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THE HAMILTONIAN STRUCTURE OF DIRAC'S EQUATION IN TENSOR FORM AND ITS FERMI QUANTIZATION

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ABSTRACT

Currently there is some interest in studying the tensor forms of the Dirac equation to elucidate the possibility of the constrained tensor fields admitting Fermi quantization. In this paper, we demonstrate that the bispinor and tensor Hamiltonian systems have equivalent Fermi quantizations. Although the tensor Hamiltonian system is noncanonical, representing the tensor Poisson brackets as commutators for the Heisenberg operators directly leads to Fermi quantization without the use of bispinors.

I. CLASSICAL DERIVATION

We apply the double covering map from bispinors to their tensor equivalents. This map[1], an extension of Cartan's spinor map [Ref. 4], maps the bispinor ψ to a constrained set of SL(2,C) x U(1) gauge potentials A_α^K and a complex scalar field ρ, where $\alpha = 0, 1, 2, 3$ is a Lorentz index and $K = 0, 1, 2, 3$. Since the Lie algebra of SL(2, C) is regarded as the complexification of the Lie algebra of SU(2), the gauge potentials A_j^α for $j = 1, 2, 3$ are complex, while the U(1) gauge potential A_o^α is real. A_α^K and ρ satisfy the following constraint:

$$A_\alpha^K A_{K\beta} = -|\rho|^2 g_{\alpha\beta} \qquad (1)$$

where K is contracted using the SU(2) $\times$ U(1) Killing metric and $g_{\alpha\beta}$ is the space-time metric.

With this constraint, the Dirac bispinor Lagrangian comes from the following Yang-Mills tensor Lagrangian L, in the limit of a large Yang-Mills coupling constant g:

$$L = -\tfrac{1}{4} \operatorname{Re} [A_K^{\alpha\beta} A_{\alpha\beta}^K] + (\overline{D_\alpha \rho})(D^\alpha \rho) - \frac{g^2}{2} |\rho + 2m|^4 \qquad (2)$$

where m denotes mass. D_α denotes the Yang-Mills covariant derivative with connection coefficients A_α^K, and $A_{\alpha\beta}^K$ is the Yang-Mills curvature tensor associated with the gauge potentials A_α^K. The indices are contracted using the Killing metric as well as the space-time metric. All bispinor observables can be derived from L using Yang-Mills formulas. Although previous authors [Refs. 5, 2] derived the tensor form of the Dirac Lagrangian, they did not put it in the gauge symmetric Yang-Mills form (2).

The Dirac equation can be derived from the Lagrangian L by ascribing to the Yang-Mills field A_α^K a large self-coupling constant g. To be consistent with observation, the fields A_α^K and ρ must couple more weakly (by the factor 1/g) with other fields. In particular, Einstein's equation becomes $G_{\alpha\beta} = kT_{\alpha\beta}$ where $G_{\alpha\beta}$ is the Einstein tensor, k is the gravitation constant, and $T_{\alpha\beta}$ is the energy-momentum tensor derived from the Lagrangian L. In the limit of large self-coupling g (neglecting terms in $T_{\alpha\beta}$ not containing g) we have $T_{\alpha\beta} = g\, T'_{\alpha\beta}$ where $T'_{\alpha\beta}$ is exactly the usual Dirac energy-momentum tensor [Ref. 5]. Hence $k' = kg$ is the observed gravitational constant, not k. Note also that in the Lagrangian L, the observed mass is $m' = mg$, not m. Then, as g tends to infinity, the Lagrangian kL is independent of g. In this limit, which we henceforth assume, we have:

$$\lim_{g \to \infty} k\, L = k' L' \qquad (3)$$

where L' is exactly equal to Dirac's bispinor Lagrangian [Ref. 5]. Thus, as previously stated, Dirac's bispinor Lagrangian is a limiting case of the Yang-Mills Lagrangian (2), in which the self-coupling constant g tends to infinity.

II. FERMI QUANTIZATION

We quantize A_α^K and ρ by defining the classical Hamiltonian to be: (Let $S \subset R^3$ be a large cube.)

[1] The Cartan map [Refs. 1, 2] maps the bispinor ψ to a triplet of complex antisymmetric tensors $F_j^{\alpha\beta}$ (where j = 1,2,3) of Carmeli class G [Ref. 3]. Such $F_j^{\alpha\beta}$ can be expressed as $F_j^{\alpha\beta} = \rho(A_o^\alpha A_j^\beta - A_j^\alpha A_o^\beta + i\varepsilon_{jkm} A_k^\alpha A_m^\beta)$ where ρ is a complex scalar and A_K^α for K = 0,1,2,3 are SL(2,C) $\times$ U(1) gauge potentials satisfying (1). (ε_{jkm} for j, k, m = 1, 2, 3 is the permutation symbol.)

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$$H = \int_S T^{oo}\, d\mathbf{x} \qquad (4)$$

where $T^{\alpha\beta}$ is the energy-momentum tensor derived from the fermion tensor Lagrangian (3). We make a classical change of variables that simplifies H. The resulting Hamiltonian equations are then formulated as Heisenberg operator equations.

Because the SL(2, C) × U(1) gauge group is not compact, H is not bounded from below. This has the consequence that any quantization of the fields A^K_α and ρ must obey the exclusion principle; otherwise fermions descend forever to lower energy states.

By the Cartan map [Refs. 1, 5] the energy-momentum tensor has an expansion of the form:

$$T^{\alpha\beta}(\mathbf{x},t) = \sum_p \sum_q T^{\alpha\beta}_{pq}(\mathbf{x})\, a_{pq}(t) \qquad (5)$$

where the sum is over all pairs of fermion modes p and q, and where $T^{\alpha\beta}_{pq}(\mathbf{x})$ are fixed functions of $\mathbf{x} \varepsilon S$, and $a_{pq}(t)$ are complex functions of time t satisfying $a_{pq} = \bar{a}_{qp}$. The bimodal expansion (5) is irreducible because it cannot be expressed in tensor terms as a sum over products of single modes, as is the case with bosons. The Hamiltonian (4) can be written in terms of the amplitudes $a_{pq}(t)$ as follows:

$$H = \sum_p \omega_p\, a_{pp} \qquad (6)$$

where ω_p is the frequency of the mode p. Note that for simplicity, the amplitudes $a_{pq}(t)$ are defined to be consistent with the hole theory.

The classical Hamiltonian equations (which are equivalent to the constrained Euler-Lagrange equations for A^K_α and ρ) are given by:

$$\frac{da_{pq}}{dt} = \{a_{pq}, H\} \qquad (7)$$

where the Poisson brackets { , } are defined for the classical amplitudes $a_{pq}(t)$ as follows:

$$\{a_{pq}, a_{p'q'}\} = -i\,(a_{pq'}\,\delta_{p'q} - a_{p'q}\,\delta_{pq'}) \qquad (8)$$

where δ_{pq} equals one if p = q and zero otherwise.

Formulas (6), (7), and (8) are noncanonical tensor Hamiltonian equations which cannot be formulated as canonical equations in tensor terms. Nevertheless, they are easily quantized by replacing the classical amplitudes $a_{pq}(t)$ with Heisenberg operators, denoted as $\hat{a}_{pq}(t)$, and the Poisson brackets (8) with (equal time) commutators [,] as follows:

$$[\hat{a}_{pq}, \hat{a}_{p'q'}] = \hat{a}_{pq'}\,\delta_{p'q} - \hat{a}_{p'q}\,\delta_{pq'} \qquad (9)$$

The Heisenberg equations become:

$$\frac{d\hat{a}_{pq}}{dt} = -i\,[\hat{a}_{pq}, \hat{H}] \qquad (10)$$

where $\hat{H}$ is the operator version of the Hamiltonian (6).

To further simplify these equations, we attempt to factor $\hat{a}_{pq}(t)$ into a product of operators:

$$\hat{a}_{pq}(t) = \hat{c}^\dagger_p(t)\,\hat{c}_q(t) \qquad (11)$$

where the dagger (†) signifies adjoint. Since they do not occur explicitly in the Hamiltonian $\hat{H}$, the new operators $\hat{c}_p(t)$ a priori could satisfy *any* relations consistent with the commutation relations (9). We exploit this arbitrariness in order to satisfy the exclusion principle, previously discussed. At time t we define:

$$\hat{c}^\dagger_p\,\hat{c}_q + \hat{c}_q\hat{c}^\dagger_p = \delta_{pq} \qquad (12)$$

All other equal time anti-commutators of $\hat{c}_p(t)$ are defined to be zero. Formulas (11) and (12) are consistent with the commutation relations (9) as required.

It is clear that equations (9), (10), (11), and (12), while derived from the tensor Hamiltonian equations, are equivalent to Fermi quantization via bispinors. Thus, the tensor Lagrangian (3) leads to Fermi quantization without the use of bispinors.

Again, without the use of bispinors, we may extend the tensor Lagrangian (3) to include the electromagnetic field. Quantization is straight forward due to the fact that the interaction term is a function of the fermion amplitudes $a_{pq}(t)$, as well as boson amplitudes $b_n(t)$.

III. QUANTUM GRAVITY

Spinor structure can be defined on a noncompact space-time manifold M by specifying, at each point xεM, a set of Pauli spin-half matrices $\sigma^\alpha_{AB'}(x)$ satisfying [Ref. 3]:

$$\sigma^\alpha_{AB'}\sigma^{\beta AB'} = g^{\alpha\beta} \qquad (13)$$

Formula (13) has a topological as well as a metric consequence. The topological consequence of (13) is that M must be parallelizable [Ref. 6]. The metric consequence is that $g_{\alpha\beta}$ is constrained as in formula (1). Since, for noncompact parallelizable space-times formulas (1) and (13) are equivalent, spinor structure is nothing but an indirect way of constraining the metric $g_{\alpha\beta}$ on such space-times. However, the tensor fields A_α^K and ρ satisfying the constraint (1) are more general than (13), since they can be defined on general space-times.

Formula (13) presents a dilemma [Ref. 7] for quantizing both gravity and the Dirac field, since the definition of the Pauli matrices $\sigma^\alpha_{AB'}$ depends on the gravitational field $g_{\alpha\beta}$. The problem is resolved by identifying the degrees of freedom in the constraint (1) as follows.

Consider a *fixed* metric $\tilde{g}_{\alpha\beta}$ on M and define Pauli matrices $\tilde{\sigma}^\alpha_{AB'}$ with respect to $\tilde{g}_{\alpha\beta}$. The metric $g_{\alpha\beta}$ on M is expressed by:

$$g_{\alpha\beta} = \tilde{g}_{\alpha\beta} + h_{\alpha\beta} \tag{14}$$

We also express the gauge potentials A_α^K by:

$$A_\alpha^K = f_\alpha^\beta \tilde{A}_\beta^K \tag{15}$$

where $\tilde{A}_\alpha^K$ satisfies the constraint (1) with respect to the fixed metric $\tilde{g}_{\alpha\beta}$. The dynamical fields are then $\tilde{A}_\alpha^K$, ρ, and $h_{\alpha\beta}$ provided that the matrix $f = f_\alpha^\beta$ can be uniquely solved as a function of $h_{\alpha\beta}$. Since $\tilde{A}_\alpha^K$ and ρ have bispinor coordinates with respect to the *fixed* spinor structure on M, the fields $\tilde{A}_\alpha^K$, ρ, and $h_{\alpha\beta}$ can be quantized as in Section II.

It remains to solve for the matrix f in formula (15) using the constraint (1). Since

$$\tilde{A}_\alpha^K \tilde{A}_{K\beta} = -|\rho|^2 \tilde{g}_{\alpha\beta} \tag{16}$$

formulas (14) and (15) give:

$$\tilde{g}_{\gamma\sigma} f_\alpha^\gamma f_\beta^\sigma = \tilde{g}_{\alpha\beta} + h_{\alpha\beta} \tag{17}$$

The solution of (17) is given by:

$$f = \sum_{n=0}^{\infty} C_n^{1/2} h^n \tag{18}$$

where C_n^m denote the binomial coefficients, and the matrix h is defined by:

$$h = h_\alpha^\beta = \tilde{g}^{\beta\gamma} h_{\gamma\alpha} \tag{19}$$

where $\tilde{g}^{\alpha\beta}$ is the inverse matrix of $\tilde{g}_{\alpha\beta}$.

For the power series (18) to converge, the eigenvalues of h must lie within the unit circle. This restricts the validity of quantum gravity to small fluctuations of $g_{\alpha\beta}$ about the fixed metric $\tilde{g}_{\alpha\beta}$.

IV. CONCLUSIONS

In this paper we have adhered to the program of first defining all fields, Bose and Fermi, as classical tensor fields, and then quantizing them using Hamilton equations and Poisson brackets. From this vantage point, the Dirac equation becomes a classical tensor equation on the same level as the electromagnetic and gravitation tensor equations. Fermions, photons, and gravitons are obtained by quantizing the degrees of freedom allowed by the tensor constraint (1). We have shown in Section III that the constraint (1) implies that we cannot, in general, separate fermion and graviton degrees of freedom, except when the power series (18) converges.

We also found that the fermion degrees of freedom require the use of noncanonical Hamilton equations (6), (7), and (8). Since the free Dirac tensor equation is completely *integrable*, we have shown that current usage of only canonical Hamilton equations is too restrictive for quantizing integrable tensor fields.

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