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ACOUSTIC FORCING OF A LIQUID DROP

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FINAL REPORT

February, 1992

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MECHANICAL AND AEROSPACE ENGINEERING DEPARTMENT

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ABSTRACT

The development of systems such as acoustic levitation chambers will allow for the positioning and manipulation of material samples (drops) in a microgravity environment. This provides the capability for fundamental studies in droplet dynamics as well as containerless processing work. Such systems utilize acoustic radiation pressure forces to position or to further manipulate (e.g., oscillate) the sample. The primary objective of this report was to determine the effect of a viscous acoustic field/tangential radiation pressure forcing on drop oscillations. To this end, the viscous acoustic field is determined. Modified (forced) hydrodynamic field equations which result from a consistent perturbation expansion scheme are solved. This is done in the separate cases of an unmodulated and a modulated acoustic field. The effect of the tangential radiation stress on the hydrodynamic field (drop oscillations) is found to manifest as a correction to the velocity field in a sublayer region near the drop/host interface. Moreover, the forcing due to the radiation pressure vector at the interface is modified by inclusion of tangential stresses.

CONTENTS

I. INTRODUCTION

IA. BACKGROUND

The development of acoustic levitation systems has provided a technology which will enable containerless processing in a microgravity environment. Such systems also provide the capability to undertake fundamental studies of droplet dynamics. Acoustic levitation devices utilize radiation pressure forces to position the sample away from the chamber walls. Specific types of systems are the single axis system (Barmatz, 1981) as well as a three-axes design (Wang et. al., 1984). In the case of a three-axes system, acoustic drivers (speakers) centered in the three orthogonal sides of a parallel piped chamber are driven at its resonant frequency. Liquid drops can then be positioned in the region in which the pressure is at a minimum (i.e., the wave pressure nodes). The acoustic levitation system provides for more capabilities than merely that of drop positioning. In addition, the drop can be made to rotate via phase lagging two of the acoustic waves. Also, drop oscillations can be induced via frequency modulation of an acoustic wave.

The quadrupole resonance of simple drops was investigated by Marston and Apfel (1980) in an experimental study. Modulated acoustic radiation pressure provided the driving force. The experimental work of Trinh, Zwern, and Wang (1982) studied the small amplitude oscillation and decay of a free (non-driven) drop. Furthermore, large-amplitude drop shape oscillations have been experimentally investigated for the free and forced cases (Trinh et. al., 1982). Both drop oscillations and break-up were studied in a visualization experiment by Marston and Goosby (1985). All these experiments were performed in a 1g gravity field. The drop itself was surrounded by an immiscible host fluid, with positioning accomplished using acoustic radiation forces.

Analytical and numerical investigations have focused primarily on free drop oscillations and decay. Miller and Scriven (1968) addressed the problem of normal mode oscillations of a viscous liquid drop immersed in a viscous host medium, and derived analytical expressions for limit cases. More recently, Prosperetti (1980) completed a numerical study of free viscous drop oscillations and decay for a range of values of host medium viscosities. Both these investigations involved a linear analysis. Also, there has been some recent work concerning the nonlinear oscillation of inviscid drops (Tsamapoulos & Brown, 1983, Natarajan & Brown, 1987).

The work of Marston (1980) was an analytical investigation into the acoustically forced fluid drop problem. It is the projection of the radiation stress tensor onto the surface of a drop in order to form the radiation pressure vector which actually accounts for the forcing terms. Marston calculated the radial component of the radiation pressure vector for a specific limit case; that in which the acoustic field is taken to be irrotational and the correct tangential acoustic boundary conditions at the drop/host medium interface were <u>not</u> enforced. Thus, no tangential radiation stresses could contribute. The drop was forced at the boundary/interface only. There was no forcing of the Navier-Stokes equations which govern the behavior of the hydrodynamic field which itself results from the action of the acoustic forcing. The only coupling between the hydrodynamics and acoustic fields is through the boundary/interface conditions.

The incorporation of viscous effects into the acoustic field has ramifications for the full hydrodynamic problem in two ways. First, the modification of the acoustic field to include viscous effects will allow for the enforcement of the tangential boundary condition on the acoustic field. This will result in the existence of tangential radiation pressure forces. These forces then enter into the boundary/interface conditions on the hydrodynamic field. Secondly, the full hydrodynamic equations themselves are forced by nonlinear terms relating to the acoustic field. The solution of the hydrodynamic field itself is then altered from that given by, say, Miller and Scriven (1968).

IB. OBJECTIVES

The primary goal of this work is to determine the effect that incorporation of the tangential radiation pressure forces has on the drop oscillations.

Additional related considerations include: (a) possible effects of a non-axisymmetric acoustic field, (b) efficient evaluation of the tangential radiation pressure vector, (c) the role of bulk viscosity in the tangential radiation stresses, and (d) extensions to compound drop forcing.

In order to meet the primary objective, it is necessary to include the effects of viscosity in the acoustic field representation. This is done in Section III of the report. Since the viscous acoustic field will affect the hydrodynamic field both through the boundary/interface conditions and via a modification of the hydrodynamic field equations themselves, these topics must be addressed. The determination of the tangential radiation pressure vector which contributes to the boundary/interface conditions is done in Section IV. The modified hydrodynamic field is investigated in Section V for the case of forcing by an unmodulated acoustic standing wave field and in Section VI for the case in which the acoustic standing wave field is modulated.

The secondary objectives are addressed in the appropriate sections (see Table of Contents). Moreover, supporting material in the Appendices serves to further elucidate these topics.

Conclusions are presented in Section VII.

II. ACOUSTIC FORCING MECHANISM AND GOVERNING EQUATIONS

IIA. ACOUSTIC FORCING MECHANISM

Certain definitions and an associated discussion will be useful, as they pertain to concepts and quantities which will be calculated/manipulated in subsequent sections.

The acoustic radiation stress tensor is given by

$$\Pi_{ij} = \tilde{p}\,\delta_{ij} + \rho \langle v_i v_j \rangle \tag{IIA.1}$$

with

$$\tilde{p} = \frac{\hat{\beta}}{2}\langle p^2 \rangle - \left(\frac{\rho}{2}\right)\langle V^2 \rangle \tag{IIA.2}$$

Adiabatic compressibility is given by $\hat{\beta}$. The brackets indicate averaging over an acoustic period. The projection of the radiation stress onto the drop surface is given by

$$\begin{aligned} \overline{pr} = {} & \left(\Pi^{i}_{rr} - \Pi^{o}_{rr}\right)\hat{e}_r \\ & + \left(\Pi^{i}_{\theta r} - \Pi^{o}_{\theta r}\right)\hat{e}_\theta + \left(\Pi^{i}_{\phi r} - \Pi^{o}_{\phi r}\right)\hat{e}_\phi \end{aligned} \tag{IIA.3}$$

Superscripts indicate the regions interior and exterior to the drop's surface. The tangential component of the radiation pressure vector is given by the $\hat{e}_\theta$ and $\hat{e}_\phi$ components of equation (IIA.3). Note the velocity and pressure fields indicate acoustic quantities.

The drop oscillation is induced through frequency modulation of the acoustic wave.

Given an acoustic frequency, ω_1 , a second wave with frequency ω_2 is required, such that the drop frequency of oscillation is given by the difference of those of the acoustic waves. That is, $\omega_{(DROP)} = \omega_1 - \omega_2$. The carrier frequency is given by the average of the two acoustic frequencies. Note that the wavelength of the acoustic carrier wave should be such that it couples reasonably to the drop diameter. The representation of the acoustic radiation pressure vector is best done through the use of spherical harmonics. That is,

$$\overline{pr} = \sum_{\ell,m} \left\{ \overline{pr}_{STATIC} + \overline{pr}\cos(\omega t + \eta) \right\}_{\ell m} Y_{\ell m}(\theta, \phi) \tag{IIA.4a}$$

The projection is

$$\left\{ (\overline{pr})_{r_{STATIC}} + (\overline{pr})_r \cos(\omega t + \eta) \right\}_{\ell m} = \int_{\Omega} Y^*_{\ell m} \langle (\overline{pr})_r \rangle \, d\Omega \tag{IIA.4b}$$

and similarly

$$\nabla_s \cdot \left\{ (\overline{pr})^{TANG}_{STATIC} + (\overline{pr})^{TANG} \cos(\omega t + \eta) \right\}_{\ell m} = \int_{\Omega} Y^*_{\ell m} \langle \nabla_t \cdot (\overline{pr})^{TANG} \rangle \, d\Omega \tag{IIA.4c}$$

Note that $d\Omega = \sin\theta d\theta d\phi$. The static part of the radiation stress vector as well as the oscillating contribution (which oscillates at the drop frequency) are given. If the acoustic wave is not modulated, there will NOT be an oscillating forcing term. Note that the tangential radiation pressure appears in an odd operator form. This is due to the fact that the unforced drop oscillations were originally determined (Miller and Scriven, 1968) using this form.

For further discussion of the radiation stress as related to drops, see Marston et. al., (1982).

IIB. CASE OF NON-AXISYMMETRIC FORGING

This section pertains to a qualitative investigation into the possible forcing of shear waves (in the hydrodynamic field) due to the acoustic field. As such, the acoustic field will be assumed to be viscous, and to generate non-zero tangential radiation stresses at the interface. As effects can be seen even if the coupling between the acoustic and hydrodynamic fields is restricted to the boundary conditions, this will be the taken to be the case. That is, no forcing of the governing equations themselves will be taken into account. Again, this is making the inviscid acoustic approximation in the governing equations of the hydrodynamic field, but <u>not</u> in the boundary conditions. This introduces the simplification that the general solutions to the unforced hydrodynamic equations are those given be Miller and Scriven (1968).

The task of this section is to investigate the result of boundary/interface forcing on the behavior of these solutions. The boundary conditions which must hold at the drop/host interface are (1) the kinematic condition, (2) the stresses are balanced, and (3) the velocity components are continuous. Both the second and third conditions generate three equations. Moreover, all the boundary conditions can be expressed using only the radial component of the velocity and vorticity fields. This can be done via operations of taking the surface divergence of the tangential balances in conditions (2) and (3), and of taking the radial component of the surface curl of these same conditions. A spherical coordinate system is used.

For clarity, the radial velocity ($\hat{u}_r$) and vorticity ($\hat{\omega}_r$) components are listed below

$$\hat{u}_r^i = \sum_\ell e^{-\lambda_\ell t} \left\{ a_1 r^{\ell-1} + a_3 \frac{1}{r} j_\ell(\hat{\gamma}_i r) \right\} Y_{\ell m}(\theta,\phi) \qquad \text{(IIB.1a)}$$

$$\hat{u}_r^o = \sum_\ell e^{-\lambda_\ell t}\left\{a_2 r^{-\ell-2} + a_4 \frac{1}{r} h_\ell^{(1)}(\hat{\gamma}_o r)\right\} Y_{\ell m}(\theta,\phi) \tag{IIB.1b}$$

$$\hat{\omega}_r^i = \sum_\ell e^{-\lambda_\ell t}\left\{b_1 \frac{1}{r} j_\ell(\hat{\gamma}_i r)\right\} Y_{\ell m}(\theta,\phi) \tag{IIB.1c}$$

$$\hat{\omega}_r^o = \sum_\ell e^{-\lambda_\ell t}\left\{b_2 \frac{1}{r} h_\ell^{(1)}(\hat{\gamma}_o r)\right\} Y_{\ell m}(\theta,\phi) \tag{IIB.1d}$$

The functions are spherical Bessel and Hankel functions of the first kind. The caret indicates the hydrodynamic velocity field. The $Y_{lm}(\theta,\phi)$ are spherical harmonics.

Note that the "a" and "b" coefficient are determined via application of the boundary conditions. In this case, the boundary conditions will involve forcing due to the radiation stresses at the interface. The forcing of the shear waves is constructed via performing the aforementioned operations involving the surface curl. This yields (at r=R)

$$b_1\left(\frac{1}{R} j_\ell(\hat{\gamma}_i R)\right) - b_2\left(\frac{1}{R} h_\ell^{(1)}(\hat{\gamma}_o R)\right) = 0 \tag{IIB.2a}$$

$$b_1 \frac{\mu_o^i}{R^2}\left\{(\ell-1) j_\ell(\hat{\gamma}_i R) - (\hat{\gamma}_i R) j_{\ell+1}(\hat{\gamma}_i R)\right\}$$

$$- b_2 \frac{\mu_o^o}{R^2}\left\{(\ell-1) h_\ell^{(1)}(\hat{\gamma}_o R) - (\hat{\gamma}_o R) h_{\ell+1}^{(1)}(\hat{\gamma}_o R)\right\}$$

$$= \frac{-1}{R \sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\, \overline{pr})_\phi - \frac{\partial}{\partial\phi}(\overline{pr}_\theta)\right\} e^{i\tilde{\eta}_{\ell m}} \qquad \text{(IIB.2b)}$$

The exponential term indicates the phase. Without the forcing term on the right hand side of equation (IIB.2b), this system will reduce to two homogeneous equations in two unknowns. The resulting dispersion relation describes the behavior of shear waves and is well known. Physically, it describes waves which are purely rotational and decaying. The presence of the forcing term is due itself to the components of the tangential radiation stress vector. Note that if axisymmetry is assumed, and $\overline{pr}_\phi$ is zero, then there is no forcing. However, if the axisymmetry of an acoustic field is not maintained in, say, an experiment, then the forcing of hydrodynamic shear waves appears possible. Of course, a full investigation would require that the hydrodynamic field equations themselves be forced in addition to the boundary/interface forcing described above.

IIC. GOVERNING EQUATIONS

The governing equations of those of fluid mechanics; the conservation of mass, momentum, and energy equations. Since the fluid (acoustic field and hydrodynamic field) is considered to be isothermal, the energy equation will simplify considerably, and result in a relationship between the pressure and density fields.

The general form of the governing equations will include contributions due to compressibility. Although the hydrodynamic field is considered incompressible, the acoustic field clearly cannot be.

The ambient mean pressure and density are constant. The acoustic field (or sound field) can be viewed as a perturbation of the ambient environment. A consistent perturbation expansion scheme will be developed whereby the hydrodynamic field will arise naturally (at second order), forced by terms which are quadratic in the acoustic field quantities.

The governing equations must be satisfied in the regions interior and exterior to the drop. In dimensional form they are given by:

(Conservation of Mass)

$$\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho + \rho \nabla \cdot \underline{u} = 0 \qquad \text{(IIC.1)}$$

(Conservation of Momentum)

$$\rho \left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right)$$

$$= -\nabla p + \mu \nabla^2 \underline{u} + \left(\frac{\mu}{3} + \mu_{\text{BULK}} \right) \nabla (\nabla \cdot \underline{u}) \qquad \text{(IIC.2)}$$

(Conservation of Energy/Equation of State)

$$p = c_0^2 \rho \qquad \text{(IIC.3)}$$

The variables p, $\underline{u}$, and ρ denote the pressure, velocity and density fields. The material properties of dynamic viscosity, μ , and bulk viscosity, μ_{BULK} , are constants. Of course, the values will be different according as to which fluids form the drop and host media. The conservation of momentum equation is the compressible Navier - Stokes equation in which no body forces are present. In equation (IIC.3), c_0 refers to the speed of sound.

III. VISCOUS ACOUSTIC FIELD

III A. DISCUSSION OF APPROACH: VELOCITY DECOMPOSITION METHOD VS. METHOD OF COMPOSITE EXPANSIONS

Attention is restricted to the case in which the acoustic field is axisymmetric. In the case of the velocity decomposition approach, the acoustic velocity field is comprised of a solenoidal and an irrotational contribution. That is,

$$\underline{V} = \nabla \times \underline{A} + \nabla\phi \tag{III.1}$$

Moreover, $\underline{A}$ is of the form (o,o,ψ) , with ψ a scalar function. Equations can be developed for ϕ and ψ . The details of such a development, and the resulting solutions, are found in Appendix I. That is, for the axisymmetric case, the problem of determining the acoustic field which incorporates the effects of viscosity is analytically very tractable.

In particular, note that arguments (of spherical Bessel functions) arise in the form of $[SQRT(i\omega/\nu_o)]R$, with ω the acoustic frequency and R the radius of the drop. Specifically, these arise in the solution for ψ . (Of course, ψ itself contributes to the boundary conditions at the drop/host interface.) The acoustic frequency is on the order of hundreds of kilohertz, and the drop radius is, in general, several millimeters. Moreover, it is necessary to allow for reasonable values of ν_o , so it is doubtful that realistic values of the kinematic viscosity would serve to decrease the magnitude of the argument.

This leads to difficulties in the numerical evaluation of the unknown coefficients (listed in the boundary/interface conditions given by equations [AI.11a - AI.11d]).

An alternate approach which avoids these difficulties is to utilize a boundary layer formulation. Moreover, this approach has the capability to elucidate the flow field structure in the region near the host/drop interface.

It is this second approach which will be employed. However, note that the first approach has been developed, also, and is discussed in detail in Appendix I.

The boundary layer approach which is to be used is termed the method of composite expansions (see Nayfeh, 1973). The details of this method as they pertain to this problem are found in succeeding sections.

III B. SOLUTION OF THE VISCOUS ACOUSTIC FIELD VIA THE METHOD OF COMPOSITE EXPANSIONS

The governing equations are those listed in Section II C. They will be nondimensionalized with respect to acoustic field variables. Let

$$(\omega^{-1})\,\hat{t} = t \qquad ; \qquad (c_o^\circ/\omega)\,\underline{\tilde{X}} = \underline{X}$$

$$(c_o^\circ)\,\underline{\tilde{V}} = \underline{V} \quad ; \quad \rho_o^\circ\,\tilde{\rho} = \rho \;; \quad \rho_o^\circ (c_o^\circ)^2\,\tilde{p} = p \qquad \text{(IIIB.1)}$$

The velocity scale is given by the sound speed in the outer medium. The acoustic frequency is used to define the time scale. The length scale follows from these two quantities. Density is nondimensionalized with respect to the ambient density in the outer medium. Tildes indicate nondimensional quantities. The nondimensionalization scheme is the same in both regions, interior and exterior to the drop.

The method of composite expansions is a generalization of the method of matched asymptotic expansions which has been used successfully in boundary layer applications (Nayfeh, 1973). The basis idea is that the viscous effects on the acoustic field are of primary importance in a region near the drop/host interface. The scale of this region is proportional to SQRT (ν/ω), with ν the kinematic viscosity and ω the frequency of the acoustic wave. In levitation systems, this quantity is quite small due to the relatively large magnitude of the acoustic frequency. The choice of the acoustic frequency is driven by the selection of the acoustic wavelength to be on the order of the drop dimension.

The full nonlinear nondimensional governing equations of compressible isothermal fluid dynamics are to be expanded in a small parameter, δ . In this analysis, δ is a formal parameter. It is interpreted as the ratio of the natural frequency of oscillation of the drop to that of the acoustic frequency. Another small parameter arises naturally in this "boundary/interface layer" approach. This second small paramter, ϵ , is equal to *SQRT* (1/*Re*) , with *Re* a Reynolds type number which is defined later in this section.

It is possible to relate the formal structure of this analysis to work done by Riley (1967) in his consideration of the flow induced by a solid body that is in oscillatory motion in an infinite viscous fluid which is otherwise motionless. Of course, the problem being considered here is very different. The fluid drop has no translatory motion. Oscillations of the drop would be due strictly to the action of a modulated acoustic field. However, a formal comparison may be made with Riley's work. He defined three parameters, R, Rs, and M^2. These are equivalent to (δ/ϵ^2), $(\delta/\epsilon)^2$, and $(1/\epsilon^2)$, respectively. The region of interest in the acoustically forced drop problem would correspond to the case of $M \gg 1,\ R,\ Rs \gg 1$.

Consider the fluid region exterior to the drop (host region). The system of governing equations (Section IIC) is nondimensionalized and then is linearized after a perturbation expansion in a small parameter, δ . The base state about which the perturbation occurs is that of zero mean motion, and constant ambient pressure and density. The lowest order perturbation is the sound field. The tildes are dropped for convenience. The resulting system (to lowest order in δ) is

$$\frac{\partial \rho_1^\circ}{\partial t} + \nabla \cdot \underline{V}_1^\circ = 0 \qquad \textbf{(IIB.2a)}$$

$$\frac{\partial \underline{v}_1^o}{\partial t} = -\nabla p_1^o + \left(\frac{1}{Re}\right)\nabla^2 \underline{v}_1^o + \left(\frac{1}{3} + \tau^o\right)\left(\frac{1}{Re}\right)\nabla(\nabla \cdot \underline{v}_1^o) \qquad \text{(IIB.2b)}$$

$$p_1^o = \rho_1^o \qquad \text{(IIIB.2c)}$$

The superscript "o" denotes the fluid region exterior to the drop. A subscript of "1" indicates sound field quantities. At this stage, higher order terms have been neglected. The nondimensional parameters which appear are

$$\frac{1}{Re} = \frac{\nu_0^o}{c_0^o\,(c_0^o/\omega_{Ac})} \quad ; \quad \tau^o = \mu^o_{BULK}/\mu_0^o$$

In the fluid region interior to the drop, the governing equations (to lowest order in small expansion parameter δ) are

$$\frac{\partial \rho^i}{\partial t} + \beta \nabla \cdot \underline{v}_1^i = 0 \qquad \text{(IIIB.3a)}$$

$$\frac{\partial \underline{v}_1^i}{\partial t} = -\left(\frac{1}{\beta}\right)\nabla p_1^i + \left(\frac{\alpha}{\beta}\right)\left(\frac{1}{Re}\right)\nabla^2 \underline{v}_1^i + \left(\frac{1}{Re}\right)\left(\frac{1}{\beta}\right)\left(\frac{\alpha}{3} + \tau^i\right)\nabla(\nabla \cdot \underline{v}_1^i) \qquad \text{(IIIB.3b)}$$

$$p_1^i = \left(\frac{c_0^i}{c_0^o}\right)^2 \rho_1^i \qquad \textbf{(IIIB.3c)}$$

with $\alpha - \mu_0^i/\mu_0^o$, $\beta - \rho_0^i/\rho_0^i$, and $\tau^i - \mu_{bulk}^i/\mu_0^o$. The system of equations (IIIB.3a-c) in the region interior to the drop (and (IIIB.2a-c) exterior to the drop) are subjected to a "boundary layer" type analysis. That is, in the interior of the drop, the fluid will be considered to be comprised of an "inner" layer and an "outer" region. Note both the "inner" and "outer" regions lie within the drop. In the "inner" region, the effects of viscosity are important, and in the "outer" region they are absent. This analysis itself is termed the method of composite expansions. The same methodology is employed in the (host) fluid exterior to the drop. See Figure IIIB.1.

The subscript "1" which heretofore has referred to sound field quantities is now dropped. <u>Subscripts will now refer to inner and outer expansion quantities</u>.

<u>Application of Method of Composite Expansions to Linearized System of Equations</u>

In the method of composite expansions, the dependent variables are expressed in terms of functions having different independent variables (in the $\hat{e}_r$ direction). One is a stretched variable. In the outer region, the variable is given, of course, by r. In the inner region, the independent variable is given by ξ (if interior to the fluid drop) or ζ (if exterior to the fluid drop). The small parameter in the expansion which is part of the method of composite expansions is ϵ. This small parameter is taken to be larger than the small parameter δ (found in the δ expansion). That is, this method is valid for the case in which the nonlinear terms arising at next order are smaller than the viscous contribution.

Although this approach is valid when the acoustic field is non-axisymmetric, the equation development will be done for the less algebraically involved case (i.e., axisymmetric).

Consider the fluid region interior to the drop. Let $r = \tilde{R} - \epsilon\xi$. The constant $\tilde{R} = a\omega/c_o^o$, and "a" is the nondimensional drop radius. The small parameter, ϵ , is taken to be sqrt(1/Re). Rewriting system (IIIB.3a-c) in terms of the inner layer variable, ξ , yields

$$\frac{\partial \rho^i}{\partial t} + \beta\left(\frac{-1}{\epsilon}\frac{\partial v_r^i}{\partial \xi} + \frac{2}{(\tilde{R}-\epsilon\xi)} v_r^i + \frac{1}{(\tilde{R}-\epsilon\xi)\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\theta^i\right)\right) = 0 \tag{IIIB.4a}$$

$$\begin{aligned}
\frac{\partial v_r^i}{\partial t} = & -\left(\frac{1}{\beta}\right)\left(\frac{-1}{\epsilon}\right)\frac{\partial p^i}{\partial \xi} \\
& + \frac{\alpha}{\beta}\epsilon^2 \left\{ \left(\frac{1}{\epsilon^2}\frac{\partial^2}{\partial \xi^2} + \frac{-(2/\epsilon)}{(\tilde{R}-\epsilon\xi)}\frac{\partial}{\partial \xi} + \frac{1}{(\tilde{R}-\epsilon\xi)^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right)\right) v_r^i \right. \\
& \left. + \frac{-2}{(\tilde{R}-\epsilon\xi)^2} v_r^i + \frac{-2}{(\tilde{R}-\epsilon\xi)^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\theta^i\right) \right\} \\
& + \left(\frac{\alpha}{3} + \tau^i\right)\frac{1}{\beta}\epsilon^2 \cdot \\
& \left(\frac{-1}{\epsilon}\frac{\partial}{\partial \xi}\left\{\frac{-1}{\epsilon}\frac{\partial v_r^i}{\partial \xi} + \frac{2}{(\tilde{R}-\epsilon\xi)} v_r^i + \frac{1}{(\tilde{R}-\epsilon\xi)\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\theta^i\right)\right\}\right)
\end{aligned} \tag{IIIB.4b}$$

$$\frac{\partial v_\theta^i}{\partial t} = -\frac{1}{\beta}\frac{1}{(\tilde{R}-\epsilon\xi)}\frac{\partial p^i}{\partial \theta}$$

$$+\frac{\alpha}{\beta}\left\{\left[\frac{\partial^2}{\partial\xi^2} + \frac{-2\epsilon}{(\tilde{R}-\epsilon\xi)}\frac{\partial}{\partial\xi} + \frac{\epsilon^2}{(\tilde{R}-\epsilon\xi)^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]v_\theta^i + \frac{2\epsilon^2}{(\tilde{R}-\epsilon\xi)^2}\frac{\partial v_r^i}{\partial\theta} - \frac{\epsilon^2}{(\tilde{R}-\epsilon\xi)^2\sin^2\theta}v_\theta^i\right\}$$

$$\left(\frac{\alpha}{3}+\tau^i\right)\frac{1}{\beta}\cdot$$

$$\frac{1}{(\tilde{R}-\epsilon\xi)}\frac{\partial}{\partial\theta}\left\{-\epsilon\frac{\partial v_r^i}{\partial\xi} + \frac{2\epsilon^2}{(\tilde{R}-\epsilon\xi)}v_r^i + \frac{\epsilon^2}{(\tilde{R}-\epsilon\xi)\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta^i)\right\} \tag{IIIB.4c}$$

The dependent variable $q^i \in \{v_r^i, v_\theta^i, p^i, \rho^i\}$ is expanded in the inner layer region as follows:

$$q^i = \left(\hat{q}_0^i|_{\tilde{R}} + \tilde{\tilde{q}}_0^i(\xi,\theta,t)\right) + \epsilon\left(\hat{q}_1^i|_{\tilde{R}} - \xi\,\hat{q}_0^{i\prime}|_{\tilde{R}} + \tilde{\tilde{q}}_1^i(\xi,\theta,t)\right) + \text{H.O.T.} \tag{IIIB.5}$$

The notation "^" indicates the outer region dependent variable. It is expanded in a Taylor series expansion in terms of the inner region variable. The "$\approx$" denotes the inner region dependent variable. Superscripts "i" indicate that these quantities are defined in the fluid region interior to the drop. The inner region functions must decay to zero as $\xi \to \infty$.

The functions which are dependent upon the outer region variable, r, can be obtained via solution of outer region system. To lowest order in ϵ, this system is simply that which represents the inviscid acoustic standing wave field. For completeness, these equations and the resulting

solution are listed below

$$\frac{\partial \hat{\rho}^i}{\partial t} + \beta \nabla \cdot \hat{\underline{v}}^i = 0 \tag{IIIB.6a}$$

$$\frac{\partial \underline{v}^i}{\partial t} = -\left(\frac{1}{\beta}\right) \nabla p^i \tag{IIIB.6b}$$

$$\hat{p}^i = \left(\frac{c_0^i}{c_0^o}\right)^2 \hat{\rho}^i \tag{IIIB.6c}$$

A solution is obtained in ϕ^i, which is $\hat{\underline{v}}^i = \nabla \hat{\phi}^i$,

$$\hat{\phi}^i(r,\theta,t) = \sum_{\ell=0}^{\infty} \hat{\hat{\delta}}_\ell \, \alpha_\ell^i \, j_\ell\left(\frac{c_0^o}{c_0^i} r\right) P_\ell(\cos\theta) \, e^{-it} \tag{IIIB.6d}$$

j_l is a spherical Bessel function. The coefficient $\hat{\hat{\delta}}_1$ is

$$\hat{\hat{\delta}}_\ell = (i)^\ell (2\ell+1) \left\{ \exp\left(i(\omega/c_0^o) h \right) + (-1)^\ell \exp\left(-i(\omega/c_0^o) h\right) \right\}$$

with h the distance from the acoustic velocity nodal plane to the center of the drop. If this distance is zero, then $\hat{\hat{\delta}}_l$ is zero for odd l. The coefficient $\hat{\hat{\delta}}_l$ should <u>not</u> be confused with the small expansion parameter δ. P_l is a Legendre polynomial. The other dependent variables in the outer region (still interior to the fluid drop) can be constructed from knowledge of $\hat{\phi}^i$.

Note that the unknown coefficient (at each value of l) is given by α_l^i .

Substitution of the expansions (IIIB.5) into the system (IIIA.4a-c) yields

$$
\begin{aligned}
&\varepsilon \frac{\partial}{\partial t}\left(\hat{\rho}^i_{0|\tilde{R}} + \tilde{\tilde{\rho}}^i_0 + \varepsilon\left[\hat{\rho}^i_{1|\tilde{R}} - \xi\,\hat{\rho}^{i\prime}_{0|\tilde{R}} + \tilde{\tilde{\rho}}^i_1\right] + \cdots \right) \\
&-\beta \frac{\partial}{\partial \xi}\left(\hat{V}^i_{r_0|\tilde{R}} + \tilde{\tilde{V}}^i_{r_0} + \varepsilon\left[\hat{V}^i_{r_1|\tilde{R}} - \xi\,\hat{V}^{i\prime}_{r_0|\tilde{R}} + \tilde{\tilde{V}}^i_{r_1}\right] + \cdots \right) \\
&+ \varepsilon\,\beta \frac{2}{(\tilde{R}-\varepsilon\xi)}\left(\hat{V}^i_{r_0|\tilde{R}} + \tilde{\tilde{V}}^i_{r_0} + \varepsilon\left[\hat{V}^i_{r_1|\tilde{R}} - \xi\,\hat{V}^{i\prime}_{r_0|\tilde{R}} + \tilde{\tilde{V}}^i_{r_1}\right] + \cdots \right) \\
&+ \frac{\varepsilon\,\beta}{(\tilde{R}-\varepsilon\xi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left\{ \hat{V}^i_{\theta_0|\tilde{R}} + \tilde{\tilde{V}}^i_{\theta_0} + \varepsilon\left[\hat{V}^i_{\theta_1|\tilde{R}} - \xi\,\hat{V}^{i\prime}_{\theta_0|\tilde{R}} + \tilde{\tilde{V}}^i_{\theta_1}\right] + \cdots \right\} \right) \\
&= 0
\end{aligned}
\qquad \text{(IIIB.7a)}
$$

$$
\frac{\partial}{\partial t}\left(\hat{V}^i_{r_0|\tilde{R}} + \tilde{\tilde{V}}^i_{r_0} + \varepsilon\left[\hat{V}^i_{r_1|\tilde{R}} - \xi\,\hat{V}^{i\prime}_{r_0|\tilde{R}} + \tilde{\tilde{V}}^i_{r_1}\right] + \cdots \right) \qquad \text{(IIIB.7b)}
$$

$$=$$

$$\left(\frac{1}{\epsilon}\right)\left(\frac{1}{\beta}\right)\frac{\partial}{\partial\xi}\Big(\hat{p}^{i}_{o}|_{\tilde{R}} + \tilde{\tilde{p}}^{i}_{o} + \epsilon\big[\hat{p}^{i}_{1}|_{\tilde{R}} - \xi\hat{p}^{i\prime}_{o}|_{\tilde{R}} + \tilde{\tilde{p}}^{i}_{1}\big] + \cdots\Big)$$

$$+\left(\frac{\alpha}{\beta}\right)\Bigg\{\Big[\frac{\partial^2}{\partial\xi^2} + \frac{-2\epsilon}{(\tilde{R}-\epsilon\xi)}\frac{\partial}{\partial\xi} + \frac{\epsilon^2}{(\tilde{R}-\epsilon\xi)^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big)\Big]\cdot$$

$$\Big(\hat{V}^{i}_{r_o}|_{\tilde{R}} + \tilde{\tilde{V}}^{i}_{r_o} + \epsilon\big[\hat{V}^{i}_{r_1}|_{\tilde{R}} - \xi\hat{V}^{i\prime}_{r_o}|_{\tilde{R}} + \tilde{\tilde{V}}^{i}_{r_1}\big] + \cdots\Big)$$

$$-\frac{\epsilon^2(2)}{(\tilde{R}-\epsilon\xi)^2}\Big(\hat{V}^{i}_{r_o}|_{\tilde{R}} + \tilde{\tilde{V}}^{i}_{r_o} + \epsilon\big[\hat{V}^{i}_{r_1}|_{\tilde{R}} - \xi\hat{V}^{i\prime}_{r_o}|_{\tilde{R}} + \tilde{\tilde{V}}^{i}_{r_1}\big] + \cdots\Big)$$

$$+\frac{-\epsilon^2(2)}{(\tilde{R}-\epsilon\xi)^2\sin\theta}\frac{\partial}{\partial\theta}\Big\{\sin\theta\Big(\hat{V}^{i}_{\theta_o}|_{\tilde{R}} + \tilde{\tilde{V}}^{i}_{\theta_o} + \epsilon\big[\hat{V}^{i}_{\theta_1}|_{\tilde{R}} - \xi\hat{V}^{i\prime}_{\theta_o}|_{\tilde{R}} + \tilde{\tilde{V}}^{i}_{\theta_1}\big] + \cdots\Big)\Big\}\Bigg\}$$

$$+\left(\frac{\alpha}{3} + \tau^{i}\right)\frac{1}{\beta}\cdot$$

$$\frac{\partial}{\partial\xi}\Bigg\{\left(\frac{\partial}{\partial\xi} + \frac{-2\epsilon}{(\tilde{R}-\epsilon\xi)}\right)\Big(\hat{V}^{i}_{r_o}|_{\tilde{R}} + \tilde{\tilde{V}}^{i}_{r_o} + \epsilon\big[\hat{V}^{i}_{r_1}|_{\tilde{R}} - \xi\hat{V}^{i\prime}_{r_o}|_{\tilde{R}} + \tilde{\tilde{V}}^{i}_{r_1}\big] + \cdots\Big)$$

$$\frac{-\epsilon}{(\tilde{R}-\epsilon\xi)\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\Big\{\hat{V}^{i}_{\theta_o}|_{\tilde{R}} + \tilde{\tilde{V}}^{i}_{\theta_o} + \epsilon\big[\hat{V}^{i}_{\theta_1}|_{\tilde{R}} - \xi\hat{V}^{i\prime}_{\theta}|_{\tilde{R}} + \tilde{\tilde{V}}^{i}_{\theta_1}\big] + \cdots\Big\}\Big)\Bigg\}$$

$$\frac{\partial}{\partial t}\left(\hat{v}^{i}_{\theta_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{\theta_0} + \varepsilon\left[\hat{v}^{i}_{\theta_1|\tilde{R}} - \xi\,\hat{v}^{i\prime}_{\theta_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{\theta_1}\right] + \cdots\right)$$

$$= -\frac{1}{\beta}\frac{1}{(\tilde{R}-\varepsilon\xi)}\frac{\partial}{\partial\theta}\left(\hat{p}^{i}_{0|\tilde{R}} + \tilde{\tilde{p}}^{i}_{0} + \varepsilon\left[\hat{p}^{i}_{1|\tilde{R}} - \xi\,\hat{p}^{i\prime}_{0|\tilde{R}} + \tilde{\tilde{p}}^{i}_{1}\right] + \cdots\right) \tag{IIIB.7c}$$

$$+ \left(\frac{\alpha}{\beta}\right)\Bigg\{ \left[\frac{\partial^2}{\partial\xi^2} - \frac{\varepsilon\,2}{(\tilde{R}-\varepsilon\xi)}\frac{\partial}{\partial\xi} + \frac{\varepsilon^2}{(\tilde{R}-\varepsilon\xi)^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\cdot$$

$$\left(\hat{v}^{i}_{\theta_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{\theta_0} + \varepsilon\left[\hat{v}^{i}_{\theta_1|\tilde{R}} - \xi\,\hat{v}^{i\prime}_{\theta_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{\theta_1}\right] + \cdots\right)$$

$$- \frac{\varepsilon^2}{(\tilde{R}-\varepsilon\xi)^2\sin^2\theta}\left(\hat{v}^{i}_{\theta_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{\theta_0} + \varepsilon\left[\hat{v}^{i}_{\theta_1|\tilde{R}} - \xi\,\hat{v}^{i\prime}_{\theta_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{\theta_1}\right] + \cdots\right)$$

$$+ \frac{2\,\varepsilon^2}{(\tilde{R}-\varepsilon\xi)^2}\frac{\partial}{\partial\theta}\left(\hat{v}^{i}_{r_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{r_0} + \varepsilon\left[\hat{v}^{i}_{r_1|\tilde{R}} - \xi\,\hat{v}^{i\prime}_{r_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{r_1}\right] + \cdots\right)\Bigg\}$$

$$+ \left(\frac{\alpha}{3} + \tau^{i}\right)\frac{1}{\beta}\cdot$$

$$\frac{1}{(\tilde{R}-\varepsilon\xi)}\frac{\partial}{\partial\theta}\Bigg\{\left(-\varepsilon\frac{\partial}{\partial\xi} + \frac{2\epsilon^2}{(\tilde{R}-\varepsilon\xi)}\right)\left(\hat{v}^{i}_{r_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{r_0} + \varepsilon\left[\hat{v}^{i}_{r_1|\tilde{R}} - \xi\,\hat{v}^{i\prime}_{r_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{r_1}\right] + \cdots\right)$$

$$+ \frac{\varepsilon^2}{(\tilde{R}-\varepsilon\xi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left\{\hat{v}^{i}_{\theta_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{\theta_0} + \varepsilon\left[\hat{v}^{i}_{\theta_1|\tilde{R}} - \xi\,\hat{v}^{i\prime}_{\theta_0|\tilde{R}} + \tilde{\tilde{v}}^{i}_{\theta_1}\right] + \cdots\right\}\right)\Bigg\}$$

Note that

$$\frac{\partial \hat{v}^i_{\theta_0|R}}{\partial t} = -\frac{1}{\beta}\frac{1}{\tilde{R}}\frac{\partial \hat{p}^i_{0|\tilde{R}}}{\partial \theta}$$

from the outer region analysis (that is, from the solution of the inviscid acoustic field interior to the drop, which is then expanded in a Taylor series expansion in terms of ξ , the inner region variable . This equation is embedded in (IIIB.7c). Similarly,

$$\epsilon\left(\frac{\partial \hat{\rho}^i_0|_{\tilde{R}}}{\partial t}\right) + \epsilon\beta\left(\hat{v}^{i\,\prime}_{r_0|\tilde{R}}\right) + \epsilon\beta\frac{2}{\tilde{R}}\hat{v}^i_{r_0|\tilde{R}} + \frac{\epsilon\beta}{\tilde{R}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\hat{v}^i_{\theta_0|\tilde{R}}\right) = 0$$

which is embedded in (IIIB.7a). Also,

$$\frac{\partial}{\partial t}\left(\hat{v}^i_{r_0|\tilde{R}}\right) = -\frac{1}{\beta}\left(\hat{p}^{i\,\prime}_0|_{\tilde{R}}\right)$$

which is found embedded in (IIIB.7b). From the remaining terms in (IIIB.7a-c), it can be seen that $\tilde{\tilde{v}}^i_\theta = 0(1) \Rightarrow \tilde{\tilde{v}}^i_{\theta_0}$; $\tilde{\tilde{v}}^i_r = 0(\epsilon) \Rightarrow \tilde{\tilde{v}}^i_{r_1}$, and $\tilde{\tilde{\rho}}^i, \tilde{\tilde{p}}^i - 0(\epsilon^2) \rightarrow \tilde{\tilde{\rho}}^i_2, \tilde{\tilde{p}}^i_2$. Then, <u>to lowest order, the inner region equations (interior to the drop) are</u>

$$-\beta \frac{\partial \tilde{\tilde{V}}_{r_1}^{i}}{\partial \xi} + \frac{\beta}{\tilde{R}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, \tilde{\tilde{V}}_{\theta_0}^{i}\right) = 0 \qquad \text{(IIIB.8a)}$$

$$\frac{1}{\beta}\frac{\partial \tilde{\tilde{p}}_2^{i}}{\partial \xi}$$

$$= \frac{\partial \tilde{\tilde{V}}_{r_1}^{i}}{\partial t} - \frac{\alpha}{\beta}\frac{\partial^2 \tilde{\tilde{V}}_{r_1}^{i}}{\partial \xi^2} - \left(\frac{\alpha}{3}+\tau^{i}\right)\frac{1}{\beta}\left\{\frac{\partial^2 \tilde{\tilde{V}}_{r_1}^{i}}{\partial \xi^2} - \frac{1}{\tilde{R}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial \tilde{\tilde{V}}_{\theta_0}^{i}}{\partial \xi}\right)\right\} \qquad \text{(IIIB.8b)}$$

$$\frac{\partial \tilde{\tilde{V}}_{\theta_0}^{i}}{\partial t} = \frac{\alpha}{\beta}\frac{\partial^2 \tilde{\tilde{V}}_{\theta_0}^{i}}{\partial \xi^2} \qquad \text{(IIIB.8c)}$$

The solution for $\tilde{\tilde{V}}_{\theta_0}^{i}$ can be readily obtained. Following that, it is straightforward to obtain $\tilde{\tilde{V}}_{r_1}^{i}$, and then $\tilde{\tilde{p}}_2^{i}$. The solution for $\tilde{\tilde{V}}_{\theta_0}^{i}$ is given as

$$\tilde{\tilde{V}}^{i}_{\theta_o}(\xi,\theta,t) = \sum_{\ell=1}^{\infty} \hat{\hat{\delta}}_\ell\, A^{i}_{BL_\ell} \exp\left(\sqrt{\frac{-\beta}{2\alpha}}\,(1+i)\xi\right) \frac{dP_\ell}{d\theta}\, e^{-it} \qquad \textbf{(IIIB.9)}$$

The determination of the acoustic field quantities in the fluid region exterior to the drop (ie., host region) may be done in a similar fashion. An inner variable, ζ is defined. Let $r = \tilde{R} + \epsilon\zeta$. Again, the small parameter ϵ equals sqrt (1/Re). Also, $\tilde{R} = a\omega/c_o^o$, with "a" the dimensional radius of the drop. As was the case in the fluid region interior to the drop, the dependent variables are to be expanded as

$$\begin{aligned} q^o = & \left(\hat{q}^o_0|_{\tilde{R}} + \tilde{\tilde{q}}^o_0(\zeta,\theta,t) \right) \\ & + \epsilon \left(\hat{q}^o_1|_{\tilde{R}} + \zeta\, \hat{q}^{o\prime}_0|_{\tilde{R}} + \tilde{\tilde{q}}^o_1(\zeta,\theta,t) \right) \\ & + \text{H.O.T.} \end{aligned} \qquad \textbf{(IIIB.10)}$$

where $q^o \in \{v_r^o, v_\theta^o, p^o, \rho^o\}$. The superscript "o" denotes the fluid region exterior to the drop. The outer region dependent variables are indicated by "^", and the inner region dependent variables are known by the "$\approx$". Functions in ζ must decay to zero as $\zeta \to \infty$. Note the Taylor series expansion form of the outer region dependent variables are expressed in terms of ζ , the inner variable.

The functions which are dependent upon (r,θ,t) can be obtained by solving the outer region system. To lowest order in ϵ , this is the inviscid (scattered) acoustic standing wave field. For completeness, the equations and resulting solution are listed

$$\frac{\partial \hat{\rho}^o}{\partial t} + \nabla \cdot \underline{\hat{v}}^o = 0 \qquad \textbf{(IIIB.11a)}$$

$$\frac{\partial \hat{\underline{v}}^{o}}{\partial t} = -\nabla \hat{p}^{o} \qquad \text{(IIB.11b)}$$

$$\hat{p}^{o} = \hat{\rho}^{o} \qquad \text{(IIIB.11c)}$$

with $\hat{\underline{v}}^{o} = \nabla \hat{\phi}^{o}_{SCT}$. A solution is obtained for $\hat{\phi}^{o}_{SCT}$, which is

$$\hat{\phi}^{o}_{SCT}(r,\theta,t) = \sum_{\ell=0}^{\infty} \alpha^{o}_{s_\ell} \hat{\hat{\delta}}_{\ell} h^{(1)}_{\ell}(r) P_{\ell}(\cos\theta) e^{-it} \qquad \text{(IIIB.11d)}$$

$h_l^{(1)}$ is a spherical Bessel function. Both P_l and $\hat{\hat{\delta}}_l$ are as given previously in this section. The unknown coefficient (for each l) is given by α^{o}_{sl}. The other dependent variables (in the outer region and exterior to the drop) may be generated from knowledge of $\hat{\phi}^{o}_{SCT}$.

The system of governing equations exterior to the drop (IIIB.2a-c) rewritten in ζ (inner variable), yields, <u>to lowest order in</u> ϵ ,

$$\frac{\partial \tilde{\tilde{V}}^{o}_{r_1}}{\partial \zeta} + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, \tilde{\tilde{V}}^{o}_{\theta_0}) = 0 \qquad \text{(IIIB.12a)}$$

$$\frac{\partial \tilde{\tilde{p}}_2^o}{\partial \zeta} = \frac{\partial \tilde{\tilde{v}}_{r_1}^o}{\partial t} - \frac{\partial^2 \tilde{\tilde{v}}_{r_1}^o}{\partial \zeta^2} - \left(\frac{1}{3} + \tau^o\right)\left(\frac{\partial^2 \tilde{\tilde{v}}_{r_1}^o}{\partial \zeta^2} + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial \tilde{\tilde{v}}_{\theta_0}^o}{\partial \zeta}\right)\right) \tag{IIIB.12b}$$

$$\frac{\partial \tilde{\tilde{v}}_{\theta_0}^o}{\partial t} = \frac{\partial^2 \tilde{\tilde{v}}_{\theta_0}^o}{\partial \zeta^2} \tag{IIIB.12c}$$

The solution for $\tilde{\tilde{v}}_{\theta_0}^o$ can be readily obtained. Following that, it is straightforward to obtain $\tilde{\tilde{v}}_{r_1}^o$ and the $\tilde{\tilde{p}}_2^o$. The solution of $\tilde{\tilde{v}}_{\theta_0}^o$ is

$$\tilde{\tilde{v}}_{\theta_0}^o(\zeta,\theta,t) = \sum_{\ell=1}^{\infty} \hat{\hat{\delta}}_\ell\, B_{\theta L_\ell}^o \exp\left(-\frac{(1+i)}{\sqrt{2}}\zeta\right)\frac{dP_\ell}{d\theta}\, e^{-it} \tag{IIIB.13}$$

The solutions for $\tilde{\tilde{v}}_{r_1}^i$, $\tilde{\tilde{v}}_{r_1}^o$, $\tilde{\tilde{p}}_2^i$, and $\tilde{\tilde{p}}_2^o$ are listed in the following. For completeness, $\tilde{\tilde{v}}_{\theta_0}^i$ and $\tilde{\tilde{v}}_{\theta_0}^o$ are included. Note that to lowest order, only the $\hat{e}_\theta$ component of velocity has a contribution in the inner variable (ζ for $\tilde{\tilde{v}}_{\theta_0}^i$, ζ for $\tilde{\tilde{v}}_{\theta_0}^o$). It is at higher orders that the other components will contribute terms which are strictly functions of the inner variable. In order to determine the total contribution at order ϵ , it would be necessary to find $\tilde{\tilde{v}}_{\theta_1}^i$ and $\tilde{\tilde{v}}_{\theta_1}^o$. In a similar fashion, in order to determine the total contribution at order ϵ^2 , it would be necessary to calculate $\tilde{\tilde{v}}^{i}_{r_2}$, $\tilde{\tilde{v}}^{o}_{r_2}$, $\tilde{\tilde{v}}^{i}_{\theta_2}$, $\tilde{\tilde{v}}^{o}_{\theta_2}$. Moreover, the "outer" region solution must be modified at this order.

These complete sets of higher order contributions are not needed. However, if there were to be interest in, say, the function $\tilde{\tilde{v}}^{i}_{\theta_1}$ (at order ε), it is noted that simply solving for them via the order ε equations which derive from (IIB.7a-7c) would <u>not</u> yield a uniformly valid expression for $\tilde{\tilde{v}}^{i}_{\theta_1}$. That is, secular terms arise. These can be eliminated via introduction of, say, a slow time scale. Such calculations are not required in the execution of this project, and so are not presented here.

It is found that

$$\tilde{\tilde{V}}^{o}_{\theta_0}(\zeta,\theta,t) = \sum_{\ell=1}^{\infty} \hat{\hat{\delta}}_{\ell}\, B^{o}_{BL_\ell} \exp\left(-\frac{(1+i)}{\sqrt{2}}\,\zeta\right) \frac{dP_\ell}{d\theta}\, e^{-it} \qquad \text{(IIIB.14a)}$$

$$\tilde{\tilde{V}}^{i}_{\theta_0}(\xi,\theta,t) = \sum_{\ell=1}^{\infty} \hat{\hat{\delta}}_{\ell}\, A^{i}_{BL_\ell} \exp\left(-\sqrt{\frac{\beta}{2\alpha}}\,(1+i)\,\xi\right) \frac{dP_\ell}{d\theta}\, e^{-it} \qquad \text{(IIIB.14b)}$$

$$\tilde{\tilde{V}}^{o}_{r_1}(\zeta,\theta,t)$$

$$= \sum_{l=1}^{\infty} \Big\{ (-1)\frac{\sqrt{2}}{2}(1-i)\, l(l+1) \cdot \hat{\hat{\delta}}_l \frac{1}{\tilde{R}} B^{o}_{BL_l} \cdot \exp\left(-\frac{(1+i)}{\sqrt{2}}\zeta\right) P_l \; e^{-it} \Big\} \tag{IIIB.14c}$$

$$\tilde{\tilde{V}}^{i}_{r_1}(\xi,\theta,t)$$

$$= \sum_{l=1}^{\infty} \Big\{ \frac{\sqrt{2}}{2}\sqrt{\frac{\alpha}{\beta}}\,(1-i)\, l(l+1) \cdot \hat{\hat{\delta}}_l \frac{1}{\tilde{R}} A^{i}_{BL_l} \cdot \exp\left(-\sqrt{\frac{\beta}{2\alpha}}(1+i)\xi\right) P_l \; e^{-it} \Big\} \tag{IIIB.14d}$$

$$\tilde{\tilde{p}}_2^{\,o}(\zeta,\theta,t)$$

$$= \sum_{\ell=1}^{\infty} \Big\{ 2\ell(\ell+1)\, \hat{\hat{\delta}}_\ell \frac{1}{\tilde{R}} B^{o}_{BL_\ell} \cdot \exp\left(-\frac{(1+i)}{\sqrt{2}}\zeta\right) P_\ell(\cos\theta)\, e^{-it} \Big\}$$

(IIIB.14e)

$$\tilde{\tilde{p}}_2^{\,i}(\xi,\theta,t)$$

$$= \sum_{\ell=1}^{\infty} \Big\{ 2\alpha\,\ell(\ell+1)\, \hat{\hat{\delta}}_\ell \frac{1}{\tilde{R}} A^{i}_{BL_\ell} \cdot \exp\left(-\sqrt{\frac{\beta}{2\alpha}}(1+i)\xi\right) P_\ell(\cos\theta)\, e^{-it} \Big\}$$

(IIIB.14f)

It is stressed that, to lowest order, the velocity field and pressure fields are given by

$$V_r^{i(0)} = \hat{V}_{r_0}^{i(0)}(r,\theta,t) + O(\epsilon)$$

$$V_\theta^{i(0)} = \hat{V}_{\theta_0}^{i(0)}(r,\theta,t) + \tilde{\tilde{V}}_{\theta_0}^{i(0)}(\xi(\zeta),\theta,t) + O(\epsilon)$$

and

$$p^{i(0)} = \hat{p}_0^{i(0)}(r,\theta,t) + O(\epsilon^2)$$

To reiterate, the "^" indicate the solutions in the outer regions, which correspond to inviscid acoustic field solutions.

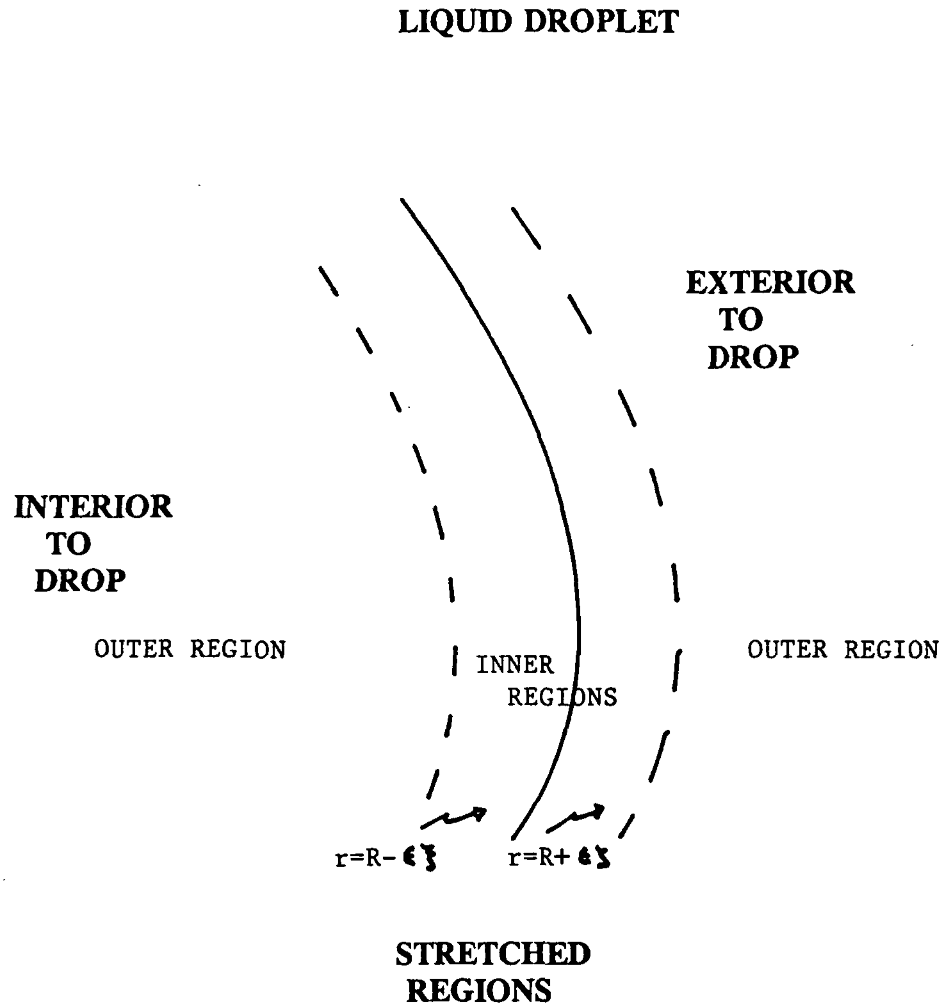

FIGURE IIIB.1

IIIC. BOUNDARY/INTERFACE CONDITIONS

Solutions have been obtained for the inner and outer regions, both interior and exterior to the drop. All solutions are finite in their respective regions of validity. The unknowns (for each value of l) are α_l^i, α_{sl}^o, $A_{bl_l}^i$, $B_{bl_l}^o$.

There are four boundary/interface conditions which must be imposed. Use of these conditions will generate a set of algebraic equations (at each l) which can be solved for the unknowns. Since the equations are nonhomogeneous, that is, they are forced by (functions of) the incident acoustic wave, this is not an eigenvalue problem. Once the coefficients are known the viscous acoustic field, both interior and exterior to the drop, is known.

The four boundary/interface conditions at $r=\tilde{R}$

$$V_r^i = V_r^o{}_{\text{(SCATTERED + INCIDENT)}} \tag{IIIC.1a}$$

$$\tau_{rr}^i = \tau_{rr}^o{}_{\text{(SCATTERED + INCIDENT)}} \tag{IIIC.1b}$$

$$V_\theta^i = V_\theta^o{}_{\text{(SCATTERED + INCIDENT)}} \tag{IIIC.1c}$$

$$\tau_{r\theta}^i = \tau_{r\theta}^o{}_{\text{(SCATTERED + INCIDENT)}} \tag{IIIC.1d}$$

The boundary/interface conditions must be applied at r-$\tilde{R}$, which corresponds to $\xi = 0$ and $\zeta = 0$. Note that in this context τ refers to the stresses. Physically, these boundary conditions represent the velocity and stress balance in the normal and tangential directions to the interface.

If the solution of interest had been restricted to just the inviscid acoustic field, only conditions (IIIC.1a-b) would be applicable. Moreover, τ_{rr} would be replaced by pressure.

To lowest order in ϵ the conditions are

$$\alpha_\ell^i \left(\frac{\ell}{\tilde{R}} j_\ell\left(\frac{c_0^o}{c_0^i}\tilde{R}\right) - \left(\frac{c_0^o}{c_0^i}\right) j_{\ell+1}\left(\frac{c_0^o}{c_0^i}\tilde{R}\right)\right) - \alpha_{s_\ell}^o \left(\frac{\ell}{\tilde{R}} h_\ell^{(1)}(\tilde{R}) - h_{\ell+1}^{(1)}(\tilde{R})\right) = A_{INC}\left(\frac{\ell}{\tilde{R}} j_\ell(\tilde{R}) - j_{\ell+1}(\tilde{R})\right) \tag{IIIC.2a}$$

$$\alpha_\ell^i \left(\beta\, j_\ell\left(\frac{c_0^o}{c_0^i}\tilde{R}\right)\right) - \alpha_{s_\ell}^o h_\ell^{(1)}(\tilde{R}) = A_{INC}\, j_\ell(\tilde{R}) \tag{IIIC.2b}$$

$$A_{BL_\ell}^i + \alpha_\ell^i \frac{1}{\tilde{R}} j_\ell\left(\frac{c_0^o}{c_0^i}\tilde{R}\right) - B_{BL_\ell}^o - \alpha_{s_\ell}^o \frac{1}{\tilde{R}} h_\ell^{(1)}(\tilde{R}) = A_{INC} \frac{1}{\tilde{R}} j_\ell(\tilde{R}) \tag{IIIC.2c}$$

$$\sqrt{\alpha\beta}\ A^{i}_{BL_l} = -B^{o}_{BL_l} \tag{IIIC.2d}$$

In (IIIC.2a-b), $l \geq 1$. If the acoustic field were inviscid only, then the acoustic field variables would be known once α^i_l and α^o_{sl} were known (at each l value). That is, a system of two algebraic nonhomogeneous equations in two unknowns would have to be solved. The forcing term is due to the presence of the <u>incident acoustic standing wave field</u>, without which there is no levitation system or flow field.

In the first boundary/interface condition, it is the radial components of the acoustic velocity which must balance at the interface $(r-\tilde{R},\ \zeta-\xi-0)$. Note that this condition involves only the contributions from the outer regions (i.e., inviscid solutions), as the viscous correction occurs at a higher order. This is also the case for the second boundary/interface condition in which the viscous contribution to the normal stresses occur at higher order.

It is in the tangential acoustic velocity balance at the interface that contributions from the "inner" region enter, along with those which represent the inviscid solution. Finally, in the tangential stress balance, only the inner region solutions contribute. Therefore, it is tangential velocity balance which provides the coupling between the inviscid "outer" system and the "inner" region corrections.

Since viscous effects are to be taken into account, conditions (IIIC.2c-d) also contribute; resulting in four equations in four unknowns which must be determined.

This solution is done numerically using parameters of interest. Such quantities include those of viscosity and density inside and outside the drop, as well as the drop radius. Note that the bulk viscosity does not appear (at lowest order) in either the outer or inner region equations.

Knowledge of the viscous acoustic field is necessary in order to determine the tangential stress component of the radiation pressure vector. This is done in the next section.

Moreover, if the acoustic field is viscous, the acoustic field will couple to the second order hydrodynamic field not only through the boundary/interface conditions, but also act as a forcing term in the hydrodynamic equations themselves. This is presented in upcoming sections for both the unmodulated acoustic standing wave field case and the modulated acoustic standing wave field case.

IIID. COMMENTS ON THE ROLE OF BULK VISCOSITY

In the previous section, it was shown that part of the contribution to the attenuation of the acoustic field is due to the presence of bulk (expansive) viscosity which enters into the governing equations. The equations governing the acoustic pressure and velocity fields are solved retaining this physical contribution, in general.

However, in the determination of the viscous correction to the acoustic field via the method of composite expansions, it is found that those terms involving the bulk viscosity only contribute at higher orders. Thus, in this approach, the bulk viscosity is not relevant.

If the vector decomposition approach had been utilized, the bulk viscosity would enter into the analytical solution. This can be seen in Appendix I (see for example, AI.6B). It acts to attenuate the acoustic wave.

However, in order to determine actual numerical values, it is necessary to know physical property data. This necessity is not restricted to the value of the bulk viscosity, but also applies to ambient medium density, shear viscosity, speed of sound in the medium, etc.

Of course, for a monatomic gas, the bulk viscosity is zero. Among the empirical formulas which give values for μ is the classical work of Greenspan (1959) concerning attenuation in diatomic gases. He found (using a polynomial fit to the data)

$$\mu = \mu\left(2\gamma\alpha - 1.333 - (\gamma-1)/Pr\right)$$

where μ (the shear viscosity) is taken to be constant, α is the empirically determined constant, γ the ratio of specific heats, and Pr the Prandtl number. For air, $\alpha = 0.8903$, and for nitrogen, $\alpha = 0.9104$.

When a specific physical system is under consideration, it is necessary to have at hand the relevant physical property data in order to determine numerical values. However, the effect of the physical phenomenon of bulk viscosity (among other contributing factors) <u>is</u> retained in the analytical formulation which involves the exact solution (rather than the approximate expansion method).

IV. TANGENTIAL RADIATION PRESSURE VECTOR

The calculation of the radial component of the radiation pressure vector was done by Marston (1980) for the case in which the acoustic field was considered to be strictly inviscid. This topic will not be readdressed in the report. That is because the additional complexity due to the viscous contribution will be evident from the full discussion of the tangential contribution to the radiation pressure vector. Also, the recalculated radial radiation pressure would be, in some sense, a correction to that done for the inviscid case. Moreover, in the case in which the acoustic field is considered to be fully inviscid, as was done by Marston, there is no contribution whatsoever to the tangential radiation stresses. Thus, no counterpart to the following work exists.

IV.A. CALCULATION OF THE TANGENTIAL RADIATION PRESSURE VECTOR

The calculation of the tangential radiation pressure vector, denoted by pr^{TANG}, involves the computation of $(\Pi^i_{\theta r} - \Pi^o_{\theta r})$ and $(\Pi^i_{\phi r} - \Pi^o_{\phi r})$, with

$$\Pi^i_{\theta r} = \rho^i \langle v^i_r v^i_\theta \rangle \quad ; \quad \Pi^i_{\phi r} = \rho^i \langle v^i_r v^i_\phi \rangle \tag{IVA.1a}$$

$$\Pi^o_{\theta r} = \rho^o \langle v^o_r v^o_\theta \rangle \quad ; \quad \Pi^o_{\phi r} = \rho^o \langle v^o_r v^o_\phi \rangle \tag{IVA.1b}$$

This obviously means that $v_r^{i,o}$, $v_\theta^{i,o}$, and $v_\phi^{i,o}$ must be calculated in regions interior and exterior to the drop interface. This has been done in the previous section. The time averages are done in the same manner as Marston (1980), and are taken over an interval of $2\pi/\omega_c$. The acoustic carrier wave frequency is denoted by ω_c. It can be shown that, in the case in which the acoustic wave was modulated, the resulting oscillation will be at the drop frequency (with a phase factor also present). In the case of the unmodulated standing acoustic wave, the averaging eliminates any time dependence.

From the results of the previous section it is clear that the acoustic velocity components are expressed as a series in "l". The r dependence is seen to involve spherical Bessel and Hankel functions. Also, the "inner" region stretched variables appear in exponential functions. The θ dependence occurs through the expressions involving Legendre and associated Legendre polynomials. These functional dependencies involve infinite series.

In particular, in nondimensional form

$$(\overline{pr}^{\,TANG})_\theta = \beta \langle V_r^i V_\theta^i \rangle - \langle V_r^o V_\theta^o \rangle \tag{IVA.2a}$$

Note that the velocity components are also nondimensional quantities. To lowest order (in ϵ) this can be written as

$$\begin{aligned}(\overline{pr}^{\,TANG})_\theta = \; & \beta \left\langle \left(\hat{V}^i_{r_0}\big|_{\tilde{R}} + O(\epsilon)\right)\left(\hat{V}^i_{\theta_0}\big|_{\tilde{R}} + \tilde{\tilde{V}}^i_{\theta_0}\big|_{\zeta=0}\right)\right\rangle \\ & - \left\langle \left(\hat{V}^o_{r_0}\big|_{\tilde{R}} + O(\epsilon)\right)\left(\hat{V}^o_{\theta_0}\big|_{\tilde{R}} + \tilde{\tilde{V}}^o_{\theta_0}\big|_{\zeta=0}\right)\right\rangle\end{aligned} \tag{IVA.2b}$$

$$= \beta \left\langle \hat{v}^{i}_{r_0|\tilde{R}} \hat{v}^{i}_{\theta_0|\tilde{R}} + \hat{v}^{i}_{r_0|\tilde{R}} \tilde{\tilde{v}}^{i}_{\theta_0|\zeta=0} \right\rangle - \left\langle \hat{v}^{o}_{r_0|\tilde{R}} \hat{v}^{o}_{\theta_0|\tilde{R}} + \hat{v}^{o}_{r_0|\tilde{R}} \tilde{\tilde{v}}^{o}_{\theta_0|\zeta=0} \right\rangle \quad \text{(IVA.2c)}$$

Recall that $< v^{i}_{r0} > = < v^{o}_{r0} >$ at the interface, and thus

$$(\overline{pr}^{\,TANG})_{\theta} = \left\langle \hat{v}^{i}_{r_0|\tilde{R}} \left(\beta \tilde{\tilde{v}}^{i}_{\theta_0|\zeta=0} - \tilde{\tilde{v}}^{o}_{\theta_0|\zeta=0} \right) + \hat{v}^{i}_{r_0|\tilde{R}} \left(\beta \hat{v}^{i}_{\theta_0|\tilde{R}} - \hat{v}^{o}_{\theta_0|\tilde{R}} \right) \right\rangle \quad \text{(IVA.2d)}$$

However, $\beta v^{i}_{\theta_0} - v^{o}_{\theta_0} = 0$ at the interface (based upon periodicity and continuity of pressure, see Marston, 1980). Therefore,

$$(\overline{pr}^{\,TANG})_{\theta} = \left\langle \hat{v}^{i}_{r_0|\tilde{R}} \left(\beta \tilde{\tilde{v}}^{i}_{\theta_0|\zeta=0} - \tilde{\tilde{v}}^{o}_{\theta_0|\zeta=0} \right) \right\rangle \quad \text{(IVA.3)}$$

In a similar fashion, it is possible to calculate $(\overline{pr}^{TANG})_{\phi}$. However, the calculation of the tangential radiation pressure vector will only involve the $\hat{e}_{\theta}$ component, as axisymmetry has been assumed. Even so, the manipulations will involve multiplication of the aforementioned velocities, each represented by an infinite series. Moreover, an integration over Θ must be performed.

If the arguments of the Bessel functions are small, simplifications will result. Only a few terms of each series then need be retained. Of course, the arguments are composed of factors which involve physical property data. Enough terms must be retained so as to insure that the solution is adequately represented.

From inspection of equation (IIA.4c), it is clear that for any projection in which the Y^*_{lm} does not have m = 0 the integral will be zero. An evaluation of such integrals will involve products of expressions comprised of Legendre polynomials, with Y^*_{lo} ultimately to be rewritten in terms of Legendre polynomials.

IV.B. SPHERICAL HARMONICS/COMPUTATIONAL CONCERNS

In order to calculate $(\overline{\mathrm{pr}}^{TANG})_\Theta$, it is necessary to calculate integrals such as

$$I = 2\pi \int_0^{\pi} Y_{\ell o}^{*} \left\{ \langle \hat{v}_{r_o}^{i,o} \tilde{\tilde{v}}_{\theta_o}^{i,o} \rangle \right\} \sin\theta \, d\theta \tag{IVB.1}$$

where $\hat{v}_{ro}^{i,o}$, $\tilde{\tilde{v}}_{\Theta o}^{i,o}$ are evaluated at $r=\tilde{R}$ ($\zeta=\xi=0$), the drop radius, and the remaining functional dependence is that of θ.

The primary interest will be in the lowest order shape oscillation, and $\hat{l}$ will be taken to be equal to two. The resulting integrals which must be evaluated will have θ-dependence of the form

$$I_a = c_a \int_0^{\pi} P_2 \left\{ P_{\ell'} \ \frac{dP_{\hat{\ell}}}{d\theta} \right\} \sin\theta \, d\theta \tag{IVB.2}$$

c_a is a constant.

As many $\hat{l}$ and l' values must be used as those used to represent the Bessel functions. Moreover, it is possible that some of the resulting integrals will be zero.

Another form in which the tangential radiation pressure forcing could appear is in that given in equation (IIA.4c). This occurs if the stress balance equations are manipulated through such operations as taking the surface divergence. In this case, the tangential derivatives of the product of, say, $<\hat{v}_{ro}^{i} \tilde{\tilde{v}}_{\theta o}^{i}>$ with respect to θ must be taken. This will be reflected in that the type of terms in the integrands of the resulting integrals will become more complicated than that in (IVB.2).

It is clear that this is a manipulation involving intensive calculation which is best done using computational resources. To that end, software development was done; the strategy is to make use of representations of Legendre polynomials and efficient computational schemes which have appeared in the literature (Press et. al., 1986).

In particular, the integrands may be represented by Chebyshev polynomials. Also, the integration of a Chebyshev series can be done without quadrature per se, as relationships between the coefficients of the integrand and of the would be integral are known. This is discussed in more detail in the MSAE thesis of Ferguson.

Of course, the Bessel functions themselves have to have been evaluated at whatever the constant value of the argument is. It is important to include enough terms in the series to satisfy that the solution is adequately represented; that is, the truncation of the terms will have to be done in order to obtain numerical results.

IV.C. EXAMPLE CALCULATION

This section will consider the forcing in the boundary/interface conditions which will occur if the description of the acoustic standing wave field includes viscous terms.

Note carefully that this section will not determine the forced hydrodynamic field. That is to be done in Section V and VI for the cases in which the acoustic standing wave field (with viscous effects included) is unmodulated and modulated, respectively.

In a sense, this is an intermediate approach to the problem of acoustic forcing of a liquid drop. The work of Marston considered the acoustic field to be inviscid and the hydrodynamic field to be strictly represented by the natural oscillating drop field determined by Miller and Scriven (1968).

The primary work of this project is to determine the viscous acoustic field and hydrodynamic field which is a solution to the forced governing equations. This hydrodynamic field will be a modification of that known to Miller and Scriven. [These are given in Sections III, V, and VI].

In this section, the acoustic field will include viscous terms. Therefore, the boundary/interface conditions will include tangential stresses. However, the hydrodynamic field will remain that of the naturally oscillating (unforced) drop.

The focus will be on the determination of the deformation to the drop due to the (modulated) acoustic field; referred to as the "static part" by Marston (1980). The problem will be done in dimensional form as to most easily compare with Marston (1980).

It is emphasized that very few experimental measurements on a variety of drop/host systems have made. Therefore, calculations will be done for the case of a 1 mm drop of p-xylene in water. This was the system consider by Marston (1980) and Marston and Apfel (1980), so some comparisons between calculations will be possible. Also, the physical property data is taken from the aforementioned sources.

The other physical system with extent experimental results is a silicone oil drop in water (Marston and Goosby, 1985).

The effect of parameter variation has been considered numerically (Ferguson, see MSAE thesis).

The radial component of the <u>hydrodynamic</u> velocity field for this "static" problem is given (in <u>dimensional</u> form) by

$$u_r^i = \sum_{\ell,m} \left(a_I r^{\ell-1} + c_I r^{\ell+1} \right) Y_{\ell m}(\theta,\phi) \tag{IVC.1a}$$

and

$$u_r^o = \sum_{\ell,m} \left(b^o r^{-\ell-2} + d^o r^{-\ell} \right) Y_{\ell m}(\theta,\phi) \tag{IVC.1b}$$

The $Y_{lm}(\Theta,\phi)$ represent spherical harmonics. However, since the problem is axisymmetric, $\frac{\partial}{\partial \phi} = 0$, and the value of m is strictly zero. The superscripts "i" and "o" indicate the regions interior and exterior to the drop.

Boundary/interface conditions are applied at r = R, with R the undisturbed drop radius. Note that only the radial component of the hydrodynamic field is given. In the original work of Miller and Scriven (1968) essentially a poloidal/toroidal field decomposition method was used in the theoretical investigation. Use of surface divergence and surface curl operators on the boundary/interface conditions resulted in a set of relationships which involved on u_r^i, u_r^o, and the radial vorticity component. (They allowed for a ϕ variation.) Moreover, the conditions involving the vorticity decouple form those involving the velocity. (The resulting conditions involving vorticity pertain to the existence of shear waves.)

In this example, only the radial (hydrodynamic) velocity components are necessary. The boundary/interface conditions of continuity of radial velocity across the interface and the kinematic condition result in

$$a_I = -c_I R^2 \tag{IVC.2a}$$

$$b^o = -d^o R^2 \tag{IVC.2b}$$

as well as a consistency check.

The continuity of the tangential component(s) of the hydrodynamic velocity is addressed using the surface divergence operator, with $\nabla_{II} \cdot \underline{u}^i - \nabla_{II} \cdot \underline{u}^o$ at r = R. Also, $\nabla_{II} - \nabla_{surface} - \nabla - \hat{e}_r \dfrac{\partial}{\partial r}$. This yields (after making use of relations is (IVC.2a-2b))

$$c_I R^\ell - d^o R^{-\ell-1} = 0 \tag{IVC.3}$$

The surface divergence of the tangential stress is given by the following equation. Note that it is forced by the surface divergence of $\overline{\overline{pr}}^{TANG}$, discussed in Section IVA. The constituents of

$\overline{pr}^{TANG}$ are the acoustic field quantities. It is

$$\mu^{i}\left\{2(1-\ell^{2})\,a_{I}\,R^{\ell-3}+(-2\ell)(\ell+2)\,c_{I}\,R^{\ell-1}\right\}Y_{\ell m}(\theta,\phi)\,e^{i\eta_R^t}$$

$$=\mu^{o}\left\{\begin{aligned}&-2\ell(\ell+2)\,b^{o}\,R^{-\ell-4}\\&+\ -2(\ell^{2}-1)\,d^{o}\,R^{-\ell-2}\end{aligned}\right\}Y_{\ell m}(\theta,\phi)\,e^{i\eta_R^t}$$

$$+\left\langle \nabla_{SURFACE}\cdot\overline{pr}^{TANG}\right\rangle e^{i\eta_{IM}^t}$$

(IVC.4a)

Henceforth, abbreviate $\nabla_{SURFACE}$ by ∇_s. The only component of $\overline{pr}^{TANG}$ will be $(\overline{pr}^{TANG})_\theta$, as there is no ϕ dependence. Moreover, m = 0. The brackets "< >" indicate time averaging (over the acoustic period).

It is understood that a summation appears in Equation (IVC.4a). Utilizing the orthogonility properties of spherical harmonic functions as well as Eqns. (IVC.2a-2b) yields, after manipulations

$$\mu^{i}\,(2l+1)\,c_{I}\,R^{l-1} \;+\; \mu^{o}(2l+1)\,d^{o}\,R^{-l-2}$$

$$= \frac{-\frac{1}{2}\left(\int_0^{2\pi}\int_0^{\pi} \sin\theta\, Y_{lm}^{*}\langle\langle \nabla_s \cdot \overline{pr}^{\,TANG} \rangle\rangle\, d\theta\, d\phi\right) e^{i(\eta_{IM}^{t} - \eta_{R}^{t})}}{\left(\int_0^{2\pi}\int_0^{\pi} \sin\theta\, Y_{lm}^{*}\, Y_{lm}\, d\theta\, d\phi\right)}$$

(IVC.4b)

The exponential represents the phase; with η_{IM}^{t} the imposed phase and η_{R}^{t} the response. Let $\eta_{DIF}^{t} = \eta_{IM}^{t} - \eta_{R}^{t}$. Also, via orthogonality properties, the integral which appears in the numerator is known to have a value of one. Therefore, Equation (IVC.4b) can be rewritten as

$$\mu^{i}\,(2l+1)\,R^{l-1}\,c_{I} \;+\; \mu^{o}(2l+1)\,d^{o}\,R^{-l-2}$$

$$= -\frac{1}{2}\left(\int_0^{2\pi}\int_0^{\pi} \sin\theta\, Y_{lm}^{*}\,(\nabla_s \cdot \overline{pr}^{\,TANG})\, d\theta\, d\phi\right)$$

$$\cdot\, \exp(i\,\eta_{DIF}^{t})$$

(IVC.4c)

The final condition is that of the normal stress balance. It is given by

$$2\mu^{o} \frac{\partial u_r^{o}}{\partial r} - 2\mu^{i} \frac{\partial u_r^{i}}{\partial r}$$

$$= \frac{\sigma}{R^2} (l+2)(l-1) \hat{k}_{l,m=0} \, Y_{lm}(\theta,\phi) \, e^{i\eta_R}$$

$$+ \left\langle -\overline{pr}^{\,RADIAL} \right\rangle e^{i\eta_{IM}}$$

(IVC.5a)

As before, a summation is implied. After substitutions, manipulations and use of orthogonality properties, this is given by

$$- 4\mu^{o} R^{-l-1} d^{o} + 4\mu^{i} R^{l} c_I$$

$$+ \frac{\sigma}{R^2} (l+2)(l-1) \hat{k}_{l,m=0}$$

$$= \left(\int_0^{2\pi} \int_0^{\pi} \sin\theta \, Y_{lm}^{*} \left\langle \overline{pr}^{\,RADIAL} \right\rangle d\theta \, d\phi \right)$$

$$\cdot \exp(i[\eta_{IM} - \eta_R])$$

(IVC.5b)

The "static" deformation is given by $\hat{\kappa}_{l,m=0}$. The surface tension/curvature term is third on the l.h.s., with σ indicating the surface tension. Let $\eta_{DIF} = \eta_{IM} - \eta_R$; with η_{IM} the phase of the forcing and η_R the phase of the response.

The system of equations can be written in matrix form:

$$\begin{pmatrix} 1 & -1 & 0 \\ (2l+1)\mu^i & (2l+1)\mu^o & 0 \\ -4\mu^o & 4\mu^i & \frac{\sigma(l+2)(l-1)}{R^3} \end{pmatrix} \begin{pmatrix} R^l c_I \\ R^{-l-1} d^o \\ \hat{K}_{l,m=0} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{-1}{2}\left(\int_0^{2\pi}\int_0^{\pi} \sin\theta\, Y^*_{l,m=0} \langle \nabla_s \cdot \overline{pr}^{TANG} \rangle d\theta\, d\phi\right) e^{i\eta^t_{DIF}} \\ \frac{1}{R}\left(\int_0^{2\pi}\int_0^{\pi} \sin\theta\, Y^*_{l,m=0} \langle \overline{pr}^{RADIAL} \rangle d\theta\, d\phi\right) e^{i\eta_{DIF}} \end{pmatrix}$$

(IVC.6)

This is a forced system in three unknowns. The expression for $\hat{\kappa}_{l,o}$ can be obtained easily via Cramer's rule. It is

$$\hat{\kappa}_{l,m=0} = \frac{1}{\left[\sigma(l+2)(l-1)/R^2\right]} \cdot$$

$$\Bigg\{ \left(\int_0^{2\pi}\int_0^{\pi} \sin\theta \, Y^*_{l,m=0} \, \langle \overline{pr}^{\,RADIAL} \rangle \, d\theta \, d\phi \right) e^{i\eta_{DIF}}$$

$$+ \frac{2(\mu^i - \mu^o)}{(2l+1)(\mu^i + \mu^o)} \cdot \left(\int_0^{2\pi}\int_0^{\pi} \sin\theta \, Y^*_{l,m=0} \, \langle \nabla_s \cdot \overline{pr}^{\,TANG} \rangle \, d\theta \, d\phi \right) e^{i\eta^t_{DIF}} \Bigg\}$$

(IVC.7)

Clearly, the tangential radiation stress will contribute to the deformation. Moreover, the incorporation of viscous effects into the acoustic field will modify the value of $<pr^{radial}>$.

Marston calculated $\hat{\kappa}_{l,o}$ for a 1 mm p-xylene drop in water, assuming an inviscid acoustic field. He took the acoustic carrier wave frequency to be 217.5 kHz. The drop's center was taken to be located at the velocity modal plane of the incident carrier wave. The incident pressure wave had amplitude of 10^5 dyn/cm^2.

His calculation of pr^{radial} was based on an approximate formula as opposed to expansion of the Bessel function series, later improved (Marston et. al., 1981). His improved formula calculated $<pr_{20}^{radial}>$ as -12 7 dyn/cm^2. Note that l = 2. Deformation was found to be on the order of 1 micron.

The calculation of $<pr^{radial}>$ in this work utilized the full series solution for the inviscid functions of y_1, p_1; the velocity and pressure fields of the acoustic wave [v_r^i, v_r^o, v_θ^i, v_θ^o, p^i, p^o]; also projected onto the $l = 2$ mode. Of course the series had to be truncated. Four terms were kept in the series (from j_o to j_3, $h_o^{(1)}$ to $h_3^{(1)}$, with j, $h^{(1)}$ representing spherical Bessel functions).

Physical property data was taken from Marston (1980). In this work, the incident wave field amplitude was expressed in terms of the acoustic velocity potential. Therefore, the value used by Marston for the acoustic pressure had to be converted to the corresponding value for the acoustic velocity. This was done.

It was found that $<\overline{pr}_{20}^{radial}>$ equaled -.12 dynes/cm^2. No viscous effects were taken into account in the calculation of the radial pressure forcing in order that its value be compared to that obtained by Marston.

In the calculation of the tangential forcing effect it was found that

$$\frac{2(\mu^i - \mu^o)}{(2l+1)(\mu^i+\mu^o)} \left(\int_0^{2\pi}\int_0^{\pi} \sin\theta \, Y_{lm=0}^{*} \langle \nabla_s \cdot \overline{pr}^{TANG} \rangle \, d\theta \, d\phi \right) e^{i\eta_{DIF}^{t}}$$

$$\{ \eta_{DIF}^{t} \equiv 0 \}$$

for $l = 2$ was equal to -0.05 dynes/cm^2. This then modifies the amount of deformation - which still remains quite small relative to the drop size.

The above number is with η'_{DIF} set to zero. That is, the phase of the hydrodynamic field response was assumed equal to that of the imposed driver - which is not likely. Thus, the magnitude of this deformation represents a maximum.

It is in Section V that the static deformation will be re-addressed. In that section, it will be shown that the newly found form of the hydrodynamic field will result in modifications (potentially very significant) to the aforementioned result.

Also, in the work of this project, expansions for the hydrodynamic field were done in terms of Legendre polynomials, as opposed to the spherical harmonics used by Miller and Scriven (1968) and by Marston (1980).

The work of Marston et. al., (1981) determined $<\overline{\mathrm{pr}}_{20}^{\mathrm{radial}}>$ to be -.127 dynes/cm^2 using an approximate scheme. The calculation of the effect of the tangential acoustic radiation pressure forcing could not be determined by Marston's fomulation.

Further discussion of details on the calculations of this section can be found in Ferguson (see Reference section).

V. HYDRODYNAMIC FIELD: FORCED BY AN UNMODULATED ACOUSTIC WAVE FIELD

In this section, the hydrodynamic field which exists (at second order in the expansion parameter δ) as a result of the unmodulated acoustic standing wave field is investigated. In this case, the steady state hydrodynamic field is simply the streaming field. Moreover, the radiation pressure vector is comprised of static contributions only. The static radiation pressure results in drop deformation. This is the situation which would occur if the acoustic levitation system were to be used solely to position the sample rather than to induce oscillations as well. It is noted, however, that if the modulated standing acoustic wave field was the acoustic field, static deformations would exist. That is, in the case of the (induced) oscillating fluid droplet, deformation of the droplet will exist. Of course, if the deformation is very small relative to the drop dimension, it need not be taken into account in the interface/boundary conditions.

It is remarked that equations at this order could be constructed for which the time dependence is exp $(\mp 2 i t)$. However, they are not of interest.

The hydrodynamic field is considered to be viscous and incompressible. It must be determined both interior and exterior to the drop. The generation of the second order (in δ) system of equations will be presented explicitly in regions both interior and exterior to the drop.

VA. HYDRODYNAMIC FIELD EXTERIOR TO THE DROP: EQUATIONS AND SOLUTIONS

The nonlinear system of governing equations given in Section IIC (by equations IIC.1 - IIC.3) is nondimensionalized via relationships given in Section IIIB (see IIIB.1).

This results in the nondimensional system

$$\frac{\partial \rho^\circ}{\partial t} + \underline{U}^\circ \cdot \nabla \rho^\circ + \rho^\circ \nabla \cdot \underline{U}^\circ = 0 \tag{VA.1a}$$

$$\rho^\circ \frac{\partial \underline{U}^\circ}{\partial t} + \rho^\circ \underline{U}^\circ \cdot \nabla \underline{U}^\circ = -\nabla p^\circ + \frac{1}{Re_{AC}} \nabla^2 \underline{U}^\circ + \left(\frac{1}{3} + \tau^\circ\right)\frac{1}{Re_{AC}} \nabla(\nabla \cdot \underline{U}^\circ) \tag{VA.1b}$$

and relationship between pressure and density which holds at order δ. Note that the "o" superscript refers to the region exterior to the drop. The velocity field is given by $\underline{U}^\circ$, the pressure and density by p° and ρ° respectively. The Reynolds-type number is denoted by Re_{AC}, with the subscript denoting that quantities relevant to the acoustic field are utilized. (This has been referred to previously as Re). Thus, $Re_{AC} = c_o^o(c_o^o/\omega_{AC})/\nu_o^o$. Again, the subscript on the frequency, ω_{AC}, indicates that it is the acoustic frequency.

Let the dependent field variables be expanded in a series in the expansion parameter δ, (Recall $\delta = \omega_{DROP}/\omega_{AC}$, with ω_{DROP} the (natural) frequency of drop oscillation). That is,

$$\underline{U}^\circ = \delta \underline{v}_1^\circ + \delta^2 \underline{u}_2^\circ \tag{VA.2a}$$

$$p^{o} = P_{o}^{o} + \delta p_{1}^{o} + \delta^{2} p_{2}^{o} \tag{VA.2b}$$

$$\rho^{o} = 1 + \delta \rho_{1}^{o} + \delta^{2} \rho_{2}^{o} \tag{VA.2c}$$

Subscripts of "1" indicate acoustic field variables, "2" indicates a hydrodynamic field variable. At order δ, the governing equations of the acoustic field are recovered. These can be found in Section IIIB; see Equations (IIIB.2a-IIIB.2c).

The hydrodynamic field equations occur at order δ^2. Recall that the hydrodyanmic field is incompressible.

Thus,

$$\nabla \cdot \underline{u}_{2}^{o} = 0 \tag{VA.3a}$$

$$\frac{\partial \underline{u}_{2}^{o}}{\partial t} + \nabla p_{2}^{o} - \frac{1}{Re_{Ac}} \nabla^{2} \underline{u}_{2}^{o} = -\left(\rho_{1}^{o} \frac{\partial \underline{v}_{1}^{o}}{\partial t} + \underline{v}_{1}^{o} \cdot \nabla \underline{v}_{1}^{o} \right) \tag{VA.3b}$$

It is clear that the term $\left(\frac{1}{3}+\tau^{o}\right)\frac{1}{Re_{Ac}}\nabla\left(\nabla \cdot \underline{u}_{2}^{o}\right)$ which occurs in the conservation of momentum equation must be zero. The quantities $\underline{v}_1^{o}$ and ρ_1^{o} are known at this order, and act as forcing (or source) terms in the conservation of momentum equation (VA.3b).

Interest is in the steady state streaming field. The time average of equations (VA.3a-3b) over a period $(2\pi/\omega_{Ac})$ must be taken. Before doing this, it is noted (using Equation IIIB.2a) that

$$\frac{\partial(\rho_1^\circ \underline{v}_1^\circ)}{\partial t} = \rho_1^\circ \frac{\partial \underline{v}_1^\circ}{\partial t} + \underline{v}_1^\circ \frac{\partial \rho_1^\circ}{\partial t}$$

$$= \rho_1^\circ \frac{\partial \underline{v}_1^\circ}{\partial t} - \underline{v}_1^\circ (\nabla \cdot \underline{v}_1^\circ)$$

(VA.4)

Therefore

$$\frac{\partial \underline{u}_2^\circ}{\partial t} + \nabla p_2^\circ + \frac{-1}{Re_{Ac}} \nabla^2 u_2^\circ$$

$$= -\left(\begin{array}{c} \frac{\partial}{\partial t}(\rho_1^\circ \underline{v}_1^\circ) + \underline{v}_1^\circ (\nabla \cdot \underline{v}_1^\circ) \\ + \underline{v}_1^\circ \cdot \nabla \underline{v}_1^\circ \end{array} \right)$$

(VA.5)

Taking the time average yields

(VA.6a)

$$\nabla \cdot \underline{u}_2^\circ = 0$$

$$\nabla p_2^\circ - \frac{1}{Re_{Ac}} \nabla^2 \underline{u}_2^\circ$$

$$= -\left(\underline{v}_1^\circ \, \overline{(\nabla \cdot \underline{v}_1^\circ)} + \underline{v}_1^\circ \cdot \overline{\nabla \underline{v}_1^\circ} + \text{COMPLEX CONJUGATE} \right)$$

(VA.6b)

The overbar indicates complex conjugation. At this step, it is now understood that $\underline{u}_2^\circ$, p_2°, and $\underline{v}_1^\circ$ are expressions that are <u>independent of time</u>. It remains to solve for (and p_2°), using system (VA.6a-6b).

Several facets of this problem will be discussed before proceeding to a solution. The first aspect involves a <u>re</u>-nondimensionalization of the variables, and hence the system of equations. The second aspect will address the nature of the forcing (or source) terms. In particular, it is recalled from Section III that the acoustic field itself (i.e. $\underline{v}_1^\circ$, p_1°) has been expressed in terms of a composite solution. The ramifications of the composite solution for this problem will be explored.

<u>Re-nondimensionalization Scheme</u>

Heretofore, the nondimensionalizations employed have utilized acoustic field reference quantities, such as ω_{Ac} and c_o°, the speed of sound in the outer medium. The solutions which have be found are, of course, nondimensional.

It is found to be more convenient at this stage to re-nondimensionalize with respect to reference quantities important in the hydrodynamic problem.

Rewriting (VA.6b) in terms of dimensional quantities yields

$$\frac{1}{(\rho_o^\circ c_o^{\circ 2})}(c_o^\circ/\omega_{AC})\,\hat{\hat{\nabla}}\hat{p}_2^\circ - \frac{1}{Re_{AC}}\frac{1}{c_o^\circ}(c_o^\circ/\omega_{AC})^2\,\hat{\hat{\nabla}}^2\hat{\underline{u}}_2^\circ$$

$$= \frac{1}{(c_o^\circ)^2}(c_o^\circ/\omega_{AC})(-1)\left(\hat{\underline{V}}_1^\circ\cdot\overline{\hat{\underline{\nabla}}\hat{V}_1^\circ} + \hat{\underline{V}}_1^\circ\,\overline{(\nabla\cdot\hat{\underline{V}}_1^\circ)} + c.c.\right) \qquad \text{(VA.7)}$$

The" ^ " indicates dimensional quantities.

The drop reference dimension is taken to be d, the reference velocity field is taken to be $(d\omega_{drop})$, and the reference pressure field is taken to be $(\rho_o^o)(d\omega_{DROP})^2$. These are used to re-nondimensionalize Equation (VA.7).

A key point is the fact that since the acoustic frequency was chosen so that (c_o^o/ω_{Ac}) would be on the order of the drop size; the drop reference dimension d can be equated to (c_o^o/ω_{Ac}).

After substitution and manipulation, the <u>re-nondimensionalized</u> hydrodynamic field governing equations are found to be

$$\nabla\cdot\underline{u}_2^\circ = 0 \qquad \text{(VA.8a)}$$

$$\nabla p_2^\circ + \frac{-1}{Re_{HYDR}}\nabla^2\underline{u}_2^\circ = (-1)\left\{\begin{matrix}\underline{V}_1^\circ\cdot\overline{\nabla\underline{V}_1^\circ} + \underline{V}_1^\circ\overline{(\nabla\cdot\underline{V}_1^\circ)}\\ + \text{COMPLEX CONJUGATE}\end{matrix}\right\} \qquad \text{(VA.8b)}$$

Note that $Re_{HYDR} - \dfrac{d(d\omega_{DROP})}{\nu_o^o}$.

Discussion of Forcing Terms

The nondimensional forcing terms seen in equation (VA.8b) (or in (VA.6b)) are composed of known acoustic field quantities. Recall that a composite solution exists for v_1°. For example, $v_{1\theta}^\circ = \hat{v}_{\theta o}^\circ(r,\theta) + \tilde{\tilde{v}}_{\theta o}^\circ(\zeta,\theta)$. Since the time dependence has been eliminated, this is a variation on results of Section III. As $\zeta \to \infty$, $\tilde{\tilde{v}}_{\theta o}^\circ \to o$. There is then an acoustic "sublayer region" (of dimension SQRT $\left(\nu_o^\circ/\omega_{Ac}\right)$) <u>outside</u> of which any terms involving $\tilde{\tilde{v}}_{\theta o}^\circ$, $\tilde{\tilde{v}}_{r1}^\circ$, or their respective complex conjugates are zero.

Therefore, outside of this acoustic sublayer region, the only terms which contribute as forcing terms are of the form

$$\left(\nabla \hat{\phi}^\circ \cdot \overline{\nabla(\nabla \hat{\phi}^\circ)} + \nabla \hat{\phi}^\circ \left(\overline{\nabla^2 \hat{\phi}^\circ}\right) \right)$$

$$+ \text{ COMPLEX CONJUGATE}$$

However, expressions of this form can be re-written in terms of the gradient of a scalar function. The ramification is that outside of the acoustic sublayer, the forcing function, written in terms of the gradient of a scalar function, can be viewed as strictly a modification to the pressure field.

Therefore, <u>outside</u> of the "acoustic sublayer" region, the curl of Equation (VA.8b) is

$$\frac{-1}{Re_{HYDR}} \nabla \times \nabla^2 \underline{u}_2^\circ = \underline{0} \qquad \textbf{(VA.9a)}$$

Let $\underline{\omega}_2^\circ\left(=\nabla \times \underline{u}_2^\circ\right)$ be the vorticity. Since the velocity field is two-dimensional, only the $\hat{e}_\phi$ component of $\underline{\omega}_2^\circ$ exists. Equation (VA.9a) can be written in terms of the vorticity as

$$\frac{1}{Re_{HYDR}} \nabla^2 \underline{\omega}_2^\circ = \underline{0} \tag{VA.9b}$$

In essence, outside of the "acoustic sublayer" region, the forcing of the hydrodynamic field is zero, and there are no sources of vorticity.

It is inside the "acoustic sublayer" region only that the hydrodynamic field equations are forced. Solutions to this forced vector equation will be found in terms of ζ, the stretched independent variable introduced in Section III (and Θ, of course). These solutions will decay as $\zeta \to \infty$, and will represent corrections to the unforced problem in the region of the drop/host medium interface.

Solution in the "Outer" Region Exterior to the Drop

The governing equation is given by (VA.9a). It is reasonable to work this problem in terms of a Stokes' stream function. Let

$$u_{2r}^\circ = \frac{1}{r^2 \sin\theta} \frac{\partial \psi^\circ}{\partial \theta} \tag{VA.10a}$$

$$u_{2\theta}^\circ = \frac{-1}{r \sin\theta} \frac{\partial \psi^\circ}{\partial r} \tag{VA.10b}$$

Substitution into Equation (VA.9b) yields

$$\frac{1}{Re_{HYDR}}\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\cdot$$
$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\psi^o = 0 \qquad \text{(VA.11a)}$$

The solution to this equation is simply Stokes flow. It is given by

$$\psi_2^o(r,\theta) = \sum_{\ell=1}^{\infty}\left(B^o r^{-\ell+2} + F^o r^{-\ell}\right)\sin\theta\, P_\ell^1(\cos\theta) \qquad \text{(VA.11b)}$$

Solution in the "Acoustic Sublayer" Region Exterior to the Drop

In this region, the independent variable r is stretched, and is represented by

$$r = \tilde{R} + \epsilon\zeta$$

Recall that ϵ = 1/SQRT (Re_{Ac}). However, the governing system of equations has been re-nondimensionalized with respect to reference quantities important in the hydrodynamic problem.

The question arises: What relationship should ϵ have to Re_{HYDR}? Recall that $\delta = \omega_{DROP}/\omega_{Ac}$. Then

$$Re_{HYDR} = \delta\,(Re_{Ac}) = \delta/\epsilon^2 \qquad \text{(VA.12a)}$$

and

$$\epsilon = \sqrt{\delta}\,/\sqrt{Re_{HYDR}} \qquad \text{(VA.12b)}$$

It is remarked that other nondimensional parameters can be developed which will relate this problem to that studied by Riley. Define

$$R_s = \delta^2 Re_{AC} = \delta^2/\epsilon^2 = \delta\, Re_{HYDR} \tag{VA.12c}$$

In this work, it is clear that $Re_{Ac} \gg 1$ and $\epsilon \ll 1$.

In order to select the order of magnitude of Re_{HYDR}, a relationship between the orders of magnitude of δ and ϵ must be stated. Moreover, selection of the order of magnitude of Re_{HYDR} will then determine the order of magnitude of R_s.

In this work, the following will be taken:

$$Re_{HYDR} = O(1) \tag{VA.12d}$$

Then it follows that $R_s = o(\delta)$. Therefore, the order of δ must be equivalent to that of ϵ^2. (Of course, Re_{Ac} may then be re-expressed as being of order (δ^{-1}).)

A formal analogy may be made with the work of Riley. This work considered the flow field resulting from a <u>solid</u> body in <u>oscillatory</u> <u>translational</u> motion. Such a problem is quite different from that investigated in this project. However, there are mathematical similarities. In his work, the parameters Rs_{RILEY}, ϵ_{RILEY}, and M_{RILEY} occur, with $Rs_{RILEY} = \epsilon^2_{RILEY} M^2_{RILEY}$. M_{RILEY} corresponds to SQRT (Re_{Ac}). ϵ_{RILEY} corresponds to δ. R_{RILEY} and Rs_{RILEY} correspond to Re_{HYDRO} and Rs, respectively. For the case of $M \gg 1$, R_{RILEY} of order 1, and $Rs_{RILEY} \ll 1$, Riley finds that the flow outside of a shear layer region is Stokes-like. Note that <u>outside</u> of the acoustic sublayer <u>in this problem</u>, the flow was found to be Stokes-flow.

Since $\epsilon = \dfrac{\sqrt{\delta}}{\sqrt{Re_{HYDR}}}$, the relationship between r and ζ can be re-expressed as

$$r = R + \frac{\sqrt{\delta}}{\sqrt{Re_{HYDR}}}\,\zeta \qquad \textbf{(VA.13)}$$

with Re_{HYDR} of order one. It now is necessary to rewrite the governing system of equations in terms of the stretched variable ζ. Note that in this acoustic sublayer region, the forcing terms on the right hand side of equation (VA.8b) will contribute. These forcing terms are comprised of functions which involve the acoustic velocity field. Recall that this field had a representation in the acoustic sublayer region which involved functions of ζ as well as the outer region field (representing the inviscid acoustic field) re-expressed in terms of a Taylor series expansion.

As was the case in the outer region, it is convenient to work in terms of a stream function. After taking the curl of equation (VA.8b) and expanding in terms of the acoustic sublayer region variable, one obtains

$$\frac{\partial}{\partial r} \Rightarrow \frac{\sqrt{Re_{HYDR}}}{\sqrt{\delta}}\,\frac{\partial}{\partial \zeta} \qquad \textbf{(VA.14)}$$

and

$$\frac{1}{Re_{HYDR}} \quad \frac{1}{(\tilde{R} + \sqrt{\delta/Re_{HYDR}}\,\zeta)} \quad \frac{1}{\sin\theta}$$

$$\cdot \left\{ \left(\frac{Re_{HYDR}}{\delta}\right) \frac{\partial^2}{\partial\zeta^2} + \frac{\sin\theta}{(\tilde{R} + \sqrt{(\delta/Re_{HYDR})}\,\zeta)^2} \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \right\}$$

$$\cdot \left\{ \left(\frac{Re_{HYDR}}{\delta}\right)^2 \frac{\partial^2}{\partial\zeta^2} + \frac{\sin\theta}{(\tilde{R} + \sqrt{(\delta/Re_{HYDR})}\,\zeta)^2} \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \right\}$$

$$\left[\Psi^{\circ}_{20} + \sqrt{\frac{\delta}{Re_{HYDR}}}\, \Psi^{\circ}_{21} + \left(\frac{\delta}{Re_{HYDR}}\right)\Psi^{\circ}_{22} + \cdots \right]$$

$$= \left\{ \sqrt{\frac{Re_{HYDR}}{\delta}} \frac{\partial}{\partial\zeta}\left((\tilde{R} + \sqrt{\delta/Re_{HYDR}}\,\zeta) \cdot \left(\right.\right.\right.$$

$$\left[\left(\hat{V}^{\circ}_{r_0|\tilde{R}} + \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2} \hat{V}^{\circ\prime}_{r_0|\tilde{R}} + \left(\frac{\delta}{Re_{HYDR}}\right) \tilde{\tilde{V}}^{\circ}_{r_1}(\zeta,\theta) + \cdots \right] \left(\sqrt{\frac{Re_{HYDR}}{\delta}}\right) \frac{\partial}{\partial\zeta} \overline{\left(\hat{V}^{\circ}_{\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{\circ}_{\theta_0}(\zeta,\theta) + \cdots \right)} \right]$$

$$+ \left[(\hat{V}^{\circ}_{\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{\circ}_{\theta_0} \cdots)\, \overline{(\tilde{R} + \sqrt{(\delta/Re_{HYDR})}\,\zeta)^{-1}}\, \overline{\frac{\partial}{\partial\theta}[\, \tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots]} \right]$$

$$+ \left[(\tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots)\, \overline{(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta)^{-1} \frac{\partial}{\partial\theta}[\, \hat{V}^{\circ}_{\theta_0|\tilde{R}} + \cdots]} \right]$$

$$+ \left[(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta)^{-1} (\hat{V}^{\circ}_{\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots)\, \overline{(\sqrt{\delta}/\sqrt{Re_{HYDR}}\, \tilde{\tilde{V}}^{\circ}_{r_1} + \cdots)} \right]$$

$$+ \left[(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta)^{-1} (\tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots)\, \overline{\left(\hat{V}^{\circ}_{r_0|\tilde{R}} + \frac{\sqrt{\delta}\,\zeta}{\sqrt{Re_{HYDR}}} \hat{V}^{\circ\prime}_{r_0|\tilde{R}} + \cdots \right)} \right]$$

$$+ \overline{\left[\hat{V}^{\circ}_{\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots \right]} \left\{ \left(\frac{Re_{HYDR}}{\delta}\right)^{1/2} \frac{\partial}{\partial\zeta}\left(\sqrt{\frac{\delta}{Re_{HYDR}}}\, \tilde{\tilde{V}}^{\circ}_{r_1} + \cdots \right) + 2(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta)^{-1} \left(\sqrt{\frac{\delta}{Re_{HYDR}}}\, \tilde{\tilde{V}}^{\circ}_{r_1} + \cdots\right) \right.$$

$$\left. + \frac{(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta)^{-1}}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta\, (\tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots) \right) \right\}$$

$$+$$

$$+ \overline{\left[\tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots \right]} \left\{ \left(\frac{Re_{HYDR}}{\delta} \right)^{1/2} \frac{\partial}{\partial \zeta} \left(\hat{V}^{\circ}_{r_0|\tilde{R}} + \sqrt{\frac{\delta}{Re_{HYDR}}}\, \zeta\, \hat{V}^{\circ\prime}_{r_0|\tilde{R}} + \cdots \right) + 2\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2} \zeta\right)^{-1} \left(\hat{V}^{\circ}_{r_0|\tilde{R}} + \cdots \right) \right.$$

$$\left. + \frac{\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2} \zeta\right)^{-1}}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \left(\hat{V}^{\circ}_{\theta_0|\tilde{R}} + \cdots \right) \right) \right\} \Bigg) \Bigg)$$

$$- \frac{\partial}{\partial\theta} \Bigg(\left[\left(\hat{V}^{\circ}_{r_0|\tilde{R}} + \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2} \left(\tilde{\tilde{V}}^{\circ}_{r_1} + \zeta\, \hat{V}^{\circ\prime}_{r_0|\tilde{R}} \right) + \cdots \right) \left(\frac{Re_{HYDR}}{\delta}\right)^{1/2} \frac{\partial}{\partial \zeta} \overline{\left(\sqrt{\frac{\delta}{Re_{HYDR}}}\ \tilde{\tilde{V}}^{\circ}_{r_1} + \cdots \right)} \right]$$

$$+ \left[\overline{\left\{ \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2} \tilde{\tilde{V}}^{\circ}_{r_1} + \cdots \right\}} \left(\frac{Re_{HYDR}}{\delta}\right)^{1/2} \frac{\partial}{\partial\zeta} \overline{\left[\hat{V}^{\circ}_{r_0|\tilde{R}} + \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2} \zeta\, \hat{V}^{\circ\prime}_{r_0|\tilde{R}} + \cdots \right]} \right]$$

$$+ \left[\overline{\left(\hat{V}^{\circ}_{\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots \right)} \left(\tilde{R} + (\delta/Re_{HYDR})^{1/2} \zeta \right)^{-1} \frac{\partial}{\partial\theta} \left(\sqrt{\frac{\delta}{Re_{HYDR}}}\ \tilde{\tilde{V}}^{\circ}_{r_1} + \cdots \right) \right]$$

$$+ \left[\overline{\left(\tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots \right)} \left(\tilde{R} + (\delta/Re_{HYDR})^{1/2} \zeta \right)^{-1} \frac{\partial}{\partial\theta} \left(\hat{V}^{\circ}_{r_0|\tilde{R}} + \cdots \right) \right]$$

$$- \left[\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2} \zeta \right)^{-1} \left(\tilde{\tilde{V}}^{\circ}_{\theta_0} + \hat{V}^{\circ}_{\theta_0|\tilde{R}} + \cdots \right) \overline{\left(\tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots \right)} \right]$$

$$- \left[\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2} \zeta \right)^{-1} \left(\tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots \right) \overline{\left(\hat{V}^{\circ}_{\theta_0|\tilde{R}} + \cdots \right)} \right]$$

$$+ \left[\left(\hat{V}^{\circ}_{r_0|\tilde{R}} + (\delta/Re_{HYDR})^{1/2} \left(\hat{V}^{\circ\prime}_{r_0|\tilde{R}} + \tilde{\tilde{V}}^{\circ}_{r_1} \right) + \cdots \right) \left(\frac{Re_{HYDR}}{\delta} \right)^{1/2} \frac{\partial}{\partial\zeta} \overline{\left((\delta/Re_{HYDR})^{1/2}\, \tilde{\tilde{V}}^{\circ}_{r_1} + \cdots \right)} \right]$$

$$+ \left[\left((\delta/Re_{HYDR})^{1/2} \left(\tilde{\tilde{V}}^{\circ}_{r_1} \right) + \cdots \right) \left(Re_{HYDR}/\delta \right)^{1/2} \frac{\partial}{\partial\zeta} \overline{\left(\hat{V}^{\circ}_{r_0|\tilde{R}} + (\delta/Re_{HYDR})^{1/2} \left(\zeta\, \hat{V}^{\circ\prime}_{r_0|\tilde{R}} \right) + \cdots \right)} \right]$$

$$+ \left[\left(\hat{V}^{\circ}_{r_0|\tilde{R}} + \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2} \left(\zeta\, \hat{V}^{\circ\prime}_{r_0|\tilde{R}} + \tilde{\tilde{V}}^{\circ}_{r_1} \right) + \cdots \right) \overline{2\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2} \zeta \right)^{-1} \left((\delta/Re_{HYDR})^{1/2}\, \tilde{\tilde{V}}^{\circ}_{r_1} + \cdots \right)} \right]$$

$$+ \left[\left\{ (\delta/Re_{HYDR})^{1/2}\, \tilde{\tilde{V}}^{\circ}_{r_1} + \cdots \right\} 2 \overline{\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2} \right)^{-1} \left(\hat{V}^{\circ}_{r_0} + \cdots \right)} \right]$$

$$+ \left[\left(\hat{V}^{\circ}_{r_0|\tilde{R}} + \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2} \left(\zeta\, \hat{V}^{\circ\prime}_{r_0|\tilde{R}} + \tilde{\tilde{V}}^{\circ}_{r_1} \right) + \cdots \right) \overline{\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2} \zeta \right)^{-1} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \left[\tilde{\tilde{V}}^{\circ}_{\theta_0} + \cdots \right] \right)} \right]$$

$$+ \left[\left\{ (\delta/Re_{HYDR})^{1/2}\, \tilde{\tilde{V}}^{\circ}_{r_1} + \cdots \right\} \overline{\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2} \zeta \right)^{-1} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \left\{ \hat{V}^{\circ}_{\theta_0|\tilde{R}} + \cdots \right\} \right)} \right]$$

$$\left. + \text{COMPLEX CONJUGATE} \right\} \qquad \textbf{(VA.15)}$$

Inspection of the leading order terms in Equation (VA.15) reveals that

$$\frac{1}{Re_{HYDR}}\,\frac{1}{\sin\theta}\,\frac{1}{\tilde{R}}\left(\frac{Re_{HYDR}}{\delta}\right)^2 \frac{\partial^4 \Psi_2^o}{\partial \zeta^4}$$

$$= \frac{(-1)}{\tilde{R}}\left\{\sqrt{\frac{Re_{HYDR}}{\delta}}\,\tilde{R}\left(\hat{V}_{r_o}^o\Big|_{\tilde{R}}\right)\sqrt{\frac{Re_{HYDR}}{\delta}}\,\overline{\frac{\partial^2 \tilde{\tilde{V}}_{\theta_o}^o}{\partial \zeta^2}} + c.c.\right\}$$

(VA.16)

This simplifies to

$$\frac{1}{Re_{HYDR}}\,\frac{1}{\tilde{R}\sin\theta}\left(\frac{Re_{HYDR}}{\delta}\right)\frac{\partial^4 \Psi_2^o}{\partial \zeta^4} = (-1)\left(\hat{V}_{r_o}^o\Big|_{\tilde{R}}\,\overline{\frac{\partial^2 \tilde{\tilde{V}}_{\theta_o}^o}{\partial \zeta^2}} + c.c.\right)$$

Therefore, it is clear that Ψ_2^o is of order $\left(\frac{\delta}{Re_{HYDR}}\right)$ in the acoustic sublayer. Then Ψ_{2o}^o and Ψ_{21}^o are zero and

$$\frac{\partial^4 \Psi_{22}^o}{\partial \zeta^4} = -(Re_{HYDR})\,\tilde{R}\sin\theta\left(\hat{V}_{r_o}^o\Big|_{\tilde{R}}\,\overline{\frac{\partial^2 \tilde{\tilde{V}}_{\theta_o}^o}{\partial \zeta^2}} + c.c.\right)$$

(VA.17)

It is necessary to solve for Ψ_{22}^o . Recall that

$$\tilde{\tilde{V}}^{\circ}_{\theta_0}(\zeta,\theta) = \sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\ell'} \, B^{\circ}_{BL_{\ell'}} \exp\left(\frac{-(1+i)}{\sqrt{2}}\zeta\right) \frac{dP_{\ell'}}{d\theta}$$

Let $\Psi^{o}_{22}(\zeta,\theta) - f(\zeta)\sin\theta\, P^{1}_{l}(\cos\theta)$, and substitute this into Equation (VA.17). Continue the manipulations by multiplying through by $P^{1}_{\mathcal{L}}(\cos\theta)$, and integrating over (o,π). This eliminates the theta dependence, and yields a forced equation for $f(\zeta)$, which is

$$\frac{d^4 f}{d\zeta^4} = Re_{HYDR}\left\{-\tilde{R}\,(I1^{\circ}_{\ell})\right\} i \exp\left(\frac{-(1+i)}{\sqrt{2}}\zeta\right)$$

$$+ Re_{HYDR}\left\{-\tilde{R}\,(I2^{\circ}_{\ell})\right\} (-i) \exp\left(-\frac{(1-i)}{\sqrt{2}}\zeta\right)$$

(VA.18)

with $\mathcal{L} = \ell$

$$I1^{\circ}_{\ell} = \left\{ \int_0^{\pi} P^{1}_{\mathcal{L}}(\cos\theta)\cdot \sin\theta \cdot \left(\sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\ell'} B^{\circ}_{BL_{\ell'}} \frac{dP_{\ell'}}{d\theta}\right)\cdot \left\{\sum_{\hat{\ell}=0}^{\infty} \overline{\left[\alpha^{\circ}_{S_{\hat{\ell}}}\, \hat{\hat{\delta}}_{\hat{\ell}} \frac{d}{dr}\left(h^{(1)}_{\hat{\ell}}(r)\right)_{1\tilde{R}} + \hat{\hat{\delta}}_{\hat{\ell}}\, A_{INC} \frac{dj_{\hat{\ell}}}{dr}\Big|_{\tilde{R}}\right]} P_{\hat{\ell}} \right\} d\theta \right\}$$

$$\cdot \frac{(2\ell+1)}{2\ell(\ell+1)}$$

and let

$$\sum_{L=1}^{\infty} I_L^{o} P_L^{1} = \left(\sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\ell'} B^{o}_{BL_{\ell'}} \frac{dP_{\ell'}}{d\theta} \right) \overline{\left(\sum_{\hat{\ell}=0}^{\infty} \left[\hat{\hat{\delta}}_{\hat{\ell}} \right] \left(\alpha^{o}_{S_{\hat{\ell}}} \frac{d}{dr} \left(h^{(1)}_{\hat{\ell}}(r) \right|_{\tilde{R}} + A_{NC} \frac{d}{dr} \left(j_{\hat{\ell}} \right|_{\tilde{R}} \right) P_{\hat{\ell}} \right)}$$

and

$$I2^{o}_{\ell} = \left\{ \int_0^{\pi} P^{1}_{\ell}(\cos\theta) \cdot \sin\theta \cdot \overline{\left(\sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\ell'} B^{o}_{BL_{\ell'}} \frac{dP_{\ell'}}{d\theta} \right)} \cdot \left(\sum_{\hat{\ell}=0}^{\infty} \delta_{\hat{\ell}} \left\{ \alpha^{o}_{S_{\hat{\ell}}} \frac{d}{dr} \left(h^{(1)}_{\hat{\ell}} \right|_{\tilde{R}} + A_{NC} \frac{d}{dr} \left(j_{\hat{\ell}} \right|_{\tilde{R}} \right\} P_{\hat{\ell}} \right) d\theta \right\} \cdot \frac{(2l+1)}{2\ell(\ell+1)}$$

It is found that

$$f_{particular}(\zeta) = (i)\left(K1^{o}_{\ell}\right)(-1)\exp\left(-\frac{(1+i)}{\sqrt{2}}\zeta\right) + (-i)\left(K2^{o}_{\ell}\right)(-1)\exp\left(-\frac{(1-i)}{\sqrt{2}}\zeta\right) \qquad \textbf{(VA.19)}$$

with $K1^{o} = Re_{HYDR}\left(-\tilde{R}\right)\left(I1^{o}_{i}\right)$ and $K2^{o} = Re_{HYDR}\left(-\tilde{R}\right)\left(I2^{o}_{i}\right)$.

Therefore, the solution for the stream function in the acoustic sublayer region is given by

$$\Psi^{\circ}_{2_2}(\varsigma,\theta) = \sum_{\ell=1}^{\infty} \sin\theta \, P'_\ell(\cos\theta) \cdot \left[\begin{array}{l} b^{\circ}_0 + b^{\circ}_1 \varsigma + b^{\circ}_2 \varsigma^2 + b^{\circ}_3 \varsigma^3 \\ + (-i) K1^{\circ}_\ell \exp\left(-\frac{(1+i)}{\sqrt{2}} \varsigma\right) \\ + (i) K2^{\circ}_\ell \exp\left(-\frac{(1-i)}{\sqrt{2}} \varsigma\right) \end{array} \right] \tag{VA.20}$$

VB. HYDRODYNAMIC FIELD INTERIOR TO THE DROP: EQUATIONS AND SOLUTIONS

The nondimensionalized nonlinear system of governing equations is given by

$$\frac{\partial \rho^i}{\partial t} + \underline{U}^i \cdot \nabla \rho^i + \rho^i \nabla \cdot \underline{U}^i = 0 \tag{VB.1a}$$

$$\rho^i \frac{\partial \underline{U}^i}{\partial t} + \rho^i \underline{U}^i \cdot \nabla \underline{U}^i = -\nabla p^i + \frac{\alpha}{Re_{AC}} \nabla^2 \underline{U}^i + \left(\frac{\alpha}{3} + \tau^i\right)\frac{1}{Re_{AC}} \nabla\left(\nabla \cdot \underline{U}^i\right) \tag{VB.1b}$$

and the relationship between pressure and density which holds at order δ. The "i" superscript refers to the region interior to the drop. The velocity field is given by $\underline{U}^i$, the pressure and density by p^i and ρ^i, respectively. The Reynolds-type number is denoted by $Re_{Ac} = \left[c_o^o\left(c_o^o/\omega_{Ac}\right)/\nu_o^o\right]$.

Let the dependent field variables be expanded in the expansion parameter $\delta\left(=\omega_{DROP}/\omega_{Ac}\right)$ as

$$\underline{U}^i = \delta \underline{v}_1^i + \delta^2 \underline{u}_2^i \tag{VB.2a}$$

$$p^i = P_0^i + \delta p_1^i + \delta^2 p_2^i \tag{VB.2b}$$

and

$$\rho^i = \beta + \delta \rho_1^i + \delta^2 \rho_2^i \tag{VB.2c}$$

The quantities having a subscript "1" are acoustic field variable, "2" indicates hydrodynamic field variables. At order δ, the governing equations of the acoustic field interior to the drop are recovered. (These can be found in Section IIIC).

The hydrodynamic field equations occur at order δ_2. Recall that the hydrodynamic field is incompressible, and thus

$$\nabla \cdot \underline{u}_2^i = 0 \tag{VB.3a}$$

$$\beta \frac{\partial \underline{u}_2^i}{\partial t} + \nabla p_2^i - \frac{\alpha}{Re_{Ac}} \nabla^2 \underline{u}_2^i$$

$$= -\left(\rho_1^i \frac{\partial \underline{v}_1^i}{\partial t} + \beta\, \underline{v}_1^i \cdot \nabla \underline{v}_1^i \right) \tag{VB.3b}$$

It is clear that $\left(\frac{\alpha}{3}+\tau^i\right) \nabla \left(\nabla \cdot \underline{u}_2^i\right)$ which appears in the conservation of momentum equation must be zero. The quantities $\underline{v}_1^i$ and ρ_1^i are known at this order, and act as forcing terms to the hydrodynamic field.

Interest is in the steady-state streaming field. The time average of equations (VB.3a-3b) over a period $(2\pi/\omega_{Ac})$ must be taken. Before doing this, it is noted that

$$\frac{\partial (\rho_1^i \underline{v}_1^i)}{\partial t} = \rho_1^i \frac{\partial \underline{v}_1^i}{\partial t} + \underline{v}_1^i \frac{\partial \rho_1^i}{\partial t}$$

$$= \rho_1^i \frac{\partial \underline{v}_1^i}{\partial t} + \underline{v}_1^i \left(- \beta\, \nabla \cdot \underline{v}_1^i \right) \tag{VB.4}$$

Therefore

$$\beta \frac{\partial \underline{u}_2^i}{\partial t} + \nabla p_2^i + \frac{-\alpha}{Re_{AC}} \nabla^2 \underline{u}_2^i = -\left(\frac{\partial}{\partial t} (\rho_1^i \underline{v}_1^i) + \beta \underline{v}_1^i (\nabla \cdot \underline{v}_1^i) \right)$$

$$- \left(\beta \underline{v}_1^i \cdot \nabla \underline{v}_1^i \right) \qquad \textbf{(VB.5)}$$

Taking the time-average yields

$$\nabla \cdot \underline{u}_2^i = 0 \qquad \textbf{(VB.6a)}$$

$$\nabla p_2^i - \frac{\alpha}{Re_{AC}} \nabla^2 \underline{u}_2^i = -\beta \left(\underline{v}_1^i \cdot \nabla \underline{v}_1^i + \underline{v}_1^i (\nabla \cdot \underline{v}_1^i) \right)$$

$$+ \text{COMPLEX CONJUGATE}$$

$$\textbf{(VB.6b)}$$

The overbar denotes complex conjugation. AT THIS STEP, IT IS UNDERSTOOD THAT u_2^i, p_2^i ,and $\underline{v}_1^i$ represent expressions that are <u>independent of time</u>.

It remains to solve for u_2^i $\left(and\ p_2^i\right)$, using system (VB.6a-6b). The methodology will proceed in the same manner as that of Section VA.

<u>Re-nondimensionalization Scheme</u>

As was the case in Section VA, it is of interest to re-nondimensionalize the hydrodynamic field equations with respect to reference quantities of relevance.

The reference length will be that of the drop dimension, d. The reference velocity is to be given by $d\omega_{DROP}$. Finally, the reference pressure field is denoted by $\rho_o^o(d\omega_{DROP})^2$.

A key point is that the reference length, related to the drop dimension, must also be the reference length for the case in which the nondimensionalization scheme is carried out with respect to acoustic-field-relevant reference quantities. These are tied together because the acoustic frequency (ω_{Ac}) itself was chosen so that (c_o^o/ω_{Ac}) would be on the order of the drop dimension. Thus, $d\sim c_o^o/\omega_{Ac}$.

The re-nondimensionalization of the governing equations yields

$$\nabla \cdot \underline{u}_2^i = 0 \tag{VB.7a}$$

$$\nabla p_2^i + \frac{-\alpha}{Re_{HYDR}} \nabla^2 \underline{u}_2^i = -\beta\left(\underline{v}_1^i \cdot \nabla \underline{v}_1^i + \underline{v}_1^i (\nabla \cdot \underline{v}_1^i) \right) + \text{COMPLEX CONJUGATE} \tag{VB.7b}$$

Note that $Re_{HYDR}\sim\left(d(d\omega_{DROP})/\nu_o^o\right)$.

<u>Discussion of Forcing Terms</u>

The nondimensional forcing terms which appear in Equation (VB.7b) are composed of acoustic field variables, which are known at this order (δ^2). Recall that a composite solution was found for $\underline{v}_1^i$. Recall, for example, that $(\underline{v}_1^i)_\theta = \hat{v}_{\theta o}^i(r,\theta) + \tilde{\tilde{v}}_{\theta o}^i(\xi,\theta)$. Since the time dependence has been eliminated, this is a variation on the results of Section III. As $\xi \rightarrow \infty$ $\tilde{\tilde{v}}_{\theta o}^i \rightarrow 0$. There is then an acoustic "sublayer region" (of dimension SQRT((ν_o^o/ω_{Ac})) <u>outside</u> of which any terms involving $\tilde{\tilde{v}}_{\theta o}^i$, $\tilde{\tilde{v}}_{r1}^i$, or their respective complex conjugates are zero.

Therefore, outside of this acoustic sublayer region, the only terms which contribute as forcing terms are of the form

$$\left\{ \begin{array}{c} \nabla\hat{\phi}^i \cdot \overline{\nabla(\nabla\hat{\phi}^i)} + \nabla\phi^i\,\overline{(\nabla^2\hat{\phi}^i)} \\ + \text{COMPLEX CONJUGATE} \end{array} \right\}$$

(Recall that $\hat{\phi}^i$ is the acoustic velocity field potential for the inviscid acoustic field in the "outer region", still interior to the drop). However, expressions of this form can be rewritten in terms of the gradient of a scalar function. The ramification is that, <u>outside</u> of the acoustic sublayer, but still interior to the drop, the forcing function, written in terms of the gradient of a scalar function, can be viewed as strictly a modification to the pressure field.

Therefore, outside of the "acoustic sublayer" region, the curl of Equation (VB.7b) is

$$-\frac{\alpha}{Re_{HYDR}} \nabla \times \nabla^2 \underline{u}_2^i = \underline{0} \tag{VB.8a}$$

Let $\underline{\omega}_2^i \left(= \nabla \times \underline{u}_2^i\right)$ be the vorticity field. Since the velocity field is two-dimensional, only the $\hat{e}_\phi$ component of $\underline{\omega}_2^i$ exists. Equation (VB.8a) can be written in terms of the vorticity as

$$\frac{\alpha}{Re_{HYDR}} \nabla^2 \underline{\omega}_2^i = \underline{0} \tag{VB.8b}$$

In essence, outside of the "acoustic sublayer" region, the forcing of the hydrodynamic field is zero, and there are <u>no</u> sources of vorticity.

It is inside the "acoustic sublayer" region only that the hydrodynamic field equations are forced. Solutions to this forced vector equation will be found in terms of ξ, the stretched independent variable, introduced in Section III (and Θ, of course). These solutions will decay as $\xi \rightarrow \infty$, and will represent corrections to the unforced problem in the "acoustic sublayer" region (located near the drop/host interface).

Solution in the "Outer" Region Interior to the Drop

The governing equation is given by (VB.8). It is reasonable to work this problem in terms of a Stokes' stream function. Let

$$u^i_{2r} = \frac{1}{r^2 \sin\theta} \frac{\partial \psi^i}{\partial \theta} \tag{VB.9a}$$

$$u^i_{2\theta} = \frac{-1}{r \sin\theta} \frac{\partial \psi^i}{\partial r} \tag{VB.9b}$$

Substitution into Equation (VB.8b) yields

$$\frac{\alpha}{Re_{HYDR}} \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right)\right] \cdot \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right)\right] \psi^i = 0 \tag{VB.10a}$$

The solution for ψ^i is simply that of Stokes' flow. It is given by

$$\psi_2^i(r,\theta) = \sum_{\ell=1}^{\infty} \left(A^i r^{\ell+3} + E^i r^{\ell+1} \right) \sin\theta \; P_\ell^1(\cos\theta) \tag{VB.10b}$$

<u>Solution in the "Acoustic Sublayer" Region Interior to the Drop</u>

In this region, the independent variable r is stretched, and is given by

$$r = \tilde{R} - \varepsilon \xi$$

Recall that $\epsilon = (SQRT(Re_{Ac}))^{-1}$. The question arises: what relationship does/should ϵ have to Re_{HYDR}?

Recall that $\delta = \omega_{DROP}/\omega_{Ac}$. Then it is seen that

$$Re_{HYDR} = \delta \, Re_{AC} = \delta / \epsilon^2 \tag{VB.11a}$$

or, alternatively,

$$\epsilon = SQRT\left(\delta / Re_{HYDR} \right) \tag{VB.11b}$$

It is noted that other nondimensional parameters can be developed which will relate the problem under consideration to the work of Riley involving a solid body in periodic motion (Riley, 1967).

Clearly, the problem under consideration differs in a number of important aspects. The center of the drop is motionless; no solid boundaries exist in the problem. Most importantly, the hydrodynamic field (fluid motion) results from the existence of the acoustic standing wave field. The analogy with the work of Riley is that which has been presented in Section VA.

Define

$$R_s = \delta^2 Re_{Ac} = \delta^2/\epsilon^2 = \delta Re_{HYDR} \tag{VB.11c}$$

In this work, it is clear that $Re_{Ac} \gg 1$ and $\epsilon \ll 1$.

In order to select the order of magnitude of Re_{HYDR}, a relationship between δ and ϵ must be established. Moreover, selection of the order of magnitude of Re_{HYDR} will then determine the order magnitude of R_s.

In this work, the following will hold

$$Re_{HYDR} = O(1) \tag{VB.11d}$$

from which it follows that $R_s = o(\delta)$.

Re-expressing ϵ in terms of Re_{HYDR}, it is seen that

$$\epsilon = \frac{\sqrt{\delta}}{\sqrt{Re_{HYDR}}}$$

so that

$$r = \tilde{R} - \sqrt{\frac{\delta}{Re_{HYDR}}}\ \xi \tag{VB.12}$$

It is now necessary to rewrite the system of governing equations in terms of the stretched variable, ξ. Note that in the "acoustic sublayer" region, the forcing terms which appear on the right hand side of (VB.7b) <u>will contribute</u>. These forcing terms are comprised of functions which involve the acoustic velocity field interior to the drop. Recall that this field had a representation in the "acoustic sublayer" region which involved functions of ξ (representing the viscous correction) as well as the outer region field (still interior to the drop) representing the <u>inviscid</u> acoustic field re-expressed in terms of a Taylor series expansion.

It is convenient to work in terms of a stream function. After taking the curl of Equation (VB.7b), and expanding in terms of the "acoustic sublayer" region variable ξ, one obtains

$$\frac{\partial}{\partial r} \Longrightarrow -\sqrt{\frac{Re_{HYDR}}{\delta}}\,\frac{\partial}{\partial \xi} \tag{VB.13}$$

and
$$\left(\frac{\alpha}{Re_{HYDR}}\right)\frac{\left(\tilde{R}-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}}{\sin\theta}$$
$$\cdot\left[\left(\frac{Re_{HYDR}}{\delta}\right)\frac{\partial^2}{\partial\xi^2}+\frac{\sin\theta}{\left(\tilde{R}-(\delta/Re_{HYDR})^{1/2}\xi\right)^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\cdot$$
$$\left[\left(\frac{Re_{HYDR}}{\delta}\right)\frac{\partial^2}{\partial\xi^2}+\frac{\sin\theta}{\left(\tilde{R}-(\delta/Re_{HYDR})^{1/2}\xi\right)^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\left(\Psi^{i}_{20}+\left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\Psi^{i}_{21}+\cdots\right)$$

$$= -\beta\left(\tilde{R}-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\Bigg\{-\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\xi}\Bigg(\left(\tilde{R}-(\delta/Re_{HYDR})^{1/2}\xi\right)\cdot\Bigg($$

$$\left[\left(\hat{v}^{i}_{r_0}\big|_{\tilde{R}}+(\delta/Re_{HYDR})^{1/2}\left(-\xi\,\hat{v}^{i\prime}_{r_0}\big|_{\tilde{R}}+\tilde{\tilde{v}}^{i}_{r_1}\right)+\cdots\right)\overline{\left(-\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\xi}\left(\tilde{\tilde{v}}^{i}_{\theta_0}+\cdots\right)\right)}\right]$$
$$+\left[(\delta/Re_{HYDR})^{1/2}\left(\tilde{\tilde{v}}^{i}_{r_1}+\cdots\right)\overline{\left(-(Re_{HYDR}/\delta)^{1/2}\frac{\partial}{\partial\xi}\left(\hat{v}^{i}_{\theta_0}\big|_{\tilde{R}}+\cdots\right)\right)}\right]$$
$$+\left[\left\{\left(\hat{v}^{i}_{r_0}\big|_{\tilde{R}}\right)+(\delta/Re_{HYDR})^{1/2}\left(\tilde{\tilde{v}}^{i}_{r_1}+-\xi\,\hat{v}^{i\prime}_{r_0}\big|_{\tilde{R}}\right)+\cdots\right\}\overline{\left(\tilde{R}-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\frac{\partial}{\partial\theta}\left(\tilde{\tilde{v}}^{i}_{\theta_0}+\cdots\right)}\right]$$
$$+\left[\left\{(\delta/Re_{HYDR})^{1/2}\tilde{\tilde{v}}^{i}_{r_1}+\cdots\right\}\overline{\left(\tilde{R}-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\frac{\partial}{\partial\theta}\left(\hat{v}^{i}_{\theta_0}\big|_{\tilde{R}}+\cdots\right)}\right]$$

$$
+\left[\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\hat V^{i}_{\theta_0|\tilde R}+\tilde{\tilde V}^{i}_{\theta_0}+\cdots\right)\overline{\left(\tilde{\tilde V}^{i}_{r_1}\left(\delta/Re_{HYDR}\right)^{1/2}+\cdots\right)}\right]
$$

$$
+\left[\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\tilde V^{i}_{\theta_0}+\cdots\right)\overline{\left(\hat V^{i}_{r_0|\tilde R}+\cdots\right)}\right]
$$

$$
+\overline{\left(\tilde{\tilde V}^{i}_{\theta_0}+\cdots\right)}\cdot\Bigg[-\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\xi}\left(\hat V^{i}_{r_0|\tilde R}+\frac{\sqrt{\delta}}{\sqrt{Re_{HYDR}}}\left(-\xi\,\hat V^{i\,\prime}_{r_0|\tilde R}+\tilde{\tilde V}^{i}_{r_1}\right)+\cdots\right)
$$

$$
+2\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\hat V^{i}_{r_0|\tilde R}+(\delta/Re_{HYDR})^{1/2}\left(\tilde{\tilde V}^{i}_{r_1}-\xi\,\hat V^{i\,\prime}_{r_0|\tilde R}\right)+\cdots\right)
$$

$$
+\frac{\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(\hat V^{i}_{\theta_0|\tilde R}+\tilde{\tilde V}^{i}_{\theta_0}+\cdots\right)\right)\Bigg]
$$

$$
+\overline{\left(\hat V^{i}_{\theta_0|\tilde R}+\cdots\right)}\cdot\Bigg[-\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\xi}\left(\left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\tilde{\tilde V}^{i}_{r_1}+\cdots\right)+2\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\tilde{\tilde V}^{i}_{r_1}+\cdots\right)
$$

$$
+\frac{\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(\tilde{\tilde V}^{i}_{\theta_0}+\cdots\right)\right)\Bigg]\Bigg)\Bigg)
$$

$$
-\frac{\partial}{\partial\theta}\Bigg(\Bigg[\overline{\left\{\hat V^{i}_{r_0|\tilde R}+\left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\left(\tilde{\tilde V}^{i}_{r_1}-\xi\,\hat V^{i\,\prime}_{r_0|\tilde R}\right)+\cdots\right\}}\left(-\sqrt{\frac{Re_{HYDR}}{\delta}}\frac{\partial}{\partial\xi}\left(\left[\sqrt{\frac{\delta}{Re_{HYDR}}}\,\tilde{\tilde V}^{i}_{r_1}+\cdots\right]\right)\right)\Bigg]
$$

$$
+\left[\overline{\left\{(\delta/Re_{HYDR})^{1/2}\,\tilde{\tilde V}^{i}_{r_1}+\cdots\right\}}\left(-\sqrt{Re_{HYDR}/\delta}\;\frac{\partial}{\partial\xi}\left(\hat V^{i}_{r_0|\tilde R}+\cdots\right)\right)\right]
$$

$$
+\left[\overline{\left(\hat V^{i}_{\theta_0|\tilde R}+\tilde{\tilde V}^{i}_{\theta_0}+\cdots\right)}\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\frac{\partial}{\partial\theta}\left[\left(\tilde{\tilde V}^{i}_{r_1}\right)+\cdots\right]\left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\right]
$$

$$
+\left[\overline{\left(\tilde{\tilde V}^{i}_{\theta_0}+\cdots\right)}\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\frac{\partial}{\partial\theta}\left(\hat V^{i}_{r_0|\tilde R}+\cdots\right)\right]
$$

$$
-\left[\overline{\left(\tilde{\tilde V}^{i}_{\theta_0}+\hat V^{i}_{\theta_0|\tilde R}+\cdots\right)}\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\tilde{\tilde V}^{i}_{\theta}+\cdots\right)\right]
$$

$$
-\left[\overline{\left(\tilde{\tilde V}^{i}_{\theta_0}+\cdots\right)}\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\hat V^{i}_{\theta_0|\tilde R}+\cdots\right)\right]
$$

$$
+\overline{\left(\left(\delta/Re_{HYDR}\right)^{1/2}\tilde{\tilde V}^{i}_{r_1}+\cdots\right)}\cdot\Bigg[-\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\xi}\left(\hat V^{i}_{r_0|\tilde R}+(\delta/Re_{HYDR})^{1/2}\left(\tilde{\tilde V}^{i}_{r_1}-\xi\,\hat V^{i\,\prime}_{r_0|\tilde R}\right)+\cdots\right)
$$

$$
+2\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\hat V^{i}_{r_0|\tilde R}+\cdots\right)
$$

$$
+\frac{\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(\tilde{\tilde V}^{i}_{\theta_0}+\hat V^{i}_{\theta_0|\tilde R}+\cdots\right)\right)\Bigg]
\qquad \text{(VB.14)}
$$

$$
+\overline{\left(\hat V^{i}_{r_0|\tilde R}+\cdots\right)}\cdot\Bigg[-\left(Re_{HYDR}/\delta\right)^{1/2}\frac{\partial}{\partial\xi}\left(\left(\delta/Re_{HYDR}\right)^{1/2}\tilde{\tilde V}^{i}_{r_1}+\cdots\right)
$$

$$
+2\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\left(\delta/Re_{HYDR}\right)^{1/2}\tilde{\tilde V}^{i}_{r_1}+\cdots\right)
$$

$$
+\frac{\left(\tilde R-(\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(\tilde{\tilde V}^{i}_{\theta_0}+\cdots\right)\right)\Bigg]\Bigg)
$$

+ COMPLEX CONJUGATE $\Bigg\}$

Inspection of the leading order terms in Equation (VB.14) reveals that

$$\frac{\alpha}{Re_{HYDR}} \frac{1}{\tilde{R}\sin\theta} \left(\frac{Re_{HYDR}}{\delta} \right)^2 \frac{\partial^4 \Psi_2^i}{\partial \xi^4}$$

$$= \frac{(-\beta)}{R} \left\{ -\left(\frac{Re_{HYDR}}{\delta} \right)^{1/2} \frac{\partial}{\partial \xi} \left(\hat{V}^i_{r_0 | \tilde{R}} \overline{\left[-\left(\frac{Re_{HYDR}}{\delta} \right)^{1/2} \frac{\partial \tilde{\tilde{V}}^i_{\theta_0}}{\partial \xi} \right]} \right) \right\}$$

$$+ \text{ COMPLEX CONJUGATE}$$

(VB.15)

This simplifies to

$$\frac{\alpha}{Re_{HYDR}} \frac{1}{\tilde{R}\sin\theta} \left(\frac{Re_{HYDR}}{\delta} \right) \frac{\partial^4 \Psi_2^i}{\partial \xi^4}$$

$$= (-\beta) \left(\hat{V}^i_{r_0 | \tilde{R}} \overline{\frac{\partial^2 \tilde{\tilde{V}}^i_{\theta_0}}{\partial \xi^2}} + \text{c.c.} \right)$$

It is clear that Ψ_2^i is of order (δ / Re_{HYDR}) in the "acoustic sublayer", such that

$$\frac{\partial^4 \Psi^i_{2_2}}{\partial \xi^4} = \frac{-\beta}{\alpha} Re_{HYDR} (\tilde{R}\sin\theta) \left(\hat{V}^i_{r_0 | \tilde{R}} \overline{\frac{\partial^2 \tilde{\tilde{V}}^i_{\theta_0}}{\partial \xi^2}} + \text{c.c.} \right) \qquad \textbf{(VB.16)}$$

It is necessary to solve for $\Psi^{i}_{22}(\xi,\theta)$. Recall that

$$\tilde{\tilde{V}}^{i}_{\theta_0}(\xi,\theta) = \sum_{\ell'=1}^{\infty} \hat{\hat{S}}_{\ell'} A^{i}_{BL_{\ell'}} \exp\left(-\sqrt{\frac{\beta}{2\alpha}}(1+i)\xi\right)\frac{dP_{\ell'}}{d\theta}$$

Let $\Psi^{i}_{22}(\xi,\theta) - g(\xi)\sin\theta\, P^{1}_{l}(\cos\theta)$, and substitute this and the expression for $\tilde{\tilde{V}}_{\theta_0}{}^{i}$ into Equation (VB.16). Continue by multiplying Equation (VB.16) through by $P^{1}_{\mathscr{L}}(\cos\theta)$, and integrating over (o, π). This eliminates the theta dependence, and renders a forced ordinary differential equation for g(ξ) which is

$$\frac{d^4 g}{d\xi^4} = Re_{HYDR}\left(\frac{\beta}{\alpha}\right)^2 \tilde{R}\,(-i)\,(I1^{i}_{\ell})\exp\left(-\sqrt{\frac{\beta}{2\alpha}}(1-i)\xi\right)$$

$$+\, Re_{HYDR}\left(\frac{\beta}{\alpha}\right)^2 \tilde{R}\,(i)\,(I2^{i}_{\ell})\exp\left(-\sqrt{\frac{\beta}{2\alpha}}(1+i)\xi\right)$$

(VB.17)

with $\mathscr{L} = \ell$

$$I1_{\mathscr{L}}^{i} = \left\{ \int_0^{\pi} P_{\mathscr{L}}^{1}(\cos\theta)\cdot\sin\theta\cdot\overline{\left(\sum_{\ell'=1}^{\infty} \hat{\delta}_{\ell'} A_{BL_{\ell'}}^{i} \frac{dP_{\ell'}}{d\theta}\right)} \cdot\left(\sum_{\hat{\ell}=0}^{\infty} \hat{\hat{\delta}}_{\hat{\ell}}\, \alpha_{\hat{\ell}}^{i} \frac{d}{dr}\left(j_{\hat{\ell}}\left(\frac{c_0^{o}}{c_0^{i}}r\right)\right)_{|\tilde{R}} P_{\hat{\ell}}(\cos\theta)\right) d\theta \right\} \cdot \frac{(2\mathscr{L}+1)}{2\mathscr{L}(\mathscr{L}+1)}$$

and let

$$\sum_{L=1}^{\infty} I_L^{i} P_L^{i} = \left(\sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\ell'} A_{BL_{\ell'}}^{i} \frac{dP_{\ell'}}{d\theta}\right) \cdot \left(\sum_{\hat{\ell}=0}^{\infty} \hat{\hat{\delta}}_{\hat{\ell}}\, \alpha_{\hat{\ell}}^{i} \frac{d}{dr}\left(j_{\hat{\ell}}\left(\frac{c_0^{o}}{c_0^{i}}r\right)\right)_{|\tilde{R}} P_{\hat{\ell}}(\cos\theta)\right)$$

with L = $\mathscr{L}$ to contribute

$$I2_{\mathscr{L}}^{i} = \left\{ \int_0^{\pi} P_{\mathscr{L}}^{1}(\cos\theta)\cdot\sin\theta\cdot\left(\sum_{\ell'=1}^{\infty} \hat{\delta}_{\ell'} A_{BL_{\ell'}}^{i} \frac{dP_{\ell'}}{d\theta}\right) \cdot\overline{\left(\sum_{\hat{\ell}=0}^{\infty} \hat{\hat{\delta}}_{\hat{\ell}}\, \alpha_{\hat{\ell}}^{i} \frac{d}{dr}\left(j_{\hat{\ell}}\left(\frac{c_0^{o}}{c_0^{i}}r\right)\right)_{|\tilde{R}} P_{\hat{\ell}}(\cos\theta)\right)} \right\} \cdot \frac{(2\mathscr{L}+1)}{2\mathscr{L}(\mathscr{L}+1)}$$

It is found that

$$g_{particular}(\xi) = i\,[K1_\ell^i]\exp\left(-\sqrt{\tfrac{\beta}{2\alpha}}(1-i)\xi\right) + (-i)[K2_\ell^i]\exp\left(-\sqrt{\tfrac{\beta}{2\alpha}}(1+i)\xi\right) \tag{VB.18}$$

with

$$K1_\ell^i = (-\tilde{R})\,Re_{HYDR}\,(I1_\ell^i)$$

and

$$K2_\ell^i = (-\tilde{R})\,Re_{HYDR}\,(I2_\ell^i)$$

Therefore, the solution for the stream function in the "acoustic sublayer" region is given by

$$\Psi_{2_2}^i(\xi,\theta) = \sum_{\ell=1}^{\infty} \sin\theta\, P_\ell^1(\cos\theta) \cdot \left| \begin{array}{l} b_0^i + b_1^i\xi + b_2^i\xi^2 + b_3^i\xi^3 \\ + (i)(K1_\ell^i)\exp\left(-\sqrt{\tfrac{\beta}{2\alpha}}(1-i)\xi\right) \\ + (-i)(K2_\ell^i)\exp\left(-\sqrt{\tfrac{\beta}{2\alpha}}(1+i)\xi\right) \end{array} \right| \tag{VB.19}$$

VC. BOUNDARY/INTERFACE CONDITIONS

The boundary/interface conditions are those appropriate for the hydrodynamic field. Moreover, the boundary/interface conditions, applied at the drop/host interface, should include only linearized terms, as the hydrodynamic problem solved at this order is linear. In general, the conditions are: the radial component of the velocity is continuous (across the interface), the tangential velocity component is continuous, the tangential stress balance holds, and the change in normal force across the interface is balanced by the surface tension/curvature term. Finally, the kinematic condition holds at the interface.

The velocity and pressure fields which enter into these conditions refer to those of the hydrodynamic field. (That is, these are given by $\underline{u}_2^o$, p_2^o, $\underline{u}_2^i$, p_2^i of Sections VA and VB.

Of course, there are surface forces which also enter into the boundary/interface conditions. These are $\left\langle\left(\overline{pr}^{radial}\right)\right\rangle$ and $\left\langle\left(\overline{pr}^{TANG}\right)_\theta\right\rangle$, which have been presented in Sections II and IV.

The velocity and/or pressure fields which describe the hydrodynamic field will include the modifications/corrections which arise as a result of the forcing by the acoustic field variables. Simply stated, these modifications are those contributions which arose in the "acoustic sublayer". Were the forcing of the hydrodynamic field governing equations to be neglected, the resulting hydrodynamic flow would be strictly that of Stokes flow. In the work presented in Sections VA and VB, it is seen that Stokes flow arises in the "outer" regions (i.e., not in the acoustic sublayer), both exterior and interior to the drop.

In the work of Marston (1980), the acoustic field was taken to be strictly inviscid. Then $\left(pr^{TANG}\right)_\theta$ did not contribute, and the hydrodynamic field itself was described entirely by Stokes flow.

It is the intention of this section to develop the boundary/interface conditions which arise in the case in which the acoustic field incorporates viscous effects. In doing this, it will be necessary to look at the velocity fields obtained in the previous two sections in more detail.

As a first step, the equilibrium interface will be defined. This will be followed by a presentation of the general form of the interface/boundary conditions. Finally, the velocity field (after some manipulations) and the pressure field determined in Sections VA and VB will be utilized in the conditions, and specific equations obtained. Of particular interest is the resulting deformation (from sphericity) of the drop.

Equilibrium Interface

Let the equilibrium interface, Fe, be defined by

$$Fe = r - \tilde{R} - \sum_{\ell=0}^{\infty} K_\ell P_\ell(\cos\theta) = 0 \qquad \text{(VC.1)}$$

with the third term (on the r.h.s.) representing the deformation.

Boundary/Interface Conditions: General Form

The general form of the boundary/interface conditions will be presented. In keeping with earlier work, there will have been nondimensionalized.

They are:

(Continuity of Radial Velocity)

$$U_r^{i} = U_r^{o} \qquad \text{at } r = \tilde{R},\ \zeta = \xi = 0 \qquad \text{(VC.2a)}$$

(Continuity of Tangential Velocity)

$$U_\theta^i = U_\theta^o \qquad \text{at} \quad r = \tilde{R},\ \zeta = \xi = 0 \tag{VC.2b}$$

(Kinematic Condition)

$$\begin{aligned} U_r^i &= 0 \qquad \text{at} \quad r = \tilde{R},\ \xi = \zeta = 0 \\ U_r^o &= 0 \qquad \text{at} \quad r = \tilde{R},\ \xi = \zeta = 0 \end{aligned} \tag{VC.2c}$$

(Tangential Force Balance)

$$\alpha \left\{ r \frac{\partial}{\partial r}\left(\frac{U_\theta^i}{r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(U_r^i\right) \right\}$$

$$- \left\{ r \frac{\partial}{\partial r}\left(\frac{U_\theta^o}{r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(U_r^o\right) \right\}$$

$$= Re_{HYDR} \left\langle \left(\overline{pr}^{\,TANG}\right)_\theta \right\rangle$$

$$= Re_{HYDR} \left\{ \left\langle \beta\, V_{r_o}^i\big|_{\tilde{R}}\, \overline{V_{\theta_o}^i} - V_{r_o}^o\big|_{\tilde{R}}\, \overline{V_{\theta_o}^o} + \text{COMPLEX CONJUGATE} \right\rangle \right\}$$

$$\text{at} \quad r = \tilde{R},\ \zeta = \xi = 0 \tag{VC.2d}$$

Recall that $\alpha - \mu_o^i/\mu_o^o$ and $\beta - \rho_o^i/\rho_o^o$. Note that $U_r^{i,o}$ and $U_\theta^{i,o}$ refer to the hydrodynamic field. The forcing due to the (Θ) tangential component of the radiation pressure vector is on the right hand side. It's form is that discussed in Section IV. That is, for example

$$V_{\theta_o}^i = \hat{V}_{\theta_o}^i\big|_{\tilde{R}} + \tilde{\tilde{V}}_{\theta_o}^i(\xi, \theta)$$

(Normal Force Balance)

$$-p^i + \frac{2\alpha}{Re_{HYDR}} \frac{\partial U_r^i}{\partial r} = -p^o + \frac{2}{Re_{HYDR}} \frac{\partial U_r^o}{\partial r} - G \nabla \cdot \hat{n} + \langle \overline{pr}^{RADIAL} \rangle$$

(VC.2e)

with $G = \left(\sigma_o / \rho_o^o \, d^3 \omega_{DROP}^2\right)$; and σ_o^o is the surface tension associated with the interface between the liquid drop and the host medium. The unit vector $\hat{n}$ is the outward pointing normal to the interface. Specifically, $\hat{n} = \nabla Fe / |\nabla Fe|$.

Equations (VC.2a-2e) represent the general form of the boundary/interface conditions which must be satisfied. Note that forcing terms, namely, the $\hat{e}_r$ and $\hat{e}_\theta$ components of the radiation pressure vector, appear in the above system of equations. This is a nonhomogeneous set of linear equations. The unknowns are the coefficients of the velocity and pressure fields. Once these are determined, the hydrodynamic flow field which only exists as a result of the acoustic forcing will be known.

Velocity and Pressure Fields

Recall that the velocity fields in both the "acoustic sublayer" region and the "outer" region exterior and interior to the drop have been found (in general form) in Sections VA and VB. However, the stream function formulation was employed. This is not convenient, as the boundary interface conditions require $U_r^{i,o}$ and $U_\theta^{i,o}$. Also, it is necessary to determine explicitly the pressure field.

Recall that

$$\psi_2^o = \sum_{\ell=1}^{\infty} \left(B^o r^{-\ell+2} + F^o r^{-\ell} \right) \sin\theta\, P_\ell^1(\cos\theta)$$

$$+ \left(\frac{\delta}{Re_{HYDR}} \right) \sum_{\ell=1}^{\infty} \left| \begin{array}{l} b_0^o + b_1^o \zeta + b_2^o \zeta^2 + b_3^o \zeta^3 \\ + (-i)(K1_\ell^o) \exp\left(-\frac{(1+i)}{\sqrt{2}}\zeta\right) \\ + (i)(K2_\ell^o) \exp\left(-\frac{(1-i)}{\sqrt{2}}\zeta\right) \end{array} \right| \sin\theta\, P_\ell^1(\cos\theta) \tag{VC.3}$$

However, as $\zeta \rightarrow \infty$, the contribution that is in the "acoustic sublayer" region must decay. This implies that $b_o^o = b_1^o = b_2^o = b_3^o = 0$.

It is a straight forward matter to determine u_{r2}^o and $u_{\theta 2}^o$. These are given by

$$u_{r_2}^o = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} (\psi_2^o)$$

$$= \sum_{\ell=1}^{\infty} \left(B^o r^{-\ell} + F^o r^{-\ell-2} \right) \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta\, P_\ell^1 \right)$$

$$+ \left(\frac{\delta}{Re_{HYDR}} \right) \frac{1}{\tilde{R}^2} \sum_{\ell=1}^{\infty} \left\{ \begin{array}{l} (-i)(K1_\ell^o) \exp\left(-\frac{(1+i)\zeta}{\sqrt{2}}\right) \\ + (i)(K2_\ell^o) \exp\left(\frac{-(1-i)\zeta}{\sqrt{2}}\right) \end{array} \right\} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta\, P_\ell^1 \right)$$

or

$$u^o_{r_2} = \sum_{\ell=1}^{\infty} \ell(\ell+1)\left(B^o r^{-\ell} + F^o r^{-\ell-2}\right) P_\ell(\cos\theta)$$

$$+ \left(\frac{\delta}{Re_{HYDR}}\right) \sum_{\ell=1}^{\infty} \frac{\ell(\ell+1)}{\tilde{R}^2} \left\{ \begin{array}{l} (-i)(K1^o_\ell)\exp\left(-\frac{(1+i)}{\sqrt{2}}\zeta\right) \\ +(i)(K2^o_\ell)\exp\left(-\frac{(1-i)}{\sqrt{2}}\zeta\right) \end{array} \right\} P_\ell(\cos\theta) \qquad \text{(VC.4)}$$

Similarly, (recalling that $\frac{\partial}{\partial r} \Rightarrow \frac{\sqrt{Re_{HYDR}}}{\delta}\frac{\partial}{\partial \zeta}$ in the acoustic sublayer),

$$u^o_{\theta_2} = \frac{-1}{r\sin\theta}\frac{\partial \psi^o_2}{\partial r}$$

$$= \sum_{\ell=1}^{\infty} \left(-(2-\ell)B^o r^{-\ell} + \ell F^o r^{-\ell-2}\right) P'_\ell(\cos\theta)$$

$$+ \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2} \sum_{\ell=1}^{\infty} \frac{1}{\tilde{R}} \left[\begin{array}{l} (-i)\left(\frac{1+i}{\sqrt{2}}\right)(K1^o_\ell)\exp\left(-\frac{(1+i)}{\sqrt{2}}\zeta\right) \\ +(i)\left(\frac{1-i}{\sqrt{2}}\right)(K2^o_\ell)\exp\left(-\frac{(1-i)}{\sqrt{2}}\zeta\right) \end{array} \right] P'_\ell(\cos\theta) \qquad \text{(VC.5)}$$

Note that the contribution to $u^o_{\theta 2}$ from the "acoustic sublayer" region is at $o(SQRT(\delta/Re_{HYDR}))$, and that to u^i_{r2} at $o(\delta/Re_{HYDR})$. Recall that Re_{HYDR} is order one.

The form of u^i_{2r} and $u^i_{2\theta}$ also proceeds from ψ^i_2 . Recall that

$$\psi^i_2 = \sum_{\ell=1}^{\infty} \left(A^i r^{\ell+3} + E^i r^{\ell+1}\right) \sin\theta \, P'_\ell(\cos\theta)$$

$$+ \left(\frac{\delta}{Re_{HYDR}}\right) \sum_{\ell=1}^{\infty} \sin\theta \, P'_\ell(\cos\theta) \left| \begin{array}{l} b^i_0 + b^i_1\xi + b^i_2\xi^2 + b^i_3\xi^3 \\ +(i)(K1^i_\ell)\exp\left(-\sqrt{\beta/2\alpha}\,(1-i)\xi\right) \\ +(-i)(K2^i_\ell)\exp\left(-\sqrt{\beta/2\alpha}\,(1+i)\xi\right) \end{array} \right| . \qquad \text{(VC.6)}$$

Since the contribution that is strictly in the acoustic sublayer must decay as $\xi \to \infty$, $b_o^i = b_1^i = b_2^i = b_3^i = o$.

Then

$$u_{r_2}^i = \sum_{\ell=1}^{\infty} (\ell)(\ell+1)\left(A^i r^{\ell+1} + E^i r^{\ell-1}\right) P_\ell(\cos\theta)$$

$$+\left(\frac{\delta}{Re_{HYDR}}\right)\sum_{\ell=1}^{\infty}\frac{\ell(\ell+1)}{\tilde{R}^2}\left\{\begin{array}{l}(i)(K1_\ell^i)\exp\left(-\sqrt{\beta/2\alpha}\,(1-i)\,\xi\right)\\ +(-i)(K2_\ell^i)\exp\left(-\sqrt{\beta/2\alpha}\,(1+i)\,\xi\right)\end{array}\right\} P_\ell(\cos\theta)$$

(VC.7)

and, recalling that in the acoustic sublayer region, $\frac{\partial}{\partial r} \Rightarrow -\sqrt{\frac{Re_{HYDR}}{\delta}}\frac{\partial}{\partial \xi}$, it is found that

$$u_{\theta_2}^i = -\sum_{\ell=1}^{\infty}\left\{(\ell+3)A^i r^{\ell+1} + (\ell+1)E^i r^{\ell-1}\right\} P_\ell^1(\cos\theta)$$

$$+\left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\sum_{\ell=1}^{\infty}\frac{1}{\tilde{R}}\left\{\begin{array}{l}(-1)\sqrt{\frac{\beta}{\alpha}}\,\frac{(1-i)}{\sqrt{2}}(i)(K1_\ell^i)\exp\left(-\sqrt{\beta/2\alpha}\,(1-i)\xi\right)\\ +(-1)\sqrt{\frac{\beta}{\alpha}}\,\frac{(1+i)}{\sqrt{2}}(-i)(K2_\ell^i)\exp\left(-\sqrt{\beta/2\alpha}\,(1+i)\xi\right)\end{array}\right\} P_\ell^1(\cos\theta)$$

(VC.8)

The specific forms of $u_{\theta 2}^i$, $u_{\theta 2}^o$, u_{r2}^i, u_{r2}^o have been determined, up to a constant. The unknown constants which remain are B^o, F^o, A^i, E^i . Of course, these depend upon "l".

(Recall that $K1_l^{i,o}$ and $K2_l^{i,o}$ are known from previous sections, and also depend upon "l".)

It remains to explicitly determine p_2^i and p_2^o. From the governing equations for the hydrodynamic field presented in Sections VA and VB, it is clear that the contribution to $p_2^{i,o}$ from the "acoustic sublayer" region cannot be greater than order $\left(\sqrt{\delta / Re_{HYDR}}\right)$ at the very most.

The determination of $p_2^{i,o}$ in the Stokes layer region can utilize the results of Miller and Scriven (1968). That is, although they solved for the hydrodynamic field of an unforced, naturally oscillating drop, the pressure for Stokes flow can be recovered by setting their oscillating frequency to zero. Then the contribution to pressure which depends on the radial coordinate r is proportional to, in the Stokes' flow region exterior to the drop,

$$\frac{\partial}{\partial r}\left[\left(r\frac{\partial^2}{\partial r^2} + 2\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r}\right)\right]\left(B^o\frac{r^{-\ell}}{r}\right)$$

This is found to be zero. A similar development can be done for the pressure in the Stokes region interior to the drop.

It is then found that, to lowest order, the contribution to the normal stress balance of the hydrodynamic field will involve terms which are the radial derivative of the radial component of the hydrodynamic field.

It is remarked that this result was also the case in the work of Marston (1980).

Boundary-Interface Conditions: Specific Equations

Substitution of the forms for $U_r^{i,o}$, $U_\theta^{i,o}$, $p_2^{i,o}$ presented in the prior subsection into Equations (VC.2a-2e) will be done.

Lowest order terms <u>only</u> will be kept. It is recalled that the hydrodynamic field itself occurs at order δ^2 in the overall perturbation scheme.

(Continuity of radial velocity)

$$A^i \tilde{R}^{l+1} + E^i \tilde{R}^{l-1} + \cancel{O(\delta/Re_{HYDR})} = B^o \tilde{R}^{-l} + F^o \tilde{R}^{-l-2} + \cancel{O(\delta/Re_{HYDR})} \tag{VC.9}$$

(Continuity of tangential velocity)

$$\begin{aligned} -(l+3)A^i \tilde{R}^{l+1} - (l+1)E^i \tilde{R}^{l-1} + O\left(\cancel{(\delta/Re_{HYDR})^{1/2}}\right) \\ = (l-2)B^o \tilde{R}^{-l} + l F^o \tilde{R}^{-(l+2)} + O\left(\cancel{(\delta/Re_{HYDR})^{1/2}}\right) \end{aligned} \tag{VC.10}$$

(Kinematic condition)

$$\begin{aligned} B^o \tilde{R}^{-l} + F^o \tilde{R}^{-l-2} + \cancel{O(\delta/Re_{HYDR})} &= 0 \\ A^i \tilde{R}^{l+1} + E^i \tilde{R}^{l-1} + \cancel{O(\delta/Re_{HYDR})} &= 0 \end{aligned} \tag{VC.11}$$

(Tangential stress)

$$\begin{aligned} &\alpha\left(-\left(\sqrt{\frac{Re_{HYDR}}{\delta}}\right)\frac{\partial U_\theta^i}{\partial \zeta} - \frac{U_\theta^i}{\tilde{R}} + \frac{1}{\tilde{R}}\frac{\partial}{\partial \theta}(U_r^i)\right)_{\zeta=0} \\ &- \left(\sqrt{\frac{Re_{HYDR}}{\delta}}\frac{\partial U_\theta^o}{\partial \zeta} - \frac{U_\theta^o}{\tilde{R}} + \frac{1}{R}\frac{\partial}{\partial \theta}(U_r^o)\right)_{\zeta=0} \\ &= Re_{HYDR}\left\langle (\overline{pr}^{TANG})_\theta \right\rangle \end{aligned} \tag{VC.12a}$$

Notice that presence of the forcing term due to the tangential component of the (time-averaged) radiation pressure vector, but only lowest order terms will be kept. The expansion for U_θ^i $\left(U_\theta^o\right)$ in terms of ξ (ζ) can be written, utilizing a Taylor series expansion, as

$$U_\theta^i = u_{\theta_2}^i(r,\theta) + \sqrt{\frac{\delta}{Re_{HYDR}}}\, u_{\theta_2}^i(\xi,\theta)$$

$$= \hat{u}_{\theta_{2_0}}^i(\tilde{R},\theta) - \xi\sqrt{\frac{\delta}{Re_{HYDR}}}\left(\hat{u}_{\theta_{2_0}}^{i\,\prime}\Big|_{\tilde{R}}\right) + \cdots$$

$$+ \sqrt{\frac{\delta}{Re_{HYDR}}}\, u_{\theta_2}^i(\xi,\theta) + \cdots$$

(VC.12b)

and

$$U_\theta^o = u_{\theta_2}^o(r,\theta) + \sqrt{\frac{\delta}{Re_{HYDR}}}\, u_{\theta_2}^o(\zeta,\theta)$$

$$= \hat{u}_{\theta_{20}}^o(\tilde{R},\theta) + \zeta\sqrt{\frac{\delta}{Re_{HYDR}}}\left(\hat{u}_{\theta_{20}}^{i\,\prime}\Big|_{\tilde{R}}\right) + \cdots$$

$$+ \sqrt{\frac{\delta}{Re_{HYDR}}}\, u_{\theta_2}^o(\zeta,\theta) + \cdots$$

(VC.12c)

Substitution of (VC.12b) and use of $U_r^{i,o} = u_{r2}^{i,o}$ yields (to lowest order in $\sqrt{\dfrac{\delta}{Re_{HYDR}}}$),

$$\sum_{\ell=1}^{\infty} \alpha \left\{ -(\ell+3)(\ell+1) A^{i} \tilde{R}^{\ell} - (\ell+1)(\ell-1) E^{i} \tilde{R}^{\ell-2} \right\} P_\ell^{1}(\cos\theta)$$

$$+ \sum_{\ell=1}^{\infty} \alpha \left(\frac{-1}{\tilde{R}}\right) \left[\begin{array}{l} (\beta/\alpha) \frac{(1-i)^2}{2} (i)(K1_\ell^{i})(1) \\ + (\beta/\alpha) \frac{(1+i)^2}{2} (-i)(K2_\ell^{i})(1) \end{array} \right] P_\ell^{1}(\cos\theta)$$

$$+ \sum_{\ell=1}^{\infty} \alpha \left\{ (\ell+3) A^{i} \tilde{R}^{\ell} + (\ell+1) E^{i} \tilde{R}^{\ell-2} \right\} P_\ell^{1}(\cos\theta)$$

$$+ \sum_{\ell=1}^{\infty} \left\{ \alpha \left\{ \ell(\ell+1) A^{i} \tilde{R}^{\ell} + \ell(\ell+1) E^{i} \tilde{R}^{\ell-2} \right\} \right\} \frac{dP_\ell}{d\theta}(\cos\theta)$$

$$- \Bigg\{ \sum_{\ell=1}^{\infty} \left\{ -\ell(\ell-2) \tilde{R}^{-\ell-1} B^{0} - \ell(\ell+2) \tilde{R}^{-(\ell+3)} F^{0} \right\} P_\ell^{1}(\cos\theta)$$

$$+ \sum_{\ell=1}^{\infty} \left(\frac{-1}{\tilde{R}}\right) \left(\begin{array}{l} (-i) \frac{(1+i)^2}{2} K1_\ell^{0}(1) \\ + (i) \frac{(1-i)^2}{2} K2_\ell^{0}(1) \end{array} \right) P_\ell^{1}(\cos\theta)$$

$$- \sum_{\ell=1}^{\infty} \left\{ -(2-\ell) B^{0} \tilde{R}^{-\ell-1} + \ell F^{0} \tilde{R}^{-\ell-3} \right\} P_\ell^{1}(\cos\theta)$$

$$+ \sum_{\ell=1}^{\infty} \left\{ \ell(\ell+1) B^{0} \tilde{R}^{-\ell-1} + \ell(\ell+1) F^{0} \tilde{R}^{-\ell-3} \right\} \frac{dP_\ell}{d\theta} \Bigg\}$$

$$= (Re_{\mathrm{HYDR}}) \left\langle (\overline{pr}^{\mathrm{TANK}})_\theta \right\rangle$$

(VC.12d)

Gathering terms yields

$$\sum_{\ell=1}^{\infty} \left\{ \alpha\, A^{i} (-2\ell)(\ell+2)\, \tilde{R}^{\ell} + \alpha\, E^{i} (-2)(\ell-1)(\ell+1)\, \tilde{R}^{\ell-2} \right\} P_\ell^{1}(\cos\theta)$$

$$+ \sum_{\ell=1}^{\infty} \left(\frac{-1}{\tilde{R}}\right)(\beta) \left\{ K1_\ell^{i} + K2_\ell^{i} \right\} P_\ell^{1}(\cos\theta)$$

$$- \left\{ \sum_{\ell=1}^{\infty} \left\{ B^{o} \tilde{R}^{-\ell-1} (-2)(\ell^2-1) + F^{o} \tilde{R}^{-\ell-3} (-2\ell)(\ell+2) \right\} P_\ell^{1}(\cos\theta) + \sum_{\ell=1}^{\infty} \left(\frac{-1}{\tilde{R}}\right) \left(K1_\ell^{o} + K2_\ell^{o}\right) P_\ell^{1}(\cos\theta) \right\}$$

$$= Re_{HYDR} \left\langle \left(\overline{pr}^{\,TANG}\right)_\theta \right\rangle \qquad \textbf{(VC.12e)}$$

Multiply through by $\left(\sin\theta \cdot P_\ell^{1}\right)$ and integrate over $(0, \pi)$ to obtain

$$\alpha \left\{ A^{i} \tilde{R}^{\ell} (-2\ell)(\ell+2) + E^{i} \tilde{R}^{\ell-2} (-2)(\ell-1)(\ell+1) \right\}$$

$$- \left\{ B^{o} \tilde{R}^{-\ell-1} (-2)(\ell^2-1) + F^{o} \tilde{R}^{-\ell-3} (-2\ell)(\ell+2) \right\}$$

$$+ \; -\frac{\beta}{\tilde{R}} \left(K1_\ell^{i} + K2_\ell^{i}\right) + \frac{1}{\tilde{R}} \left(K1_\ell^{o} + K2_\ell^{o}\right)$$

$$= Re_{HY} \left(\frac{2\ell+1}{2\ell(\ell+1)}\right) \int_0^{\pi} P_\ell^{1}(\cos\theta) \cdot \sin\theta \cdot \left\langle \left(\overline{pr}^{\,TANG}\right)_\theta \right\rangle d\theta \qquad \textbf{(VC.12f)}$$

The right hand side of Equation (VC.12f) can be written as

$$\frac{(2\ell+1)}{2\ell(\ell+1)} \; Re_{HYDR} \int_0^{\pi} \mathcal{P}_\ell^1(\cos\theta)\,\sin\theta \left\{ \left\langle \hat{V}^i_{r_0|\tilde{R}} \left(\beta \tilde{\tilde{V}}^i_{\theta_0} - \tilde{\tilde{V}}^o_{\theta_0} \right) \right\rangle_{\varsigma=\xi=0} \right\} d\theta$$

using Equation (IVA.3). Note that

$$\left\langle \hat{V}^i_{r_0|\tilde{R}} \left(\beta \tilde{\tilde{V}}^i_{\theta_0} - \tilde{\tilde{V}}^o_{\theta_0} \right) \right\rangle_{\varsigma=\xi=0}$$

$$= \left\{ \hat{V}^i_{r_0|\tilde{R}} \left(\overline{\beta \tilde{\tilde{V}}^i_{\theta_0}} \right)_{\xi=0} - \hat{V}^i_{r_0|\tilde{R}} \left(\overline{\tilde{\tilde{V}}^o_{\theta_0}} \right)_{\varsigma=0} + \text{COMPLEX CONJUGATE} \right\}$$

(VC.13)

Also recall that

$$\tilde{\tilde{V}}^o_{\theta_0}(\varsigma,\theta) = \sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\ell'} \, B^o_{BL_{\ell'}} \exp\left(-\frac{(1+i)}{\sqrt{2}} \, \varsigma \right) \frac{dP_{\ell'}}{d\theta}$$

and

$$\tilde{\tilde{V}}^i_{\theta_0}(\xi,\theta) = \sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\ell'} \, A^i_{BL_{\ell'}} \exp\left(-\sqrt{\frac{\beta}{2\alpha}} \, (1+i)\,\xi \right) \frac{dP_{\ell'}}{d\theta}$$

Therefore, the right hand side of Equation (VC.12f) can be further expanded as

$$\frac{(2\ell+1)}{2\ell(\ell+1)} \cdot \Bigg\{ \beta \, Re_{HYDR} \int_0^{\pi} P^1_{\mathcal{L}}(\cos\theta) \cdot \sin\theta \cdot \left[\sum_{\hat{\ell}=0}^{\infty} \hat{\hat{\delta}}_{\hat{\ell}} \, \alpha^{i}_{\hat{\ell}} \, \frac{d}{dr}\left(j_{\hat{\ell}}\left(\frac{c_0^o}{c_0^i} r\right)\right)\Big|_{\tilde{R}} P_{\hat{\ell}}(\cos\theta) \right] \cdot$$

$$\cdot \overline{\left\{ \sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\ell'} \, A^{i}_{BL_{\ell'}} \frac{dP_{\ell'}}{d\theta} \right\}} \, d\theta$$

$$+ \text{ COMPLEX CONJUGATE}$$

$$- \beta \, Re_{HYDR} \int_0^{\pi} P^1_{\mathcal{L}}(\cos\theta) \cdot \sin\theta \cdot \left\{ \sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\hat{\ell}} \, \alpha^{i}_{\hat{\ell}} \, \frac{d}{dr}\left(j_{\hat{\ell}}\left(\frac{c_0^o}{c_0^i} r\right)\right)\Big|_{\tilde{R}} P_{\hat{\ell}}(\cos\theta) \right\} \cdot$$

$$\cdot \overline{\left\{ \sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\ell'} \, B^{o}_{BL_{\ell'}} \frac{dP_{\ell'}}{d\theta} \right\}} \, d\theta$$

$$- \text{ COMPLEX CONJUGATE} \Bigg\}$$

However, certain integrals are equivalent to pre-defined quantities. Therefore, the r.h.s. of Equation (VC.2f) can be further expressed using the definitions of $(I1_l^i)$, $(I2_l^i)$, $(I1_l^o)$, $(I2_l^o)$ and the continuity of the radial component of velocity of the acoustic filed across the interface. This yields

(R.H.S.) of (Eq. VC.12f)

$$\left\{ Re_{HYDR} \right\} \left(\beta\, I1_\ell^i + \beta\, I2_\ell^i - I1_\ell^o - I2_\ell^o \right)$$

(VC.14)

Therefore,

$$\alpha \left\{ (-2\ell)(\ell+2)\,\tilde{R}^{\ell}\, A^i + (-2)(\ell^2-1) E^i\, \tilde{R}^{\ell-2} \right\}$$
$$- \left\{ B^o \tilde{R}^{-\ell-1} (-2)(\ell^2-1) + F^o\, \tilde{R}^{-\ell-3} (-2\ell)(\ell+2) \right\}$$
$$- \frac{\beta}{\tilde{R}} (K1_\ell^i) - \frac{\beta}{\tilde{R}} (K2_\ell^i) + \frac{1}{\tilde{R}} K1_\ell^o + \frac{1}{\tilde{R}} K2_\ell^o$$
$$= \qquad Re_{HYDR} \left\{ \beta I1_\ell^i + \beta I2_\ell^i - I1_\ell^o - I2_\ell^o \right\}$$

(VC.15)

Recall that

$$K1_\ell^o = (-\tilde{R})\, Re_{HYDR}\, I1_\ell^o \quad ; \quad K2_\ell^o = (-\tilde{R})\, Re_{HYDR}\, I2_\ell^o$$

$$K1_\ell^i = (-\tilde{R})\, Re_{HYDR}\, I1_\ell^i \quad ; \quad K2_\ell^i = (-\tilde{R})\, Re_{HYDR}\, I2_\ell^i$$

Therefore, the tangential stress balance in the hydrodynamic field is given as

$$\alpha \left[A^i \tilde{R}^{\ell} (-2\ell)(\ell+2) + E^i \tilde{R}^{\ell-2} (-2)(\ell^2-1) \right]$$
$$- \left[B^o \tilde{R}^{-\ell-1} (-2)(\ell^2-1) + F^o \tilde{R}^{-\ell-3} (-2\ell)(\ell+2) \right]$$
$$- \beta \left(\frac{1}{\tilde{R}}\right) K1_{\ell}^{i} - \beta\left(\frac{1}{\tilde{R}}\right) K2_{\ell}^{i} + \frac{1}{\tilde{R}} K1_{\ell}^{o} + \frac{1}{\tilde{R}} K2_{\ell}^{o}$$
$$= Re_{HYDR} \left\{ \beta I1_{\ell}^{i} + \beta I2_{\ell}^{i} - I1_{\ell}^{o} - I2_{\ell}^{o} \right\}$$
$$= -\frac{1}{\tilde{R}} \left\{ \beta K1_{\ell}^{i} + \beta K2_{\ell}^{i} - K1_{\ell}^{o} - K2_{\ell}^{o} \right\} \tag{VC.16}$$

Clearly, the contribution to the tangential stress balance which arose due to the modified flow field (different from that of Stokes' flow) cancels the component of the forcing (radiation pressure) vector. This is for the very restricted subcase in which the response to the forcing has <u>zero</u> phase lag with respect to the forcing. This results in

$$\alpha\left\{(-2\ell)(\ell+2) A^i \tilde{R}^{\ell} - 2(\ell^2 -1) E^i \tilde{R}^{\ell-2}\right\}$$
$$- \left\{ B^o \tilde{R}^{-\ell-1} (-2)(\ell^2-1) + F^o \tilde{R}^{-\ell-3} (-2\ell)(\ell+2)\right\} \tag{VC.17}$$

Clearly, use of the kinematic condition could serve to simplify this expression.

Thus, for Equations (VC.9), (VC.10), VC.11), and (VC.17) represent the specific boundary/interface conditions. It remains to construct the specific form of the normal force balance condition.

(Normal Force Balance)
From the previous discussion

$$\frac{2\alpha}{Re_{HYDR}} \frac{\partial v_r^i}{\partial r} = \frac{2}{Re_{HYDR}} \frac{\partial v_r^o}{\partial r} - G\, \nabla \cdot \hat{\underline{n}} - \langle \overline{pr}^{\,RADIAL} \rangle$$

(VC.18)

To lowest order (in the hydrodynamic field balance) this becomes

$$\frac{2\alpha}{Re_{HYDR}} \sum_{\ell=0}^{\infty} \left\{ (\ell+1) A^i \tilde{R}^{\ell} + (\ell-1) E^i \tilde{R}^{\ell-2} \right\} \ell(\ell+1)\, P_\ell(\cos\theta)$$

$$= \frac{2}{Re_{HYDR}} \sum_{\ell=0}^{\infty} (\ell)\, (\ell+1)\, P_\ell(\cos\theta) \left\{ -\ell B^o \tilde{R}^{-\ell-1} - (\ell+2) F^o \tilde{R}^{-\ell-3} \right\}$$

$$- \frac{G}{\tilde{R}^2} \sum_{\ell=0}^{\infty} (\ell+2)(\ell-1) K_\ell\, P_\ell(\cos\theta) + \langle \overline{pr}^{\,RADIAL} \rangle \qquad \text{(VC.19a)}$$

The theta dependence can be eliminated after multiplication through via $P_\ell(\cos\theta)\sin\theta$ and integration over (o,π) . This yields

$$\frac{2\alpha}{Re_{HYDR}} \left((\ell+1) A^i \tilde{R}^{\ell} + (\ell-1) E^i \tilde{R}^{\ell-2} \right)$$

$$- \frac{2}{Re_{HYDR}} \left(-\ell B^o \tilde{R}^{-\ell-1} - (\ell+2) F^o \tilde{R}^{-\ell-3} \right) + \frac{G}{\tilde{R}^2} \frac{(\ell+2)(\ell-1) K_\ell}{\ell(\ell+1)}$$

(VC.19b)

$$= \frac{(2\ell+1)}{2\ell(\ell+1)} \int_0^{\pi} P_\ell(\cos\theta) \sin\theta\, \langle \overline{pr}^{\,RADIAL} \rangle\, d\theta$$

Let the right hand side of (VC.19b) be renamed as

$$\frac{\left(\overline{pr}_{\text{FORCING}}\right)_{\text{radial}}}{\ell(\ell+1)} = \frac{(2\ell+1)}{2}\int_0^{\pi} P_\ell(\cos\theta)\cdot\sin\theta\,\langle \overline{pr}^{\,\text{RADIAL}}\rangle\, d\theta$$

(VC.20)

The boundary/interface conditions applied at the order of the hydrodynamic field and forced by the radiation pressure vector components at the surface $(r=\tilde{R},\ \zeta=\xi=o)$ are listed below for convenience. They are

$$A^i\,\tilde{R}^{\ell+1} + E^i\,\tilde{R}^{\ell-1} = B^o\,\tilde{R}^{-\ell} + F^o\,\tilde{R}^{-\ell-2} \qquad \text{(VC.21a)}$$

$$-(\ell+3)A^i\,\tilde{R}^{\ell+1} - (\ell+1)E^i\,\tilde{R}^{\ell-1} = (\ell-2)B^o\,\tilde{R}^{-\ell} + \ell\,\tilde{R}^{-(\ell+2)}F^o \qquad \text{(VC.21b)}$$

$$B^o\,\tilde{R}^{-\ell} + F^o\,\tilde{R}^{-\ell-2} = 0 \qquad \text{(VC.21c)}$$

$$A^i\,\tilde{R}^{\ell+1} + E^i\,\tilde{R}^{\ell-1} = 0 \qquad \text{(VC.21d)}$$

$$\alpha\,\ell(\ell+2)\tilde{R}^{\ell}A^i + \alpha\,\tilde{R}^{\ell-2}(\ell-1)(\ell+1)E^i$$

$$= (\ell^2-1)B^o\,\tilde{R}^{-\ell-1} + F^o\,\tilde{R}^{-\ell-3}(\ell)(\ell+2) \qquad \text{(VC.21e)}$$

and

$$\frac{2\alpha}{Re_{HYDR}}\left((\ell+1)A^i\tilde{R}^{\ell} + (\ell-1)E^i\tilde{R}^{\ell-2}\right)$$

$$+\frac{2}{Re_{HYDR}}\left(\ell B^o\tilde{R}^{-\ell-1} + (\ell+2)\tilde{R}^{-\ell-3}F^o\right)$$

$$+\frac{G}{\tilde{R}^2}\frac{(\ell+2)(\ell-1)}{\ell(\ell+1)}K_\ell = \frac{(\overline{pr})_{forcing\ radial}}{\ell(\ell+1)} \qquad \textbf{(VC.21f)}$$

The unknown coefficients are: κ_l, A^i, B^o, E^i, F^o. (The two kinematic conditions are not necessary - that is, only one should be used. Either (VC.21c) or (VC.21d) can be utilized).

It is possible to eliminate E^i and F^o. This yields a system of nonhomogeneous linear algebraic equations for the unknowns κ_l, A^i, and B^o. These are

$$\tilde{R}^{\ell}A^i + -\tilde{R}^{-\ell-1}B^o = 0 \qquad \textbf{(VC.22a)}$$

$$2(2\ell+1)\alpha\tilde{R}^{\ell}A^i + 2(2\ell+1)\tilde{R}^{-\ell-1}B^o = 0 \qquad \textbf{(VC.22b)}$$

and

$$\frac{4\alpha}{Re_{HYDR}}\tilde{R}^{\ell}A^i + \frac{-4}{Re_{HYDR}}\tilde{R}^{-\ell-1}B^o + \frac{G(\ell+2)(\ell-1)}{\tilde{R}^2\,\ell(\ell+1)}K_\ell$$

$$= \frac{(\overline{pr}_{forcing})_{radial}}{\ell(\ell+1)} \qquad \textbf{(VC.22c)}$$

Final Comments

It is clear that

$$\begin{pmatrix} 1 & -1 & 0 \\ \alpha & 1 & 0 \\ \dfrac{4\alpha}{Re_{HYDR}} & \dfrac{-4\alpha}{Re_{HYDR}} & \dfrac{G(\ell+2)(\ell-1)}{\tilde{R}^2 \ell(\ell+1)} \end{pmatrix} \begin{pmatrix} \tilde{R}^{\ell} A^{i} \\ \tilde{R}^{-\ell-1} B^{o} \\ K_{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (\overline{pr}_{forcing})_{radial} / \ell(\ell+1) \end{pmatrix}$$

Therefore

$$K_{\ell} = \frac{(\overline{pr}_{forcing})_{radial}}{\left(\dfrac{G(\ell+2)(\ell-1)}{\tilde{R}^2} \right)} \tag{VC.23}$$

K_l is the coefficient of the deformation, and is directly proportional to the radial component of the radiation pressure vector.

The tangential stress has been included in the calculations. First, the viscous acoustic field was determined. This allowed the tangential $(\hat{e}_\theta)$ component of the radiation pressure vector to be calculated. Secondly, the modification of the hydrodynamic field due to source terms in the governing equation was determined. The source terms are caused directly by the inclusion of viscous effects on the acoustic field. Solution of the governing equations shows there to be enhanced flow in the "acoustic sublayer" region, particularly of the u_θ component. The flow correction due to this enhanced flow is $o\left(\frac{\sqrt{\delta}}{\sqrt{Re_{HYDR}}}\right)$ for u_θ, which is the largest such component in the acoustic sublayer region. Stokes' flow is the flow in the "outer" regions, both interior and exterior to the drop.

In the application of the boundary/interface conditions, only lowest order contributions (at the level of the hydrodynamic field) are included. Of course, in the tangential stress balance condition, this results in the inclusion of velocity gradient terms (of u_θ) from the acoustic sublayer region.

It was found that these contributions to the stress balance cancel the forcing due to the tangential component of the radiation pressure vector for the special subcase in which there is no phase lag of the hydrodynamic field with respect to the initial forcing.

It is clear that incorporation of viscous effects into the acoustic field and investigation of the forced hydrodynamic field shows that the velocity field has a level of complexity which is not apparent if only the inviscid acoustic field is considered. It is only through doing the "viscous" problem that the structure of the velocity field in the "acoustic sublayer" regions - which border the drop/host interface can be elucidated. The hydrodynamic (streaming) field due to an unmodulated acoustic standing wave field has been considered. In the case of a modulated standing wave field, static (streaming) contributions to the hydrodynamic field would exist as well as the oscillating contributions. However, there would be a phase lag between forcing a response, in general, so that the deformation would be due not only to the radial component of the radiation pressure vector, but to the tangential component as well.

VI. HYDRODYNAMIC FIELD: FORCED BY A MODULATED ACOUSTIC STANDING WAVE FIELD

This section details the structure of the hydrodynamic field which exists (at second order in the expansion parameter δ) strictly as a result of the modulated acoustic standing wave field. In this case, the resulting hydrodynamic field is oscillatory in time. The drop itself is undergoing (small amplitude) shape oscillations due to the forcing at the natural frequency of the drop.

Let $\omega_{AC} = \omega' = \omega'' + \omega_{DROP}$, with ω'' the acoustic frequency of a second acoustic wave. Clearly, $\omega' - \omega'' = \omega_{DROP}$, the drop oscillation frequency. The frequencies of the acoustic waves are orders of magnitude larger than that of the natural frequency of the drop. The time-averaging of the governing equations over the period $\left(\frac{2\pi}{\omega_{AC}}\right)$ will result in a set of time-dependent equations. However, the frequency of the resulting time dependence is at the "beat" frequency, which is that of drop's natural frequency of oscillation. (Refer to Miller and Scriven, 1968, for a discussion of natural (free) drop oscillations.)

The hydrodynamic field is considered to be viscous and incompressible It must be determined both interior and exterior to the drop. The generation of the governing equations for both regions will be presented explicitly.

It is remarked that the methodology of this Section is quite simialr to that of Section V. Of course, the time dependence introduces an additional level of complexity. However, some of the discussions, such as the need for re-nondimensionalization, apply in both cases. Where possible, it is the intention to avoid reduplication providing clarity is not compromised. Moreover, an expanded discussion as to the motivation behind several of the methological steps will be included.

VIA. HYDRODYNAMIC FIELD EXTERIOR TO THE DROP: EQUATIONS AND SOLUTIONS

The nonlinear system of governing equations given previously in Section IIC (see Eqs. IIC.1-3) is nondimensionalized through utilization of the relationships given in Section IIIB (see IIIB.1). Acoustic field reference quantities were used in the nondimensionalization scheme.

The resulting sytem is

$$\frac{\partial \rho^{o}}{\partial t} + \underline{U}^{o} \cdot \nabla \rho^{o} + \rho^{o} \nabla \cdot \underline{U}^{o} = 0 \qquad \text{(VIA.1a)}$$

$$\rho^{o} \frac{\partial \underline{U}^{o}}{\partial t} + \rho^{o} \underline{U}^{o} \cdot \nabla \underline{U}^{o} = -\nabla p^{o} + \frac{1}{Re_{AC}} \nabla^{2} \underline{U}^{o} + \left(\frac{1}{3} + \tau^{o}\right) \frac{1}{Re_{AC}} \nabla (\nabla \cdot \underline{U}^{o})$$

(VIA.1b)

Also, the relationship between pressure and density holds at $o(\delta)$. The "o" superscript refers to the region exterior to the drop. The velocity, pressure and density fields are given by $\underline{u}^{o}$, p^{o}, and ρ^{o}, respectively. Re_{AC} is a Reynolds-type number, and is given as $c_{o}^{o}(c_{o}^{o}/\omega_{AC})/\nu_{o}^{o}$, with ω_{AC} the acoustic frequency.

Let the dependent variables be expanded in a series in the expansion parameter δ. (Note that $\delta - \omega_{DROP}/\omega_{AC}$, with ω_{DROP} representing the natural frequency of oscillation of the drop.)

That is,

$$\underline{U}^o = \delta \underline{v}_1^o + \delta^2 \underline{u}_2^o \tag{VIA.2a}$$

$$p^o = P_o^o + \delta p_1^o + \delta^2 p_2^o \tag{VIA.2b}$$

$$\rho^o = 1 + \delta \rho_1^o + \delta^2 \rho_2^o \tag{VIA.2c}$$

Acoustic field variables are denoted by the "1" subscript $\left(\underline{v}_1^o,\ p_1^o,\ \rho_1^o\right)$, and hydrodynamic quantities by "2" $\left(\underline{u}_2^o,\ p_2^o\right)$. At order δ, the governing equations of the acoustic field are recovered (see Section IIIB, Equations IIIB.2a-2c).

It is at order δ^2 that the hydrodynamic field occurs. It owes its existence to the (modulated) acoustic standing wave field. The hydrodynamic field is taken to be incompressible.

Resulting governing equations are then

$$\nabla \cdot \underline{u}_2^o = 0 \tag{VIA.3a}$$

$$\frac{\partial \underline{u}_2^o}{\partial t} + \nabla p_2^o - \frac{1}{Re_{AC}} \nabla^2 \underline{u}_2^o = -\left(\rho_1^o \frac{\partial \underline{v}_1^o}{\partial t} + \underline{v}_1^o \cdot \nabla \underline{v}_1^o \right) \tag{VIA.3b}$$

Clearly, the term $\left(\frac{1}{3}+\tau^{o}\right) \frac{1}{Re_{HYDR}} \nabla\left(\nabla \cdot \underline{u}_2^{o}\right)$ which formally appears at this order is zero due to the incompressibility of the hydrodynamic field.

The quantities $\underline{v}_1^{o}$ *and* ρ_1^{o} are known at this order. They act as forcing (or source) terms in the conservation of momentum Equation (VIA.3b).

Now; the acoustic field represented by $\underline{v}_1^{o}$ (and, of course, p_1^{o}) is a <u>modulated</u> standing wave field. For example,

$$\hat{\phi}^{o}_{SCT} = \hat{\phi}^{\prime}_{SCT} + \hat{\phi}^{\prime\prime}_{SCT} \tag{VIA.4}$$

with Φ^{o}_{sct} (the scattered) velocity potential for the acoustic wave. The "′" refers to the acoustic wave of frequency $\omega'(-\omega_{AC})$ and "″" refers to the wave of frequency ω''.

The time average of Equations (VIA3a-3b) over a period of $\left(\frac{2\pi}{\omega_{AC}}\right)$ must be taken. This will result in having a time-dependent source term (i.e., the right hand side), of a form to be exhibited explicitly. A solution for $\underline{u}_2^{o}$ $(and\ p_2^{o})$ will be sought with this same time-dependence. Taking the time average of terms on the right hand side yields

$$\begin{aligned} \langle RHS \rangle = \; & \overline{\hat{\rho}_1^{o}}\; \hat{\underline{v}}_1^{o}(i) \left\{ 2 + 2\cos(\delta t + \eta'' - \eta') \right\} \\ & + \hat{\rho}_1^{o}\; \overline{(i \hat{\underline{v}}_1^{o})} \left\{ 2 + 2\cos(\delta t + \eta'' - \eta') \right\} \\ & + \left(\overline{\hat{\underline{v}}_1^{o}} \cdot \nabla \hat{\underline{v}}_1^{o} \right)\left(2 + 2\cos(\delta t + \eta'' - \eta') \right) \\ & + \left(\hat{\underline{v}}_1^{o} \cdot \overline{\nabla \hat{\underline{v}}_1^{o}} \right)\left(2 + 2\cos(\delta t + \eta'' - \eta') \right) \end{aligned} \tag{VIA.5}$$

$\delta = (\omega_{DROP} / \omega_{AC})$

In this expression only, the caret "^" indicates a time independent general quantity. Both η'' and η' indicate phases. Note that there is a steady state contribution to the forcing as well as the cos ($\delta t+\eta''-\eta'$) time dependent forcing. The steady state forcing was addressed via consideration of the unmodulated acoustic standing wave field, in Section V. It will not be considered in this section. Rather, the focus will be solely on the response of the hydrodynamic field to the oscillatory forcing.

The governing equations, after being time-averaged, are

$$\nabla \cdot \underline{u}_2^{\circ} = 0 \tag{VIA.6a}$$

$$\frac{\partial \underline{u}_2^{\circ}}{\partial t} + \nabla p_2^{\circ} - \frac{1}{Re_{AC}} \nabla^2 \underline{u}_2^{\circ}$$

$$= -2 \left\{ \underline{v}_1^{\circ} \cdot \overline{\nabla \underline{v}_1^{\circ}} + \rho_1^{\circ} \left(\overline{i \underline{v}_1^{\circ}} \right) + \text{COMPLEX CONJUGATE} \right\} \cdot$$

$$\cos(\delta t + \eta'' - \eta')$$

$$(\delta = \omega_{DROP} / \omega_{AC}) \tag{VIA.6b}$$

Note that Equations (VIA.6a-6b) are nondimensional, and have been nondimensionalized with respect to acoustic field reference quatitiies. The periodic temporal forcing contains "δt"; with $\delta \ll 1$. However, at this stage, the hydrodynamic field is of interest. A re-nondimensionalization scheme will be performed following the methodology discussed in Section V.

Re-nondimensionalization

From a physical point of view, it is the hydrodynamic field which is the focal point. It is the hydrodynamic field which exists only as a result of the forcing. This is clearly shown in Equations (VIA.6a-6b). However, in these equations all quantities have been nondimensionalized with respect to reference quantities which are not relevant to an oscillating drop.

The re-nondimensionalization acts as a renormalization, with the result that the oscillating drop problem is viewed from the standpoint of the hydrodynamic field as opposed to that of the acoustic field.

Rewrite (VIA.6b) in terms of dimensional quantities as follows

$$\left(\frac{1}{c_0^\circ \omega_{AC}}\right)\frac{\partial \hat{\underline{u}}_2^\circ}{\partial \hat{t}} + \left(\frac{1}{\rho_0^\circ c_0^{\circ 2}}\right)(c_0^\circ/\omega_{AC})\, \hat{\nabla}(\hat{p}_2^\circ)$$

$$+ \frac{-1}{Re_{AC}}\left(\frac{1}{c_0^\circ}\right)(c_0^\circ/\omega_{AC})^2\ \hat{\nabla}^2 \hat{\underline{u}}_2^\circ$$

$$= \frac{1}{(c_0^\circ)^2}(c_0^\circ/\omega_{AC})(-2)\left\{\begin{array}{c} \hat{\underline{v}}_1^\circ \cdot \overline{\nabla \hat{\underline{v}}_1^\circ} + \hat{\rho}_1^\circ\ \overline{(i\, \hat{\underline{v}}_1^\circ)} \\ + \text{COMPLEX CONJUGATE} \end{array}\right\}$$

$$\cdot \cos(\delta\omega_{AC}\hat{t} + \eta'' - \eta')$$

(VIA.7)

The " " indicate dimensional quantities. Renondimensionalize as follows:

$$\omega_{DROP}^{-1}\, t = \hat{\hat{t}} \qquad ; \qquad d\,\underline{x} = \hat{\hat{\underline{x}}}$$

$$d\,\omega_{DROP}\,\underline{u} = \hat{\hat{\underline{u}}} \qquad ; \qquad \rho_o^{\circ}(d\,\omega_{DROP})^2\, p = \hat{\hat{p}} \tag{VIA.8}$$

The reference length is that of the drop dimension, d. This, in fact, provides the coupling to the acoustic field; $d \sim \frac{c_o^{\circ}}{\omega_{AC}}$. The natural frequency of oscillation of the drop provides the time scale.

Utilizing the re-nondimensionalization scheme yields, after manipulations

$$\nabla \cdot \underline{\mathcal{U}}_2^{\circ} = 0 \tag{VIA.9a}$$

$$\frac{\partial \underline{\mathcal{U}}_2^{\circ}}{\partial t} + \nabla \mathcal{P}_2^{\circ} - \frac{1}{Re_{HYDR}} \nabla^2 \underline{\mathcal{U}}_2^{\circ}$$

$$= -2 \left(\underline{v}_1^{\circ} \cdot \overline{\nabla \underline{v}_1^{\circ}} + \rho_1^{\circ}\,\overline{(i\underline{v}_1^{\circ})} + \text{COMPLEX CONJUGATE} \right) \cos(t + \eta'' - \eta')$$

(VIA.9b)

with $Re_{HYDR} \sim (d^2 \omega_d / \nu_o^{\circ})$. The quantities are now nondimensional with respect to hydrodynamic field quantities. Moreover, $\mathcal{U}_2^{\circ}$ *and* $\mathcal{P}_2^{\circ}$ represent the hydrodynamic/drop oscillation field quantities. The system represented by (VIA.9a-9b) is linear. Of course, the right hand side of (VIA.9b) contains nonlinear forcing terms, but these are known quantities. It proves to be more convenient to work in terms of

$$\nabla \cdot \underline{\mathcal{U}}_2^\circ = 0 \tag{VIA.10a}$$

$$\frac{\partial \underline{\mathcal{U}}_2^\circ}{\partial t} + \nabla \mathcal{P}_2^\circ - \frac{1}{Re_{HYDR}} \nabla^2 \underline{\mathcal{U}}_2^\circ$$

$$= (-2)\left(\underline{v}_1^\circ \cdot \overline{\nabla \underline{v}_1^\circ} + \rho_1^\circ \overline{(i\underline{v}_1^\circ)} + c.c.\right) e^{i(t+\eta''-\eta')} \tag{VIA.10b}$$

Let

$$\underline{\mathcal{U}}_2^\circ = \underline{u}_2^\circ(r,z,\theta) \exp(i(t+\eta''-\eta'))$$

$$\mathcal{P}_2^\circ = p_2^\circ(r,z,\theta) \exp(i(t+\eta''-\eta')) \tag{VIA.11}$$

Substitution of (VIA.11) into system (VIA.10a-10b) yields

$$\nabla \cdot \underline{u}_2^\circ = 0 \tag{VIA.12a}$$

$$i\underline{u}_2^o + \nabla p_2^o - \frac{1}{Re_{HYDR}}\nabla^2 \underline{u}_2^o = (-2)\left(\underline{v}_1^o \cdot \overline{\nabla \underline{v}_1^o} + \rho^o\left(\overline{i\underline{v}_1^o}\right) + c.c.\right) \qquad \text{(VIA.12b)}$$

It is the system (VIA.12) which must be solved for $\underline{u}_2^o$ (and p_2^o).

Discussion of Forcing Terms and Resulting Decomposition

The forcing terms on the right hand side are of the form $\left\{\underline{v}_1^o \cdot \overline{\nabla \underline{v}_1^o} + \rho_1^o \overline{\left(i\underline{v}_1^o\right)} + c.c.\right\}$, and represent real quantities. In particular,

$$\underline{v}_1^o = \hat{\underline{v}}_1^o(r,\theta) + \approx{\underline{v}}_1^o(\zeta,\theta)$$

with $\hat{\underline{v}}_1^o$ referring to the velocity field in the outer region (represented by $\nabla\hat{\phi}^o$), exterior to the drop and with $\approx{\underline{v}}_1^o(\zeta,\theta)$ referring to the correction to the acoustic velocity field in the "acoustic sublayer" region. As $\zeta \to \infty$, quantities which depend on ζ will go to zero.

If the curl of Equation (VIA.12b) is taken, the quantities on the right hand side which contain only terms representing the inviscid outer flow will not contribute. That is, if the vorticity equation is constructed, it will be unforced in the outer region.

With this in mind, decompose as follows

$$\underline{u}_2^o = \hat{\underline{u}}_2^o(r,\theta) + \tilde{\underline{u}}_2^o(\zeta,\theta)$$

$$p_2^o = \hat{p}_2^o(r,\theta) + \tilde{p}_2^o(\zeta,\theta) \qquad \text{(VIA.13)}$$

Solution in the "Outer" Region Exterior to the Drop

Taking the curl of Equation (VIA.12b) - and the restricting attention to the "outer" region exterior to the drop (i.e. no remaining ζ dependence), one obtains

$$\frac{\partial \underline{\omega}_2^{\circ}}{\partial t} - \frac{1}{Re_{HYDR}} \nabla^2 \underline{\omega}_2^{\circ} = \underline{0} \qquad \text{(VIA.14)}$$

with $\underline{\omega}_2^{\circ}$ the vorticity in the outer region exterior to the drop. Notice that there are no sources of vorticity; in this region the hydrodynamic field is unforced. However, this is precisely the problem that was investigated by Miller and Scriven (1968), although their analysis was dimensional and the notation differs somewhat from the present exposition.

The solutions can then be written for both the velocity and pressure fields in this region.

$$\hat{u}^{\circ}_{r_2}(r,\theta) = \sum_{\ell} \left(a_2^{\circ} r^{-\ell-2} + a_4^{\circ} \frac{1}{r} h_\ell^{(1)}(\lambda r) \right) P_\ell(\cos\theta) \qquad \text{(VIA.15a)}$$

$$\hat{u}^{\circ}_{\theta_2}(r,\theta) = \sum_{\ell} \frac{1}{\ell(\ell+1)} \left\{ \begin{array}{l} -\ell\, a_2^{\circ} r^{-\ell-2} \\ + a_4^{\circ} \left\{ \frac{(\ell+1)}{r} h_\ell^{(1)}(\lambda r) - \lambda h_{\ell+1}^{(1)}(\lambda r) \right\} \end{array} \right\} \frac{dP_\ell}{d\theta} \qquad \text{(VIA.15b)}$$

$$\hat{p}_2^{o}(r,\theta) = \sum_{\ell} \frac{1}{Re_{HYDR}} \frac{(\Delta^2)}{\ell(\ell+1)} \left(-\ell\, a_2^{o}\, r^{-\ell-1}\right) P_\ell(\cos\theta)$$ (VIA.15c)

with $\Delta^2 = (-i\ Re_{HYDR})$.

Solution in the "Acoustic Sublayer" Region Exterior to the Drop

In this region, the independent variable r is stretched, and the independent (radial) variable ζ is related to r as

$$r = \tilde{R} + \varepsilon \zeta$$

Recall that $\epsilon = \frac{1}{\sqrt{(Re_{AC})}}$. However, the system of governing equations has been re-nondimensionalized with respect to hydrodynamic field reference quantities.

A relationship between $\epsilon\,(=\frac{1}{Re_{AC}})$ and Re_{HYDR} must be stated.

Recall that $\delta = \omega_{DROP}/\omega_{AC}$. Then

$$Re_{HYDR} = \delta\,(Re_{AC}) = \delta/\epsilon^2$$ (VIA.16a)

and

$$\epsilon = \sqrt{\delta}\,/\sqrt{Re_{HYDR}}$$ (VIA.16b)

One can also define a third "Reynolds number-like" parameter, R_s, such that

$$R_S = \delta^2 Re_{AC} = \delta^2/\epsilon^2 = \delta Re_{HYDR} \tag{VIA.16c}$$

In this work, it is clear that $Re_{AC} \gg 1$ and $\epsilon \ll 1, \delta \ll 1$. If the order of magnitude relationship between δ and ϵ is specified, an order of magnitude for R_s will follow.

The remarks concerning the relationship of this forced drop problem to the work of Riley (1967), which involved a solid body in periodic motion which were presented in Section V carry over to this Section. Therefore, they will not be repeated here.

Select $Re_{HYDR} = 0(1)$. Since $\epsilon \sim \sqrt{\delta}/\sqrt{Re_{HYDR}}$, the relationship between r and δ can be re-expressed as

$$r = \tilde{R} + \sqrt{\frac{\delta}{Re_{HYDR}}}\,\zeta \tag{VIA.17}$$

with Re_{HYDR} of order one. It is now necessary to rewrite the governing system of equations in the variable ζ. Note that in this region, the forcing terms on the right hand side of (VIA.12b) will contribute. The forcing terms themselves are comprised of functions which involve the acoustic velocity field.

The expansion will be done in terms of the primitive variables as opposed to a stream function. Since the conservation of momentum equation is a vector equation, the $\hat{e}_r$ and $\hat{e}_\theta$ component equations will be expanded in the acoustic sublayer region. Writing the $\hat{e}_r$ and $\hat{e}_\theta$ component equations yields:

($\hat{e}_r$ **cons. of momentum)**

$$i\, u^{o}_{2r} + \frac{\partial p^{o}_{2}}{\partial r}$$

$$- \frac{1}{Re_{HYDR}} \left\{ \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta} \right) \right) u^{o}_{2r} - \frac{2}{r^2} u^{o}_{2r} - \frac{2}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, u^{o}_{2\theta} \right) \right\}$$

$$= (-2) \left\{ \rho^{o}_{1} \overline{(i\, V^{o}_{1r})} + \hat{V}^{o}_{1r} \overline{\frac{\partial V^{o}_{1r}}{\partial r}} + \frac{V^{o}_{1\theta}}{r} \overline{\frac{\partial V_{1r}}{\partial \theta}} - \frac{V_{\theta 1} \overline{V_{\theta 1}}}{r} + \text{COMPLEX CONJUGATE} \right\}$$

(VIA.18a)

($\hat{e}_\theta$ cons. of momentum)

$$i\,u^{\circ}_{2\theta} + \frac{1}{r}\frac{\partial p^{\circ}_2}{\partial\theta}$$

$$-\frac{1}{Re_{HYDR}}\left\{\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right)u^{\circ}_{2\theta} + \frac{2}{r^2}\frac{\partial u^{\circ}_{2r}}{\partial\theta} - \frac{u^{\circ}_{2\theta}}{r^2\sin^2\theta}\right\}$$

$$= -2\left\{\rho^{\circ}_1\,\overline{\left(i\,V^{\circ}_{1\theta}\right)} + V^{\circ}_{1r}\,\overline{\frac{\partial\,(V^{\circ}_{1\theta})}{\partial r}} + \frac{V^{\circ}_{1\theta}}{r}\,\overline{\frac{\partial V^{\circ}_{1\theta}}{\partial\theta}} + \frac{V^{\circ}_{1\theta}\,\overline{V^{\circ}_{1r}}}{r} + \text{COMPLEX CONJUGATE}\right\}$$

(VIA.18b)

Now, expand in terms of

$$\tilde{u}^{\circ}_{r2} = \tilde{u}^{\circ}_{2r_0} + \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\tilde{u}^{\circ}_{2r_1} + \left(\frac{\delta}{Re_{HYDR}}\right)\tilde{u}^{\circ}_{2r_2} + \cdots \qquad \text{(VIA.19a)}$$

$$\tilde{u}^{\circ}_{2\theta} = \tilde{u}^{\circ}_{2\theta_0} + \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2} \tilde{u}^{\circ}_{2\theta_1} + \left(\frac{\delta}{Re_{HYDR}}\right) \tilde{u}^{\circ}_{2\theta_2} + \cdots \tag{VIA.19b}$$

with

$$\frac{\partial}{\partial r} \Rightarrow \sqrt{\frac{Re_{HYDR}}{\delta}}\, \frac{\partial}{\partial \zeta} \tag{VIA.19c}$$

The right hand side forcing terms in (VIA.18a-b) (written terms of ζ) are known from knowledge of $\underline{v}^{\circ}_1$ (and ρ°_1), which are the acoustic field quantities.

Therefore,

($\hat{e}_r$ cons. momentum - expanded)

$$i\left(\tilde{u}^{\circ}_{2r_0} + (\delta/Re_{HYDR})^{1/2}\, \tilde{u}^{\circ}_{2r_1} + \cdots \right)$$

$$+ \left(Re_{HYDR}/\delta\right)^{1/2} \frac{\partial}{\partial \zeta}\left(\tilde{p}^{\circ}_{2_0} + (\delta/Re_{HYDR})\, \tilde{p}^{\circ}_{2_1} + \cdots \right)$$

$$-\frac{1}{Re_{HYDR}} \Bigg\{ \Bigg[\left(\frac{Re_{HYDR}}{\delta}\right)\frac{\partial^2}{\partial \zeta^2} + 2\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta\right)^{-1}\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial \zeta}$$

$$+ \frac{\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta\right)^{-2}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) - 2\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta\right)^{-2}\Bigg] \cdot$$

$$\left(\tilde{u}^{\circ}_{2r_0} + (\delta/Re_{HYDR})^{1/2}\, \tilde{u}^{\circ}_{2r_1} + \cdots \right)$$

$$+$$

$$-\frac{2\left(\tilde{R}+(\delta/Re_{HYDR})^{1/2}\zeta\right)^{-2}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left\{\tilde{u}^{\circ}_{2\theta_0}+\tilde{u}^{\circ}_{2\theta_1}\left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}+\cdots\right\}\right)\Bigg\}$$

$$=(-2)\Bigg\{\overline{\left(\hat{\rho}^{\circ}_{10}|_{\tilde{R}}+\cdots\right)}\left[(i)\left(\left(\delta/Re_{HYDR}\right)^{1/2}\tilde{\tilde{V}}^{\circ}_{1r_1}+\cdots\right)\right]$$

$$+\overline{\left((\delta/Re_{HYDR})\tilde{\tilde{\rho}}^{\circ}_{12}+\cdots\right)}\left[(i)\left\{\hat{V}^{\circ}_{1r_0}|_{\tilde{R}}+\left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\left(\zeta\hat{V}^{\circ\prime}_{1r_0}|_{\tilde{R}}+\tilde{\tilde{V}}^{\circ}_{1r_1}\right)+\cdots\right\}\right]$$

$$+\overline{\left(\hat{V}^{\circ}_{1r_0}|_{\tilde{R}}+\cdots\right)}\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\zeta}\left((\delta/Re_{HYDR})^{1/2}\tilde{\tilde{V}}^{\circ}_{1r_1}+\cdots\right)$$

$$+\overline{\left((\delta/Re_{HYDR})^{1/2}\tilde{\tilde{V}}^{\circ}_{1r_1}+\cdots\right)}\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\zeta}\left(\hat{V}^{\circ}_{1r_0}|_{\tilde{R}}+\left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\left(\tilde{\tilde{V}}^{\circ}_{1r_1}+\zeta\hat{V}^{\circ\prime}_{1r_0}|_{R}\right)+\cdots\right)$$

$$+\left(\tilde{R}+(\delta/Re_{HYDR})^{1/2}\zeta\right)^{-1}\overline{\left(\hat{V}^{\circ}_{1\theta_0}|_{\tilde{R}}+\cdots\right)}\left[\frac{\partial}{\partial\theta}\left((\delta/Re_{HYDR})^{1/2}\left(\tilde{\tilde{V}}^{\circ}_{1r_1}\right)+\cdots\right)\right]$$

$$+\left(\tilde{R}+(\delta/Re_{HYDR})^{1/2}\zeta\right)^{-1}\overline{\left(\tilde{\tilde{V}}^{\circ}_{1\theta_0}+\cdots\right)}\left[\frac{\partial}{\partial\theta}\left(\hat{V}^{\circ}_{1r_0}|_{\tilde{R}}+(\delta/Re_{HYDR})^{1/2}\left(\zeta\hat{V}^{\circ\prime}_{1r_0}|_{\tilde{R}}+\tilde{\tilde{V}}^{\circ}_{1r_1}\right)+\cdots\right)\right]$$

$$-\left(\tilde{R}+(\delta/Re_{HYDR})^{1/2}\zeta\right)^{-1}\overline{\left(\hat{V}^{\circ}_{1\theta_0}|_{\tilde{R}}+\cdots\right)}\left(\tilde{\tilde{V}}^{\circ}_{1\theta_0}+\cdots\right)$$

$$-\left(\tilde{R}+(\delta/Re_{HYDR})^{1/2}\zeta\right)^{-1}\overline{\left(\tilde{\tilde{V}}^{\circ}_{1\theta_0}+\cdots\right)}\left(\hat{V}^{\circ}_{1\theta_0}|_{\tilde{R}}+\tilde{\tilde{V}}^{\circ}_{1\theta_0}+\cdots\right)$$

$$+\text{ COMPLEX CONJUGATE}\Bigg\}$$

(VIA.20a)

The $\underline{v}_1^o$ and ρ_1^o quantities represent the acoustic field. They are expanded in the acoustic sublayer region; and contain both terms which decay to zero out of the sublayer region (i.e. as $\zeta \to \infty$) and terms which represent the inviscid acoustic field quantities re-expressed in a Taylor series expansion in terms of ζ in the sublayer region. The subscript "1" on $\underline{v}_1^o$ and ρ_1^o is kept in order to distinguish clearly these acoustic field quantities from the hydrodynamic field quantities, which are indicated by a "2" subscript. It should be clear that there is not any time dependence present in Equations (VIA.18a-18b) through (VIA.20a-20b). For more details concerning the acoustic field quantities, see Section III.

The expanded form of the $\hat{e}_\theta$ conservation of momentum equation will be exhibited. The above remarks also apply to this equation.

($\hat{e}_\theta$ **cons. momentum - expanded)**

$$
\begin{aligned}
& i\left(\tilde{u}^o_{2\theta_0} + (\delta/Re_{HYDR})^{1/2}\,\tilde{u}^o_{2\theta_1} + \cdots\right) \\
& + \left(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta\right)^{-1}\frac{\partial}{\partial\theta}\left(\tilde{p}^o_{20} + (\delta/Re_{HYDR})^{1/2}\,\tilde{p}^o_{21} + \cdots\right) \\
& \frac{-1}{Re_{HYDR}}\left\{\left[\left(\frac{Re_{HYDR}}{\delta}\right)\frac{\partial^2}{\partial\zeta^2} + \frac{2}{\left(R + (\delta/Re_{HYDR})^{1/2}\zeta\right)}\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\zeta}\right.\right. \\
& \qquad \left. + \frac{\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta\right)^{-2}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) - \frac{\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta\right)^{-2}}{\sin^2\theta}\right]\cdot \\
& \qquad \left(\tilde{u}^o_{\theta 2_0} + (\delta/Re_{HYDR})^{1/2}\,\tilde{u}^o_{\theta 2_1} + \cdots\right) \\
& \qquad \left. + 2\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta\right)^{-2}\frac{\partial}{\partial\theta}\left\{\tilde{u}^o_{2r_0} + (\delta/Re_{HYDR})^{1/2}\,\tilde{u}^o_{2r_1} + \cdots\right\}\right\} \\
& =
\end{aligned}
$$

$$
\begin{aligned}
(-2)\Bigg\{ & \overline{\left(\hat{\rho}^{o}_{10|\tilde{R}} + \cdots\right)}\left(i\left(\tilde{\tilde{V}}^{o}_{1\theta_0} + \cdots\right)\right) + \overline{\left(\left(\delta/Re_{HYDR}\right)\tilde{\tilde{\rho}}^{o}_{12} + \cdots\right)}\left(i\left[\left(\hat{V}^{o}_{1\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{o}_{1\theta_0}\right) + \cdots\right]\right) \\
& + \overline{\left(\hat{V}^{o}_{r0|\tilde{R}} + \cdots\right)}\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\zeta}\left\{\tilde{\tilde{V}}^{o}_{1\theta_0} + \cdots\right\} \\
& \qquad + \overline{\left(\left(\delta/Re_{HYDR}\right)^{1/2}\tilde{\tilde{V}}^{o}_{1r_1} + \cdots\right)}\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\zeta}\left\{\hat{V}^{o}_{1\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{o}_{1\theta_0} + \cdots\right\} \\
& + \left(\tilde{R} + \left(\delta/Re_{HYDR}\right)^{1/2}\zeta\right)^{-1}\overline{\left(\hat{V}^{o}_{1\theta_0|\tilde{R}} + \cdots\right)}\frac{\partial}{\partial\theta}\left(\tilde{\tilde{V}}^{o}_{1\theta_0} + \cdots\right) \\
& + \left(\tilde{R} + \left(\delta/Re_{HYDR}\right)^{1/2}\right)^{-1}\overline{\left(\tilde{\tilde{V}}^{o}_{1\theta_0} + \cdots\right)}\frac{\partial}{\partial\theta}\left(\hat{V}^{o}_{1\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{o}_{1\theta_0} + \cdots\right) \\
& + \left(\tilde{R} + \left(\delta/Re_{HYDR}\right)^{1/2}\zeta\right)^{-1}\overline{\left(\hat{V}^{o}_{1\theta_0|\tilde{R}} + \cdots\right)}\left(\left(\delta/Re_{HYDR}\right)^{1/2}\tilde{\tilde{V}}^{o}_{1r_1} + \cdots\right) \\
& + \left(\tilde{R} + \left(\delta/Re_{HYDR}\right)^{1/2}\zeta\right)^{-1}\overline{\left(\tilde{\tilde{V}}^{o}_{1\theta_0} + \cdots\right)}\left(\hat{V}^{o}_{1r_0|\tilde{R}} + \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\left(\zeta\,\hat{V}^{o\prime}_{1r_0|\tilde{R}} + \tilde{\tilde{V}}^{o}_{1r_1}\right) + \cdots\right) \\
& \qquad\qquad + \text{COMPLEX CONJUGATE}\Bigg\}
\end{aligned}
$$

(VIA.20b)

Note that in both (VIA.20a-20b), the terms on the right hand side involve products, as least one factor which involves only the viscous correction to the acoustic field. Also, it is necessary to expand the conservation of mass equation in order to relate $\tilde{u}^o_{r_2}$ and $\tilde{u}^o_{\theta_2}$. It is

(cons. mass-expanded)

$$\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2} \frac{\partial}{\partial \zeta}\left(\tilde{u}^o_{ar_0} + (\delta / Re_{HYDR})^{1/2} \tilde{u}^o_{ar_1} + \cdots \right)$$

$$+ 2\left(\tilde{R} + (\delta / Re_{HYDR})^{1/2} \zeta\right)^{-1} \left(\tilde{u}^o_{ar_0} + (\delta / Re_{HYDR})^{1/2} \tilde{u}^o_{ar_1} + \cdots \right)$$

$$+ \frac{\left(\tilde{R} + (\delta / Re_{HYDR})^{1/2} \zeta\right)^{-1}}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \left\{ \tilde{u}^o_{a\theta_0} + \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2} \tilde{u}^o_{a\theta_1} + \cdots \right\}\right) = 0$$

(VIA.21)

It is clear from Equations (VIA.20a-20b) that a balance can be achieved between the forcing terms and the hydrodynamic field terms if $\tilde{u}^o_{a\theta}$ is selected to be of order $\dfrac{\sqrt{\delta}}{\sqrt{Re_{HYDR}}}$.

Physically, the hydrodynamic field is forced by the acoustic field. The objective of the mathematics is to have the lowest order hydrodynamic field terms balanced by the lowest order forcing terms.

Inspection of (VIA.20b) shows that this is the case if the highest derivative of the tangential component of the hydrodynamic field balances the radial derivative (normal to the interface) of the tangential component of the acoustic field, convected by the radial component of the inviscid acoustic velocity field. Moreover, it must be that $\tilde{u}^{o}_{2\theta_0}$ is zero, and that the balance occurs with $\tilde{u}^{o}_{2\theta_1}(\zeta,\theta)$.

This has ramifications for the $\tilde{u}^{o}_{2r}$ component. From Equation (VIA.21), it is clear that $\tilde{u}^{o}_{2r_2}$ is the lowest order contribution to $\tilde{u}^{o}_{2r}$.

From (VIA.20b),

$$\frac{-1}{Re_{HYDR}} \frac{\partial^2 \tilde{u}^{o}_{2\theta_1}}{\partial \zeta^2} = -2\left\{ \hat{V}^{o}_{1r_0}\Big|_{\tilde{R}} \overline{\frac{\partial \tilde{\tilde{V}}^{o}_{1\theta_0}}{\partial \zeta}} + \text{COMPLEX CONJUGATE} \right\}$$

(VIA.22)

which is

$$\frac{\partial^2 \tilde{u}^{o}_{2\theta_1}}{\partial \zeta^2} = 2(Re_{HYDR})\left\{\left(\sum_{\hat{l}=0}^{\infty}\left(\alpha^{o}_{S\hat{l}}\,\hat{S}_{\hat{l}}\frac{d}{dr}\left(h^{(1)}_{\hat{l}}(r)\right)_{\tilde{R}} + A_{NC}\,\hat{\hat{S}}_{\hat{l}}\frac{d}{dr}\left(j_{\hat{l}}(r)\right)_{\tilde{R}}\right)P_{\hat{l}}\right)\right.$$

$$\cdot\overline{\left(\sum_{l'=1}^{\infty}\hat{\hat{S}}_{l'}B^{o}_{BL_{l'}}\frac{dP_{l'}}{d\theta}\right)}$$

$$\left.\cdot\left(-(1-i)/\sqrt{2}\right)\exp\left(\frac{-(1-i)}{\sqrt{2}}\zeta\right)\right\} +$$

$$2(R_{e_{HYDR}})\left\{\overline{\left(\sum_{\hat{\ell}=0}^{\infty}\hat{\hat{\delta}}_{\hat{\ell}}\left[\alpha^{\circ}_{s_{\hat{\ell}}}\frac{d}{dr}\left(h_{\hat{\ell}}^{(1)}\right|_{\tilde{R}}+A_{INC}\frac{d}{dr}\left(j_{\hat{\ell}}(r)\right|_{\tilde{R}}\right)P_{\hat{\ell}}(\cos\theta)\right)}\right.$$

$$\cdot\left(\sum_{\ell'=1}^{\infty}\left(\hat{\hat{\delta}}_{\ell'}\,B^{\circ}_{BL_{\ell'}}\,\frac{dP_{\ell'}}{d\theta}\right)\right)$$

$$\cdot\left(\frac{-(1+i)}{\sqrt{2}}\right)\exp\left(-\frac{(1+i)}{\sqrt{2}}\,\zeta\right)$$

(VIA.23)

It is possible to re-express the coefficient in θ as expansions in $\frac{dP_L}{d\theta}$; that is,

$$\sum_{L=1}^{\infty}C_L\frac{dP_L}{d\theta}=\left(\sum_{\hat{\ell}=0}^{\infty}\hat{\delta}_{\hat{\ell}}\left(\alpha^{\circ}_{s_{\hat{\ell}}}\frac{d}{dr}\left(h_{\ell}^{(1)}\right|_{\tilde{R}}+A_{INC}\frac{d}{dr}\left(j_{\hat{\ell}}(r)\right|_{\tilde{R}}\right)P_{\hat{\ell}}\right)$$

$$\cdot\overline{\left(\sum_{\ell'=1}^{\infty}\hat{\hat{\delta}}_{\ell'}\,B^{\circ}_{BL_{\ell'}}\,\frac{dP_{\ell'}}{d\theta}\right)}$$

(VIA.24a)

and

$$\sum_{L=1}^{\infty} \overline{C}_L \frac{dP_L}{d\theta} = \overline{\left(\sum_{\hat{\ell}=0}^{\infty} \hat{\hat{\delta}}_{\hat{\ell}} \left(\alpha^{\circ}_{s_{\hat{\ell}}} \frac{d}{dr} \left(h^{(1)}_{\hat{\ell}} \big|_{\tilde{R}} \right) + A_{INC} \frac{d}{dr} \left(j_{\hat{\ell}} \big|_{\tilde{R}} \right) \right) P_{\hat{\ell}}(\cos\theta) \right)}$$

$$\cdot \left(\sum_{\ell'=1}^{\infty} \hat{\hat{\delta}}_{\ell'} B^{\circ}_{BL_{\ell'}} \frac{dP_{\ell'}}{d\theta} \right)$$

(VIA.24b)

so

$$\frac{\partial^2 \tilde{u}^{\circ}_{2\theta_1}}{\partial s^2} = 2(Re_{HYDR})\,(-1)\frac{(1-i)}{\sqrt{2}} \left(\sum_{L=1}^{\infty} C_L \frac{dP_L}{d\theta} \right)$$

$$\cdot \exp\left(-(1-i)/\sqrt{2}\; s\right)$$

$$+\, 2(Re_{HYDR})\,(-1)\frac{(1+i)}{\sqrt{2}} \left(\sum_{L=1}^{\infty} \overline{C}_L \frac{dP_L}{d\theta} \right)$$

$$\cdot \exp\left(-(1+i)/\sqrt{2}\; s\right) \qquad \text{(VIA.25)}$$

Let

$$\tilde{u}^{\circ}_{2\theta_1} = \sum_{L=1}^{\infty} g^{\circ}(s) \frac{dP_L}{d\theta} \qquad \text{(VIA.26)}$$

Then, using (VIA.26) in (VIA.25), and multiplying through by $\left(\frac{dP_q}{d\theta}\right)\sin\theta$, and integrating yields,

$$\frac{d^2 g^\circ}{d\zeta^2} \left(\int_0^\pi \sin\theta \, \frac{dP_\ell}{d\theta} \frac{dP_L}{d\theta} \, d\theta \right)$$

$$= 2 Re_{HYDR} \, (-1) \frac{(1-i)}{\sqrt{2}} \int_0^\pi \left(\sum_{L=1}^{\infty} C_L \frac{dP_L}{d\theta} \right) \frac{dP_\ell}{d\theta} \sin\theta \, d\theta$$

$$\cdot \exp\left(-(1-i)/\sqrt{2} \; \zeta \right)$$

$$+ \, 2 Re_{HYDR} \, (-1) \frac{(1+i)}{\sqrt{2}} \int_0^\pi \left(\sum_{L=1}^{\infty} \overline{C_L} \frac{dP_L}{d\theta} \right) \frac{dP_\ell}{d\theta} \sin\theta \, d\theta$$

$$\cdot \exp\left(-(1+i)/\sqrt{2} \; \zeta \right)$$

(VIA.27a)

Let

$$C1_\ell^\circ = \frac{(2\ell+1)}{2\ell(\ell+1)} \left[\int_0^\pi \left(\sum_{L=1}^{\infty} C_L \frac{dP_L}{d\theta} \right) \frac{dP_\ell}{d\theta} \sin\theta \, d\theta \right]$$

(VIA.27b)

and

$$C2_\ell^\circ = \frac{(2\ell+1)}{2\ell(\ell+1)} \left[\int_0^\pi \left(\sum_{L=1}^{\infty} \overline{C_L} \frac{dP_L}{d\theta} \right) \frac{dP_\ell}{d\theta} \sin\theta \, d\theta \right]$$

(VIA.27c)

which yields

$$\frac{d^2 g^\circ}{d\zeta^2} = 2(Re_{HYDR})(-1)\frac{(1-i)}{\sqrt{2}}(c1_\ell^\circ)\exp\left(-\frac{(1-i)}{\sqrt{2}}\zeta\right)$$

$$+ 2(Re_{HYDR})(-1)\frac{(1+i)}{\sqrt{2}}(c2_\ell^\circ)\exp\left(-\frac{(1+i)}{\sqrt{2}}\zeta\right)$$

(VIA.27d)

The solution to (VIA.27d) can be written as

$$\tilde{u}^\circ_{2\theta_1}(\zeta,\theta) = \sum_{\ell=1}^{\infty}\left\{\begin{array}{l} d_0^\circ + d_1^\circ\zeta \\ + (Re_{HYDR})(-1)\sqrt{2}(1+i)(c1_\ell^\circ) \\ \quad \cdot \exp(-(1-i)/\sqrt{2}\,\zeta) \\ + (Re_{HYDR})(-1)\sqrt{2}(1-i)(c2_\ell^\circ) \\ \quad \cdot \exp(-(1+i)/\sqrt{2}\,\zeta) \end{array}\right\}\frac{dP_\ell}{d\theta}$$

and (VIA.28a)

$$\tilde{U}^\circ_{2\theta_1} = \tilde{u}^\circ_{2\theta_1}\exp(it + \eta'' - \eta')$$

(VIA.28b)

VIB. HYDRODYNAMIC FIELD INTERIOR TO THE DROP: EQUATIONS AND SOLUTIONS

The nonlinear system of governing equations (see Section IIC) is nondimensionalized with respect to acoustic field reference quantities. The resulting system, governing flow interior to the drop is:

$$\frac{\partial \rho^i}{\partial t} + \underline{U}^i \cdot \nabla \rho^i + \rho^i \nabla \cdot \underline{U}^i = 0 \qquad \textbf{(VIB.1a)}$$

$$\rho^i \frac{\partial \underline{U}^i}{\partial t} + \rho^i \underline{U}^i \cdot \nabla \underline{U}^i = -\nabla p^i + \frac{\alpha}{Re_{AC}} \nabla^2 \underline{U}^i + \left(\frac{\alpha}{3} + \tau^i\right) \frac{1}{Re_{AC}} \nabla (\nabla \cdot \underline{U}^i) \qquad \textbf{(VIB.1b)}$$

with $\alpha = \mu^i_o/\mu^o_o$ and $\tau^i = \mu^i_{BULK}/\mu^o_o$. The "i" superscript refers to the region interior to the drop. (Also, the relationship between pressure and density holds at order δ .) The velocity, pressure, and density fields are given by $\underline{U}^i$, p^i, and ρ^i , respectively. Re_{AC} is a Reynolds-type number, and equals $c^o_o(c^o_o/\omega_{AC})/\nu^o_o$, with ω_{AC} the acoustic wave frequency.

Let the dependent variables be expanded in a series in the expansion parameter, δ . (Note that $\delta = \omega_{DROP}/\omega_{AC}$,with ω_{DROP} indicating the natural frequency of oscillation of the drop.) That is

$$\underline{U}^i = \delta \underline{v}_1^i + \delta^2 \underline{u}_2^i \tag{VIB.2a}$$

$$p^i = P_0^i + \delta p_1^i + \delta^2 p_2^i \tag{VIB.2b}$$

$$\rho^i = \beta + \delta \rho_1^i + \delta^2 \rho_2^i \tag{VIB.2c}$$

Acoustic variables are denoted by the "1" subscript. ($\underline{v}_1^i$, p_1^i, ρ_1^i), and the hydrodynamic quantities by "2" $\left(\underline{u}_2^i, p_2^i\right)$. At order δ , the governing equations of the acoustic field are recovered (see Section III).

It is at order δ^2 that the hydrodynamic field occurs. It owes its existence to the (modulated) acoustic standing wave field.

The hydrodynamic field is taken to be incompressible. Resulting governing equations are then

$$\nabla \cdot \underline{u}_2^i = 0 \tag{VIB.3a}$$

$$\beta \frac{\partial \underline{u}_2^i}{\partial t} + \nabla p_2^i - \frac{1}{Re_{AC}} \nabla^2 \underline{u}_2^i$$

$$= -\left(\rho_1^i \frac{\partial \underline{v}_1^i}{\partial t} + \beta \underline{v}_1^i \cdot \nabla \underline{v}_1^i \right)$$

(VIB.3b)

Clearly, the term $\left(\frac{\alpha}{3}+\tau^i\right)\frac{1}{Re_{AC}}\nabla\left(\nabla \cdot u_2^i\right)$ which formally appears at this order is zero due to the incompressibility of the hydrodynamic field.

The quantities $\underline{v}_1{}^i$ and ρ_1^i are known at this order. They act as forcing or source terms in the conservation of momentum Equation (VIB.3b).

The acoustic field represented by $\underline{v}_1{}^i$ (and p_1^i, ρ_1^i) is a <u>modulated</u> acoustic standing wave field. For example,

$$\hat{\phi}^i = \hat{\phi}^{i\,\prime} + \hat{\phi}^{i\,\prime\prime}$$

(VIB.4)

with $\hat{\phi}^i$ the velocity potential for the acoustic standing wave. The "$'$" refers to the acoustic wave of frequency $\omega'(=\omega_{AC})$, and "$''$" refers to the wave of frequency ω'' .

The time average of Equations (VIB.3a-3b) over a period of $2\pi/\omega_{AC}$ must be taken. This will result in having a time-dependent source term (i.e., the right hand side), of a form to be exhibited explicitly. A solution for u_2^i (and P_2^i) will be sought with this same time dependence.

Taking the time-average of terms on the right hand side yields

$$\langle RHS \rangle = \hat{\rho}^{i} \, \overline{(i \underline{v}_1^{i})} \left\{ 2 + 2\cos(\delta t + \eta'' - \eta') \right\}$$

$$+ \overline{\hat{\rho}_1^{i}} \, (i \underline{v}_1^{i}) \left\{ 2 + 2\cos(\delta t + \eta'' - \eta') \right\}$$

$$+ \beta \left(\underline{v}_1^{i} \cdot \overline{\nabla \underline{v}_1^{i}} \right) \left(2 + 2\cos(\delta t + \eta'' - \eta') \right)$$

$$+ \beta \left(\overline{\underline{v}_1^{i}} \cdot \nabla \underline{v}_1^{i} \right) \left(2 + 2\cos(\delta t + \eta'' - \eta') \right)$$

$$\overline{\delta} = \omega_{DROP} / \omega_{AC} \qquad \textbf{(VIB.5)}$$

<u>In this expression only, the caret "^" indicates a time independent, general quantity</u>. Both η'' and η' indicate phases. Note that there is a steady state contribution to the forcing as well as the $\cos(\delta t + \eta'' - \eta')$ time-varying dependent forcing. The steady state forcing was addressed via consideration of the <u>unmodulated</u> acoustic standing wave field (see Section V). It will not be considered in this section. Rather, the focus will be solely on the response of the hydrodynamic field to the oscillatory forcing.

The governing equations, after being time-averaged, yield

$$\nabla \cdot \underline{u}_2^{i} = 0 \qquad \textbf{(VIB.6a)}$$

$$\beta \frac{\partial \underline{u}_2^i}{\partial t} + \nabla p_2^i - \frac{\alpha}{Re_{AC}} \nabla^2 \underline{u}_2^i$$

$$= -2 \left\{ \beta\, \underline{V}_1^i \cdot \overline{\nabla \underline{V}_1^i} + \rho_1^i\, \overline{(i \underline{V}_1^i)} + \text{COMPLEX CONJUGATE} \right\} \cdot \cos (\delta t + \eta'' - \eta') \qquad \textbf{(VIB.6b)}$$

Note that Equations (VIB.6a-6b) are nondimensional, and have been nondimensionalized with respect to acoustic field reference quantities. The periodic temporal forcing contains "δt", with $\delta \ll 1$. However, at this stage, the hydrodynamic field is of interest. A re-nondimensionalization scheme will be performed following the methodology discussed in Section V.

<u>Re-nondimensionalization</u>

The hydrodynamic field is the focal point. It is this flow which exists as a result of the acoustic forcing. This is clearly shown in Equations (VIB.6a-6b). However, in those equations, all quantities have been nondimensionalized with respect to reference quantities which are <u>not</u> relevant to an oscillating drop. In particular, the "δt" in the periodic time dependence indicates a very slow scale - yet the drop oscillation time should <u>define</u> the time scale of the hydrodynamic field.

The re-nondimensionalization acts as a re-normalization, with the result that the oscillating drop problem is viewed from the standpoint of the hydrodynamic field as opposed to the acoustic field.

Rewrite (VIB.6b) in terms of dimensional quantities as follows

$$\frac{1}{C_o^\circ \omega_{AC}} \beta \frac{\partial \hat{\underline{u}}_2^i}{\partial \hat{t}} + \frac{1}{\rho_o^\circ C_o^{\circ 2}} (C_o^\circ/\omega_{AC}) \, \hat{\nabla} \, \hat{p}_2^i$$

$$- \frac{\alpha}{Re_{AC}} \frac{1}{C_o^\circ} (C_o^\circ/\omega_{AC})^2 \, \hat{\nabla}^2 \, \hat{\underline{u}}_2^i$$

$$= \frac{1}{C_o^{\circ 2}} (C_o^\circ/\omega_{AC}) \, (-2) \left\{ \begin{array}{l} \beta \, \hat{\underline{v}}_1^i \cdot \nabla \overline{\hat{\underline{v}}_1^i} + \hat{\rho}_1^i \, \overline{(i \, \hat{\underline{v}}_1^i)} \\ + \text{COMPLEX CONJUGATE} \end{array} \right\}$$

$$\cdot \cos\left(\delta \omega_{AC} \hat{t} + \eta'' - \eta' \right)$$

(VIB.7)

The "^" indicates dimensional quantities. Re-nondimensionalize as follows:

$$\omega_{DROP}^{-1} \, t = \hat{t} \quad ; \quad d \, \underline{x} = \hat{\underline{x}}$$

$$d \omega_{DROP} \, \underline{u} = \hat{\underline{u}} \quad ; \quad \rho_o^\circ (d \omega_{DROP})^2 \, p = \hat{p}$$

(VIB.8)

The reference length is that of the drop dimension. This, in fact, <u>provides the coupling to the</u>

acoustic field; $d \sim c_o^o/\omega_{AC}$. The natural frequency of the drop oscillation provides the time scale.

Utilizing the re-nondimensionalization scheme yields, after manipulations

$$\nabla \cdot \underline{u}_2^i = 0 \tag{VIB.9a}$$

$$\beta \frac{\partial \underline{u}_2^i}{\partial t} + \nabla P_2^i - \frac{\alpha}{Re_{HYDR}} \nabla^2 \underline{u}_2^i = -2 \left\{ \beta (\underline{v}_1^i \cdot \overline{\nabla \underline{v}_1^i}) + \rho_1^i \, \overline{(i \underline{v}_1^i)} + \text{COMPLEX CONJUGATE} \right\} \cdot \cos (t + \eta'' - \eta') \tag{VIB.9b}$$

with $Re_{HYDR} = (d^2 \omega_d / \nu_o^o)$. The quantities are now nondimensional with respect to acoustic field quantities. Moreover, $\underline{u}_2^i$ and ρ_2^i represent the hydrodynamic/drop oscillation field quantities.

The system represented by (VIB.9a-9b) is linear. Of course, the right hand side of (VIB.9b) contains nonlinear forcing terms, but these are known quantities. It proves to be more convenient to work in terms of

$$\nabla \cdot \underline{u}_2^i = 0 \tag{VIB.10a}$$

$$\beta \frac{\partial \underline{\mathcal{U}}_2^i}{\partial t} + \nabla \mathcal{P}_2^i - \frac{\alpha}{Re_{HYDR}} \nabla^2 \underline{\mathcal{U}}_2^i$$

$$= -2 \left\{ \begin{array}{l} \beta(\underline{v}_1^i \cdot \overline{\nabla \underline{v}_1^i}) + \rho_1^i(\overline{i \underline{v}_1^i}) \\ + \text{COMPLEX CONJUGATE} \end{array} \right\} \exp(i(t + \eta'' - \eta')) \tag{VIB.10b}$$

Let

$$\underline{\mathcal{U}}_2^i = \underline{u}_2^i(r, \xi, \theta) \exp(i(t + \eta'' - \eta'))$$

$$\mathcal{P}_2^i = p_2^i(r, \xi, \theta) \exp(i(t + \eta'' - \eta')) \tag{VIB.11}$$

Substitution of (VIB.11) into system (VIB.10a-10b) yields

$$\nabla \cdot \underline{u}_2^i = 0 \tag{VIB.12a}$$

$$i\beta \underline{u}_2^i + \nabla p_2^i - \frac{\alpha}{Re_{HYDR}} \nabla^2 \underline{u}_2^i$$

$$= (-2) \left(\begin{array}{l} \beta \underline{v}_1^i \cdot \overline{\nabla \underline{v}_1^i} + (\rho_1^i)(\overline{i \underline{v}_1^i}) \\ + \text{COMPLEX CONJUGATE} \end{array} \right) \tag{VIB.12b}$$

It is the system (VIB.12) which must be solved for u_2^i (and p_2^i).

Discussion of Forcing Terms and Resulting Decomposition

The forcing terms on the right hand side are of the form $\left\{\beta \underline{v}_1^i \, \overline{\nabla \underline{v}_1^i} + \rho_1^i \overline{(i\underline{v}_1^i)} + c.c.\right\}$,

and represent real quantities. In particular, the field variables $\underline{v}_1^i$ and ρ_1^i are functions of (r,θ) and ξ . For example

$$\underline{v}_1^i = \hat{\underline{v}}_1^i(r,\theta) + \tilde{\tilde{\underline{v}}}_1^i(\xi,\theta)$$

with $\hat{\underline{v}}_1^i$ referring to the velocity field in the outer region interior to the drop and with $\tilde{\tilde{\underline{v}}}_1^i$ (ξ,θ) referring to the correction to the acoustic velocity field in the "acoustic sublayer" region. As $\xi \rightarrow \infty$, quantities which depend on ξ will decay to zero.

If the curl of Equation (VIB.12b) is taken the quantities on the right hand side which contain only terms representing inviscid outer flow will not contribute. That is, if the (linearized) vorticity equation is constructed, it will be an unforced equation in the outer region.

With this in mind, decompose as follows:

$$\underline{u}_2^i = \hat{\underline{u}}_2^i(r,\theta) + \tilde{\underline{u}}_2^i(\xi,\theta)$$

$$p_2^i = \hat{p}_2^i(r,\theta) + \tilde{p}_2^i(\xi,\theta) \qquad \textbf{(VIB.13)}$$

Solution in the "Outer" Region Interior to the Drop

Taking the curl of Equation (VIB.12b) - and restricting attention to the "outer" region exterior to the drop (i. e., no remaining ξ dependence), one obtains

$$\frac{\partial \underline{\omega}_2^i}{\partial t} - \frac{\alpha}{Re_{HYDR}} \nabla^2 \underline{\omega}_2^i = \underline{0}$$

(VIB.14)

with $\underline{\omega}_2^i$ the vorticity in the outer region interior to the drop. Note ther are no sources of vorticity; in this region the hydrodynamic field is unforced.

However, this is the problem investigated by Miller and Scriven (1968), albeit their analysis was dimensional, and differed somewhat from the present exposition.

The solutions can then be written down for both the velocity and pressure fields in this region. They are:

$$\hat{u}_{2r}^i(r,\theta) = \sum_\ell \left(a_1^i r^{\ell-1} + a_3^i j_\ell(\hat{\lambda} r) \right) P_\ell(\cos\theta) \qquad \text{(VIB.15a)}$$

$$\hat{u}_{2\theta}^i(r,\theta) = \sum_\ell \frac{1}{\ell(\ell+1)} \left\{ \begin{array}{l} (\ell+1) a_1^i r^{\ell-1} \\ + a_3^i \left[\frac{(\ell+1)}{r} j_\ell(\hat{\lambda} r) - \hat{\lambda} j_{\ell+1}(\hat{\lambda} r) \right] \end{array} \right\} \frac{dP_\ell}{d\theta}$$

(VIB.15b)

and

$$\hat{p}_2^i(r,\theta) = \sum_{\ell} \frac{-1}{\ell(\ell+1)}\left((\ell+1)\, a_1^i r^\ell\right) P_\ell(\cos\theta) \tag{VIB.15c}$$

with

$$\hat{\Delta}^2 = \left(-i\, Re_{HYDR}\beta/\alpha\right)$$

Solution in the "Acoustic Sublayer" Region Interior to the Drop

In this region, the independent variable r is stretched, with the independent (radial) variable ξ related to r as $r = \tilde{R} - \varepsilon\,\xi$.

Recall the $\epsilon = 1/\mathrm{SQRT}(Re_{AC})$. This system of governing equations has been re-nondimensionalized with respect to the hydrodynamic field reference quantities. A relationship between ϵ and Re_{HYDR} must be stated. Recall that

$$Re_{HYDR} = \delta\,(Re_{AC}) = \delta/\epsilon^2 \tag{VIB.16a}$$

and

$$\epsilon = \sqrt{\delta / Re_{HYDR}} \tag{VIB.16b}$$

One can also define a third Reynolds like number, R_s, with

$$R_s = \delta^2 Re_{AC} = \delta^2/\epsilon^2 = \delta Re_{HYDR} \tag{VIB.16c}$$

In this work, $Re_{AC} \gg 1$, and $\epsilon \ll 1$, $\delta \ll 1$. Now Re_{HYDR} is taken to be order one, and so $R_s \ll 1$.

The remarks concerning the relationship of this forced problem to the work of Riley (1967), which involved a solid body undergoing periodic motion; which was presented in Section V carry over to this section. They will not be repeated here.

Since $Re_{HYDR} = O(1)$, $\epsilon = \delta/ \ Re_{HYDR}$ is then of order SQRT (δ). The relationship between r and ξ is given by

$$r = \tilde{R} - \frac{\sqrt{\delta}}{\sqrt{Re_{HYDR}}} \xi \tag{VIB.17}$$

with Re_{HYDR} of order one. It is now necessary to rewrite the system of governing equations in the stretched variable ξ. Note that in this region, the forcing terms on the right hand side of (VIB.12b) <u>will contribute</u>. The forcing terms themselves are comprised of functions which involve the acoustic velocity field.

The expansion will be done in terms of primitive variables as opposed to the stream function. Since the conservation of momentum equation is a vector equation, the $\hat{e}_r$ and $\hat{e}_\theta$ component equations will be expanded separately in the acoustic sublayer region. Writing the $\hat{e}_r$ and $\hat{e}_\theta$ component equations yields:

($\hat{e}_r$ cons. momentum)

$$
\begin{aligned}
&\beta\left(i\, u^{i}_{2r}\right) + \frac{\partial p^{i}_{2}}{\partial r} \\
&\quad - \frac{\alpha}{Re_{HYDR}} \left\{ \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) \right] \left(u^{i}_{2r} \right) - \frac{2}{r^2} u^{i}_{2r} - \frac{2}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, u^{i}_{2\theta}\right) \right\} \\
&= (-2) \left\{ \beta\left(V^{i}_{1r}\, \overline{\frac{\partial V^{i}_{1r}}{\partial r}} + \frac{V^{i}_{1\theta}}{r}\, \overline{\frac{\partial V^{i}_{1r}}{\partial \theta}} - \frac{V^{i}_{1\theta}\, \overline{V^{i}_{1\theta}}}{r} \right) + \rho^{i}_{1}\, \overline{\left(i\, V^{i}_{1r}\right)} + \text{COMPLEX CONJUGATE} \right\}
\end{aligned}
$$

(VIB.18a)

($\hat{e}_\theta$ cons. momentum)

$$
\begin{aligned}
&\beta\left(u^{i}_{2\theta}\right) + \frac{1}{r}\frac{\partial p^{i}_{2}}{\partial \theta} \\
&\quad - \frac{\alpha}{Re_{HYDR}} \left\{ \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) \right] \left(u^{i}_{2\theta} \right) + \frac{2}{r^2}\frac{\partial u^{i}_{2r}}{\partial\theta} - \frac{u^{i}_{2\theta}}{r^2 \sin^2\theta} \right\} \\
&= (-2) \left\{ \beta\left(V^{i}_{1r}\, \overline{\frac{\partial V^{i}_{1\theta}}{\partial r}} + \frac{V^{i}_{1\theta}}{r}\, \overline{\frac{\partial V^{i}_{1\theta}}{\partial \theta}} + \frac{V^{i}_{1\theta}\, \overline{V^{i}_{1r}}}{r} \right) + \rho^{i}_{1}\, \overline{\left(i\, V^{i}_{1\theta}\right)} + \text{COMPLEX CONJUGATE} \right\}
\end{aligned}
$$

(VIB.18b)

Now, expand as

$$u_{2r}^{i} = \tilde{u}_{2r_0}^{i} + (\delta/Re_{HYDR})^{1/2}\,\tilde{u}_{2r_1}^{i} + (\delta/Re_{HYDR})\,\tilde{u}_{2r_2}^{i} + \cdots \qquad \text{(VIB.19a)}$$

$$u_{2\theta}^{i} = \tilde{u}_{2\theta_0}^{i} + (\delta/Re_{HYDR})^{1/2}\,\tilde{u}_{2\theta_1}^{i} + (\delta/Re_{HYDR})\,\tilde{u}_{2\theta_2}^{i} + \cdots \qquad \text{(VIB.19b)}$$

with

$$\frac{\partial}{\partial r} \Longrightarrow -\sqrt{\frac{Re_{HYDR}}{\delta}}\;\xi \qquad \text{(VIB.19c)}$$

A similar expansion is done for p_2^i. All functions overbarred with a "$\sim$" are dependent upon ξ and θ. The right hand side forcing terms in (VIB.18a-18b) (written in terms of ξ) are known from knowledge of $\underline{v}_1^i$ (and ρ_1^i), which are acoustic field quantities.

Therefore,

($\hat{e}_r$ cons. momentum - expanded)

$$\beta(i)\left(\tilde{u}_{2r_0}^{i} + (\delta/Re_{HYDR})^{1/2}\,\tilde{u}_{2r_1}^{i} + \cdots\right)$$

$$+ \left(Re_{HYDR}/\delta\right)^{1/2}\left(-\frac{\partial}{\partial\xi}\left[\tilde{p}_{2_0}^{i} + (\delta/Re_{HYDR})^{1/2}\,\tilde{p}_{2_1}^{i} + (\delta/Re_{HYDR})\,\tilde{p}_{2_2}^{i} + \cdots\right]\right)$$

$$+$$

$$
-\frac{\alpha}{Re_{HYDR}} \Bigg\{ \Bigg[\left(\frac{Re_{HYDR}}{\delta}\right) \frac{\partial^2}{\partial \xi^2} - 2\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-1} \left(\frac{Re_{HYDR}}{\delta}\right)^{1/2} \frac{\partial}{\partial \xi}
$$

$$
+ \frac{\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-2}}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} \quad\right) - 2\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-2} \Bigg] \cdot
$$

$$
\left\{ \tilde{u}^{\hat{i}}_{2r_0} + (\delta/Re_{HYDR})^{1/2} \tilde{u}^{\hat{i}}_{2r_1} + (\delta/Re_{HYDR}) \tilde{u}^{\hat{i}}_{2r_2} + \cdots \right\}
$$

$$
+ \; - \frac{2\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-2}}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta\left[\tilde{u}^{\hat{i}}_{2\theta_0} + (\delta/Re_{HYDR})^{1/2}\tilde{u}^{\hat{i}}_{2\theta_1} + \cdots\right]\right) \Bigg\}
$$

$$
= -2 \Bigg\{ \left(\overline{(\delta/Re_{HYDR})\tilde{\tilde{\rho}}^{\hat{i}}_{1_2} + \cdots}\right)\left(i\left[\hat{V}^{\hat{i}}_{r_0|\tilde{R}} + (\delta/Re_{HYDR})^{1/2}\left(-\xi\, \hat{V}^{\hat{i}\,\prime}_{r_0|\tilde{R}} + \tilde{\tilde{V}}^{\hat{i}}_{1r_1}\right) + \cdots\right]\right)
$$

$$
+ \left(\overline{\hat{\rho}^{\hat{i}}_{1_0|\tilde{R}} + \cdots}\right)\left(i\left[(\delta/Re_{HYDR})^{1/2}\, \tilde{\tilde{V}}^{\hat{i}}_{1r_1} + \cdots\right]\right)
$$

$$
+ \beta\left(\overline{\hat{V}^{\hat{i}}_{1r_0|\tilde{R}} + \cdots}\right)\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\left(\frac{-\partial}{\partial\xi}\left[(\delta/Re_{HYDR})^{1/2}\, \tilde{\tilde{V}}^{\hat{i}}_{1r_1} + \cdots\right]\right)
$$

$$
+ \beta\left(\overline{(\delta/Re_{HYDR})^{1/2}\, \tilde{\tilde{V}}^{\hat{i}}_{1r_1} + \cdots}\right)\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\left(\frac{-\partial}{\partial\xi}\left[\hat{V}^{\hat{i}}_{1r_0|\tilde{R}} + (\delta/Re_{HYDR})^{1/2}\left(-\xi\,\hat{V}^{\hat{i}\,\prime}_{1r_0|\tilde{R}} + \tilde{\tilde{V}}^{\hat{i}}_{1r_1}\right) + \cdots\right]\right)
$$

$$
+ \beta\overline{\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\hat{V}^{\hat{i}}_{1\theta_0|\tilde{R}} + \cdots\right)} \frac{\partial}{\partial\theta}\left((\delta/Re_{HYDR})^{1/2}\, \tilde{\tilde{V}}^{\hat{i}}_{1r_1} + \cdots\right)
$$

$$
+ \beta\overline{\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\tilde{\tilde{V}}^{\hat{i}}_{1\theta_0} + \cdots\right)} \frac{\partial}{\partial\theta}\left(\hat{V}^{\hat{i}}_{1r_0|\tilde{R}} + (\delta/Re_{HYDR})\left(-\xi\,\hat{V}^{\hat{i}\,\prime}_{1r_0|\tilde{R}} + \tilde{\tilde{V}}^{\hat{i}}_{1r_1}\right) + \cdots\right)
$$

$$
- \beta\left(\overline{\hat{V}^{\hat{i}}_{1\theta_0|\tilde{R}} + \cdots}\right)\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\tilde{\tilde{V}}^{i}_{1\theta_0} + \cdots\right)
$$

$$
- \beta\left(\overline{\tilde{\tilde{V}}^{\hat{i}}_{1\theta_0} + \cdots}\right)\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\hat{V}^{\hat{i}}_{1\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{i}_{1\theta_0} + \cdots\right)
$$

$$
+ \text{COMPLEX CONJUGATE} \Bigg\}
$$

(VIB.20a)

The $\underline{v}_1{}^i$ and $\rho_1{}^i$ quantities represent the acoustic field. They are expanded in the acoustic sublayer region; and contain both terms which decay to zero as $\xi \to \infty$ and terms which represent the inviscid field quantities re-expressed in a Taylor serves expansion in terms of ξ in the sublayer region. The subscript "1" on $\underline{v}_1{}^i$ and $\rho_1{}^i$ is kept in order to distinguish clearly these acoustic field quantities from those of the hydrodynamic field, indicated by a "2" subscript.

Note that the terms on the right hand side involve products, at least one factor of which involves only the viscous correction to the acoustic field. It should be clear that there is no time-dependence in Equations (VIB.18a-18b) through (VIB.20a-20b). For more details on the acoustic field quantities, see Section III.

The expanded form of the $\hat{e}_\theta$ - conservation of momentum equation is needed at this point. It is

($\hat{e}_\theta$ cons. momentum - expanded)

$$\beta(i)\left(\tilde{u}^{\hat{i}}_{2\theta_0} + (\delta/Re_{HYDR})^{1/2}\,\tilde{u}^{\hat{i}}_{2\theta_1} + (\delta/Re_{HYDR})\,\tilde{u}^{\hat{i}}_{2\theta_2} + \cdots\right)$$

$$+\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\right)^{-1}\frac{\partial}{\partial\theta}\left(\left[\tilde{p}^{\hat{i}}_{20} + (\delta/Re_{HYDR})^{1/2}\,\tilde{p}^{\hat{i}}_{21} + (\delta/Re_{HYDR})\,\tilde{p}^{\hat{i}}_{22} + \cdots\right]\right)$$

$$-\frac{\alpha}{Re_{HYDR}}\left\{\left[\left(\frac{Re_{HYDR}}{\delta}\right)\frac{\partial^2}{\partial\xi^2} - 2\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\xi}\right.\right.$$

$$\left. + \frac{\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-2}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) - \frac{\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-2}}{\sin^2\theta}\right]\cdot$$

$$\left(\tilde{u}^{\hat{i}}_{2\theta_0} + (\delta/Re_{HYDR})^{1/2}\,\tilde{u}^{\hat{i}}_{2\theta_1} + (\delta/Re_{HYDR})\,\tilde{u}^{\hat{i}}_{2\theta_2} + \cdots\right) \quad +$$

$$- \left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-2} \frac{\partial}{\partial\theta}\left(\tilde{u}^{i}_{2\theta_0} + (\delta/Re_{HYDR})^{1/2}\tilde{u}^{i}_{2\theta_1} + \cdots\right) \Bigg\}$$

$$= (-2)\Bigg\{ \overline{\left(\hat{\rho}^{i}_{10|\tilde{R}} + \cdots\right)} \left(i\left[\tilde{\tilde{V}}^{i}_{1\theta_0} + \cdots\right]\right)$$

$$+ \overline{\left((\delta/Re_{HYDR})\,\tilde{\tilde{\rho}}^{i}_{1_2} + \cdots\right)}\left(i\left[\hat{V}^{i}_{1\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{i}_{1\theta_0} + \cdots\right]\right)$$

$$+ \beta\,\overline{\left(\hat{V}^{i}_{1r_0|\tilde{R}} + \cdots\right)}\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\left(\frac{-\partial}{\partial\xi}\left[\tilde{\tilde{V}}^{i}_{1\theta_0} + \cdots\right]\right)$$

$$+ \beta\,\overline{\left((\delta/Re_{HYDR})^{1/2}\,\tilde{\tilde{V}}^{i}_{1r_1} + \cdots\right)}\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\left(\frac{-\partial}{\partial\xi}\left[\hat{V}^{i}_{1\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{i}_{1\theta_0} + \cdots\right]\right)$$

$$+ \beta\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\overline{\left(\hat{V}^{i}_{1\theta_0|\tilde{R}} + \cdots\right)}\frac{\partial}{\partial\theta}\left(\tilde{\tilde{V}}^{i}_{1\theta_0} + \cdots\right)$$

$$+ \beta\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\overline{\left(\tilde{\tilde{V}}^{i}_{1\theta_0} + \cdots\right)}\frac{\partial}{\partial\theta}\left(\hat{V}^{i}_{1\theta_0|\tilde{R}} + \tilde{\tilde{V}}^{i}_{1\theta_0} + \cdots\right)$$

$$+ \beta\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\overline{\left(\hat{V}^{i}_{1\theta_0|\tilde{R}} + \cdots\right)}\left((\delta/Re_{HYDR})^{1/2}\,\tilde{\tilde{V}}^{i}_{1r_1} + \cdots\right)$$

$$+ \beta\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)^{-1}\overline{\left(\tilde{\tilde{V}}^{i}_{1\theta_0} + \cdots\right)}\left(\hat{V}^{i}_{1r_0|\tilde{R}} + \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2}\left(-\xi\,\hat{V}^{i\,\prime}_{1r_0|\tilde{R}} + \tilde{\tilde{V}}^{i}_{1r_1}\right) + \cdots\right)$$

$$+ \text{COMPLEX CONJUGATE} \Bigg\}$$

(VIB.20b)

Finally, it is necessary to expand the conservation of mass equation in order to relate $\tilde{u}^{i}_{2r}$ to $\tilde{u}^{i}_{2\theta}$. It is

(cons. mass - expanded)

$$-\left(\frac{Re_{HYDR}}{\delta}\right)^{1/2}\frac{\partial}{\partial\xi}\left(\tilde{u}^{i}_{2r_0} + \left(\delta/Re_{HYDR}\right)^{1/2}\tilde{u}^{i}_{2r_1} + \cdots\right)$$

$$+\; 2\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\,\xi\right)^{-1}\left(\tilde{u}^{i}_{2r_0} + (\delta/Re_{HYDR})^{1/2}\,\tilde{u}^{i}_{2r_1} + \cdots\right)$$

$$+\; \frac{\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\,\xi\right)^{-1}}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\left[u^{i}_{2\theta_0} + \left(\delta/Re_{HYDR}\right)^{1/2}\tilde{u}^{i}_{2\theta_1} + \cdots\right]\right)$$

$$= 0$$

(VIB.21)

It is clear from Equations (VIB.20a-20b) that a balance can be achieved between the forcing terms and the hydrodynamic field terms if $u^{i}_{2\theta}$ is of order ($\delta/$ Re_{HYDR}).

Physically, the hydrodynamic field is forced by the acoustic field via convective acceleration type terms; that is, the forcing terms are quadratic in the acoustic field contribution. The objective of the mathematics is then to have the lowest order hydrodynamic field terms balanced by the lowest order forcing terms.

Inspection of (VIB.20b) shows that this is the case if the highest derivative of the tangential component of the hydrodynamic field balances the radial derivative (normal to the interface) of the tangential velocity component of the acoustic field, convected by the radial component of the inviscid acoustic velocity field. Moreover, it must be that $u^{\hat{i}}_{2\theta_0}$ is zero, and that the balance occurs with $u^{\hat{i}}_{2\theta_1}$.

This has ramifications for the $u^{\hat{i}}_{2r}$ component. From Equation (VIB.21), it is clear that $u^{\hat{i}}_{2r_2}$ is the lowest order contribution to $u^{\hat{i}}_{2r}$.

From (VIB.20b)

$$\frac{-\alpha}{Re_{HYDR}} \frac{\partial^2 \tilde{u}^{\hat{i}}_{2\theta_1}}{\partial \xi^2} = -2\left(\beta (-1) \hat{v}^{\hat{i}}_{1r_0}\Big|_{\tilde{R}} \overline{\frac{\partial \tilde{\tilde{v}}^{\hat{i}}_{\theta 1_0}}{\partial \xi}} + \text{COMPLEX CONJUGATE} \right)$$

(VIB.22)

which, when the right side is written out, is

$$\frac{\partial^2 \tilde{u}^{\hat{i}}_{2\theta_1}}{\partial \xi^2} = \left(-\frac{Re_{HYDR}\ 2\beta}{\alpha} \right) \left(\sum_{\hat{\ell}=0}^{\infty} \hat{\delta}_{\hat{\ell}}\ \alpha^{\hat{i}}_{\hat{\ell}} \frac{d}{dr}\left(j_{\hat{\ell}}\left(\frac{c_0^\circ}{c_0^{\hat{i}}} r \right) \right)\Big|_{\tilde{R}} P_{\hat{\ell}}(\cos\theta) \right)$$

$$\cdot \overline{\left(\sum_{\ell'=1}^{\infty} \hat{\delta}_{\ell'}\ A^{\hat{i}}_{BL_{\ell'}} \frac{dP_{\ell'}}{d\theta} \right)}$$

$$\cdot \left(-\sqrt{\frac{\beta}{\alpha}} \frac{(1-i)}{\sqrt{2}} \right)$$

$$\cdot \exp\left(-\sqrt{\beta/2\alpha}\ (1-i)\ \xi \right)$$

$$+$$

$$\left(-\frac{Re_{HYDR}\ 2\beta}{\alpha}\right)\left(\sum_{\hat{l}=0}^{\infty}\hat{\hat{\delta}}_{\hat{l}}\ \alpha_{\hat{l}}^{i}\ \frac{d}{dr}\left(j_{\hat{l}}\left(\frac{C_0^o}{C_0^i}r\right)_{|\hat{R}}\ P_{\hat{l}}(\cos\theta)\right)\right)$$

$$\cdot\left(\sum_{l'=1}^{\infty}\hat{\hat{\delta}}_{l'}\ A^{i}_{BL_{l'}}\ \frac{dP_{l'}}{d\theta}\right)\left(-\sqrt{\frac{\beta}{\alpha}}\ \frac{(1+i)}{\sqrt{2}}\right)$$

$$\cdot\ \exp\left(-\sqrt{\beta/2\alpha}\ (1+i)\ \xi\right) \tag{VIB.23}$$

It is possible to re-express the coefficients in θ as expansions in $\frac{dP_l}{d\theta}$; that is,

$$\sum_{L=1}^{\infty}C_L^{i}\ \frac{dP_L}{d\theta}$$

$$= \left(\sum_{\hat{l}=0}^{\infty}\hat{\hat{\delta}}_{\hat{l}}\ \alpha_{\hat{l}}^{i}\ \frac{d}{dr}\left(j_{l}\left(\frac{C_0^o}{C_0^i}r\right|_{\hat{R}}\ P_{\hat{l}}\ \right)\right)\left(\sum_{l'=1}^{\infty}\hat{\hat{\delta}}_{l'}\ A^{i}_{BL_{l'}}\ \frac{dP_{l'}}{d\theta}\right) \tag{VIB.24a}$$

and

$$\sum_{L=1}^{\infty}\overline{C}_L^{i}\ \frac{dP_L}{d\theta}$$

$$= \overline{\left(\sum_{\hat{l}=0}^{\infty}\hat{\hat{\delta}}_{\hat{l}}\ \alpha_{\hat{l}}^{i}\ \frac{d}{dr}\left(j_{\hat{l}}\left(\frac{C_0^o}{C_0^i}r\right|_{\hat{R}}\ P_{\hat{l}}(\cos\theta)\right)\right)}\left(\sum_{l'=1}^{\infty}\hat{\hat{\delta}}_{l'}\ A^{i}_{BL_{l'}}\ \frac{dP_{l'}}{d\theta}\right) \tag{VIB.24b}$$

Therefore,

$$\frac{\partial^2\tilde{U}^{i}_{\theta a_1}}{\partial\xi^2} = \left(-2\frac{\beta}{\alpha}Re_{HYDR}\right)\left(\sum_{L=1}^{\infty}C_L\frac{dP_L}{d\theta}\right)\left(-\sqrt{\frac{\beta}{\alpha}}\ \frac{(1-i)}{\sqrt{2}}\right)$$

$$\cdot\ \exp\left(-\sqrt{\beta/2\alpha}\ (1-i)\ \xi\right)$$

$$+\left(-2\frac{\beta}{\alpha}\ Re_{HYDR}\right)\left(\sum_{L=1}^{\infty}\ \overline{C}_L\frac{dP_L}{d\theta}\right)\left(-\sqrt{\frac{\beta}{\alpha}}\ \frac{(1+i)}{\sqrt{2}}\right)$$

$$\cdot\ \exp\left(-\sqrt{\beta/2\alpha}\ (1+i)\ \xi\right) \tag{VIB.25}$$

Let

$$\tilde{u}^{i}_{\theta 2_1} = \sum_{L=1}^{\infty} h^{i}(\xi) \frac{dP_L}{d\theta}$$

(VIB.26)

After a substitution of (VIB.26) into (VIB.25), multiplying through by (d $P_{\mathcal{L}}/d\theta$)sin θ, and integration over (o,π),

$$\frac{d^2 h^{i}(\xi)}{d\xi^2} \int_0^{\pi} \sin\theta \, \frac{dP_{\mathcal{L}}}{d\theta} \frac{dP_L}{d\theta} \, d\theta$$

$$= \left(-2\frac{\beta}{\alpha} Re_{HYDR}\right)\left(-\sqrt{\frac{\beta}{\alpha}} \frac{(1-i)}{\sqrt{2}}\right) \int_0^{\pi} \sin\theta \, \frac{dP_{\mathcal{L}}}{d\theta} \left(\sum_{L=1}^{\infty} C_L \frac{dP_L}{d\theta}\right) d\theta$$

$$\cdot \exp\left(-\sqrt{\beta/2\alpha}\,(1-i)\,\xi\right)$$

$$+ \left(-2\frac{\beta}{\alpha} Re_{HYDR}\right)\left(-\sqrt{\frac{\beta}{\alpha}} \frac{(1+i)}{\sqrt{2}}\right) \int_0^{\pi} \sin\theta \, \frac{dP_{\mathcal{L}}}{d\theta} \left(\sum_{L=1}^{\infty} \bar{C}_L \frac{dP_L}{d\theta}\right) d\theta$$

$$\cdot \exp\left(-\sqrt{\beta/2\alpha}\,(1+i)\,\xi\right)$$

(VIB.27a)

Let

$$C1_\ell^i = \frac{(2\ell+1)}{2\ell(\ell+1)} \left(\int_0^\pi \sin\theta \frac{dP_\ell}{d\theta} \left(\sum_{L=1}^{\infty} C_L \frac{dP_L}{d\theta} \right) d\theta \right)$$

(VIB.27b)

and

$$C2_\ell^i = \frac{(2\ell+1)}{2\ell(\ell+1)} \left(\int_0^\pi \sin\theta \frac{dP_\ell}{d\theta} \left(\sum_{L=1}^{\infty} \bar{C}_L \frac{dP_L}{d\theta} \right) d\theta \right)$$

(VIB.27c)

which yields

$$\frac{d^2 h^i}{d\xi^2} = Re_{HYDR}\, 2 \left(\frac{\beta}{\alpha}\right)\sqrt{\frac{\beta}{\alpha}}\, \frac{(1-i)}{\sqrt{2}} \left(C1_\ell^i\right)$$

$$\cdot \exp\left(-\sqrt{\beta/2\alpha}\,(1-i)\,\xi\right)$$

$$+ Re_{HYDR}\, 2 \left(\frac{\beta}{\alpha}\right)\sqrt{\frac{\beta}{\alpha}}\, \frac{(1+i)}{\sqrt{2}} \left(C2_\ell^i\right)$$

$$\cdot \exp\left(-\sqrt{\beta/2\alpha}\,(1+i)\,\xi\right)$$

(VIB.27d)

The solution to (VIB.27d) is found to be

$$\tilde{u}^{i}_{\theta 2_1}(\xi,\theta) = \sum_{l=1}^{\infty} \left| \begin{array}{l} d_0^{i} + d_1^{i}\,\xi \\ + Re_{HYDR}\sqrt{2\,\frac{\beta}{\alpha}}\,(1+i)\; c1_l^{i} \\ \quad \cdot \exp\left(-\sqrt{\beta/2\alpha}\,(1-i)\,\xi\right) \\ + Re_{HYDR}\sqrt{2\,\frac{\beta}{\alpha}}\,(1-i)\; c2_l^{i} \\ \quad \cdot \exp\left(-\sqrt{\beta/2\alpha}\,(1+i)\,\xi\right) \end{array} \right| \frac{dP_l}{d\theta}$$

(VIB.28a)

and

$$\tilde{\mathcal{U}}^{i}_{\theta 2_1} = \tilde{u}^{i}_{\theta 2_1}(\xi,\theta)\, \exp\left(i\,(t + \eta'' - \eta')\right) \qquad \text{(VIB.28b)}$$

VIC. BOUNDARY/INTERFACE CONDITIONS

The boundary/interface conditions are those that must be applied at the hydrodynamic field level. Of course, these conditions themselves will be linearized, as the hydrodynamic problem itself is linear. The boundary/interface conditions which must be applied are: (a) kinematic condition, (b) continuity of velocity field components across the drop/host medium interface, (c) tangential stress balance across interface, and (d) normal stress balance (including surface tension/curvature contribution) across the interface.

The velocity and pressure fields which are utilized are those of $\underline{u}_2^i$, $\underline{u}_2^o$, p_2^i, p_2^o, given in Sections VIA and VIB.

In addition, the surface forces due to the radiation pressure vector will enter into the boundary/interface conditions. Recall that these are the projection of the acoustic radiation stress tensor upon the surface of the drop. (That is, $\overline{pr}^{RADIAL}$ and $\overline{pr}^{TANG}$, which have been presented in Sections II and IV.)

The velocity (and pressure) field which comprise the hydrodynamic field do include contributions which arise as the result of forcing (of the governing equations) by acoustic field quantities. These contributions arise in the acoustic sublayer. If their contribution to the hydrodynamic problem velocity field were to be neglected, the resulting hydrodynamic flow field would be only that of an oscillating drop (previously investigated by Miller and Scriven, 1968). In the work presented in Sections VIA and VIB, it has been seen that the flow in the "outer" regions (both interior and exterior to the drop) is precisely that of an oscillating drop. Of course, in the work presented in these sections, the contribution to the velocity field arising in the acoustic sublayer region is determined, also.

In the work of Marston (1980), the acoustic field was taken to be strictly inviscid. Then a non-zero contribution to $(\overline{\mathrm{pr}}^{\mathrm{TANG}})_\theta$ does not exist. Moreover, the hydrodynamic governing equations themselves are not forced. Therefore, only $(\overline{\mathrm{pr}}^{\mathrm{radial}})$ will force the oscillating drop solutions in the boundary/interface conditions in the case in which the acoustic field is strictly inviscid.

It is the goal of this section to develop the boundary/interface conditions which apply in the case in which the acoustic field incorporates viscous effects. In doing this, it will be necessary to look at the velocity fields obtained in the previous two sections in more detail.

As a first step, the equilibrium interface will be defined. This will be followed by a presentation of the general form of the interface/boundary conditions. Finally the velocity and pressure fields pertaining to the forced hydrodynamic problem will be utilized in the conditions, and specific equations obtained.

Equilibrium Interface

The equilibrium interface is defined by

$$F_e = r - \tilde{R} - \sum_{\ell=0}^{\infty} K_\ell P_\ell(\cos\theta)\, e^{i(t+\eta''-\eta')} = 0 \qquad \text{(VIC.1)}$$

The third term (following the first equality) represents the time-dependent oscillation of the interface.

Boundary/Interface Conditions: General Form

The general form of the boundary/interface conditions will be presented. In keeping with earlier work, these have been nondimensionalized. They are:

(Continuity of Radial Velocity)

$$U_r^i e^{i(t+\eta''-\eta')} = U_r^o e^{i(t+\eta''-\eta')}$$

$$\text{at} \quad r=\tilde{R},\ \zeta=\xi=0 \qquad \text{(VIC.2a)}$$

(Continuity of Tangential Velocity)

$$U_\theta^i e^{i(t+\eta''-\eta')} = U_\theta^o e^{i(t+\eta''-\eta')}$$

$$\text{at} \quad r=\tilde{R},\ \zeta=\xi=0 \qquad \text{(VIC.2b)}$$

(Kinematic Condition)

$$\frac{\partial F_e}{\partial t} + U_r^i e^{i(t+\eta''-\eta')} = 0$$

or

$$\frac{\partial F_e}{\partial t} + U_r^o e^{i(t+\eta''-\eta')} = 0$$

$$\text{at} \quad r=\tilde{R},\ \xi=\zeta=0 \qquad \text{(VIC.2c)}$$

(Tangential Force Balance)

$$\alpha\left(r\frac{\partial}{\partial r}\left(\frac{U_\theta^i}{r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}(U_r^i)\right)e^{i(t+\eta''-\eta')}$$

$$-\left(r\frac{\partial}{\partial r}\left(\frac{U_\theta^o}{r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}(U_r^o)\right)e^{i(t+\eta''-\eta')} =$$

$$= Re_{HYDR} \left\langle (\overline{pr}^{TANG})_\theta \right\rangle$$

$$= Re_{HYDR}\, \beta \left(\hat{V}^i_{1r_0}\big|_{\tilde{R}} \overline{V^i_{1\theta_0}} + \text{COMPLEX CONJUGATE} \right) e^{i(t+\eta''-\eta')}$$

$$- Re_{HYDR} \left(\hat{V}^o_{1r_0} \overline{V^o_{1\theta_0}} + \text{COMPLEX CONJUGATE} \right) e^{i(t+\eta''-\eta')}$$

$$\text{at } r = \tilde{R},\ \zeta = \xi = 0 \qquad \text{(VIC.2d)}$$

Recall that $\alpha = \mu_o^i/\mu_o^o$ and $\beta = \rho_o^i/\rho_o^o$. U_r^i and U_r^i refer to the time-independent factor in the hydrodynamic field. The forcing due to the tangential component of the radiation pressure vector is shown (on the r.h.s.). Its form is that discussed in Section IV. The "^" over $\hat{V}_{1r_0}^{i,o}$ refers to the contribution due only to the inviscid acoustic field. Clearly, $V_{1\theta_0}^{i,o}$, the $\hat{e}_\theta$ component of the acoustic field, includes viscous effects. That is to say, for example,

$$V^i_{1\theta_0} = \hat{V}^i_{1\theta_0}\big|_{\tilde{R}} + \tilde{\tilde{V}}^i_{1\theta_0}$$

(Normal Force Balance)

$$\left(-p^i + \frac{2\alpha}{Re_{HYDR}} \frac{\partial U_r^i}{\partial r} \right) e^{i(t+\eta''-\eta')} = \left(-p^o + \frac{2}{Re_{HYDR}} \frac{\partial U_r^o}{\partial r} \right) e^{i(t+\eta''-\eta')}$$

$$- G\, \nabla \cdot \underline{\hat{n}}$$

$$+ \left\langle \overline{pr}^{RADIAL} \right\rangle \qquad \text{(VIC.2e)}$$

with $G = (\sigma_o/\rho_o^{\circ}\, d^3\, \omega_{DROP}^2)$, and σ_o° the surface tension associated with the interface between the liquid drop and the host medium. The unit vector $\hat{\underline{n}}$ is the outward pointing normal to the interface. Specifically, $\hat{\underline{n}} = \nabla Fe / |\nabla Fe|$.

Equations (VIC.2a-2e) represent the general form of the boundary/interface conditions which must be satisfied. Note that forcing terms, namely, the $\hat{e}_r$ and $\hat{e}_\theta$ components of the radiation pressure vector, appear in the above system of equations. This is a nonhomogeneous set of linear equations. The unknowns are the coefficients of the velocity and pressure fields. Once these are determined, the hydrodynamic flow field, which exists as a result of the acoustic forcing, will be known.

<u>**Velocity and Pressure Fields**</u>

Recall that the velocity fields in both the "acoustic sublayer" region and "outer" region interior and exterior to the drop have been found - up to the unknown coefficienct - in Sections VIA and VIB. Also, the pressure field in the "outer" region (interior and exterior to the drop) was determined explicitly (also up to a constant.) The pressure field in the "acoustic sublayer" region is of higher order in $(\sqrt{\delta / Re_{HYDR}})$ and will not contribute.

Recall that

$$u^{o}_{2r}\, e^{i(t+\eta''-\eta')} = \sum_{\ell} \left(a_2^{o}\, r^{-\ell-2} + a_4^{o} \frac{1}{r} h_\ell^{(1)}(\alpha r) \right) P_\ell(\cos\theta)\, e^{i(t+\eta''-\eta')} + O(\delta / Re_{HYDR})$$

(VIC.3)

with $s^2 = (-i\ Re_{HYDR})$.

and

$$u^{o}_{2\theta}\, e^{i(t+\eta''-\eta')} = \sum_{\ell} \frac{1}{\ell(\ell+1)} \left\{ \begin{array}{l} -\ell a^{o}_{2}\, r^{-\ell-2} \\ + a^{o}_{4} \left(\frac{(\ell+1)}{r} h^{(1)}_{\ell}(sr) - s h^{(1)}_{\ell+1}(sr) \right) \end{array} \right\} \frac{dP_\ell}{d\theta}\, e^{i(t+\eta''-\eta')}$$

$$+ \left(\frac{\delta}{Re_{HYDR}} \right)^{1/2} \sum_{\ell} \left| \begin{array}{l} d^{o}_{0} \qquad\qquad + d^{o}_{1} \zeta \\ + Re_{HYDR}\, (-1) \sqrt{2}\, (1+i)\, (c1^{o}_{\ell})\, \exp\left(-(1-i)/\sqrt{2}\, \zeta\right) \\ + (Re_{HYDR})(-1) \sqrt{2}\, (1-i)(c2^{o}_{\ell})\, \exp\left(-(1+i)/\sqrt{2}\, \zeta\right) \end{array} \right| \frac{dP_\ell}{d\theta}\, e^{i(t+\eta''-\eta')}$$

(VIC.4a)

However, as $\zeta \to \infty$, the contribution that is in the "acoustic sublayer" region must decay. This implies that $d_0^{\circ} = d_1^{\circ} \equiv 0$. Therefore,

$$u^{o}_{2\theta}\, e^{i(t+\eta''-\eta')} = \sum_{\ell} \frac{1}{\ell(\ell+1)} \left\{ \begin{array}{l} -\ell a^{o}_{2}\, r^{-\ell-2} \\ + \\ a^{o}_{4} \left(\frac{(\ell+1)}{r} h^{(1)}_{\ell}(sr) - s h^{(1)}_{\ell+1}(sr) \right) \end{array} \right\} \frac{dP_\ell}{d\theta}$$

$$\cdot \exp(i(t+\eta''-\eta'))$$

$$+ \left(\frac{\delta}{Re_{HYDR}} \right)^{1/2} \sum_{\ell} \left\{ \begin{array}{l} 2 Re_{HYDR}\, (-1) \frac{(1+i)}{\sqrt{2}}\, (c1^{o}_{\ell})\, \exp\left(-\frac{(1-i)}{\sqrt{2}} \zeta\right) \\ + 2 Re_{HYDR}\, (-1) \frac{(1-i)}{\sqrt{2}}\, (c2^{o}_{\ell})\, \exp\left(-\frac{(1+i)}{\sqrt{2}} \zeta\right) \end{array} \right\} \frac{dP_\ell}{d\theta}\, e^{i(t+\eta''-\eta')}$$

(VIC.4b)

The pressure field exterior to the drop is given by

$$p_2^o e^{i(t+\eta''-\eta')} = \sum_\ell \frac{1}{Re_{HYDR}} \frac{(\Delta^2)}{\ell(\ell+1)} \left(-\ell a_2^o r^{-\ell-1}\right) P_\ell(\cos\theta)\, e^{i(t+\eta''-\eta')}$$

$$+ O\left([\delta/Re_{HYDR}]^{3/2}\right) \qquad \textbf{(VIC.5)}$$

Note that the contribution to the pressure is essentially due to the "outer" region solution.

The velocity and pressure fields interior to the drop must be utilized also. These are

$$u_{2r}^i e^{i(t+\eta''-\eta')}$$

$$= \sum_\ell \left(a_1^i r^{\ell-1} + a_3^i \frac{1}{r} j_\ell(\hat{\Delta} r)\right) P_\ell(\cos\theta)\, e^{i(t+\eta''-\eta')} + O(\delta/Re_{HYDR}) \qquad \textbf{(VIC.6)}$$

with

$$\hat{\Delta}^2 = (-i\, Re_{HYDR}\, \beta/\alpha)$$

$$u_{2\theta}^i e^{i(t+\eta''-\eta')} = \sum_\ell \left\{\frac{1}{\ell(\ell+1)}\right\} \left\{ \begin{array}{l} (\ell+1) a_1^i r^{\ell-1} \\ + \\ a_3^i \left(\frac{(\ell+1)}{r} j_\ell(\hat{\Delta} r) - \hat{\Delta} j_{\ell+1}(\hat{\Delta} r)\right) \end{array} \right\} \frac{dP_\ell}{d\theta} e^{i(t+\eta''-\eta')}$$

$$+ \left(\frac{\delta}{Re_{HYDR}}\right)^{1/2} \sum_\ell \left\{ \begin{array}{l} d_0^i + d_1^i \xi \\ + 2Re_{HYDR} \sqrt{\frac{\beta}{\alpha}} \frac{(1+i)}{\sqrt{2}} (c1_\ell^i) \exp\left(-\sqrt{\frac{\beta}{2\alpha}}(1-i)\xi\right) \\ + 2Re_{HYDR} \sqrt{\frac{\beta}{\alpha}} \frac{(1-i)}{\sqrt{2}} (c2_\ell^i) \exp\left(-\sqrt{\frac{\beta}{2\alpha}}(1+i)\xi\right) \end{array} \right\} \frac{dP_\ell}{d\theta} e^{i(t+\eta''-\eta')}$$

(VIC.7a)

The requirement that the solution which involves the "acoustic sublayer" variable, ξ, decay as $\xi \to \infty$ implies that $d_o^i = d_1^i \equiv 0$. Therefore

$$u_{2\theta}^{i}\, e^{i(t+\eta''-\eta')}$$

$$= \sum_{\ell} \frac{1}{\ell(\ell+1)} \left\{ \begin{array}{l} (\ell+1)\, a_1^i\, r^{\ell-1} \\ + \\ a_3^i \left(\frac{(\ell+1)}{r} j_\ell(\hat{\Delta} r) - \hat{\Delta}\, j_{\ell+1}(\hat{\Delta} r) \right) \end{array} \right\} \frac{dP_\ell}{d\theta}\, e^{i(t+\eta''-\eta')}$$

$$+ \left(\frac{\delta}{Re_{HYDR}} \right)^{1/2} \sum_{\ell} \left\{ \begin{array}{l} 2 Re_{HYDR} \sqrt{\frac{\beta}{\alpha}}\, \frac{(1+i)}{\sqrt{2}} (c1_\ell^i) \exp\left(-\sqrt{\frac{\beta}{2\alpha}} (1-i)\, \xi \right) \\ + \\ 2 Re_{HYDR} \sqrt{\frac{\beta}{\alpha}}\, \frac{(1-i)}{\sqrt{2}} (c2_\ell^i) \exp\left(-\sqrt{\frac{\beta}{2\alpha}} (1+i)\, \xi \right) \end{array} \right\} \frac{dP_\ell}{d\theta}$$

$$\cdot\, e^{i(t+\eta''-\eta')}$$

(VIC.7b)

Finally, the pressure field is given by

$$p_2^{i}\, e^{i(t+\eta''-\eta')}$$

$$= \sum_{\ell} \frac{\alpha}{\ell(\ell+1)} \left((\ell+1) \frac{\hat{\Delta}^2}{Re_{HYDR}} \right) a_1^i\, r^{\ell}\, P_\ell(\cos\theta)\, e^{i(t+\eta''-\eta')} \qquad \text{(VIC.8)}$$

$$+\ O\left([\delta / Re_{HYDR}]^{3/2} \right)$$

Boundary Interface Conditions: Specific Equations

It is necessary to substitute the expressions for $u_{2r}^{i,o}$, $u_{2\theta}^{i,o}$, $p_2^{i,o}$ found in Equations (VIC.3-VIC.8) into the general form of the boundary/interface conditions given by Equations (VIC.2a-2d) in order to obtain the specific system.

Lowest order terms only will be kept.

(Continuity of Radial Velocity)

$$a_1^{i}\,\tilde{R}^{\ell-1} + a_3^{i}\,\frac{1}{\tilde{R}}\,j_\ell(\hat{\Delta}\tilde{R}) = a_2^{o}\,\tilde{R}^{-\ell-2} + a_4^{o}\,\frac{1}{\tilde{R}}\,h_\ell^{(1)}(\Delta\tilde{R})$$

$$\text{at } r=\tilde{R}\,,\ \zeta=\xi=0$$

(VIC.9)

with $\Delta^2 = (-i\,Re_{HYDR})$; $\hat{\Delta}^2 = (-i\,Re_{HYDR}\cdot\beta/\alpha)$

(Continuity of Tangential Velocity)

$$-\ell\,a_2^{o}\,\tilde{R}^{-\ell-2} + \frac{a_4^{o}}{\tilde{R}}\left\{(\ell+1)\,h_\ell^{(1)}(\Delta\tilde{R}) - \Delta\tilde{R}\,h_{\ell+1}^{(1)}(\Delta\tilde{R})\right\} + \cancel{O\left((\delta/Re_{HYDR})^{1/2}\right)}$$

$$= (\ell+1)\,a_1^{i}\,\tilde{R}^{\ell-1} + \frac{a_3^{i}}{\tilde{R}}\left\{(\ell+1)\,j_\ell(\hat{\Delta}\tilde{R}) - \hat{\Delta}\tilde{R}\,j_{\ell+1}(\hat{\Delta}\tilde{R})\right\} + \cancel{O\left((\delta/Re_{HYDR})^{1/2}\right)}$$

$$\text{at } r=\tilde{R},\ \zeta=\xi=0$$

(VIC.10)

(Kinematic Condition)

$$-i K_\ell + a_2^\circ \tilde{R}^{-\ell-2} + a_4^\circ \frac{1}{\tilde{R}} h_\ell^{(1)}(\hat{\lambda}\tilde{R}) = 0$$

$$\text{at } r = \tilde{R},\ \zeta = \xi = 0$$

(VIC.11)

(Tangential Stress Balance)

This requires some further manipulation in order to see clearly the substitution steps. Expanding on Equation (VIC.2d) yields

$$\alpha \left\{ \begin{array}{l} r \frac{\partial}{\partial r}\left(\hat{u}_{2\theta}^{i}\right) - \frac{1}{r}\hat{u}_{2\theta}^{i} - \left(\frac{Re_{HYDR}}{\delta}\right)^{1/2} \frac{\partial}{\partial \xi}\tilde{u}_{2\theta}^{i} \\ - \frac{\tilde{u}_{2\theta}^{i}}{\left(\tilde{R} - (\delta/Re_{HYDR})^{1/2}\xi\right)} + \frac{1}{r}\frac{\partial \hat{u}_{2r}^{i}}{\partial \theta} + \frac{1}{r}\frac{\partial \tilde{u}_{2r}^{i}}{\partial \theta} \end{array} \right\}_{r=\tilde{R},\ \xi=0} e^{i(t+\eta''-\eta')}$$

$$- \left\{ \begin{array}{l} r \frac{\partial}{\partial r}\left(\hat{u}_{2\theta}^{o}\right) - \frac{1}{r}\hat{u}_{2\theta}^{o} + \left(\frac{Re_{HYDR}}{\delta}\right)^{1/2} \frac{\partial}{\partial \zeta}\tilde{u}_{2\theta}^{o} \\ - \frac{\tilde{u}_{2\theta}^{o}}{\left(\tilde{R} + (\delta/Re_{HYDR})^{1/2}\zeta\right)} + \frac{1}{r}\frac{\partial \hat{u}_{2r}^{o}}{\partial \theta} + \frac{1}{r}\frac{\partial \tilde{u}_{2r}^{o}}{\partial \theta} \end{array} \right\}_{r=\tilde{R},\ \zeta=0} e^{i(t+\eta''-\eta')}$$

$$= Re_{HYDR} \left\langle \left(\overline{pr}^{\,TANG}\right)_\theta \right\rangle$$

(VIC.12a)

Of course, after substitution, only the lowest order terms are kept. This leads to

$$
\Bigg(\alpha \left\{ \frac{(\ell-1)}{\ell} a_1^i \tilde{R}^{\ell-2} \right\} \frac{\partial P_\ell}{\partial \theta} e^{i(t+\eta''-\eta')}
$$

$$
+ \alpha \frac{a_3^i}{\ell(\ell+1)} \left\{ \frac{-(\ell+1)}{\tilde{R}^2} j_\ell(\hat{\Delta}\tilde{R}) + \frac{(\ell+1)}{\tilde{R}^2} \left(\ell j_\ell(\hat{\Delta}\tilde{R}) - \hat{\Delta}\tilde{R} j_{\ell+1}(\hat{\Delta}\tilde{R}) \right) - \hat{\Delta}^2 j_\ell(\hat{\Delta}\tilde{R}) + \frac{\hat{\Delta}(\ell+2)}{\tilde{R}^2} j_{\ell+1}(\hat{\Delta}\tilde{R}) \right\} \frac{dP_\ell}{d\theta} e^{i(t+\eta''-\eta')}
$$

$$
+ \alpha \left(\frac{-1}{\ell} \right) \frac{1}{\tilde{R}} a_1^i \tilde{R}^{\ell-1} \frac{dP_\ell}{d\theta} e^{i(t+\eta''-\eta')}
$$

$$
+ \alpha \left(\frac{-1}{\tilde{R}} \right) \frac{a_3^i}{\ell(\ell+1)} \left\{ \frac{(\ell+1)}{\tilde{R}} j_\ell(\hat{\Delta}\tilde{R}) - \hat{\Delta} j_{\ell+1}(\hat{\Delta}\tilde{R}) \right\} \frac{dP_\ell}{d\theta} e^{i(t+\eta''-\eta')}
$$

$$
+ \alpha \left(\frac{1}{\tilde{R}} \right) \left(a_1^i \tilde{R}^{\ell-1} + a_3^i \frac{1}{\tilde{R}} j_\ell(\hat{\Delta}\tilde{R}) \right) \frac{dP_\ell}{d\theta} e^{i(t+\eta''-\eta')}
$$

$$
- 2\, Re_{HYDR} \left(\frac{\beta}{\alpha} \right)^{1/2} \frac{(1+i)}{\sqrt{2}} (c1_\ell^i) \left\{ (-1) \left(\frac{\sqrt{\beta}}{\sqrt{\alpha}} \right) \frac{(1-i)}{\sqrt{2}} \right\} \frac{dP_\ell}{d\theta} e^{i(t+\eta''-\eta')}
$$

$$
- 2\, Re_{HYDR} \left(\frac{\beta}{\alpha} \right)^{1/2} \frac{(1-i)}{\sqrt{2}} (c2_\ell^i) \left\{ (-1) \left(\frac{\beta}{\alpha} \right)^{1/2} \frac{(1+i)}{\sqrt{2}} \right\} \frac{dP_\ell}{d\theta} e^{i(t+\eta''-\eta')} \Bigg) -
$$

$$
\Bigg(\left\{ \frac{(\ell+2)}{(\ell+1)} \, a_2^{\circ} \, \tilde{R}^{-\ell-3} \right\} \frac{dP_\ell}{d\theta} \, e^{i(t+\eta''-\eta')}
$$

$$
+ \frac{a_4^{\circ}}{\ell(\ell+1)} \left\{ \begin{array}{l} -\frac{(\ell+1)}{\tilde{R}^2} h_\ell^{(1)}(\Delta\tilde{R}) + \frac{(\ell+1)}{\tilde{R}^2} \left(\ell h_\ell^{(1)}(\Delta\tilde{R}) - \Delta\tilde{R} \, h_{\ell+1}^{(1)}(\Delta\tilde{R}) \right) \\[2ex] -\Delta^2 h_\ell^{(1)}(\Delta\tilde{R}) + \frac{\Delta}{\tilde{R}} h_{\ell+1}^{(1)}(\Delta\tilde{R}) \end{array} \right\} \frac{dP_\ell}{d\theta} \, e^{i(t+\eta''-\eta')}
$$

$$
+ \left\{ \frac{a_2^{\circ}}{(\ell+1)} \, \tilde{R}^{-\ell-3} \right\} \frac{dP_\ell}{d\theta} \, e^{i(t+\eta''-\eta')}
$$

$$
+ \frac{a_4^{\circ}}{\ell(\ell+1)} (-1) \left\{ \frac{(\ell+1)}{\tilde{R}^2} h_\ell^{(1)}(\Delta\tilde{R}) - \frac{1}{\tilde{R}} h_{\ell+1}^{(1)}(\Delta\tilde{R}) \right\} \frac{dP_\ell}{d\theta} \, e^{i(t+\eta''-\eta')}
$$

$$
+ \left(a_2^{\circ} \, \tilde{R}^{-\ell-3} + a_4^{\circ} \frac{1}{\tilde{R}^2} h_\ell^{(1)}(\Delta\tilde{R}) \right) \frac{dP_\ell}{d\theta} \, e^{i(t+\eta''-\eta')}
$$

$$
+ \left\{ 2 \, \mathrm{Re}_{\mathrm{HYDR}} \, (-1) \frac{(1+i)}{\sqrt{2}} \, (c1_\ell^{\circ}) \, (-1) \frac{(1-i)}{\sqrt{2}} \right\} \frac{dP_\ell}{d\theta} \, e^{i(t+\eta''-\eta')}
$$

$$
+ \left\{ 2 \, \mathrm{Re}_{\mathrm{HYDR}} \, (-1) \frac{(1-i)}{\sqrt{2}} \, (ca_\ell^{\circ}) \, (-1) \frac{(1+i)}{\sqrt{2}} \right\} \frac{dP_\ell}{d\theta} \, e^{i(t+\eta''-\eta')} \Bigg)
$$

$$
= \mathrm{Re}_{\mathrm{HYDR}} \left\langle \left(\overline{pr}^{\,\mathrm{TANG}} \right)_\theta \right\rangle \tag{VIC.12b}
$$

It is understood that there is a summation (over l) in the above equation.

This can be simplified to yield

$$\Bigg(\left\{ \alpha\, a_1^i\, \tilde{R}^{\ell-2}\, \frac{2(\ell-1)}{\ell} \right\} \frac{dP_\ell}{d\theta}\, e^{i(t+\eta''-\eta')}$$

$$+ \left\{ \frac{\alpha\, a_3^i}{\ell(\ell+1)} \left(j_\ell(\hat{\Delta}\tilde{R}) \left\{ \frac{2(\ell^2-1)}{\tilde{R}^2} - \hat{\Delta}^2 \right\} + \frac{2\hat{\Delta}}{\tilde{R}} j_{\ell+1}(\hat{\Delta}\tilde{R}) \right) \right\} \frac{dP_\ell}{d\theta}\, e^{i(t+\eta''-\eta')}$$

$$+ \left\{ \alpha\, 2\, Re_{HYDR} \left(\frac{\beta}{\alpha}\right) (c1_\ell^i + ca_\ell^i) \right\} \frac{dP_\ell}{d\theta}\, e^{i(t+\eta''-\eta')} \Bigg)$$

$$- \Bigg(\left\{ a_2^o\, \tilde{R}^{-\ell-3} \left(\frac{2(\ell+2)}{(\ell+1)} \right) \right\} \frac{dP_\ell}{d\theta}\, e^{i(t+\eta''-\eta')}$$

$$+ \left\{ \frac{a_4^o}{\ell(\ell+1)} \left(\frac{h_\ell^{(1)}(\Delta\tilde{R})}{\tilde{R}^2} \left\{ 2(\ell^2-1) - \Delta^2\tilde{R}^2 \right\} + \frac{2\Delta}{\tilde{R}} h_{\ell+1}^{(1)}(\Delta\tilde{R}) \right) \right\} \frac{dP_\ell}{d\theta}\, e^{i(t+\eta''-\eta')}$$

$$+ \left\{ 2\, Re_{HYDR} (c1_\ell^o + ca_\ell^o) \right\} \frac{dP_\ell}{d\theta}\, e^{i(t+\eta''-\eta')} \Bigg)$$

$$= Re_{HYDR} \left\langle (\overline{pr}^{TANG})_\theta \right\rangle \qquad \text{(VIC.12c)}$$

$$\text{at } r=\tilde{R},\ \zeta=\xi=0$$

But

$$\left\langle \left(\overline{pr}^{TANG}\right)_\theta \right\rangle \Big|_{r=\tilde{R},\ \zeta=\xi=0}$$

$$= 2\, e^{i(t+\eta''-\eta')} \cdot$$

$$\left\{ \begin{array}{l} \beta \sum_{\hat{\ell}=0}^{\infty} \left(\hat{\delta}_{\hat{\ell}}\, \alpha^{i}_{\hat{\ell}}\, \frac{d}{dr}\left(j_{\hat{\ell}}\left(\frac{\omega}{c_0}r\right)\Big|_{\tilde{R}}\, P_{\hat{\ell}}(\cos\theta)\right)\right) \overline{\left(\sum_{\ell'=1}^{\infty} \hat{\delta}_{\ell'}\, A^{i}_{BL_{\ell'}}\, \frac{dP_{\ell'}}{d\theta} \right)} \\ - \left(\sum_{\hat{\ell}=0}^{\infty} \hat{\delta}_{\hat{\ell}} \left\{ \alpha^{o}_{S\hat{\ell}}\, \frac{d}{dr}\left(h^{(1)}_{\hat{\ell}}(r)\Big|_{\tilde{R}} + A_{INC}\frac{d}{dr}\left(j_{\hat{\ell}}(r)\Big|_{\tilde{R}} \right\} P_{\hat{\ell}}(\cos\theta) \right) \cdot \right. \\ \qquad\qquad \overline{\left(\sum_{\ell'=1}^{\infty} \hat{\delta}_{\ell'}\, B^{o}_{BL_{\ell'}}\, \frac{dP_{\ell'}}{d\theta} \right)} \\ + \text{COMPLEX CONJUGATE} \end{array} \right\}$$

(VIC.13a)

$$= 2\, e^{i(t+\eta''-\eta')}$$

$$\cdot \left\{ \begin{array}{l} \beta \left(\sum_{L=1}^{\infty} C^{i}_{L} \frac{dP_L}{d\theta} + \sum_{L=1}^{\infty} \overline{C}^{i}_{L} \frac{dP_L}{d\theta} \right) \\ - \left(\sum_{L=1}^{\infty} C^{o}_{L} \frac{dP_L}{d\theta} + \sum_{L=1}^{\infty} \overline{C}^{o}_{L} \frac{dP_L}{d\theta} \right) \end{array} \right\}$$

(VIC.13b)

Therefore, multiplying by $\frac{dP_{\mathcal{L}}}{d\theta}$, and integrating over (o, π) yields, after utilizing orthogonality properties,

$$\Bigg(\alpha a_1^i \tilde{R}^{\ell-2} \frac{2(\ell-1)}{\ell} \int_0^\pi \frac{dP_\ell}{d\theta} \sin\theta \frac{dP_\ell}{d\theta} d\theta$$

$$+ \frac{\alpha a_3^i}{\ell(\ell+1)} \left\{ \frac{j_\ell(\hat{\Delta}\tilde{R})}{\tilde{R}^2} \left(2(\ell^2-1) - \hat{\Delta}^2\tilde{R}^2 \right) + \frac{2\hat{\Delta}}{\tilde{R}} j_{\ell+1}(\hat{\Delta}\tilde{R}) \right\}$$

$$\cdot \int_0^\pi \frac{dP_\ell}{d\theta} \sin\theta \frac{dP_\ell}{d\theta} d\theta$$

$$+ 2(Re_{HYDR})\, \beta\, (C1_\ell^i + C2_\ell^i) \int_0^\pi \frac{dP_\ell}{d\theta} \sin\theta \frac{dP_\ell}{d\theta} d\theta \Bigg)$$

$$- \Bigg(a_2^o \tilde{R}^{-\ell-3} \frac{2(\ell+2)}{(\ell+1)} \int_0^\pi \frac{dP_\ell}{d\theta} \sin\theta \frac{dP_\ell}{d\theta} d\theta$$

$$+ \frac{a_4^o}{\ell(\ell+1)} \left\{ \frac{h_\ell^{(1)}(\Delta\tilde{R})}{\tilde{R}^2} \left(2(\ell^2-1) - \Delta^2\tilde{R}^2 \right) + \frac{2\Delta}{\tilde{R}} h_{\ell+1}^{(1)}(\Delta\tilde{R}) \right\}$$

$$\cdot \int_0^\pi \frac{dP_\ell}{d\theta} \sin\theta \frac{dP_\ell}{d\theta} d\theta$$

$$+ 2\, Re_{HYDR} (C1_\ell^o + C2_\ell^o) \int_0^\pi \frac{dP_\ell}{d\theta} \sin\theta \frac{dP_\ell}{d\theta} d\theta \Bigg)$$

$$= \left(Re_{HYDR}\, 2\beta \int_0^\pi \left(\frac{dP_\ell}{d\theta} \sin\theta \right) \left(\sum_{L=1}^\infty C_L^i \frac{dP_L}{d\theta} \right) d\theta + \text{COMPLEX CONJUGATE} \right)$$

$$- \left(Re_{HYDR} (2) \int_0^\pi \left(\frac{dP_\ell}{d\theta} \sin\theta \right) \left(\sum_{L=1}^\infty C_L^o \frac{dP_L}{d\theta} \right) d\theta + \text{COMPLEX CONJUGATE} \right) \quad \text{(VIC.14)}$$

at $r = \tilde{R}$, $\zeta = \xi = 0$

However, recall that

$$C1_{\ell}^{i} = \frac{(2\ell+1)}{2\ell(\ell+1)} \int_0^{\pi} \left(\sum_{L=1}^{\infty} c_L^{i} \frac{dP_L}{d\theta} \right) \frac{dP_\ell}{d\theta} \sin\theta \, d\theta$$

$$C2_{\ell}^{i} = \frac{(2\ell+1)}{2\ell(\ell+1)} \int_0^{\pi} \left(\sum_{L=1}^{\infty} \overline{C}_L^{i} \frac{dP_L}{d\theta} \right) \frac{dP_\ell}{d\theta} \sin\theta \, d\theta$$

and

$$C1_{\ell}^{o} = \frac{(2\ell+1)}{2\ell(\ell+1)} \int_0^{\pi} \left(\sum_{L=1}^{\infty} c_L^{o} \frac{dP_L}{d\theta} \right) \frac{dP_\ell}{d\theta} \sin\theta \, d\theta$$

$$C2_{\ell}^{o} = \frac{(2\ell+1)}{2\ell(\ell+1)} \int_0^{\pi} \left(\sum_{L=1}^{\infty} \overline{C}_L^{o} \frac{dP_L}{d\theta} \right) \frac{dP_\ell}{d\theta} \sin\theta \, d\theta$$

Therefore, taking $\mathscr{L} = \ell$

$$\left(\frac{2\ell(\ell+1)}{(2\ell+1)}\right)\Bigg(\alpha a_1^i \tilde{R}^{\ell-2} \frac{2(\ell-1)}{\ell} + \frac{\alpha a_3^i}{\ell(\ell+1)}\left\{ \frac{j_\ell(\hat{\Delta}\tilde{R})}{\tilde{R}^2}\left(2(\ell^2-1) - \hat{\Delta}^2\tilde{R}^2\right) + \frac{2\hat{\Delta}}{\tilde{R}} j_{\ell+1}(\hat{\Delta}\tilde{R})\right\} + 2\,Re_{HYDR}\,\beta\,(Cl_\ell^i + Ca_\ell^i) \Bigg)$$

$$-\left(\frac{2\ell(\ell+1)}{(2\ell+1)}\right)\Bigg(a_2^o \tilde{R}^{-\ell-3} \frac{2(\ell+2)}{(\ell+1)} + \frac{a_4^o}{\ell(\ell+1)}\left\{ \frac{h_\ell^{(1)}(\Delta\tilde{R})}{\tilde{R}^2}\left(2(\ell^2-1) - \Delta^2\tilde{R}^2\right) + \frac{2\Delta}{\tilde{R}^2} h_{\ell+1}^{(1)}(\Delta\tilde{R})\right\} + 2\,Re_{HYDR}\,(Cl_\ell^o + Ca_\ell^o) \Bigg)$$

$$= 2\beta\,Re_{HYDR}\,(Cl_\ell^i + Ca_\ell^i)\left(\frac{2\ell(\ell+1)}{(2\ell+1)}\right) - 2\,Re_{HYDR}\,(Cl_\ell^o + Ca_\ell^o)\left(\frac{2\ell(\ell+1)}{(2\ell+1)}\right) \qquad \text{(VIC.15)}$$

The factor [2l(l+1)/(2l+1)] appears in each term (and can be cancelled). This yields

$$
\left(\alpha \frac{(2)(\ell-1)}{\ell} \tilde{R}^{\ell-2}\right) a_1^i
$$

$$
+ \frac{\alpha}{\ell(\ell+1)} \left\{ \frac{j_\ell(\hat{\lambda}\tilde{R})}{\tilde{R}^2} \left(2(\ell^2-1) - \hat{\lambda}^2 \tilde{R}^2 \right) + \frac{2\hat{\lambda}}{\tilde{R}} j_{\ell+1}(\hat{\lambda}\tilde{R}) \right\} a_3^i
$$

$$
+ \cancel{2\beta Re_{HYDR}\, C1_\ell^i} + \cancel{2\beta Re_{HYDR}\, C2_\ell^i}
$$

$$
= \left(\frac{2(\ell+2)\tilde{R}^{-\ell-3}}{(\ell+1)} \right) a_2^o
$$

$$
+ \frac{1}{\ell(\ell+1)} \left\{ \frac{h_\ell^{(1)}(\lambda\tilde{R})}{\tilde{R}^2} \left(2(\ell^2-1) - \lambda^2 \tilde{R}^2 \right) + \frac{2\lambda}{\tilde{R}} h_{\ell+1}^{(1)}(\lambda\tilde{R}) \right\} a_4^o
$$

$$
+ \cancel{2\, Re_{HYDR}\, C1_\ell^o} + \cancel{2\, Re_{HYDR}\, C2_\ell^o}
$$

$$
+ \left\{ \cancel{2\beta Re_{HYDR}\, C1_\ell^i} + \cancel{2\beta Re_{HYDR}\, C2_\ell^i} - \cancel{2\, Re_{HYDR}\, C1_\ell^o} - \cancel{2\, Re_{HYDR}\, C2_\ell^o} \right\} \qquad \textbf{(VIC.20)}
$$

Note that the contributions due to the velocity field gradient in the acoustic sublayer, which are "$2\, \beta\, Re_{HYDR}\, (c1_l^i + c2_l^i)$" on the left hand side and "$2\, Re_{HYDR}\, (c1_l^o + c2_l^o)$" on the right hand side are exactly cancelled by the tangential component of the radiation pressure vector which appears as "$2\, \beta\, Re_{HYDR}\, (c1_l^i + c2_l^i) - 2\, Re_{HYDR}\, (c1_l^o + c2_l^o)$" on the right hand side!

Therefore, the net effect is that the tangential stress resulting from the velocity gradient in the acoustic sublayer region variable cancels the tangential component of the radiation pressure vector at the interface. Both of these effects owe their existence to the inclusion of viscous effects in the acoustic field.

As far as this particular boundary/interface condition is concerned, there is no change from the case in which the acoustic field itself was taken to be strictly inviscid.

It must be noted that the phase of the response has been set to be equal to the phase of the forcing. If there were to be a phase lag, then the exact cancellation would not occur.

Rather, let $\eta_{\mathrm{IMPOSED}} = \eta'' - \eta'$ and let $\eta_{\mathrm{RESPONSE}} = \eta_R$ be unspecified. In the case in which $\eta_R = \eta_{\mathrm{IMPOSED}}$, total cancellation occurs. However, if $\eta_R \neq \eta_{\mathrm{IMPOSED}}$ ($= \eta_{\mathrm{IM}}$), then the tangential stress balance will be given by

$$\left(\alpha \frac{2(\ell-1)}{\ell} \tilde{R}^{\ell-2} \right) a_1^i$$

$$+ \frac{\alpha}{\tilde{R}^2 \ell(\ell+1)} \left(j_\ell(\hat{\delta}\tilde{R}) \left[2(\ell^2-1) - \hat{\delta}^2 \tilde{R}^2 \right] + 2\hat{\delta}\tilde{R}\, j_{\ell+1}(\hat{\delta}\tilde{R}) \right)$$

$$- \left(\frac{2(\ell+2)}{(\ell+1)} \tilde{R}^{-\ell-3} \right) a_2^o$$

$$- \frac{1}{(\ell)(\ell+1)\tilde{R}^2} \left(h_\ell^{(1)}(\delta\tilde{R}) \left[2(\ell^2-1) - \delta^2 \tilde{R}^2 \right] + 2\delta\tilde{R}\, h_{\ell+1}^{(1)}(\delta\tilde{R}) \right)$$

$$= -2(Re_{HYDR})\beta(C1_\ell^i + C2_\ell^i) + 2Re_{HYDR}(C1_\ell^o + C2_\ell^o) \qquad \text{(VIC.17)}$$

$$+ 2Re_{HYDR}\Big[\beta(C1_\ell^i + C2_\ell^i)\exp(i(\eta_{IM} - \eta_R)) - (C1_\ell^o + C2_\ell^o) \cdot \exp(i(\eta_{IM} - \eta_R))\Big]$$

That is, if there is a phase lag between the forcing and the response, then the tangential stress balance becomes a forced equation (in unknowns (a_2°, a_4°, a_1°, a_3^{i})), with

Forcing term

$$= 2\, Re_{HYDR} \cdot \left(c1_\ell^{\circ} + c2_\ell^{\circ} - \beta (c1_\ell^{i} + c2_\ell^{i}) \right) \cdot \left[1 - \exp\left(i (\eta_{IM} - \eta_R) \right) \right]$$

(VIC.18)

(Normal Force Balance)

$$-p_2^{i}\, e^{i(t+\eta_R)} + \frac{2\alpha}{Re_{HYDR}} \left(\frac{du_{2r}^{i}}{dr} + O(\delta/Re_{HYDR})^{1/2} \right) e^{i(t+\eta_R)}$$

$$= -p_2^{\circ}\, e^{i(t+\eta_R)} + \frac{2}{Re_{HYDR}} \left(\frac{du_{2r}^{\circ}}{dr} + O(\delta/Re_{HYDR})^{1/2} \right) e^{i(t+\eta_R)}$$

$$- G\, \nabla \cdot \underline{\hat{n}} + \hat{\Delta pr}^{RADIAL}\, e^{i(t+\eta_{IM})}$$

(VIC.19a)

with $(\hat{\overline{pr}}^{radial})$ the time independent contribution of the radial component of the radiation pressure vector.

Notice that in this section, the possibility of a phase lag between the forcing term and the response is acknowledged explicitly.

After substitution, the normal force balance equation becomes

$$-\left(\frac{\alpha\,\hat{\Delta}^2}{l\ Re_{HYDR}}\tilde{R}^l\right) a_1^i\ P_l(\cos\theta)\, e^{i(t+\eta_R)}$$

$$+\frac{2\alpha}{Re_{HYDR}}\left\{(l-1)\tilde{R}^{l-2}a_1^i + a_3^i\left(\frac{(l-1)}{\tilde{R}^2}j_l(\hat{\Delta}\tilde{R}) - \frac{\hat{\Delta}}{\tilde{R}}j_{l+1}(\hat{\Delta}\tilde{R})\right)\right\} P_l(\cos\theta)\, e^{i(t+\eta_R)}$$

$$= \left(\frac{\Delta^2}{Re_{HYDR}}\frac{\tilde{R}^{-l-1}}{(l+1)}\right) a_2^o\ P_l(\cos\theta)\, e^{i(t+\eta_R)}$$

$$+\frac{2}{Re_{HYDR}}\left\{-(l+2)\tilde{R}^{-l-3}a_2^o + \frac{a_4^o}{\tilde{R}^2}\left((l-1)h_l^{(1)}(\Delta\tilde{R}) - \Delta\tilde{R}\,h_{l+1}^{(1)}(\Delta\tilde{R})\right)\right\} P_l(\cos\theta)\, e^{i(t+\eta_R)}$$

$$+\frac{G}{\tilde{R}^2}(l+2)(l-1)\,k_l\,P_l(\cos\theta)\, e^{i(t+\eta_R)}$$

$$+\ (\hat{\overline{pr}}^{RADIAL})\, e^{i(t+\eta_{IM})}$$

(VIC.19b)

It is understood that there is a summation sign (over l) in the above equation. Recall that β^2 = -i Re_{HYDR} and $\hat{\beta}^2$ = -i Re_{HYDR} (β/α).

Multiply (VIC.19b) through by (P_{ℓ} sin θ) and integrate over (o,π). Use of orthogonality properties yields

$$\left(-\frac{\alpha}{\ell}\frac{\beta^2}{Re_{HYDR}}\tilde{R}^{\ell}\right)a_1^i$$

$$+\frac{2\alpha}{Re_{HYDR}}(\ell-1)\tilde{R}^{\ell-2}a_1^i$$

$$+\frac{2\alpha}{Re_{HYDR}}\frac{1}{\tilde{R}^2}\left((\ell-1)j_\ell(\hat{\Delta}\tilde{R})-\hat{\Delta}\tilde{R}\,j_{\ell+1}(\hat{\Delta}\tilde{R})\right)a_3^i$$

$$-\left(\frac{\Delta^2}{Re_{HYDR}}\right)\frac{\tilde{R}^{-\ell-1}}{(\ell+1)}a_2^o \quad + 2(\ell+2)\tilde{R}^{-\ell-3}a_2^o$$

$$-\left(\frac{2}{Re_{HYDR}}\right)\frac{1}{\tilde{R}^2}\left((\ell-1)h_\ell^{(1)}(\Delta\tilde{R})-\Delta\tilde{R}\,h_{\ell+1}^{(1)}(\Delta\tilde{R})\right)a_4^o$$

$$-\frac{G(\ell+2)(\ell-1)K_\ell}{\tilde{R}^2}$$

$$=\left(\int_0^\pi \hat{pr}^{RADIAL}P_\ell(\cos\theta)\cdot\sin\theta\,d\theta\right)e^{i(\eta_{IM}-\eta_R)}\frac{(2\ell+1)}{2}$$

$$=\hat{pr}^{RADIAL}e^{i(\eta_{IM}-\eta_R)}$$

(VIC.19c)

Clearly, this can be simplified to

$$a_1^i \ \tilde{R}^l \left(i\frac{\beta}{\alpha} + \frac{2(l-1)\alpha}{Re_{HYDR}} \frac{1}{\tilde{R}^2} \right)$$

$$+ \ a_3^i \ \frac{2\alpha}{Re_{HYDR}} \left(\frac{1}{\tilde{R}^2}\right) \left\{ (l-1)\, j_l(\hat{\lambda}\tilde{R}) - \hat{\lambda}\tilde{R}\, j_{l+1}(\hat{\lambda}\tilde{R}) \right\}$$

$$+ \ a_2^o \ \tilde{R}^{-l-1} \left(\frac{i}{Re_{HYDR}} \frac{1}{(l+1)} + \frac{2(l+2)}{\tilde{R}^2} \right) a_2^o$$

$$+ \ a_4^o \left(\frac{2}{Re_{HYDR}}\right) \left(\frac{1}{\tilde{R}^2}\right) \left((l-1)\, h_l^{(1)}(\lambda\tilde{R}) - \lambda\tilde{R}\, h_{l+1}(\lambda\tilde{R}) \right)$$

$$- \ \frac{G}{\tilde{R}^2} (l+2)(l-1)\, K_l \ = \ \hat{p}r^{RADIAL} \ exp\left(i(\eta_{IM} - \eta_R) \right)$$

(VIC.19d)

with $s^2 = -i\, Re_{HYDR}$ and $\mathfrak{s}^2 = -i\, Re_{HYDR}\, (\beta/\alpha)$.

Looking more closely at $\hat{pr}^{radial}$, it is seen to be the projection of the radial component of the radiation pressure vector onto the "P_ℓ" mode (or Y_{l_0} mode, if spherical harmonics required). It is of the form

$$\hat{pr}^{RADIAL} = 2\int_0^\pi P_\ell(\cos\theta)\cdot\sin\theta\cdot \left\{ \begin{aligned} &\frac{\beta}{2}\left(\hat{v}^{i}_{1r_0|\tilde{R}}\overline{\hat{v}^{i}_{1r_0|\tilde{R}}} + v_{1\theta_0}v_{1\theta_0} + c.c.\right) \\ &+\frac{\hat{\beta}^{i}}{2}\left(p_1^{i}\overline{p_1^{i}} + c.c.\right) \\ &-\frac{1}{2}\left(\hat{v}^{o}_{1r|\tilde{R}}\overline{\hat{v}^{o}_{1r_0|\tilde{R}}} + v_{1\theta_0}v_{1\theta_0} + c.c.\right) \\ &-\frac{\hat{\beta}^{o}}{2}\left(p_1^{o}\overline{p_1^{o}} + c.c.\right) \end{aligned} \right\} d\theta \cdot \left(\frac{2l+1}{2}\right)$$

(VIC.20)

with $\hat{\beta}^{i,o}$ the adiabatic compressibility, and the subscript "1" indicating the acoustic field. Now, to this order of approximation; $v_{1r}^{i,o}$ remains the same as if only the inviscid acoustic field were taken into account. It is in $v_{1\theta}^{i,o}$ that the viscous correction to the acoustic field will contribute.

Therefore, although the <u>form</u> of the forcing term is the same as in the case in which the acoustic field is inviscid, it is in fact "corrected" by inclusion of the terms due to viscous acoustic field contributions.

It is remarked that the drop is taken to be spherical. The actual deformation to the drop was determined in Section V. Calculations indicated that for a millimeter drop, the deformation from sphericity is on the order of microns; and is therefore neglected as a possible modifying agent in the boundary/interface conditions.

Final Comments

It is helpful to the purpose of further discussion to reprise the final form of the boundary/interface conditions in one subsection. They are given by

$$\tilde{R}^{\ell} a_1^i + j_\ell(\hat{\lambda}\tilde{R})\, a_3^i - \tilde{R}^{-\ell-1} a_2^o - h_\ell^{(1)}(\lambda\tilde{R})\, a_4^o = 0 \qquad \text{(VIC.21a)}$$

$$\text{at } r = \tilde{R},\ \zeta = \xi = 0$$

$$(\ell+1)\tilde{R}^{\ell} a_1^i + \left((\ell+1) j_\ell(\hat{\lambda}\tilde{R}) - \hat{\lambda}\tilde{R}\, j_{\ell+1}(\hat{\lambda}\tilde{R}) \right) a_3^i$$

$$+ \left(\ell \tilde{R}^{-\ell-1}\right) a_2^o - \left((\ell+1) h_\ell^{(1)}(\lambda\tilde{R}) - \lambda\tilde{R}\, h_{\ell+1}^{(1)} \right) a_4^o = 0$$

$$\text{at } r = \tilde{R},\ \zeta = \xi = 0 \qquad \text{(VIC.21b)}$$

$$-i K_\ell \tilde{R} + \tilde{R}^{-\ell-1} a_2^o + \frac{1}{\tilde{R}} h_\ell^{(1)}(\lambda\tilde{R})\, a_4^o = 0 \qquad \text{(VIC.21c)}$$

$$\text{at } r = \tilde{R},\ \zeta = \xi = 0$$

$$\left\{ \frac{\alpha\, 2(\ell-1)}{\ell} \tilde{R}^{\ell} \right\} a_1^i$$

$$+ \frac{\alpha}{\ell(\ell+1)} \left\{ \begin{array}{l} j_\ell(\hat{\Delta}\tilde{R}) \left(2(\ell^2-1) - \hat{\Delta}^2 \tilde{R}^2 \right) \\ + 2\tilde{R}\hat{\Delta}\, j_{\ell+1}(\hat{\Delta}\tilde{R}) \end{array} \right\} a_3^i$$

$$+ \left(- \frac{2(\ell+2)}{(\ell+1)} \tilde{R}^{-\ell-1} \right) a_2^o$$

$$- \frac{1}{(\ell)(\ell+1)} \left\{ \begin{array}{l} h_\ell^{(1)}(\Delta\tilde{R}) \left(2(\ell^2-1) - \Delta^2 \tilde{R}^2 \right) \\ + 2\Delta\tilde{R}\, h_{\ell+1}^{(1)}(\Delta\tilde{R}) \end{array} \right\} a_4^o$$

$$= \tilde{R}^2 \left(2 Re_{HYDR} \right) \left(1 - \exp(i[\eta_{IM} - \eta_R]) \right) \left(c1_\ell^o + c2_\ell^o - \beta(c1_\ell^i + c2_\ell^i) \right)$$

$$\text{at } r = \tilde{R},\ \zeta = \xi = 0 \qquad \text{(VIC.21d)}$$

$$\left\{ \frac{i\beta}{\ell} + \frac{2\alpha(\ell-1)}{Re_{HYDR}} \frac{1}{\tilde{R}^2} \right\} \tilde{R}^{\ell} a_1^i + \left\{ \frac{i}{Re_{HYDR}} \frac{1}{(\ell+1)} + \frac{2(\ell+2)}{\tilde{R}^2} \right\} \tilde{R}^{-\ell-1} a_2^o$$

$$+ \frac{2\alpha}{Re_{HYDR}} \frac{1}{\tilde{R}^2} \left((\ell-1) j_\ell(\hat{\Delta}\tilde{R}) - \hat{\Delta}\tilde{R}\, j_{\ell+1}(\hat{\Delta}\tilde{R}) \right)$$

$$- \frac{2}{Re_{HYDR}} \frac{1}{\tilde{R}^2} \left((\ell-1) h_\ell^{(1)}(\Delta\tilde{R}) - \Delta\tilde{R}\, h_{\ell+1}^{(1)}(\hat{\Delta}\tilde{R}) \right) - \frac{G_1(\ell+2)(\ell-1)}{\tilde{R}^2} k_\ell \qquad \text{(VIC.21e)}$$

$$= \hat{pr}^{RADIAL} \exp(i[\eta_{IM} - \eta_R]) \qquad \text{at } r = \tilde{R},\ \zeta = \xi = 0$$

Clearly, this is a forced system in five unknowns: $a_{1\ell}^{i}$, $a_{3\ell}^{i}$, $a_{2\ell}^{o}$, $a_{4\ell}^{o}$, and κ_l. If the forcing terms are set equal to zero, the problem reduces to that of free drop oscillations addressed by Miller and Scriven (1968).

In comparing, it must be noted that they did not nondimensionalize. Also, they found it more convenient to work in terms of surface divergence conditions. (However, their results can be recovered from those listed in Equations (VIC.21a-21e), provided the forcing terms are set equal to zero, the system re-dimensionalized, and some manipulations performed.)

If there is <u>no</u> phase lag between the time periodic forcing and the time periodic response; then <u>only</u> the normal force balance equation is forced. However, even in this case the results do differ from those in which the acoustic field is considered strictly inviscid - that is, the viscous acoustic field would modify the forcing in Equation VIC.21e).

To summarize, the inclusion of viscosity in the acoustic field produces three effects:

1.) in the "acoustic sublayer" region, the velocity field is enhanced

2.) the tangential stress balance equation, taken at the drop/host interface, $r = \tilde{R}$, $\xi = \zeta = 0$. (linearized case), is forced by the (projection of the) tangential component of the radiation pressure vector <u>provided</u> there is a phase difference between the time periodic forcing and the time periodic response.

and

3.) $V_{1\theta_o}^{ip}$ which occurs in the forcing of the normal stress balance is modified by inclusion of viscosity in the acoustic field.

VII. CONCLUSIONS

The focus of this project has been to elucidate the nature of the flow field (both interior and exterior to the drop) for the case in which acoustic radiation pressure forcing due to the tangential radiation stress is incorporated.

It is clear from the physics of this problem that viscous effects in the acoustic field must be included in order to have a non-zero tangential radiation pressure contribution. Prior analytical work in this area is represented by the major work of Marston (1980), and several other related later papers (Marston et. al., 1981, 1982). In this prior work, the acoustic field was assumed to be inviscid. Thus, no tangential stresses could be taken into acount due to the nature of the formulation.

Physically, it is recognized that the flow field of the oscillating drop only exists if the acoustic standing wave field itself is modulated, with the difference between the acoustic wave frequencies that of the natural oscillation frequency of the drop. In the mathematical formulation of this situation, prior work has noted that, were a consistent formulation to be developed, there would be no forcing of the hydrodynamic field vorticity equation since the acoustic field was taken to be inviscid. Therefore, the previous analytical formulation assumed that the hydrodynamic flow field would be adequately represented by the known solution of the freely oscillating drop; and simply proceeded to force this solution by adding the forcing term the boundary/interface conditions.

In addition to the primary goal of this research project which did determine the viscous acoustic field and elucidate the nature of the resulting hydrodynamic field forcing, certain side topics were investigated briefly. These include the radial forcing only of a compound drop with core/shell of the same density and compressibility (presented in Appendix II) and a discussion of the effects of non-axisymmetric acoustic forcing. It was seen that shear waves could be excited were the acoustic forcing to be non-axisymmetric.

Work done in this project has determined the viscous correction to the acoustic field. These results are presented in Section III. It is remarked that in order to accomplish this task, an acoustic boundary layer/method of composite expansion technique was used.

The acoustic boundary layer/composite expansion approach was developed as an alternative to an exact anlytical method approach involving an acoustic field decomposition that was tried initially. The results of this first approach are listed in Appendix I. Briefly, this first approach was not useful because it was too difficult to implement the resulting boundary/interface conditions numerically - some terms proved to be extremely small relative to others in the equation. It is this result which pointed in the direction of a boundary layer/composite type expansion approach.

Results obtained for the acoustic field with viscous effects (Section III) included show that the solutions for the "inner" region <u>decay exponentially</u> as the inner region variable goes to infinity. An inner region solution was found for both regions interior and exterior to the drop. Since the drop is liquid as opposed to a solid sphere, the perhaps more typical case of imposing the no-slip condition is replaced by solving a set of boundary/interface conditions. One interesting result of this process was that the boundary/interface conditions of (1) the radial component of the acoustic velocity field balance at the interface and (2) the normal force balance at the interface were unchanged from what they had been for the case in which the acoustic field is strictly inviscid. (Of course, in the work of this project, nondimensionalizations have been done. This must be remembered in any comparisons with the literature.) The inclusion of viscous effects in the acoustic field is manifest in the boundary/interface acoustic field conditions of (1) tangential $(\hat{e}_\theta)$ component of the acoustic velocity field balance at the interface, and (2) the tangential stress balance at the interface. It was determined that the tangential stress balance is due only to terms which arise via incorporation of the viscous effects. In the tangential velocity balance, both terms which arise due to viscous effects (i.e., the inner region contribution) and to the inviscid formulation (i.e., the outer region contribution) enter. It is through the solution of these boundary/interface conditions that the unknown coefficients are found, and the acoustic field characterized.

Of the inner region dependent field quantities, only the tangential component of the acoustic velocity is of order one (in the inner region, it decays to zero as the inner region independent variable goes to infinity). All others of lower order.

The results of Section III which give the viscous correction effects to the acoustic field, although interesting in their own right, are absolutely necessary in order to determine the hydrodynamic field. It is in Sections V and VI that the structure of the hydrodynamic field itself is elucidated.

Before addressing the hydrodynamic field structure, the ramifications of incorporating viscosity into the acoustic field description on the tangential radiation stress tensor and on the radiation stress vector were explored in Section IV. The form of the tangential radiation stress was exhibited.

It must be kept in mind that few examples in the literature exist on real or even supposed experimental systems. The work of Marston and others has utilized p-xylene drops in water and silicone oil drops in water. The calculations of relevant quantities in the inviscid acoustic field approximation - are minimal in the existing literature. A caluclation due to Marston (1980) and later improved/corrected by Marston et. al., (1981) for a p-xylene drop in water exists. It is this result which forms the basis for a quantitative comparison. The work in Section IV has proceeded to calculate the forcing terms in the boundary interface conditions only. This is an intermediate step which may be done before actually solving for the new hydrodynamic field itself.

The comparison between the calculations of this project for a one mm drop of p-xylene in water and those of Marston (1980, 1981) were done in Section IV. The results between the work of this project and Marston's were found to be quite close for the calculation of the radial component of the radiation pressure vector, which was on the order of (.1) dynes/cm^2 for a carrier wave amplitude (in pressure) of 10^5 dynes/cm^2. This comparison utilized terms found in the inviscid acoustic field approximation only. In the calculation of the effect of the tangential stress upon the drop deformation, it is necessary to utilize the viscous acoustic field. It was found that effect is of the order of between (.01) to (0.1) dynes/cm^2. The formulation of Marston (1980, 1981) could not and does not include a viscous acoustic field, and so did not calculate this quantity.

It is possible to do a limited parameter study of "theoretical" systems. Because no comparisons exist with experiments, the results are not being presented here, but will be found in the MS thesis (by Ferguson).

The results of Section IV show that the viscous acoustic field contributes tangential stresses which have an effect upon drop deformation, as well as other quantities.

It is the actual structure of the hydrodynamic field which has been probed in Sections V and VI.

In the work of Marston, the flow field is taken to be that of a freely oscillating/decaying viscous drop which is forced via the boundary/interface conditions. No modification of the hydrodynamic field itself occurs due to the forcing of the Navier-Stokes equation by functions quadratic in acoustic field quantities. This entirely consistent with the level of approximation done by Marston; but such an approach cannot incorporate tangential forcing, which is the primary goal of this project.

A formal expansion scheme in the small parameter δ, which represented $(\omega_{DROP}/\omega_{ACOUSTIC})$ was utilized in obtaining the forced hydrodynamic field incompressible Navier-Stokes equation.

It is noted that a re-normalization was performed at this stage. This involved a re-nondimensionalization with respect to hydrodynamic field reference quantities as opposed to acoustic field reference quantities. Following this, the following restriction was made that $Re_{HYDRO}\left(-\ \omega_{DROP}\ d^2/\nu_o\right)$ is of order one.

A detailed discussion relating this to streaming Reynolds number as well as Re_{AC} was done in Sections V and VI. The relationship of this work to that of Riley was explored.

The resulting forced equations were solved. It was seen that the forcing of the equations by acoustic field quantities only existed in an acoustic sublayer region; outside of this region the flow field is given by either Stokes flow (if the acoustic field was unmodulated) or the freely oscillating drop flow field of Miller and Scriven (if the acoustic field was modulated).

This, then, represents the modification in the hydrodynamic field due to acoustic field forcing. Moreover, is is seen that the correction to the hydrodynamic velocity field is smaller than order one in the acoustic sublayer region. Moreover, these contributions decay as the "acoustic sublayer" region is left. It is the tangential component of velocity that is the largest in this sublayer region, and therefore the tangential component of the hydrodynamic field which receives the most enhancement from the acoustic forcing. However, this largest enhancement is smaller than order one.

The application of boundary/interface conditions at this hydrodynamic field level produces a very interesting result.

The boundary/interface conditions at the hydrodynamic field level are the kinematic condition, the velocity component balance at the interface, and the stress balances at the interface. It is the normal force balance across the interface which includes the surface tension/curvature contribution. These quantities are all evaluated at the interface.

There are also contributions due to the radiation pressure vector; the radial component of which forces the normal force balance equation. It is the $\hat{e}_\theta$ tangential component of the radiation pressure vector which forces the tangential stress balance equation.

The radiation pressure terms have an associated (constant representing the) phase, and the hydrodynamic field variables have an associated phase; in general these are not equal.

The effect of the viscous acoustic field contributes most dramatically to the tangential stress balance. Not only do the tangential velocity components resulting from acoustic field forcing in the sublayer appear in this equation, but also the tangential component of the radiation pressure vector serves to force this equation. Of course, the tangential velocity components associated with the standard oscillating drop solution also contribute.

After the evaluation at the interface, it is seen that the velocity field modification terms due to the acoustic forcing in the sublayer are the same but opposite in sign from the terms involved in the tangential radiation pressure forcing - up to the phase angle. If the phase of the response (i.e., the hydrodynamic field) is the same as the phase of the forcing, then the contributions would identically cancel! This would result in the effect of the acoustic forcing being present only through the radial component of the radiation pressure vector. (Of course, this radial component itself is recalculated to include the viscous contributions of the acoustic field, but the form would not change from that used by Marston.)

However, it is not reasonable in general to suppose a response (i.e., the hydrodynamic field) which does not have a phase lag with respect to the forcing. The smaller this lag, the more the tangential forcing term is reduced. The final system given in Section VI may be solved for values of the physical parameter of interest.

Finally, although there is enhancement of the flow field in the "acoustic sublayer" region, this enhancement is less than order one. Also, outside of this sublayer region, the flow is that given by Miller and Scriven (1968), and assumed by Marston (1980); i.e., the flow field of the oscillating drop.

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APPENDIX I:

DETAILS ON VELOCITY DECOMPOSITION APPROACH

This section presents in detail the determination of the viscous acoustic field by means of the velocity decomposition approach.

The acoustic field which incorporates the effects of viscosity must be solved for in the regions interior and exterior to the drop. Quantities in the respective regions will be designated by superscripts i and o respectively.

To facilitate solution of the acoustic field equations, the velocity vector is decomposed as

$$\underline{V}_1^{i} = \nabla \phi^{i} + \nabla \times \underline{A}^{i} \tag{AI.1a}$$

$$\underline{V}_1^{o} = \nabla \phi^{o} + \nabla \times \underline{A}^{o} \tag{AI.1b}$$

into irrotational and solenoidal parts. The vector potential is denoted by $\underline{A}^i$ (and $\underline{A}^o$). In order to render that analytical calculations tractable, axisymmetry is assumed. Then $\underline{A}^i = (0,0,\psi^i)$, with $\psi^i(r,\theta,t)$ a scalar function which is independent of ϕ^i. This is done in a similar fashion for $\underline{A}^o$. The vector potential has a structure akin to that of a toroidal field in the familiar poloidal/toroidal vector filed decomposition approach. However, ψ does not generate $\underline{A}$ in the usual sense.

It is noted that exterior to the drop the acoustic field must account for the scattered as well as the incident wave contributions. (This is in addition to whatever contribution exists due to the presence of viscosity.) Thus, $\phi^o = \phi^o_{sct} + \phi^o_{inc}$.

Substitution of the velocity field decomposition into the linearized governing equations and further manipulations yields

Exterior region

$$\frac{\partial^2 \phi^o_{SCT}}{\partial t^2} - (c^o_o)^2 \left\{ 1 + \frac{1}{(c^o_o)^2} \left\{ \frac{4}{3} \nu^o_o + \frac{\mu^o_{BULK}}{\rho^o_o} \right\} \frac{\partial}{\partial t} \right\} \nabla^2 \phi^o_{SCT} = 0 \tag{AI.2a}$$

$$\frac{\partial \underline{A}^o}{\partial t} = \nu^o_o \nabla^2 \underline{A}^o \tag{AI.2b}$$

Interior region

$$\frac{\partial^2 \phi^i}{\partial t^2} - (c^i_o)^2 \left\{ 1 + \frac{1}{(c^i_o)^2} \left\{ \frac{4}{3} \nu^i_o + \frac{\mu^i_{BULK}}{\rho^i_o} \right\} \frac{\partial}{\partial t} \right\} \nabla^2 \phi^i = 0 \tag{AI.3a}$$

$$\frac{\partial \underline{A}^i}{\partial t} = \nu^i_o \nabla^2 \underline{A}^i \tag{AI.3b}$$

The incident wave is known strictly in terms of $\phi_{INC}{}^o$. In order to deternmine the scattered wave field, both $\underline{A}_s{}^o$ and $\phi_s{}^o$ must be determined. This is also the case when determining the velocity field in the interior of the drop.

Note the factor which appears (with i,o superscripts) in Equations AI.2a and AI.3a. This indicates that the effects of (shear) viscosity (μ) and bulk viscosity (μ_B) appear in the longitudinally propagating part of the acoustic wave. These effects serve to attenuate the wave. Moreover, the bulk viscosity does not contribute to the solenoidal part of the velocity field. This is evident by considering Equations AI.2b and AI.3b. These equations are vector diffusion equations, with the shear viscosity responsible for the diffusion. This will be discussed further.

The incident traveling wave can be expressed as

$$\phi^{o}_{INC}(r,\theta,t) = A_{INC} \sum_{\ell=0}^{\infty} \left\{ (i)^{\ell} (2\ell+1)\, j_{\ell}(K_o r)\, P_{\ell}(\cos\theta)\, e^{-i\omega t} \right\} \tag{AI.4}$$

where ω is the acoustic frequency (carrier frequency in the case of the modulated wave). This wave is proceeding from $-\infty$ to the drop in the direction of the polar axis. Ultimately, the interest is in the acoustic standing wave field. This can be expressed as the superposition of two (oppositely) traveling wave fields.

Assuming a form of ϕ_s^o (or ϕ^i) to be an expansion in Legendre polynomials, with harmonic time dependence

$$\phi^{o}_{SCT}(r,\theta,t) = \sum_{\ell=0}^{\infty} \hat{\alpha}^{o}_{S\ell}\, P_{\ell}(\cos\theta)\, \hat{\phi}^{o}(r)\, e^{-i\omega t} \tag{AI.5}$$

the equation governing the longitudinally propagating part of the wave reduces to

$$\frac{d^2 \hat{\phi}^{o}_{SCT}}{dr^2} + \frac{2}{r}\frac{d\hat{\phi}^{o}_{SCT}}{dr} + \left(K_o^2 - \frac{\ell(\ell+1)}{r^2}\right)\hat{\phi}^{o}_{SCT} = 0 \tag{AI.6a}$$

with

$$K_0^2 = \frac{(\omega/c_0^\circ)^2}{\left\{1 - \frac{i\omega}{(c_0^\circ)^2}\left[\frac{4}{3}\nu_0^\circ + \frac{\mu_{BULK}^\circ}{\rho_0^\circ}\right]\right\}} \tag{AI.6b}$$

A change of superscripts will denote the equation governing ϕ^i. The solution for the exterior region is given by

$$\phi_{SCT}^\circ(r,\theta,t) = \sum_{l=0}^{\infty} \hat{\alpha}_{s_l}^\circ\, h_l^{(1)}(K_0 r)\, P_l(\cos\theta)\, e^{-i\omega t} \tag{AI.6c}$$

with $h_l^{(1)}(K_o\, r)$ indicating the outwardly propagating wave. In the interior of the drop, the solution is given by

$$\phi^i(r,\theta,t) = \sum_{l=0}^{\infty} \hat{\alpha}_l^i\, j_l(K_i r)\, P_l(\cos\theta)\, e^{-i\omega t} \tag{AI.6d}$$

Since axisymmetry has been assumed, there will be only one contribution to the vector potential equations, the $\hat{e}_\phi$ component. This yields a scalar equation

$$\frac{\partial \psi_s^\circ}{\partial t} - \nu_0^\circ\left(\nabla^2\psi_s - \frac{\psi_s^\circ}{r^2\sin^2\theta}\right) = 0 \tag{AI.7a}$$

governing the shear wave exterior to the drop. Replacement of the o by i superscripts, together with the physical properties data, gives the equation governing the shear wave interior to the drop. The exterior solution is given by

$$\psi_s^o(r,\theta,t) = \sum_{\ell=1}^{\infty} \hat{A}_{s\ell}^o \, h_\ell^{(1)}(\gamma_o r) \, P_\ell^1(\cos\theta) \, e^{-i\omega t} \tag{AI.7b}$$

where γ = SQRT (i ω ρ_o^o/μ_o^o) and P_ℓ^1 is the associated Legendre polynomial.

The interior solution is given by

$$\psi^i(r,\theta,t) = \sum_{\ell=1}^{\infty} \hat{A}_\ell^i \, j_\ell(\gamma_i r) \, P_\ell^1(\cos\theta) \, e^{-i\omega t} \tag{AI.7c}$$

Since the standing wave field is of interest, these solutions will be superimposed with waves travelling in the other direction. This can be indicated by a change in the multiplicative constants. Let

$$\delta_\ell = (i)^\ell (2\ell+1) \left\{ e^{iK_o h} + e^{-iK_o h} \right\} \tag{AI.7d}$$

with h the distance from the acoustic velocity nodal plane to the drop's center. (This will be taken to be zero; then, for odd values of 1, δ_l will be zero.)

with
$$\hat{\alpha}^{o}_{s_\ell} = \delta_\ell\, \alpha^{o}_{s_\ell} \qquad \hat{A}^{o}_{s_\ell} = \delta_\ell\, A^{o}_{s_\ell}$$
$$\hat{\alpha}^{i}_{\ell} = \delta_\ell\, \alpha^{i}_{\ell} \qquad \hat{A}^{i}_{\ell} = \delta_\ell\, A^{i}_{\ell}$$

The general form of the solutions which satisfy conditions of finiteness in the respective regions has been determined. It remains to solve for the multiplicative constants, which can be done through the imposition of the appropriate conditions at the drop/host medium interface. The conditions on the acoustic field are that (1) the pressures balance, (2) the radial and tangential components of velocity are continuous (respectively), and (3) the shear stress balances at the interface. This yields four conditions for each 1 value, and there are four unknown constants existing for each 1 value. It remains to determine the pressure in order to construct all the necessary conditions.

The pressure can be determined quite readily. It is a solution of

$$\frac{\partial p^{i,o}}{\partial t} = -\,(c_o^{i,o})^2\, \rho_o^{i,o}\, \nabla^2 \phi^{i,o} \tag{AI.9a}$$

The pressure in the exterior and interior regions is given by

$$p^{o}(r,\theta,t) = (-i\rho_o^{o} c_o^{o\,2}/\omega) \sum_\ell \left\{ \begin{array}{l} \alpha^{o}_{s_\ell}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2}\right) h_\ell^{(1)}(K_o r) \\ + A_{INC}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2}\right) j_\ell(K_o r) \end{array} \right\} P_\ell(\cos\theta)\, e^{-i\omega t} \cdot \delta_\ell \tag{AI.9b}$$

$$p^{i}(r,\theta,t) = (-i\rho_o^{i} c_o^{i\,2}/\omega) \sum_\ell \left\{ \delta_\ell\, \alpha^{i}_\ell \left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2}\right\} j_\ell(K_i r)\, P_\ell(\cos\theta)\, e^{-i\omega t} \right\} \tag{AI.9c}$$

For completeness, the velocity components interior and exterior to the drop are listed below. These are

$$u_r^i(r,\theta,t) = \sum_\ell \delta_\ell \alpha_\ell^i \frac{d}{dr}\left(j_\ell(\kappa_i r)\right) P_\ell(\cos\theta) e^{-i\omega t}$$

$$+ \sum_\ell \delta_\ell A_\ell^i \left(\frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta P_\ell')\right) j_\ell(\gamma_i r)\, e^{-i\omega t} \qquad \text{(AI.10a)}$$

$$u_\theta^i(r,\theta,t) = \sum_\ell \delta_\ell \alpha_\ell^i \left(\frac{j_\ell(\kappa_i r)}{r}\right)\frac{dP_\ell}{d\theta} e^{-i\omega t} \qquad \text{(AI.10b)}$$

$$+ \sum_\ell \delta_\ell A_\ell^i (-1)\frac{1}{r}\frac{d}{dr}\left(r\, j_\ell(\kappa_i r)\right) P_\ell'(\cos\theta) e^{-i\omega t}$$

$$u_{r_{SCT}}^o(r,\theta,t) = \sum_\ell \delta_\ell \alpha_{s_\ell}^i \frac{d}{dr}\left(h_\ell^{(1)}(\kappa_o r)\right) P_\ell(\cos\theta) e^{-i\omega t} \qquad \text{(AI.10c)}$$

$$+ \sum_\ell \delta_\ell A_{s_\ell}^o \left(\frac{h_\ell^{(1)}(\gamma_o r)}{r}\right)\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta P_\ell')\, e^{-i\omega t}$$

$$u_{\theta_{SCT}}^o(r,\theta,t) = \sum_\ell \delta_\ell \alpha_{s_\ell}^o \left(\frac{h_\ell^{(1)}(\kappa_o r)}{r}\right)\frac{dP_\ell}{d\theta} e^{-i\omega t} \qquad \text{(AI.10d)}$$

$$+ \sum_\ell \delta_\ell A_{s_\ell}^o (-1)\frac{1}{r}\frac{d}{dr}\left(r\, h_\ell^{(1)}(\gamma_o r)\right) P_\ell'(\cos\theta) e^{-i\omega t}$$

The conditions at the interface are then given by

$$\alpha_\ell^i \left\{\frac{\ell}{R} j_\ell(\kappa_i R) - \kappa_i\, j_{\ell+1}(\kappa_i R)\right\} + A_\ell^i\ \ell(\ell+1)\frac{j_\ell(\gamma_i R)}{R}$$

$$= \alpha_{s_\ell}^o \left\{\frac{\ell}{R} h_\ell^{(1)}(\kappa_o R) - \kappa_o\, h_{\ell+1}^{(1)}(\kappa_o R)\right\} + A_{s_\ell}^o\ \ell(\ell+1)\frac{h_\ell^{(1)}(\gamma_o R)}{R}$$

$$+ A_{INC}\left\{\frac{\ell}{R} j_\ell(\kappa_o R) - \kappa_o\, j_{\ell+1}(\kappa_o R)\right\} \qquad \text{(AI.11a)}$$

$$\alpha_\ell^i \frac{j_\ell(K_i R)}{R} - A_\ell^i\left\{\frac{(\ell+1)}{R} j_\ell(\gamma_i R) - \gamma_i j_{\ell+1}(\gamma_i R)\right\}$$

$$= \alpha_{s_\ell}^o \frac{h_\ell^{(1)}(K_i R)}{R} - A_{s_\ell}^o\left\{\frac{(\ell+1)}{R} h_\ell^{(1)}(\gamma_o r) - \gamma_o h_{\ell+1}^{(1)}(\gamma_o R)\right\} + A_{INC}\frac{j_\ell(K_o R)}{R} \qquad \textbf{(AI.11b)}$$

$$\alpha_\ell^i \mu_o^i \left\{\frac{2(\ell-1)}{R^2} j_\ell(K_i R) - \frac{2}{R} j_{\ell+1}(K_i R)\right\} + A_\ell^i \frac{\mu_o^i}{R^2}\left\{\left[2(\ell^2-1) - \gamma_i^2 R^2\right] j_\ell(\gamma_i R) + 2\gamma_i R\, j_{\ell+1}(\gamma_i R)\right\}$$

$$= \alpha_{s_\ell}^o \mu_o^o \left\{\frac{2(\ell-1)}{R^2} h_\ell^{(1)}(K_o R) - \frac{2}{R} h_{\ell+1}^{(1)}(K_o R)\right\}$$

$$+ A_{s_\ell}^o \frac{\mu_o^o}{R^2}\left\{\left[2(\ell^2-1) - \gamma_o^2 R^2\right] h_\ell^{(1)}(\gamma_o R) + 2\gamma_o R\, h_{\ell+1}^{(1)}(\gamma_o R)\right\}$$

$$+ A_{INC}\, \mu_o^o \left\{\frac{2(\ell-1)}{R^2} j_\ell(K_o R) - \frac{2}{R} j_{\ell+1}(K_o R)\right\}$$

(AI.11c)

$$\alpha_\ell^i \left(i\rho_o^i c_o^{i\,2}/\omega\right)\left\{\ell(\ell+1)\left(1/R^2 - 1\right) j_\ell(K_i R) - K_i^2 j_{\ell+1}(K_i R)\right\}$$

$$+ \alpha_\ell^i \frac{(2\mu_o^i)}{R^2}\left\{\left[\ell(\ell+1) - K_i^2 R^2\right] j_\ell(K_i R) + 2K_i R\, j_{\ell+1}(K_i R)\right\}$$

$$+ A_\ell^i \frac{2\mu_o^i}{R^2}\left\{\ell(\ell+1)\left\{(\ell-1) j_\ell(\gamma_i R) - \gamma_i R\, j_{\ell+1}(\gamma_i R)\right\}\right\}$$

$$= \alpha_{s_\ell}^o \left(i\rho_o^o c_o^{o\,2}/\omega\right)\left\{\ell(\ell+1)\left(1/R^2 - 1\right) h_\ell^{(1)}(K_o R) - K_o^2 h_{\ell+1}^{(1)}(K_o R)\right\}$$

$$+ \alpha_{s_\ell}^o \frac{2\mu_o^o}{R^2}\left\{\left[\ell(\ell+1) - K_o^2 R^2\right] h_\ell^{(1)}(K_o R) + 2K_o R\, h_{\ell+1}^{(1)}(K_o R)\right\}$$

$$+ A_{s_\ell}^o \frac{2\mu_o^o}{R^2}\left\{\ell(\ell+1)\left\{(\ell-1) h_\ell^{(1)}(\gamma_o R) - \gamma_o R\, h_{\ell+1}^{(1)}(\gamma_o R)\right\}\right\} \qquad \textbf{(AI.11d)}$$

$$+ A_{INC}\left(i\rho_o^o c_o^{o\,2}/\omega\right)\left\{\ell(\ell+1)\left(1/R^2 - 1\right) j_\ell(K_o R) - K_o^2 j_{\ell+1}(K_o R)\right\}$$

$$+ A_{INC} \frac{2\mu_o^o}{R^2}\left\{\left[\ell(\ell+1) - K_o^2 R^2\right] j_\ell(K_o R) + 2K_o R\, j_{\ell+1}(K_o R)\right\}$$

APPENDIX II:

ACOUSTIC FORCING (UNMODULATED) OF COMPOUND FLUID DROP: SUBCASE OF CORE/SHELL VISCOSITIES DIFFERENT

This work considers the effect of acoustic forcing upon a compound drop system. The compound drop system itself consists of a core fluid of density ρ_o^i, viscosity μ_o^i, surrounded by a shell fluid (density ρ_o^s, viscosity μ_o^s) embedded in a host medium (with ρ_o^o, μ_o^o). In general, the surface tension values at the inner and outer interfaces will differ. Only a specialized sub case of the general compound fluid drop system is considered. Attention is restricted to the case in which the core and shell fluids have the same density and compressibility, but differ in viscosity.

The acoustic wave providing for the forcing of the drop is considered to be inviscid. That is, the viscous effects will be restricted to the hydrodynamic field. The acoustic wave then sees the outer interface as a boundary between two media since the acoustic wave is taken to be inviscid, the only contributing term in the radiation pressure vector will be the radial component.

The case of the unmodulated acoustic wave is considered. This will result in drop deformation. The problem is to determine the deformation and compare it to what would exist for a simple drop.

Linearized governing equations for the hydrodynamic field reflect the steady state nature of this problem. Manipulations on the conservation of momentum equation will yield the following system, expressed in the radial components of velocity,vorticity (of the hydrodynamic field)

$$\nabla^2(\nabla^2(r u_r^j)) = 0 \qquad j \in \{i, s, o\} \qquad \text{(AII.1a)}$$

$$\nabla^2(r\omega_r^j) = 0 \qquad j \in \{i, s, o\} \qquad \text{(AII.1b)}$$

That is, the steady state forced hydrodynamic field problem can avail itself of the same decomposition as that used by Miller and Scriven in their analysis of the unforced oscillating drop problem. The boundary/interface conditions can be expressed strictly in terms of the aforementioned radial components.

Since the only forcing component from the acoustic field is in the radial direction, the necessary boundary conditions will be expressed in terms of the radial velocity only. Solutions to the system are found to be

$$u_r^i(r,\theta,\phi) = \sum_{l,m} \left(a_I r^{l-1} + c_I r^{l+1} \right) Y_{lm}(\theta,\phi) \tag{AII.2a}$$

$$u_r^s(r,\theta,\phi) = \sum_{l,m} \left(a_s r^{l-1} + c_s r^{l+1} + b_s r^{-l-2} + d_s r^{-l} \right) Y_{lm}(\theta,\phi) \tag{AII.2b}$$

$$u_r^o(r,\theta,\phi) = \sum_{l,m} \left(b_o r^{-l-2} + d_o r^{-l} \right) Y_{lm}(\theta,\phi) \tag{AII.2c}$$

The boundary/interface conditions which must be imposed are the kinematic condition, continuity of velocity components across the interface, and that the normal and tangential stress balances at the interface. This is at both the inner and outer interfaces. However, the additional (acoustic) forcing in the radial direction will enter into the normal stress balance at the outer interface. It is this contribution which will force the system, and result in deformation.

Imposition of the boundary conditions leads to the following system of equations

$$2c_I = a_s(\ell-1)(s/\tau)^2 + c_s(\ell+1) - b_s(\ell+2)(\tau/s)^{2\ell+3} - d_s\,\ell\,(\tau/s)^{2\ell+1} \tag{AII.3a}$$

$$2d_o = a_s(\ell-1)(1/\tau s)^2 + c_s(\ell+1) - b_s(\ell+2)(\tau s)^{-2\ell-3} - d_s\,\ell\,(\tau s)^{-2\ell-1} \tag{AII.3b}$$

$$a_s + c_s(s/\tau)^2 + (\tau/s)^{2\ell+1} b_s + d_s(\tau/s)^{2\ell-1} = 0 \tag{AII.3c}$$

$$a_s + c_s(\tau s)^2 + b_s(\tau s)^{-2\ell-1} + d_s(\tau s)^{-2\ell+1} = 0 \tag{AII.3d}$$

$$c_I\, 2\mu_o^I(2\ell+1) = \mu_o^s \left\{ a_s\, 2(\ell^2-1)(\tau/s)^2 + c_s\, 2\ell(\ell+2) + b_s\, 2\ell(\ell+2)(\tau/s)^{2\ell+3} + d_s(2)(\ell^2-1)(\tau/s)^{2\ell+1} \right\} \tag{AII.3e}$$

$$b_o\, 2\mu_o^o(2\ell+1) = \mu_o^s \left\{ 2(1-\ell^2)(\tau s)^{2\ell-1} a_s - c_s\, 2\ell(\ell+2)(\tau s)^{-2\ell+1} - b_s\, 2\ell(\ell+2)(1/\tau s)^2 - d_s\, 2(\ell^2-1) \right\} \tag{AII.3f}$$

$$
c_I \, \mu_o^i \left(\frac{b}{\ell}\right)(s/\tau)^{\ell} + 2\mu_o^s (\ell-1)(s/\tau)^{\ell-2} a_s
$$

$$
+ c_s \frac{2\mu_o^s}{\ell}(\ell^2-\ell-3)(s/\tau)^{\ell} - b_s \, 2(\ell+2)\mu_o^s (\tau/s)^{2\ell+3}
$$

$$
-d_s \frac{2\mu_o^s}{(\ell+1)}(\ell^2+3\ell-1)(\tau/s)^{\ell+1} = \frac{\sigma^i (\ell+2)(\ell-1)\zeta^i}{(s/\tau)^2}
$$

(AII.3g)

$$
c_s \left(\frac{\mu_o^s}{R}\right)(-2)(\ell^2-\ell-3)(\tau s)^{\ell} + \frac{d_s^o}{(\ell+1)}(\ell^2+3\ell-1)(\tau s)^{-\ell-1}(2\mu_o^s)
$$

$$
+ a_s(-2\mu_o^s)(\ell-1)(\tau s)^{\ell-2} + b_s \, 2\mu_o^s (\ell+2)(\tau s)^{-\ell-3}
$$

$$
+ \frac{b}{(\ell+1)}\mu_o^o (\tau s)^{-\ell-1} d_o = \frac{\sigma_o (\ell+2)(\ell-1)\zeta^o}{(\tau s)^2} - \langle (\overline{pr})^{\text{RADIAL}} \rangle_{\text{projected}} \quad \text{(AII.3h)}
$$

With s = SQRT ($R_I R_o$) and τ = SQRT (R_o/R_I). This system of equations is a set of forced equations in the unknown coefficients: c_I, a_s, b_s, c_s, d_s, d_o, ζ^i, ζ^o. The deformation at the outer interface is indicated by ζ^o. Note that this is a reduction from the tenth order algebraic system which would have been expected if the compound drop were oscillating (either free or forced). This is due to the repetition of relationships in two interface equations.

This is an (8 x 8) nonhomogeneous linear system which can be solved by standard numerical means. Although not efficient numerically, it is clear that an application of Cramer's rule would lead to a solution for ζ^o, which is of interest. This solution would be in the form of a numerical value. Application of the symbolic mathematics manipulator REDUCE (Hearn, 1987), leads to an expression for ζ^{o}, which is

$$\zeta^{o} = \left(\frac{R_o^2}{\sigma^{o}(\ell+2)(\ell-1)} \right) \langle \overline{p_r}^{\,radial} \rangle_{projected} \tag{AII.4}$$

This is the result obtained for the deformation of the interface of the simple drop. Note that the inner surface tension does not contribute. At first, this appears to be a surprising result. However, upon reflection it is recalled that the core and shell region fluids differ only in their respective viscosities, and that the unmodulated acoustic wave responsible for the forcing is taken to be inviscid.