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Coupled Fluid-Structure Interaction
Part I—Theory
Part II—Application

Carlos A. Felippa
University of Colorado
Boulder, Colorado

and

Roger Ohayon
Office National D'Études et de Recherches
Aérospatiales
Chatillon, France

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COUPLED FLUID-STRUCTURE INTERACTION
PART I, THEORY

CARLOS A. FELIPPA

Department of Aerospace Engineering and
Center for Space Structures and Controls
University of Colorado, Boulder, CO 80309-0429, USA

ROGER OHAYON

Office National D'Études et de Recherches Aérospatiales
BP 72, 92322 Chatillon, France

ABSTRACT

A general three-field variational principle is obtained for the motion of an acoustic fluid enclosed in a rigid or flexible container by the method of canonical decomposition applied to a modified form of the wave equation in the displacement potential. The general principle is specialized to a mixed two-field principle that contains the fluid displacement potential and pressure as independent fields. Semidiscrete finite-element equations of motion based on this principle are displayed.

1. INTRODUCTION

An elastic container (the *structure*) is totally or partly filled with a compressible liquid or gas (the *fluid*). The fluid structure system is initially in static equilibrium in a steady body force field such as gravity or centrifugal forces. We consider *small departures from equilibrium* that result in forced or free vibratory motions. To analyze these motions the fluid is treated as a linear acoustic fluid, *i.e.*, compressible but irrotational and inviscid. The purpose of the present work is

1. To derive variational equations of motion based on a mixed variational principle for the fluid subsystem.

2. To obtain semidiscrete equation of motions following spatial discretization of the coupled problem by the finite element method.

The derivation of the mixed variational principle for the fluid is based on the method of *canonical equations* advocated by Oden and Reddy [1] for mechanical applications. The most general dynamical principle derived in this paper contains three primary variables: pressure-momentum vector, dilatation-velocity vector, and displacement potential.

The general principle is specialized to a two-field functional of Reissner type that has pressure and displacement potential as primary variables, as well as a free coefficient α that parametrizes the application of the divergence theorem. The coupled variational equations are then discretized by the finite element method. Linkage with the elastic container is made to establish coupled semidiscrete equations of motion for a rigid container and for the fluid-structure system. By appropriate selection of the coefficient α a number of discrete formulations of the elastoacoustic problem results. The computational implications of these selections are briefly discussed.

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Table 1 Notation for Fluid States

Quantities	Domain	Original	Reference	Current	Transient
Displacements	V	0	$\mathbf{d}^0$	$\mathbf{d}^t$	$\mathbf{d} = \mathbf{d}^t - \mathbf{d}^0$
Velocities	V	0	$\dot{\mathbf{d}}^0$	$\dot{\mathbf{d}}^t$	$\dot{\mathbf{d}} = \dot{\mathbf{d}}^t - \dot{\mathbf{d}}^0$
Boundary displacements*	S	0	d_n^0	d_n^t	$d_n = d_n^t - d_n^0$
Displacement potential	V	0	ψ^0	ψ^t	$\psi = \psi^t - \psi^0$
Pressures	V	0	p^0	p^t	$p = p^t - p^0$
Body forces	V	0	$\mathbf{b} = \nabla\beta$	$\mathbf{b} = \nabla\beta$	
Density	V	ρ	ρ	ρ	
*Positive along outward normal					

2. GOVERNING EQUATIONS

The three-dimensional volume domain occupied by the fluid is denoted by V. This volume is assumed to be simply connected. The fluid boundary S consists generally of two portions

$$S : S_d \cup S_p. \tag{1}$$

S_d is the interface with the container at which the normal displacement d_n is prescribed (or found as part of the coupled fluid-structure problem) whereas S_p is the "free surface" at which the pressure p is prescribed (or found as part of the "fluid slosh" problem). If the fluid is fully enclosed by the container, as is necessarily the case for a gas, then S_p is missing and $S \equiv S_d$. The domain is referred to a Cartesian coordinate system (x_1, x_2, x_3) grouped in vector $\mathbf{x}$.

The fluid is under a body force field $\mathbf{b}$ which is assumed to be the gradient of a *time independent* potential $\beta(\mathbf{x})$, *i.e.* $\mathbf{b} = \nabla\beta$. All displacements are taken to be infinitesimal and thus the fluid density ρ is invariant.

We consider three states or configurations: *original*, from which displacements, pressures and forces are measured, *current*, where the fluid is in dynamic equilibrium at time t, and *reference*, which is obtained in the static equilibrium limit of slow motions. *Transient* motions are the difference between current and reference states. It should be noted that in many situations the original configuration is not physically attainable. Table 1 summarizes the notation used in relation to these states.

2.1 Field Equations

The governing equations of the acoustic fluid are the momentum, state and continuity equations. They are stated below for the current configuration, and specialized to the reference configuration later. The momentum (balance) equation expresses Newton's second law for a fluid particle:

$$\rho\ddot{\mathbf{d}}^t = -\nabla p^t + \mathbf{b} = -\nabla p^t + \nabla\beta. \tag{2}$$

The continuity equation may be combined with the linearized equation of state to produce the *constitutive equation* that expresses the small compressibility of a liquid:

$$p^t = -K\nabla \mathrm{d}^t = -\rho c^2 \nabla \mathrm{d}^t, \qquad (3)$$

where K is the bulk modulus and $c = \sqrt{K/\rho}$ the fluid sound speed. If the fluid is incompressible, $K, c \to \infty$. This relation is also applicable to nonlinear elastic fluids such as gases undergoing small excursions from the reference state, if the constitutive equation is linearized there so that $K = \rho_0 (dp/d\rho)_0$.

The boundary conditions are

$$d_n^t = \tilde{d}_n^t \quad \text{on} \quad S_d, \qquad p^t = \tilde{p}^t \quad \text{on} \quad S_p, \qquad (4)$$

where $\tilde{d}_n^t$ is either prescribed or comes from the solution of an auxiliary problem as in fluid-structure interaction, and $\tilde{p}$ may be either prescribed or a function of d_n and b, as in the surface-wave ("slosh") problem.

2.2 Integral Abbreviations

In the sequel the following abbreviations for the volume and surface integrals are used:

$$(f)_V \overset{\text{def}}{=} \int_V f \, dV, \qquad [g]_S \overset{\text{def}}{=} \int_S g \, dS, \qquad [g]_{S_d} \overset{\text{def}}{=} \int_{S_d} g \, dS, \quad \text{etc.} \qquad (5)$$

That is, domain-subscripted parentheses (square brackets) are used to abbreviate volume (surface) integrals. Abbreviations for function innerproducts are illustrated by

$$(f,g)_V \overset{\text{def}}{=} \int_V fg \, dV, \qquad (f,g)_{V \times t} \overset{\text{def}}{=} \int_{t_0}^{t_1} \int_V fg \, dV \, dt, \qquad [f,g]_{S_d \times t} \overset{\text{def}}{=} \int_{t_0}^{t_1} \int_{S_d} fg \, dS \, dt, \quad \text{etc.} \qquad (6)$$

3. THE DISPLACEMENT POTENTIAL

3.1 The Reference State

Taking the curl of both sides of (2) yields

$$\operatorname{curl} \ddot{\mathrm{d}}^t = 0. \qquad (7)$$

The general integral of this equation for a simply connected domain is

$$\mathrm{d}^t = \nabla \psi^t + \mathbf{a} + \mathbf{b}t, \qquad (8)$$

where $\psi^t = \psi^t(\mathbf{x},t)$ is the displacement potential, $\mathbf{a} = \mathbf{a}(\mathbf{x})$ and $\mathbf{b} = \mathbf{b}(\mathbf{x})$ are time-independent vector functions, and t denotes the time. If accelerationless motions (for example, rigid body motions) are precluded by the boundary conditions, a and b vanish. Replacing $\ddot{\mathrm{d}}^t = \nabla \ddot{\psi}^t$ into the momentum equation (2) we get

$$\nabla p^t = -\rho \nabla \ddot{\psi}^t + \nabla \beta, \qquad (9)$$

which spatially integrated gives

$$p^t = -\rho \ddot{\psi}^t + \beta + C(t), \qquad (10)$$

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where the scalar $C(t)$ is not spatially dependent. Next integrate the constitutive equation (3) over V and apply the divergence theorem to ∇d:

$$(p^t)_V + (\rho c^2 \nabla d^t)_V = (p^t)_V + [\rho c^2 d_n^t]_S = 0. \tag{11}$$

Inserting p^t from (10) into the above equation furnishes a condition on $C(t)$, which gives

$$C(t) = -\frac{\rho c^2}{v}[d_n^t]_S + \frac{\rho}{v}(\ddot{\psi}^t)_V - \frac{1}{v}(\beta)_V = -\frac{\rho c^2}{v}[d_n^t]_S + \rho\overline{\ddot{\psi}^t} - \overline{\beta}, \tag{12}$$

where $v = (1)_V$ is the fluid volume and $\overline{f} = (f)_V/v$ denotes the volume average of a function f defined over V. Substituting $C(t)$ into (10) we get

$$p^t = -\rho(\ddot{\psi}^t - \overline{\ddot{\psi}^t}) + (\beta - \overline{\beta}) - \frac{\rho c^2}{v}[d_n^t]_S. \tag{13}$$

In the static limit of very slow motions, the inertia terms may be neglected and we recover the reference solution

$$p^0 = (\beta - \overline{\beta}) - \frac{\rho c^2}{v}[d_n^0]_S. \tag{14}$$

For an incompressible fluid $[d_n]_S = 0$ but $c \to \infty$; thus it would be incorrect to conclude that $p^0 = \beta - \overline{\beta}$. A counterexample to this effect is provided in [2].

3.2 Transient Motions

Subtracting the constitutive relations at the current and reference states we get

$$p = -\rho c^2 \nabla^2 \psi = \rho c^2 s, \tag{15}$$

where $s = -\nabla^2 \psi$ is called, following Lamb [3], the condensation. Subtracting (14) from (13) yields

$$p = -\rho(\ddot{\psi} - \overline{\ddot{\psi}}) - \frac{\rho c^2}{v}[d_n]_S. \tag{16}$$

On equating (15) and (16) we get modified forms of the wave equation that account for mean boundary surface motions:

$$s = \nabla^2 \psi = \frac{\ddot{\psi} - \overline{\ddot{\psi}}}{c^2} + \frac{1}{v}[d_n]_S, \qquad \text{or} \qquad c^2(\nabla^2 \psi - \overline{\nabla^2\psi}) = \ddot{\psi} - \overline{\ddot{\psi}}. \tag{17}$$

The second form follows from $-v\overline{s} = [d_n]_S$, which is a consequence of the divergence theorem. For an incompressible fluid, $c \to \infty$ and $[d_n]_S = 0$, and from the first of (17) we recover the Laplace equation $\nabla^2 \psi = 0$.

3.3 Adjusting the Displacement Potential

If the transient displacement potential is modified by a function of time:

$$\psi = \hat{\psi} + P(t), \tag{18}$$

where $\hat{\psi}$ is the potential of (8)-(17), we may chose $P(t)$ so that $c^2\bar{\bar{\psi}} = \overline{\nabla^2\psi} = -\bar{s}$ for any t. Then we obtain the classical wave equation

$$c^2\nabla^2\psi = \bar{\psi}, \qquad \text{or} \qquad \left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\psi = 0. \tag{19}$$

In the sequel it is assumed that this adjustment has been made. If so, $C(t)$ vanishes and (16) reduces to

$$p = -\rho\bar{\psi}. \tag{20}$$

4. MIXED VARIATIONAL PRINCIPLES

4.1 Canonical Decomposition

In this section we derive multifield variational principles for the fluid domain following the canonical decomposition method advocated by Oden and Reddy [1]. This method is applicable to self-adjoint boundary value problems (BVP) of the form

$$Au = f \qquad \text{in } D \tag{21}$$

where u is the unknown function, f the data, A a symmetric linear operator, and D the domain of existence of the solution. For time-dependent problems D is the tensor product of the time domain (typically 0 to t) and the volume V. To apply this method, the operator A is factored as

$$Au = T^*ETu = f, \tag{22}$$

where T and E are linear operators in V and T^* is the adjoint of T. This is called a *canonical decomposition*. This decomposition may be represented as the operator composition sequence

$$Tu = e, \qquad Ee = \sigma, \qquad T^*\sigma = f, \tag{23}$$

where e and σ denote intermediate field variables in D. The three equations (23) are called the kinematic, constitutive and balance equations, respectively, in mechanical applications. The canonical representation of boundary conditions on the surface $S = S_u \cup S_\sigma$ is

$$B_S u_S = g \quad \text{on} \quad S_u, \qquad B_S^*\sigma_S = h \qquad \text{on} \quad S_\sigma. \tag{24}$$

where B_S and B_S^* are surface operators, g and h denote boundary data, and $u_S = \gamma_S u$ and $\sigma_S = \Gamma_S\sigma$ are extensions of u and σ to the boundary S. The extension operators γ_S and δ_S often involve normal derivatives.

4.2 The Wave Equation

The classical wave equation (19) is *not* a good basis for the canonical decomposition (22). Its principal drawback is that the pressure field does not appear naturally as an intermediate variable in (23). A more convenient form for our purposes is obtained by taking the Laplacian of both sides of (19), and multiplying through by the density ρ:

$$\rho\nabla^2(\ddot\psi - c^2\nabla^2\psi) = 0, \quad \text{whence} \quad A = \rho\nabla^2\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right), \quad f = 0. \tag{25}$$

A suitable canonical decomposition is $A = \mathbf{T}\mathbf{E}\mathbf{T}^T$, where

$$\mathbf{T} = \begin{bmatrix} i\nabla\frac{\partial}{\partial t} \\ -\nabla^2 \end{bmatrix}, \qquad \mathbf{E} = \rho\begin{bmatrix} \mathbf{I} & 0 \\ 0 & c^2 \end{bmatrix}, \qquad \mathbf{T}^* = \begin{bmatrix} -i\nabla\frac{\partial}{\partial t} & \nabla^2 \end{bmatrix} = -\mathbf{T}^T, \tag{26}$$

in which $i = \sqrt{-1}$. Boldface symbols are used for $\mathbf{T}$ and $\mathbf{E}$ because these are 4×1 and 4×4 matrices, respectively. The operator product sequence (23) becomes

$$\mathbf{e} = \mathbf{T}\psi = \begin{bmatrix} i\nabla\dot\psi \\ -\nabla^2\psi \end{bmatrix} = \begin{bmatrix} i\mathbf{v} \\ s \end{bmatrix}, \quad \sigma = \mathbf{E}\mathbf{e} = \begin{bmatrix} i\rho\nabla\dot\psi \\ -\rho c^2\nabla^2\psi \end{bmatrix} = \begin{bmatrix} i\mathbf{m} \\ p \end{bmatrix}, \quad \mathbf{T}^*\sigma = \rho\nabla^2\ddot\psi - \rho c^2\nabla^4\psi = 0. \tag{27}$$

The intermediate fields e and σ are 4×1 column vectors. These vectors are partitioned into their temporal and spatial derivative subvectors for convenience in subsequent manipulations. Note that the transient pressure p appears naturally as the spatial component of σ. The temporal components of e and σ are the complex velocity $i\mathbf{v}$ and complex specific momentum $i\mathbf{m}$, respectively.

The boundary portions S_u and S_σ of (24) are relabeled S_d and S_p, respectively, to match the notation (1). Boundary and initial conditions may be stated in generic form as

$$\mathbf{B}\psi(\mathbf{x},t) = \mathbf{g}(\mathbf{x},t) \text{ on } S_d, \quad \mathbf{B}^*\sigma(\mathbf{x},t) = h(\mathbf{x},t) \text{ on } S_p, \quad \mathbf{d}(\mathbf{x},t_0) = \mathbf{d}_0(\mathbf{x}), \quad \mathbf{d}(\mathbf{x},t_1) = \mathbf{d}_1(\mathbf{x}). \tag{28}$$

Here $\mathbf{B}$ and $\mathbf{B}^*$ are time-independent 4×1 and 1×4 vectors, respectively, related to the canonical $\mathbf{B}_S$ and $\mathbf{B}_S^*$ operators of (24) by $\mathbf{B} = \mathbf{B}_S\gamma_S$ and $\mathbf{B}^* = \mathbf{B}_S^*\Gamma_S$, where γ_S (a scalar) and Γ_S (a 4×4 matrix) are boundary extension operators for ψ and σ, respectively. Comparison with (4) and the use of Green's function reveals that

$$\mathbf{B}_S^T = -\mathbf{B}_S^* = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{g}^T = \begin{bmatrix} 0 & 0 & 0 & \ddot d_n \end{bmatrix}, \quad \gamma_S = \frac{\partial}{\partial n}, \quad \Gamma_S = \mathbf{I}, \quad h = -\tilde p. \tag{29}$$

4.3 Three Field Principle

The most general variational principle for the canonical decomposition (26) allows the three fields: ψ, e, and σ, to be varied independently. The principle may be stated as $\delta L(\psi, \mathbf{e}, \sigma) = 0$, where the functional L is [2]

$$L(u, \mathbf{e}, \sigma) = L_V + L_S = \tfrac{1}{2}(\mathbf{E}\mathbf{e}, \mathbf{e})_{V \times t} + (\sigma, \mathbf{T}\psi - \mathbf{v})_{V \times t} - (f, \psi)_{V \times t} + (\sigma_S, \mathbf{B}\psi - \mathbf{g})_{S_d \times t} - (h, \psi_S)_{S_p \times t}, \tag{30}$$

where L_V and L_S collect volume and surface terms, respectively. On inserting (27-29) into (30) we get

$$L_V = \tfrac{1}{2}(\mathbf{E}\mathbf{e},\mathbf{e})_{V \times t} + (\boldsymbol{\sigma}, \mathbf{T}\psi - \mathbf{e})_{V \times t} = \int_{t_0}^{t_1} \int_V \left[\tfrac{1}{2}\rho(-\mathbf{v}^T\mathbf{v} + c^2 s^2) - \mathbf{m}^T(\nabla\dot{\psi} - \mathbf{v}) - p(\nabla^2\psi + s) \right] dV\, d$$

$$L_S = (\boldsymbol{\sigma}_S, \mathbf{B}\psi - \mathbf{g})_{S_d \times t} - (h, \psi_S)_{S_p \times t} = \int_{t_0}^{t_1} \left[\int_{S_d} p(\frac{\partial\psi}{\partial n} - \bar{d}_n)\, dS + \int_{S_p} \tilde{p}\frac{\partial\psi}{\partial n}\, dS \right] dt.$$

(31)

whereas $(f, \psi)_{V \times t}$ vanishes.

4.4 Two Field Principles

A two field principle of Reissner type can be derived from the functional L by enforcing the inverse constitutive equations $\mathbf{e} = \mathbf{E}^{-1}\boldsymbol{\sigma}$ a priori. The resulting principle, which allows ψ and $\boldsymbol{\sigma}$ to be varied simultaneously, is $\delta R(\psi, \boldsymbol{\sigma}) = 0$, where

$$R(\psi, \boldsymbol{\sigma}) = R_V + R_S = -\tfrac{1}{2}(\mathbf{E}^{-1}\boldsymbol{\sigma}, \boldsymbol{\sigma})_{V \times t} + (\boldsymbol{\sigma}, \mathbf{T}\psi)_{V \times t} - (f, \psi)_{V \times t} + (\boldsymbol{\sigma}_S, \mathbf{B}\psi - \mathbf{g})_{S_d \times t} - (h, \psi_S)_{S_p \times t}.$$

(32)

where $R_S = L_S$ and

$$R_V(\psi, \boldsymbol{\sigma}) = -\tfrac{1}{2}(\mathbf{E}^{-1}\boldsymbol{\sigma}, \boldsymbol{\sigma})_{V \times t} + (\boldsymbol{\sigma}, \mathbf{T}\psi)_{V \times t} = \int_{t_0}^{t_1} \int_V \left(\frac{1}{2\rho}\mathbf{m}^T\mathbf{m} - \frac{p^2}{2\rho c^2} - \mathbf{m}^T\nabla\dot{\psi} - p\nabla^2\psi \right) dV\, dt.$$

(33)

The specific momentum disappears as an independent field if we enforce $\mathbf{m} = \rho\nabla\dot{\psi}$ a priori, whereupon the functional R becomes a function of ψ and p only and the volume term contracts to

$$R_V(\psi, p) = \int_{t_0}^{t_1} \int_V \left(-\tfrac{1}{2}\rho(\nabla\dot{\psi})^T\nabla\dot{\psi} - \tfrac{1}{2}\frac{p^2}{\rho c^2} - p\nabla^2\psi \right) dV\, dt.$$

(34)

To check $R = R_V(\psi, p) + R_S$ we form its first variation:

$$\delta R = -\left(\rho\nabla^2\ddot{\psi} + \nabla^2 p, \delta\psi \right)_{V \times t} - \left(\frac{1}{\rho c^2}p + \nabla^2\psi, \delta p \right)_{V \times t} + \left[\rho\nabla^2\ddot{\psi} + \nabla p, \delta\psi \right]_{S \times t}$$
$$- \left[p - \tilde{p}, \delta\frac{\partial\psi}{\partial n} \right]_{S_p \times t} - \left[\frac{\partial\psi}{\partial n} - \bar{d}_n, \delta p \right]_{S_d \times t} - \left[\rho\nabla\dot{\psi}, \delta\frac{\partial\psi}{\partial n} \right]_{t_0}^{t_1}.$$

(35)

Setting $\delta R = 0$ provides the field equations and boundary conditions.

4.5 Parametrization

A one parameter family of variational principles can be obtained by transforming all or part of the last term in (34), viz. $p\nabla^2\psi$, by the divergence theorem (Green's first formula for the Laplace operator)

$$\int_V p\nabla^2\psi\, dV + \int_V \nabla\psi\nabla p\, dV = \int_S p\frac{\partial\psi}{\partial n}\, dS = \int_{S_d} p\frac{\partial\psi}{\partial n}\, dS + \int_{S_p} \tilde{p}\frac{\partial\psi}{\partial n}\, dS.$$

(36)

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Let $0 \leq \alpha \leq 1$ be the portion of that term to be transformed. Insert $p\nabla^2\psi = \alpha p\nabla^2\psi + (1-\alpha)p\nabla^2\psi$ in (35) and apply (36) to $\alpha p\nabla^2\psi$ to get

$$R_{\alpha V} = \int_{t_0}^{t_1}\left[\int_V \left(\tfrac{1}{2}\rho(\nabla\dot{\psi})^T\nabla\dot{\psi} - \tfrac{1}{2}\frac{p^2}{\rho c^2} + \alpha\nabla p\nabla\psi - (1-\alpha)p\nabla^2\psi\right)dV - \alpha\int_{S_d} p\frac{\partial\psi}{\partial n}\,dS - \alpha\int_{S_p}\frac{\partial\psi}{\partial n}\tilde{p}\,dS\right]dt. \tag{37}$$

Finally, replace the Laplacian $\nabla^2\psi$ left over in (37) by $c^{-2}(\ddot{\psi})$ to arrive at the parametrized two-field functional

$$R_\alpha(\psi, p) = R_{\alpha V} + R_S = \int_{t_0}^{t_1}\left[\int_V\left(-\tfrac{1}{2}\rho(\nabla\dot{\psi})^T\nabla\dot{\psi} - \tfrac{1}{2}\frac{p^2}{\rho c^2} + \alpha\nabla p\nabla\psi - (1-\alpha)c^{-2}p\ddot{\psi}\right)dV\right. \tag{38}$$
$$\left. + \int_{S_d} p\left[(1-\alpha)\frac{\partial\psi}{\partial n} - \tilde{d}_n\right]dS + \int_{S_p}(1-\alpha)\frac{\partial\psi}{\partial n}\tilde{p}\,dS\right]dt.$$

The highest spatial derivative index for both primary variables ψ and p is 1, except if $\alpha = 0$, in which case it is only 0 for p. The two interesting limit cases are of course $\alpha = 0$ and $\alpha = 1$, for which

$$R_0(\psi, p) = \int_{t_0}^{t_1}\left[\int_V\left(-\tfrac{1}{2}\rho(\nabla\dot{\psi})^T\nabla\dot{\psi} - \tfrac{1}{2}\frac{p^2}{\rho c^2} - c^{-2}p\ddot{\psi}\right)dV + \int_{S_d} p\left(\frac{\partial\psi}{\partial n} - \tilde{d}_n\right)dS + \int_{S_p}\frac{\partial\psi}{\partial n}\tilde{p}\,dS\right]dt, \tag{39}$$

$$R_1(\psi, p) = \int_{t_0}^{t_1}\left[\int_V\left(-\tfrac{1}{2}\rho(\nabla\dot{\psi})^T\nabla\dot{\psi} - \tfrac{1}{2}\frac{p^2}{\rho c^2} + \nabla p\nabla\psi\right)dV - \int_{S_d} p\tilde{d}_n\,dS\right]dt. \tag{40}$$

5. FINITE ELEMENT DISCRETIZATION

5.1 Discretization of R_α

In the following we derive semidiscrete finite-element equations of motion based on the R_α functional (38). The volume V is subdivided into fluid finite elements. Over each fluid element the state is represented by the primary variables ψ and p, which are defined as functions of position in the usual shape-function interpolation procedure. The finite element interpolation in V may be expressed as

$$\psi(\mathbf{x}, t) = \mathbf{N}_\psi^T(\mathbf{x})\,\mathbf{\Psi}(t), \qquad p(\mathbf{x}, t) = \mathbf{N}_p^T(\mathbf{x})\,\mathbf{p}(t), \tag{41}$$

where $\mathbf{\Psi}$ and $\mathbf{p}$ are computational column vectors that contain nodal values of ψ and p, respectively, and $\mathbf{N}_\psi$ and $\mathbf{N}_p$ are corresponding arrays of dimensionless shape functions. The specified displacement over S_d is interpolated by

$$\tilde{d}_n(\mathbf{x}, t) = \mathbf{n}^T d(\mathbf{x}, t) = \mathbf{n}^T\mathbf{N}_d^T(\mathbf{x})\tilde{\mathbf{d}}, \tag{42}$$

where $\mathbf{n}$ is the external-normal unit vector on S_d, $\mathbf{N}_d$ contains the displacement shape functions of the enclosing container, and $\tilde{\mathbf{d}}$ contains nodal displacement values. For the moment these displacements will be assumed to be prescribed, hence the superposed tilde. Inserting (41)-(42) into the

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functional (38) yields the semidiscrete quadratic form

$$R_\alpha(\boldsymbol{\Psi}, \mathbf{p}) = -\tfrac{1}{2}\rho\dot{\boldsymbol{\Psi}}^T\mathbf{H}\dot{\boldsymbol{\Psi}} - \frac{1}{2\rho}\mathbf{p}^T\mathbf{G}\mathbf{p} + \alpha\boldsymbol{\Psi}^T\mathbf{F}\mathbf{p} + (1-\alpha)\left[\boldsymbol{\Psi}^T\mathbf{W}\mathbf{p} - \dot{\boldsymbol{\Psi}}^T\mathbf{D}\mathbf{p} + \boldsymbol{\Psi}^T\mathbf{f}_\psi\right] - \mathbf{p}^T\mathbf{T}^T\ddot{\mathbf{d}}, \quad (43)$$

where

$$\mathbf{H} = \int_V \nabla\mathbf{N}_\psi^T\nabla\mathbf{N}_\psi\,dV = \mathbf{H}^T, \qquad \mathbf{F} = \int_V \nabla\mathbf{N}_p^T\nabla\mathbf{N}_\psi\,dV, \qquad \mathbf{G} = \int_V c^{-2}\mathbf{N}_p^T\mathbf{N}_p\,dV = \mathbf{G}^T,$$

$$\mathbf{D} = \int_V c^{-2}\mathbf{N}_\psi^T\mathbf{N}_p\,dV, \quad \mathbf{W} = \int_{S_4} (\nabla_n\mathbf{N}_\psi)^T\mathbf{N}_p\,dS, \quad \mathbf{T}^T = \int_{S_4} \mathbf{N}_p^T\,\mathbf{n}^T\mathbf{N}_d\,dS, \quad \mathbf{f}_\psi = \int_{S_p} \tilde{p}\,\nabla_n\mathbf{N}_\psi\,dS.$$

$$(44)$$

The integration with respect to time is dropped as it has no effect on the variation process described below.

5.2 Continuity Requirements

The interelement continuity requirements of the shape functions of ψ and p depend on the index of the highest spatial derivatives that appears in R_α. If $\alpha \neq 0$, this index is 1 for both ψ and p and consequently C^0 continuity is required. It is then natural to take the *same* shape functions for both variables:

$$\mathbf{N}_\psi \equiv \mathbf{N}_p \tag{45}$$

with both vectors $\boldsymbol{\Psi}$ and $\mathbf{p}$ of equal dimension and evaluated at the same nodes. Then some of the matrices in (44) coalesce as

$$\mathbf{H} = \mathbf{F}, \qquad \mathbf{G} = \mathbf{D}. \tag{46}$$

The case $\alpha = 0$ is exceptional in that no spatial derivatives of p appear. One can then chose C^{-1} (discontinuous) pressure shape functions; for example, constant over each fluid element. If this is done, obviously

$$\mathbf{N}_\psi \neq \mathbf{N}_p \tag{47}$$

because ψ must be C^0 continuous. Furthermore the dimensions of $\mathbf{p}$ and $\boldsymbol{\Psi}$ will not be generally the same.

5.3 The Rigid-Container Equations of Motion

Since R_α contains time derivatives of of order up to 2 in $\boldsymbol{\Psi}$, the appropriate Euler-Lagrange variational equation is

$$\delta R_\alpha = \left(\frac{\partial R_\alpha}{\partial \boldsymbol{\Psi}} - \frac{\partial}{\partial t}\frac{\partial R_\alpha}{\partial \dot{\boldsymbol{\Psi}}} + \frac{\partial^2}{\partial t^2}\frac{\partial R_\alpha}{\partial \ddot{\boldsymbol{\Psi}}}\right)\delta\boldsymbol{\Psi} + \frac{\partial R_\alpha}{\partial \mathbf{p}}\delta\mathbf{p} = 0, \tag{48}$$

which applied to (43) yields

$$\left[\rho\mathbf{H}\ddot{\boldsymbol{\Psi}} + \alpha\mathbf{F}\mathbf{p} - (1-\alpha)\mathbf{D}\dot{\mathbf{p}} + (1-\alpha)\mathbf{W}\mathbf{p} + (1-\alpha)\mathbf{f}_\psi\right]\delta\boldsymbol{\Psi} = 0,$$

$$\left[-\rho^{-1}\mathbf{G}\mathbf{p} + \alpha\mathbf{F}^T\boldsymbol{\Psi} - (1-\alpha)\mathbf{D}^T\dot{\boldsymbol{\Psi}} + (1-\alpha)\mathbf{W}^T\boldsymbol{\Psi} - \mathbf{T}^T\ddot{\mathbf{d}}\right]\delta\mathbf{p} = 0. \tag{49}$$

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These equations can be represented in partitioned matrix form as

$$\begin{bmatrix} \rho H & -(1-\alpha)D \\ -(1-\alpha)D^T & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\Psi} \\ \ddot{p} \end{Bmatrix} + \begin{bmatrix} 0 & J \\ J^T & -\rho^{-1}G \end{bmatrix} \begin{Bmatrix} \Psi \\ p \end{Bmatrix} = \begin{Bmatrix} -(1-\alpha)f_\psi \\ T^T\tilde{d} \end{Bmatrix}, \qquad (50)$$

where $J = (1-\alpha)W + \alpha F$.

5.4 The Flexible-Container Equations of Motion

If the fluid is enclosed in a flexible container, the boundary displacements $\tilde{d}$ are no longer prescribed on S_d but must be incorporated in the problem by including them on the left hand side of the equations of motion. We shall only consider here the case in which the container is modelled as a *linear undamped* structure for which the standard mass/stiffness semidiscrete equation of motion is

$$M\ddot{d} + Kd = f_d + Tp, \qquad (51)$$

where M is the mass matrix, K the tangent stiffness matrix at the reference state, Tp is the pressure force on the structure, and f_d is the externally applied force on the structure. Note that K *in general must account for prestress effects through the geometric stiffness*. Combining (50) and (51) we get the coupled system

$$\begin{bmatrix} M & 0 & 0 \\ 0 & \rho H & -(1-\alpha)D \\ 0 & -(1-\alpha)D^T & 0 \end{bmatrix} \begin{Bmatrix} \ddot{d} \\ \ddot{\Psi} \\ \ddot{p} \end{Bmatrix} + \begin{bmatrix} K & 0 & -T \\ 0 & 0 & J \\ -T^T & J^T & -\rho^{-1}G \end{bmatrix} \begin{Bmatrix} d \\ \Psi \\ p \end{Bmatrix} = \begin{Bmatrix} f_d \\ -(1-\alpha)f_\psi \\ 0 \end{Bmatrix}. \qquad (52)$$

If $\alpha = 0$,

$$\begin{bmatrix} M & 0 & 0 \\ 0 & \rho H & D \\ 0 & D^T & 0 \end{bmatrix} \begin{Bmatrix} \ddot{d} \\ \ddot{\Psi} \\ \ddot{p} \end{Bmatrix} + \begin{bmatrix} K & 0 & -T \\ 0 & 0 & W \\ -T^T & W^T & -\rho^{-1}G \end{bmatrix} \begin{Bmatrix} d \\ \Psi \\ p \end{Bmatrix} = \begin{Bmatrix} f_d \\ -f_\psi \\ 0 \end{Bmatrix}. \qquad (53)$$

There is little than can be done beyond this point, as the shape functions for p and ψ will be generally different. Although the pressure may be constant over each element, no static condensation of p is possible in the dynamic case.

If $\alpha = 1$,

$$\begin{bmatrix} M & 0 & 0 \\ 0 & \rho H & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{d} \\ \ddot{\Psi} \\ \ddot{p} \end{Bmatrix} + \begin{bmatrix} K & 0 & -T \\ 0 & 0 & F \\ -T^T & F^T & -\rho^{-1}G \end{bmatrix} \begin{Bmatrix} d \\ \Psi \\ p \end{Bmatrix} = \begin{Bmatrix} f_d \\ 0 \\ 0 \end{Bmatrix}. \qquad (54)$$

Note that all these systems are *symmetric*.

5.5 Identical Shape Functions

Further progress in the case $\alpha = 1$ can be made if we assume, as discussed in §5.2, that the shape functions for p and ψ coincide. Taking then (46) into account, (54) simplifies to

$$\begin{bmatrix} M & 0 & 0 \\ 0 & \rho H & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{d} \\ \ddot{\Psi} \\ \ddot{p} \end{Bmatrix} + \begin{bmatrix} K & 0 & -T \\ 0 & 0 & H \\ -T^T & H & -\rho^{-1}G \end{bmatrix} \begin{Bmatrix} d \\ \Psi \\ p \end{Bmatrix} = \begin{Bmatrix} f_d \\ 0 \\ 0 \end{Bmatrix}. \qquad (55)$$

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The second matrix equation gives $\rho\mathbf{H}\ddot{\boldsymbol{\Psi}} + \mathbf{H}\mathbf{p} = 0$. Since $\mathbf{H}$ is nonnegative definite we must have

$$\mathbf{p} = -\rho\ddot{\boldsymbol{\Psi}}. \tag{56}$$

This is the discrete analog of the continuous relation (20) for the dynamic overpressure. It is possible to rearrange (54) into several equivalent symmetric forms; for example,

$$\begin{bmatrix} \mathbf{M} & \rho\mathbf{T} & \mathbf{0} \\ \rho\mathbf{T}^T & -\rho\mathbf{H} & \mathbf{G} \\ \mathbf{0} & \mathbf{G} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \rho^{-1}\mathbf{G} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{57}$$

plus two more symmetric forms not shown here. Note that (55) and (57) are *not* equivalent for the static limit, since the static matrix of the latter becomes singular. In fact, only (55) handles that condition in a numerically stable way.

5.6 Static Condensation

If (56) is used to eliminate the pressure vector from (55) we obtain

$$\begin{bmatrix} \mathbf{M} & \rho\mathbf{T} \\ \mathbf{0} & \mathbf{G} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\boldsymbol{\Psi}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ -\mathbf{T}^T & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\Psi} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ \mathbf{0} \end{Bmatrix}. \tag{58}$$

Conversely, eliminating the displacement potential vector gives

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \rho\mathbf{T}^T & \mathbf{G} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & -\mathbf{T} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ \mathbf{0} \end{Bmatrix}. \tag{59}$$

Unlike the previous equations, both (58) and (59) are *unsymmetric*. Thus the elimination of a field variable, be it p or ψ, causes symmetry to be lost.

6. CONCLUDING REMARKS

Displacement-potential formulations are of practical interest in fluid-structure transient-response and vibration problems since they form the basis for effective numerical computations. For some recent applications see [4-10] and references therein. The preceding treatment unifies a number of previous continuum-based and algebraic statements [7-12] of the coupled problem. It may be extended in the following directions:

(1) Elastoacoustic free vibrations of the coupled system. The symmetry properties of the algebraic eigenproblem are studied in [8-13].

(2) Gravity waves and free-surface "slosh" modes. An additional field, the elevation η, appears over S_p; but at the same time the fluid may be taken as incompressible. See *e.g.* [8].

(3) The inhomogeneous wave equation $c^2\nabla^2\psi - \ddot{\psi} = f$, $f \neq 0$, when the body force field $\mathbf{b}(\mathbf{x}, t)$ is time-dependent and $\nabla^2\mathbf{b} \neq 0$. Additional forcing terms, including $\mathbf{f}_p$, appear in the equations of motion. These are of interest for rotating fluids.

(4) Retaining the specific momentum $\mathbf{m}$ as independent field.

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Acknowledgements

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COUPLED FLUID-STRUCTURE INTERACTION
PART II, APPLICATION

ROGER OHAYON

Office National D'Études et de Recherches Aérospatiales
BP 72, 92322 Chatillon, France

CARLOS A. FELIPPA

Department of Aerospace Engineering Sciences
and Center for Space Structures and Controls
University of Colorado
Boulder, Colorado 80309-0429, USA

SUMMARY

The equations of motion for an acoustic fluid enclosed in a moving or flexible container are studied. It is shown that the determination of the reference state must account for the surface-integrated effect of the wall motions. The governing equation of transient motions about this state in the displacement potential does not generally reduce to the classical wave equation unless special adjustments are made. The results are relevant to finite element formulations based on the displacement potential.

1 Problem Description

The results presented in this Note were obtained in the course of a wider study by Felippa and Ohayon (1988) of variational methods for transient motions and vibrations of acoustic fluids held in flexible and/or moving containers. These partial results merit special attention on two counts. First, they extend the classical dynamic equations of acoustic fluids to include wall motions as well as the static limit in a consistent manner. Second, they are relevant to finite element implementations that have not accounted for the correction terms described herein. Computations that are particularly affected by these corrections involve liquid masses subject to prescribed dynamic motions, such as tanks and reservoirs under seismic ground-motion excitations, and rocket fuel tanks under launch conditions.

The general problem is as follows. A container (the *structure*) is totally or partly filled with a compressible, homogeneous liquid or gas (the *fluid*). Although the container is generally flexible, the rigid-but-moving container case is not excluded. The fluid is modelled as an acoustic medium (the linearly-compressible generalization of an ideal fluid). We consider dynamic motions about a static reference state, which will be determined as part of the study.

If the container is rigid and fixed, the reference state is well known: the static equilibrium solution in which the pressure is equal to the hydrostatic pressure, and the displacements may be taken as zero. The acoustic motions about this position are governed by the homogeneous wave equation in the displacement potential. But if the container walls move, we show that a correction term that depends on the mean boundary motion appears. The reference state is affected, and the resulting transient vibration problem is no longer given by the classical wave equation unless special adjustments are made.

A boundary-integral term representing the mean container motion was introduced by Aganovic (1981) for the surface-wave problem of an incompressible fluid posed in terms of the velocity potential. Ohayon (1987) considered similar terms in the displacement potential formulation. The general forms presented in this Note for a compressible are believed to be new.

2 The Acoustic Fluid

The three-dimensional volume occupied by the fluid is denoted by V. This volume is assumed to be simply connected. The fluid boundary S consists generally of two portions

$$S : S_d \cup S_p. \tag{1}$$

Table 1   Notation for Fluid States

Quantities	Domain	Original	Reference	Current	Transient
Displacements	V	0	$\mathbf{d}^0$	$\mathbf{d}^t$	$\mathbf{d} = \mathbf{d}^t - \mathbf{d}^0$
Velocities	V	0	$\dot{\mathbf{d}}^0$	$\dot{\mathbf{d}}^t$	$\dot{\mathbf{d}} = \dot{\mathbf{d}}^t - \dot{\mathbf{d}}^0$
Boundary displacements*	S	0	d_n^0	d_n^t	$d_n = d_n^t - d_n^0$
Displacement potential	V	0	ψ^0	ψ^t	$\psi = \psi^t - \psi^0$
Pressures	V	0	p^0	p^t	$p = p^t - p^0$
Body forces	V	0	$\mathbf{b} = \nabla\beta$	$\mathbf{b} = \nabla\beta$	
Density	V	ρ	ρ	ρ	

* Positive along outward normal

S_d is the interface with the container at which the normal displacement d_n is prescribed (or found as part of the coupled fluid-structure problem) whereas S_p is the "free surface" at which the pressure p is prescribed (or found as part of the "slosh" problem). If the fluid is fully enclosed by the container, as is necessarily the case for a gas, then S_p is missing and $S \equiv S_d$. The domain is referred to a Cartesian coordinate system (x_1, x_2, x_3) grouped in vector $\mathbf{x}$.

The fluid is under a body force field $\mathbf{b}$ which is assumed to be the gradient of a *time independent* potential $\beta(\mathbf{x})$, *i.e.* $\mathbf{b} = \nabla\beta$. All displacements are taken to be infinitesimal and thus the fluid density ρ is invariant.

We consider three states or configurations: *original*, from which displacements, pressures and forces are measured, *current*, where the fluid is in dynamic equilibrium at time t, and *reference*, which is obtained in the static equilibrium limit of slow motions. *Transient* motions are the difference between current and reference states. It should be noted that in many situations the original configuration is not physically attainable. Table 1 summarizes the notation used in relation to these states.

Field Equations.

The governing equations of the acoustic fluid are the momentum, state and continuity equations. They are stated below for the current configuration, and specialized to the reference configuration later. The momentum (balance) equation expresses Newton's second law for a fluid particle:

$$\rho \ddot{\mathbf{d}}^t = -\nabla p^t + \mathbf{b} = -\nabla p^t + \nabla\beta. \tag{2}$$

The continuity equation may be combined with the linearized equation of state to produce the *constitutive equation* that expresses the small compressibility of a liquid:

$$p^t = -K\nabla \mathbf{d}^t = -\rho c^2 \nabla \mathbf{d}^t, \tag{3}$$

where K is the bulk modulus and $c = \sqrt{K/\rho}$ the fluid sound speed. If the fluid is incompressible, $K, c \to \infty$. This relation is also applicable to nonlinear elastic fluids such as gases undergoing small excursions from the reference state, if the constitutive equation is linearized there so that $K = \rho_0 (dp/d\rho)_0$.

The boundary conditions are

$$d_n^t = \tilde{d}_n^t \quad \text{on} \quad S_d, \qquad p^t = \tilde{p}^t \quad \text{on} \quad S_p, \tag{4}$$

where $\tilde{d}_n^t$ is either prescribed or comes from the solution of an auxiliary problem as in fluid-structure interaction, and $\tilde{p}$ may be either prescribed or a function of d_n and b, as in the surface-wave ("slosh") problem.

3 The Reference State

Taking the curl of both sides of (2) yields

$$\text{curl } \ddot{\mathbf{d}}^t = 0. \tag{5}$$

The general integral of this equation for a simply connected domain is

$$\mathbf{d}^t = \nabla \psi^t + \mathbf{a} + \mathbf{b}t, \tag{6}$$

where $\psi^t = \psi^t(\mathbf{x}, t)$ is the displacement potential, $\mathbf{a} = \mathbf{a}(\mathbf{x})$ and $\mathbf{b} = \mathbf{b}(\mathbf{x})$ are time-independent vector functions, and t denotes the time. If accelerationless motions (for example, rigid body motions) are precluded by the boundary conditions, a and b vanish. Replacing $\ddot{\mathbf{d}}^t = \nabla \ddot{\psi}^t$ into the momentum equation (2) we get

$$\nabla p^t = -\rho \nabla \ddot{\psi}^t + \nabla \beta, \tag{7}$$

which spatially integrated gives

$$p^t = -\rho \ddot{\psi}^t + \beta + C(t), \tag{8}$$

where the scalar $C(t)$ is not spatially dependent. Next integrate the constitutive equation (3) over V and apply the divergence theorem to $\nabla \mathbf{d}$:

$$(p^t)_V + (\rho c^2 \nabla \mathbf{d}^t)_V = (p^t)_V + [\rho c^2 d_n^t]_S = 0. \tag{9}$$

Inserting p^t from (8) into the above equation furnishes a condition on $C(t)$, giving

$$C(t) = -\frac{\rho c^2}{v}[d_n^t]_S + \frac{\rho}{v}(\ddot{\psi}^t)_V - \frac{1}{v}(\beta)_V = -\frac{\rho c^2}{v}[d_n^t]_S + \rho\overline{\ddot{\psi}^t} - \overline{\beta}, \tag{10}$$
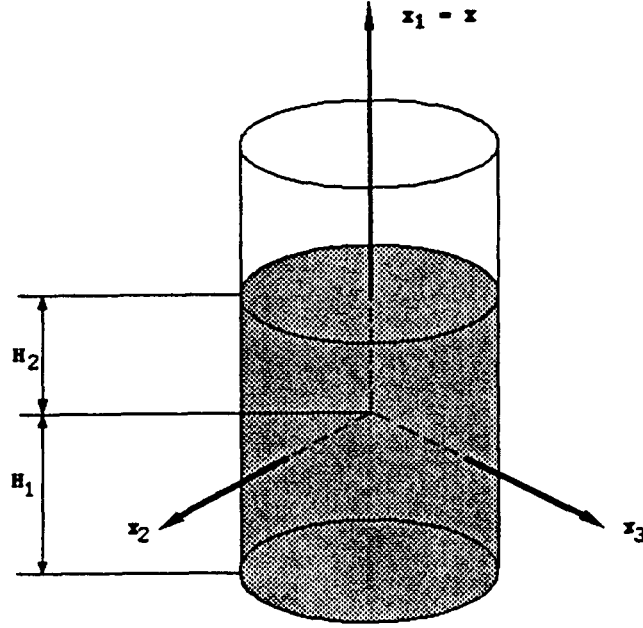
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Figure 1. Cylindrical fluid container in gravity field

where $v = (1)_V$ is the fluid volume and $\overline{f} = (f)_V/v$ denotes the volume average of a function f defined over V. Substituting $C(t)$ into (9) we get

$$p^t = -\rho(\ddot{\psi}^t - \overline{\ddot{\psi}^t}) + (\beta - \overline{\beta}) - \frac{\rho c^2}{v}[d_n^t]_S. \tag{11}$$

In the static limit the inertia terms may be neglected and we recover the reference solution

$$p^0 = (\beta - \overline{\beta}) - \frac{\rho c^2}{v}[d_n^0]_S. \tag{12}$$

For an incompressible fluid, $[d_n]_S = 0$ but $c \to \infty$; thus it would be incorrect to conclude that $p^0 = \beta - \overline{\beta}$. To illustrate this point, consider a rigid cylindrical container of cross section area A, filled with liquid up to height $H = H_1 + H_2$. The origin of the Cartesian system $(x_1,\ x_2,\ x_3)$ is placed at H_2 below the free surface, with $x_1 \equiv x$ upwards and normal to that surface; see Figure 1. The body force is the gravity field $\mathbf{b} = (-\rho g, 0, 0)$; thus $\beta = -\rho g x + B$, B being an arbitrary constant.

In passing from the original configuration under zero body force to the reference configuration under gravity, the free surface moves downwards by the amount

$$d_n|_{x=H_2} = \eta^0 = -\frac{\rho g H^2}{2K} = -\frac{g H^2}{2c^2}. \tag{13}$$

Evaluaton of (12) gives

$$p^0 = \rho g\left(x - \tfrac{1}{2}(H_2 - H_1)\right) + B - B + \frac{\rho c^2 A \eta^0}{AH} = \rho g(H_2 - x), \tag{14}$$

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which is the correct hydrostatic pressure if $\delta << H$. If one passes to the incompressible limit, $c \to \infty$ and $\delta \to 0$, but $c^2\delta$ remains fixed and equal to $gH^2/2$. Note that subtracting $\bar{\beta}$ eliminates B. The associated displacement field is easily calculated to be

$$\mathbf{d}^0 = \left\{ \begin{matrix} d_1^0 \\ d_2^0 \\ d_3^0 \end{matrix} \right\} = \left\{ \begin{matrix} -\frac{g}{2c^2}[H^2 - (x - H_2)^2]0 \\ 0 \\ 0 \end{matrix} \right\} + \mathbf{d}_{rot}^0, \tag{15}$$

where $\mathbf{d}_{rot}^0$ is an arbitrary divergence-free rotational motion that satisfies the boundary conditions.

4 Transient Motions

Subtracting the constitutive relations at the current and reference states we get

$$p = -\rho c^2 \nabla^2 \psi = \rho c^2 s, \tag{16}$$

where $s = -\nabla^2\psi$ is the condensation. Subtracting (12) from (11) yields

$$p = -\rho(\ddot{\psi} - \overline{\ddot{\psi}}) - \frac{\rho c^2}{v}[d_n]_S. \tag{17}$$

On equating (16) and (17) we get modified forms of the wave equation that account either for nonzero mean boundary surface motions,

$$s = \nabla^2\psi = \frac{\ddot{\psi} - \overline{\ddot{\psi}}}{c^2} + \frac{1}{v}[d_n]_S, \tag{18}$$

or nonzero mean dilatation,

$$c^2(\nabla^2\psi - \overline{\nabla^2\psi}) = c^2(\nabla^2\psi + \bar{s}) = \ddot{\psi} - \overline{\ddot{\psi}}. \tag{19}$$

The second form follows from $-v\bar{s} = [d_n]_S$, which is a consequence of the divergence theorem. For an incompressible fluid, $c \to \infty$ and $\bar{s} = [d_n]_S = 0$, and either form one recovers the Laplace equation $\nabla^2\psi = 0$.

Adjusting the Displacement Potential.

If the transient displacement potential is modified by a function of time:

$$\psi = \hat{\psi} + P(t), \tag{20}$$

where $\hat{\psi}$ is the potential of (6)-(19), we may chose $P(t)$ so that $c^2\overline{\ddot{\psi}} = \overline{\nabla^2\psi} = -\bar{s}$ for any t. [$P(t)$ may be found by integrating $c^2\ddot{\hat{\psi}} - \nabla^2\psi$ twice in time.] We then recover the classical wave equation

$$c^2\nabla^2\psi = \ddot{\psi}. \tag{21}$$

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If this adjustment has been made, $C(t)$ vanishes and (17) reduces to

$$p = -\rho\ddot{\psi}. \tag{22}$$

As an example, consider again the container of Figure 1 in which $H_1 = 0$ for convenience. At $t \leq 0$ the container is in the reference state of rest. At $t \geq 0$ it is subjected to a prescribed constant-velocity motion of the bottom surface,

$$\bar{d}_n(t)\Big|_{x=0} = -\alpha t, \quad t \geq 0, \tag{23}$$

positive upwards for $\alpha > 0$. The unknown free-surface vertical displacement is $\eta(t)$, also positive upwards. As all quantities become independent of x_2 and x_3, the governing equation is one dimensional:

$$\psi'' = \frac{\ddot{\psi} - \ddot{\bar{\psi}}}{c^2} + \frac{1}{H}(-\alpha t + \eta), \tag{24}$$

where primes denote derivatives with respect to $x_1 \equiv x$. The solution for $0 \leq t \leq H/c$ is

$$\eta = 0, \qquad \psi(x,t) = \begin{cases} \alpha t x + \frac{\alpha}{2c}x^2 + P(t), & x \leq ct, \\ P(t) - \frac{1}{2}\alpha ct^2, & x \geq ct. \end{cases} \tag{25}$$

The adjustment condition gives $P(t) = -\frac{1}{2}\alpha ct^2$ and consequently $\psi = 0$ for $x \geq ct$. Hence for finite c we have $\psi'' = c^2\ddot{\psi}$. If $c \to \infty$, the solution approaches the rigid body motion $\eta = \alpha t$.

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Carlos A. Felippa and Roger Ohayon

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

University of Colorado
Department of Aerospace Engineering
Boulder, Colorado 80309

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A general three-field variational principle is obtained for the motion of an acoustic fluid enclosed in a rigid or flexible container by the method of canonical decomposition applied to a modified form of the wave equation in the displacement potential. The general principle is specialized to a mixed two-field principle that contains the fluid displacement potential and pressure as independent fields. Semidiscrete finite-element equations of motion based on this principle are derived in Part I, while application sample cases are shown in Part II.

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