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# The Effect of Acceleration Versus Displacement Methods on Steady-State Boundary Forces

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D. S. McGhee

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**The Effect of Acceleration  
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Methods on Steady-State  
Boundary Forces**

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National Aeronautics and  
Space Administration  
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Scientific and Technical  
Information Program



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## TECHNICAL PAPER

# THE EFFECT OF ACCELERATION VERSUS DISPLACEMENT METHODS ON STEADY-STATE BOUNDARY FORCES

## I. INTRODUCTION

When a substructure model is reduced by the Craig-Bampton method, a number of degrees-of-freedom (DOF) are retained as physical DOF's to provide interface to other substructures. When more DOF's are retained in this interface than are actually required, the model is said to be over constrained. The results of this, when using the displacement method, are typically an inaccurate distribution of boundary forces. This inaccuracy also occurs when there are justifiably many interface DOF's which result in an indeterminate interface. When the acceleration method is used, this inaccuracy is overcome. However, many people do not fully understand this method and the many ways of implementing it, so its implementation is sometimes haphazard.

In the space shuttle payload community there has been an increase in the use of over-constrained payload models. This has been, mainly, to afford easy recovery of relative deflection data between the payload and the shuttle. While there has also been an increase in the use of the acceleration method for the recovery of payload displacements and forces, the displacement method remains the method used for recovering system displacements and forces.

The purpose of this study is to describe the acceleration method and investigate the problem of indeterminate or over-constrained interface substructures.

This report will first look at the acceleration and displacement methods from the example of a simple 2-DOF problem. This will define the methods and describe the similarities and differences. Generalizations will then be made to larger systems. A simple two-dimensional, two-beam problem will then be explored and conclusions drawn. Recommendations for areas of study with space shuttle payload systems will be made.

## II. DYNAMIC BASICS

Before discussing the acceleration method, it is useful to look at a simple 2-DOF system as shown in figure 1.

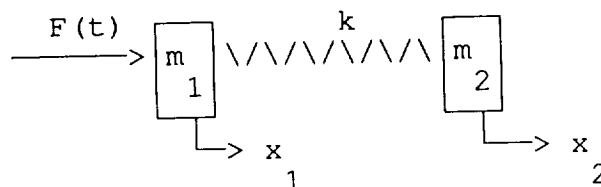


Figure 1. Two-DOF system.

The equation of motion for this system is:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix}. \quad (1)$$

The eigenvalues and eigenvectors can be determined and normalized (appendix A) such that the problem simplifies to:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{k(m_1+m_2)}{m_1 m_2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{m_1+m_2}} & \frac{1}{\sqrt{m_1+m_2}} \\ \sqrt{\frac{m_2}{m_1(m_1+m_2)}} & \frac{-m_1}{m_2} \sqrt{\frac{m_2}{m_1(m_1+m_2)}} \end{bmatrix} \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} F(t) \frac{1}{\sqrt{m_1+m_2}} \\ F(t) \frac{m_2}{\sqrt{m_1(m_1+m_2)}} \end{Bmatrix}, \quad (2)$$

$$\omega_1^2 = 0 \quad ; \quad \omega_2^2 = \frac{k(m_1+m_2)}{m_1 m_2}. \quad (3a, 3b)$$

$\omega_1$  is the frequency of the rigid body mode; i.e., both masses move in unison with no deflection in the spring.  $\omega_2$  is the elastic mode frequency of the two masses moving relative to each other deflecting the spring. A rigid-body mode will cause no internal forces since there is no deflection in the spring.

Here, an assumption will be made about the applied force,  $F(t)$ . It will be assumed to be a linear force since this assumption is typically made for each time step in a numerical solution. Therefore:

$$F(t) = F_s t + F_c \quad (4)$$

where  $s$  stands for slope and  $c$  for constant.

The solutions to the now uncoupled equations in equation (2) are then:

$$q_1 = \left[ \frac{F_s t^3}{6} + \frac{F_c t^2}{2} + C_1 t + C_2 \right] \frac{1}{\sqrt{m_1+m_2}}, \quad (5a)$$

$$q_2 = \left[ \frac{F(t)}{\omega_2^2} + (A \cos \omega_2 t + B \sin \omega_2 t) \right] \sqrt{\frac{m_2}{m_1(m_1+m_2)}}, \quad (5b)$$

$$\ddot{q}_1 = F(t) \frac{1}{\sqrt{m_1+m_2}}, \quad (5c)$$

$$\ddot{q}_2 = \left[ -(\omega_2^2 A \cos \omega_2 t + \omega_2^2 B \sin \omega_2 t) \right] \sqrt{\frac{m_2}{m_1(m_1+m_2)}}. \quad (5d)$$



The detailed steps for this solution are shown in appendix B. The constants,  $C_1$ ,  $C_2$ ,  $A$ , and  $B$  are determined by initial conditions.

One can see here that  $q_1$  is the generalized displacement of the first mode which is the rigid body mode. One can also see that  $q_2$  is the generalized displacement of the second or elastic mode. This mode relates to the two masses moving relative to each other, and one can see that it has two terms. The first term is the particular or steady-state solution which is the response due to the steady-state equilibrium between the applied and inertia forces. The second term is the dynamic response due to the harmonic transient. This second term would damp out if damping were included in the problem.

In regard to the generalized accelerations, one can see that  $\ddot{q}_1$  is the generalized rigid body acceleration due to the applied force and that  $\ddot{q}_2$  is the harmonic transient acceleration of the two masses relative to one another.

Now, the physical displacements and accelerations,  $x_1$ ,  $x_2$ ,  $\ddot{x}_1$ , and  $\ddot{x}_2$ , can be recovered by multiplying the modal displacements and accelerations by the eigenvectors.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [\phi] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{m_1+m_2}} & \sqrt{\frac{m_2}{m_1(m_1+m_2)}} \\ \frac{1}{\sqrt{m_1+m_2}} & \frac{-m_1}{m_2} \sqrt{\frac{m_2}{m_1(m_1+m_2)}} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}, \quad (6)$$

$$\begin{aligned} x_1 = & \left[ \frac{F_s t^3}{6} + \frac{F_c t^2}{2} + C_1 t + C_2 \right] \frac{1}{m_1+m_2} + \frac{F(t)}{\omega_2^2} \frac{m_2}{m_1(m_1+m_2)} \\ & + (A \cos \omega_2 t + B \sin \omega_2 t) \frac{m_2}{m_1(m_1+m_2)}, \end{aligned} \quad (7a)$$

$$\begin{aligned} x_2 = & \left[ \frac{F_s t^3}{6} + \frac{F_c t^2}{2} + C_1 t + C_2 \right] \frac{1}{m_1+m_2} + \frac{F(t)}{\omega_2^2} \frac{-1}{(m_1+m_2)} \\ & + (A \cos \omega_2 t + B \sin \omega_2 t) \frac{-1}{(m_1+m_2)}, \end{aligned} \quad (7b)$$

$$\ddot{x}_1 = \frac{F(t)}{(m_1+m_2)} - (\omega_2^2 A \cos \omega_2 t + \omega_2^2 B \sin \omega_2 t) \frac{m_2}{m_1(m_1+m_2)}, \quad (7c)$$

$$\ddot{x}_2 = \frac{F(t)}{(m_1+m_2)} + (\omega_2^2 A \cos \omega_2 t + \omega_2^2 B \sin \omega_2 t) \frac{1}{(m_1+m_2)}. \quad (7d)$$

From these, it can be seen that the displacements, like the generalized displacements are made up of rigid-body, steady-state, and transient terms. In fact, the rigid body terms for  $x_1$  and  $x_2$  are identical, which they should be since the body is moving as a whole. It can also be seen that the steady-state acceleration term is the classic force divided by mass and is the same for  $\ddot{x}_1$  and  $\ddot{x}_2$  since it is a rigid-body acceleration.

These then are the basic equations for the response of a single 2-DOF system. When the physical displacements computed this way are used in loads recovery, the method is called the displacement method.

### III. ACCELERATION METHOD

As seen in the previous section, the displacement method uses the eigenvectors to compute the physical displacements. Another way to compute the physical displacements is to use the physical accelerations computed above and to re-solve the system equations for the physical displacements. Reiterating the system equation:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix}. \quad (1)$$

Solving for the displacements:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}^{-1} \left[ \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix} - \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \right]. \quad (8)$$

Now, inverting  $\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$  is impossible, since it is free-free and the determinate is zero, however, it is realized that the accelerations were computed free-free and that any time the applied forces are in equilibrium with the inertia forces. Thus, one should be able to ground the system determinately, invert the stiffness matrix, and not induce any additional forces. The 2-DOF system can be grounded determinately at mass 1 by adding a small stiffness ( $k_2$ ) to the  $K_{11}$  term of stiffness matrix. The displacements calculated in this way will be called  $y_1$  and  $y_2$  to differentiate the two methods. Doing this, the following is obtained:

$$\begin{aligned} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} &= \begin{bmatrix} k+k_2 & -k \\ -k & k \end{bmatrix}^{-1} \left[ \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix} - \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \right] \\ &= \begin{bmatrix} \frac{1}{k_2} & \frac{1}{k_2} \\ \frac{1}{k_2} & \frac{k+k_2}{k k_2} \end{bmatrix} \left[ \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix} - \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \right] \\ &= \begin{Bmatrix} \frac{F(t)}{k_2} \\ \frac{F(t)}{k_2} \end{Bmatrix} - \begin{bmatrix} \frac{m_1}{k_2} & \frac{m_2}{k_2} \\ \frac{m_1}{k_2} & \frac{m_2(k+k_2)}{k k_2} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}, \quad (9) \end{aligned}$$

$$\begin{aligned} y_1 &= \frac{F(t)}{k_2} - \frac{m_1}{k_2} \ddot{x}_1 - \frac{m_2}{k_2} \ddot{x}_2 = \frac{F(t)}{k_2} - \frac{F(t) m_1}{k_2 (m_1+m_2)} + (\omega_2^2 A \cos \omega_2 t + \omega_2^2 B \sin \omega_2 t) \frac{m_2}{k_2 (m_1+m_2)} \\ &\quad - \frac{F(t) m_2}{k_2 (m_1+m_2)} - (\omega_2^2 A \cos \omega_2 t + \omega_2^2 B \sin \omega_2 t) \frac{m_2}{k_2 (m_1+m_2)} = 0, \quad (10) \end{aligned}$$

$$\begin{aligned}
y_2 &= \frac{F(t)}{k_2} - \frac{m_1}{k_2} \ddot{x}_1 - \frac{m_2(k+k_2)}{k k_2} \ddot{x}_2 = \frac{F(t)}{k_2} - \frac{F(t) m_1}{k_2 (m_1+m_2)} \\
&+ (\omega_2^2 A \cos \omega_2 t + \omega_2^2 B \sin \omega_2 t) \frac{m_2}{k_2 (m_1+m_2)} - \frac{F(t) m_2 (k+k_2)}{k k_2 (m_1+m_2)} \\
&- (\omega_2^2 A \cos \omega_2 t + \omega_2^2 B \sin \omega_2 t) \frac{m_2 (k+k_2)}{k k_2 (m_1+m_2)} \\
&= - \frac{F(t) m_2}{k (m_1+m_2)} - (\omega_2^2 A \cos \omega_2 t + \omega_2^2 B \sin \omega_2 t) \frac{m_2}{k (m_1+m_2)} \\
&= - \frac{F(t) m_2}{k (m_1+m_2)} - (A \cos \omega_2 t + B \sin \omega_2 t) \frac{1}{m_1} . \tag{11}
\end{aligned}$$

From this it can be seen that displacement  $y_1$  is equal to 0 which indicates that the ground spring that was added has not deflected, so the assumption of no added forces was correct. Also, it can be seen that the rigid body term is gone. This is understandable since the system was grounded, and it does not affect loads calculations since new forces were not added and rigid-body displacements caused no internal forces. One can also see that  $y_2$  is exactly equal to the difference between  $x_2$  and  $x_1$  calculated from the displacement method. For loads calculations, it is the relative displacements which are important so  $y_2$  suits the purpose adequately. And finally,  $y_2$  still contains the steady-state and transient terms.

This is the heart of the acceleration method. The displacements are found in terms of the accelerations.

#### IV. EFFECTS OF MODAL TRUNCATION

At first glance, the benefit of the acceleration method is hidden, since the same results have been obtained with much more effort. However, the benefit becomes apparent when modes are truncated. If, as above, all modes are used, the two methods are identical. Suppose the elastic mode is truncated from the 2-DOF problem, from equations (5), (6), and (7):

$$x_1 = \left[ \frac{F_s t^3}{6} + \frac{F_c t^2}{2} + C_1 t + C_2 \right] \frac{1}{m_1+m_2} , \tag{12a}$$

$$x_2 = \left[ \frac{F_s t^3}{6} + \frac{F_c t^2}{2} + C_1 t + C_2 \right] \frac{1}{m_1+m_2} , \tag{12b}$$

$$\ddot{x}_1 = \frac{F(t)}{(m_1+m_2)} ; \tag{12c}$$

$$\ddot{x}_2 = \frac{F(t)}{(m_1+m_2)} . \tag{12d}$$

And, if the new  $\ddot{x}_1$  and  $\ddot{x}_2$  are substituted into the equations for  $y_1$  and  $y_2$  (equations (9), (10), and (11):

$$y_1 = 0 , \quad (13a)$$

$$y_2 = -\frac{F(t) m_2}{k (m_1+m_2)} . \quad (13b)$$

Now, it can be seen that  $x_1$  and  $x_2$  have retained only the rigid-body displacement terms which have been noted to be of little use in loads calculations. However,  $y_2$  has retained the steady-state term which can be a major load contributor. So, it is the accuracy of the steady-state displacement term which is the benefit of the acceleration method.

## V. GENERALIZATION OF RESULTS

If one looks closely at the results from the 2-DOF problem in the previous section, some generalizations can be made to multi-DOF problems. First, ignoring rigid-body displacement, the modal displacements are:

$$q_{j\text{elastic}} = \frac{1}{\omega_j^2} \sum_{i=1}^n \phi_{ij} F_i(t) + A \sin (\omega_j t) + B \cos (\omega_j t) , \quad (14)$$

where:

$j$  = mode number

$i$  = physical force DOF number

$n$  = number of physical DOF's.

The modal accelerations are:

$$\ddot{q}_{j\text{rigid}} = \sum_{i=1}^n \phi_{ij} F_i(t) , \quad (15a)$$

$$\ddot{q}_{j\text{elastic}} = -\omega_j^2 A \sin (\omega_j t) - \omega_j^2 B \cos (\omega_j t) . \quad (15b)$$

The detailed steps for these solutions are shown in appendix C.

Now, the physical displacements are:

$$x_p = \sum_{j=1}^m \phi_{pj} q_j = \sum_{j=1}^m \phi_{pj} \left[ \frac{1}{\omega_j^2} \sum_{i=1}^n \phi_{ij} F_i(t) + A \sin (\omega_j t) + B \cos (\omega_j t) \right] , \quad (16)$$

where:

$j$  = mode number

$i$  = physical force DOF number

$p$  = physical displacement DOF number

$m$  = number of elastic modes.

The physical accelerations are:

$$\ddot{x}_p = \sum_{j=1}^r \phi_{pj} \left[ \sum_{i=1}^n \phi_{ij} F_i(t) \right] + \sum_{j=1}^m \phi_{pj} \left[ -\omega_j^2 A \sin(\omega_j t) - \omega_j^2 B \cos(\omega_j t) \right], \quad (17)$$

where:

$j$  = mode number

$i$  = physical force DOF number

$p$  = physical displacement DOF number

$m$  = number of elastic modes

$r$  = number of rigid body modes.

Since it has been determined that it is only the steady-state terms which the acceleration method helps, these terms can be isolated as follows:

$$x_{p_{\text{steady state}}} = \sum_{j=1}^m \phi_{pj} \left[ \frac{1}{\omega_j^2} \sum_{i=1}^n \phi_{ij} F_i(t) \right], \quad (18)$$

$$\ddot{x}_{p_{\text{steady state}}} = \sum_{j=1}^r \phi_{pj} \left[ \sum_{i=1}^n \phi_{ij} F_i(t) \right]. \quad (19)$$

So, one can see that the steady-state displacement is determined by the summation of contributions from each elastic mode. As modes are truncated, this summation will contain more error. However, the steady-state acceleration term which causes this displacement is dependent only on the summation of rigid-body modal contributions. These rigid-body modes are rarely truncated. Therefore, no matter how many elastic modes are truncated, steady-state displacements computed in terms of this acceleration will contain no error. Of course the transient term of the acceleration is still a summation, so the error introduced to the transient displacements by model truncation is unaffected by solution in terms of acceleration.

## VI. IMPACT ON COMPUTATION OF FORCES

As has been shown, the truncation of mode shapes of the boundary can lead to errors in the displacements computed by the displacement method. The error to the displacement vector  $\{x\}$  can be called  $\{e\}$ . Then one can find the error in the forces computed from these displacements since:

$$\{F\} = [K] \{x\} , \quad (20)$$

so,

$$\{F\}_{\text{error}} = [K] \{e\} . \quad (21)$$

For a simple 3-DOF problem, if the interface is determinate, then the DOF's are relatively uncoupled:

$$\begin{bmatrix} k & -k & 0 \\ -k & k+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} k e_1 & -k e_2 & \\ -k e_1 + (k+k_2) e_2 & -k_2 e_3 & \\ -k e_2 & +k e_3 & \end{Bmatrix} . \quad (22)$$

If they were indeterminate or coupled then:

$$\begin{bmatrix} k & -k & \boxed{C} \\ -k & k+k_2 & -k_2 \\ \boxed{C} & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} k e_1 & -k e_2 & + \boxed{C e_3} \\ -k e_1 + (k+k_2) e_2 & -k_2 e_3 & \\ \boxed{C e_1} & -k_2 e_2 & + k_2 e_3 \end{Bmatrix} . \quad (23)$$

So, as shown by these additional terms, the more coupled or indeterminate the system the more a force computed for 1 DOF is affected by error in the displacements at other DOF's. It should also be noted that because of the nature of the stiffness matrix these magnification effects on the displacement error can be quite large, thus producing large force errors from seemingly insignificant displacement errors. The acceleration method produces extremely accurate displacements which overcomes this problem.

## VII. APPLICATION TO A SIMPLE BEAM

A somewhat more complicated problem can now be defined for investigation. Take a simple two-dimensional beam with element nodes having translational and rotational DOF's (fig. 2). Two of these discrete physical models can be coupled together. Thus, as shown in figures 3 and 4, each model can be divided into interior and boundary DOF's, coupled together at either 2 or 4 translational DOF's and driven by a force.

The equations of motion for these beams and system can be written:

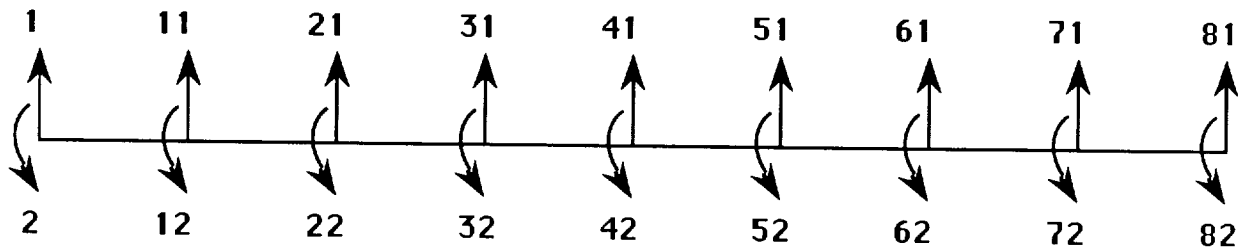


Figure 2. Simple beam DOF's.

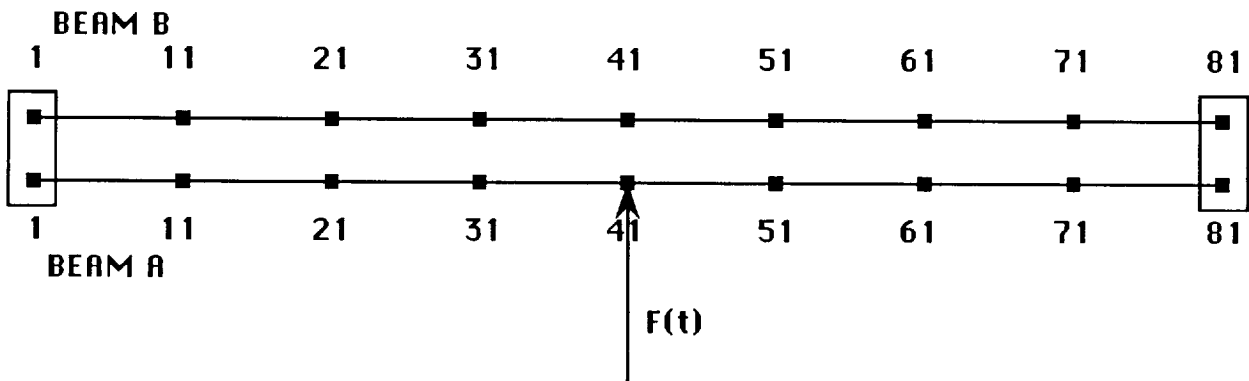


Figure 3. Two beams coupled at 2 DOF's.

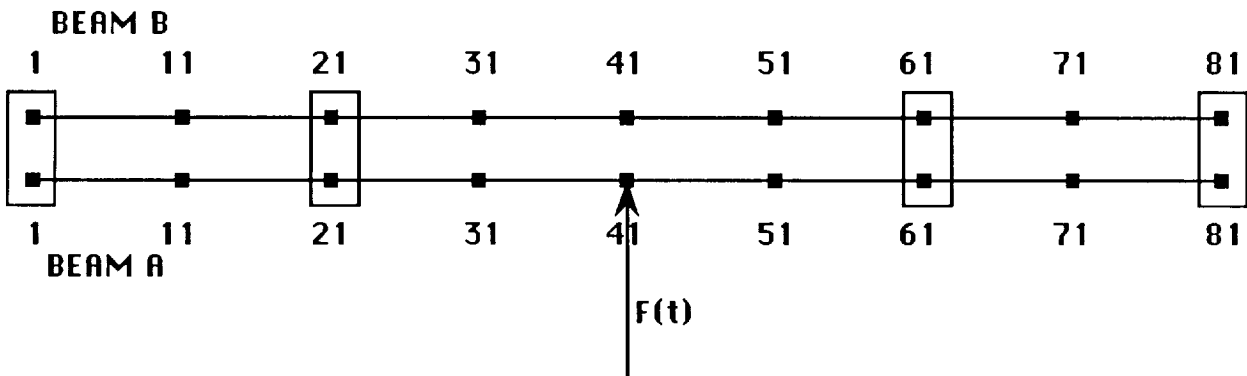


Figure 4. Two beams coupled at 4 DOF's.

$$\begin{bmatrix} M_{ii} & M_{ib} \\ M_{bi} & M_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{X}_i \\ \ddot{X}_b \end{Bmatrix}_B + \begin{bmatrix} K_{ii} & K_{ib} \\ K_{bi} & K_{bb} \end{bmatrix} \begin{Bmatrix} X_i \\ X_b \end{Bmatrix}_B = \begin{Bmatrix} 0 \\ F_b \end{Bmatrix}, \quad (24)$$

$$\begin{bmatrix} M_{bb} & M_{bi} \\ M_{ib} & M_{ii} \end{bmatrix} \begin{Bmatrix} \ddot{X}_b \\ \ddot{X}_i \end{Bmatrix}_A + \begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii} \end{bmatrix} \begin{Bmatrix} X_b \\ X_i \end{Bmatrix}_A = \begin{Bmatrix} F_b \\ F \end{Bmatrix}, \quad (25)$$

$$[M] \begin{Bmatrix} \ddot{X}_{iB} \\ \ddot{X}_b \\ \ddot{X}_{iA} \end{Bmatrix} + [K] \begin{Bmatrix} X_{iB} \\ X_b \\ X_{iA} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix}, \quad (26)$$

here:

$i$  = interior DOF

$b$  = boundary DOF.

Equation (26) can then be coupled by a eigenvalue problem.

$$\begin{Bmatrix} X_{iB} \\ X_b \\ X_{iA} \end{Bmatrix} = [\phi] \{q\}, \quad (27)$$

$$[I] \{\ddot{q}\} + [\omega^2] \{q\} = [\phi]^T \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix}. \quad (28)$$

These uncoupled equations can then be solved for  $\{q\}$  and  $\{\ddot{q}\}$ .

The physical displacements can then be recovered by the displacement or acceleration method. Equation (27) is the displacement method, and the acceleration method is shown in equation (29). The displacements from the acceleration method will again be called  $Y$  to differentiate them.

$$\begin{Bmatrix} Y_{iB} \\ Y_b \\ Y_{iA} \end{Bmatrix} = [K^*]^{-1} \left[ \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix} - [M] \begin{Bmatrix} \ddot{X}_{iB} \\ \ddot{X}_b \\ \ddot{X}_{iA} \end{Bmatrix} \right]. \quad (29)$$

Here, an adjustment must be made so that a comparison can be made between the two methods. In the acceleration method, because the stiffness matrix is grounded, the displacement of the ground DOF is zero and all other displacements are relative to it. In the displacement method, assuming rigid body displacements are zeroed, all displacements are relative to the center of gravity. Therefore, a correction is made to the displacement method displacements. If the displacement method displacement of the acceleration method grounded DOF is distributed by a rigid-body transformation and then subtracted from the displacement method displacements, a one-to-one comparison can be made between the two methods.



Now, as has already been noted, the acceleration method is only useful in gaining accuracy in the displacements caused by the steady-state accelerations. Therefore, it is desirable to isolate this displacement to eliminate confusion with other effects. As seen in the acceleration method, it is easy to separate the steady-state, rigid-body accelerations, and therefore displacements, from the transients by simply partitioning  $\{\ddot{q}\}$ . This separation is not possible in the displacement method. However, it has been shown that if both methods are truncated the same, then the transient portions are equal. Therefore, the steady-state and transient portions of the displacements can be computed by the acceleration method. This transient portion can then be subtracted from the displacement method solution to obtain a displacement method steady-state solution. The two methods can then be compared without confusion by truncation effects on the transient portions.

### VIII. RESULTS FOR THE SIMPLE BEAM PROBLEM

For the problem of two discrete physical beam models coupled together as described in section VII, the effect of truncation in the system eigenvalue problem, equation (27), was investigated. The particular aspect investigated was that of interface forces between the substructures or beams as computed by equations (24) and (25). The results were compared for both the acceleration and displacement methods and for severely truncated modes (all rotational modes truncated) and no modal truncation. These results are tabulated in table 1. Keep in mind that only the steady-state response is compared and that the only truncation is at the system eigenvalue level.

As can be seen, and expected, there is excellent comparison between the two methods when all the modes are used. The acceleration method also agrees exactly whether modes are truncated or not. However, the displacement method with truncated modes is greatly in error. In fact, the error in the applied force on beam A is 90 to 95 percent.

### IX. CRAIG-BAMPTON REDUCTION

An investigation of what happens if beam B is reduced by the Craig-Bampton method is shown in the following. The coordinate transformation for this reduction is:

$$\begin{Bmatrix} X_i \\ X_b \end{Bmatrix} = \begin{bmatrix} \phi_C & -K_{ii}^{-1} K_{ib} \\ 0 & I \end{bmatrix} \begin{Bmatrix} Q_i \\ X_b \end{Bmatrix}. \quad (30)$$

Equation (24) can then be replaced by:

$$\begin{bmatrix} I & \overline{M}_{ib} \\ \overline{M}_{bi} & \overline{M}_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{Q}_i \\ \ddot{X}_b \end{Bmatrix} + \begin{bmatrix} \omega^2 & 0 \\ 0 & \overline{K}_{bb} \end{bmatrix} \begin{Bmatrix} Q_i \\ X_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_b \end{Bmatrix}. \quad (31)$$

Table 1. Effect of acceleration versus displacement methods on steady-state boundary forces.

BEAM A	ERROR				REFERENCE VALUE
	abs(abs(REFERENCE)-abs)				
	acceleration method all modes	displacement method all modes	acceleration method truncated	displacement method truncated	
2 coupled DOF's					
DOF 1	0.00000000	0.00000001	0.00000000	0.02689317	-0.16666667
DOF 41	0.00000000	0.00000005	0.00000000	0.47039463	0.49999997
DOF 81	0.00000000	0.00000001	0.00000000	0.02689319	-0.16666665
4 coupled DOF's					
DOF 1	0.00000000	0.00000019	0.00000000	0.00602957	-0.00722064
DOF 21	0.00000000	0.00000011	0.00000000	0.02826142	-0.15944602
DOF 41	0.00000000	0.00000000	0.00000000	0.44786349	0.50000000
DOF 61	0.00000000	0.00000000	0.00000000	0.02826125	-0.15944602
DOF 81	0.00000000	0.00000000	0.00000000	0.00602981	-0.00722064

BEAM B	ERROR				REFERENCE VALUE
	abs(abs(REFERENCE)-abs)				
	acceleration method all modes	displacement method all modes	acceleration method truncated	displacement method truncated	
2 coupled DOF's					
DOF 1	0.00000000	0.00000000	0.00000000	0.01147362	0.16666667
DOF 41	0.00000000	0.00000004	0.00000000	0.00540434	0.00000000
DOF 81	0.00000000	0.00000001	0.00000000	0.01147363	0.16666665
4 coupled DOF's					
DOF 1	0.00000000	0.00000030	0.00000000	0.00057095	0.00722064
DOF 21	0.00000000	0.00000063	0.00000000	0.02842032	0.15944602
DOF 41	0.00000000	0.00000004	0.00000000	0.00320943	0.00000000
DOF 61	0.00000000	0.00000000	0.00000000	0.02841970	0.15944602
DOF 81	0.00000000	0.00000000	0.00000000	0.00057068	0.00722064

The coupled system is then:

$$[\overline{M}] \begin{Bmatrix} \ddot{Q}_i B \\ \ddot{X}_b \\ \ddot{X}_i A \end{Bmatrix} + [\overline{K}] \begin{Bmatrix} Q_i B \\ X_b \\ X_i A \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix}. \quad (32)$$

The displacements  $\begin{Bmatrix} Q_i B \\ X_b \\ X_i A \end{Bmatrix}$  can be determined by the same methods as equations (27), (28), and (29). The interface forces for beam B are computed from equation (31).

In this study, four different Craig-Bampton boundary conditions were considered. The first two were an over-constrained case where all translational DOF's of beam B were

retained as boundary DOF's; one was for no Craig-Bampton modal truncation and another for severely truncated Craig-Bampton modes. The third case considered was a Craig-Bampton reduction from the physical model down to the coupling DOF's. Craig-Bampton modes were truncated. And finally, the fourth case was a two-step reduction first to the translational DOF's with truncation and then to the boundary DOF's.

## X. RESULTS OF CRAIG-BAMPTON REDUCTION SCHEMES

For all four Craig-Bampton cases described in section IX, it was, again, the interface forces that were studied. The acceleration method with and without system modal truncation and the displacement method using all modes showed excellent agreement. This data is presented in appendix D.

For all four cases, the only method that indicated error was the displacement method using truncated system modes. These results are tabulated in table 2. Keep in mind that this data is only for the displacement method using truncated system modes, and that only the steady-state response is considered. The following conclusions can be drawn from this data:

The severity of truncation of the Craig-Bampton modes does not affect steady-state interface force error. It is the truncation of modes describing the motion across the boundary (i.e., system modes) that causes the error.

Over-constrained Craig-Bampton models, when used with the displacement method, introduce sizeable error in the steady-state interface forces.

Boundary-constrained Craig-Bampton models, when used with the displacement method, provide accurate steady-state interface forces. It is noted that they are not as accurate as those computed by the acceleration method.

For boundary-constrained Craig-Bampton models, the error for indeterminate interfaces is greater than for determinate interfaces.

A two-step reduction to the boundary provides just as accurate steady-state interface forces as a single-step reduction.

Finally, a discrete physical model is the limit in over constraint. A discrete physical model can be considered to be over constrained by retaining all DOF's. The error due to system mode truncation is substantial since more information is truncated than in Craig-Bampton models.

Table 2. Effect of Craig-Bampton reduction schemes on steady-state boundary forces (truncated displacement method).

BEAM A Physical		ERROR abs(abs(REFERENCE)-abs)				REFERENCE VALUE
		Physical Beam A				
		Beam B over constrained no truncation	Beam B over constrained truncated	Beam B 1-step to boundary truncated	Beam B 2-step to boundary truncated	acceleration method all modes
2 coupled DOF's						
	DOF 1	0.02689333	0.02689277	0.02689245	0.02689188	-0.16666667
	DOF 41	0.47039469	0.47039462	0.47039451	0.47039449	0.49999997
	DOF 81	0.02689316	0.02689277	0.02689206	0.02689189	-0.16666665
4 coupled DOF's						
	DOF 1	0.00602957	0.00602840	0.00602595	0.00602551	-0.00722064
	DOF 21	0.02826125	0.02826261	0.02825282	0.02825380	-0.15944602
	DOF 41	0.44786349	0.44786334	0.44786317	0.44786309	0.50000000
	DOF 61	0.02826125	0.02826261	0.02825274	0.02825380	-0.15944602
	DOF 81	0.00602962	0.00602892	0.00602571	0.00602551	-0.00722064

BEAM B Craig-Bampton		ERROR abs(abs(REFERENCE)-abs)				REFERENCE VALUE
		Physical Beam A				
		Beam B over constrained no truncation	Beam B over constrained truncated	Beam B 1-step to boundary truncated	Beam B 2-step to boundary truncated	acceleration method all modes
2 coupled DOF's						
	DOF 1	0.01335352	0.01335329	0.00000000	0.00000000	0.16666667
	DOF 41	0.00547892	0.00547904	n/a	n/a	0.00000000
	DOF 81	0.01335353	0.01335329	0.00000001	0.00000002	0.16666665
4 coupled DOF's						
	DOF 1	0.00046798	0.00046793	0.00001097	0.00001097	0.00722064
	DOF 21	0.02862931	0.02863001	0.00001101	0.00001103	0.15944602
	DOF 41	0.00327701	0.00327707	n/a	n/a	0.00000000
	DOF 61	0.02862883	0.02862996	0.00001098	0.00001099	0.15944602
	DOF 81	0.00046812	0.00046814	0.00001095	0.00001097	0.00722064

## XI. CONCLUSIONS AND RECOMMENDATIONS

This study has described the acceleration and displacement methods for use in the recovery of coupled-system boundary forces. A simple 2-DOF system has been used for illustration. The effect of the choice of method for use with indeterminate or over-constrained boundaries has been investigated. It has specifically looked at results from a simple two-dimensional beam problem using both methods.

In the space shuttle payload community, there has been an increase in the use of over-constrained payload models. Much work has been done on the effects of Craig-Bampton modal truncation on system displacements and forces, however little work has been done on system-level modal truncation (i.e., modes across the boundary). The findings of this study indicate that the effect of this system-level truncation is significant. This may be particularly true for the 35-Hz system cutoff frequency that is required by the space shuttle. From this study's findings, the following recommendations can be made:

The effect of using displacement method recovery for over-constrained payload models with the space shuttle directed 35-Hz system cutoff needs to be studied.

Until this recommended study is conducted, either the acceleration method should be used or a secondary Craig-Bampton reduction to only the attaching interfaces should be done for payload-coupled loads analyses.

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## APPENDIX A

### Eigenvalues and Eigenvectors of a 2-DOF System

The homogeneous equation of motion for a 2-DOF system is:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \quad (\text{A-1})$$

Assuming a sinusoidal response:

$$\begin{aligned} x_1 &= A \cos(\omega t - \alpha) & x_2 &= A \cos(\omega t - \alpha) \\ \ddot{x}_1 &= -\omega^2 A \cos(\omega t - \alpha) & \ddot{x}_2 &= -\omega^2 A \cos(\omega t - \alpha) \end{aligned} \quad (\text{A-2})$$

Substituting into equation (A-1):

$$-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{bmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \quad (\text{A-3})$$

Equation (A-3) has a nontrivial solution only if the determinate is zero, i.e.,

$$\begin{vmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{vmatrix} = (k - \omega^2 m_1)(k - \omega^2 m_2) - (-k)(-k) = \omega^2 (\omega^2 m_1 m_2 - k(m_1 + m_2)) = 0 . \quad (\text{A-4})$$

The roots of equation (A-4) are:

$$\omega_1^2 = 0 , \quad (\text{A-5a})$$

$$\omega_2^2 = \frac{k(m_1 + m_2)}{m_1 m_2} . \quad (\text{A-5b})$$

To define the eigenvectors, the top half of equation (A-3) is:

$$(k - \omega^2 m_1)A - kB = 0 .$$

Defining:

$$C = \frac{B}{A} = \frac{(k - \omega^2 m_1)}{k} = 1 - \omega^2 \frac{m_1}{k} , \quad (\text{A-6})$$

$$C_1 = 1 - 0 = 1 \quad (\text{A-7a})$$

$$C_2 = 1 - \frac{k(m_1 + m_2)}{m_1 m_2} \frac{m_1}{k} = \frac{-m_1}{m_2} . \quad (\text{A-7b})$$

Therefore, the eigenvectors:

$$[\phi] = \begin{bmatrix} 1 & 1 \\ 1 & \frac{-m_1}{m_2} \end{bmatrix}. \quad (\text{A-8})$$

These eigenvectors are then normalized such that  $\phi^T M \phi$  equals unity:

$$[\phi]_{\text{normalized}} = \begin{bmatrix} \frac{1}{\sqrt{m_1+m_2}} & \sqrt{\frac{m_2}{m_1(m_1+m_2)}} \\ \frac{1}{\sqrt{m_1+m_2}} & \frac{-m_1}{m_2} \sqrt{\frac{m_2}{m_1(m_1+m_2)}} \end{bmatrix}. \quad (\text{A-9})$$



## APPENDIX B

### Solution of Uncoupled Equations of Motion of a 2-DOF System

The uncoupled equations of motion for a 2-DOF system are:

$$\ddot{q}_1 + 0 q_1 = \frac{F(t)}{\sqrt{m_1+m_2}} = (F_s t + F_c) \frac{1}{\sqrt{m_1+m_2}} , \quad (\text{B-1})$$

$$\ddot{q}_2 + \omega^2 q_2 = (F_s t + F_c) \sqrt{\frac{m_2}{m_1(m_1+m_2)}} . \quad (\text{B-2})$$

The solutions to these equations will take the form:

$$q = C_1 t^3 + C_2 t^2 + C_3 t + C_4 + C_5 \sin \omega t + C_6 \cos \omega t , \quad (\text{B-3a})$$

$$\dot{q} = 3C_1 t^2 + 2C_2 t + C_3 + \omega C_5 \cos \omega t - \omega C_6 \sin \omega t , \quad (\text{B-3b})$$

$$\ddot{q} = 6C_1 t + 2C_2 - \omega^2 C_5 \sin \omega t - \omega^2 C_6 \cos \omega t . \quad (\text{B-3c})$$

In equation (B-1)  $\omega = 0$ , therefore:

$$6C_1 t + 2C_2 = (F_s t + F_c) \frac{1}{\sqrt{m_1+m_2}} , \quad (\text{B-4})$$

$$C_1 = \frac{F_s}{6} \frac{1}{\sqrt{m_1+m_2}} , \quad (\text{B-5a})$$

$$C_2 = \frac{F_c}{2} \frac{1}{\sqrt{m_1+m_2}} . \quad (\text{B-5b})$$

Therefore:

$$q_1 = \left[ \frac{F_s t^3}{6} + \frac{F_c t^3}{2} + C_3 + C_4 \right] \frac{1}{\sqrt{m_1+m_2}} . \quad (\text{B-6})$$

Substituting equations (B-3) into equation (B-2):

$$\omega^2 C_1 t^3 + \omega^2 C_2 t^2 + (\omega^2 C_3 + 6C_1) t + (\omega^2 C_4 + 2C_2) = (F_s t + F_c) \sqrt{\frac{m_2}{m_1(m_1+m_2)}} , \quad (\text{B-7})$$

$$C_1 = 0 , \quad (\text{B-8a})$$

$$C_2 = 0 , \quad (\text{B-8b})$$

$$\omega^2 C_3 t + \omega^2 C_4 = (F_s t + F_c) \sqrt{\frac{m_2}{m_1(m_1+m_2)}} , \quad (\text{B-9})$$

$$C_3 = \frac{F_s}{\omega^2} \sqrt{\frac{m_2}{m_1(m_1+m_2)}} , \quad (\text{B-10a})$$

$$C_4 = \frac{F_c}{\omega^2} \sqrt{\frac{m_2}{m_1(m_1+m_2)}} , \quad (\text{B-10b})$$

therefore:

$$q_2 = \left[ \frac{F(t)}{\omega_2^2} + (A \cos \omega_2 t + B \sin \omega_2 t) \right] \sqrt{\frac{m_1}{m_1(m_1+m_2)}} . \quad (\text{B-11})$$

Constants  $A$  and  $B$  in equation (B-11) and constants  $C_3$  and  $C_4$  in equation (B-6) are determined from the initial conditions.

## APPENDIX C

### Solution of Uncoupled Equations of Motion of a General System

The uncoupled equation of motion for a general system is:

$$\ddot{q}_j + \omega_j^2 q_j = \sum_{i=1}^n \phi_{ij} F_i(t) \quad , \quad (\text{C-1})$$

where:

$j$  = mode number

$i$  = physical DOF number

$n$  = number of physical DOF's.

If linear forces are assumed, then the solution to this equation will take the form:

$$q = C_1 t^3 + C_2 t^2 + C_3 t + C_4 + C_5 \sin \omega t + C_6 \cos \omega t \quad , \quad (\text{C-2a})$$

$$\dot{q} = 3C_1 t^2 + 2C_2 t + C_3 + \omega C_5 \cos \omega t - \omega C_6 \sin \omega t \quad , \quad (\text{C-2b})$$

$$\ddot{q} = 6C_1 t + 2C_2 - \omega^2 C_5 \sin \omega t - \omega^2 C_6 \cos \omega t \quad . \quad (\text{C-2c})$$

For rigid-body modes  $\omega_j^2 = 0$ , therefore:

$$\ddot{q}_j = \sum_{i=1}^n \phi_{ij} F_i = \sum_{i=1}^n \phi_{ij} (F_{si} t + F_{ci}) = 6C_1 t + 2C_2 \quad , \quad (\text{C-3})$$

$$C_1 = \frac{1}{6} \sum_{i=1}^n \phi_{ij} F_{si} \quad , \quad (\text{C-4a})$$

$$C_2 = \frac{1}{2} \sum_{i=1}^n \phi_{ij} F_{ci} \quad . \quad (\text{C-4b})$$

So, for rigid-body motion:

$$q_{j\text{rigid}} = \frac{t^3}{6} \sum_{i=1}^n \phi_{ij} F_{si} + \frac{t^2}{2} \sum_{i=1}^n \phi_{ij} F_{ci} + C_3 t + C_4 \quad , \quad (\text{C-5})$$

$$\ddot{q}_{j\text{rigid}} = t \sum_{i=1}^n \phi_{ij} F_{si} + \sum_{i=1}^n \phi_{ij} F_{ci} = \sum_{i=1}^n \phi_{ij} F_i(t) \quad . \quad (\text{C-6})$$

For elastic-body modes:

$$\ddot{q}_j + \omega_j^2 q_j = \sum_{i=1}^n \phi_{ij} F_i . \quad (\text{C-7})$$

Substituting equations (C-2) into equation (C-7):

$$\omega_j^2 C_1 t^3 + \omega_j^2 C_2 t^2 + (\omega_j^2 C_3 + 6C_1)t + (\omega_j^2 C_4 + 2C_2) = \sum_{i=1}^n \phi_{ij} (F_{si}t + F_{ci}) , \quad (\text{C-8})$$

$$C_1 = 0 , \quad (\text{C-9a})$$

$$C_2 = 0 , \quad (\text{C-9b})$$

$$\omega^2 C_3 t + \omega^2 C_4 = \sum_{i=1}^n \phi_{ij} (F_{si}t + F_{ci}) , \quad (\text{C-10})$$

$$C_3 = \frac{1}{\omega_j^2} \sum_{i=1}^n \phi_{ij} F_{si} , \quad (\text{C-11a})$$

$$C_4 = \frac{1}{\omega_j^2} \sum_{i=1}^n \phi_{ij} F_{ci} . \quad (\text{C-11b})$$

So, for elastic motion:

$$q_{j\text{elastic}} = \frac{1}{\omega_j^2} \sum_{i=1}^n \phi_{ij} F_i(t) + A \sin(\omega_j t) + B \cos(\omega_j t) , \quad (\text{C-12})$$

$$\ddot{q}_{j\text{elastic}} = -\omega_j^2 A \sin(\omega_j t) - \omega_j^2 B \cos(\omega_j t) . \quad (\text{C-13})$$

Constants  $A$  and  $B$  in equations (C-12) and (C-13) and constants  $C_3$  and  $C_4$  in equation (C-5) are determined from the initial conditions.

## APPENDIX D

### Analysis Results

#### EFFECT OF ACCELERATION vs DISPLACEMENT METHODS ON STEADY STATE BOUNDARY FORCES

(Beam B - Craig-Bampton - over constrained - no truncation)

BEAM A Physical		ERROR abs(abs(REFERENCE)-abs)				REFERENCE VALUE
		acceleration method all modes	displacement method all modes	acceleration method truncated	displacement method truncated	acceleration method all modes
2 coupled DOF's						
	DOF 1	0.00000000	0.00000016	0.00000000	0.02689333	-0.16666667
	DOF 41	0.00000000	0.00000001	0.00000000	0.47039469	0.49999997
	DOF 81	0.00000000	0.00000002	0.00000000	0.02689316	-0.16666665
4 coupled DOF's						
	DOF 1	0.00000000	0.00000042	0.00000000	0.00602957	-0.00722064
	DOF 21	0.00000000	0.00000000	0.00000000	0.02826125	-0.15944602
	DOF 41	0.00000000	0.00000000	0.00000000	0.44786349	0.50000000
	DOF 61	0.00000000	0.00000000	0.00000000	0.02826125	-0.15944602
	DOF 81	0.00000000	0.00000000	0.00000000	0.00602962	-0.00722064

BEAM B Craig-Bampton		ERROR abs(abs(REFERENCE)-abs)				REFERENCE VALUE
		acceleration method all modes	displacement method all modes	acceleration method truncated	displacement method truncated	acceleration method all modes
2 coupled DOF's						
	DOF 1	0.00000000	0.00000001	0.00000000	0.01335352	0.16666667
	DOF 41	0.00000000	0.00000003	0.00000000	0.00547892	0.00000000
	DOF 81	0.00000000	0.00000002	0.00000000	0.01335353	0.16666665
4 coupled DOF's						
	DOF 1	0.00000000	0.00000030	0.00000000	0.00046798	0.00722064
	DOF 21	0.00000000	0.00000049	0.00000000	0.02862931	0.15944602
	DOF 41	0.00000000	0.00000003	0.00000000	0.00327701	0.00000000
	DOF 61	0.00000000	0.00000000	0.00000000	0.02862883	0.15944602
	DOF 81	0.00000000	0.00000000	0.00000000	0.00046812	0.00722064

**EFFECT OF ACCELERATION vs DISPLACEMENT METHODS ON  
STEADY STATE BOUNDARY FORCES**

(Beam B - Craig-Bampton - over constrained - truncated)

<b>BEAM A</b> Physical		ERROR				REFERENCE VALUE
		abs(abs(REFERENCE)-abs)				
		acceleration method all modes	displacement method all modes	acceleration method truncated	displacement method truncated	acceleration method all modes
2 coupled DOF's						
	DOF 1	0.00000000	0.00000000	0.00000000	0.02689278	-0.16666667
	DOF 41	0.00000000	0.00000004	0.00000000	0.47039459	0.49999997
	DOF 81	0.00000000	0.00000001	0.00000000	0.02689277	-0.16666665
4 coupled DOF's						
	DOF 1	0.00000000	0.00000045	0.00000000	0.00602840	-0.00722064
	DOF 21	0.00000000	0.00000000	0.00000000	0.02826261	-0.15944602
	DOF 41	0.00000000	0.00000000	0.00000000	0.44786334	0.50000000
	DOF 61	0.00000000	0.00000000	0.00000000	0.02826261	-0.15944602
	DOF 81	0.00000000	0.00000000	0.00000000	0.00602892	-0.00722064

<b>BEAM B</b> Craig-Bampton		ERROR				REFERENCE VALUE
		abs(abs(REFERENCE)-abs)				
		acceleration method all modes	displacement method all modes	acceleration method truncated	displacement method truncated	acceleration method all modes
2 coupled DOF's						
	DOF 1	0.00000000	0.00000000	0.00000000	0.01335330	0.16666667
	DOF 41	0.00000000	0.00000005	0.00000000	0.00547904	0.00000000
	DOF 81	0.00000000	0.00000000	0.00000000	0.01335329	0.16666665
4 coupled DOF's						
	DOF 1	0.00000000	0.00000014	0.00000000	0.00046793	0.00722064
	DOF 21	0.00000000	0.00000005	0.00000000	0.02863001	0.15944602
	DOF 41	0.00000000	0.00000003	0.00000000	0.00327707	0.00000000
	DOF 61	0.00000000	0.00000000	0.00000000	0.02862996	0.15944602
	DOF 81	0.00000000	0.00000000	0.00000000	0.00046814	0.00722064

**EFFECT OF ACCELERATION vs DISPLACEMENT METHODS ON  
STEADY STATE BOUNDARY FORCES**

(Beam B - Craig-Bampton - 1 step to boundary - truncated)

BEAM A Physical		ERROR abs(abs(REFERENCE)-abs)				REFERENCE VALUE
		acceleration method all modes	displacement method all modes	acceleration method truncated	displacement method truncated	acceleration method all modes
2 coupled DOF's						
	DOF 1	0.00000000	0.00000043	0.00000000	0.02689245	-0.16666667
	DOF 41	0.00000000	0.00000003	0.00000000	0.47039448	0.49999997
	DOF 81	0.00000000	0.00000003	0.00000000	0.02689207	-0.16666665
4 coupled DOF's						
	DOF 1	0.00000000	0.00000000	0.00000000	0.00602595	-0.00722064
	DOF 21	0.00000000	0.00000006	0.00000000	0.02825282	-0.15944602
	DOF 41	0.00000000	0.00000000	0.00000000	0.44786317	0.50000000
	DOF 61	0.00000000	0.00000000	0.00000000	0.02825274	-0.15944602
	DOF 81	0.00000000	0.00000008	0.00000000	0.00602571	-0.00722064

BEAM B Craig-Bampton		ERROR abs(abs(REFERENCE)-abs)				REFERENCE VALUE
		acceleration method all modes	displacement method all modes	acceleration method truncated	displacement method truncated	acceleration method all modes
2 coupled DOF's						
	DOF 1	0.00000000	0.00000000	0.00000000	0.00000000	0.16666667
	DOF 41	n/a	n/a	n/a	n/a	0.00000000
	DOF 81	0.00000000	0.00000000	0.00000000	0.00000000	0.16666665
4 coupled DOF's						
	DOF 1	0.00000000	0.00000000	0.00000000	0.00001097	0.00722064
	DOF 21	0.00000000	0.00000004	0.00000000	0.00001101	0.15944602
	DOF 41	n/a	n/a	n/a	n/a	0.00000000
	DOF 61	0.00000000	0.00000000	0.00000000	0.00001098	0.15944602
	DOF 81	0.00000000	0.00000003	0.00000000	0.00001095	0.00722064

**EFFECT OF ACCELERATION vs DISPLACEMENT METHODS ON  
STEADY STATE BOUNDARY FORCES**

(Beam B - Craig-Bampton - 2 step to boundary - truncated)

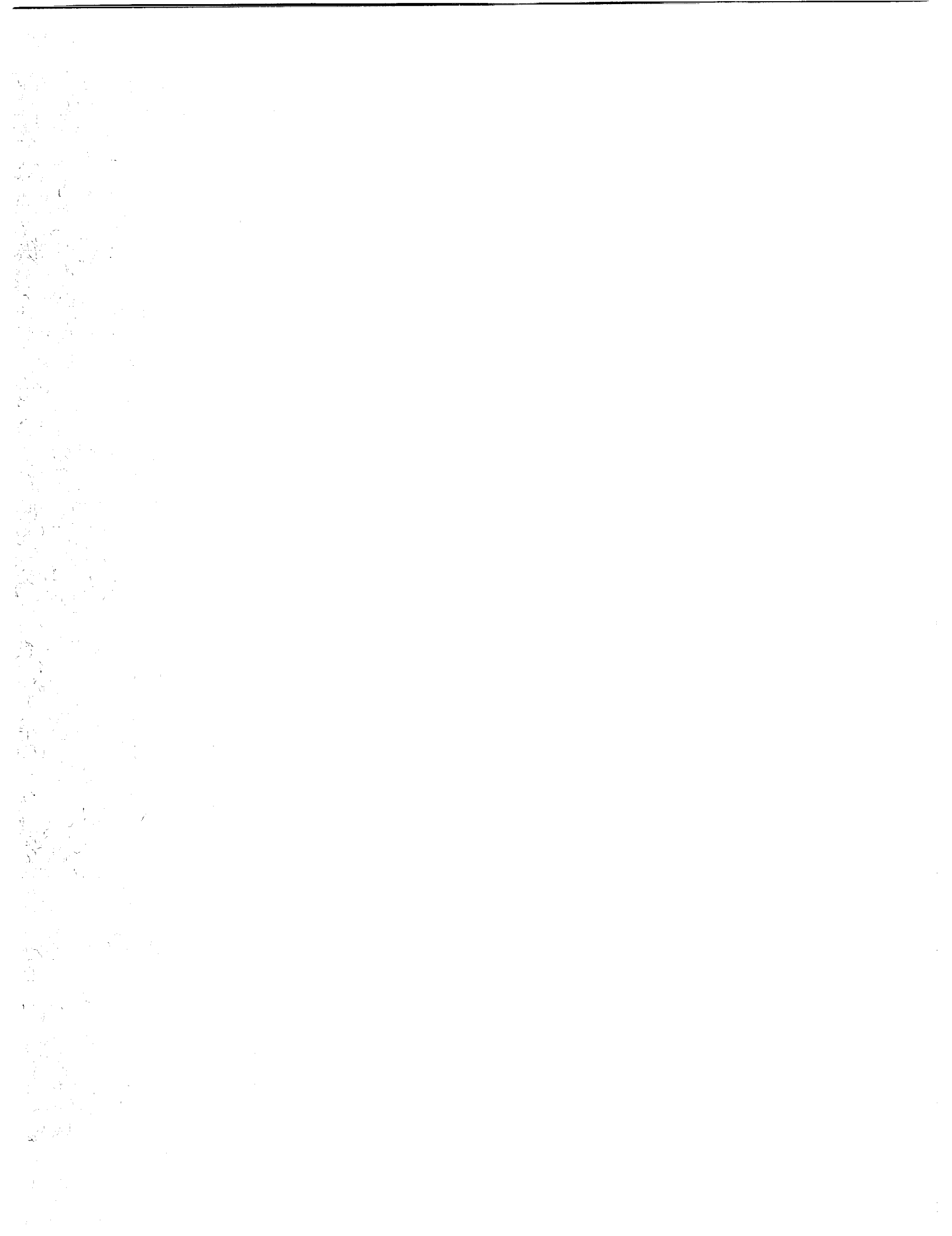
<b>BEAM A</b> Physical		ERROR				REFERENCE
		abs(abs(REFERENCE)-abs)				VALUE
		acceleration method all modes	displacement method all modes	acceleration method truncated	displacement method truncated	acceleration method all modes
2 coupled DOF's						
	DOF 1	0.00000000	0.00000000	0.00000000	0.02689188	-0.16666667
	DOF 41	0.00000000	0.00000004	0.00000000	0.47039445	0.49999997
	DOF 81	0.00000000	0.00000003	0.00000000	0.02689191	-0.16666665
4 coupled DOF's						
	DOF 1	0.00000000	0.00000000	0.00000000	0.00602551	-0.00722064
	DOF 21	0.00000000	0.00000000	0.00000000	0.02825380	-0.15944602
	DOF 41	0.00000000	0.00000000	0.00000000	0.44786309	0.50000000
	DOF 61	0.00000000	0.00000000	0.00000000	0.02825380	-0.15944602
	DOF 81	0.00000000	0.00000000	0.00000000	0.00602551	-0.00722064

<b>BEAM B</b> Craig-Bampton		ERROR				REFERENCE
		abs(abs(REFERENCE)-abs)				VALUE
		acceleration method all modes	displacement method all modes	acceleration method truncated	displacement method truncated	acceleration method all modes
2 coupled DOF's						
	DOF 1	0.00000000	0.00000000	0.00000000	0.00000000	0.16666667
	DOF 41	n/a	n/a	n/a	n/a	0.00000000
	DOF 81	0.00000000	0.00000000	0.00000000	0.00000000	0.16666665
4 coupled DOF's						
	DOF 1	0.00000000	0.00000000	0.00000000	0.00001097	0.00722064
	DOF 21	0.00000000	0.00000006	0.00000000	0.00001103	0.15944602
	DOF 41	n/a	n/a	n/a	n/a	0.00000000
	DOF 61	0.00000000	0.00000002	0.00000000	0.00001099	0.15944602
	DOF 81	0.00000000	0.00000000	0.00000000	0.00001097	0.00722064





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13. ABSTRACT (Maximum 200 words) <p>When a substructure model is reduced by the Craig-Bampton method, a number of degrees-of-freedom (DOF's) are retained as physical DOF's to provide interface to other substructures. When more DOF's are retained in this interface than are actually required, the model is said to be over constrained. The result of this, when using the displacement method, is typically an inaccurate distribution of boundary forces. This inaccuracy also occurs when there are justifiably many interface DOF's which result in an indeterminate interface. When the acceleration method is used, this inaccuracy is overcome. However, many people do not fully understand this method and the many ways of implementing it, and so its implementation is sometimes haphazard.</p> <p>This study describes the acceleration and displacement methods for use in the recovery of coupled system boundary forces. A simple 2-DOF system has been used for illustration. The effect of the choice of method for use with indeterminate or over-constrained boundaries has been investigated. It has specifically looked at results from a simple two-dimensional beam problem using both methods.</p> <p>In the space shuttle payload community there has been an increase in the use of over-constrained payload models. This has been, mainly, to afford easy recovery of relative deflection data between the payload and the shuttle. While there has also been an increase in the use of the acceleration method for the recovery of payload displacements and forces, the displacement method remains the method used for recovering system displacements and forces. Much work has been done on the effects of Craig-Bampton modal truncation on system displacements and forces, however, little work has been done on system level modal truncation (i.e., modes across the boundary). The findings of this study indicate the effect of this system level truncation is significant. This may be particularly true for the 35-Hz system cutoff frequency that is required by the space shuttle. From this study's findings, recommendations for areas of study with space shuttle payload systems are made.</p>				
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