}{\partial z} \delta(z) \sin \psi] \qquad (17)$$

where M_D is the magnetic dipole moment. The dipole is located at the origin of the (GSM) coordinate system.

Analytic solutions of (15) which include the dipole term (17) exist only for a very specific pressure function, namely,

$$P(\alpha) = \frac{1}{2\mu_0} k^2 \alpha^2 \tag{18}$$

and boundary conditions specified on a rectangular magnetopause [e.g., Voigt, 1986]. Thus we decided to develop a so-called Grad-Shafranov solver for high beta-plasma, i.e., a numerical scheme that solves the equilibrium equation (15) for general pressure functions $P(\alpha)$, and for realistic magnetopause geometries and boundary conditions (more details follow in Subsections II.4.1 and II.4.2).

For solving the equilibrium equation (15) it is convenient to split the magnetic flux function into two separate components,

$$\alpha = \alpha_{dipole} + \alpha_j \tag{19}$$

where α_{dipole} is the flux function for the Earth's dipole field and α_j is the flux function for the plasma currents. The flux function for the dipole satisfies the relation

$$\nabla^2 \alpha_{dipole} = -\mu_0 j_{dipole} \tag{20}$$

Thus equation (15) reduces to

$$\nabla^2 \alpha_j + \mu_0 \frac{dP}{d\alpha} = 0 \tag{21}$$

If we use the pressure function (18), for example, then equation (21) reads

$$\nabla^2 \alpha_j + \mu_0 k^2 \alpha_j = -\mu_0 k^2 \alpha_{dipole} \tag{22}$$

The solution of (22) involves solving a matrix inversion problem of type (14), where S is the right side of (22). At the far-tail boundary, we have assumed for simplicity that

$$\frac{\partial \alpha_j}{\partial \xi^1} = 0 \tag{23}$$

The "closed" magnetosphere, i.e., $B_n = 0$ at the magnetopause, requires Dirichlet boundary conditions which are specified by the simple condition $\alpha_j = -\alpha_{dipole}$.

More interesting is a generalized "open" magnetosphere. General non-zero boundary conditions can be formulated in terms of

$$\mathbf{n} \cdot \mathbf{B} = B_n = f(x_{mp}) \tag{24a}$$

where (x_{mp}) denotes locations at the magnetopause. Using the relations (5), (9), and (12a), and noting that $\mathbf{B} = \nabla\alpha \times \mathbf{y}$ we obtain the magnetic normal component (24a) at the magnetopause in the form

$$B_n = - \frac{\dfrac{\partial\alpha_j}{\partial\xi^1}}{(g\,g^{22})^{1/2}} \tag{24b}$$

The integration of (24b) gives the boundary values for the magnetic flux function,

$$\alpha_j(\xi^1) = -\,\alpha_{dipole} - \int_{\xi^0}^{\xi^1} B_n\,(g\,g^{22})^{1/2}\,d\xi^{1'} \tag{25}$$

where ξ^0 denotes the location where $\alpha = 0$. ξ^0 is chosen to be the subsolar point. Following the theory of Toffoletto and Hill [1989] for the "open" magnetosphere, the magnetic normal component, $B_n = f(x_{mp})$, can be specified by the relation

$$\nabla \times (\,\mathbf{v_t} \times \mathbf{B_n}) = 0 \tag{26}$$

which is subject to an input velocity function

$$v_t = v_{sw}\,\frac{z}{z_\infty} + v_a\left(1 - \frac{z}{z_\infty}\right) \tag{27}$$

where v_{sw} is the solar-wind velocity, v_a is the Alfvén velocity, and z_∞ is the far-tail radius. Equation (27) approximates the tangential magnetosheath flow away from the subsolar point plus an additional merging outflow which is of the order of the Alfvén speed. In two-dimensions, (26) reduces to the expression

$$v_t\,B_n = v_{sw}\,B_{n\infty} = constant \tag{28}$$

$B_{n\infty}$ is the normal component in the far-tail. Then B_n becomes

$$B_n(z) = \frac{B_{n\infty}}{\dfrac{z}{z_\infty} + \dfrac{v_a}{v_{sw}}\left(1 - \dfrac{z}{z_\infty}\right)} \tag{29}$$

The magnitude of the constant $B_{n\infty}$ can be estimated from the polar cap co-latitude φ_0, the tail length L, and dipole moment M as

$$B_{n\infty} \approx \frac{2\,M\,\sin\varphi_0}{L} \tag{30}$$

because a constant $(\varphi_0\,B_n)$ is much smaller than the corresponding normal component magnitude for a three-dimensional magnetosphere. The corresponding expression for (30) in three dimensions is given by

$$B_{n\infty} \approx \frac{\pi\,M\,\sin^2\varphi_0}{2\,L\,R}$$

(31)

where R is the tail radius. The ratio of the normal component is then

$$B_{n\infty}^{2\text{-D}}\,/\,B_{n\infty}^{3\text{-D}} \approx \frac{\pi\,\sin\varphi_0}{4\,R}$$

(32)

which is much smaller than unity.

4. A Bouquet of Physical Results:

With the Grad-Shafranov solver in place, we were able to address a number of interesting physical questions regarding the influence of the plasma pressure and of magnetopause boundary conditions on the equilibrium magnetotail structure.

4.1 The Closed Magnetosphere:

The "closed" magnetosphere, $f(x_{mp}) = 0$, remains as a special case in a class of more general solutions to the boundary conditions (24a). That simple boundary condition allows us to demonstrate quite nicely the influence of the plasma pressure and the magnetopause shape on the magnetotail configuration.

Influence of the magnetotail plasma pressure:

In a feasibility study for this proposal, we chose for the plasma pressure the functional form (18). The parameter k in (18) is a measure of the plasma pressure. As the thermal pressure increases (k increases), owing to an unspecified plasma supply mechanism, tail currents develop and stretch the magnetotail field lines accordingly. Of course, the pressure function (18) can hardly be justified from a thermodynamic point of view. Thus we began trying more realistic pressure functions which will allow us to investigate series of magnetospheric equilibrium states that are compatible with magnetotail convection. This will then allow us to study the underlying thermodynamics of the magnetosphere.

Influence of the magnetopause shape:

We found that the magnetopause geometry strongly influences the overall magnetotail structure; the field lines change from a more dipolar configuration to a very stretched tail configuration without changing the plasma pressure function $P(\alpha)$. We were able to demonstrate that the solar wind pressure itself, via its influence on the magnetopause geometry, affects the very structure of the magnetotail plasma currents. This important effect can be investigated on the level of the MHD equilibrium theory; it does not reveal itself in traditional empirical models that only satisfy Maxwell's equations.

4.2 The Open Magnetosphere:

It is well known that many aspects of magnetospheric dynamics have to be modeled by means of an "open" magnetosphere, i.e., for general magnetic boundary conditions $f(x_{mp})$ in (24a). From the very beginning, we therefore coded the new Grad-Shafranov solver (see Section II.3) such that general boundary conditions can be included.

Influence of the magnetopause boundary conditions:

The "open" magnetosphere possesses a small southward B_z component along the tail magnetopause. Such a B_z component was calculated according to the theory of Toffoletto and Hill [1989] we sketched briefly in equations (24) to (32). We were able to demonstrate that a southward turning of the IMF can, indeed, stretch the magnetotail and increase the tail current density accordingly. This effect is extremely important for the problem of substorm dynamics.

References:

Thompson, J. F., Z. U. A. Warsi, and C. W. Mastin, *Numerical Grid Generation, Foundations and Applications,* North-Holland, Amsterdam, 1985.

Toffoletto, F. R., and T. W. Hill, Mapping of the solar wind electric field to the Earth's polar caps, *J. Geophys. Res.,* **94**, 329-347, 1989.

Voigt, G.-H., A mathematical magnetospheric field model with independent physical parameters, *Planet. Space Sci.,* **29**, 1-20, 1981.

Voigt, G.-H., The shape and position of the plasma sheet in Earth's magnetosphere, *J. Geophys. Res.,* **89**, 2169-2179, 1984.

Voigt, G.-H., Magnetospheric equilibrium configurations and slow adiabatic convection, in *Solar Wind-Magnetosphere Coupling,* edited by Y. Kamide and J. A. Slavin, pp. 233-273, Terra Scientific Publishing Co., Tokyo, 1986.