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## Summary

A near-wall four-equation turbulence model is developed for the calculations of high-speed compressible turbulent boundary layers. The four equations are the  $k$ - $\epsilon$  equations and the  $\overline{\theta^2}$ - $\epsilon_\theta$  equations. These equations are used to define the turbulent diffusivities for momentum and heat fluxes, thus allowing the assumption of dynamical similarity between momentum and heat transport to be relaxed. The Favre-averaged equations of motions are solved in conjunction with the four transport equations for  $k$ ,  $\epsilon$ ,  $\overline{\theta^2}$  and  $\epsilon_\theta$ . Calculations are compared with measurements and with another model predictions where the assumption of a constant turbulent Prandtl number is invoked. Compressible flat plate turbulent boundary layers with both adiabatic and constant temperature wall boundary conditions are considered. Cases where the free-stream Mach number as high as 10 and where the wall temperature ratio as low as 0.2 are calculated. Results for the range of low Mach numbers and temperature ratios investigated are essentially the same as those obtained using an identical near-wall  $k$ - $\epsilon$  model and the assumption of  $Pr_t = 0.9$ . One reason could be that the model constants used in the  $\overline{\theta^2}$  and  $\epsilon_\theta$  equations have not been optimized to give the best results for incompressible and compressible flows. In general, the numerical predictions are in very good agreement with measurements and there are significant improvements in the predictions of mean flow properties at high Mach numbers. Present results further show that the calculated  $Pr_t$  for all cases investigated varies rapidly from about 0.5 at the wall to a maximum of approximately 1.6 in the near-wall region; however, it quickly settles to a constant value of 0.9 beyond  $y_w^+ \geq 200$ . Therefore, the calculations lend credence to the  $Pr_t = 0.9$  assumption invoked by other researchers.

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## Nomenclature

### English Letters

$a_k, b_k$	coefficients in the expansion for $k^+$ in the near-wall region
$a_{uv}, b_{uv}$	coefficients in the expansion for $\overline{uv}^+$ in the near-wall region
$a_{\theta^2}, b_{\theta^2}$	coefficients in the expansion for $\theta^{+2}$ in the near-wall region
$a_{v\theta}, b_{v\theta}$	coefficients in the expansion for $\overline{v\theta}^+$ in the near-wall region
$a_{\varepsilon\theta}, b_{\varepsilon\theta}$	coefficients in the expansion for $\varepsilon_\theta^+$ in the near-wall region
A	model constant taken to be 45
B	constant in law-of-the-wall
$C_{1\lambda}$	model constant taken to be 0.1
$C_{d1}$	model constant taken to be 1.8
$C_{d2}$	model constant taken to be zero
$C_{d3}$	model constant taken to be 0.72
$C_{d4}$	model constant taken to be 2.2
$C_{d5}$	model constant taken to be 0.8
$C_f$	skin friction coefficient, $= 2\tau_w/\rho_\infty U_\infty^2$
$(C_f)_i$	skin friction coefficient for an incompressible flow
$C_p$	specific heat at constant pressure
$C_{\varepsilon 1}$	model constant taken to be 1.5
$C_{\varepsilon 2}$	model constant taken to be 1.83
$C_\mu$	model constant taken to be 0.096
$C_\lambda$	model constant taken to be 0.11
$C_{\theta^2}$	model constant taken to be 0.11
$C_{\varepsilon\theta}$	model constant taken to be 0.11
$f_{w,2}$	near-wall damping function for $\varepsilon$ equation
$f_{w,\varepsilon\theta}$	near-wall damping function for $\varepsilon_\theta$ equation
$f_\mu$	near-wall damping function for turbulent momentum diffusivity
$f_\lambda$	near-wall damping function for turbulent heat diffusivity
H	instantaneous total enthalpy, $= C_p T + \frac{1}{2} U_k U_k$
k	turbulent kinetic energy
$k^+$	normalized k, $= k/u_\tau^2$
M	Mach number
$M_t$	Mach number based on friction velocity, $= u_\tau/(\gamma R T_w)^{1/2}$
p	instantaneous pressure

$p'$	Reynolds fluctuating pressure
$Pr$	molecular Prandtl number
$Pr_t$	turbulent Prandtl number
$R$	universal gas constant
$R_t$	turbulent Reynolds number, $= k^2/\bar{\nu} \epsilon$
$R_\theta$	Reynolds number based on momentum thickness
$T$	instantaneous temperature
$U_i$	$i$ th component of the instantaneous velocity
$U, V$	instantaneous velocity components along $x$ and $y$ , respectively
$u_i$	$i$ th component of the Favre fluctuating velocity
$u, v$	Favre fluctuating velocity components along $x$ and $y$ , respectively
$u^+$	normalized mean $U$ velocity, $= \langle U \rangle / u_\tau$
$u_\tau$	friction velocity, $= (\tau_w / \bar{\rho}_w)^{1/2}$
$\overline{uv}^+$	normalized turbulent shear stress, $= \langle uv \rangle / u_\tau^2$
$\bar{v}\theta^+$	normalized turbulent heat flux, $= \langle v\theta \rangle / U_\infty \Theta_\infty$
$x, y$	coordinates along stream and normal directions
$y^+$	normalized $y$ coordinate, $= yu_\tau/\bar{\nu}$
$y_w^+$	normalized $y$ coordinate, $= yu_\tau/\bar{\nu}_w$

#### Greek Letters

$\alpha$	thermal conductivity
$\alpha_t$	turbulent heat diffusivity
$\gamma$	specific heat ratio
$\delta$	boundary layer thickness
$\epsilon$	solenoidal dissipation rate of $k$
$\epsilon_\theta$	dissipation rate of temperature variance
$\tilde{\epsilon}$	dissipation rate defined as $\epsilon - 2\bar{\nu}(\partial\sqrt{k}/\partial y)^2$
$\tilde{\epsilon}_\theta$	dissipation rate defined as $\epsilon_\theta - \bar{\alpha}(\partial\sqrt{\theta^2}/\partial y)^2$
$\epsilon^*$	dissipation rate defined as $\epsilon - 2\bar{\nu}k/y^2$
$\epsilon_\theta^*$	dissipation rate defined as $\epsilon_\theta - \bar{\alpha}\theta^2/y^2$
$\epsilon^+$	normalized dissipation rate, $= \epsilon \bar{\nu}_w / u_\tau^4$
$\epsilon_\theta^+$	normalized dissipation rate, $= \epsilon_\theta \bar{\nu}_w / u_\tau^2 \Theta_\infty^2$
$\theta$	Favre fluctuating temperature
$\theta_\tau$	friction temperature
$\theta^2$	temperature variance



$\theta^{+2}$	normalized temperature variance, = $\langle \theta^2 \rangle / \Theta_\infty^2$
$\Theta$	mean component of temperature
$\kappa$	von Karman constant
$\mu$	fluid viscosity
$\mu_t$	turbulent viscosity
$\nu$	fluid kinematic viscosity
$\nu_t$	turbulent kinematic viscosity, = $\mu_t / \rho$
$\xi$	near-wall correction to $\varepsilon$ equation
$\xi_{\varepsilon\theta}$	near-wall correction to $\varepsilon_\theta$ equation
$\rho$	instantaneous fluid density
$\rho'$	Reynolds fluctuating density
$\sigma_k$	model constant taken to be 0.75
$\sigma_\varepsilon$	model constant taken to be 1.45
$\sigma_{\theta^2}$	model constant taken to be 0.75
$\sigma_{\varepsilon\theta}$	model constant taken to be 1.45
$\tau$	shear stress
$\omega_i$	fluctuating vorticity

#### Subscripts

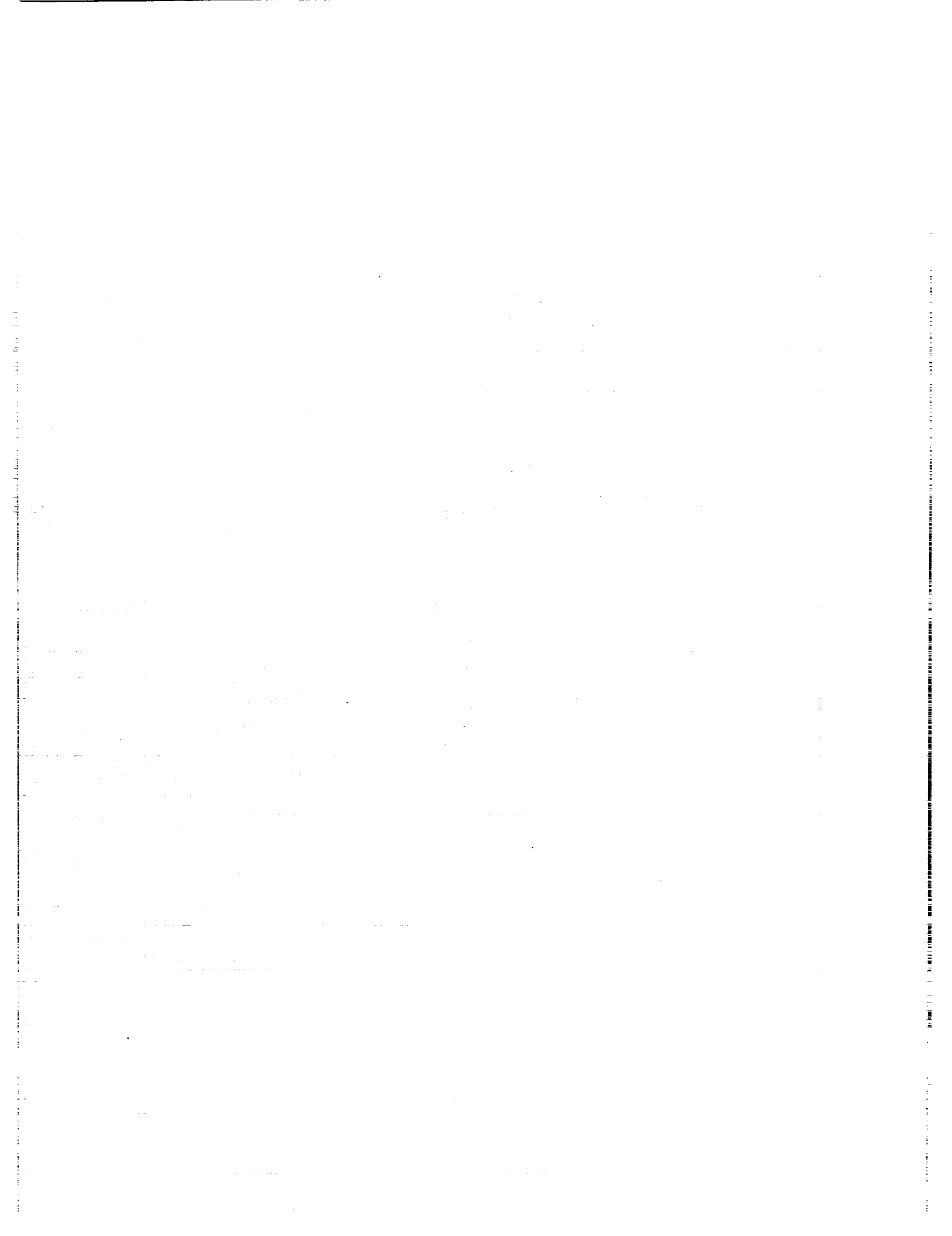
aw	adiabatic wall
r	reference condition
w	wall
$\infty$	free-stream condition

#### Overbars

—	time-averaged quantities
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#### Brackets

$\langle \rangle$	Favre-averaged quantities
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## 1. Introduction

In non-isothermal turbulent flow calculations, turbulent momentum and heat fluxes need modeling if the governing equations are to be closed. If, in addition, the flow is compressible, the modeling of these fluxes are complicated by the presence of a variable mean and fluctuating density in the governing equations. Conventional approach is to neglect the effects of the fluctuating density and to propose models for the momentum fluxes while an additional assumption is made to relate the heat fluxes to the modeled momentum fluxes. Proposals for the incompressible momentum fluxes range from one-equation to second-order closure models (Speziale 1991). Most closure schemes for compressible flows invoke Morkovin's (1962) hypothesis of dynamical field similarity between compressible and incompressible flows. This assumption, therefore, allows the direct extension of incompressible models to account for compressibility effects. In addition, the assumption of dynamical similarity between turbulent heat and momentum transport is invoked and this permits the specification of a constant turbulent Prandtl number in the closure schemes (van Driest 1951; Anderson and Lewis 1971; Bradshaw 1977; Wilcox 1988; Speziale and Sarkar 1991; Aupoix and Cousteix 1991). Under these assumptions compressibility effects are accounted for by the mean density alone. As a result, the ability of conventional models to reliably predict compressible turbulent boundary-layer flows for Mach numbers  $M_\infty \geq 5$  has been called into question (Bradshaw et al. 1991; Huang et al. 1992).

Attempts to relax some of these assumptions have been made recently. For example, Zhang et al. (1992) propose a compressible near-wall k- $\epsilon$  model where all additional dilatational terms are systematically derived and accounted for in the governing equations. Therefore, they are able to assess the validity and extent of Morkovin's hypothesis. Their analysis reveals that, if the near-wall model is internally consistent and asymptotically correct, Morkovin's hypothesis is essentially valid for adiabatic wall with  $M_\infty$  as high as 10. Consequently, the predictions in this Mach number range are in very good agreement with measurements. On the other hand, the model predictions of  $C_f$  in the case of cooled-wall boundary layers are not as good. The reason may not

be the breakdown of Morkovin's hypothesis but rather the consequence of the assumption of a constant turbulent Prandtl number. The present study makes a first attempt to assess this postulate, and proposes to relax the assumption of a constant turbulent Prandtl number in the modeling of compressible turbulent boundary-layer flows. A near-wall four-equation model is suggested as an alternative; two equations each to model the turbulent heat and momentum fluxes. The following approach is adopted in the formulation of this near-wall four-equation model.

In Zhang et al.'s (1992) near-wall  $k$ - $\epsilon$  model, the compressible dissipation function is split into a solenoidal part, which is not sensitive to changes of compressibility indicators, and a dilatational part, which is directly affected by these changes (Sarkar et al. 1989). This procedure, therefore, isolates terms in the  $k$  equation with explicit dependence on compressibility so that they can be modeled accordingly. An equation that governs the transport of the solenoidal dissipation rate with additional terms that are explicitly dependent on compressibility effects is derived similarly. A model (Sarkar et al. 1989) with an explicit dependence on the turbulent Mach number is adopted for the dilatational dissipation rate. Thus formulated, all near-wall incompressible flow models could be expressed in terms of the solenoidal dissipation rate and straight-forwardly extended to compressible flows. As a result, the incompressible equations are recovered correctly in the limit of constant density and vanishing turbulence Mach number. A number of near-wall two-equation models are available (Myong and Kasagi 1990; Deng and Piquet 1991; Karlsson et al. 1991; Michelassi et al. 1991; So et al. 1991a; Yang and Shih 1991). However, none is as widely tested for asymptotic consistency as the model of So et al. (1991a) who have validated their model against such benchmark data as the direct numerical simulations of channel flows (Kim et al. 1987; Mansour et al. 1988), of flat plate boundary-layer flows (Spalart 1988) and of Couette flows (Lee and Kim 1991) as well as experimental measurements (Klebanoff 1955; Nishino and Kasagi 1989). The results are in excellent agreement with data and have been reported by So et al. (1991b) and by Zhang and So (1991). In view of this, Zhang et al. (1992) adopt the near-wall model of So et al. (1991a) and extend it directly to compressible flows. Their results show that

compressible flat plate turbulent boundary layers can be predicted correctly up to  $M_\infty = 10$  for both adiabatic and cooled wall boundary conditions. Therefore, this suggests that the near-wall k- $\epsilon$  model of Zhang et al. (1992) should be adopted for the momentum fluxes in the present study.

If a constant turbulent Prandtl number is not assumed, consistent with the momentum flux model, a near-wall heat flux model has to be proposed. Near-wall modeling of heat fluxes is not as well developed; nevertheless, a second-order closure has been proposed by Lai and So (1990a) and a  $\overline{\theta^2}$ - $\epsilon_\theta$  model has been put forward by Nagano and Kim (1988) for non-isothermal incompressible flows. However, none has been formulated for compressible flows. The work of Zhang et al. (1992) points to the importance of having an internally consistent and asymptotically correct near-wall model for compressible flows. Therefore, if an incompressible near-wall model is to be extended to compressible boundary layers, its asymptotic behavior near a wall has to be analysed first. This analysis has been carried out for the  $\overline{\theta^2}$ - $\epsilon_\theta$  model by Sommer et al. (1992) and their results show that the  $\overline{\theta^2}$ - $\epsilon_\theta$  model of Nagano and Kim (1988) fails to correctly reproduce the asymptotic behavior of the temperature variance and its dissipation rate. Modifications along the line of So et al.'s (1991a) analysis of the  $\epsilon$  equation have been proposed and a correction function for the  $\epsilon_\theta$  equation has been derived by extending the coincidence condition of Shima (1988) to the analysis of the  $\epsilon_\theta$  equation. Thus derived, the new near-wall  $\overline{\theta^2}$ - $\epsilon_\theta$  model for heat fluxes is found to correlate well with direct simulation data (Kim and Moin 1989; Kasagi et al. 1991) and experimental measurements (Johnk and Hanratty 1962; Hishida et al. 1986). In particular, the asymptotic near-wall behavior of the direct simulation data is reproduced correctly for both constant temperature and constant heat flux wall boundary conditions. This near-wall  $\overline{\theta^2}$ - $\epsilon_\theta$  model is extended to compressible flows in the present study by following the approach used by Zhang et al. (1992) to develop the compressible k- $\epsilon$  model.

The four equations thus formulated are used to calculate compressible flat plate turbulent boundary layers with adiabatic and constant temperature wall boundary conditions. Comparisons with well documented experimental measurements (Fernholz and Finley 1977) as well as with the

model calculations of Zhang et al. (1992) are carried out. Furthermore, the model is used to calculate the variation of skin friction coefficient with wall temperature for a cooled wall at a fixed free-stream Mach number and the result is compared with the van Driest II formula (Kline et al. 1981). The validity and extent of the constant turbulent Prandtl number assumption is then assessed by comparing the results of the four-equation model with those of the k- $\epsilon$  model (Zhang et al. 1992).

In the following, the governing equations for compressible boundary layers are given in Section 2. The modeled equations for  $k$  and  $\epsilon$  as derived by Zhang et al. (1992) are presented in Section 3. In Section 4, the near-wall modeled equations for  $\overline{\theta^2}$  and  $\epsilon_\theta$  as derived by Sommer et al. (1992) are given together with their extension to compressible flows. Model validations and comparisons with measurements are discussed in Section 5. Finally, the conclusions of this study are presented in Section 6.

## 2. The Compressible Boundary-Layer Equations

The mean equations of motions for compressible turbulent boundary layers can be derived from the instantaneous Navier-Stokes equations by applying Favre-averaging and then invoking the Prandtl boundary-layer approximations to simplify the resulting averaged equations. Favre decomposition is invoked for all variables except  $p$  and  $\rho$  where conventional Reynolds decomposition is assumed. In other words,

$$U_i = \langle U_i \rangle + u_i \quad , \quad (1)$$

$$T = \langle \Theta \rangle + \theta \quad , \quad (2)$$

$$p = \bar{p} + p' \quad , \quad (3)$$

$$\rho = \bar{\rho} + \rho' \quad . \quad (4)$$

When these decompositions are substituted into the Navier-Stokes equations and time averaging is applied, a set of turbulent mean flow equations are obtained. The boundary-layer approximations and the assumption of negligible fluctuations in fluid properties, such as  $\mu$ ,  $C_p$ , etc, are used to further simplify these equations. Since the pressure field is constant for flat plate boundary layers, the resulting compressible turbulent boundary-layer equations can be written as (Wilcox 1988):

$$\frac{\partial}{\partial x} (\bar{\rho} \langle U \rangle) + \frac{\partial}{\partial y} (\bar{\rho} \langle V \rangle) = 0 \quad , \quad (5)$$

$$\bar{\rho} \langle U \rangle \frac{\partial \langle U \rangle}{\partial x} + \bar{\rho} \langle V \rangle \frac{\partial \langle U \rangle}{\partial y} = \frac{\partial}{\partial y} \left[ (\bar{\mu} + \bar{\mu}_t) \frac{\partial \langle U \rangle}{\partial y} \right] \quad , \quad (6)$$

$$\begin{aligned} \bar{\rho} \langle U \rangle \frac{\partial \langle H \rangle}{\partial x} + \bar{\rho} \langle V \rangle \frac{\partial \langle H \rangle}{\partial y} = \\ \frac{\partial}{\partial y} \left[ \left( \frac{\bar{\mu}}{Pr} + \frac{\bar{\mu}_t}{Pr_t} \right) \frac{\partial \langle H \rangle}{\partial y} + \left\{ \bar{\mu} \left( 1 - \frac{1}{Pr} \right) + \bar{\mu}_t \left( 1 - \frac{1}{Pr_t} \right) \right\} \langle U \rangle \frac{\partial \langle U \rangle}{\partial y} \right] \quad , \quad (7) \end{aligned}$$

where Cartesian x-y coordinates have been used and, consistent with conventional wisdom, the temperature equation is converted into the total enthalpy equation. The mean equation of state is assumed to be given by  $\bar{p} = \bar{p} R \langle \Theta \rangle$  and Sutherland's law is used to evaluate the mean fluid viscosity (Zhang et al. 1992). Therefore, once  $\bar{\mu}_t$  and  $Pr_t$  are known, (5) - (7) can be solved to give the velocity and temperature fields inside the boundary layers.

In writing down these equations, gradient transport has been assumed for both the turbulent momentum and heat fluxes. Therefore, if  $\bar{\mu}_t$  is taken to be given by  $\bar{\rho}(\bar{\nu}_t)$ , then the turbulent fluxes can be written as:

$$-\bar{\rho}\langle uv \rangle = \bar{\rho} \bar{\nu}_t \frac{\partial \langle U \rangle}{\partial y} , \quad (8)$$

$$-\bar{\rho}\langle v\theta \rangle = \bar{\rho} \bar{\alpha}_t \frac{\partial \langle \Theta \rangle}{\partial y} . \quad (9)$$

It should be pointed out that even though the equations are written in terms of a turbulent Prandtl number  $Pr_t$ , they do not imply constant  $Pr_t$ . The equations are simply written in this form for convenience and to comply with conventional format (Wilcox 1988; Zhang et al. 1992). Here,  $Pr_t = \bar{\nu}_t / \bar{\alpha}_t$  and the turbulent diffusivities are defined as:

$$\bar{\nu}_t = C_\mu f_\mu k^2 / \varepsilon , \quad (10)$$

$$\bar{\alpha}_t = C_\lambda f_\lambda k [k \bar{\theta}^2 / \varepsilon \varepsilon_\theta]^{1/2} , \quad (11)$$

where the damping functions are to be defined when closure models for the momentum and heat fluxes are discussed in the next two sections. In this form,  $Pr_t$  varies according to the variations of  $\bar{\nu}_t$  and  $\bar{\alpha}_t$ . In the following, near-wall turbulence models are proposed for  $\bar{\nu}_t$  and  $\bar{\alpha}_t$  so that their variations across the boundary layers can be determined together with other properties of the boundary-layer flows.



It should be pointed out that a rigorous derivation of (7) will give an additional turbulent kinetic energy term on the right hand side of (7) as pointed out by Zhang et al. (1992). In the present formulation, this term is omitted consistent with the work of other researchers (Wilcox 1988). Besides, Zhang et al. (1992) have demonstrated that the neglect of this term has a positive effect on the prediction of skin friction for compressible boundary-layer flows with a highly cooled-wall.

The boundary conditions for  $\langle U \rangle$  and  $\langle V \rangle$  are no slip at the wall and  $\langle U \rangle$  approaches  $U_\infty$  in the free stream. As for  $\langle H \rangle$ , its free-stream value is given by  $H_\infty = C_p \Theta_\infty + U_\infty^2/2$ , while its wall value is taken to be either that of an adiabatic wall or a specified constant  $\langle H \rangle$ .

### 3. Near-Wall k-ε Turbulence Model for $\bar{v}_t$

In adopting and extending the near-wall k-ε model of So et al. (1991a) to compressible flows, Zhang et al. (1992) made two assumptions to simplify the formulation. The first assumption is to split the compressible dissipation function into a solenoidal part according to the suggestion of Sarkar et al. (1989), so that the solenoidal dissipation is insensitive to compressibility effects and, therefore, approaches its incompressible limit correctly. A second assumption is the simultaneous neglect of fluctuating density and temperature at the wall. Physically, this assumption is not quite valid. However, it does permit an asymptotic analysis of the near-wall behavior of the compressible k and ε equations to be carried out in a manner similar to that proposed by Lai and So (1990b) and So et al. (1991a) for incompressible flows. Consequently, it is found that the near-wall k-ε model of So et al. (1991a) can be extended to compressible flows and the dilatational effects in the near-wall region can be accounted for by the varying mean density alone. Since Zhang et al. (1992) have found that the additional dilatational terms have little effect on compressible boundary-layer calculations in the Mach number range of  $0 \leq M_\infty \leq 10$  and wall temperature ratio range of  $0.2 \leq T_w/T_{aw} \leq 1.0$ , as a first attempt, it is prudent to calculate turbulent heat fluxes using a near-wall model where the additional dilatational terms are neglected. In view of this, the near-wall k-ε model of Zhang et al. (1992) without the dilatational terms are adopted for the present study. The modeled k-ε equations for boundary-layer flows can be written as:

$$\bar{\rho} \langle U \rangle \frac{\partial k}{\partial x} + \bar{\rho} \langle V \rangle \frac{\partial k}{\partial y} = \frac{\partial}{\partial y} \left[ \left( \bar{\mu} + \frac{\bar{\mu}_t}{\sigma_k} \right) \frac{\partial k}{\partial y} \right] + \bar{\mu}_t \left( \frac{\partial \langle U \rangle}{\partial y} \right)^2 - \bar{\rho} \epsilon, \quad (12)$$

$$\bar{\rho} \langle U \rangle \frac{\partial \epsilon}{\partial x} + \bar{\rho} \langle V \rangle \frac{\partial \epsilon}{\partial y} = \frac{\partial}{\partial y} \left[ \left( \bar{\mu} + \frac{\bar{\mu}_t}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial y} \right] + C_{\epsilon 1} \frac{\epsilon}{k} \bar{\mu}_t \left( \frac{\partial \langle U \rangle}{\partial y} \right)^2 - C_{\epsilon 2} \bar{\rho} \left( \frac{\epsilon \tilde{\epsilon}}{k} \right) + \xi, \quad (13)$$

where  $\xi$ , the near-wall function proposed for the  $\epsilon$  equation, and  $\epsilon$ , the solenoidal dissipation of  $k$ , are defined by

$$\xi = f_{w,2} \bar{\rho} \left[ -2 \frac{\epsilon \tilde{\epsilon}}{k} + 1.5 \frac{\epsilon^{*2}}{k} \right] \quad (14a)$$

$$\bar{\rho} \epsilon = \overline{\mu \omega_i \omega_i} \quad (14b)$$

Here,  $f_{w,2}$  is a damping function that asymptotes to one at the wall and zero far away from the wall. It is defined as  $f_{w,2} = e^{-(R_t/64)^2}$ . The boundary conditions for  $k$  and  $\epsilon$  are specified to be zero in the free stream. At the wall,  $k = 0$  is assumed, while  $\epsilon$  is taken to be given by  $2\bar{v}_w (\partial\sqrt{k}/\partial y)_w^2$ .

Once  $k$  and  $\epsilon$  are known, they can be used to evaluate  $\bar{v}_t$  according to (10) and hence  $\bar{\mu}_t = \bar{\rho}(\bar{v}_t)$ . The damping function  $f_\mu$  associated with  $\bar{v}_t$  is given by So et al. (1991a) as

$$f_\mu = (1 + 3.45/\sqrt{R_t}) \tanh(y^+/115) \quad (15)$$

This damping function behaves correctly as the wall is approached, i.e.  $f_\mu$  goes like  $y^{-1}$  as  $y$  approaches zero. In other words, the modeled turbulent shear stress again behaves like  $y^3$  near a wall similar to its exact behavior.

#### 4. Near-Wall $\overline{\theta^2}$ - $\epsilon_\theta$ Turbulence Model for $\overline{\alpha_t}$

A detailed derivation of the non-isothermal incompressible near-wall  $\overline{\theta^2}$ - $\epsilon_\theta$  model has been given by Sommer et al. (1992). Consequently, there is no need to repeat the derivation here. However, some major differences between the  $k$ - $\epsilon$  equations and the  $\overline{\theta^2}$ - $\epsilon_\theta$  equations should be pointed out. The first is in the modeling of the generation and destruction terms in the  $\epsilon_\theta$  equation. Since thermal and velocity time scales are of equal importance in turbulent heat transfer, both time scales are used in the modeling of the generation and destruction terms. A second difference is in the wall boundary conditions. Constant heat flux as well as constant temperature wall boundary condition can be specified for non-isothermal flows. Therefore, these differences have to be taken into account in the derivation of the near-wall correction for the  $\epsilon_\theta$  equation. Sommer et al. (1992) incorporate the coincidence condition of Shima (1988) to treat the  $\epsilon_\theta$  equation and derive a near-wall correction in a manner similar to that used by So et al. (1991a) in their derivation of  $\xi$ . Thus formulated, the  $\overline{\theta^2}$  and  $\epsilon_\theta$  equations behave correctly as a wall is approached; at least to the lowest order of  $y$ .

The near-wall  $\overline{\theta^2}$  and  $\epsilon_\theta$  equations of Sommer et al. (1992) for a non-isothermal incompressible turbulent boundary-layer flow can now be written as:

$$\begin{aligned} \overline{U} \frac{\partial \overline{\theta^2}}{\partial x} + \overline{V} \frac{\partial \overline{\theta^2}}{\partial y} &= \frac{\partial}{\partial y} \left( \alpha \frac{\partial \overline{\theta^2}}{\partial y} \right) + \frac{\partial}{\partial y} \left( \alpha_t \frac{\partial \overline{\theta^2}}{\partial y} \right) \\ &+ 2\alpha_t \left( \frac{\partial \overline{\Theta}}{\partial x} \right)^2 + 2\alpha_t \left( \frac{\partial \overline{\Theta}}{\partial y} \right)^2 - 2\epsilon_\theta, \end{aligned} \quad (16)$$

$$\begin{aligned} \overline{U} \frac{\partial \epsilon_\theta}{\partial x} + \overline{V} \frac{\partial \epsilon_\theta}{\partial y} &= \frac{\partial}{\partial y} \left( \alpha \frac{\partial \epsilon_\theta}{\partial y} \right) + \frac{\partial}{\partial y} \left( \alpha_t \frac{\partial \epsilon_\theta}{\partial y} \right) + C_{d1} \frac{\epsilon_\theta}{\theta^2} P_\theta \\ &+ C_{d2} \frac{\epsilon}{k} P_\theta + C_{d3} \epsilon_\theta \frac{P}{k} - C_{d4} \frac{\tilde{\epsilon}_\theta}{\theta^2} \epsilon_\theta - C_{d5} \frac{\tilde{\epsilon}}{k} \epsilon_\theta + \xi_{\epsilon\theta}, \end{aligned} \quad (17)$$

where  $P$  is the turbulent production of  $k$  defined as  $P = -\overline{u'v'}(\partial\overline{U}/\partial y)$  and  $P_\theta$  is the production of temperature variance due to mean temperature gradient and is given by  $P_\theta = -[\overline{u'\theta}(\partial\overline{\Theta}/\partial x) + \overline{v'\theta}(\partial\overline{\Theta}/\partial y)]$ . The near-wall correction,  $\xi_{\varepsilon\theta}$ , and the viscous dissipation of  $\overline{\theta^2}$ ,  $\varepsilon_\theta$ , are defined as:

$$\xi_{\varepsilon\theta} = f_{w,\varepsilon\theta} \left( (C_{d4} - 4) \frac{\tilde{\varepsilon}_\theta}{\theta^2} \varepsilon_\theta + C_{d5} \frac{\tilde{\varepsilon}}{k} \varepsilon_\theta - \frac{\varepsilon_\theta^{*2}}{\theta^2} + (2 - C_{d1} - \text{Pr}C_{d2}) \frac{\varepsilon_\theta}{\theta^2} P_\theta^* \right) , \quad (18a)$$

$$\varepsilon_\theta = \alpha \frac{\overline{\partial\theta \partial\theta}}{\partial x_k \partial x_k} . \quad (18b)$$

Here,  $P_\theta^* = -\overline{u'\theta}(\partial\overline{\Theta}/\partial x)$ , where  $\partial\overline{\Theta}/\partial x$  is constant for constant wall heat flux boundary condition. The presence of this term in  $\xi_{\varepsilon\theta}$  is required in order to balance the term involving  $\partial\overline{\Theta}/\partial x$  in the  $\overline{\theta^2}$  and  $\varepsilon_\theta$  equations in the near-wall region for constant wall heat flux boundary condition. It should be pointed out that for constant wall temperature, the term is identically zero because  $\partial\overline{\Theta}/\partial x = 0$ . The damping function  $f_{w,\varepsilon\theta}$  is introduced to guarantee that  $\xi_{\varepsilon\theta}$  vanishes far away from the wall. This way, the high-Reynolds-number form of the equations is recovered correctly.

Consistent with So et al.'s (1991a) approach for the  $k$ - $\varepsilon$  model, Sommer et al. (1992) suggest the following form for the damping function, or  $f_{w,\varepsilon\theta} = \exp[-(\text{Re}_t/80)^2]$ . Once  $\overline{\theta^2}$  and  $\varepsilon_\theta$  are known,  $\alpha_t$  can be determined by assuming gradient transport similar to that invoked for  $v_t$ . However, it should be pointed out that both thermal and velocity time scales are involved in the transport of heat. Therefore,  $\alpha_t$  should be defined as

$$\alpha_t = C_\lambda f_\lambda k [\overline{k\theta^2}/\varepsilon\varepsilon_\theta]^{1/2} , \quad (19)$$

where the damping function  $f_\lambda$  is given by

$$f_\lambda = \left[ f_{w,\varepsilon\theta} \frac{C_{1\lambda}}{\sqrt[4]{\text{Re}_t}} \right] + [1 - \exp(-y^+/A^+)]^2 . \quad (20)$$

The boundary conditions for  $\overline{\theta^2}$  and  $\epsilon_\theta$  are again specified as zero in the free stream and vanishing  $\overline{\theta^2}$  at the wall. As for  $\epsilon_\theta$ , its value at the wall is given by  $\alpha_w(\partial\sqrt{\overline{\theta^2}}/\partial y)^2$ .

The incompressible near-wall  $\overline{\theta^2}$ - $\epsilon_\theta$  model for  $\alpha_t$  can be extended to compressible flows in the following manner. Again, fluctuating temperature and density are assumed to go to zero simultaneously at the wall and fluctuating fluid properties are neglected. Therefore, all fluid properties, such as  $\mu$ ,  $C_p$ , etc., can be replaced by their time-averaged values and the following near-wall expansions can be assumed for the fluctuating quantities. These are:

$$u = a_1y + a_2y^2 + \dots, \quad (21a)$$

$$v = b_1y + b_2y^2 + \dots, \quad (21b)$$

$$\theta = c_1y + c_2y^2 + \dots, \quad (21c)$$

$$\rho' = d_1y + d_2y^2 + \dots, \quad (21d)$$

where the a's, b's, c's and d's are random functions of  $x$ ,  $z$  and  $t$ . As pointed out by Bradshaw (1974),  $\theta$  and  $\rho'$  cannot go to zero simultaneously at the wall; otherwise it would lead to a zero wall  $p'$ , which is not physically possible. In general,  $\theta$  is taken to be zero at the wall but  $\rho'$  is not. Here,  $\rho'$  is also assumed to be zero at the wall, however, its value away from the wall is finite. Therefore, this assumption represents an improvement over Morkovin's hypothesis (1962) which neglects the influence of fluctuating density altogether in the whole flow. Under this assumption and with the help of the continuity equation for  $\rho'$  and (21), it can be easily shown that  $b_1$  is identically zero irrespective of the wall thermal boundary conditions. This means that the near-wall asymptotic analysis of So et al. (1991a) can be used to examine the exact and modeled compressible  $\overline{\theta^2}$  and  $\epsilon_\theta$  equations in the near-wall region. The result is similar to that given by Sommer et al. (1992) for the incompressible case. Therefore, the incompressible form of the near-wall  $\overline{\theta^2}$  and  $\epsilon_\theta$  equations can be directly extended to compressible flows just like the case made for the  $k$  and  $\epsilon$  equations.

In view of this, the compressible  $\overline{\theta^2}$  and  $\varepsilon_\theta$  equations can be written as:

$$\begin{aligned} \overline{\rho} \langle U \rangle \frac{\partial \langle \theta^2 \rangle}{\partial x} + \overline{\rho} \langle V \rangle \frac{\partial \langle \theta^2 \rangle}{\partial y} &= \frac{\partial}{\partial y} \left( \overline{\rho} \overline{\alpha} \frac{\partial \langle \theta^2 \rangle}{\partial y} \right) + \frac{\partial}{\partial y} \left( \overline{\rho} \overline{\alpha}_t \frac{\partial \langle \theta^2 \rangle}{\partial y} \right) \\ &+ 2\overline{\rho} \overline{\alpha}_t \left( \frac{\partial \langle \theta \rangle}{\partial x} \right)^2 + 2\overline{\rho} \overline{\alpha}_t \left( \frac{\partial \langle \theta \rangle}{\partial y} \right)^2 - 2\overline{\rho} \varepsilon_\theta , \end{aligned} \quad (22)$$

$$\begin{aligned} \overline{\rho} \langle U \rangle \frac{\partial \varepsilon_\theta}{\partial x} + \overline{\rho} \langle V \rangle \frac{\partial \varepsilon_\theta}{\partial y} &= \frac{\partial}{\partial y} \left( \overline{\rho} \overline{\alpha} \frac{\partial \varepsilon_\theta}{\partial y} \right) + \frac{\partial}{\partial y} \left( \overline{\rho} \overline{\alpha}_t \frac{\partial \varepsilon_\theta}{\partial y} \right) + C_{d1} \frac{\varepsilon_\theta}{\theta^2} \overline{\rho} \overline{\alpha}_t \left( \frac{\partial \langle \theta \rangle}{\partial x} \right)^2 \\ &+ C_{d1} \frac{\varepsilon_\theta}{\theta^2} \overline{\rho} \overline{\alpha}_t \left( \frac{\partial \langle \theta \rangle}{\partial y} \right)^2 + C_{d2} \frac{\varepsilon}{k} \overline{\rho} \overline{\alpha}_t \left( \frac{\partial \langle \theta \rangle}{\partial y} \right)^2 + C_{d3} \frac{\varepsilon_\theta}{k} \overline{\mu}_t \left( \frac{\partial \langle U \rangle}{\partial y} \right)^2 \\ &- C_{d4} \frac{\tilde{\varepsilon}_\theta}{\theta^2} \overline{\rho} \varepsilon_\theta - C_{d5} \frac{\tilde{\varepsilon}}{k} \overline{\rho} \varepsilon_\theta + \xi_{\varepsilon\theta} , \end{aligned} \quad (23)$$

where the near-wall correction function,  $\xi_{\varepsilon\theta}$ , and the viscous dissipation of  $\langle \theta^2 \rangle$ ,  $\varepsilon_\theta$ , are given by

$$\xi_{\varepsilon\theta} = f_{w,\varepsilon\theta} \overline{\rho} \left( (C_{d4} - 4) \frac{\tilde{\varepsilon}_\theta}{\theta^2} \varepsilon_\theta + C_{d5} \frac{\tilde{\varepsilon}}{k} \varepsilon_\theta - \frac{\varepsilon_\theta^{*2}}{\theta^2} + (2 - C_{d1} - C_{d2} \text{Pr}) \frac{\varepsilon_\theta}{\theta^2} P_\theta^* \right) , \quad (24a)$$

$$\varepsilon_\theta = \overline{\alpha} \left\langle \frac{\partial \theta}{\partial x_k} \frac{\partial \theta}{\partial x_k} \right\rangle . \quad (24b)$$

The boundary conditions for  $\langle \theta^2 \rangle$  and  $\varepsilon_\theta$  are the same as those specified in the incompressible case. However, at the wall,  $\varepsilon_\theta$  is given by  $\overline{\alpha}_w (\partial \sqrt{\overline{\theta^2}} / \partial y)^2$ .

## 5. Results and Discussion

The governing equations (5) - (7), (12), (13), (22) and (23) are solved using the boundary-layer code of Anderson and Lewis (1971) with appropriate modifications made to the computer program. Exact boundary conditions at the wall for the turbulence quantities are used because, with near-wall corrections proposed in (14) and (24), the equations can be integrated directly to the wall and no approximations need be applied to the numerical solutions of the equations in the near-wall region. Three different cases are selected from the data file compiled by Fernholz and Finley (1977). Two cases have adiabatic wall boundary conditions and they are labeled as cases 53011302 and 73050504 by Fernholz and Finley (1977). The free-stream Mach numbers of these two cases are 4.544 and 10.31, respectively, and the corresponding  $R_\theta$  are 5,532 and 15,074. The wall temperature ratio for these two cases is  $\Theta_w/\Theta_r = 1.0$ , where  $\Theta_r$  is assumed to be the recovery temperature for adiabatic wall boundary condition and is taken to be  $\Theta_{aw}$  for cooled wall boundary condition. Since the fluid medium of Case 53011302 is air, Sutherland's law can be used to evaluate viscosity and  $Pr = 0.74$  is specified. On the other hand, helium is used in the experiments of Case 73050504, therefore,  $Pr = 0.7$ . A power law as suggested by Fernholz and Finley (1977) is used to calculate viscosity in this case. The third case is specified by  $\Theta_w/\Theta_{aw} = 0.92$ ,  $M_\infty = 5.29$  and  $R_\theta = 3,939$ . Air is also the working fluid in this case, therefore, Sutherland's law can again be used to calculate viscosity and  $Pr = 0.74$ . The cases chosen span a wide range of  $M_\infty$  and  $R_\theta$ . In addition to comparing with measurements, the present results are also compared with the calculations of a near-wall k- $\epsilon$  model where a constant  $Pr_t$  is assumed. The model adopted is identical to solving (12) and (13) with  $Pr_t = 0.9$ . Finally, another set of calculations is carried out so that the results can be compared with the van Driest II formula (Kline et al. 1981) used to estimate the variation of  $C_f$  with  $\Theta_w/\Theta_{aw}$  for a fixed  $M_\infty$ . This set of calculations is carried out at  $M_\infty = 5.0$  and  $Pr = 0.74$ . It should be pointed out that all calculations are carried out to the same  $R_\theta$  as the experiments so that proper comparisons can be made with the measurements.



Since turbulence measurements from these three cases are not available for comparisons, an alternative check on the correctness of the model is to compare the near-wall asymptotics predicted by the two- and four-equation models. This approach is justified because So et al. (1991a) and Zhang et al. (1992) have demonstrated that the near-wall asymptotics deduced from the k- $\epsilon$  model are in excellent agreement with direct numerical simulations. Furthermore, Sommer et al. (1992) have also validated the near-wall behavior of the  $\overline{\theta^2}$  and  $\epsilon_\theta$  equations using direct numerical simulation results. Therefore, a comparison of the present predictions with those of the k- $\epsilon$  model can help establish the validity of the four-equation model for near-wall calculations of compressible flows. Once the near-wall behavior is properly established, the present calculations are used to assess the assumption of  $Pr_t = 0.9$  adopted in Zhang et al.'s (1992) and other researchers' (Aupiox and Cousteix 1991; Wilcox 1988) calculations.

According to So et al. (1991a) and Sommer et al. (1992), near a wall, the quantities  $k$ ,  $\langle uv \rangle$ ,  $\epsilon$ ,  $\langle \theta^2 \rangle$ ,  $\langle v\theta \rangle$  and  $\epsilon_\theta$  can be expanded in terms of  $y$ . After proper normalization using wall variables, the expansions can be written as (So et al. 1991a; Sommer et al. 1992):

$$k^+ = a_k(y_w^+)^2 + b_k(y_w^+)^3 + \dots, \quad (25)$$

$$\overline{uv}^+ = a_{uv}(y_w^+)^3 + b_{uv}(y_w^+)^4 + \dots, \quad (26)$$

$$\epsilon^+ = 2a_k + 4b_k y_w^+ + \dots, \quad (27)$$

$$\theta^{+2} = a_\theta(y_w^+)^2 + b_\theta(y_w^+)^3 + \dots, \quad (28)$$

$$\overline{v\theta}^+ = a_{v\theta}(y_w^+)^3 + b_{v\theta}(y_w^+)^4 + \dots, \quad (29)$$

$$\epsilon_\theta^+ = a_{\epsilon\theta} + b_{\epsilon\theta} y_w^+ + \dots \quad (30)$$

The accuracy in which the near-wall asymptotics, such as  $a_k$ ,  $a_{uv}$ ,  $a_\theta$ ,  $a_{v\theta}$  and  $a_{\epsilon\theta}$ , can be predicted is a measure of the correctness of the four-equation model. Furthermore,  $k^+/\epsilon^+(y_w^+)^2$  and  $\theta^{+2}/\epsilon_\theta^+(y_w^+)^2$  approach exactly 0.5 and  $Pr$ , respectively, at the wall. So et al. (1991a) and Zhang et

al. (1992) have shown that the k- $\epsilon$  model gives reasonable values for  $a_k$  and  $a_{uv}$  compared to direct numerical simulations and an exact value of 0.5 is calculated for  $k^+/\epsilon^+(y_w^+)^2$  at  $M_\infty$  as high as 10. This means that the k and  $\epsilon$  equations are asymptotically correct and internally consistent. Since the same validation has been carried out by Sommer et al. (1992) for the  $\overline{\theta^2}$  and  $\epsilon_\theta$  equations for incompressible flows, the present objective is to demonstrate that the values calculated for  $a_\theta$ ,  $a_{v\theta}$  and  $a_{\epsilon\theta}$  are reasonable and that  $\theta^{+2}/\epsilon_\theta^+(y_w^+)^2$  is evaluated to be identical to the Pr assumed for the compressible calculations. Therefore, (22) and (23) can be shown to be asymptotically correct and internally consistent for incompressible as well as compressible flows.

The mean velocity and temperature results are presented in Figs. 1 - 6 with the adiabatic wall cases shown in Figs. 1, 2, 4 and 5 and the cooled wall case given in Figs. 3 and 6. Three different ways of plotting the velocity results are presented; the conventional semi-log plot (Figs. 1 and 3a), the linear plot (Figs. 2 and 3b) and the semi-log plot in van Driest (1951) coordinates (Fig. 4). On the other hand, only linear plots of the mean temperature are shown in Figs. 5 and 6 because a friction temperature cannot be suitably defined for adiabatic wall boundary condition, therefore, a semi-log plot of the temperature profile is not appropriate. The rationale for presenting the mean velocity in these three different forms can be explained as follows. Firstly, the conventional semi-log plot for compressible flows has a density effect included in the definitions of  $u^+$  and  $y_w^+$ , therefore, the true velocity profile prediction cannot be directly compared with measurements. Secondly, errors in the predictions of the mean temperature and hence the mean density can occur in such a way that they tend to mask the discrepancy in the semi-log plots of the mean velocity. In view of this, it is also necessary to compare the mean velocity and temperature in linear plots so that their actual agreement with measurements can be thoroughly analysed. Thirdly, van Driest suggests stretching  $u^+$  further by a density ratio so that a new  $u_c^+$  can be defined as

$$u_c^+ = \int_0^{u^+} \left( \frac{\bar{\rho}}{\rho_w} \right)^{1/2} du^+ \quad . \quad (31)$$

With this new coordinate, the compressible law-of-the-wall as deduced by van Driest (1951) and simplified by Bradshaw (1977) can be written as

$$u_c^+ = \frac{1}{0.41} \ln y_w^+ + 5.2 + 95M_t^2 \quad . \quad (32)$$

This form differs from the conventional law-of-the wall which is given by

$$u^+ = \frac{1}{0.41} \ln y_w^+ + B \quad , \quad (33)$$

where the constant B is a function of Mach number for compressible flat plate turbulent boundary layers. When the velocity results are plotted in terms of  $u^+$  and  $u_c^+$ , the validity of (32) and (33) can be evaluated. Thus presented, the mean properties can be thoroughly compared and their agreement or lack thereof with measurements and other model calculations can be analysed. Finally, the calculated  $C_f$  for the different cases is tabulated in Table 1 for comparison with data.

It can be seen from these plots that the model calculations are in good agreement with measurements and the predictions of the k-ε model. Furthermore, these results are essentially identical at low Mach numbers for both adiabatic (Figs. 1a, 2a and 5a) and cooled wall (Figs. 3 and 6) conditions. The only difference appears to occur in the case of  $M_\infty = 10.31$ , where the four-equation model gives a significant improvement in the predictions of both mean velocity and temperature (Figs. 1b, 2b and 5b) compared to those given by the k-ε model. This improvement is due to a better estimate of the turbulent Prandtl number near the wall. More will be said about this when the turbulence properties in the near-wall region are examined. Zhang et al. (1992) have demonstrated that the calculated  $u^+$  can be described fairly well by the conventional law-of-the-wall (33) and the constant B thus deduced is approximately 4.7 for the three cases examined. The

present results are in agreement with their conclusion. Therefore, the variable turbulent Prandtl number formulation has little or no effect on the log region of the boundary-layer flow. Plots of the mean velocities in van Driest coordinates for the case with  $M_\infty = 4.544$  are shown in Fig. 4 together with a plot of (32). A line parallel to (32) can be drawn through some of the data points; however, the intercept thus deduced is different from that given in (32). On the other hand, the calculated profiles from the two different models are in very good agreement with data over a much wider range of  $y_w^+$  and the slope of the log-law thus determined yields a von Karman constant  $\kappa = 0.35$  which is significantly smaller than a value of 0.41 quoted by Bradshaw (1977). Finally, the calculated  $C_f$ 's are compared with data in Table 1 and, in general, the four-equation model gives a slight improvement over that of the k- $\epsilon$  model.

The turbulence properties in the near-wall region are plotted in Figs. 7 - 12. Only the profiles of  $k^+$ ,  $\epsilon^+$ ,  $\overline{uv}^+$  and  $\overline{v\theta}^+$  are compared. The results for the adiabatic wall cases are presented in Figs. 7, 8, 10 and 11 while those for the cooled wall case are shown in Figs. 9 and 12. At low Mach numbers, the predictions of these properties by the two different models are essentially identical. This conclusion applies to both adiabatic (Figs. 7a, 8a, 10a and 11) and cooled wall (Figs. 9 and 12) boundary conditions. The only difference comes in the predictions of the case where  $M_\infty = 10.31$ . In general, the present calculations of  $k^+$ ,  $\epsilon^+$  and  $\overline{uv}^+$  are slightly lower than those predicted by the k- $\epsilon$  model. One of the reason is the variable turbulent Prandtl number. Plots of  $Pr_t$  across the inner region of the boundary layers are shown in Figs. 13 and 14. The turbulent Prandtl number is seen to vary rapidly in the region very near the wall. It increases from a wall value of about 0.5 to a maximum of approximately 1.6 and then decreases to about 0.75 before settling back to a value of 0.9 at  $y_w^+ = 200$ . Thereafter,  $Pr_t$  remains fairly constant at 0.9. This shows that all variations of  $Pr_t$  occur in the region,  $0 \leq y_w^+ \leq 200$ . Consequently, it is not surprising to find that differences in  $k^+$ ,  $\epsilon^+$  and  $\overline{uv}^+$  between the two model calculations also take place in this region for the case where  $M_\infty = 10.31$ . In conclusion, it can be said that variable

$Pr_t$  effect on the calculated properties is small and this observation is also supported by a comparison of the near-wall asymptotics which are tabulated in Table 2.

In general, the  $a_k$  values calculated by the four-equation model are slightly lower than those deduced from the  $k$ - $\epsilon$  model and the differences are insignificant. The other quantities calculated by the two different models are essentially the same. The four-equation model again yields a  $k^+/\epsilon^+(y_w^+)^2 = 0.5$  for all three cases considered, thus showing that it is asymptotically correct and internally consistent as far as the  $k$  and  $\epsilon$  equations are concerned even though they are coupled to the  $\overline{\theta^2}$  and  $\epsilon_\theta$  equations through the mean velocity and mean density fields. Since a friction temperature cannot be defined for adiabatic wall boundary condition, the free-stream temperature has to be used to normalize  $\langle \theta^2 \rangle$  and  $\epsilon_\theta$ . As a result, their calculated near-wall asymptotics are at least one order of magnitude smaller than those for  $k$  and  $\epsilon$ . The calculated  $a_{v\theta}$  from the two different models are of the same order with those deduced from the four-equation model generally higher than those determined from the  $k$ - $\epsilon$  model. Furthermore, the  $\overline{\theta^2}$  and  $\epsilon_\theta$  equations are also asymptotically correct and internally consistent because the calculations of  $\overline{\theta^2}/\epsilon_\theta^+(y_w^+)^2$  are identical to the  $Pr$  assumed for each case. Once the reliability of the near-wall asymptotics has been established, the calculated variations of  $Pr_t$  become credible. For all three cases considered, the  $Pr_t$  calculated is constant beyond  $y_w^+ = 200$  and its value is 0.9. This result lends credence to the  $Pr_t = 0.9$  assumption invoked by other researchers (Wilcox 1988; Aupoix and Cousteix 1991; Zhang et al. 1992) and suggests that it is essentially valid for the range of  $M_\infty$  and  $\Theta_w/\Theta_{aw}$  considered.

Zhang et al. (1992) have shown that the  $k$ - $\epsilon$  model gives excellent prediction of the variation of  $C_f/(C_f)_i$  with  $M_\infty$  in the range  $0 \leq M_\infty \leq 10$  for adiabatic wall boundary condition compared to the van Driest II formula (Kline et al. 1981). Since the present model is in good agreement with the  $k$ - $\epsilon$  model in its prediction of  $C_f$  for adiabatic wall boundary condition, there is no further need to verify the validity of the four-equation model for its ability to calculate correctly the variation of  $C_f/(C_f)_i$  with  $M_\infty$  in the range  $0 \leq M_\infty \leq 10$ . On the other hand, its ability to predict the variation of  $C_f/(C_f)_i$  with  $\Theta_w/\Theta_{aw}$  for a fixed  $M_\infty$  has to be verified. A comparison of the

present calculations at a constant  $M_\infty = 5.0$  with the results deduced from van Driest II formula (Kline et al. 1981) and the k- $\epsilon$  model is tabulated in Table 3. As suggested by Kline et al. (1981), the calculations are carried out to  $R_\theta = 10^4$  and  $(C_f)_i = 2.70 \times 10^{-3}$  is assumed. The results reveal that there are essentially very little difference between the present calculations and those obtained from the k- $\epsilon$  model. Both sets of predictions are in good agreement with the van Driest II formula. Therefore, the ability of the four-equation model to calculate  $C_f$  correctly for cooled wall boundary conditions is established.

## 6. Conclusions

A near-wall four-equation turbulence model has been developed for the calculations of compressible flat plate turbulent boundary layers with constant heat flux and constant temperature wall boundary conditions. The four equations consist of the transport equations for  $k$ ,  $\epsilon$ ,  $\overline{\theta^2}$  and  $\epsilon_\theta$ . These equations are modified for near-wall flow calculations so that they can be integrated directly to the wall and the exact boundary conditions at the wall can be satisfied. The modifications of So et al. (1991a) for the  $k$  and  $\epsilon$  equations are adopted and extended directly to compressible flows. Similar modifications for the  $\overline{\theta^2}$  and  $\epsilon_\theta$  equations have been carried out for the incompressible form of these equations (Sommer et al. 1992) and they are extended to compressible flows by invoking Morkovin's (1962) hypothesis. Thus formulated, the four-equation model is internally consistent and asymptotically correct near a wall, and there is no need to assume a constant turbulent Prandtl number because the turbulent heat flux can be estimated from a knowledge of  $\overline{\theta^2}$  and  $\epsilon_\theta$  by assuming gradient heat transport.

The near-wall four-equation turbulence model is used to calculate compressible flat plate turbulent boundary layers with free-stream Mach numbers as high as 10 and with adiabatic and cooled wall boundary conditions. The calculations are compared with measurements and with the predictions of a near-wall  $k$ - $\epsilon$  model where  $Pr_t = 0.9$  is assumed. Three cases have been calculated; two with adiabatic wall boundary condition and one with constant wall temperature. The free-stream Mach numbers for these cases vary from a low of 4.544 to a high of 10.31 and the wall temperature ratio varies from 0.2 to 1. Good agreement with measurements and the  $k$ - $\epsilon$  model calculations is obtained. In general, a variable turbulent Prandtl number formulation improves the calculated properties; particularly for high free-stream Mach numbers. At  $M_\infty = 10.31$ , there are significant improvements in the predictions of the mean velocity and mean temperature compared to the  $k$ - $\epsilon$  model. The turbulent Prandtl number thus calculated has a wall value of about 0.5 for all cases considered. It increases sharply to approximately 1.6 away from the wall and then settles

down to 0.9 at a  $y_w^+ = 200$ . These results, therefore, verify that the  $Pr_t = 0.9$  assumption invoked by past researchers is valid for compressible flat plate turbulent boundary layers for the range of  $M_\infty$  and  $\Theta_w/\Theta_{aw}$  examined in this study.



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Table 1. Comparison of calculated and measured  $C_f$ .

Source	$C_f \times 10^3$		
	$M_\infty = 4.544$	$M_\infty = 10.31$	$M_\infty = 5.29$
Data (Fernholz and Finley 1977)	1.26	0.24	1.31
k- $\epsilon$ model (Zhang et al. 1992)	1.320	0.220	1.262
Four-equation model	1.35	0.24	1.28

Table 2. Comparisons of the near-wall asymptotics deduced from the two different models.

Near-wall asymptotics	k- $\epsilon$ model (Zhang et al. 1992)			Four-equation model		
	$M_\infty = 4.544$	$M_\infty = 10.31$	$M_\infty = 5.29$	$M_\infty = 4.544$	$M_\infty = 10.31$	$M_\infty = 5.29$
$a_k$	0.0836	0.0771	0.0788	0.0824	0.0739	0.0772
$a_{uv} \times 10^4$	6.76	6.74	6.14	6.75	6.30	6.10
$k^+/\epsilon^+(y_w^+)^2$	0.50	0.50	0.50	0.50	0.50	0.50
$a_\theta$	-	-	-	0.00310	0.23415	0.00547
$a_{\epsilon\theta}$	-	-	-	0.00419	0.33450	0.00739
$a_{v\theta} \times 10^5$	- 0.118	- 1.310	- 0.059	- 0.20	- 1.69	- 0.10
$\theta^{+2}/\epsilon_\theta^+(y_w^+)^2$	-	-	-	0.74	0.70	0.74

Table 3. Comparisons of calculated  $C_f/(C_f)_i$  for cooled wall boundary condition at  $M_\infty = 5.0$ .

$\Theta_w/\Theta_{aw}$	$C_f/(C_f)_i$		
	van Driest II (Kline et al.1981)	k- $\epsilon$ model (Zhang et al. 1992)	Four-equation model
0.2	$0.58 \pm 0.058$	0.52	0.52
0.4	$0.49 \pm 0.049$	0.47	0.49
0.6	$0.43 \pm 0.043$	0.45	0.46
0.8	$0.39 \pm 0.039$	0.42	0.42
1.0	$0.35 \pm 0.035$	0.38	0.38

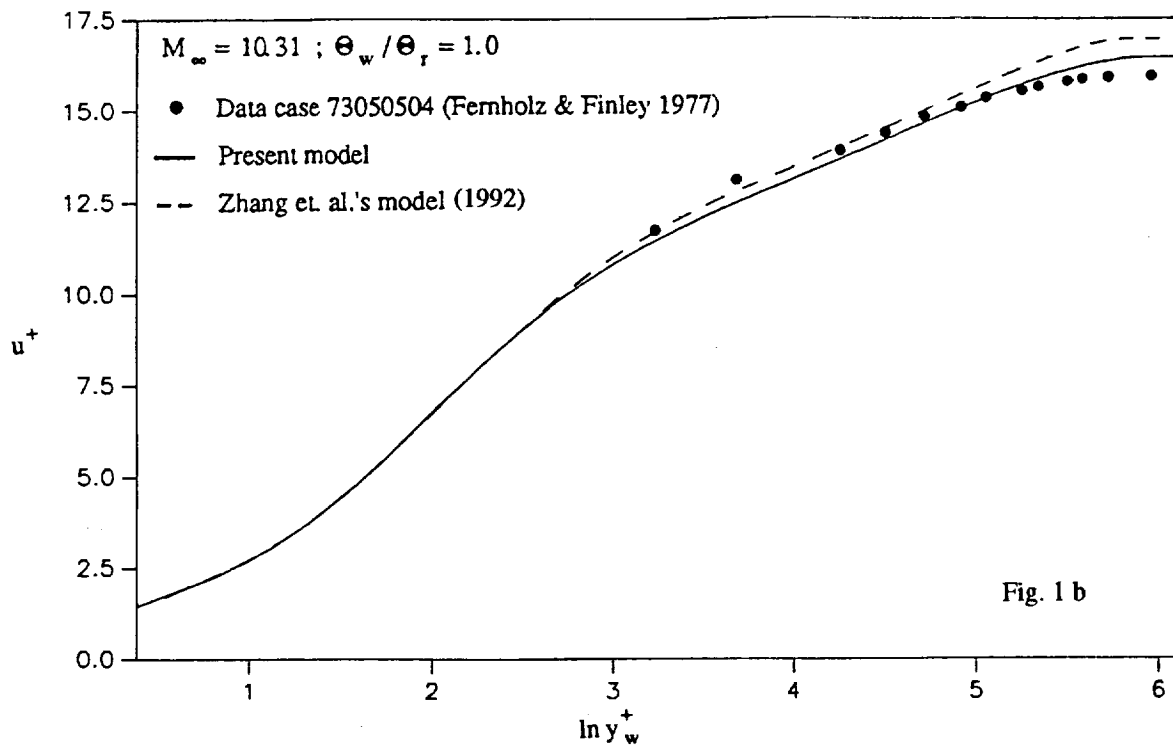
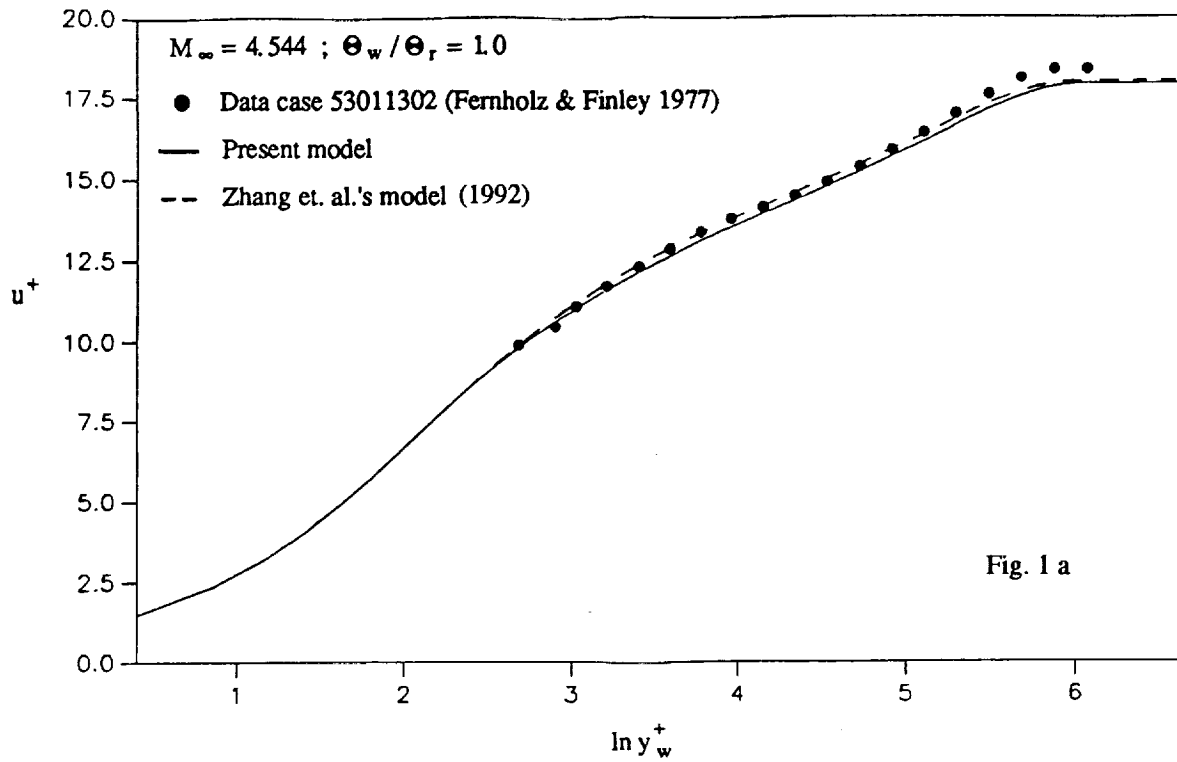


Figure 1. Comparison of the mean velocity in semi-log plots for adiabatic wall boundary condition: (a)  $M_\infty = 4.544$ , (b)  $M_\infty = 10.31$ .

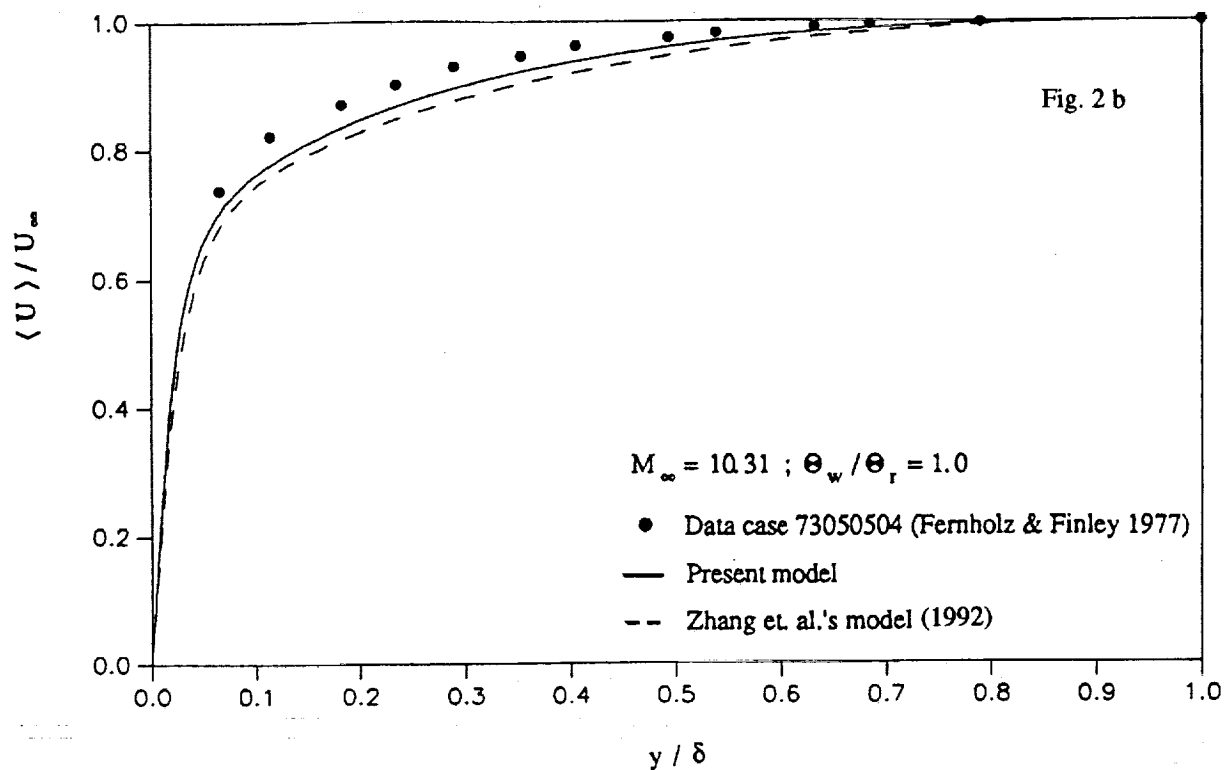
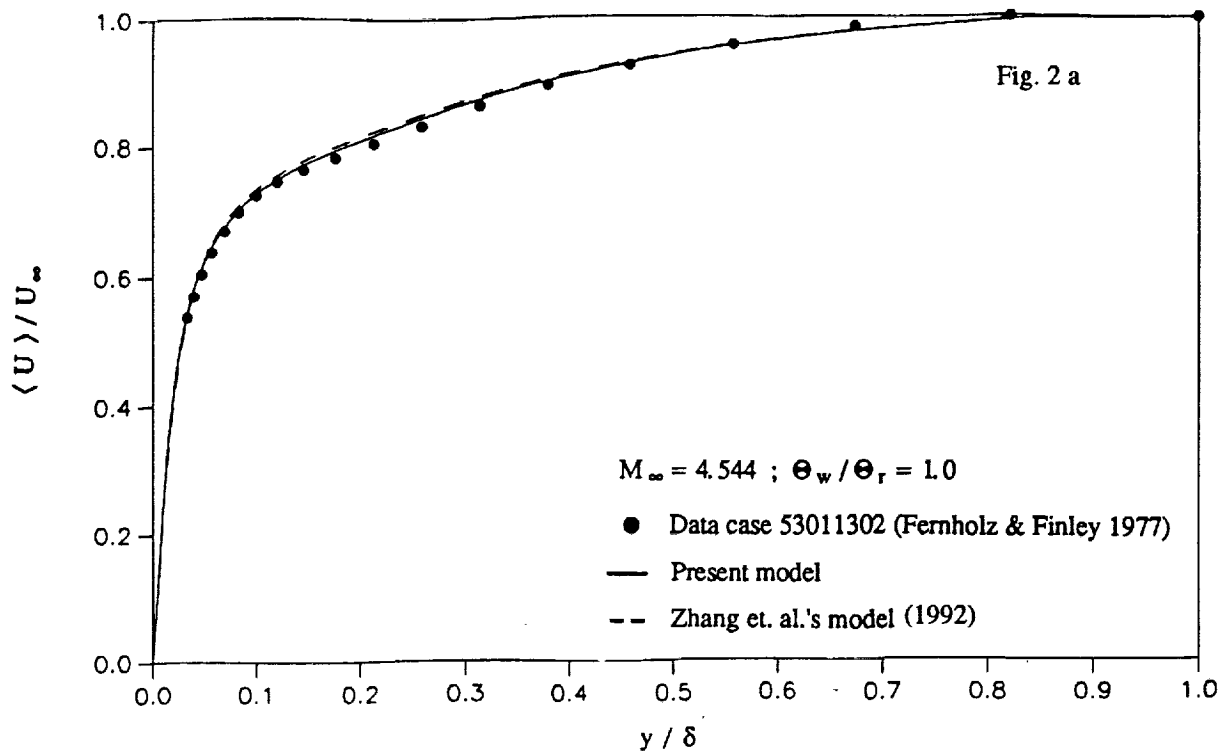


Figure 2. Comparison of the mean velocity in linear plots for adiabatic wall boundary condition: (a)  $M_\infty = 4.544$ , (b)  $M_\infty = 10.31$ .



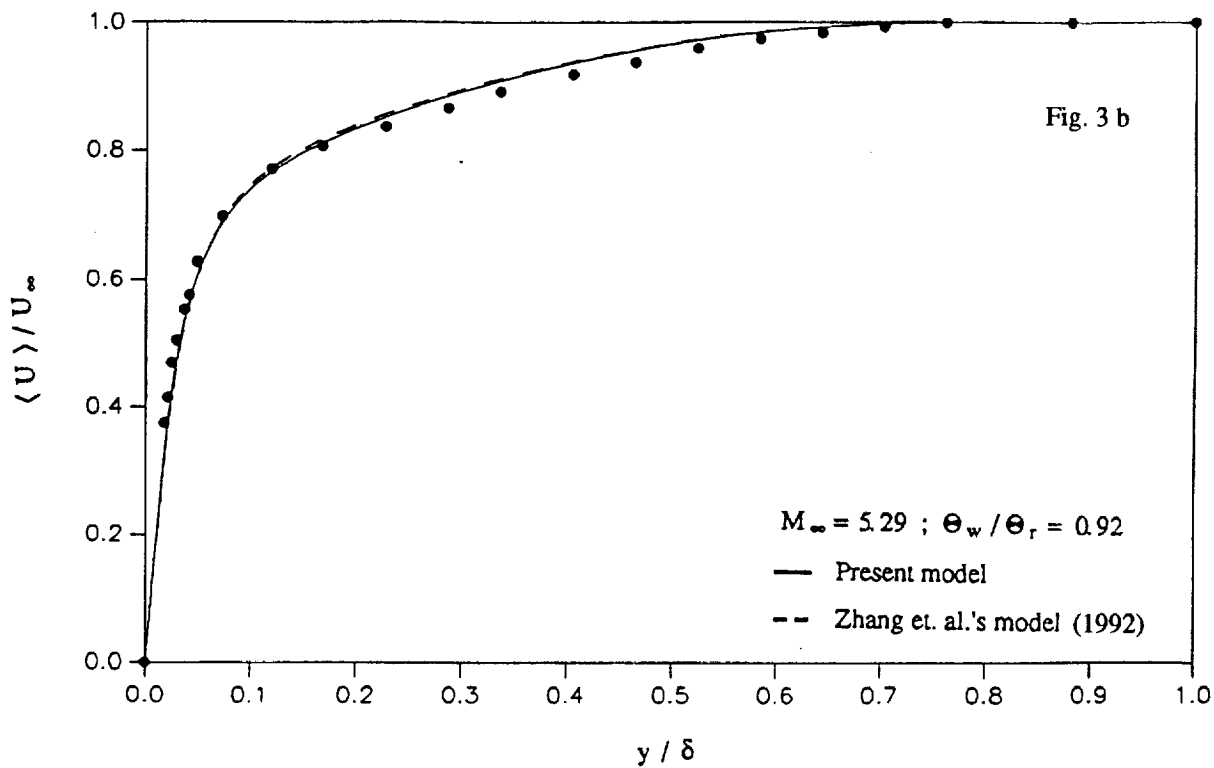
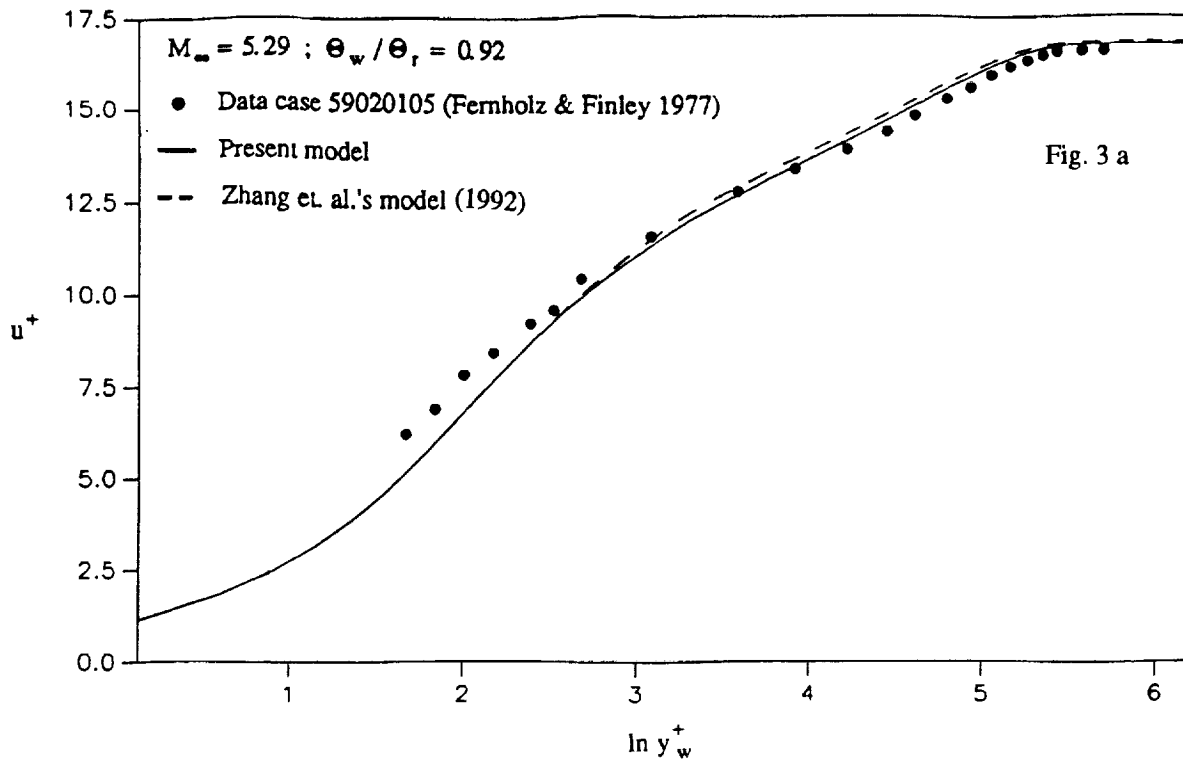


Figure 3. Comparison of the mean velocity for cooled wall condition at  $M_\infty = 5.29$ : (a) semi-log plot, (b) linear plots.

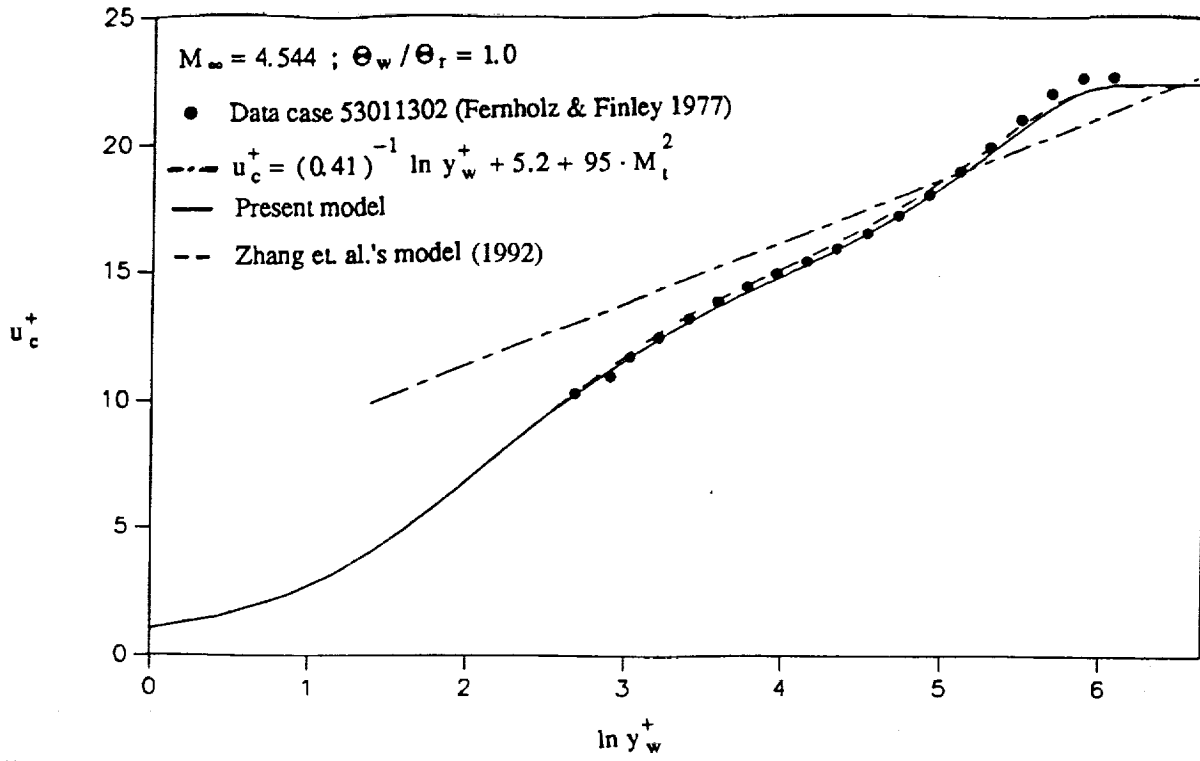


Figure 4. Comparison of the mean velocity in van Driest compressible law-of-the-wall plots for  $M_\infty = 4.544$ .

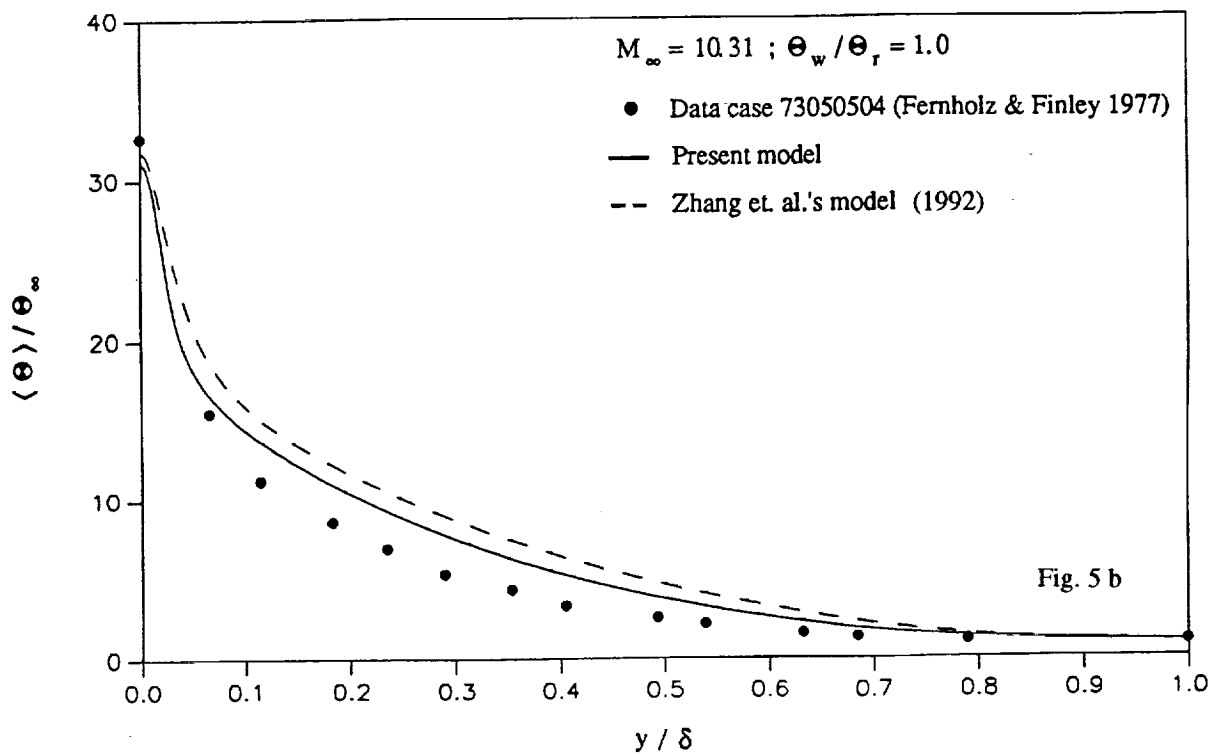
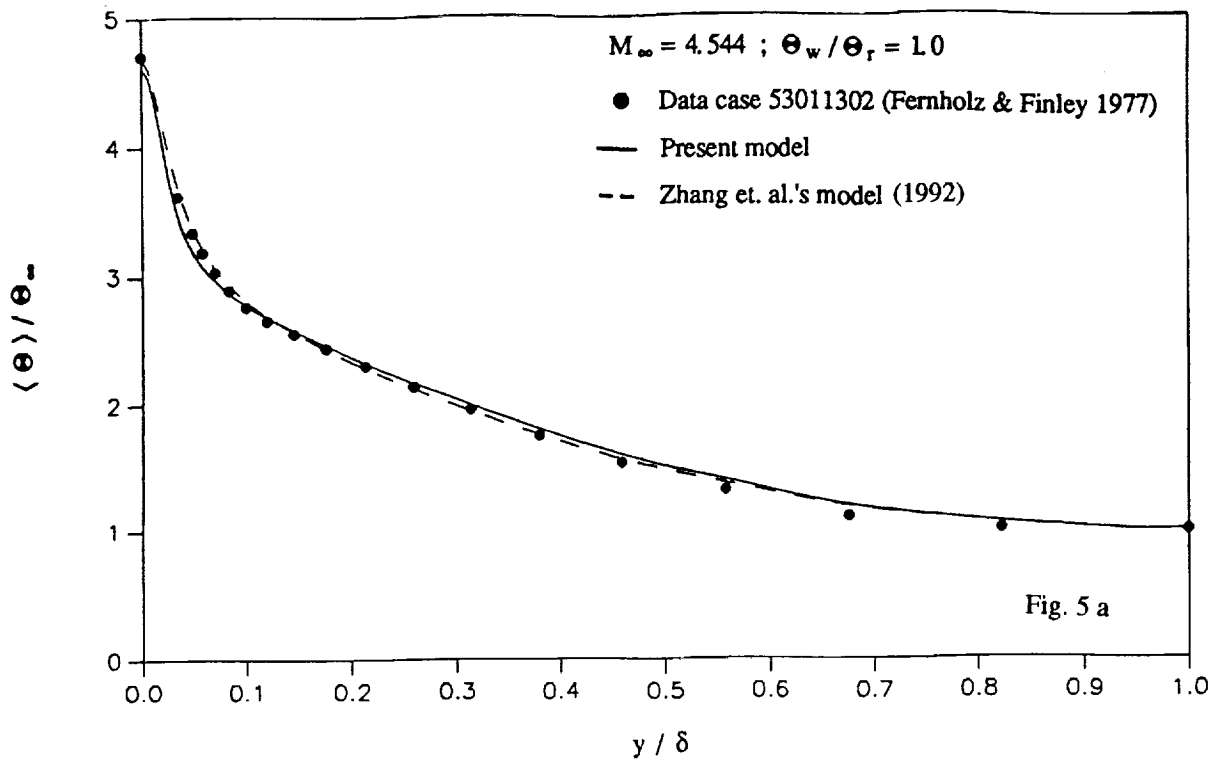


Figure 5. Comparison of the mean temperature in linear plots for adiabatic wall boundary condition: (a)  $M_\infty = 4.544$ , (b)  $M_\infty = 10.31$ .

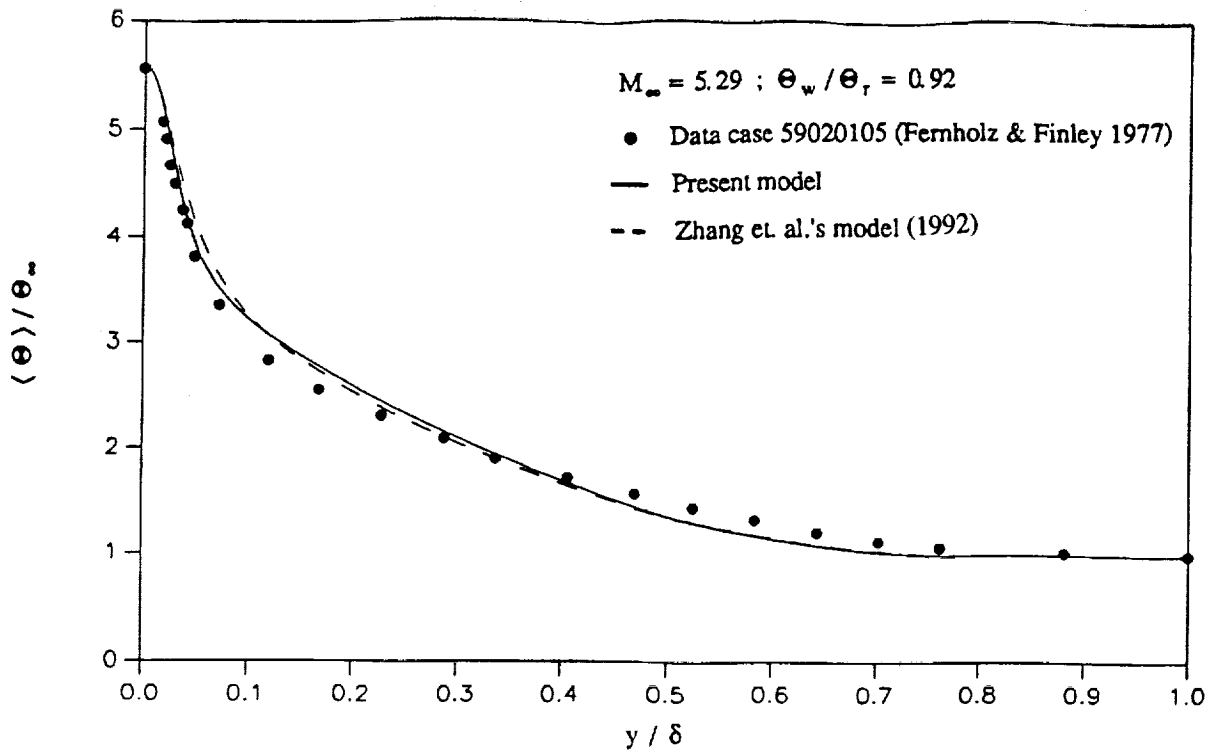


Figure 6. Comparison of the mean temperature in linear plots for cooled wall boundary condition at  $M_\infty = 5.29$ .

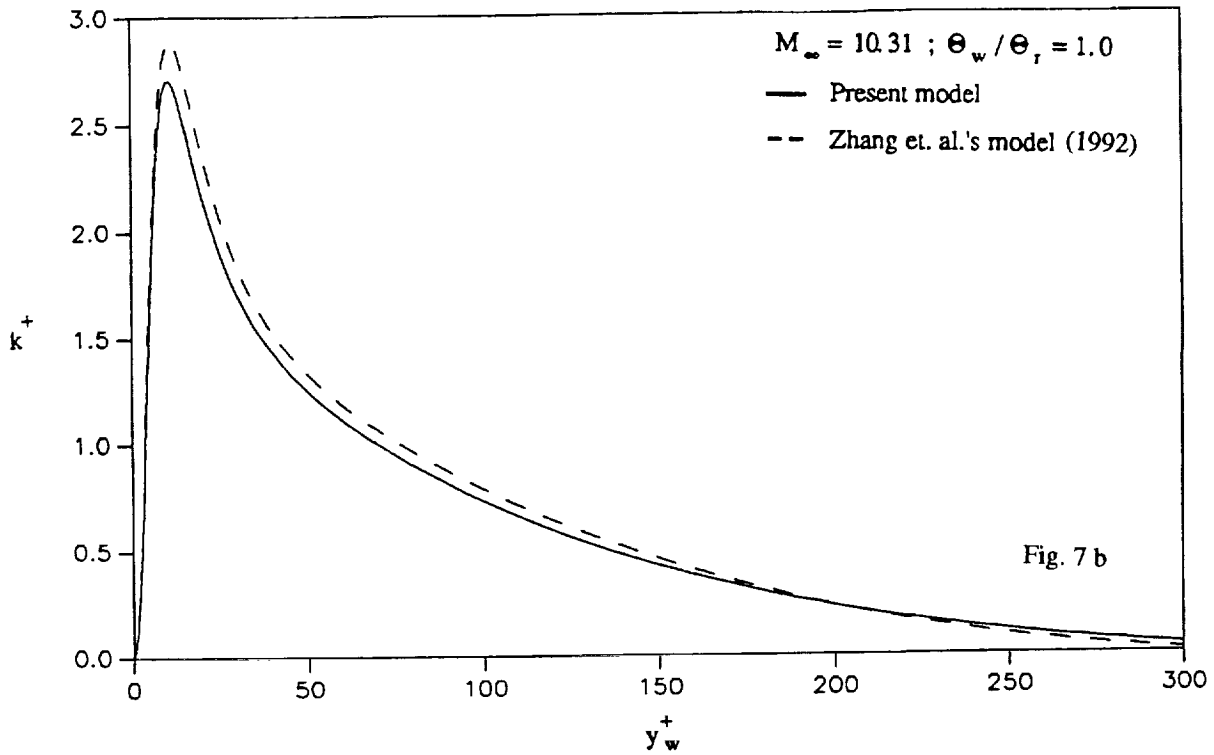
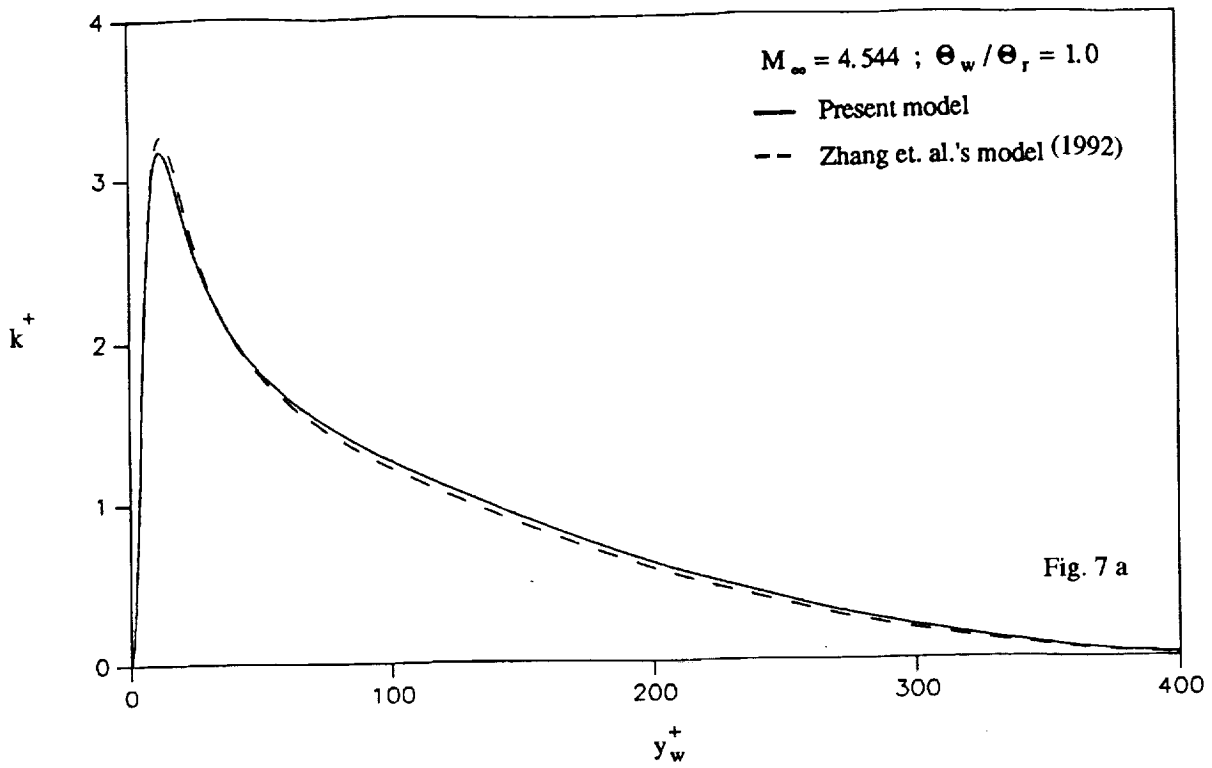


Figure 7. Comparison of  $k^+$  for adiabatic wall boundary condition:  
 (a)  $M_\infty = 4.544$ , (b)  $M_\infty = 10.31$ .

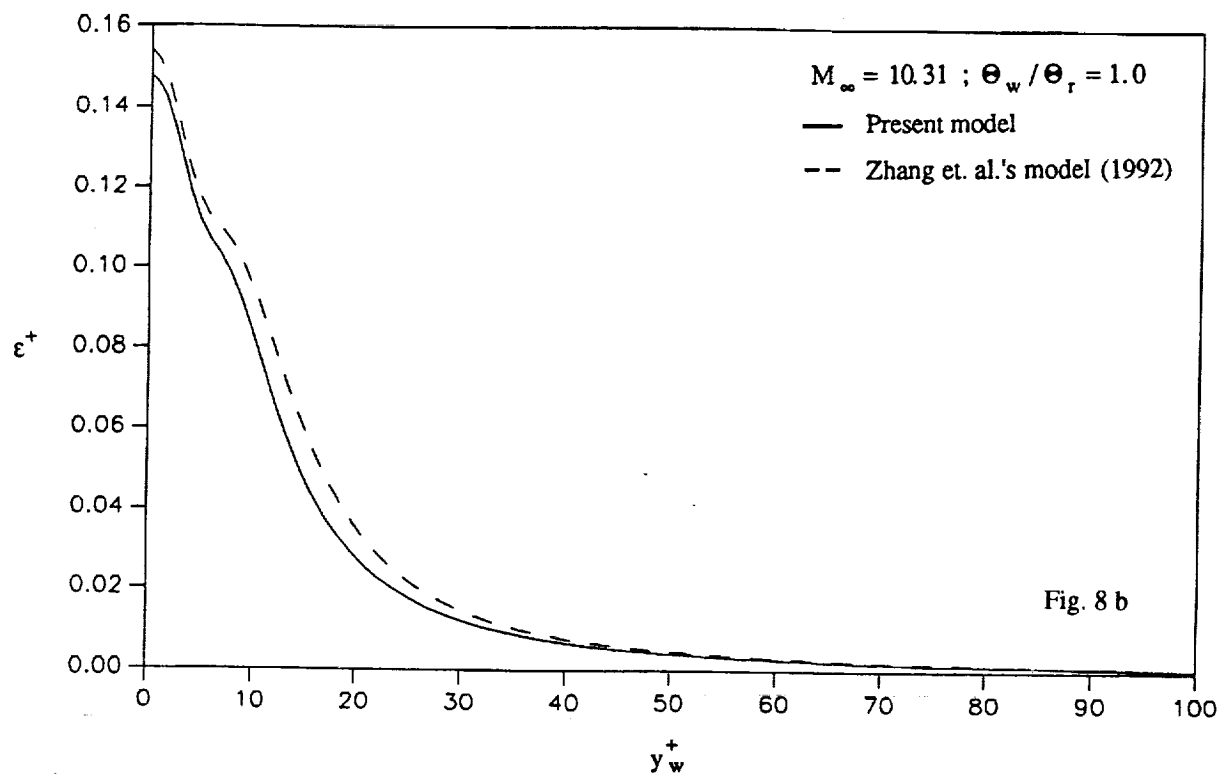
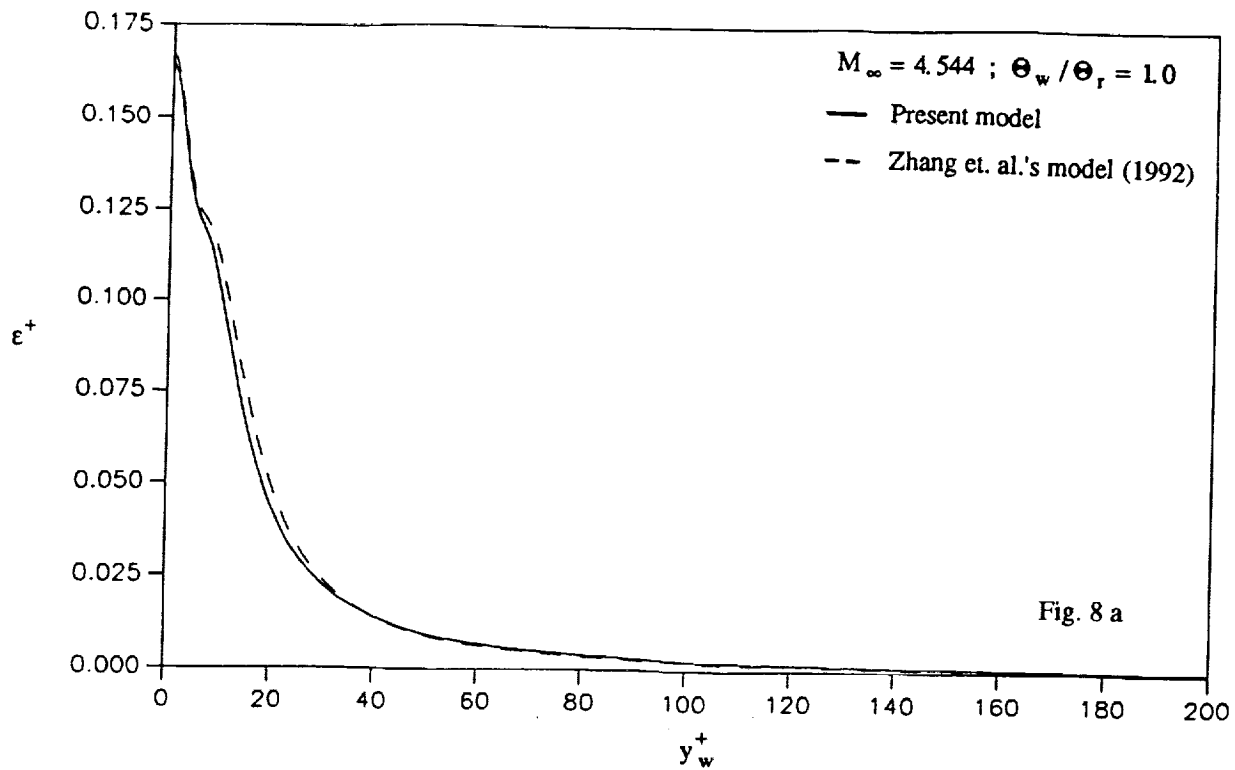


Figure 8. Comparison of  $\epsilon^+$  for adiabatic wall boundary condition: (a)  $M_\infty = 4.544$ , (b)  $M_\infty = 10.31$ .

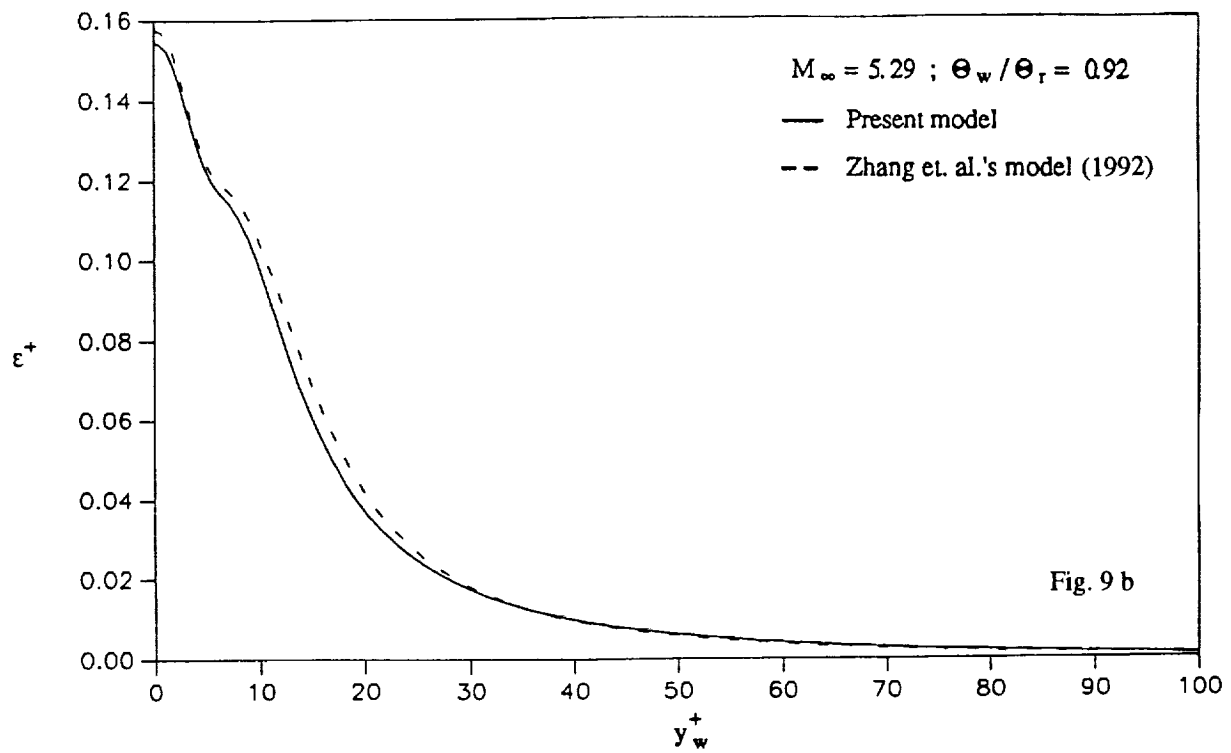
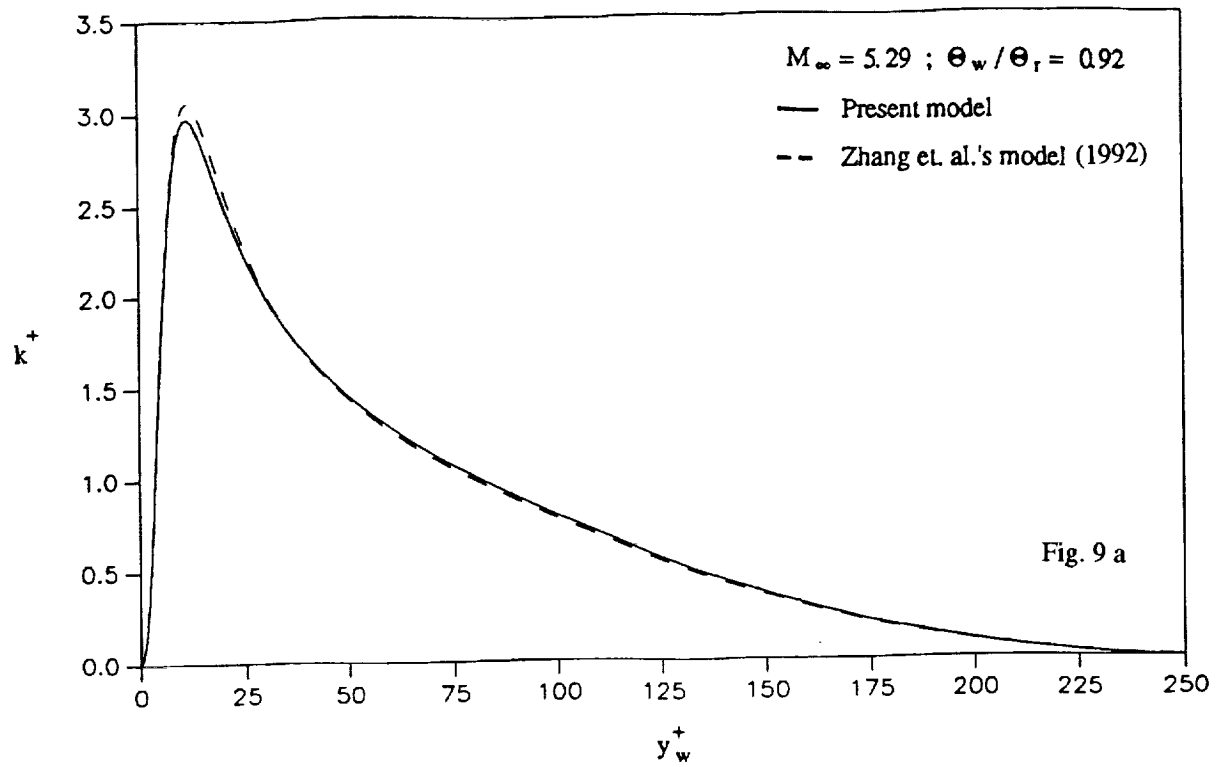


Figure 9. Comparison of  $k^+$  and  $\epsilon^+$  for cooled wall boundary condition at  $M_\infty = 5.29$ .

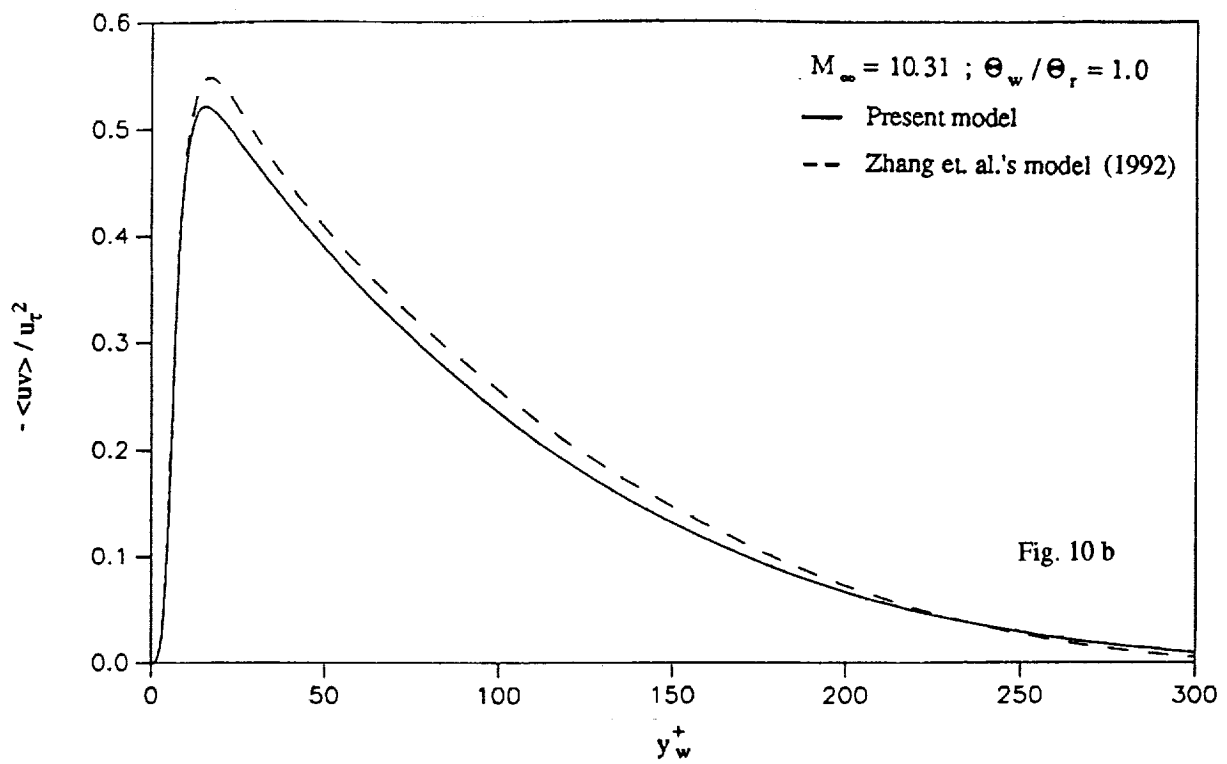
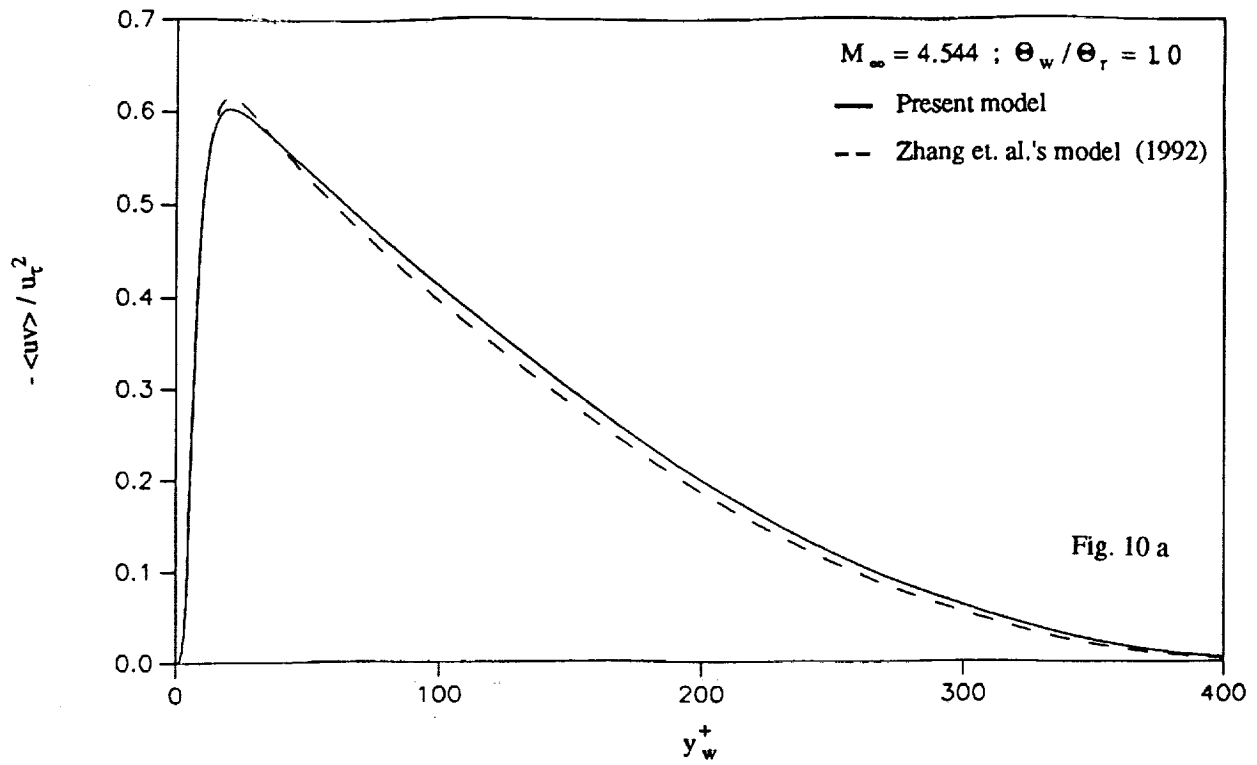


Figure 10. Comparison of  $\langle uv \rangle$  for adiabatic wall boundary condition:  
 (a)  $M_\infty = 4.544$ , (b)  $M_\infty = 10.31$ .



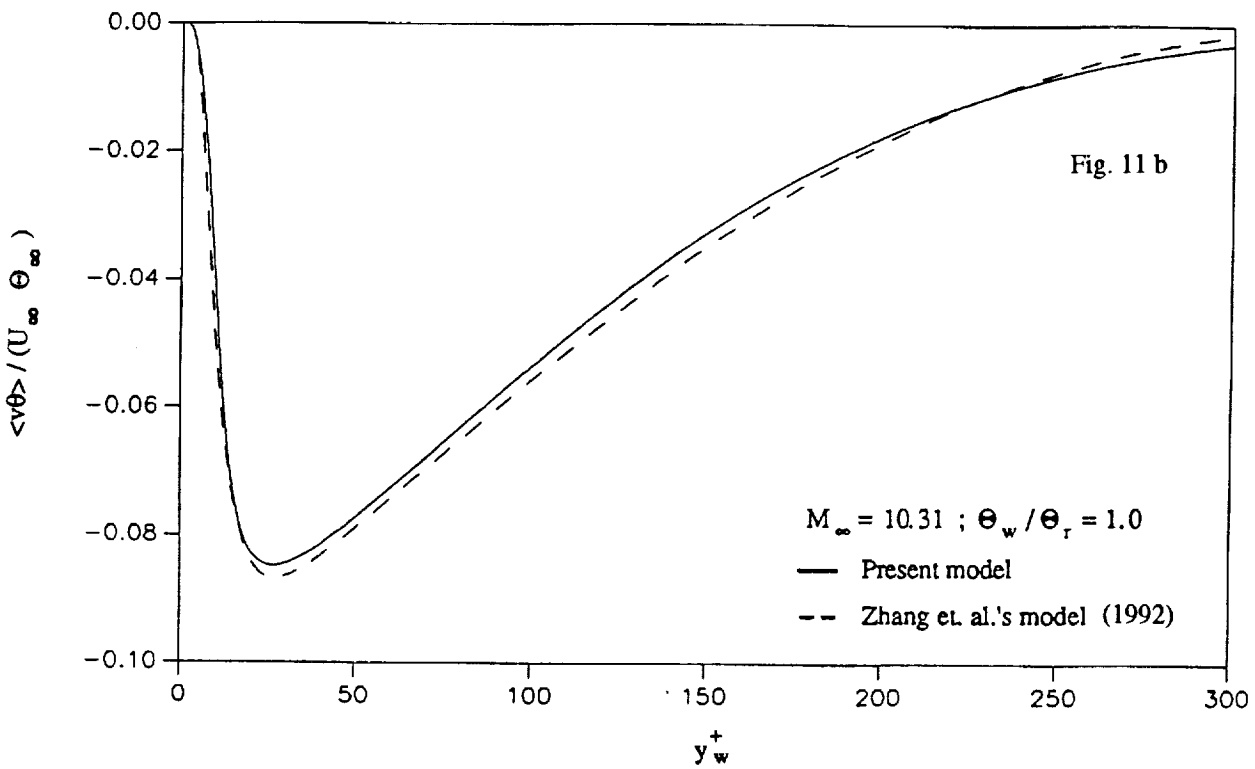
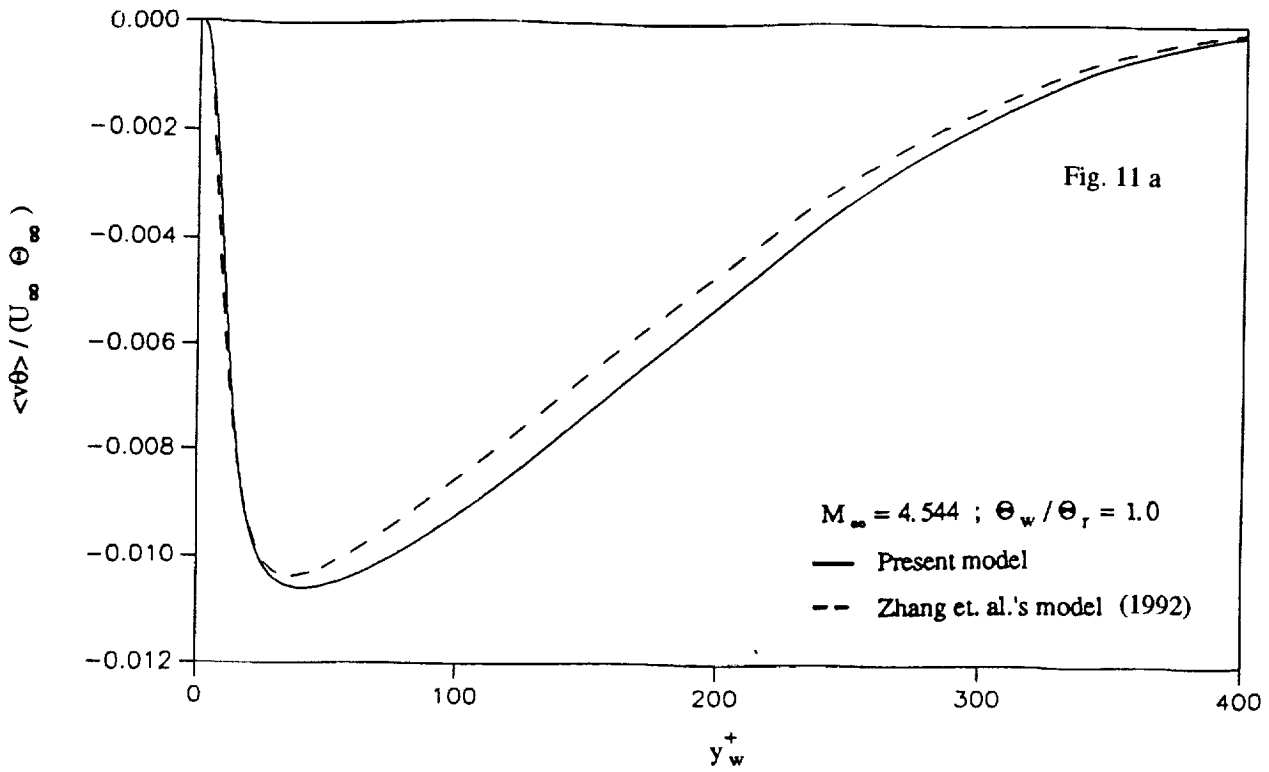


Figure 11. Comparison of  $\langle v\theta \rangle$  for adiabatic wall boundary condition: (a)  $M_\infty = 4.544$ , (b)  $M_\infty = 10.31$ .

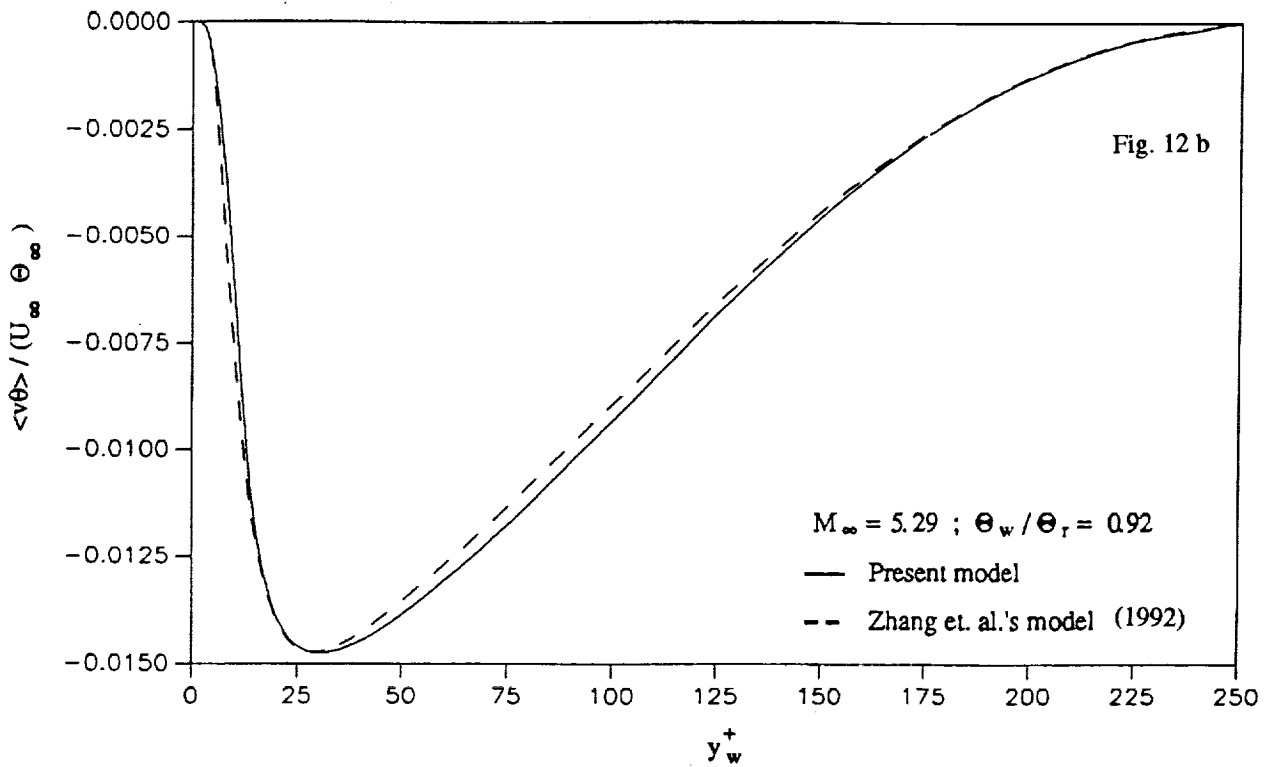
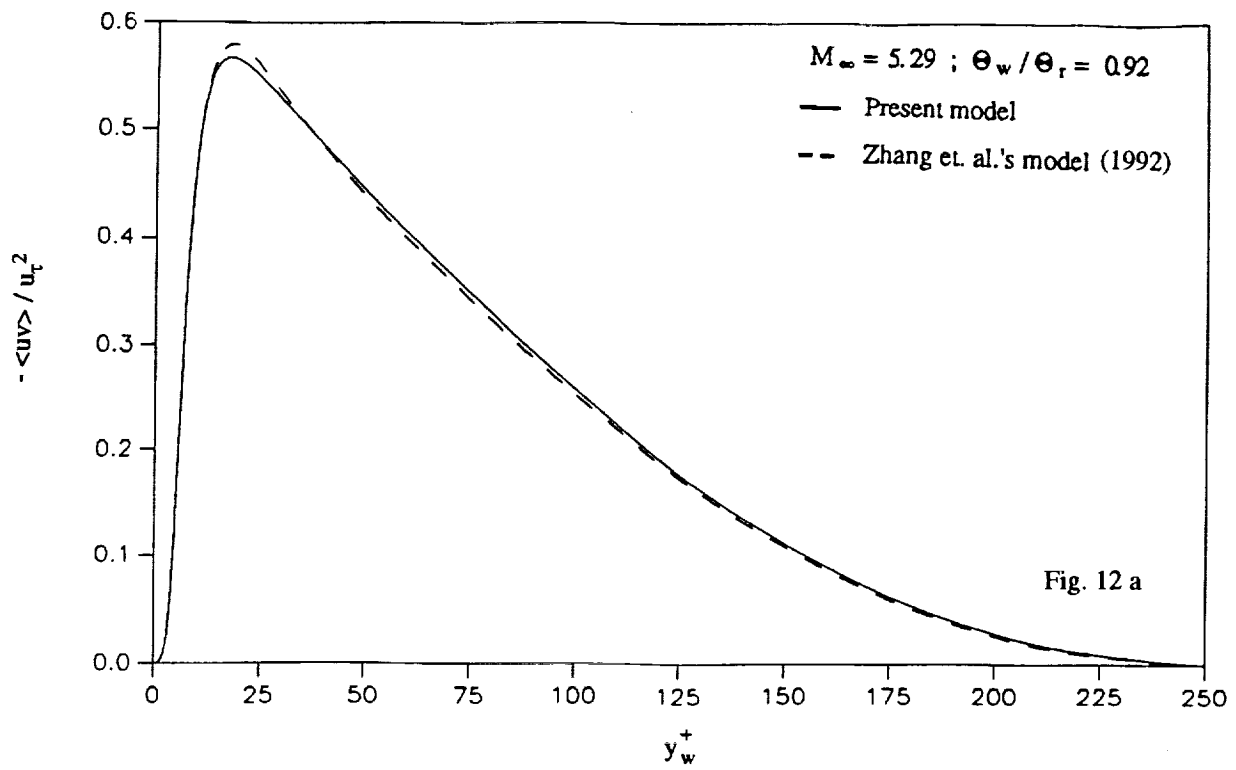


Figure 12. Comparison of (a)  $\langle uv \rangle$  and (b)  $\langle v\theta \rangle$  for cooled wall boundary condition at  $M_\infty = 5.29$ .

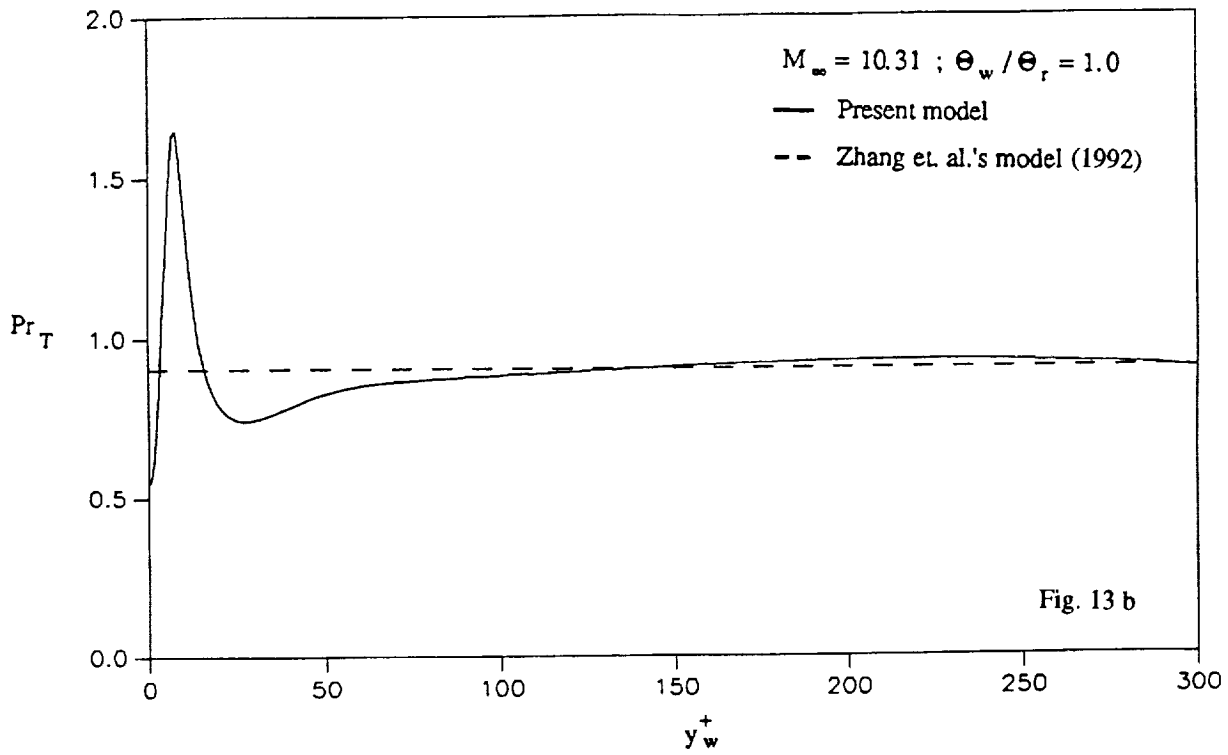
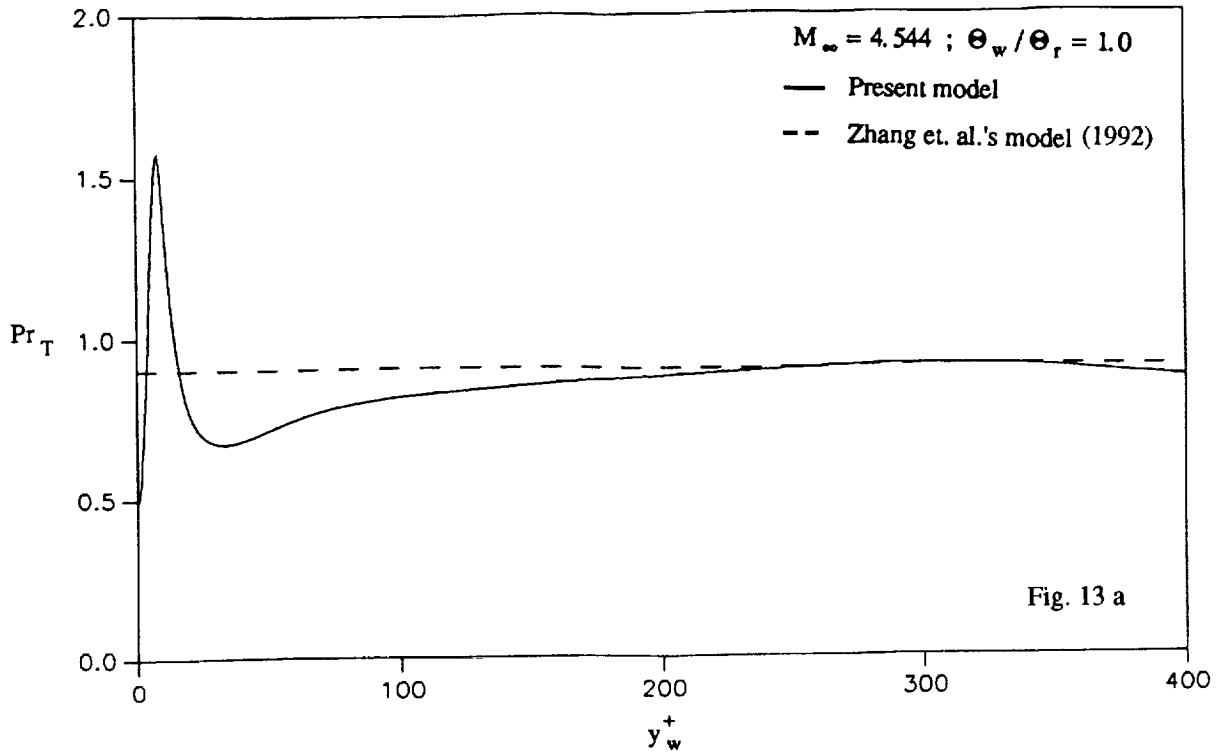


Figure 13. Variation of  $Pr_t$  across the boundary layer for adiabatic wall boundary condition: (a)  $M_\infty = 4.544$ , (b)  $M_\infty = 10.31$ .

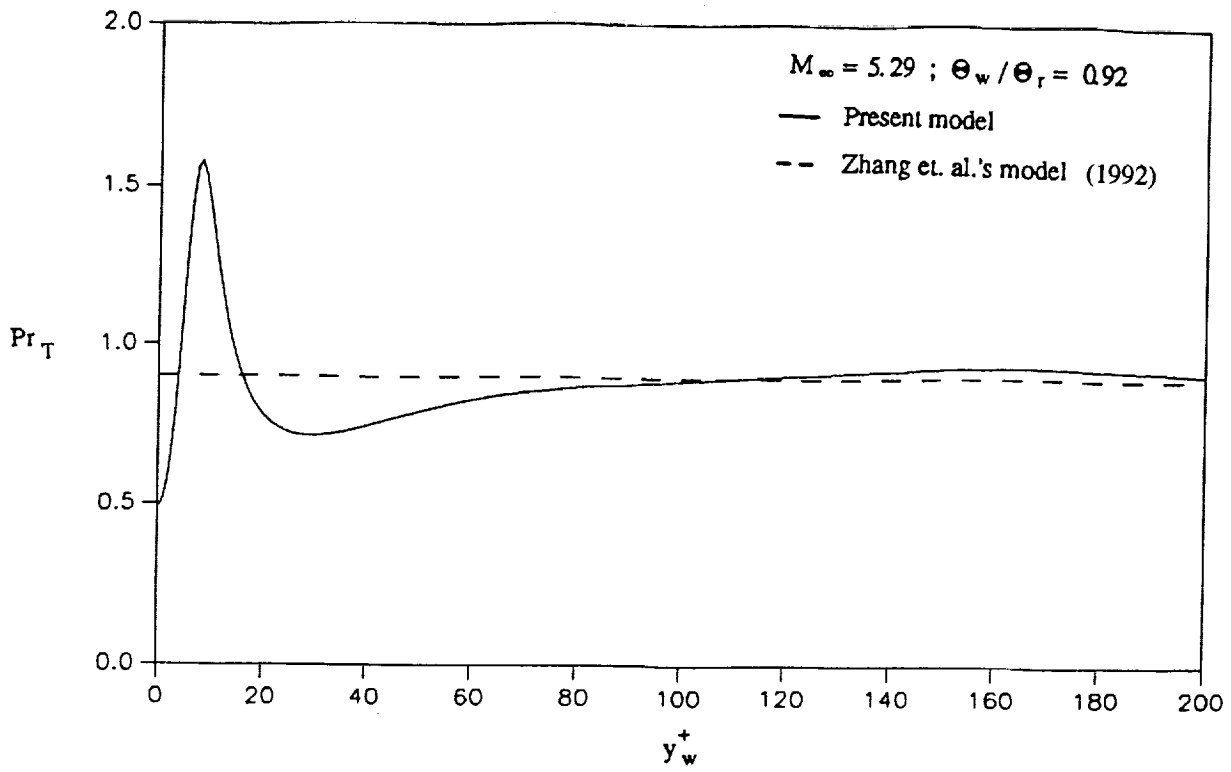


Figure 14. Variation of  $Pr_t$  across the boundary layer for cooled wall boundary condition at  $M_\infty = 5.29$ .



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13. ABSTRACT (Maximum 200 words) A near-wall turbulence model involving the transport equations for $k$ , $\epsilon$ , $\theta$ and $\epsilon_\theta$ is developed for the calculations of hypersonic compressible turbulent boundary layers. Therefore, there is no need to assume dynamical similarity between momentum and heat transport. The Favre-averaged equations of motions are solved together with these four transport equations. Calculations are compared with measurements having both adiabatic and constant temperature wall boundary conditions and with another model prediction where the turbulent Prandtl number, $Pr_t$ , is assumed constant. Results for Mach numbers up to 5 and temperature ratios as low as 0.6 are essentially the same as those obtained using an identical near-wall $k-\epsilon$ model and assuming $Pr_t = 0.9$ . In general, the numerical predictions are in very good agreement with measurements and there are significant improvements in the predictions of mean flow properties at Mach numbers higher than 5. Present results further show that the calculated $Pr_t$ for all cases investigated varies rapidly from $\sim 0.5$ at the wall to a maximum of about 1.6 in the near-wall region; however, it quickly settles to 0.9 beyond $y_w^+ \geq 200$ . Therefore, the model lends credence to the $Pr_t = 0.9$ assumption invoked by other researchers.				
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