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FIELD QUANTIZATION AND SQUEEZED STATES  
GENERATION IN RESONATORS WITH TIME-DEPENDENT  
PARAMETERS

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The problem of electromagnetic field quantization is usually considered in text-books under the assumption that the field occupies some empty box. The case when the box is filled with a uniform dielectric medium was considered in (Refs.1,2). The quantization of the field in the medium consisting of two uniform dielectrics with different permittivities was studied in (Refs.3-5). The case of an arbitrary inhomogeneous dielectric medium was investigated in (Refs.6,7) and especially in (Refs.8,9). However, in all mentioned papers the properties of the medium were believed time-independent. Here we want to consider the most general case of non-uniform and time-dependent media. Earlier this problem was investigated in (Ref.10), but its authors considered only approximate solutions of the Heisenberg equations for field operators in the case of small polarization of the medium. Our approach differs from that of (Ref.10) and enables to study the case when non-uniform time-dependent dielectric medium is confined in some space region with time-dependent boundaries.

The basis of the subsequent consideration is the system of Maxwell's equations in linear passive time-dependent dielectric and magnetic medium without sources:

$$\begin{aligned} \operatorname{rot} E &= -1/c \partial B / \partial t, & \operatorname{rot} H &= 1/c \partial D / \partial t, \\ \operatorname{div} D &= 0, & \operatorname{div} B &= 0, \\ D &= \epsilon(r, t) E, & B &= \mu(r, t) H. \end{aligned} \quad (1)$$

Introducing the vector potential according to the relations:

$$B = \operatorname{rot} A, \quad E = -1/c \partial A / \partial t \quad (2)$$

and imposing gauge conditions

$$\operatorname{div}(\epsilon \partial A / \partial t) = 0, \quad \varphi = 0 \quad (3)$$

we can replace the system of the first-order equations (1) with the single second-order one:

$$\operatorname{rot}(1/\mu \operatorname{rot} A) = -1/c^2 \partial / \partial t (\epsilon \partial A / \partial t) = 1/c \partial D / \partial t. \quad (4)$$

One can check that the vector equation (4) coincides with the set of Euler's equations

$$\partial/\partial t \partial L/\partial(\partial_T A_0) + \partial/\partial x_s \partial L/\partial(\partial_s A_0) - \partial L/\partial A_0 = 0 \quad (5)$$

for the Lagrangian density

$$L = 1/2 [ \epsilon(r,t)(\partial A/\partial t)^2/c^2 - (\text{rot}A)^2/\mu(r,t) ] \quad (6)$$

in the case of quite arbitrary time and space dependences of the dielectric and magnetic permittivities. Then introducing the canonically conjugated variable

$$P = \partial L/\partial(\partial_T A) = \epsilon(r,t)/c^2 \partial A/\partial t = -1/c D \quad (7)$$

one can construct the Hamiltonian density

$$H = P \partial A/\partial t - L = 1/2 [ D^2/\epsilon(r,t) + B^2/\mu(r,t) ] \quad (8)$$

which leads again to eq.(4). But in the general case the expression (8) is by no means the energy of the system due to possible time-dependences of the coefficients. This fact complicates the quantization procedure. The usual procedure consists in introducing the field expansion over mode functions

$$D(r,t) = \Sigma q_N(t)u_N(r), \quad B(r,t) = \Sigma p_N(t)v_N(r). \quad (9)$$

Substituting these expansions into the Hamiltonian density (8) and integrating it over the space variables one gets usually (due to certain orthogonality properties of functions  $u_N$  and  $v_N$ ) a sum of independent oscillator-like Hamiltonians

$$H = \int H(r,t)d^3r = 1/2 \Sigma (\mu_N p_N^2 + \eta_N q_N^2). \quad (10)$$

After this the coefficients  $p_N$  and  $q_N$  are proclaimed operators satisfying canonical commutation relations, so that the fields become quantized. But this sketch of standard quantization scheme shows distinctly that it can be used only in the case when the solutions of Maxwell's equations can be factorised into the products of two functions: one dependent only of time, and another dependent on space coordinates only. In the general case of non-uniform and time-dependent medium such solutions do not exist, and the usual scheme of quantization is impossible. This means in particular, that we cannot obtain any Hamiltonian and, consequently, any unitary evolution operator. Therefore the Schrödinger picture does not exist in the general case. But the Heisenberg description is still possible. It can be introduced as a generalisation of the approach used earlier by Moore (Ref.11) for the field quantization in the empty space region confined with moving boundaries.

First of all we notice the important property of equation (4): it admits a time-independent scalar product of any two different solutions in the following form:

$$((A,B)) = -1/2 i \int d^3r \epsilon(r,t) [A\partial B^*/\partial t - B^*\partial A/\partial t]. \quad (11)$$

It is essential that the dielectric permittivity is a real function, i.e. the medium is assumed lossless. Besides, the vector potential has to turn into zero at the surfaces confining the integration domain. The case of moving boundaries (considered in Ref.11) is included to general situation automatically.

Suppose that before some instant of time (let it be  $t=0$ ) both the medium and the boundaries were time-independent. Then solutions of (4) could be factorized:

$$A(r,t) = g(r)\exp(-i\omega t), \quad (12)$$

$$\text{rot}[1/\mu(r) \text{rot}g] - \omega^2/c^2 \epsilon(r)g = 0. \quad (13)$$

The scalar product (11) was proportional to the usual scalar product:

$$((A,B)) = 1/2(\omega_A + \omega_B)\exp[i(\omega_B - \omega_A)t](g_B, g_A), \quad (14)$$

$$(g_B, g_A) = \int d^3r \epsilon(r)g_B^* g_A. \quad (15)$$

But it is known that solutions of eq.(13) form the complete orthogonal with respect to scalar product (15) set of vector functions. Therefore any real vector field can be decomposed over this set of functions:

$$A(r,t) = \Sigma[a_N g_N(r)\exp(-i\omega_N t) + a_N^* g_N^*(r)\exp(i\omega_N t)]. \quad (16)$$

Comparing (14) and (15) we can see that the basis functions can be normalised in such way that they will satisfy the relations

$$((A_N, A_M)) = \delta_{NM}, \quad ((A_N, A_M^*)) = 0. \quad (17)$$

After the instant when the properties of the medium became time-dependent the basis functions change their explicit expressions, but the scalar products (17) will not change, and instead of (16) we can write the decomposition

$$A(r,t) = \Sigma[a_N A_N(r,t) + a_N^* A_N^*(r,t)]. \quad (18)$$

Then we proclaim the (time-independent) coefficients of this expansion operators satisfying bosonic commutation relations and thus obtain the quantized field from a classical one.

If in some time the medium will become again time-independent then the physical states will be described with monochromatic mode functions of the type (12), which will not coincide in general with the basis functions of expansion (18). Therefore we will have two different expansions of the field operator: expansion (18) over the states corresponding to the physical photons in remote past and expansion like (16) over the physical states ari-

sing in future. Designating the "physical" states with the superscript "zero", we can expand each set of basis functions into a series with respect to another one:

$$A_N = \sum_M [\alpha_{NM} A_M^{(0)} + \beta_{NM} A_M^{(0)*}]. \quad (19)$$

The corresponding expansion of "new" creation and annihilation operators over the set of "old" ones is as follows:

$$\hat{a}_M^{(0)} = \sum_N [\hat{a}_N \alpha_{NM} + \hat{a}_N^* \beta_{NM}]. \quad (20)$$

The initial state of the quantized field was determined with respect to the set of "old" operators (without the superscript "zero"). Then using expansion (20) we can calculate all quantum statistical characteristics of the field in the final state. Taking into account conditions (17) and the evident properties of the scalar product (11)

$$((A, B)) = ((B, A))^* = -((B^*, A^*)) \quad (21)$$

one can express the coefficients of expansions (19) or (20) as follows:

$$\alpha_{NM} = ((A_N, A_M^{(0)})), \quad \beta_{NM} = ((A_N^*, A_M^{(0)}))^*. \quad (22)$$

The quantization scheme sketched above can be applied to the most general situation of an arbitrary space-time nonuniform medium and moving boundaries. However, the explicit calculation of the mode functions and coefficients of the canonical transformation (20) can be performed only for rather simple special cases. The first of them corresponds to the media with factorized electric and magnetic permittivities:

$$\epsilon(r, t) = \bar{\epsilon}(r)\chi(t), \quad \mu(r, t) = \bar{\mu}(r)\nu(t) \quad (23)$$

(the boundaries do not move). Then mode functions can be also sought for in a factorized form:

$$A(r, t) = g(r)\xi(t) \quad D(r, t) = \bar{\epsilon}(r)g(r)\eta(t). \quad (24)$$

Let us demand the function  $g(r)$  to satisfy the following equation

$$\text{rot}(1/\bar{\mu} \text{rot}g) = k^2 \bar{\epsilon}(r)g, \quad k = \text{const}. \quad (25)$$

Then eqs. (2) and (4) result in the following ordinary differential equations for time-dependent factors of the vector potential and electric displacement:

$$\dot{\eta} = k^2 c \xi / \nu(t) \quad \dot{\xi} = -c \eta / \chi(t) \quad (26)$$

Eqs. (26) resemble equations of motion of an oscillator with time-dependent mass and frequencies. The role of generalized coordinate is played by the electric displacement time-dependent factor, while the vector potential time-dependent factor plays the

role of generalised momentum. Eqs. (26) can be replaced by the following second-order differential equation:

$$\dot{\eta} + \gamma(t)\dot{\eta} + \Omega^2(t)\eta = 0, \quad \gamma = \dot{\nu}/\nu, \quad \Omega^2 = k^2 c^2 / \nu(t) \chi(t). \quad (27)$$

We shall consider the field inside a resonator. Then solutions of eq.(25) can be chosen real vector functions satisfying the orthogonality conditions:

$$\int d^3r \epsilon(r) g_K(r) g_L(r) = k^2 \delta_{KL}. \quad (28)$$

Complex solutions of eq.(27) can be normalized as follows:

$$\nu(t) [\dot{\eta}\eta^* - \dot{\eta}^*\eta] = -2i. \quad (29)$$

This means that we choose the solution of eq.(27) in the stationary case in the form of

$$\eta_0(t) = (\nu_0 \Omega_0)^{-1/2} \exp(-i\Omega_0 t). \quad (30)$$

Due to (28) coefficients (22) are not equal to zero only for coinciding indices (intermode interactions are absent), so we can omit the indices. Taking into account eqs.(11), (26), (30) one can represent these coefficients as follows:

$$\alpha = 1/2(\nu_0/\Omega_0)^{1/2}(\Omega_0\eta + i\dot{\eta})\exp(i\Omega_0 t) \quad (31)$$

$$\beta = 1/2(\nu_0/\Omega_0)^{1/2}(\Omega_0\eta - i\dot{\eta})\exp(i\Omega_0 t). \quad (32)$$

Let us introduce the quadrature components and their variances as follows:

$$\hat{X}_1 = (2)^{-1/2}(\hat{a}_0 + \hat{a}_0^*) \quad \hat{X}_2 = i(2)^{-1/2}(\hat{a}_0^* - \hat{a}_0) \quad (33)$$

$$\sigma_{1j} = 1/2\langle \hat{X}_1 \hat{X}_j + \hat{X}_j \hat{X}_1 \rangle - \langle \hat{X}_1 \rangle \langle \hat{X}_j \rangle. \quad (34)$$

Suppose for simplicity that initially the field was in the coherent quantum state. Taking into account eq.(20) one can easily obtain the following expressions:

$$\sigma_{11} = 1/2|\alpha + \beta|^2 = 1/2\nu_0\Omega_0|\eta|^2, \quad (35)$$

$$\sigma_{22} = 1/2|\alpha - \beta|^2 = 1/2\nu_0\Omega_0^{-1}|\dot{\eta}|^2, \quad (36)$$

$$\sigma_{12} = \text{Im}(\alpha\beta^*) = 1/2\nu_0\text{Re}(\dot{\eta}\eta^*). \quad (37)$$

We see that time-dependent medium transforms an initially coherent state to a "correlated quantum state" characterized by a nonzero covariance (37) and unequal variances (35) and (36).

This state minimizes the generalized uncertainty relation by Schrodinger and Robertson (Refs.12,13):

$$\sigma_{11}\sigma_{22} - \sigma_{12}^2 \geq 1/4 \quad (38)$$

(the equality takes place in the case under study due to eq. (29)). For the detailed review of various forms of uncertainty relations see (Ref.14). Properties of correlated and squeezed quantum states were investigated in (Refs.15-19).

Let us consider as an example the case of a parametric excitation when the properties of the medium harmonically oscillate with twice frequency with respect to some (resonance) mode. This can be achieved, for example, by means of changing the density of the medium. Since the magnetic effects are extremely weak, we can write

$$\Omega^2(t) = \Omega_0^2 (1 + \alpha \cos 2\Omega_0 t), \quad \gamma = 0. \quad (39)$$

We look for the solution of eq.(27) in the form

$$\eta(t) = (\nu_0 \Omega_0)^{-1/2} [u(t) \exp(i\Omega_0 t) + v(t) \exp(-i\Omega_0 t)] \quad (40)$$

with slowly varying time dependent amplitudes. Substituting (39) and (40) into (27) and performing averaging over fast oscillations we arrive at the equations (neglecting the second order derivatives of slowly varying amplitudes)

$$\dot{u} = i\Omega_0 \alpha v/4, \quad \dot{v} = -i\Omega_0 \alpha u/4 \quad (41)$$

whose solutions are

$$u(t) = \cosh(\Omega_0 \alpha t/4), \quad v(t) = -i \sinh(\Omega_0 \alpha t/4). \quad (42)$$

The variances (35), (36) oscillate with the twice resonance frequency, but their ratio (the so called squeezing coefficient) is confined at every instant between the values

$$\exp(-\Omega_0 \alpha t) \leq \Omega_0^2 \sigma_{11} / \sigma_{22} \leq \exp(\Omega_0 \alpha t). \quad (43)$$

Certain inequalities for the squeezing coefficients can be found for arbitrary time dependence of the frequency in eq.(27). Considering again nonmagnetic medium one can prove the inequalities (Refs.19,20)

$$[(1-R^{1/2})/(1+R^{1/2})]^2 \leq \Omega_0^2 \sigma_{11} / \sigma_{22} \leq [(1+R^{1/2})/(1-R^{1/2})]^2 \quad (44)$$

where  $R$  is the energy reflection coefficient from the effective potential barrier corresponding to eq.(27).

We would like to emphasize once again that we have used the Heisenberg picture for the description of the quantized electromagnetic field. However, since for a factorizable medium (23) Maxwell's equations can be derived from the Hamiltonian (8) or (10), the Schrodinger description is also possible in this case. We shall illustrate it for the most simple case when

$$v(t) = 1, \quad \chi = \chi(ht), \quad \chi(0) = 1. \quad (45)$$

Here  $h$  is a characteristic frequency of the medium properties changing. The method used below can be easily applied to a more general case. Let  $c = 1$  and the dimensionless time  $t_0 = kt$ . So (27) results in

$$\dot{\eta} + \omega^2(\alpha t_0)\eta = 0, \quad (46)$$

where

$$\omega^2(\alpha t_0) = \omega^2(ht) = 1/\chi(ht), \quad \alpha = h/k. \quad (47)$$

The quantization of a harmonic oscillator with a variable frequency is done by introducing integrals of motion operators (Ref. 17):

$$\hat{a}(t_0) = i(2)^{-1/2} (y(t_0)\hat{\xi} - \dot{y}(t_0)\hat{\eta}), \quad \hat{a}^*(t_0) = [\hat{a}(t_0)]^*. \quad (48)$$

Here  $y(t_0)$  is a "ruling solution" of eq.(46) satisfying the time-independent condition

$$\dot{y}(t_0)y^*(t_0) - \dot{y}^*(t_0)y(t_0) = 2i \quad (49)$$

to ensure the following commutation relation

$$[\hat{a}(t_0), \hat{a}^*(t_0)] = 1. \quad (50)$$

We want to stress the difference in signs of right-hand sides of (29) and (49) due to the difference between Heisenberg and Schrodinger pictures. For the integrals of motion to coincide at  $t_0 = 0$  with creation and annihilation operators, it is necessary (in accordance with (49)) to choose the initial conditions

$$y(0) = 1, \quad \dot{y}(0) = i. \quad (51)$$

After the instant  $t_0 = t_f$ , when the medium properties stop changing, "new" creation and annihilation operators are to be introduced:

$$\hat{a}_{(0)} = i(2\omega(t_f))^{-1/2} (\hat{\xi} - i\hat{\eta}\omega(t_f)). \quad (52)$$

Then the expansion of "new" operators over "old" ones is

$$\hat{a}_{(0)} = \alpha\hat{a}(t_f) + \beta\hat{a}^*(t_f) \quad (53)$$

and we can get

$$\beta(t_f) = 1/2 (\omega(t_f))^{-1/2} (\omega(t_f)y(t_f) + i\dot{y}(t_f)). \quad (54)$$

Creation and annihilation operators mixing results in a change of occupation numbers in a given mode. Generation of photons from vacuum due to the medium properties change is worth considering. If at  $t_0 = 0$  the number of photons was zero then at an instant  $t_0$

$>0$  the average number of photons is

$$\bar{n} = \beta^*(t_0)\beta(t_0). \quad (55)$$

Yablonovitch (Ref.21) stated that photons will have a thermal distribution at a temperature proportional to the rate of the medium properties change, i.e. proportional to  $\hbar$ . Then for high photon energies (proportional to  $k$ ) there will be

$$\bar{n} \sim \exp(-\text{const } k/\hbar). \quad (56)$$

The solution of (39) at a little  $t_0$  can be found for an arbitrary law of change of  $\chi(t)$ . At  $t_0 = 0$  we shall have

$$d/dt_0 ((\omega)^{1/2} \beta) = 1/2 d\omega(\alpha t_0)/dt_0 \quad (57)$$

$$\beta \sim t_0/2 d(\omega(\alpha t_0))/dt_0 \quad (58)$$

$$\bar{n} \sim (t/2 d(\omega(\hbar t))/dt). \quad (59)$$

For any instant  $t$  a decomposition of the solution can be found in the limit  $\alpha \rightarrow 0$  (for high photon frequencies) by means of the method of multy-scale asymptotic decompositions (Ref.22) for an arbitrary law of change:

$$y = (\omega(\alpha t_0))^{-1/2} \{ \exp(is) + \alpha [E(\alpha t_0)\exp(is) + F(\alpha t_0)\exp(-is)] + \dots \} \quad (60)$$

where

$$s = 2/\alpha [\exp(\alpha t_0/2) - 1] \quad (61)$$

and functions  $E$  and  $F$  are determined by

$$F = \text{const} \quad (62)$$

$$dE/d(\alpha t_0) = -i/(4\omega^2) d^2\omega/d(\alpha t_0)^2 + i3/(8\omega^3) [d\omega/d(\alpha t_0)]^2$$

with initial conditions

$$E(0) + F(0) = 0, \quad i(E(0) - F(0)) = 1/2 d\omega/d(\alpha t_0). \quad (63)$$

For the case  $\chi = \exp(-\hbar t)$ , which approximately describes dielectric permittivity falling achievable in experiments (Ref.21), one can find the exact solution - a linear combination of two Hankel's functions  $H_0^{(1)}[2/\alpha \exp(\alpha t_0/2)]$  and  $H_0^{(2)}[2/\alpha \exp(\alpha t_0/2)]$ . Nevertheless an asymptotic decomposition is still more useful and can be expressed in an explicit form:

$$y = \exp(-\alpha t_0/4)\exp(is) + \alpha \exp(-\alpha t_0/4) \{ (-i/16) \times [1 + \exp(-\alpha t_0/2)]\exp(is) + i/8 \exp(-is) \} + \dots \quad (65)$$



Now we have

$$\beta = i\alpha/8 [\exp(-is) - \exp(-\alpha t_0/2)\exp(is)] \quad (66)$$

and the number of photons

$$\bar{n} = \alpha^2/64 [1 + \exp(-\alpha t_0) - 2\cos(2s)\exp(-\alpha t_0/2)] \quad (67)$$

This number oscillates with a growing frequency and a decreasing amplitude and in the limit  $t_0 \rightarrow \infty$  tends to a constant

$$\bar{n} \rightarrow h^2/64k^2.$$

As we can see it is not in agreement with the statement (38) from (Ref.21). The energy of photons in the mode is

$$\bar{n}\omega k \rightarrow h^2/64k \exp(ht/2), \quad t \rightarrow \infty \quad (69)$$

and it grows without a limit. Also at any time the sum of energies of all modes diverges. This can be explained by the fact that it is impossible to decrease dielectric permittivity to zero for nondispersive media; thus the assumption that it does not depend on a frequency is not valid for high frequencies. In the other limit  $\alpha = h/k \rightarrow \infty$  we introduce another dimensionless time  $t_1 = \alpha t_0$ . The solution expansion over  $\theta = 1/\alpha$  (not valid for large  $t_1$ ) is

$$y = 1 + \theta i t_1/2 + \theta^2/4 [t_1 + 1 - \exp t_1] + \dots \quad (70)$$

and the first term for the number of photons does not depend on the mode frequency:

$$\beta = \exp(t_1/2) - 1 + \theta i/2 [t_1 \exp(t_1/2) + 1 - \exp t_1] \quad (71)$$

$$\bar{n} = 1/2 [\cosh(ht/2) - 1]. \quad (72)$$

Another example of time dependent resonator which can be solved is an empty resonator with a moving ideal wall. Moore (Ref. 11) proposed the following complete orthonormal set of solutions (in the special case of a single space dimension, i.e. confining with the modes with linear polarisation parallel to the wall surface):

$$A_N(x,t) = (4\pi N)^{-1/2} \{ \exp[-i\pi NR(t-x)] - \exp[-i\pi NR(t+x)] \} \quad (73)$$

$(c = 1).$

These functions depend on the solution of the functional equation ( $L(t)$  is the position of the moving wall, another wall is assumed to be at rest)

$$R(t+L(t)) = R(t-L(t)) + 2. \quad (74)$$

An approximate solution of this equation in the case of the small velocities of the wall was found by Moore (Ref.11) and later used in (Refs.23,24). However, that solution is not valid in the case of parametric resonance, when

$$L(t) = L_0[1 + a \sin(2\Omega_0 t)], \quad |a| \ll 1, \quad \Omega_0 = \pi/L_0 \quad (75)$$

(the resonance at the lowest resonator eigenfrequency). The corresponding solution for small values of the percentage modulation was found in (Ref.25):

$$R(\xi) = \xi/L_0 \{1 - a \sin(2\Omega_0 \xi) + a^2[\sin^2(\Omega_0 \xi) + 1/4\Omega_0 \xi \sin(2\Omega_0 \xi)] + \dots\}. \quad (76)$$

Eqs.(11),(22),(45) result in the following expressions for the transformation coefficients (21):

$$\left. \begin{matrix} \alpha_{NM} \\ \beta_{NM} \end{matrix} \right\} = 1/2 (M/N)^{1/2} \begin{cases} t/L_0 + 1 \\ t/L_0 - 1 \end{cases} \exp\{i\pi[-NR(L_0 x) \pm Mx]\} dx. \quad (77)$$

The calculations are rather simple for not very large values of time, when the second-order correction in (76) remains small. Then the following simple formula for variances can be found (Ref.25):

$$\left. \begin{matrix} \sigma_{11} \\ \sigma_{22} \end{matrix} \right\} = 1/2 \exp(\pm 1/2 \pi a N), \quad N \gg 1, \quad |aN| \ll 1 \quad (78)$$

where  $N$  is the number of semiperiods of wall's vibrations. One can check that the maximum squeezing coefficients given by eqs. (43) and (78) coincide for equal values of percentage modulation in two different methods of exciting the field via the parametric resonance.

Now let us consider the long-time asymptotics for the  $R$ -function under the condition  $\varepsilon/L_0 \ll 1$ ,  $\varepsilon t \gg L_0^2$  for an arbitrary periodic motion of the wall  $L(t) = L_0 + \varepsilon f(t)$ . We shall choose the solution of eq.(74) in the following form:

$$R(t) = \sum_{N=0}^{\infty} \varepsilon^N R_N(t). \quad (79)$$

Substituting expansion (79) to eq.(74) we attain

$$\sum_{N=0}^{\infty} \varepsilon^N R_N[t+L_0+\varepsilon f(t)] = \sum_{N=0}^{\infty} \varepsilon^N R_N[t-L_0-\varepsilon f(t)] + 2.$$

Developing both sides of this equation into power series we have

$$\sum_{N=0}^{\infty} \sum_{K=0}^{\infty} \varepsilon^N R_N^{(K)}(t+L_0) \varepsilon^K f^K(t)/K! = \sum_{N=0}^{\infty} \sum_{K=0}^{\infty} \varepsilon^N R_N^{(K)}(t-L_0) \times \\ \times (-1)^K \varepsilon^K f^K(t)/K! + 2.$$

It is convenient to use another summation index  $M=N+K$ :

$$\sum_{M=0}^{\infty} \sum_{K=0}^M \varepsilon^M [R_{M-K}^{(K)}(t+L_0) - (-1)^K R_{M-K}^{(K)}(t-L_0)] f^{(K)}(t)/K! = 2.$$

From this equation we obtain the following system of equations for the functions  $R_M(t)$  ( $M=0,1,\dots$ ):

$$\sum_{K=0}^M [R_{M-K}^{(K)}(t+L_0) - (-1)^K R_{M-K}^{(K)}(t-L_0)] f^{(K)}(t)/K! = 2\delta_{M0}. \quad (80)$$

Further we consider the simplest law of motion (75) with  $\varepsilon = aL_0$  and the frequency  $\omega_0 = Q\Omega_0$ . We use the Fourier-transformation method to solve eq.(80):

$$\begin{aligned} \mathcal{F}(\omega) &= \int \exp(i\omega t) f(t) dt, & f(t) &= \int \exp(-i\omega t) \mathcal{F}(\omega) d\omega / 2\pi, \\ 1/2\pi \int \exp(i\omega \tau) dt &= \delta(\omega), \\ \int \exp(i\omega t) f(t+L_0) dt &= \exp(-i\omega L_0) \mathcal{F}(\omega), \\ \int \exp(i\omega t) f^{(N)}(t) dt &= (-i\omega)^N \mathcal{F}(\omega). \end{aligned} \quad (81)$$

Then we get from (80) the following integral equation:

$$\begin{aligned} \sum_{N=1}^M 1/N! \int du (-i\omega)^N R_{M-N}(u) [\exp(-i\omega L_0) - (-1)^N \exp(i\omega L_0)] \times \\ \times \Phi_{\omega-u}(\sin^N(\omega_0 t)) = 2\pi\delta(\omega)\delta_{M0} \end{aligned} \quad (82)$$

$$\begin{aligned} \Phi_{\omega}(\sin^N \omega_0 t) &= \int \exp(i\omega t) \sin^N(\omega_0 t) dt = \\ &= 1/(2\pi)^{N-1} \sum_{J=0}^N C_N^J (-1)^J \delta(\omega + (N-2J)\omega_0). \end{aligned} \quad (83)$$

Substituting expression (83) to eq.(82) we can easily make integration over  $u$  and arrive at the equation

$$\begin{aligned} -R_M(\omega) 2i \sin(\omega L_0) &= 2\delta_{M0} \delta(\omega) - \sum_{N=1}^M \sum_{J=0}^N (-1)^{N+J} / 2^N C_N^J \times \\ &\times [\omega + (N-2J)\omega_0]^N R_{M-N}[\omega + (N-2J)\omega_0] \times \\ &\times [\exp(-i\omega L_0) - (-1)^N \exp(i\omega L_0)] (-1)^{MN}. \end{aligned} \quad (84)$$

Taking into account the formula for derivatives of  $\delta$ -function

$$\delta^{(N)}(x) = -N\delta^{(N-1)}(x)/x,$$

one can easily find the expression for  $R_0(\omega)$

$$R_0(\omega) = 2\pi\delta'(\omega)/iL_0. \quad (85)$$

Then making inverse Fourier-transformation we have

$$R_0(t) = t/L_0. \quad (86)$$

To find the long-time asymptotics we will seek for the solution of eq.(84) in the form of a sum over  $\delta$ -functions

$$R_N(\omega) = \sum_{K=0}^M a_K^N \delta^{(N)}(\omega + (M-2K)\omega_0) \quad (87)$$

This choice corresponds to the representation of the function  $R(t)$  in the form of a power series with respect to the parameter  $(\epsilon t/L^2)$  and neglecting terms like  $\epsilon^J t^N$  with  $J > N$ . Then only terms corresponding to  $N=1$  are significant in eq.(84):

$$-R_N(\omega) 2i \sin(\omega L_0) = 2\delta_{N0} \delta(\omega) + (-1)^0 \cos(\omega L_0) \times \\ \times [(\omega + \omega_0) R_{N-1}(\omega + \omega_0) - (\omega - \omega_0) R_{N-1}(\omega - \omega_0)]. \quad (88)$$

Taking into account the expression for  $R_0(\omega)$  (85) we get

$$R_1(\omega) = 2\pi(-1)^0/2L_0 \cotan(\omega L_0) [(\omega + \omega_0)\delta'(\omega + \omega_0) - \\ - (\omega - \omega_0)\delta'(\omega - \omega_0)] = 2\pi(-1)^0 [\delta'(\omega + \omega_0) - \delta'(\omega - \omega_0)]/2L_0^2. \quad (89)$$

With respect to expansion (87) we have

$$a_0^0 = 2\pi/iL_0, \quad a_0^1 = -a_1^1 = 2\pi(-1)^0/2L_0^2.$$

Substituting expansion (87) we obtain the following recurrence relation for  $N \geq 2$

$$a_K^N = (-1)^0 i\omega_0/2NL_0 [(N-2K-1)a_K^{N-1} - (N-2K+1)a_{K-1}^{N-1}]. \quad (90)$$

Let us introduce the notation  $a_K^{2M} = i\bar{a}_K^{2M}$ ,  $\bar{a}_K^{2M+1} = a_K^{2M+1}$ , then

$$N\bar{a}_K^N = (-1)^N \alpha [\bar{a}_K^{N-1}(N-2K-1) - \bar{a}_{K-1}^{N-1}(N-2K+1)], \\ \alpha = (-1)^0 \omega_0/2L_0. \quad (91)$$

Making inverse Fourier-transformation of (87) we get ( $N \geq 2$ ):

$$R_N(t) = 1/2\pi \sum_{K=0}^N a_K^N (it)^N \exp(it(N-2K)\omega_0)$$

Then it is easy to see that we have got the same expression for  $R_1(t)$  that was given in eq.(76). Now we consider the sum

$$\Phi(t) = \sum_{N=2}^{\infty} \epsilon^N R_N(t) = \sum_{N=2}^{\infty} \epsilon^N (it)^N \sum_{K=0}^N a_K^N \exp(it\omega_0(N-2K)) p^K / 2\pi \quad (92)$$

for  $p=1$ . Taking into account the evident symmetry condition  $\bar{a}_K^N = -\bar{a}_{N-K}^N$  we obtain

$$\Phi = 2 \sum_{N=2}^{\infty} z^N (-1)^n \sum_{K=0}^n \bar{a}_K^N \sin(\omega_0(N-2K)t) p^K / 2\pi, \quad z = \epsilon t, \quad (93)$$

where  $n=[N/2]$  and  $[ ]$  is the entire part of a number. Taking into account that  $(-1)^n = (-1)^{N(N-1)/2}$  we introduce the notation  $\Phi = -2\text{Im}F$ . With the help of the recurrence relation (91) we get the differential equation for the function  $F$

$$\partial F / \partial z = \alpha [p \exp(-i\omega_0 t) - \exp(i\omega_0 t)] [z \partial F / \partial z - 2p \partial F / \partial p +$$

$$+ (z\alpha/\pi Q)\exp(i\omega_0 t)] \quad (94)$$

with the initial condition  $F(z=0, p) = 0$ . Its solution is

$$F = -(z\alpha/\pi Q)\exp(i\omega_0 t) + \Psi,$$

where  $\Psi$  satisfies the equation

$$\partial\Psi/\partial z = \alpha[p\exp(-i\omega_0 t) - \exp(i\omega_0 t)][z\partial\Psi/\partial z - 2p\partial\Psi/\partial p] + (\alpha/\pi Q)\exp(i\omega_0 t). \quad (95)$$

The particular solution of this equation is as follows:

$$\Psi_0 = -(1/2\pi Q) \ln[p\exp(-i\omega_0 t)/(p\exp(-i\omega_0 t) - \exp(i\omega_0 t))]. \quad (96)$$

Then the general solution of eq.(95) is the sum of  $\Psi_0$  and an arbitrary solution of the uniform equation (95) with  $b=0$ . The uniform equation has the first integral:

$$C = \exp(-i\omega_0 t/2) \{zp^{1/2} - 1/2\alpha \ln[(p^{1/2}\exp(-i\omega_0 t/2) - \exp(i\omega_0 t/2))/(p^{1/2}\exp(-i\omega_0 t/2) + \exp(i\omega_0 t/2))]\}.$$

Then we get the general solution of eq.(95) in the form of  $\Psi = \Psi_0 + f(C)$ , where  $f(C)$  is an arbitrary function of the first integral. From the condition  $\Psi(z=0) = 0$  we can determine the form of the function  $f$ :

$$f(x) = (1/2\pi Q) \ln\{[1 + \exp(-2\alpha\exp(i\omega_0 t/2)x)]^2 / 4\exp(-2\alpha\exp(i\omega_0 t/2)x)\}.$$

After some algebraic transformations we find the function  $\Psi$

$$\Psi = -(1/2\pi Q) \ln\{4p\exp(-i\omega_0 t)\exp(-2\alpha z) / [\cos(\omega_0 t/2) + 2i\sin(\omega_0 t/2)\exp(-2\alpha z)]^2\}.$$

Taking into account the first and the second terms of the expansion of  $R(t)$  in a set of  $\varepsilon t$  we obtain the final expression:

$$R(t) = t/L_0 - (2/\pi Q) \operatorname{Im} \ln[1 + \xi + \exp(i\omega_0 t)(1-\xi)], \quad (97)$$

where a notation  $\xi = \exp[(-1)^{0+1}\omega_0 \varepsilon t/L_0]$  is introduced. Now we can compute some characteristics of the electromagnetic field in a cavity in the presumption  $\varepsilon t/L_0^2 \gg 1$ ,  $\varepsilon \ll L_0$ . Let us evaluate the number of photons which will be generated in the resonator with moving walls in the long time limit. For this purpose we must evaluate the integral (77)

$$\beta_{NM} = 1/2 (M/N)^{1/2} \int_{N-1}^{N+1} \exp[-i\pi((M+N)x + Nf(x))],$$

where the function  $f(x)$  is given by (97):

$$f(x) = -2/\pi Q \operatorname{Im} \ln[1+\xi+\exp(i\pi Qx)(1-\xi)].$$

Let us consider the case when  $Q=2p$  is an even number, then  $\xi \ll 1$ . Due to  $\epsilon x/L_0 \ll 1$  we can consider  $f(x)$  as a periodical function with the period of oscillations  $T=2/Q$ . Then

$$\left. \begin{array}{l} \beta_{NM} \\ \alpha_{NM} \end{array} \right\} = (M/N)^{1/2} \sum_{K=0}^{Q-1} \exp[-i\pi(N+M)2K/Q] \sum_{N-1}^{N-1+2/Q} \exp[-i\pi \times \\ \times ((N+M)x+Nf(x))] dx. \quad (98)$$

Analyzing the structure of  $f(x)$  we can approximate it by three linear functions as follows:

$$f(x) = \begin{cases} -(1-\delta Q)(x-N+1), & N-1 < x < N-1+1/Q-\delta, \\ (1/Q-2\delta)(x-N+1-1/Q)/\delta, & N-1+1/Q-\delta < x < N-1+1/Q+\delta, \\ -(1-\delta Q)(x-N+1-2/Q), & N-1+1/Q+\delta < x < N-1+2/Q, \end{cases}$$

$$\delta = 2\xi^{1/2}/\pi q.$$

Then one can easily calculate integrals in (98) and obtain the general expression for the coefficients  $\alpha_{NM}$  and  $\beta_{NM}$

$$\alpha_{NM} = 2(M/N)^{1/2} (-1)^{(N-1)(N-M)} / [\pi(\delta NQ-M)] \sin[(\delta NQ-M)\pi/Q] \times \\ \times \exp[i\pi(N-M)(Q-1)/Q] \sin[\pi(N-M)] / \sin[\pi(N-M)/Q], \quad (99)$$

$$\beta_{NM} = 2(M/N)^{1/2} (-1)^{(N-1)(N+M)} / [\pi(\delta NQ+M)] \sin[(\delta NQ+M)\pi/Q] \times \\ \times \exp[i\pi(N+M)(Q-1)/Q] \sin[\pi(N+M)] / \sin[\pi(N+M)/Q].$$

Hereafter we consider only the main resonance of  $Q=2$ . After some algebraic transformations we get the following expression for the coefficient  $|\beta_{NM}|^2$ :

$$|\beta_{NM}|^2 = 4M/(N\pi^2) [1-(-1)^M \cos(2N\delta\pi)] [1+(-1)^{M+N}] / \\ / (M+2N\delta)^2. \quad (100)$$

To find the total number of photons in the mode with number  $M$  we need to calculate the sum over  $N$ . First, let us evaluate the following auxiliary sum:

$$S(z, x) = \sum_{N=1}^{\infty} \cos(xN) / [N(z+N)^2], \quad (101)$$

where  $x=2\delta\pi \ll 1$  and  $z=M/2\delta \gg 1$ . Then we have

$$S(z, x) = \int_0^{\infty} y \sum_{N=1}^{\infty} \cos(Nx) \exp[-y(z+N)] / N dy = \\ = 1/z^3 - 1/2 \int_0^{\infty} y \exp(-zy) \ln(2\cosh y - 2\cos x) dy. \quad (102)$$

Since  $x \ll 1$  and the main region of integration  $y \leq 1/z \ll 1$ , we can expand  $\cosh y$  and  $\cos x$  into power series of  $y$  and  $x$ . Thus up to the second order terms we have

$$S(z,x) = 1/z^3 - 1/2 \int_0^{\infty} y \exp(-zy) \ln(y^2+x^2) dy. \quad (103)$$

The last integral can be easily evaluated, if one takes into account the inequalities  $y \leq 1/z \ll x \ll 1$ . Therefore

$$S(z,x) = -\ln x/z^2 + O(z^{-3}). \quad (104)$$

The similar sum (see (100))

$$\sum_{N=1}^{\infty} \cos(Nx)/[N(z+N)] (-1)^N$$

can be obtained from (102) by means of the replacement  $x \rightarrow x+\pi$ . Then we have in (103)  $\ln(4+y^2-x^2) < \ln|x^2|$  for  $x \ll 1$  and  $y \ll 1$ , so that the corresponding terms can be omitted. The main contribution to the sum due to the first term (with unity in the numerator) in expression (100) is proportional to

$$S(z,0) = -\int_0^{\infty} y \exp(-yz) \ln[1-\exp(-y)] \approx \ln z/z^2. \quad (105)$$

Thus the number of photons generated in the  $M$ -th mode is

$$P_M = \sum_{N=1}^{\infty} |\beta_{NM}|^2 \approx 4[\ln(M/2\delta) - (-1)^M \ln(1/2\delta\pi)]/(M\pi^2). \quad (106)$$

Since in the considered case ( $Q=2$ )  $\delta(t) = \exp(-\pi\epsilon t/L_0^2)/\pi$ , we get the following rate of photon generation for  $\epsilon t/L_0^2 \gg 1$ :

$$dP_M/dt = 4a\Omega_0[1-(-1)^M]/(\pi^2 M). \quad (107)$$

Here  $\Omega_0 = \pi/L_0$  is the main eigenfrequency of the resonator,  $a = \epsilon/L_0$  is the dimensionless amplitude of oscillations of the wall (which vibrates at the frequency  $2\Omega_0$ ). Eq. (107) is valid in fact only for not very large numbers of excited modes  $M$  (due to limitations arising in approximations made before). Besides, in real situation we should limit the time  $t$  by the relaxation time of the resonator  $\tau$  (due to the dissipation on the walls). Then the maximum number of photons generated in the  $M$ -th mode equals approximately

$$P_M^{\max} \sim 4/(\pi^2 M) [2aQ(M)/M + O(\ln(M+1))], \quad (108)$$

where  $Q(M)$  is the quality factor of the resonator's  $M$ -th mode. This formula is valid provided  $aQ/M \gg \ln(M+1)$ .

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