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# On the Measurement of Time for the Quantum Harmonic Oscillator

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Abstract

A generalization of previous treatments of quantum phase[1] is presented. Restrictions on the class of realizable phase statistics are thereby removed, thus permitting "phase wavefunction collapse" (and other advantages). This is accomplished by exciting the auxilary mode of the measurement apparatus in a time-reversed fashion. The mathematical properties of this auxilary mode are studied in the hope that they will lead to an identification of a physical apparatus which can realize the quantum phase measurement.

## 1. The SG phase statistics

A satisfactory description of the phase of the quantum harmonic oscillator has recently been achieved by considering the realizable measurement[1] of the non-Hermitian Susskind-Glogower (SG) phase operator[2]

$$\widehat{e^{i\phi}} \equiv (\hat{n}+1)^{-1/2}\hat{a}. \tag{1}$$

Although it is not Hermitian, the SG operator *does* correspond to a realizable quantum measurement. It's measurement statistics, however, *can not* be calculated from the familiar Hermitian operator rules (e.g. moments calculated via $\langle\psi|(\widehat{e^{i\phi}})^k|\psi\rangle, k = 1,2,\ldots$ do not correspond to the SG operator's realizable measurement statistics). We have demonstrated a variety of ways in which the measurement statistics of the SG operator can be accessed [1],[3]. Perhaps the simplest of these is to form the phase wavefunction

$$\psi(\phi) \equiv \langle\phi|\psi\rangle, \tag{2}$$

from which the phase probability distribution, $p(\phi) = |\psi(\phi)|^2/2\pi$, and it's associated moments follow directly. This proceedure is justified formally by the fact that the infinite energy eigenkets of the SG operator

$$|\phi\rangle \equiv \sum_{n=0}^{\infty} e^{in\phi}|n\rangle \tag{3}$$

resolve the identity, i.e. $1 = \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}|\phi\rangle\langle\phi|$. This permits the extremely useful phase representation of an arbitrary quantum state:

$$|\psi\rangle = \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\langle\phi|\psi\rangle|\phi\rangle, \tag{4}$$

analogous to the familiar number representation of a state:

$$|\psi\rangle = \sum_{n=0}^{\infty}\langle n|\psi\rangle|n\rangle. \tag{5}$$

The number-ket expansion coefficients, $\psi_n \equiv \langle n|\psi\rangle$, may be viewed as a wavefunction in discrete n-space. The Fourier transform relationship of the number and phase wavefunctions

$$\psi(\phi) = \sum_{n=0}^{\infty}\psi_n e^{-in\phi} \tag{6}$$

$$\psi_n = \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\psi(\phi)e^{in\phi} \tag{7}$$

demonstrates the complementarity of photon number and quantum phase.

Position and momentum are familiar examples of complementary quantities, whose wavefunction representations, $\psi(x) \equiv \langle x|\psi\rangle$ and $\Phi(p) \equiv \langle p|\psi\rangle$, are also related via Fourier transform. Indeed, several relations among $\psi_n$ and $\psi(\phi)$ are reminiscient of those encountered in Schrodinger's wave mechanics. Analogous to the position representation of the momentum operator, $\hat{p} \rightarrow -i\hbar\frac{d}{dx}$, for example, we have a phase representation of the number operator, $\hat{n} \rightarrow i\frac{d}{d\phi}$, viz:

$$\langle\psi|(\hat{n})^k|\psi\rangle = \tag{8}$$

$$\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\psi^*(\phi)(i\frac{d}{d\phi})^k\psi(\phi) \tag{9}$$



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(where $k = 0, 1, 2...$). These relations yield the correct form of the number/phase uncertainty principle[3]:

$$(\Delta n^2)(\Delta\phi^2) \geq \frac{1}{4}(1 - 2\pi p(\pi))^2. \quad (10)$$

Since we are dealing with a single harmonic oscillator (of frequency $\omega$), phase is related to time ($\phi = \omega t$) in a mod-$2\pi$ sense, and number is directly related to energy ($\hat{n} = \hat{H}/\hbar\omega - 1/2$). In this sense, the above constitutes a rigorous energy/time uncertainty principle for the quantum harmonic oscillator.

The class of realizable SG phase statistics, however, is restricted (by a Paley-Wiener theorem) due to the fact that $\psi(\phi)$ is a one-sided Fourier series, i.e. this restriction stems from the absence of "negative number states" ($\psi_n = 0\ \forall n < 0$). One aspect of this restriction is that $\psi(\phi)$ is prohibited from identicaly vanishing over a non-zero interval — thus, delta-functions in phase are not allowed. In as much as we may desire a "wavefunction collapse" view of a phase measurement, the SG statistics appear to be incomplete. This dilema, however, can be resolved by generalizing an alternate route (the product space formalism) to the SG statistics.

Fundamental to the realizable measurement of any non-Hermitian operator is the existance of an auxilary noise source. Zero-point fluctuations from this auxilary mode prevent a simultaneous, perfectly precise, measurement of the noncommuting real and imaginary parts of the non-Hermitian operator (so that the uncertainty principle is not violated). We can study the interaction of our original system of interest (Hilbert space $\mathcal{H}_s$) with this auxilary system (Hilbert space $\mathcal{H}_a$) by working in the product space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_a$. The *extension* of the SG operator onto $\mathcal{H}$ is [3]

$$\hat{\Gamma} \equiv (\widehat{e^{i\phi}})_s \otimes \hat{V}_a + \hat{V}_s \otimes (\widehat{e^{i\phi}})^\dagger_a, \quad (11)$$

where $\hat{V} \equiv |0\rangle\langle 0|$. This extension has eigenkets (of non-zero eigenvalue), $\hat{\Gamma}|\phi\rangle' = e^{i\phi}|\phi\rangle'$, given by

$$|\phi\rangle' = |0\rangle_s|0\rangle_a + \sum_{n_s=1}^{\infty} e^{in_s\phi}|n_s\rangle|0\rangle_a \quad (12)$$

$$+ \sum_{n_a=1}^{\infty} e^{-in_a\phi}|0\rangle_s|n_a\rangle_a. \quad (13)$$

These reside on a subset, $\mathcal{H}'$, of $\mathcal{H}$ which is defined by the property $n_s n_a = 0$. When the auxilary mode is in the vacuum state ($n_a = 0$), the $\hat{\Gamma}$ measurement yields the SG statistics and their attendant Paley-Wiener restriction.

## 2. Beyond the SG statistics

We can go *beyond* the SG statistics by exciting the auxilary mode to create an arbitrary state on $\mathcal{H}'$:

$$|\psi\rangle' \equiv \sum_{n_s=0}^{\infty} \psi_{n_s,0}|n_s\rangle_s|0\rangle_a \quad (14)$$

$$+ \sum_{n_a=1}^{\infty} \psi_{0,n_a}|0\rangle_s|n_a\rangle_a. \quad (15)$$

For simplicity, let $N \equiv n_s - n_a$, $\psi_N \equiv \psi_{N,0}\,(\forall N \geq 0)$, and $\psi_N \equiv \psi_{0,-N}\,(\forall N < 0)$. The *generalized phase wavefunction*,

$$\psi'(\phi) \equiv {}'\langle\phi|\psi\rangle' = \sum_{N=-\infty}^{\infty} \psi_N e^{-iN\phi}, \quad (16)$$

is a two-sided Fourier series. The the Paley-Wiener restriction is removed and $\psi'(\phi)$ can "collapse" to a delta-function. The fact that the class of $\psi'(\phi)$ is more general than (and includes) the class of $\psi(\phi)$ should prove useful for various optimizations. Indeed, Shapiro[4] has pointed out that error-free communication could *in principle* be achieved by exploiting the newly aquired generality described herein.

Provided that niether of our two modes is purely in the vacuum state, the excitation which creates a state on $\mathcal{H}'$ is not arbitrary in that the $n_s n_a = 0$ property creates an entanglement. Thus, in general, the original system and auxilary modes are not statistically independent on $\mathcal{H}'$, i.e. $|\psi\rangle' \neq |\psi_s\rangle_s|\psi_a\rangle_a$. Denoting the probability that a measurement of $\hat{n}_s$ yields the outcome n by $|\psi^s_n|^2$, we see that $|\psi^s_n|^2 = |\psi_{n,0}|^2\ (\forall n \geq 1)$, whereas $|\psi^s_0|^2 = |\psi_{0,0}|^2 + \sum_{n=1}^{\infty} |\psi_{0,n}|^2$ (similarly for



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$|\psi_n^a|^2$). In spite of the lack of statistical independence, we can therefore assign any individual probability distributions for $n_s$ and $n_a$ that we wish, *provided* that

$$|\psi_{0,0}|^2 = (|\psi_0^s|^2 + |\psi_0^a|^2 - 1) \geq 0 \quad (17)$$

is satisfied.

The auxilary mode can be interpreted as a time-reversed mode in the following sense. Consider the case of the auxilary mode being in the vacuum state ($n_a = 0$). Denote the initial state by $|\psi_0\rangle'$. The state (in the Schrodinger picture) after time evolution of an amount $\tau$ is

$$|\psi_\tau\rangle' = e^{-i\hat{n}_s\omega\tau}|\psi_0\rangle', \quad (18)$$

so that the relation of the phase representations of the initial and delayed states is simply

$$\psi_\tau'(\phi) = \psi_0'(\phi + \omega\tau) \;\; (n_a = 0). \quad (19)$$

Now consider the case of the original system being in the vacuum state ($n_s = 0$). The Schrodinger picture of the delayed version of an initial state $|\psi_0\rangle'$ is

$$|\psi_\tau\rangle' = e^{-i\hat{n}_a\omega\tau}|\psi_0\rangle'. \quad (20)$$

The initial and delayed phase representations for this case are related by

$$\psi_\tau'(\phi) = \psi_0'(\phi - \omega\tau) \;\; (n_s = 0). \quad (21)$$

Thus the two modes are time-reversed in that, under time evolution, the $n_a \geq 1$ portion of the generalized phase wavefunction "moves backwards" with respect to the $n_s \geq 1$ portion.

Consistent with the time-reversal property, the auxilary mode serves the topological role of a "negative energy" mode in Hilbert space. The SG operator is a pure lowering operator which **stops** at the vacuum:

$$\widehat{e^{i\phi}}|n\rangle = |n-1\rangle \;\; (n \geq 1) \quad (22)$$

$$\widehat{e^{i\phi}}|0\rangle = 0. \quad (23)$$

It cannot lower below the vacuum since we have not allowed negative number (negative energy) states for the quantum harmonic oscillator. It's extension, $\hat{Y}$, however, lowers the original system mode number

$$\hat{Y}|n_s\rangle_s|0\rangle_a = |n_s - 1\rangle_s|0\rangle_a \;\; (n_s \geq 1), \quad (24)$$

then **continues** through the vacuum

$$\hat{Y}|0\rangle_s|0\rangle_a = |0\rangle_s|1\rangle_a, \quad (25)$$

and raises the auxilary mode number

$$\hat{Y}|0\rangle_s|n_a\rangle_a = |0\rangle_s|n_a + 1\rangle_a. \quad (26)$$

*Topologicaly*, it is as if $\hat{Y}$ continues to lower below the vacuum into the auxilary ("negative energy") mode. The visualization of this behavioral aspect can be facilitated by simply relabeling the $\mathcal{H}'$ number states according to the value of $N \equiv n_s - n_a$.

The auxilary mode has to be an irrevocable part of the physical apparatus which realizes the quantum phase measurement (so that the uncertainty principle is satisfied and so that the phase wavefunction can collapse). All of the aforementioned mathematical properties must be physicaly realized in the measurement apparatus. These restrictions should prove useful in determining an apparatus which will realize the quantum phase measurement.

## 3. REFERENCES


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