OBSERVATIONAL AND THEORETICAL INVESTIGATIONS
IN SOLAR SEISMOLOGY

IN-92-CR
85174
P-40

NASA Grant NAGW-1677

Semiannual Reports No. 1, 2, 3, and Final Report

For the period 1 March 1989 through 31 August 1990

and

For the period 1 March 1989 through 28 February 1991

Principal Investigator

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March 1992

Prepared for

National Aeronautics and Space Administration

Washington, D.C. 20546

Smithsonian Institution
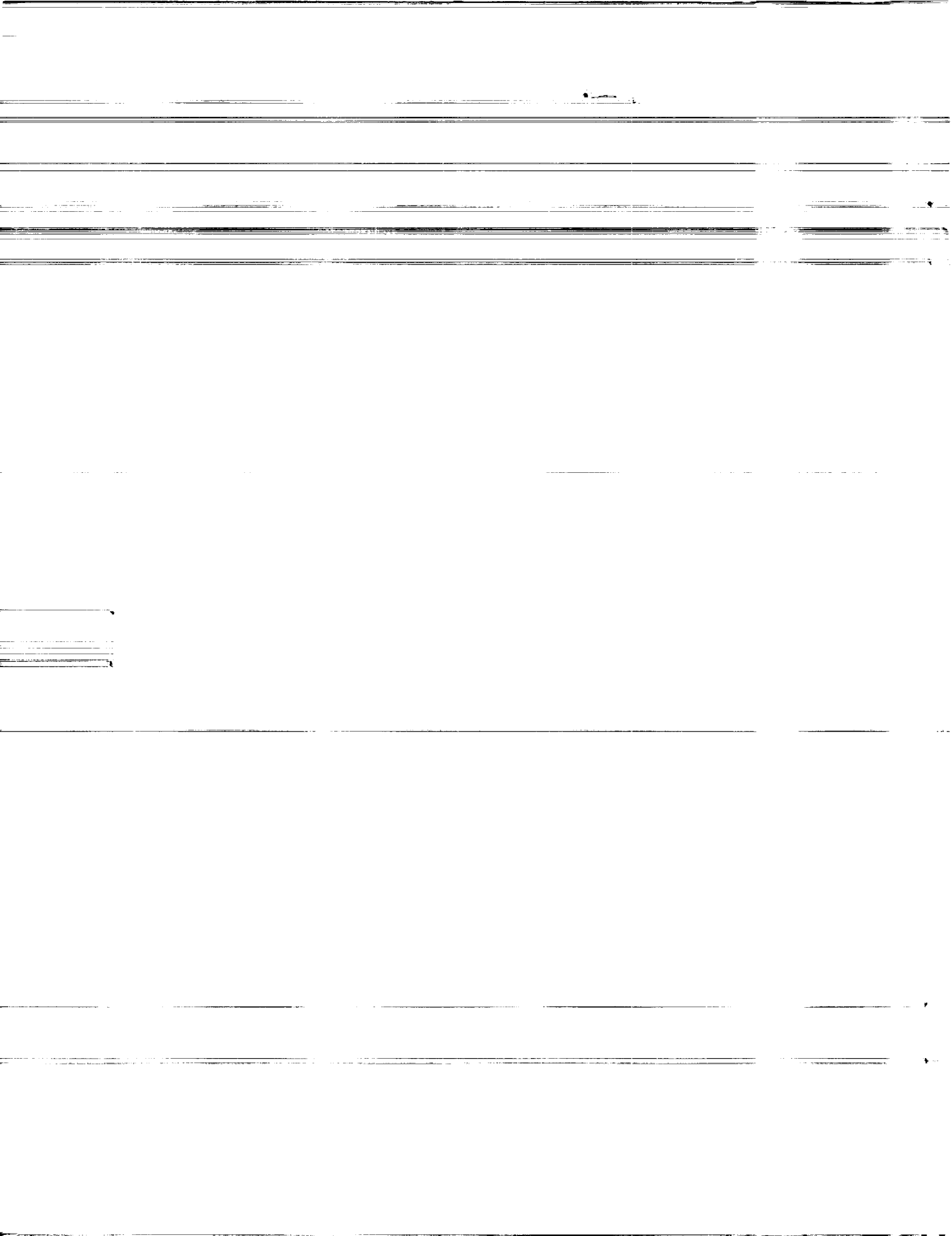Astrophysical Observatory
Cambridge, Massachusetts 02138

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Observational and Theoretical Investigations
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I. INTRODUCTION

This is the final report on a project to develop a theoretical basis for interpreting solar oscillation data in terms of the interior dynamics and structure of the Sun. Work funded under this project was carried out principally by Dr. Michael Ritzwoller, Dr. Pawan Kumar, and Dr. Sylvain Korzennik, under the overall guidance of the Principal Investigator, from March 1989 through August 1991. In sections II, III, and IV we discuss their three related areas of work, and in section V give references to four papers written with support from this grant. An Appendix gives complete copies of two of these papers and abstracts and summaries of the other two.

II. STUDIES OF THE HELIOSEISMIC SIGNATURES OF DIFFERENTIAL ROTATION AND CONVECTION IN THE SOLAR INTERIOR.

The now-traditional method to infer differential rotation in the solar interior is to expand the rotation rate $\Omega(r,\theta)$ as a power series in $\cos^k\theta$ or in $P_k(\cos\theta)$, and relate the expansion coefficients to the observed helioseismic mode splittings. These splittings may themselves may be expanded in Legendre polynomials of m/l, where m and l are the azimuthal order and degree, respectively, of the oscillation modes. This expansion has the appeal that because different terms of the expansion are orthogonal over the sphere, observational errors in determining the different terms of the expansion are nearly independent. The internal angular velocity, or differential rotation, is then determined by inversion of the splitting coefficients. This approach has the difficulties that (1) it is difficult to generalize to non-axisymmetric flows, (2) it is computationally cumbersome, and (3) there is crosstalk between the various terms in the expansion of Ω.

Ritzwoller and Eugene Lavely, with partial support from this grant, have developed a unified approach to the helioseismic forward and inverse problem of differential rotation (Ritzwoller and Lavely 1991). In this approach the differential rotation is represented as the axisymmetric component of a more general toroidal flow field. A better choice of basis functions for differential rotation allows determination of a set of vector spherical harmonic expansion coefficients for the rotation that are decoupled so that each degree of differential rotation can be estimated independently from all other degrees.

Lavely and Ritzwoller, also with partial support from this grant, have carried out a fundamental study of the effect of global-scale steady-state convection on helioseismic oscillations (Lavely and Ritzwoller 1992). They have derived the basic theory governing the influence of convection and associated structural asphericities on oscillation frequencies, without the usual assumptions of an axisymmetric model. They represention the eigenfunctions of a reference spherical model with vector spherical harmonics, and employed

quasi-degenrate perturbation theory to derive general matrix elements governing mode coupling and splitting caused by convection and structural asphericities. This formalism may be applied to models of giant-cell convection, allowing determination of whether such flows bias recently estimated differential rotation profiles. Or, it could be used to test hypothesized pole-equator differences in temperature near the top of the convection zone as a source of observed even-degree splitting coefficients.

A complete copy of the first of the two above-referenced papers, and the abstract and summary of the second, are included in the appendix to this Final Report.

III. WAVE GENERATION BY TURBULENT CONVECTION.

Dr. Pawan Kumar, supported in part by this grant, has studied the generation of solar p-mode oscillations by turbulent convection in collaboration with P. Goldreich (Goldreich and Kumar, 1990). They used a simplified model of an adiabatically stratified upper solar convection zone overlaid by a convectively stable isothermal atmosphere, and studied the rate at which convective energy is converted into energy of trapped p-modes, f-modes, g-modes, and travelling acoustic waves. They found that wave generation is concentrated at the top of the convection zone where the turbulent Mach number M_t peaks. The efficiency η of power input into trapped p and f modes, and into travelling acoustic waves, was found to vary as $\eta \approx M_t^{7.5}$, and that into g modes was found to vary as M. For p-modes the energy input depends on frequency as ω^α where $\alpha = (2m^2 + 7m - 3)/(m+3)$, and m is the polytropic index. This agrees with the observed finding of Libbrecht (1988,ESA SP-286, p 3) that the power input varies as ω^8, if we set $m \approx 4$; in fact this value of m is close to the polytropic index that fits the density profile in the upper solar convection zone. The agreement is strong evidence that wave emission by turbulent convection actually is the process by which p-modes are excited in the solar atmosphere.

Also, Dr. Kumar, with partial support by this grant, collaborated in preparation of a review of theories of excitation of oscillation modes in the Sun (Cox et al 1991). This review contains a summary of Kumar and Goldreich's work on 3-mode coupling, which shows that such mode couplings, which had previously been considered a good candidate for limiting the energies of overstable p-modes, are not strong enough for that purpose. That result casts doubt on the idea that the modes are excited by overstability, and therefore provides further support for mode excitation by coupling with the motions of convection.

Copies or summaries of these two papers are also included in the Appendix.

IV. STUDY OF ANTIPODAL SUNSPOT IMAGING AN ACTIVE REGION TOMOGRAPHY

During the last stages of this grant, we were joined by Dr. Sylvain Korzennik. Dr Korzennik has investigated, under partial support of the grant, the signatures of magnetic field structures in helioseismic data. Specifically, he carried out a search in Mt. Wilson data for the helioseismic signature of active regions and sunspots at their antipodal point; analogous signals have been seen in geo-seismology, and might be expected in helioseismology, because (a) acoustic power is known to be absorbed in sunspots, and (b)

helioseismic waves emanating from a point should interfere constructively at its antipodal position if the wave lifetime is long enough to establish global coherence and the wavelength short enough to prevent blurring of the antipodal "image". Korzennik's analysis of high wavenumber data (2 arcsec per pixel) showed no observable antipodal signature from several data sets where moderate-size sunspots were known to exist on the invisible hemisphere of the sun. This suggests that there is a lack of global coherence of high wavenumber acoustic modes. However, the analysis is still in process.

The observed acoustic energy deficit in active regions suggests that "active region tomography" will be an important tool for study of the subsurface structure of magnetic active regions, particularly with very high-resolution (high wavenumber) data such as may be expected from the SOI investigation on SOHO. Dr. Korzennik, with support from this grant, began an effort to develop the theoretical and analytical tools to characterize the acoustic field fully, with particular attention to the phase relations between velocity and intensity perturbations associated with the waves. This work is now being continued, with support from the NASA SOI investigation (P. Scherrer, PI), of which Professor Noyes is a co-investigator.

Publication of the results of Dr. Korzennik's work on antipodal imaging of sunspots and active region tomography is currently in preparation.

V. REFERENCES

The following four papers, cited in the text, were all partially suported by Grant NAGW-1677. (Acknowledgements to the grant are included explicitly in the last two references, but due to an oversight only implicitly in the first two. Nevertheless, the grant did provide financial support to Dr. Kumar for the preparation of the first two publications while he was a visiting scientist at SAO in 1989.)

Cox, A.N., Chitre, S.M., Frandsen,S., and Kumar, P., 1991,"Oscillation Mode Excitation", in *Solar Interior and Atmosphere*, A.N.Cox, W.C. Livingson, and M.S.Mathews, eds., p. 618-660 (U. Ariz. Press, Tucson).

Goldreich, P. and Kumar, P. 1990, "Wave Generation by Turbulent Convection", Astroph. J. **363**, 694-704.

Lavely, E. M. and Ritzwaller, M. H., 1992, "The Effect of Global-Scale, Steady-State Convection and Elastic-Gravitational Asphericities on Helioseismic Oscillations, Phil. Trans., Ser A., *in press*.

Ritzwaller, M. H. and Lavely, E. M., 1991, "A Unified Approach to the Helioseismic Forward and Inverse Problems of Differential Rotation", Astroph. J. **369**, 557-566.

APPENDIX

Reprints (or abstracts and summaries) of papers written with support from NAGW-1677.

A UNIFIED APPROACH TO THE HELIOSEISMIC FORWARD AND INVERSE PROBLEMS OF DIFFERENTIAL ROTATION

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ABSTRACT

We present a general, degenerate perturbation theoretic treatment of the helioseismic forward and inverse problems for solar differential rotation. Our approach differs from previous work in two principal ways. First, in the forward problem, we represent differential rotation as the axisymmetric component of a general toroidal flow field using vector spherical harmonics. The choice of these basis functions for differential rotation over previously chosen ad hoc basis functions (e.g., trigonometric functions or Legendre functions) allows the solution to the forward problem to be written in an exceedingly simple form (eqs. [32]–[37]). More significantly, their use results in inverse problems for the set of radially dependent vector spherical harmonic expansion coefficients, which represent rotational velocity, that decouple so that each degree of differential rotation can be estimated independently from all other degrees (eqs. [56] & [61]–[63]). Second, for use in the inverse problem, we express the splitting caused by differential rotation as an expansion in a set of orthonormal polynomials that are intimately related to the solution of the forward problem (eqs. [5] and [54]). The orthonormal polynomials are Clebsch-Gordon coefficients and the estimated expansion coefficients are called splitting coefficients. The representation of splitting with Clebsch-Gordon coefficients rather than the commonly used Legendre polynomials results in an inverse problem in which each degree of differential rotation is related to a single splitting coefficient (eq. [56]). The combined use of the vector spherical harmonics as basis functions for differential rotation and the Clebsch-Gordon coefficients to represent splitting provides a unified approach to the forward and inverse problems of differential rotation which will greatly simplify inversion. We submit that the mathematical and computational simplicity of both the forward and inverse problems afforded by our approach argues persuasively that helioseismological investigations would be well served if the current ad hoc means of representing differential rotation and splitting would be replaced with the unified methods presented in this paper.

Subject headings: Sun: oscillations — Sun: rotation

1. INTRODUCTION

An acoustic mode of oscillation of a spherically symmetric, nonrotating, adiabatic, static solar model without magnetic fields is typically identified by a trio of quantum numbers that represents its displacement field: n, the radial order; l, the spherical harmonic degree; and m, the azimuthal order of the mode. Because of the rotational symmetries of this model, the modes of oscillation are $2l + 1$ degenerate. That is, the frequencies of the $2l + 1$ modes with different m values but with the same n and l values are identical. These modes are said to form a multiplet. The real Sun is not so simple. Of particular relevance for this paper is the fact that the Sun is rotating and is deforming internally so that, for example, the surface rate of rotation at the solar equator is greater than at the poles. This phenomenon is known as differential rotation.

A number of ways have been chosen to represent differential rotation mathematically. We will argue in this paper that a representation with exceptionally nice consequences is the solar rotational velocity $v_{rot}(r, \theta, \phi)$, defined to be the axisymmetric component of general toroidal flow fields in the solar interior. A heretofore more popular, and perhaps more conceptually appealing, way of looking at this is that the Sun is rotating differentially and that the rotation rate $\Omega(r, \theta)$ is itself a function of both radial position and colatitude. Rotational velocity $v_{rot}(r, \theta, \phi)$ and rotation rate $\Omega(r, \theta)$ are simply related by

$$v_{rot}(r, \theta, \phi) = \Omega(r, \theta, \phi) \times r = \Omega(r, \theta)r \sin(\theta)\hat{\phi} \tag{1}$$

where $\Omega = \Omega\hat{z}$, $\hat{z}$ is the unit vector which points along the axis of rotation, and r is the position vector from the center of the Sun to position (r, θ, ϕ). The coordinates $r = (r, \theta, \phi)$ are spherical polar coordinates (where θ is colatitude) and $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ denote unit vectors in the coordinate directions. However represented, solar differential rotation lifts the degeneracy of the solar acoustic or p-modes, splitting the frequencies of oscillation of the Sun. This phenomenon is, without doubt, the largest contributor to the splitting of solar oscillations.

Observations of split solar p-mode frequencies date from Claverie et al. (1981). The quantity and quality of new measurements have been steadily improving (e.g., Gough 1982; Hill, Bos, & Goode 1982; Duvall & Harvey 1984; Duvall, Harvey, & Pomerantz

1986; Brown 1985; Libbrecht 1986. 1989; Brown & Morrow 1987; Tomczyk 1988; Rhodes et al. 1990). Since p-modes are dominantly split by differential rotation. a first-order understanding of the data requires the estimation of models of rotation that predict the data accurately. Furthermore. an understanding of the solar angular momentum budget requires accurate models of differential rotation (Gilman, Morrow, & DeLuca 1989). For these reasons. the inverse problem of inferring the radial and latitudinal dependence of differential rotation from observed split frequencies has been a major focus of helioseismology. The reader is referred to Brown et al. (1989), Christensen-Dalsgaard, Schou, & Thompson (1990), and Thompson (1990) for clear discussions of the current state of this venture.

A necessary preface to tackling the inverse problem is the solution of the forward problem: i.e., given a model of solar differential rotation. determine the split p-mode frequencies. This problem was first solved by Cowling & Newing (1949) and subsequent treatments can be found in Ledoux (1951), Hansen. Cox. & Van Horn (1977), Gough (1981), and Brown (1985). Naturally, the form of the solution of the forward problem is a function of the basis functions chosen to represent differential rotation. Since differential rotation is axisymmetric and even about the equatorial plane, it admits a very simple mathematical representation.

In helioseismology, there are two popular ways chosen to represent differential rotation, both of which are based on polynomial expansions of rotation rate Ω rather than rotational velocity v_{rot}. The more popular of these parameterizations is to expand $\Omega(r, \theta)$ in even powers of $\cos \theta$ (see Brown et al. 1989):

$$\Omega(r, \theta) = \sum_{k=0,2,4,\ldots}^{\infty} \bar{\Omega}_k(r) \cos^k \theta \tag{2}$$

where $\bar{\Omega}_0(r)$ describes the bulk rotation rate of the Sun and the $\bar{\Omega}_k(r)$ for $k > 0$ describe the radial dependence of latitudinally dependent differential rotation. The popularity of this representation probably derives from tradition, since it was used in early studies of differential rotation made from observations of the solar surface (e.g., Howard & Harvey 1970). Thus, the use of these basis functions eased comparison with direct observations of differential rotation. It is generally recognized that a problem with this parameterization is that the basis functions are not orthogonal. This is not really an obstacle from a forward theoretic perspective, but is troublesome inverse theoretically since future inversions for higher degree components must also redo the lower degree components since, strictly speaking, they are not independent of one another. The use of any set of orthogonal basis functions overcomes this problem. In particular, Legendre polynomials in cosine colatitude have seen some application (e.g., Korzennik et al. 1988):

$$\Omega(r, \theta) = \sum_{k=0,2,4,\ldots}^{\infty} \Omega_k(r) P_k(\cos \theta) . \tag{3}$$

Although the parameterizations of differential rotation given by equations (2) and (3) are intuitively simple and allow straightforward comparison to other kinds of observations, their use leads to unfortunate consequences that can be entirely circumvented with a more judicious choice of basis functions. Problems with these basis functions include the following. (1) They do not generalize easily to general nonaxisymmetric flows. (2) They do not yield conveniently to the elegant generalized spherical harmonic formalism of Phinney & Burridge (1973) and are, therefore, computationally cumbersome. There is a more significant problem arising from the way observers choose to represent splitting data as a Legendre polynomial expansion in (m/l):

$$\omega_{nl}^m = \omega_{nl} + l \sum_{i=1}^{M} {}_n a_{li} P_i(m/l) . \tag{4}$$

The expansion coefficients ${}_n a_{li}$ are commonly called splitting coefficients. The use of equations (2) or (3) as basis functions for differential rotation applied to splitting data represented with equation (4) leads to a serious practical problem troubling inversion: namely, (3) they generate a coupled set of inverse problems in which for a given k the estimation of $\Omega_k(r)$ or $\bar{\Omega}_k(r)$ depends on $\Omega_{k'}(r)$ or $\bar{\Omega}_{k'}(r)$, respectively, for all even $k' > k$.

Problem 3 has been tackled in a number of ways. including: (a) by estimating $\Omega_k(r)$ or $\bar{\Omega}_k(r)$ for all even $k \leq k_{max}$ simultaneously, where $k_{max} = 4$ usually (e.g., Thompson 1990); (b) by estimating each $\Omega_k(r)$ or $\bar{\Omega}_k(r)$ recursively by solving first for each even k' where $k_{max} > k' > k$ (e.g., Brown et al. 1989); (c) by forming recombined basis functions that allow, in the high l limit, the inverse problems for distinct k to decouple (e.g., Korzennik et al. 1988); and (d) by replacing equation (4) with an alternative representation of splitting measurements relative to which recombined basis functions decouple in the inverse problems for distinct k (Durney, Hill, & Goode 1988). There are problems with each of these approaches. Approaches a and b generate models that at different degrees k have correlated errors. In addition. approach b necessarily performs the recursion in the direction opposite from how a stable and robust recursive technique should be applied. A robust recursive technique would first estimate the features of the model that have the largest expression in the data. In this case. these are the longest wavelength features of the model (i.e., small k). Then these should be used in the estimation of shorter wavelength model components (i.e., higher k) that affect the data more subtly. Approach b does the opposite of this. By estimating the shorter wavelength features first, it propagates errors from the more poorly constrained to the better constrained features of the model. Approach c has a limited range of applicability and approach d requires observers to summarize their data in a form they consider suboptimal.

The common problem with all of these approaches to problem 3 is that the basis functions that have been chosen to represent both splitting and differential rotation have been ad hoc. In this paper we take a different approach. We define the inverse problem explicitly in terms of the form of the solution to the forward problem. We show that there exists a natural set of basis functions with which to represent differential rotation such that the inverse problems for different degrees of differential rotation decouple with respect to data represented in the usual way (eq. [4]). Consequently, joint and recursive inversions as well as asymptotic approximations can be entirely circumvented. These basis functions are simply the vector spherical harmonic components of v_{rot}. In

addition, vector spherical harmonics generalize easily to nonaxisymmetric flow fields and yield to the formalism of Phinney & Burridge; thus, their use also addresses problems (1) and (2). We also show that the inverse problem is simplified further if splitting data are expressed using a set of natural basis functions intimately related to the solution of the forward problem. These orthonormal functions are the Clebsch-Gordon coefficients β_{il}^m with which the $2l + 1$ frequencies of a single split multiplet would be represented as follows:

$$\omega_{nl}^m = \omega_{nl} + \sum_{i=1}^{M} {}_nb_{li}\beta_{il}^m .$$

(5)

The expansion coefficients ${}_nb_{li}$ represent a set of new splitting coefficients.

Our approach to the forward problem is motivated by the approach geophysicists have taken to determine the splitting and coupling of terrestrial oscillations caused by aspherical perturbations in the elastic moduli and density of the Earth (e.g., Dahlen 1968, 1969; Luh 1973, 1974; Woodhouse & Dahlen 1978; Woodhouse 1980; Woodhouse & Girnius 1982). An approximation that has proven useful in terrestrial applications is to allow modes to couple only if they share the same radial order n and harmonic degree l. This means that if two modes are not degenerate in the absence of asphericities, they will not be considered potential coupling partners in the presence of the asphericity. In this case, it is appropriate to use degenerate perturbation theory to compute the split frequencies. This approximation is known to geophysicists as the isolated multiplet approximation since it is accurate if the $2l + 1$ modes composing a multiplet are isolated in complex frequency from modes composing other multiplets. In terrestrial applications this approximation has proven to be highly useful and quite accurate for calculating split frequencies but is not as useful for computing modal displacements. To compute modal displacements accurately, quasi-degenerate perturbation theory has been used by geophysicists in which modes are allowed to couple even if they are only nearly degenerate with respect to the spherical Earth model. For the Sun, the number of significant accidental near degeneracies between modes from different multiplets satisfying the selection rules that govern coupling for differential rotation is vanishingly small. Thus, solar multiplets can accurately be considered isolated in complex frequency and degenerate perturbation theory can be used to compute the split frequencies. The solution to the forward problem we present is accurate to first order in Ω/ω in the absence of accidental degeneracies, where Ω is the differential rotational frequency and ω is a modal frequency. Due to the spacing of p-modes between and along dispersion branches, the contribution to splitting caused by quasi-degenerate coupling between modes from different multiplets is an effect of higher order than first in Ω/ω. We will discuss this further in a future contribution.

In § 2, we defined the new basis functions for differential rotation mathematically and relate them to the previously used basis functions given by equations (2) and (3). In § 3, we present the solution to the forward problem for differential rotation using degenerate perturbation theory. The solution is expressed in terms of Wigner 3-j symbols which are straightforward to compute numerically. For low-degree differential rotation, we present analytical expressions for the Wigner 3-j symbols in terms of simple polynomials in m and l. In § 4, we discuss the use of Clebsch-Gordon coefficients for representing splitting data, by using equation (5) as an alternative to equation (4). These coefficients form an orthonormal basis set and are simply related to the Wigner 3-j symbols found in the solution of the forward problem. The use of the Clebsch-Gordon coefficients to represent splitting provides a unified approach to the data analysis and inverse problems. In § 5, we derive the form of the inverse problems relating the basis functions for v_{rot} both to the new splitting coefficients b_i and to the traditional splitting coefficients a_i. In both formulations, a single degree of the vector spherical harmonic expansion of rotational velocity can be related to a linear combination of the splitting coefficients. However, by using the Clebsch-Gordon coefficients, the sum is particularly simple, reducing to a single term. For completeness, we present formulae for converting solutions back to the rotation rate basis functions for differential rotation. Finally, to simplify the recommended use of the Clebsch-Gordon coefficients β_{il}^m for representing splitting, algebraic expressions in m and l for low-degree i coefficients are presented in the Appendix.

2. VECTOR SPHERICAL HARMONIC REPRESENTATION OF DIFFERENTIAL ROTATION

It is useful to decompose a general stationary, laminar velocity field $v(r, \theta, \phi)$ into poloidal P and toroidal T components:

$$v(r, \theta, \phi) = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} [P_s^t(r, \theta, \phi) + T_s^t(r, \theta, \phi)] .$$

(6)

The poloidal and toroidal components can be fully characterized by the radius dependent vector spherical harmonic expansion coefficients $u_s^t(r)$, $v_s^t(r)$, and $w_s^t(r)$:

$$P_s^t(r, \theta, \phi) = u_s^t(r)Y_s^t(\theta, \phi)\hat{r} + v_s^t(r)\nabla_1 Y_s^t(\theta, \phi) ,$$

(7)

$$T_s^t(r, \theta, \phi) = - w_s^t(r)\hat{r} \times \nabla_1 Y_s^t(\theta, \phi) ,$$

(8)

where the surface gradient operator, ∇_1, is given by

$$\nabla_1 = r[\nabla - \hat{r}(\hat{r} \cdot \nabla)] = \hat{\theta}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{\sin\theta}\frac{\partial}{\partial\phi} .$$

(9)

The function $Y_s^t(\theta, \phi)$ is a spherical harmonic of degree s and azimuthal order t defined using the convention of Edmonds (1960) as

$$Y_s^t = (-1)^t\left[\frac{2s + 1}{4\pi}\frac{(t - s)!}{(t + s)!}\right]^{1/2} P_s^t(\cos\theta)e^{it\phi} .$$

(10)

where $Y_s^{-t} = (-1)^t [Y_s^{-t}]^*$, P_s^t are associated Legendre functions. and the asterisk represents complex conjugation. The normalization constants in equation (10) have been chosen such that

$$\int_0^{2\pi} \int_0^{\pi} [Y_s^t(\theta, \varphi)]^* Y_s^t(\theta, \phi) \sin \theta \, d\theta \, d\phi = \delta_{tt} \delta_{ss},$$ (11)

where integration is over the unit sphere. The poloidal coefficients $u_s^t(r)$ and $v_s^t(r)$ are not independent if the anelastic condition $[\nabla \cdot (\rho v) = 0]$ is imposed.

The differential rotation velocity field $v_{rot}(r, \theta)$ simply corresponds to the odd-degree. zonal part of the toroidal flow field in equation (8) which can be written:

$$v_{rot}(r, \theta) = -\sum_{s=1,3,5,\ldots}^{\infty} w_s^0(r) \partial_\theta Y_s^0 \hat{\phi},$$ (12)

where, for example:

$$\partial_\theta Y_1^0 = -\frac{1}{2}\left(\frac{3}{\pi}\right)^{1/2} \sin \theta,$$ (13)

$$\partial_\theta Y_3^0 = -\frac{3}{4}\left(\frac{7}{\pi}\right)^{1/2} \sin \theta(5 \cos^2 \theta - 1),$$ (14)

$$\partial_\theta Y_5^0 = -\frac{15}{16}\left(\frac{11}{\pi}\right)^{1/2} \sin \theta(21 \cos^4 \theta - 14 \cos^2 \theta + 1).$$ (15)

The relationship between the expansion coefficients $w_s^0(r)$ with $\bar\Omega_k(r)$ and $\Omega_k(r)$ can be determined by equating the representations in equations (1) and (12). expanding each side in terms of irreducible trigonometric functions. and equating the approropriate groups of terms. This procedure yields

$$w_1^0(r) = 2\sqrt{\frac{\pi}{3}}\, r\left[\bar\Omega_0(r) + \frac{1}{5}\bar\Omega_2(r) + \frac{3}{35}\bar\Omega_4(r)\right] = 2\sqrt{\frac{\pi}{3}}\, r\left[\Omega_0(r) - \frac{1}{5}\Omega_2(r)\right],$$ (16)

$$w_3^0(r) = 2\sqrt{\frac{\pi}{7}}\, r\left[\frac{2}{15}\bar\Omega_2(r) + \frac{28}{315}\bar\Omega_4(r)\right] = 2\sqrt{\frac{\pi}{7}}\, r\left[\frac{1}{5}\Omega_2(r) - \frac{1}{9}\Omega_4(r)\right],$$ (17)

$$w_5^0(r) = 2\sqrt{\frac{\pi}{11}}\, r\left[\frac{8}{315}\bar\Omega_4(r)\right] = 2\sqrt{\frac{\pi}{11}}\, r\left[\frac{1}{9}\Omega_4(r)\right],$$ (18)

where we have truncated the sums at $k = 4$ in equations (2) and (3) and at $s = 5$ in equation (12).

3. THE FORWARD PROBLEM

The equations governing the effect of differential rotation on solar oscillation frequencies can be presented naturally in either of two reference frames: a frame corotating with the average angular velocity of the Sun or an inertial frame roughly identifiable with the frame of observation. Since differential rotation is stationary relative to both the corotating and inertial frames, solutions to these equations will separate in both frames. For simplicity of use by observers, we choose the inertial frame in which to represent and solve the following equations. As a consequence. v_{rot} will be considered to include the average rotational velocity of the Sun.

Let $k = (n, l)$; then the displacement field for p-modes in the presence of an axisymmetric flow field separates spatially and temporally:

$$s(r, t) = s_k^m(r) e^{i\omega_k^m t}.$$ (19)

We have chosen to introduce an unfortunate notational conflict and let t represent time (as well as the azimuthal order of a convective flow field). The radial eigenfunctions are defined as follows:

$$s_k^m(r) = {}_n U_l(r) Y_l^m(\theta, \phi)\hat{r} + {}_n V_l(r) \nabla_1 Y_l^m(\theta, \phi),$$ (20)

where ${}_n U_l(r)$ and ${}_n V_l(r)$ denote, respectively, the radial and horizontal eigenfunctions for harmonic degree l and radial order n. Hereafter, we drop the subscripts n and l in equation (20) and use instead $U = {}_n U_l(r)$, and $V = {}_n V_l(r)$. The eigenfunctions satisfy an orthogonality condition given by

$$\int \rho_0 s_k^m {}^* \cdot s_k^m \, d^3r = N\delta_{m'm}\delta_{n'n}\delta_{l'l},$$ (21)

where ρ_0 is the density of the equilibrium solar model. and

$$N = \int_0^{R_\odot} \rho_0(U^2 + L^2 V^2) r^2 \, dr,$$ (22)

where $L^2 = l(l + 1)$. The scalar constant N depends on the normalization of the eigenfunctions U and V.

We seek equations governing the influence of differential rotation on p-mode frequencies. The equation of motion for the oscillations of a spherically symmetric, nonrotating, adiabatic, static solar model without magnetic fields is given by

$$\mathscr{L}(s_k) - \rho_0 \hat{c}_t^2 s_k = 0 \; . \tag{23}$$

where $\mathscr{L}$ is a linear, self-adjoint spatial integro-differential operator subject to certain idealized boundary conditions. This equation can be rewritten using equation (19) as

$$\mathscr{L}(s_k) + \rho_0 \omega_k^2 s_k = 0 \; . \tag{24}$$

If we perturb the above described model by adding a rotational velocity field v_{rot}, the equation of motion is altered. In particular, the local time derivative $\hat{c}_t$ must be generalized to the material time derivative D_t and the displacements and frequencies of the modes must be perturbed. Thus, we must make the substitutions

$$\hat{c}_t \to D_t = \hat{c}_t + v_{rot} \cdot \nabla \; . \tag{25}$$

$$s_k \to s_k + \delta s_k \; , \tag{26}$$

$$\omega_k \to \omega_k + \delta \omega \; , \tag{27}$$

where ω_k is the degenerate frequency of the unperturbed multiplet. Lynden-Bell & Ostriker (1967) showed that the advection of the velocity field v_{rot} by the displacement eigenfunctions s_k can be ignored.

Making the above substitutions into equation (23), retaining only terms first-order in v_{rot}, $\delta \omega$, and δs_k, and using equation (24) to eliminate terms, we obtain the perturbed equation of motion

$$\mathscr{L}(\delta s_k) - 2i\omega_k \rho_0 v_{rot} \cdot \nabla s_k + \rho_0 \omega_k^2 \delta s_k + \rho_0 \delta \omega^2 s_k = 0 \; . \tag{28}$$

In accordance with our discussion in § 1, we will assume the isolated multiplet approximation in the remainder of this paper and apply degenerate perturbation theory to calculate the split frequencies due to differential rotation. Under this approximation, we expand the perturbed displacement field s_k in terms of the $(2l + 1)$ eigenfunctions of a single multiplet of a spherically symmetric solar model as follows

$$s_k = \sum_{m=-l}^{l} a_m s_k^m(r) \; . \tag{29}$$

Inserting equation (29) into equation (28), taking the inner product of the resulting expression with $s_k^m{}^*$, integrating over the volume of the Sun, and using equation (24) again, we obtain

$$\int s_k^m{}^* \cdot \mathscr{L}(\delta s_k) d^3r - \int \mathscr{L}(s_k^m{}^*) \cdot \delta s_k d^3r + \sum_{m=-l}^{l} a_m \left\{ \delta \omega^2 \int \rho_0 s_k^m{}^* \cdot s_k^m d^3r - 2i\omega_k \int \rho_0 s_k^m{}^* \cdot v_{rot} \cdot \nabla s_k^m d^3r \right\} = 0 \; . \tag{30}$$

The first two terms in equation (30) cancel since for an adiabatic solar model $\mathscr{L}$ is self-adjoint. Using this fact, together with the orthonormality of the eigenfunctions given by equation (21), we obtain from equation (30) the shift in squared frequency caused by differential rotation:

$$N \delta \omega_m^2 = 2i\omega_k \int \rho_0 s_k^m{}^* \cdot v_{rot} \cdot \nabla s_k^m d^3r \; , \tag{31}$$

where we have used the fact that the right-hand side of equation (31) vanishes unless $m' = m$ since v_{rot} is axisymmetric.

Substituting into equation (31) from equations (20) and (12) and using the fact that $\delta \omega^2 \cong 2\omega_k \delta \omega$, we find that for $-l \leq m \leq l$:

$$\omega_k^m = \omega_k + \delta \omega_m = \omega_{nl} + \sum_{s=1,3,5,\dots} \gamma_{sl}^m {}_nc_{sl} \; . \tag{32}$$

where

$$\gamma_{sl}^m = (2l + 1)L^2 \left(\frac{2s+1}{4\pi} \right)^{1/2} H_s^l H_s^m F_s^2 \; . \tag{33}$$

$$_nc_{sl} = \int_0^{R_\odot} w_s^0(r) \, _nK_{ls}(r) r^2 \, dr \; . \tag{34}$$

$$_nK_{ls}(r) = -\rho_0 r^{-1} \{ U^2 + L^2 V^2 - [2UV + \tfrac{1}{2}s(s+1)V^2] \}/N \; . \tag{35}$$

and where we have defined

$$H_s^m F_s = (-1)^{(l-m)} \begin{pmatrix} s & l & l \\ 0 & m & -m \end{pmatrix} \; . \tag{36}$$

$$F_s = \left[\frac{(2l-s)!}{(2l+s+1)!} \right]^{1/2} \tag{37}$$

The gradient operator in equation (31) acts on both the scalar components and unit vectors of s_k^m which yield, respectively, what might loosely be called the advection $(U^2 + L^2 V^2)$ and Coriolis $[2UV + \tfrac{1}{2}s(s+1)V^2]$ contributions to the integral kernel $_nK_{ls}$.

The derivation of equation (33) is greatly simplified by use of the generalized spherical harmonic formalism of Phinney & Burridge (1973). It is beyond the scope of this paper to describe their formalism in any detail. However, Lavely & Ritzwoller (1991) do describe the formalism as it applies to the problem of splitting caused by a general convective flow field of which, of course, differential rotation is simply a special case. The form of equation (33), written in terms of Wigner 3-j symbols, follows from the Phinney & Burridge formalism. The product of the Wigner 3-j symbols is simply related to the integral of three generalized spherical harmonics over the unit sphere (see, e.g., Edmonds 1960). It is worth noting that the appearance of the 1 and -1 in the lower row of the 3-j symbol represented by H_s^1 is related to the gradient in equation (31). This reveals one of the beautiful aspects of the generalized spherical harmonic formalism, that gradients translate simply to index raising in the 3-j symbols.

Another attractive feature of expressing the solution to the forward problem (eq. [33]) in terms of Wigner 3-j symbols is the immediacy of selections rules which result, in part, from properties of the 3-j symbols. Under the isolated multiplet approximation these selection rules are that the frequency perturbation caused by an axisymmetric flow of degree s is nonzero only if (1) the flow is toroidal, (2) s is odd, and (3) $0 \leq s \leq 2l$. As a consequence, we have written the sum in equation (32) over only odd s. Thus, under the isolated multiplet approximation, only zonal toroidal flows with odd degree less than or equal to twice the degree of the multiplet contribute to the splitting. For the sake of accuracy we should point out that selection rule (1) does not derive from a property of the 3-j symbols. Rather it results from the fact that the integral kernel for a poloidal flow field under the isolated multiplet approximation is identically zero. We do not show this here since we are explicitly considering only differential rotation which is purely toroidal. However, this is demonstrated by Lavely & Ritzwoller (1991).

Equation (32) together with equations (33)–(37) completely specify the forward problem. We have chosen to write equation (32) in a way that has proven useful in geophysical applications (e.g., Ritzwoller, Masters, & Gilbert 1986, 1988; Giardini, Li, and Woodhouse 1988) where the coefficients $_sc_{si}$ would be recognized as splitting function coefficients or as interaction coefficients. We argue that these coefficients, being linearly related to the model parameters w_s^0, are what should be estimated in any analysis of the data aiming to infer differential rotation. We discuss an alternative method of estimating the $_sc_{is}$ coefficients in § 4 and their relation to the splitting coefficients, $_sa_{ti}$ and $_sb_{ti}$ in § 5.

It is hoped that a major product of this paper will be formulae that are simple and efficient to use both in the forward and inverse problems of differential rotation. The coefficients γ_{si}^m in equation (33) can be computed numerically and all the results in this paper could be simply stated in terms of them. Numerical methods for computing 3-j symbols are discussed and programs are tabulated in Zare (1988). However, for ease of use we will rewrite the 3-j symbols for $s \leq 5$ in terms of polynomials in l and m by using the recursion relation of Schulten & Gordon (1975). If desired, it is straightforward to extend these formulae to $s > 5$ by repeated application of the recursion relation. Setting $j_1 = s$, $j_2 = j_3 = l$, $m_1 = 0$, $m_2 = m$, and $m_3 = -m$ in equation (5a) of Schulten & Gordon (1975) and using equations (36) and (37), we obtain

$$H_{s+1}^m = \frac{1}{s+1} \left\{ 2(2s+1)mH_s^m - s[(2l+1)^2 - s^2]H_{s-1}^m \right\} .$$
(38)

To initiate the recursion, the polynomial forms of the Wigner 3-j symbols appearing on the right-hand side of equation (38) for $s = 1$ are required, and can be found in Table 2 of Edmonds (1960). We find by using equation (38) that for $s \leq 5$:

$$H_0^m = 1 ,$$
(39)

$$H_1^m = 2m .$$
(40)

$$H_2^m = 6m^2 - 2L^2 ,$$
(41)

$$H_3^m = 20m^3 - 4(3L^2 - 1)m .$$
(42)

$$H_4^m = 70m^4 - 10(6L^2 - 5)m^2 + 6L^2(L^2 - 2) ,$$
(43)

$$H_5^m = 252m^5 - 140(2L^2 - 3)m^3 + [20L^2(3L^2 - 10) + 48]m .$$
(44)

Algebraic expressions for H_6^m–H_{11}^m have been tabulated in the Appendix. Using equations (39)–(44), γ_{si}^m for $s \leq 5$ can be written:

$$\gamma_{1i}^m = \left(\frac{3}{4\pi}\right)^{1/2} m .$$
(45)

$$\gamma_{2i}^m = \left(\frac{5}{4\pi}\right)^{1/2} \frac{(3m^2 - L^2)(3 - L^2)}{4L^2 - 3} .$$
(46)

$$\gamma_{3i}^m = \frac{3}{2}\left(\frac{7}{4\pi}\right)^{1/2} \frac{-10m^3 + (6L^2 - 2)m}{4L^2 - 3} .$$
(47)

$$\gamma_{4i}^m = \frac{3}{8}\left(\frac{9}{4\pi}\right)^{1/2} \frac{[70m^4 - 10(6L^2 - 5)m^2 + 6L^2(L^2 - 2)](L^2 - 10)}{(4L^2 - 3)(4L^2 - 15)} .$$
(48)

$$\gamma_{5i}^m = \frac{15}{16}\left(\frac{11}{4\pi}\right)^{1/2} \frac{252m^5 - 140(2L^2 - 3)m^3 + [20L^2(3L^2 - 10) + 48]m}{(4L^2 - 3)(4L^2 - 15)} .$$
(49)

4. ORTHONORMAL BASIS FUNCTIONS FOR REPRESENTING SPLITTING DATA

Observed frequency splittings are typically represented in terms of a Legendre polynomial expansion with argument m/l, where the expansion coefficients are the splitting coefficients $_na_{li}$ as in equation (4). (For example. Libbrecht 1989 measured the splitting coefficients of 723 p multiplets in the range $5 \leq l \leq 60$.) The sum in equation (4) has usually been truncated at $M = 5$ since the inclusion of higher orders does not significantly improve the fit to the data (Brown et al. 1989). Apparently, Legendre polynomials have been chosen to represent the splitting since they are well known basis functions and Legendre functions are orthogonal over the continuous interval $[-1, 1]$:

$$\int_{-1}^{1} P_i(x)P_{i'}(x)dx = \frac{2}{2i+1} \delta_{ii'} .$$

(50)

Legendre functions are not orthonormal, but can easily be orthonormalized. There are two problems with their use. First, they are not an orthogonal basis set for representing discrete data such as split frequencies. If used to represent discrete data they are only approximately "orthogonal." However, the accuracy of this approximation improves with harmonic degree l as the sampling of the interval $[-1, 1]$ becomes finer. Second, and much more significantly, their use complicates the inverse problem. As we will show in § 5 (eqs. [61]–[63]), by representing split frequencies with splitting coefficients $_na_{li}$ based on Legendre polynomials, the interaction coefficients $_nc_{li}$ are not related to a single splitting coefficient, but are related to a linear combination of the $_na_{li}$.

The purpose of this section is to point out that there exists an orthonormal set of basis functions over the discrete interval $-l \leq m \leq l$, and that these functions are simply related to the γ_{ls}^m functions. These basis functions are Clebsch-Gordon coefficients. Furthermore, in § 5, we show that the use of Clebsch-Gordon coefficients to represent splitting further simplifies the inverse problem (eq. [56]).

Inspection of equation (32) suggests that a natural, alternative way to represent splitting caused by differential rotation would be to use the functions γ_{sl}^m as basis functions rather than Legendre polynomials. For fixed l, the γ_{sl}^m functions are orthogonal in s. Their orthogonality relation can be deduced from the orthogonality property of Wigner 3-j symbols given by equation (3.7.8) of Edmonds (1960). We rewrite this for our purposes as

$$\sum_{m=-l}^{l} \begin{pmatrix} s & l & l \\ 0 & m & -m \end{pmatrix} \begin{pmatrix} s' & l & l \\ 0 & m & -m \end{pmatrix} = \frac{\delta_{ss'}}{2s+1} .$$

(51)

Combining this with equations (33) and (36), we obtain the orthogonality relation for the γ_{sl}^m functions:

$$\sum_{m=-l}^{l} \gamma_{sl}^m \gamma_{s'l}^m = \frac{\delta_{ss'}}{2s+1} \frac{G_s G_{s'}}{F_s F_{s'}} ,$$

(52)

where we have defined

$$G_s = \gamma_{sl}^m / H_s^m .$$

(53)

We note that G_s is independent of m since the m-dependence of γ_{sl}^m is given by H_s^m as can be seen in equation (33).

Since the terms on the right-hand side of equation (52) can vary widely in size with s, especially for high l, the γ_{sl}^m should not be used as basis functions for the splitting. However, equations (51) and (52) suggest a set of basis functions which are orthonormal on the discrete interval $-l \leq m \leq l$. These orthonormal functions β_{sl}^m are simply related to the γ_{sl}^m functions as follows:

$$\beta_{sl}^m = \frac{(2s+1)^{1/2}F_s}{G_s} \gamma_{sl}^m = (2s+1)^{1/2}F_s H_s^m .$$

(54)

where

$$\sum_{m=-l}^{l} \beta_{sl}^m \beta_{s'l}^m = \delta_{ss'} .$$

(55)

Equation (54) can be evaluated explicitly in terms of polynomials in m and l for $s \leq 5$ by use of equations (39)–(44) and (37). Higher degree β_{sl}^m can be computed by using the formulae provided in the Appendix. By equation (3.7.3) of Edmonds (1960), it can be seen that the β_{sl}^m coefficients are simply Clebsch-Gordon coefficients. Equation (5) would then be used to represent splitting.

5. THE INVERSE PROBLEM

Once the new splitting coefficients $_nb_{sl}$ have been estimated using equation (5), the inverse problem for differential rotation can be reconstructed immediately since the $_nb_{sl}$ coefficients are simply related to the $_nc_{sl}$ coefficients as follows:

$$c_s = \int_0^{R_\odot} w_s^0(r)K_s(r)r^2 dr = \frac{(4\pi)^{1/2}}{(2l+1)L^2 H_s^1 F_s} b_s ,$$

(56)

where we have employed the notational simplification $c_s = _nc_{sl}$ and $b_s = _nb_{sl}$. Given estimates of the interaction coefficients c_s, equation (56) defines a linear inverse problem for differential rotation.

Although estimating the new splitting coefficients b_s with the orthonormal basis functions β_{ls}^m is more stable than current methods and results in an exceedingly simple inverse problem (eq. [56]), some observers may choose to retain the way splitting data have traditionally been represented. Therefore, we also seek expressions which relate the estimated splitting coefficients $_n a_{il}$ to the interaction coefficients $_n c_{sl}$. We do this only for $s = 1, 3, 5$ here, although the method we employ can be used to produce higher degree results. First, identify equations (32) and (4):

$$l \sum_{i=1}^{5} a_i P_i(m/l) = \sum_{s=1,3,5} G_s H_s^m c_s ,$$ (57)

where we have set $a_i = _n a_{il}$ and we have defined G_s in equation (53). Then, simply equating terms with the same odd power of m yields:

$$a_1 - \tfrac{1}{2}a_3 + \tfrac{15}{8}a_5 = 2G_1 c_1 + (4 - 12L^2)G_3 c_3 + [20L^2(3L^2 - 10) + 48]G_5 c_5 ,$$ (58)

$$a_3 - \tfrac{-}{2}a_5 = 8l^2 G_3 c_3 - 56l^2(2L^2 - 3)G_5 c_5 ,$$ (59)

$$a_5 = 32l^4 G_5 c_5 .$$ (60)

Substituting the polynomial representations for γ_{sl}^m (eqs. [45]–[49]) and H_s^m (eqs. [39]–[44]) into G_s, the following identities can be deduced from equations (58)–(60):

$$c_1 = \int_0^{R_\odot} w_1^0(r)K_1(r)r^2 \, dr = 2\sqrt{\frac{\pi}{3}} \left[a_1 + \frac{a_3}{2l}\left(3 - \frac{1}{l}\right) + \frac{a_5}{2l}\left(3 + \frac{7}{2l} - \frac{27}{4l^2} + \frac{9}{4l^3}\right) \right] .$$ (61)

$$c_3 = \int_0^{R_\odot} w_3^0(r)K_3(r)r^2 \, dr = -\sqrt{\frac{\pi}{7}}\,\frac{(2l - 1)(2l + 3)}{3l^2}\left[a_3 + \frac{a_5}{2l}\left(7 - \frac{21}{2l}\right) \right] .$$ (62)

$$c_5 = \int_0^{R_\odot} w_5^0(r)K_5(r)r^2 \, dr = \sqrt{\frac{\pi}{11}}\,\frac{(2l - 3)(2l - 1)(2l + 3)(2l + 5)}{15l^4}\,[a_5] .$$ (63)

Equations (61)–(63) constitute three independent inverse problems for the radial functions $w_1^0(r)$, $w_3^0(r)$, and $w_5^0(r)$. Equation (63) follows immediately from equation (60). Equation (62) was obtained by substituting equation (63) into equation (59) and solving for c_3. Equation (61) was obtained by substituting equations (63) and (62) into equation (58) and solving for c_1. For clarity, it should be pointed out that the number of terms on the right-hand sides of equations (61)–(63) depends on the accuracy with which the split frequencies are estimated. As frequency estimates become more accurate, the number of terms will increase.

This separation into three independent inverse problems, each uniquely identified with a single harmonic degree s of differential rotation, has been made possible by use of the vector spherical harmonic basis functions given by equations (6) and (12). The corresponding inverse problems for $\hat{\Omega}_k$ and Ω_k do not separate so nicely. The advantage of using the Clebsch-Gordon coefficients as basis functions for splitting is readily apparent by comparing equation (56) with equations (61)–(63). The use of the Clebsch-Gordon coefficients reduces the linear combination on the right-hand side of equations (61)–(63) to a single term in equation (56).

It is beyond the scope of this paper to discuss the large and well-known variety of approaches to linear inverse problems. The reader is directed to the following papers which discuss approaches to these problems in some detail: Backus & Gilbert (1967, 1968, 1970); Parker (1977); Christensen-Dalsgaard et al. (1990). Once $w_1^0(r)$, $w_3^0(r)$, and $w_5^0(r)$ have been estimated from equation (56) or from equations (61)–(63) by whatever inverse method has been chosen, $\hat{\Omega}_k$ and Ω_k for $k = (0, 2, 4)$ can be computed a posteriori, if desired, by equations (16)–(18):

$$\Omega_0(r) = \frac{1}{\sqrt{4\pi r}} \left[\sqrt{3}w_1^0(r) + \sqrt{7}w_3^0(r) + \sqrt{11}w_5^0(r) \right] ,$$ (64)

$$\Omega_2(r) = \frac{5}{\sqrt{4\pi r}} \left[\sqrt{7}w_3^0(r) + \sqrt{11}w_5^0(r) \right] ,$$ (65)

$$\Omega_4(r) = \frac{9}{\sqrt{4\pi r}} \left[\sqrt{11}w_5^0(r) \right] ,$$ (66)

$$\bar{\Omega}_0(r) = \frac{1}{\sqrt{4\pi r}} \left[\sqrt{3}w_1^0(r) - \tfrac{1}{2}\sqrt{7}w_3^0(r) + \tfrac{15}{8}\sqrt{11}w_5^0(r) \right] ,$$ (67)

$$\bar{\Omega}_2(r) = \frac{15}{2\sqrt{4\pi r}} \left[\sqrt{7}w_3^0(r) - \tfrac{-}{2}\sqrt{11}w_5^0 \right] ,$$ (68)

$$\bar{\Omega}_4(r) = \frac{315}{8\sqrt{4\pi r}} \left[\sqrt{11}w_5^0(r) \right] .$$ (69)

From equations (61)–(63), it can be easily seen that in the limit of large l, the expansion coefficients $w_1^0(r)$, $w_3^0(r)$, and $w_5^0(r)$ depend only on a_1, a_3, and a_5, respectively. Thus, in the large l limit, these equations can be rewritten:

$$c_1 \approx 2\left(\frac{\pi}{3}\right)^{1/2} a_1 \,. \tag{70}$$

$$c_3 \approx -\frac{4}{3}\left(\frac{\pi}{7}\right)^{1/2} a_3 \,. \tag{71}$$

$$c_5 \approx \frac{16}{15}\left(\frac{\pi}{11}\right)^{1/2} a_5 \,. \tag{72}$$

6. SUMMARY AND CONCLUSIONS

At its inception, the motivation for this paper was that there did not appear to have emerged a fully coherent, unified treatment of the forward and inverse problems for differential rotation. As a result a number of practical problems have beset inversions for differential rotation. In this paper, we present a general formulation of both the forward and inverse problems for differential rotation as well as specific formulae useful to observers with the intent of simplifying solar differential rotation inversion. There are two main points of the paper. (1) If differential rotation is represented by rotational velocity, defined as the zonal, odd degree part of the vector spherical harmonic decomposition of a general convective field in the solar interior, rather than the commonly used rotation rate represented with ad hoc basis functions, then several significant problems currently facing inversions for differential rotation disappear. In particular, the inverse problems for different degrees of rotational velocity are linear and decouple, so that independent inversions for each degree of structure can be performed without approximation. (2) The inverse problems are significantly simplified further if Clebsch-Gordon coefficients are used as basis functions to represent splitting. The Clebsch-Gordon basis functions are genuinely orthonormal on the discrete interval $-l \le m \le l$ and regression matrices comprising them are optimally well-conditioned. As a consequence, we highly recommend that the vector spherical harmonic representation of rotational velocity and the Clebsch-Gordon coefficient representation of splitting be adopted to replace the ad hoc representations employed heretofore. We have presented formulae relating the interaction coefficients c_s to both the new splitting coefficients b_s as well as to the traditional splitting coefficients a_s.

Geophysical experience has shown that it is useful to tabulate splitting data in terms of the splitting function or interaction coefficients c_s since these coefficients are linearly related to the structures producing the splitting. However, in the Sun, there are a number of different kinds of axisymmetric asphericities that could produce splitting. In addition to differential rotation, phenomena which have been discussed as potentially large enough to affect p-mode frequencies measurably include lateral density variations caused by large-scale temperature variations, asphericities in the figure of the Sun, and large-scale, dominantly quadrupolar, magnetic fields. Each of these mechanisms affects splitting differently. However, with the possible exception of the poloidal component of magnetic fields (D. Gough, 1990, personal communication), in each case the Wigner-Eckart theorem (Edmonds 1960) guarantees that the solution to the forward problem can be written in a form identical to that of equation (32), but with the γ_{sl}^m coefficients differing in detail for each mechanism of splitting. Thus, an inspection of equation (33) reveals that the interaction coefficients c_s should not be tabulated since they differ in detail among the various sources of splitting. However, the γ_{sl}^m coefficients for the various sources of splitting are similar in that each is linearly related to the same Clebsch-Gordon coefficient. Consequently, we recommend tabulating the new splitting coefficients b_s estimated relative to the Clebsch-Gordon coefficient representation of splitting. For each of the sources of splitting mentioned here, these coefficients are linearly and simply related to the appropriate basis functions representing that mechanism. (For differential rotation we have shown that the appropriate basis functions are given by eq. [12].) Of course, if multiple sources are causing splitting, then models of the various mechanisms would have to be estimated simultaneously.

Throughout the paper, we have attempted to present formulae that would prove to be easy to use. In conclusion, it is worth presenting a brief review of the most important of these. The differential rotation basis function v_{rot} is defined in equations (8) and (12). The general solution to the forward problem is given by equation (32) and the equations immediately following. The specific form of the forward problem for $s \le 5$ is presented in equations (45)–(49) where polynomials in m, l, and s replace the Wigner 3-j symbols of the general solution. [Formulae useful for calculating the solution to the forward problem analytically for higher degrees ($6 \le s \le 11$) are provided in the Appendix.] Equation (34) is the basis for the inverse problem with respect to the $_nc_{sl}$ coefficients. The orthonormal basis functions (Clebsch-Gordon coefficients) to represent splitting are defined in equation (54), are tabulated in the Appendix, and the inverse problem with these coefficients is given by equation (56). Relative to the $_na_{sl}$ coefficients, the inverse problems are given by equations (61)–(63). Once v_{rot} has been estimated, the rotation rate can be computed, if desired, with equations (64)–(69).

APPENDIX

We have argued in this paper for the following representation of splitting data:

$$\omega_{nl}^m = \omega_{nl} + \sum_{s=1}^{M} {}_n b_{ls} \beta_{sl}^m , \tag{A1}$$

where ${}_n b_{ls}$ are the new splitting coefficients and β_{sl}^m are Clebsch-Gordon coefficients defined as

$$\beta_{sl}^m = (2s+1)^{1/2} F_s H_s^m , \tag{A2}$$

$$F_s = \left[\frac{(2l-s)!}{(2l+s+1)!} \right]^{1/2} . \tag{A3}$$

Expressions for H_s^m for $1 \le s \le 5$ are given by equations (40)–(44) in the body of the text and for $6 \le s \le 11$ below by equations (A4)–(A9). We note that with the H_s^m coefficients provided below, it is straightforward to calculate the γ_{sl}^m coefficients for $a > 5$ by use of equation (33).

The H_s^m coefficients for $6 \le s \le 11$ are given by

$$H_6^m = 924m^6 - 420m^4(3L^2 - 7) + 84m^2(5L^4 - 25L^2 + 14) - 20L^2(L^4 - 8L^2 + 12) , \tag{A4}$$

$$H_7^m = 3432m^7 - 1848m^5(3L^2 - 10) + 168m^3(15L^4 - 105L^2 + 101) - 8m(35L^6 - 385L^4 + 882L^2 - 180) , \tag{A5}$$

$$H_8^m = 12870m^8 - 12012m^6(2L^2 - 9) + 2310m^4(6L^4 - 56L^2 + 81)$$
$$- 12m^2(210L^6 - 3045L^4 + 9898L^2 - 4566) + 70L^2(L^6 - 20L^4 + 108L^2 - 144) . \tag{A6}$$

$$H_9^m = 48620m^9 - 17160m^7(6L^2 - 35) + 12012m^5(6L^4 - 72L^2 + 145) - 440m^3$$
$$\times (42L^6 - 777L^4 + 3402L^2 - 2630) + 12m(105L^8 - 2660L^6 + 18844L^4 - 36528L^2 + 6720) , \tag{A7}$$

$$H_{10}^m = 184756m^{10} - 145860m^8(3L^2 - 22) + 12012m^6(30L^4 - 450L^2 + 1199) - 2860m^4$$
$$\times (42L^6 - 966L^4 + 5481L^2 - 6248) + 132m^2(105L^8 - 3290L^6 + 29680L^4 - 78900L^2 + 32208)$$
$$- 252L^2(L^8 - 40L^6 + 508L^4 - 2304L^2 + 2880) , \tag{A8}$$

$$H_{11}^m = 705432m^{11} - 1847560m^9(L^2 - 9) + 58344m^7(30L^2 - 550L^2 + 1869) - 120120m^5$$
$$\times (6L^6 - 168L^4 + 1199L^2 - 1873) + 1144m^3(105L^8 - 3990L^6 + 44730L^4 - 156200L^2 + 105228)$$
$$- 24m(231L^{10} - 11165L^8 + 174328L^6 - 1006764L^4 + 1771440L^2 - 302400) . \tag{A9}$$

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THE EFFECT OF GLOBAL-SCALE, STEADY-STATE CONVECTION AND ELASTIC-GRAVITATIONAL ASPHERICITIES ON HELIOSEISMIC OSCILLATIONS

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ABSTRACT

In this paper we derive a theory, based on quasi-degenerate perturbation theory, that governs the effect of global-scale, steady-state convection and associated static asphericities in the elastic-gravitational variables (adiabatic bulk modulus κ, density ρ, and gravitational potential ϕ) on helioseismic eigenfrequencies and eigenfunctions and present a formalism with which this theory can be applied computationally. Eigenfunctions are simply seismic displacement patterns. The theory rests on three formal assumptions: (1) that convection is temporally steady in a frame corotating with the Sun, (2) that accurate eigenfrequencies and eigenfunctions can be determined by retaining terms in the seismically perturbed equations of motion only to first order in p-mode displacement, and (3) that we are justified in retaining terms only to first-order in convective velocity as well (this assumption is tantamount to the requirement that the convective flow is anelastic). The most physically unrealistic assumption is (1), and we view the results of this paper as the first steps toward a more general theory governing the seismic effects of time-varying fields. Although the theory does not govern the seismic effects of nonstationary flows, it can be used to approximate the effects of unsteady flows on the seismic wavefield if the flow is varying smoothly in time. The theory does not attempt to model seismic modal amplitudes since these are governed, in part, by the exchange of energy between convection and acoustic motions which is not a part of this theory.

The basic reference model that will be perturbed by rotation, convection, structural asphericities, and seismic oscillations is a spherically symmetric, non-rotating, non-magnetic, isotropic, static solar model that, when subject to seismic oscillations, oscillates adiabatically. We call this the SNRNMAIS model. An acoustic mode of the SNRNMAIS model is denoted by $k = (n, l, m)$, where n is the radial order, l is the harmonic degree, and m is the azimuthal order of the mode.

The main result of the paper is the general matrix element $H^{m'm}_{n'l'nl}$ for steady-state convection satisfying the anelastic condition with static structural asphericities. It is written in terms of the radial, scalar eigenfunctions of the SNRNMAIS model, resulting in equations (92)-(112). We prove Rayleigh's principle in our derivation of quasi-degenerate perturbation theory, which, as a by-product yields the the general matrix element. Within this perturbative method, modes need not be exactly degenerate in the SNRNMAIS model in order to couple, only nearly so. General matrix elements compose the Hermitian supermatrix $Z_{k'k}$. The eigenvalues of the supermatrix are the eigenfrequency perturbations of the convecting, aspherical model and the eigenvector components of $Z_{k'k}$ are the expansion coefficients in the linear combination forming the perturbed eigenfunctions in which the eigenfunctions of the SNRNMAIS model act as basis functions. We show how helioseismic synthetic seismograms can be computed using the supermatrix.

The properties of the Wigner 3-j symbols and the reduced matrix elements composing $H^{m'm}_{n'l'nl}$ produce selection rules governing the coupling of SNRNMAIS modes that hold even for time-varying flows. We state selection rules for both quasi-degenerate and degenerate perturbations theories. For example, within degenerate perturbation theory, only odd-degree s toroidal flows and even degree structural asphericities, both with $s \leq 2l$, will affect the splitting and coupling of acoustic modes with harmonic degree l. In addition, the frequency perturbations caused by a toroidal flow display odd symmetry with respect to the degenerate frequency when plotted from the minimum to the maximum perturbation.

We consider the special case of differential rotation, the odd-degree, axisymmetric, toroidal component of general convection, and present the general matrix element and selection rules under quasi-degenerate perturbation theory. We argue that due to the spacing of modes that satisfy the selection rules, quasi-degenerate coupling can, for all practical purposes, be neglected in modeling the effect of low-degree differential rotation on helioseismic data. In effect, modes that can couple through differential rotation are too far separated in frequency to couple strongly. This

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s not the case when we consider nonaxisymmetric flows and asphericities. In this case. near degeneracies will regularly occur, and couplings can be relatively strong, especially among SNRNMAIS modes within the same multiplet.

All derivations are performed and all solutions are presented in the corotating frame. Equation (18) shows how to transform the eigenfrequencies and eigenfunctions in the corotating frame into an inertial frame. The transformation has the effect that each eigenfunction in the inertial frame is itself time-varying. That is, a mode of oscillation, which is defined to have a single frequency in the corotating frame, becomes multiply periodic in the inertial frame.

I. INTRODUCTION

Helioseismic images of the acoustic velocity field of the Sun are providing new and unique information about solar structure and dynamics. To continue to utilize effectively the information provided by the continually improving data sets will require a thorough understanding of the way in which a number of solar structures and processes affect helioseismic data. It is upon such an understanding of these forward problems that any future inversions will rest.

We consider here the helioseismic effect of one such solar process: convection. In particular, the purpose of this paper is to present a theory that governs the effect of large-scale, steady-state convection, with associated asphericities in the structural elastic-gravitational variables (adiabatic bulk modulus κ, density ρ, and gravitational potential ϕ), on helioseismic oscillations. Many studies have been completed concerning differential rotation, the long-wavelength axisymmetric component of convection (e.g., Duvall and Harvey 1984: Brown 1985: Duvall et al. 1986: Libbrecht 1986. 1989: Brown and Morrow 1987: Rhodes et al. 1990: Brown et al. 1989: Thompson 1990: and Ritzwoller and Lavely 1991). However, to date. studies of the seismic effects of non-axisymmetric convection are rather sparse. In an asymptotic treatment. Gough and Toomre (1983) calculated the frequency shift of an acoustic mode due to advection by a purely horizontal flow. Brown (1984) calculated the influence of turbulent convection on modal degenerate frequencies. The scattering of sound by an isolated, steady laminar compact vortex was considered by Bogdan (1989). Hill (1983, 1988, 1989) has used a ray-theoretic method and has attempted to infer horizontal convective velocities near the solar surface using helioseismic data. All of these studies make restrictive assumptions about the geometry of the flow field including either that the flows are horizontal in a plane-parallel medium or demonstrate cylindrical symmetry, and none attempts to model wavefront distortion and deflection caused by convection. In summary, to the best of our knowledge no general theory for the effect of convection on helioseismic oscillations currently exists.

The theory presented in this paper differs from these previous studies in the following ways. (1) Our theory is nonasymptotic. In principle, the results are accurate for all wavelengths and frequencies of helioseismic oscillation. (2) It is derived within a spherical geometry. Previous investigations that modeled convective effects within a nonspherical geometry are appropriate for short-wavelength convection but inappropriate for the largest scales of convection which are the subject of this paper. (3) The theory presented here makes no assumptions about the geometry of the flow. We represent general non-axisymmetric flow fields comprising both poloidal and toroidal components in terms of vector spherical harmonics, which are complete basis functions for a vector field in a sphere. (4) Our approach is modal-theoretic rather than ray-theoretic. From a traveling wave perspective this means that both wavefront deformation as well as the perturbation in local sound speed by convection are modelled. In modal-theoretic language. convection results in modal coupling as well as splitting.

In a later paper we will implement the theory presented in this paper using a numerical simulation of large-scale convection and discuss the observational consequences of the theory. In particular, we will show that the helioseismic frequencies. displacement patterns. and line-widths of an aspherical solar model are appreciably altered relative to the corresponding quantities calculated from a model with differential rotation alone.

a) Modal Notation and Terminology

The basic reference model to which all subsequent structural perturbations and processes will be added is a solar model that is spherically symmetric, nonrotating, nonmagnetic, isotropic, and static, subject to adiabatic acoustic oscillations. We refer to this as the SNRNMAIS solar model. An acoustic mode of oscillation of any solar model is defined to be a characteristic spatial displacement pattern that oscillates with a single frequency.

An acoustic mode of a SNRNMAIS model is uniquely identified by a single triple of quantum numbers (n,l,m) that denote. respectively, the radial order, harmonic degree. and azimuthal order of the mode. A modal frequency for such a model is simply the degenerate frequency of the multiplet $_nS_l$ that comprises the $(2l+1)$ modes with identical n and l values. Any symmetry-breaking agent such as rotation, magnetic fields. or convection will lift this $(2l+1)$ degeneracy and split the frequencies of the modes composing the multiplet. We call any model with such a symmetric-breaking perturbation a non-SNRNMAIS model. A major goal of this paper is to provide formulae with

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which to calculate the modal eigenfunctions and eigenfrequencies of both the SNRNMAIS and non-SNRNMAIS solar models. As we shall later see, the perturbations of the non-SNRNMAIS solar model will be assumed to have small magnitude and be stationary in a frame corotating with the Sun. If the symmetry-breaking agent is axisymmetric, as is differential rotation, then to a good approximation the spatial structure of each mode will remain specified by the same triplet of quantum numbers. For a general, non-axisymmetric perturbation such as the convective fields we consider here, the perturbed eigenfunction (or spatial displacement pattern) of each mode is a linear combination of the eigenfunctions of the SNRNMAIS model. We call this phenomenon oscillation-oscillation coupling or interaction to distinguish it from oscillation-convection coupling, the exchange of energy between seismic and convective modes. The acoustic modes that are said to couple as a result of a convective flow or an asphericity in the elastic-gravitational variables are SNRNMAIS modes. The modes of the non-SNRNMAIS solar model do not couple.

Two modes of the SNRNMAIS solar model whose spatial eigenfunctions are orthogonal are said to be isolated from one another. Two or more modes that are not isolated from one another can couple when the reference model is perturbed either by a structural perturbation or a convective flow. A multiplet composed of modes whose combined eigenspace is orthogonal to the combined eigenspace of the modes composing all other multiplets is said to be isolated or self-coupled. The degree of coupling between SNRNMAIS modes is a function of a number of factors, among which are the strength of the asphericity or convective field producing the coupling, the proximity of the eigenfrequencies of the modes, the relation between the geometries of the perturbing field and the oscillations which is encoded in a set of analytical angular selection rules, and the similarity of the radial eigenfunctions of the two modes. When two SNRNMAIS modes $k = (n, l, m)$ and $k' = (n', l', m')$ couple, the strength of interaction is described by the general matrix element $H_{n'l'm'}^{nlm}$. The matrix $H_{n'n''}$ composed of all the general matrix elements for the multiplets $_nS_l$ and $_{n'}S_{l'}$ is of dimension $(2l' + 1) \times (2l + 1)$ and is called the general matrix. The square general matrix $H_{nn,ll}$ is called the splitting matrix and governs self-coupling. The eigenfrequencies of non-isolated modes that couple within or across n or l are the eigenvalues of an assemblage of block diagonal splitting matrices and off-block diagonal general matrices. The entire assemblage is called the supermatrix $Z_{k'k}$.

Since the seismic modes of the SNRNMAIS solar model are spheroidal, their spatial vector eigenfunctions $s_k(\mathbf{r})$ (or displacement patterns) may be written in the form

$$s_k(\mathbf{r}) = {}_nU_l(r)Y_l^m(\theta, \phi)\hat{\mathbf{r}} + {}_nV_l(r)\nabla_1 Y_l^m(\theta, \phi) \tag{1}$$

where $_nU_l(r)$ and $_nV_l(r)$ are the scalar radial eigenfunctions for harmonic degree l and radial order n. With the gravitational potential scalar eigenfunction, $_nO_l(r)$, and its radial derivative, $_nU_l(r)$ and $_nV_l(r)$ and their radial derivatives form the set of scalar radial eigenfunctions. The coordinates (r, θ, ϕ) are spherical polar coordinates (where θ is colatitude) and $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$ denote unit vectors in the coordinate directions. The surface gradient operator is given by

$$\nabla_1 = r(\nabla - \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \nabla)). \tag{2}$$

The function Y_l^m is a spherical harmonic of degree l and azimuthal order m defined using the convention of Edmonds (1960):

$$\int_0^{2\pi} \int_0^\pi [Y_{l'}^{m'}(\theta, \phi)]^* Y_l^m(\theta, \phi) \sin\theta \, d\theta \, d\phi = \delta_{m'm}\delta_{l'l} \tag{3}$$

where integration is over the unit sphere. Henceforth, we drop the subscripts n and l in equation (1) and use instead $U =_n U_l(r)$, $U' =_{n'} U_{l'}(r)$ and so on. The SNRNMAIS spatial vector eigenfunctions satisfy an orthogonality condition given by

$$\int \rho_0 s_{k'}^* \cdot s_k d^3\mathbf{r} = N\delta_{m'm}\delta_{n'n}\delta_{l'l} \tag{4}$$

where

$$N = \int_0^{R_\odot} \rho_0[UU' + l(l+1)VV']r^2 dr, \tag{5}$$

and $d^3\mathbf{r} = r^2 \sin\theta \, d\theta \, d\phi \, dr$. Henceforth, an integral sign without limits, as in equation (4), will denote a three-dimensional integration over the volume of the solar model. The scalar normalization constant N depends on the normalization convention of the eigenfunctions U and V.

Perturbation theoretic techniques are usually employed to calculate split acoustic mode eigenfrequencies and perturbed eigenfunctions. In this paper, we will show how quasi-degenerate perturbation theory can be applied to determine these quantities for a convecting solar model with associated aspherical perturbations in the structural

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elastic-gravitational parameters density ρ and adiabatic bulk modulus κ. In particular, we will use the eigenfunctions of the SNRNMAIS model as basis functions to represent the perturbed eigenfunctions $\vec{s}_j(r,t)$:

$$\vec{s}_j(r,t) = \left[\sum_{\kappa \in K} a^j_\kappa s_\kappa(r) \right] e^{i\omega_j t} \tag{6}$$

We will show how to determine the appropriate eigenspace K required to represent the perturbed eigenfunction, will derive the expansion coefficient a^j_κ for each component of the eigenspace to calculate the perturbed eigenfunction, and will derive expressions for the split eigenfrequency of the mode $\omega_j = \omega_{ref} + \delta\omega_j$.

The major theoretical result of the paper is analytical expressions for the general matrix elements that compose the supermatrix (or splitting matrix in the case of self-coupling). The perturbed modal frequency $\delta\omega_j$ is simply an eigenvalue of the supermatrix (or splitting matrix), and the expansion coefficients are simply the eigenvector components a^j_κ. We wish to emphasize at this point that we will not attempt to present a theory that accurately predicts modal amplitudes, but only modal eigenfrequencies and perturbed eigenfunctions (or displacement patterns). The formal assumptions of the theory discussed in §I.b will reflect this point.

b) Assumptions and their Implications

Although the theory we present in this paper is more general than previous work, its application is restricted both by practical considerations and by the set of assumptions upon which it is formally based. The major practical limitation is that the convective structures considered should be global in extent. For example, although it is theoretically possible to represent a single small-scale convective vortex in terms of vector spherical harmonics, there are better representations and doing so would probably be a misuse of this theory. Thus, though the theory holds for all but very short wavelength, turbulent convection, it will be most usefully applied to long wavelength flows. There is a caveat: spatially repetitive small-scale structures, such as the solar granulation, can be well represented by vector spherical harmonics and are not beyond the practical limitations of this theory. Another practical limitation is that the theory is not particularly useful for g-modes since they have very small amplitudes in the convection zone. Therefore, though the theory governs the effect of convection on g-modes our discussion will center on p-modes.

Much more restrictive are the following set of formal assumptions. (1) The convection is steady in time. As we will discuss in §II, this assumption is necessary for the perturbed equations of motion to separate. The asphericities in the structural elastic-gravitational variables will also be assumed to be time invariant. (2) We retain terms in the seismically perturbed equations of motion only to first-order in p-mode displacement and quantities that depend on it. Thus, we derive and use linearized equations of motion. (3) We also retain terms in the seismically perturbed equations of motion only to first-order in convective velocity. This is done so that acoustic oscillations and convection do not exchange energy and to this extent can be considered independently. This is tantamount to the requirement that the convective flow field is anelastic. We will discuss briefly the implications of each of these assumptions in turn. Arguments are presented to justify assumptions (2) and (3) in §I.e.

(1) If convection is steady in time, each identically directed acoustic wave that propagates through a given region will experience the same convective effect. In particular, multiply orbiting waves propagating along near great-circles will experience a constructively accumulating effect in that region. In this case, the split modal frequency associated with the propagating wave will be time invariant. If the convective state changes appreciably during the time it takes an acoustic wave to execute a single orbit, then the convective effect will vary between orbits. In fact, the effect may destructively accumulate. Consequently, modal frequencies would be time varying, leading to an effective spectral line-broadening. This line-broadening is not a part of the theory presented in this paper and the seismic effect of aspects of convection that are rapidly evolving in time cannot be determined from the results presented here. Of particular significance is the fact that the effect of the shearing of sectoral or banana cell modes of convection by differential rotation cannot be modeled within this theory. Rather, the results in this paper represent the first steps toward constructing a more general theory that governs time-varying fields.

Although the results in this paper are correctly applied only to steady-state convection, they may be most useful if seen to provide instantaneous frequencies and displacement patterns for a time-varying convective field. These instantaneous frequencies would be accurate over the lifetime of the convection cell which, for long-lived modes of convection, may be appreciable. In this case, the steady-state assumption would amount to a short-time approximation. For example, since the shearing of convective patterns takes time to develop, the results presented here are applicable until the shearing effects accumulate. The numerical simulations of Glatzmaier and Gilman (1981, 1982) show that some components of flow have lifetimes on the order of weeks. Furthermore, there are certain observable solar features,

in particular active longitudes and coronal holes. that appear to evolve relatively unsheared by differential rotation. If these features are somehow anchored at depth in convective structures. then their existence is further evidence for a relatively stable component of flow deep in the convection zone. Evidence for the existence of solar giant cells is discussed in §I.d.)

From the view of the steady-state assumption as a short-time approximation. it is straightforward to implement a numerical formalism to approximate the time-varying seismic wavefield if we asume that the variations in convection are temporally smooth. We would calculate a time sequence of instantaneously valid eigenfrequencies and eigenvectors in a coarse set of time knots where at each knot the flow field is assumed to be stationary. We would then interpolate the eigenfrequencies and eigenvectors onto a finer time grid and allow the wavefield to evolve continuously through each of the intervals between the knots.

(2) Neglecting higher order terms than first in the seismically perturbed quantities amounts to neglecting seismic self-advective effects evidenced through acoustic three-mode coupling. In particular. the self-advection of the displacement field is neglected which is tantamount to assuming that the total seismic displacement in a region is much smaller than the displacements produced by convection during the passage of a wave. (We also neglect all source terms such as entropy and internal energy fluctuations caused by the oscillations.) As we will discuss in §§I.c and I.e. the accuracy of this assumption improves with depth. The application of the theory will be most accurate for seismic paths below the strongly super-adiabatic layer near the solar surface where turbulence is most vigorous.

3) We neglect all terms second-order in the convective velocity. There are two main types of second-order terms that we discard. advective terms and Reynold's stresses. Discarding the former amounts to assuming that convective velocities are relatively small. Ignoring Reynold's stresses. which are proportional to the Laplacian of the convective velocity. requires that convective wavelengths be relatively large. and implies that convection-oscillation coupling is neglected so that there is no mechanism by which convection and the acoustic oscillations can exchange energy. In particular. we assume that convective flows do not generate seismic waves and. therefore. we require that the flows satisfy the unperturbed continuity equation commonly called the anelastic condition. This condition eliminates potential sources. sinks. and cavitation in the flow field. Thus. we view convection as a sort of passive background on which acoustic oscillations are superposed. It deforms seismic wavefronts and perturbs local sound speeds, but does not exchange energy with acoustic waves. The assumption that second-order terms in convective velocity and the Reynold's stresses can be ignored is poor near the surface but. as with formal assumption (2). improves with depth below the photosphere.

In summary, the implications of these assumptions are that the convective fields to which the theory is applicable should be global in extent. relatively long wavelength, steady in time or at least relatively long-lived. and well below the photosphere. Giant-cell convection satisfies these criteria and provides the best target for the application of the theory presented herein. In the remainder of this section we will discuss solar convection. review the evidence for the existence of giant-cell convection. and attempt to justify the use of linearized equations of motion to determine the seismic effect of giant-cells.

c) Solar Convection and its Seismic Effects

Observation of the distinct cellular motions of granules and supergranules suggests that there are preferred scales of motion for thermal convection. The common picture of convection is that the Sun contains a multiplicity of scales of motion ranging from the Kolmogoroff microscales at the short end to differential rotation which is global in extent. At intermediate length scales. convective modes are thought to be organized into granules. supergranules. giant cells. and energy-bearing eddies. Temporal scales also range from a few minutes for granule overturn times to weeks for the largest scale of giant cells deep in the convection zone. Goldreich and Kumar (1988) present a recent review of turbulence. Bray et al. (1984) and Gilman (1987) provide overviews of the physics and morphology of granules. supergranules, and giant cells.

In order to discuss qualitatively the likely general characteristics of convection below the photosphere, we look to mixing-length theory for guidance. In the mixing-length picture of convection one would take the mixing length, Mach number, and the time and velocity scales of convection to be given, respectively, by

$$H \sim \alpha H_p \tag{7}$$

$$M \sim \left[\frac{g F_z H Q \rho^{1/2}}{4\kappa^{3/2} c_p T} \right]^{1/3} \tag{8}$$

$$v_H \sim c M \tag{9}$$

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where $H_\omega = P/(\rho g)$ is the pressure scale height, α is the ratio of the mixing length to H_ω, P is pressure, T is temperature, g is the gravity, c_p is the specific heat at constant pressure, and c is the sound speed. We have set $\gamma = (1 - \beta)3\beta^{-1}$ where β is the ratio of the gas pressure to the total pressure. The convective flux can be calculated using

$$F_c = \frac{L_T}{4\pi r^2}\left[\frac{\nabla_* - \nabla}{\nabla_r}\right]$$
(11)

where $L_\odot$ is the solar luminosity. We have used equation (14.58) of Cox and Giuli (1968) to obtain equation (8). Figure 1 is a plot of the characteristic length, velocity, and time scales of convection predicted by equations (7), (9), and (10) using the solar model of Podsiadlowski (1989) with α taken to be 1.305. The predicted time and velocity scales near the surface correspond well with observations of solar granulation.

Convection at all depths in the convection zone will affect helioseismic oscillations. The solar p-modes have scales that range in size from the smallest to the largest of the convective motions and the dominant modal frequencies coincide with the characteristic overturn times for convective motions near the solar surface. Since the energy and the characteristic length and time scales of convection vary with depth, the physics of interaction between acoustic modes and convection will necessarily also vary. For example, granule and sub-granule scale motions are thought to be the source of the acoustic oscillations (Goldreich and Kumar 1988). To model the total effect of convection at this scale on the acoustic oscillations would be very difficult as it would involve modeling convection-oscillation coupling in addition to oscillation-oscillation coupling. As the formal assumptions indicate, we have set for ourselves a simpler task: to model the effect of deeper, long-wavelength convection such as giant cells that, we argue in §I.e, exchange very little energy with acoustic oscillations.

Though, as Figure 1 shows, it is likely that the characteristic temporal and spatial scales of convection vary continuously across the convection zone, convective processes can be thought to be segregated into two concentric shells (an outer shell and an inner shell), with convection in each shell being dominated by distinct processes. In §I.e we attempt to quantify the extent of each shell; here we discuss qualitatively the characteristics of the convection and its likely seismic effects in each shell.

The outer shell occupies the top few scale heights where the acoustic and convective physics are most complex. Convection is most vigorous in this shell, being highly turbulent and with relatively short characteristic convective lifetimes and length scales. In this shell, the convective velocity is an appreciable fraction of the local sound speed ($M \simeq 0.3$), the time scales of the turbulence and of the acoustic radiation are commensurable, and the amplitudes of the p-modes and the convective waves are largest. Goldreich and Keely (1977a, b) and Goldreich and Kumar (1988, 1989) calculated the amplitudes and energies of the p-modes under the assumption they are excited by stochastic turbulent convection. Results of Goldreich and Kumar (1988) show that seismic wave emission and absorption in the Sun principally take place through interaction with turbulence in the top few scale heights of the convection zone. We define the radial extent of the outer shell as the region of significant interaction between the p-modes and convection. We argue in §I.e that this region also marks the extent of significant three-mode coupling and attempt to approximate its depth extent as well.

In the outer shell, convective cells evolve rapidly (Stein and Nordlund 1989; Title et al. 1989). If, in addition, cells are distributed isotropically in space, then they will produce little accumulated splitting effect on globally propagating waves. There will be local acoustic effects, but the isotropic assumption guarantees that the net global effect on frequency will be small. However, acoustic modal amplitudes, line-widths, and degenerate frequencies will be affected by outer shell processes (e.g. Brown 1984; Christensen-Dalsgaard and Fransden 1983; Christensen-Dalsgaard et al. 1989) such as convection-oscillation coupling, three-mode coupling (Kumar and Goldreich 1989), and radiative damping.

The inner shell is much larger than the outer shell and lies directly beneath it, occupying, as we argue below, more than $\sim$ 99.8% in radius of the convection zone. By definition, the emission and absorption of seismic waves by turbulence in this shell is negligible, and convection-oscillation coupling can be ignored accurately. Consequently, the anelastic condition can be applied. Furthermore, p-mode amplitudes are much smaller than in the outer shell and the solar gas in this shell is optically thick so radiative damping is negligible. Thus, the contribution to the interaction coefficient describing three-mode coupling in the inner shell is relatively small (Kumar and Goldreich 1989). Therefore, we argue that splitting and the global distortion of seismic wavefronts dominantly result from convection that is relatively coherent temporally and spatially. If long-lived, long wavelength features of convection do exist, they would possess characteristic signatures in p-mode frequencies, line-widths, and displacement patterns

that could be computed from the theory presented herein. In principle, once we have identified these signatures, their observation would place constraints on the causative convective structures. For example, we show in a future contribution that because of the way helioseismic images are reduced and analyzed, the effect of aspherical structure is to broaden line-widths. This effect can be significant for modes with low intrinsic damping rates. The value of helioseismological constraints such as these would be enhanced by the fact that large-scale convection has been linked with the dynamic structure of the differential rotation (e.g. Gilman 1987) and with the solar dynamo (e.g. Stix 1981). In addition, magnetic activity observed at the solar surface probably is controlled by its subsurface expression that is intimately involved with flows at depth.

Next, we address two questions in §§I.d and I.e. (1) What is the evidence that large-scale convection exists in the inner shell? (2) What is the extent of the outer shell where we do not accurately model the convective effect of convection?

d) On the Existence of Giant-Cells

A problem for the utility of the theory presented here is that giant cells have not been unambiguously observed at the solar surface. If they do exist at the surface of the Sun, their amplitudes are less than 10 ms^{-1} (Howard and Labonte 1980; Labonte et al. 1981; Brown and Gilman 1984). Nevertheless, the evidence for their existence is strong, though circumstantial. (1) First, the Sun displays a number of features that are suggestive of sustained large-scale motions (Gilman 1987). These include persistent large-scale patterns in the solar magnetic field, the coronal holes which survive several solar rotation periods without being sheared apart by differential rotation, and the existence of active longitudes where new active regions preferentially arise. (2) Second, the observed distinct cellular convection may continue well below the surface. In the mixing length picture of convection (e.g., Fig. 1), the scale of the convective eddies is set by the pressure scale-height so that one would expect layers of convection with monotonically increasing vertical scale with depth. In addition, both linear and nonlinear models (e.g., Gough et al. 1976) have shown that even when the fluid is compressible, and the stratification includes several scale heights, convection spanning the entire unstable layer is favored. Thus, for the Sun, patterns of motion with horizontal dimensions up to the depth of the convection zone (i.e. $\lambda \sim 200,000$ km or harmonic degrees of $l \sim 20$) would be expected. (3) Third, the space-lab experiment of thermal convection (Hart et al. 1986) and the numerical simulations of Glatzmaier (1984) and Gilman and Miller (1986) have suggested that large and sustained patterns of motion may exist in the Sun with scales approaching the depth of the convection zone. (4) Fourth, Hill (1988) constructed three-dimensional spectra (k_x, k_y, ω) of helioseismic images of small rectangular regions near the solar equator and discovered relatively large-scale horizontal, poleward flows of approximately 100 m/s that may be the surface expression of giant-cells. (5) Finally, a possible explanation of the small vertical velocities of the supergranules and the absence of a strong signature of giant-cells in the data of Howard and Labonte (1980) may be found in the work of Latour et al. (1981) and van Ballegooijen (1986). Latour et al. (1981) found that buoyancy breaking in A-type stars may occur in upward-directed flows that have horizontal scales large compared to the pressure scale height of the region into which they penetrate. This leads to lateral deflection and strong horizontal shearing motions. If this result applies as well to G-type stars such as the Sun, it may provide the explanation for the lack of surface observations of giant-cells. In addition, van Ballegooijen (1986) found that density stratification screens out periodic components of the near surface flow pattern in his convection model so that periodic motions that exist at depth would not be observed at the surface.

e) Justification of Linearization for Application to Giant Cell Convection

We now attempt to quantify the extent of the outer shell, defined to be that region where convection-oscillation coupling is appreciable. The extent of energy exchange between oscillations and turbulent convection depends on their relative time and velocity scales. Perhaps the best available measure of the coupling between convection and acoustic oscillations is the flux of energy F_p pumped into to the acoustic modes from the convective motions. Goldreich and Kumar (1989) derive an expression for F_p given by

$$F_p = M^{\frac{13}{2}} F_c, \tag{12}$$

where F_c is the convective flux given by equation (11). F_p depends primarily on the Reynolds stresses.

An inspection of Figure 2, which plots the radial dependence of F_p, reveals that convection-oscillation coupling is relatively insignificant below the top $\sim .15\%$ of the convection zone. Thus, as a mechanism of oscillation-convection coupling, Reynolds stresses and entropy fluctuations act far more efficiently in the top few scale heights than in the deeper layers where the characteristic velocities are smaller and the length scales are larger.

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Kumar and Goldreich (1989) also discuss the effect of nonlinear interactions among solar acoustic modes. They argue that these interactions are strongest in the outermost layers of the Sun. Indeed, an inspection of their Figure 3 indicates that the coupling coefficients are sensitive to three-mode interactions only in the outer $\sim .2\%$ by radius of the convection zone. Consequently, we infer that three-mode interactions can be ignored in the determination of seismic effects of convective flows below this depth.

In criticism of the linearization in both convective velocity and seismic displacement, it might be suggested on intuitive grounds that a theory governing the effect of convection on seismic waves must be accurate along the entire path of the seismic wave, and since all seismic waves propagate through the outer shell the theory must be general enough to govern outer shell physics. This would certainly be true if we were interested in describing all of the seismic effects of convection. However, as discussed in §I.e, turbulent convection and other nonlinear processes in the outer shell will dominantly affect modal amplitudes and degenerate frequencies. We are only interested here in determining split frequencies and perturbed eigenfunctions (or displacement patterns) of acoustic modes. Consequently, outer shell physics will be subsequently ignored.

In conclusion, we define the outer shell to have a depth of $\sim .2\%$ of the convection zone and we argue that the seismic effect of convection can be modeled accurately with a linearized theory for flows within the the inner $\sim 99.8\%$ of the convection zone by radius.

f) Overview

In §II we discuss reference frames and the separability of the equations of motion, and present a means of transferring the theoretical results presented in this paper from a frame corotating with the Sun to an inertial frame that can be roughly identified as the observer's frame. In §III we present mathematical representations for convection and for the asphericities in the elastic-gravitational variables. The equations of motion governing the seismic oscillations in the presence of a steady-state global-scale velocity field and the associated static structural perturbations to density and bulk modulus are derived in §IV. We derive in §V the quasi-degenerate perturbation theory needed to calculate the influence of a velocity field and structural perturbations on solar oscillations. In §VI, we derive the general matrix elements that determine the displacement field and split frequencies caused by an anelastic model of convection represented with scalar and vector spherical harmonics by using the perturbation operator derived in §IV and the perturbation theory derived in §V. The application of quasi-degenerate perturbation theory to the acoustic modes of the Sun and the derivation of the general matrix elements are presented in §VI. In §VII we discuss properties of the supermatrix. In §VIII, we consider differential rotation. In §IX we show how the the supermatrix may be used to generate theoretical seismograms. The principal conclusions of the paper are summarized in §X.

The system of equations that governs the modal eigenfucntions and eigenfrequencies of the SNRNMAIS solar model is presented in Appendix A. The equation of motion of the non-SNRNMAIS solar model is derived in Appendix B. In Appendix C, we present a mathematical method adapted from Phinney and Burridge (1973) that considerably simplifies the application of differential operators to vector and tensor fields in a spherical geometry which are common in helioseismology. This technique is used to calculate the general matrices presented in §VI. Appendix D contains a discussion of the incorporation of the anelastic condition into the general matrix and a derivation of the matrix elements for the Coriolis force, centripetal acceleration, and for general convection. Appendix E presents detailed expressions for the matrix elements for aspherical perturbations in the elastic-gravitational variables.

[Sections II-IX omitted.]

X. SUMMARY AND CONCLUSIONS

The purpose of this paper has been to derive a theory that governs the effect of steady-state convection and associated asphericities in the elastic-gravitational variables (adiabatic bulk modulus κ, density ρ, and gravitational potential ϕ) on seismic frequencies and displacement patterns and to present a formalism with which this theory can be applied computationally. The theory is not intended to predict modal amplitudes since these are governed, in part, by the exchange of energy between convection and seismic waves, which is excluded by our theory since it is linear and since the convective flow is defined to be anelastic. To the best of our knowledge, every global-scale study of the helioseismic effect of convection or structural asphericities, to date, has assumed an axisymmetric model. We have made no such assumption, and have represented convective flow (a vector field) and structural asphericities (a scalar field) with general global basis functions, vector and scalar spherical harmonics, respectively. We also represent the eigenfunctions

of the spherical reference model (the SNRNMAIS solar model) with vector spherical harmonics. These representations allow us to employ quasi-degenerate perturbation theory in a straightforward manner to derive the general matrix elements $H_{n'l',nl}^{m'm}$ that govern the modal coupling and splitting caused by convection and the structural asphericities. We present formulae for the general matrix elements explicitly in terms of the scalar eigenfunctions of the SNRNMAIS solar model. Thus, the use of this theory requires only the following quantities: (1) a SNRNMAIS solar model ($\kappa(r)$ and $\rho(r)$), (2) the seismic scalar eigenfunctions of the SNRNMAIS solar model ($_nU_l(r)$, $_nU_l(r)$, $_nV_l(r)$, $_nV_l(r)$, $_n\delta o_l(r)$ and $_n\delta o_l(r)$), and (3) the spherical harmonic representation of convection ($u_s^t(r)$, $v_s^t(r)$, and $w_s^t(r)$) and/or asphericities in the elastic-gravitational variables ($\delta\kappa_s^t(r)$ and $\delta\rho_s^t(r)$) at each radial knot of the SNRNMAIS model. The general matrix elements compose the Hermitian supermatrix Z, whose eigenvalues are the eigenfrequency perturbations of the general non-SNRNMAIS solar model and whose eigenvector components are the expansion coefficients in the linear combination forming the perturbed eigenfunctions (or displacement patterns) in which SNRNMAIS eigenfunctions are basis functions.

Optimally, the next stage of this research would be the application of this formalism to a realistic global model of long-wavelength convection with associated asphericities in the elastic-gravitational variables. A major aspect of this effort would be the determination of the accuracy of degenerate perturbation theory relative to quasi-degenerate perturbation theory. We have argued in §VIII that due to the spacing of modes that satisfy the selection rules, quasi-degenerate coupling can, for all practical purposes, be neglected in modeling the effect of differential rotation on helioseismic data. In effect, modes that can couple through differential rotation are too far separated in frequency to couple strongly. This is not the case when we consider nonaxisymmetric flows and asphericities. In this case, near degeneracies will regularly occur, and couplings can be relatively strong especially among SNRNMAIS modes within the same multiplet. However, since solar convection is dominantly axisymmetric, complete hybridization of modes would be rare, and the perturbed mode would retain many of the characteristics of a mode of the SNRNMAIS model. Most importantly, a perturbed eigenfunction, on average, would resemble a slightly perturbed SNRNMAIS eigenfunction: i.e., a single spherical harmonic. However, these perturbations to the eigenfunctions and eigenfrequencies will be systematic and it should prove interesting to investigate the cumulative effect on splitting data. In particular, one could compute the perturbed eigenfrequencies for a given model of giant-cell convection (e.g., Gilman and Glatzmaier 1981; Glatzmaier and Gilman 1981, 1982; Glatzmaier 1984) and then invert for the input differential rotation profile using currently standard methods that assume that nonaxisymmetric components of flow are nonexistent. If the flow model were realistic, one would uncover any bias in the recently estimated differential rotation profiles.

Another use of the theory would be to determine whether general asphericities in the elastic-gravitational variables could appreciably affect helioseismic data. For example, Kuhn et al. (1988) observed a surface temperature variation of several degrees Celsius from the solar south pole to the solar equator and hypothesized that this or a similar structure may be responsible for the non-zero even-degree frequency splitting coefficients. Given an equation of state, these temperature variations could be expressed in terms of the perturbations $\delta\rho_s^0(r)$ and $\delta\kappa_s^0(r)$. Although the depth extent of the observed temperature variation is unknown, different hypothesized depth structures could be constructed. Using the theory presented here, the general matrix elements and, hence, the splitting caused by each temperature model could be computed and used to test the hypothesis of Kuhn et al. (1988).

The major constraint on the application of the theory presented here is that we have assumed that the convective flows and asphericities are stationary in time. Consequently, we view this paper as the first step toward a more general theory governing time-varying flows. Nevertheless, a number of the consequences of the theory will hold for time-varying flows as well. Most importantly, the selection rules listed in §VII will hold for non-stationary flows. For example, under self-coupling (or within degenerate perturbation theory), by Selection Rules 1_{sc} and 3_{sc}, only odd-degree s toroidal flows and even degree structural asphericities with $s \leq 2l$ will affect the splitting and coupling of acoustic modes with harmonic degree l.

In closing, since this paper is long, it is worthwhile to present a road map through the major results. Modal notation and terminology are discussed in §I.a and model notation and terminology are presented in §IV.b.i. The major assumptions of the theory are presented and discussed in §I.b, and are justified in §§I.c and I.e. The mathematical representation of convection is in equations (23)-(25) and the representation of the elastic-gravitational variables is in equations (29)-(31) and (35)-(37). The general equation of motion is equation (B.11) and the equation of motion for the perturbed model with first-order perturbations including rotation, ellipticity in the structural variables, centripetal force, convective flow, and asphericities in the elastic-gravitational variables is equation (50). The general forms of the general matrix element and the supermatrix are shown in equations (67) and (68), respectively, and the general matrix element for the perturbations listed in the previous sentence is in equation (77). The explicit form of the general matrix elements suitable for computation, written in terms of the scalar eigenfunctions of the SNRNMAIS

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solar model, can be found in equation (92) with notation and the integral kernels defined in equations (93)-(112). We consider this the main result of this paper. Three selection rules governing coupling are listed in equations (120), (122), and (123), with the self-coupling form of the selection rules in equations (121), (122), and (124). The Diagonal Sum Rule and the Superdiagonal Sum Rule are stated and proved in §VII.d. The general matrix element and selection rules for differential rotation are in equations (137) and (141), respectively. All results of the paper are presented in a frame corotating with the Sun. Equation (18) can be used to construct the perturbed eigenfrequencies and eigenfunctions of a non-SNRNMAIS solar model in an inertial frame. The Generalized Spherical Harmonic formalism, which was used to derive the explicit form of the general matrix elements, is discussed in Appendix C. The incorporation of anelasticity constraint into the general matrix element for convection is the subject of Appendix D.
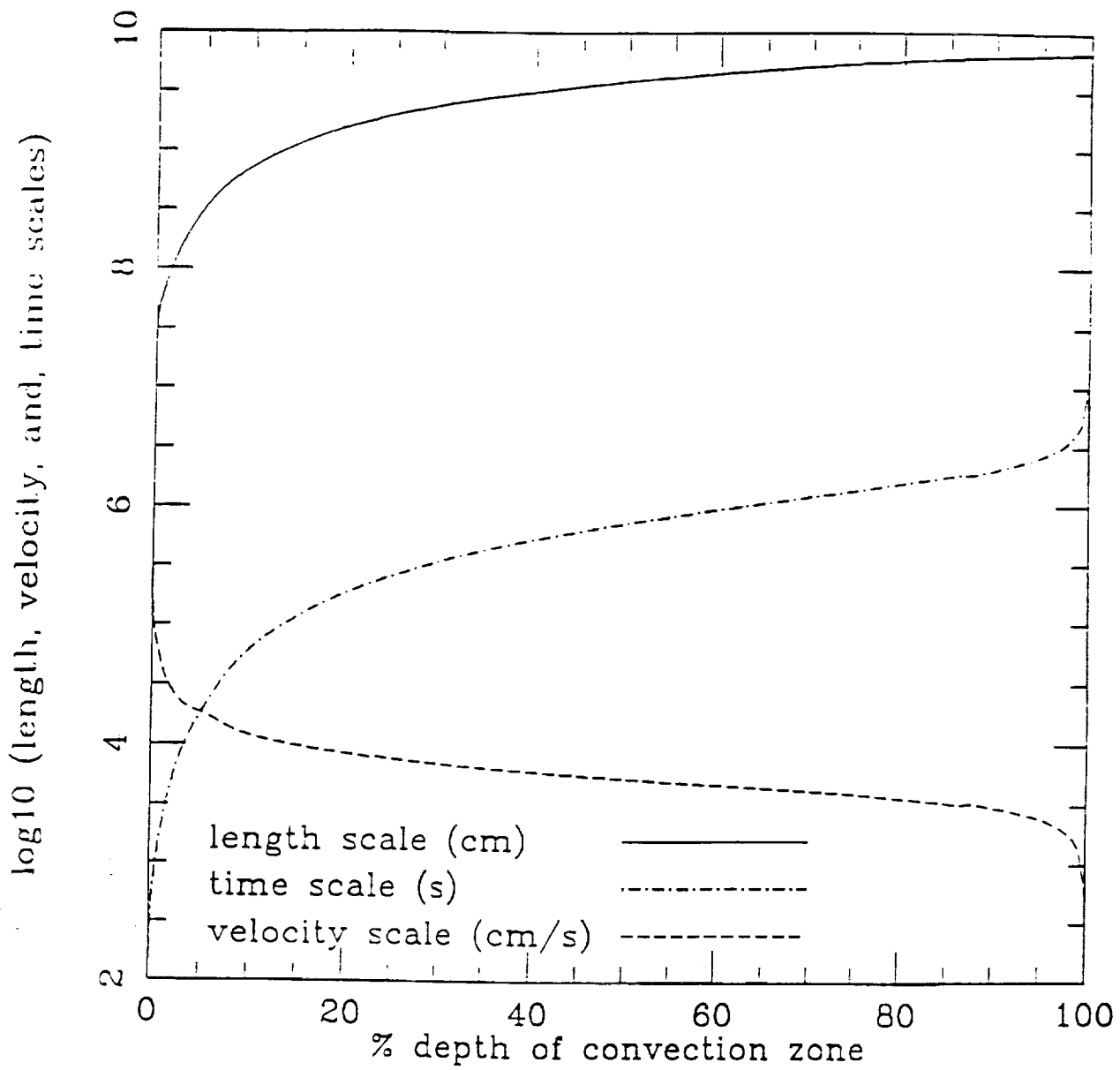
ACKNOWLEDGEMENTS

Figure. 1. - Characteristic length, velocity, and time scales of convective eddies as predicted by mixing length theory (eqs. [7], [9] and [10]) plotted as a function of depth. The solar model of Podsiadlowski (1989) was used to calculate these quantities.
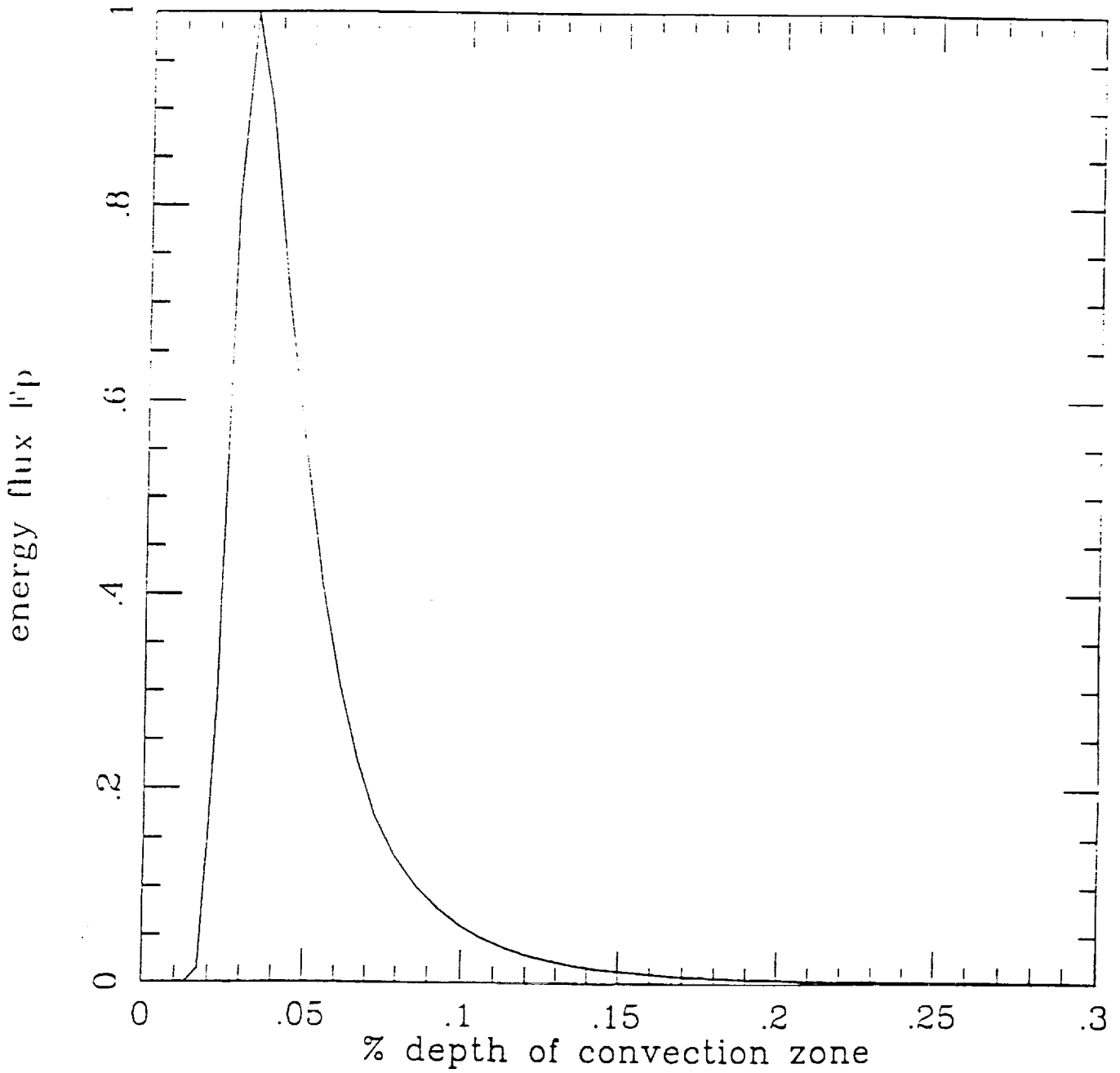
Figure. 2. - A plot of $F_p = M^{\frac{11}{4}} F_e$ which is the flux of energy pumped into the acoustic modes from the convective motions (see eq. [12]). We have normalized F_p by its peak value. We use the radial dependence of F_p to argue that coupling between convection and p modes is significant only in the top $\sim 0.2\%$ of the convection zone.

WAVE GENERATION BY TURBULENT CONVECTION

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ABSTRACT

We consider wave generation by turbulent convection in a plane parallel, stratified atmosphere that sits in a gravitational field, g. The atmosphere consists of two semi-infinite layers, the lower adiabatic and polytropic and the upper isothermal. The adiabatic layer supports a convective energy flux given by mixing length theory; $F_c \sim \rho v_H^3$, where ρ is mass density and v_H is the velocity of the energy bearing turbulent eddies.

Acoustic waves with $\omega > \omega_{ac}$ and gravity waves with $\omega < 2k_h H_i \omega_b$ propagate in the isothermal layer whose acoustic cutoff frequency, ω_{ac}, and Brunt-Väisälä frequency, ω_b, satisfy $\omega_{ac}^2 = \gamma g/4H_i$ and $\omega_b^2 = (\gamma - 1)g/\gamma H_i$, where γ and H_i denote the adiabatic index and scale height. The atmosphere traps acoustic waves in upper part of the adiabatic layer (p-modes) and gravity waves on the interface between the adiabatic and isothermal layers (f-modes). These modes obey the dispersion relation

$$\omega^2 \approx \frac{2}{m} g k_h \left(n + \frac{m}{2} \right),$$

for $\omega < \omega_{ac}$. Here, m is the polytropic index, k_h is the magnitude of the horizontal wave vector, and n is the number of nodes in the radial displacement eigenfunction; $n = 0$ for f-modes.

Wave generation is concentrated at the top of the convection zone since the turbulent Mach number, $M = v_H/c$, peaks there; we assume $M_t \ll 1$. The dimensionless efficiency, η, for the conversion of the energy carried by convection into wave energy is calculated to be $\eta \sim M_t^{15/2}$ for p-modes, f-modes, and propagating acoustic waves, and $\eta \sim M_t$ for propagating gravity waves. Most of the energy going into p-modes, f-modes, and propagating acoustic waves is emitted by inertial range eddies of size $h \sim M_t^{3/2} H_t$ at $\omega \sim \omega_{ac}$ and $k_h \sim 1/H_t$. The energy emission into propagating gravity waves is dominated by energy bearing eddies of size $\sim H_t$ and is concentrated at $\omega \sim v_t/H_t \sim M_t \omega_{ac}$ and $k_h \sim 1/H_t$.

We find the power input to individual p-modes, $\dot{E}_p$, to vary as $\omega^{(2m^2 + 7m - 3)/(m + 3)}$ at frequencies $\omega \ll v_t/H_t$. Libbrecht has shown that the amplitudes and linewidths of the solar p-modes imply $\dot{E}_p \propto \omega^8$ for $\omega \ll 2 \times 10^{-2}$ s^{-1}. The theoretical exponent matches the observational one for $m \approx 4$, a value obtained from the density profile in the upper part of the solar convection zone. This agreement supports the hypothesis that the solar p-modes are stochastically excited by turbulent convection.

Subject headings: convection — Sun: atmosphere — Sun: oscillations — turbulence — wave motions

I. INTRODUCTION

Lighthill (1952) wrote the seminal paper on the generation of acoustic waves by turbulence in homogeneous fluids. Stein (1967) extended Lighthill's techniques to stratified fluids and also treated the emission of gravity waves. We reconsider Stein's problem for a more realistic model atmosphere and relate the turbulent spectrum to the convective energy flux via the Kolmogorov scaling and the mixing length hypothesis. Our goal is to estimate efficiencies for the conversion of the convective energy flux into both trapped and propagating waves. We treat mode excitation but not mode damping. Thus, we cannot estimate the energies of trapped modes which depend upon the balance between these two effects.

The plan of our paper is as follows. In § II we describe the model atmosphere and its eigenmodes. Next, in § III, we derive expressions for the rates at which individual modes gain energy from turbulent convection. In § IV, we estimate the total emissivities for the different wave types, p-modes, f-modes, propagating acoustic waves, and propagating gravity waves. A comparison of our results with those obtained in earlier studies, and a discussion of their implications, is given in § V.

II. ATMOSPHERE AND EIGENMODES

a) Static Atmosphere

Our model atmosphere is plane parallel, sits in a constant gravitational field, g, and consists of two semi-infinite layers, the lower adiabatic and polytropic and the upper isothermal. The pressure, p, density, ρ, and temperature, T, are continuous across the interface between the two layers. In the lower layer the adiabatic and polytropic indices are related by $\Gamma = 1 + 1/m$. The adiabatic

index in the upper layer, γ, may differ from Γ. The z coordinate measures *depth* below the level at which the adiabatic layer would terminate in the absence of the isothermal layer. We denote quantities evaluated at the top of the adiabatic layer by a subscript t. Parameters in the isothermal layer are distinguished by a subscript i. Note, the ratio of the sound speeds $c_i/c_t = (\gamma/\Gamma)^{1/2}$.

In the adiabatic layer the thermodynamic variables exhibit a power-law behavior with depth:

$$p = p_t\left(\frac{z}{z_t}\right)^{m+1} , \qquad \rho = \rho_t\left(\frac{z}{z_t}\right)^m , \qquad T = T_t\left(\frac{z}{z_t}\right) . \tag{1}$$

The sound speed, c, and the pressure scale height, H, satisfy $c^2 = gz/m$ and $H = z/(m+1)$.

The isothermal atmosphere is still simpler: $T = T_i$, $c = c_i$, and $H = H_i$ are all constant, whereas p and ρ are proportional to $\exp(z/H_i)$.

b) Normal Modes

We choose the Eulerian enthalpy perturbation, $Q \equiv p_1/\rho$, as the dependent variable in the linear wave equations. These read

$$\frac{d^2Q}{dz^2} + \frac{m}{z}\frac{dQ}{dz} + \left(\frac{\omega^2}{c^2} - k_h^2\right)Q = 0 , \tag{2}$$

in the adiabatic layer, and

$$\frac{d^2Q}{dz^2} + \frac{1}{H_i}\frac{dQ}{dz} + \left[\frac{\omega^2}{c_i^2} - k_h^2\left(1 - \frac{\omega_b^2}{\omega^2}\right)\right]Q = 0 , \tag{3}$$

in the isothermal layer (Kumar and Goldreich 1989). Here, ω is the wave frequency and k_h is the horizontal wavevector ($l \equiv k_h R_\odot$).

The displacement vector, ξ, is related to Q by

$$\xi_h = i\frac{k_h}{\omega^2}Q , \qquad \xi_z = \frac{1}{\omega^2}\frac{\partial Q}{\partial z} , \tag{4}$$

in the adiabatic layer, and by

$$\xi_h = i\frac{k_h}{\omega^2}Q , \qquad \xi_z = \frac{1}{(\omega^2 - \omega_b^2)}\left[\frac{\partial Q}{\partial z} + \frac{(\gamma - 1)}{\gamma H_i}Q\right] , \tag{5}$$

in the isothermal layer.

The normal modes are obtained by solving equations (2) and (3) subject to $Q \to 0$ as $z \to \infty$, Q and ξ_z continuous across the interface at z_i, and the appropriate boundary conditions as $z \to -\infty$. The continuity of ξ_h follows from that of Q.

The modes are classified as trapped or propagating, and as composed of acoustic or gravity waves. The adiabatic layer supports acoustic waves, but not gravity waves. Moreover, it refracts acoustic waves upward. Thus, propagating modes must be traveling waves in the isothermal atmosphere.

Solutions of the wave equation in the isothermal atmosphere are proportional to $\exp(-\kappa_\pm z)$, with

$$\kappa_\pm = \left\{\frac{1}{2H_i} \pm i\sqrt{\left[\left(\frac{\omega}{\omega_{ac}}\right)^2 - 1\right]\frac{1}{(2H_i)^2} + \left[\left(\frac{\omega_b}{\omega}\right)^2 - 1\right]k_h^2}\right\} , \tag{6}$$

where ω_{ac} and ω_b are the acoustic cutoff and Brunt-Väisälä frequencies:

$$\omega_{ac}^2 = \frac{\gamma g}{4H_i} , \tag{7}$$

and

$$\omega_b^2 = \frac{(\gamma - 1)g}{\gamma H_i} . \tag{8}$$

Thus $\omega_b^2 = 4(\gamma - 1)\omega_{ac}^2/\gamma^2$. There are two branches to the dispersion curve for traveling waves. For $2k_h H_i \ll 1$, these are a high frequency, acoustic wave, branch with $\omega > \omega_{ac}$, and a low-frequency, gravity-wave, branch with $\omega < 2k_h H_i \omega_b$.

Wave excitation by turbulent convection is concentrated in the upper adiabatic layer where the convective velocity peaks. We seek analytic expressions for the normalized eigenfunctions in this region. Since the dominant interactions are proportional to $\partial^2 Q/\partial z^2$ (see § IIIb), we explicitly evaluate this quantity for each mode. In doing so, we drop factors of order unity including, in places, γ, Γ, and m.

i) Trapped Modes

Trapped modes correspond to evanescent solutions in the isothermal layer and are restricted to a discrete set of eigenfrequencies for fixed k_h. In the limit that the adiabatic layer extends to vanishing surface pressure, the eigenfunctions may be expressed in terms of associated Laguerre polynomials and the dispersion relation reads

$$\omega^2 = \frac{2}{m}gk_h\left(n + \frac{m}{2}\right) , \tag{9}$$

where the integer n denotes the number of nodes in the radial displacement eigenfunction (Christensen-Dalsgaard 1980; Christensen-Dalsgaard and Gough 1980). Trapped acoustic modes, or p-modes, correspond to $n \neq 0$. Modes with $n = 0$ are surface gravity waves, or f-modes. Trapped g-modes with $n \neq 0$ do not exist since the adiabatic layer is neutrally stratified, that is, its Brunt-Väisälä frequency vanishes. Equation (9) remains a good approximation for $\omega < \omega_{ac}$ even with finite surface pressure.

Only the physical solution, the one that grows less rapidly with height in the isothermal layer, is normalizable. The normalization condition reads

$$I \equiv \omega^2 \int_{-\infty}^{\infty} dz\, \rho \boldsymbol{\zeta}_\omega \cdot \boldsymbol{\zeta}_{\omega'}^* = \delta_{\omega, \omega'}, \tag{10}$$

at fixed k_h. For modes with $2k_h H_i \ll 1$, most of the contribution to the energy integral comes from the adiabatic layer. This enables us to reexpress the normalization condition, using equation (2), in terms of the enthalpy perturbation as

$$I \approx \int_{z_t}^{\infty} dz\, \frac{\rho}{c^2} Q_\omega Q_{\omega'}^* = \delta_{\omega, \omega'}. \tag{11}$$

For $\omega = \omega'$, this integral evaluates the potential energy of a trapped mode in the adiabatic layer. The potential energy is equal to the kinetic energy for all modes. This accounts for the relation between equations (10) and (11).

1. P-Modes

A p-mode is a standing acoustic wave trapped between an upper reflecting layer at z_1, where $\omega/c(z_1) = 1/2H(z_1)$, and a lower turning point at z_2, where $\omega/c(z_2) = k_h$. The requirement that there be an upper reflecting layer restricts p-modes to frequencies below ω_{ac}.

It is easily shown that

$$\frac{z_1}{z_t} \sim \left(\frac{\omega_{ac}}{\omega}\right)^2, \tag{12}$$

and

$$\frac{z_2}{z_1} \sim \left(n + \frac{m}{2}\right)^2. \tag{13}$$

Outside the interval $z_1 \lesssim z \lesssim z_2$, the mode is evanescent. Both Q and ζ increase slowly with height above z_1. Below z_2 the k_h term in equation (2) dominates and $Q \propto \exp\left(-k_h z\right)$.

We study the p-mode eigenfunctions in the dual limit $\omega \ll \omega_{ac}$ and $2k_h H_t \ll 1$. In a polytropic layer with vanishing surface pressure, the eigenfunctions are solutions of equation (2) that are analytic at $z = 0$. These solutions may be expressed in terms of associated Laguerre polynomials. When the polytropic layer is overlane by an isothermal layer, the eigenfunctions include a contribution from the solution that is singular at $z = 0$. However, the boundary conditions at the interface between the two layers ensure that the contribution from the singular solution is small for $\omega \ll \omega_{ac}$.

We can approximate the eigenfunction in the region of propagation, $z_1 \ll z \ll z_2$, by the WKB solution

$$Q \sim \left(\frac{z_t}{z}\right)^{(m-1)/2} B_p \sin\left[2\omega\left(\frac{mz}{g}\right)^{1/2} + \phi_p\right]. \tag{14}$$

Below the lower turning point at z_2, the eigenfunction is exponentially small. In the evanescent zone above z_1 the atmosphere responds stiffly. Thus B_p is approximately equal to the surface amplitude, $Q(z_t)$, for $\omega \ll \omega_{ac}$.

The z derivaties of Q in the evanescent region enter into the expressions we derive for wave generation. For $\omega \ll \omega_{ac}$, $\partial/\partial z$ has magnitude $\omega^2/g \sim (\omega/\omega_{ac})^2 H^{-1}$, as follows directly from equation (2). This equation has a singular point at $z = 0$, and its regular solution is given by a power series in $\omega^2 z/g$. This verifies our assertion about the magnitude of $\partial/\partial z$. Of course, the polytropic atmosphere does not extend to $z = 0$. However, this is of little consequence for the eigenfunctions that become evanescent well below $z = z_t$.

Given the properties of the eigenfunction described above, it follows from the normalization equation (11) that

$$B_p^2 \sim \frac{z_t^m \omega^{2(m-1)} k_h}{g^{(m-2)} \rho_t}. \tag{15}$$

Evaluating $\partial^2 Q/\partial z^2$ we obtain

$$\frac{\partial^2 Q}{\partial z^2} \sim \left(\frac{\omega^2}{g}\right)^2 B_p, \tag{16}$$

for $z_t \leq z \ll z_1$.

2. F-Modes

Direct substitution into equations (2) and (3) verifies that $Q = B_f \exp\left(-k_h z\right)$, with $\omega^2 = g k_h$, is an exact solution of the wave equations in both the adiabatic and the isothermal layers. Moreover, ζ_z formed from equations (4) and (5) is continuous across z_t. This family of normal modes consists of gravity waves confined near the surface of the convection zone; they are known as f-modes.

The f-modes are incompressible, $\nabla \cdot \xi = 0$, which accounts for their simple dispersion relation. The amplitude, B_f, is determined from the normalization equation (11) to be

$$B_f^2 \sim \frac{z_t^{-m}\omega^{2(m-1)}k_h}{g^{(m-2)}\rho_t} . \tag{17}$$

For all z,

$$\frac{\partial^2 Q}{\partial z^2} = k_h^2 B_f \exp(-k_h z) . \tag{18}$$

ii) Propagating Waves

Modes that propagate in the isothermal layer have continuous spectra. They are chosen to have no net flux in the isothermal layer; that is, they are composed of pairs of inward- and outward-propagating waves of equal amplitude. This choice ensures that propagating modes have real frequencies and are orthogonal to trapped modes. These modes are normalized such that

$$\omega^2 \int_{-\infty}^{\infty} dz\, \rho \xi_\omega \cdot \xi_{\omega'}^* = \delta(\omega - \omega') , \tag{19}$$

at fixed k_h. The upper limit on the integral in equation (19) may be taken to be z_t, since the contribution from the adiabatic layer is finite, and therefore negligible.

1. Acoustic Waves

These modes have $\omega > \omega_{ac}$ and propagate in the isothermal atmosphere and in the upper part of the adiabatic layer. They are evanescent below the lower turning point at $z_2 \sim \omega^2/gk_h^2$. We deduce the properties of the eigenfunctions in the joint limit $\omega \gg \omega_{ac}$ and $k_h \ll \omega/c_i$.

In the isothermal layer

$$Q = C_a \sin\left[K_z(z_t - z) + \zeta_a\right] \exp\left[\frac{(z_t - z)}{2H_i}\right] , \tag{20}$$

where $K_z \approx \omega/c_i$. Application of the normalization condition given by equation (19) to equation (20) yields

$$C_a^2 \sim \frac{g^{1/2} z_t^{1/2}}{\rho_t} . \tag{21}$$

We approximate the eigenfunctions in the adiabatic layer by the WKB solutions

$$Q \sim \left(\frac{z_t}{z}\right)^{(m-1)/2} B_a \sin\left[2\omega\left(\frac{m}{g}\right)^{1/2}(z^{1/2} - z_t^{1/2}) + \phi_a\right] , \tag{22}$$

for $z_t \leq z \ll z_2$. The continuity of Q and ξ_z across z_t is used to relate B_a and ϕ_a to C_a and ζ_a. The phase, ϕ_a, is determined by the condition that $Q \propto \exp(-k_h z)$ for $z \to \infty$. For ω just above ω_{ac}, $B_a(\omega, k_h)$ displays sharp ridges along extensions of the p-mode dispersion curves. These correspond to resonances for the scattering of incoming waves by the atmosphere. These ridges flatten for $\omega \gg \omega_{ac}$ and

$$B_a^2 \sim \frac{\Gamma C_a^2}{[1 + (\Gamma - \gamma)\cos^2 \phi_a]} . \tag{23}$$

For later use we record

$$\frac{\partial^2 Q}{\partial z^2} \approx -\frac{m\omega^2}{gz} Q , \tag{24}$$

for $z_t \leq z \ll z_2$.

2. Gravity Waves

Gravity modes with $\omega < 2k_h H_i \omega_b$ propagate in the isothermal atmosphere but are evanescent in the neutrally stable adiabatic layer. We detail their properties in the double limit $\omega \ll 2k_h H_i \omega_b$ and $2k_h H_i \ll 1$.

In the isothermal atmosphere

$$Q = C_g \sin\left[K_z(z_t - z) + \zeta_g\right] \exp\left[\frac{(z_t - z)}{2H_i}\right] , \tag{25}$$

where $K_z \approx (\omega_b/\omega)k_h$.

The amplitude C_g is determined by the normalization equation (19) to be

$$C_g^2 \sim \frac{g^{1/2}}{\rho_t z_t^{1/2} k_h} . \tag{26}$$

In the adiabatic layer, for $z_t \leq z \ll k_h^{-1}$, the last term in the wave equation (2) is much smaller than the first and second terms and may be ignored. The reduced wave equation yields

$$Q \approx B_g \left(\frac{z_t}{z}\right)^{(m-1)} + D_g .$$

(27)

The ratio D_g/B_g is determined by fitting Q in the upper part of the adiabatic layer to $Q \propto \exp(-k_h z)$ at $z \gg k_h^{-1}$. For $2k_h H_t \ll 1$, $D_g/B_g \ll 1$.

The continuity of Q and ζ_z across z_t is used to relate B_g and D_g to C_g and ζ_g. We find

$$\tan \zeta_g \sim -\frac{\omega k_h z_t^{3/2}}{g^{1/2}} .$$

(28)

The small value of $\tan \zeta_g$ is due to the change in orientation of the velocity field from almost vertical in the top of the adiabatic layer to almost horizontal in the isothermal layer. From equation (28) it follows that

$$B_g^2 \sim \frac{z_t^{5/2} \omega^2 k_h}{g^{1/2} \rho_t} .$$

(29)

For later use we note that

$$\frac{\partial^2 Q}{\partial z^2} \sim \frac{m(m-1)Q}{z^2} ,$$

(30)

holds for $z \ll k_h^{-1}$.

c) Turbulent Convection

In the absence of a reliable theory for turbulent convection, we are guided by the mixing length hypothesis. According to this hypothesis, the convective energy flux, F_c, is carried by turbulent eddies whose dimensions are of order the local pressure scale height, $H(z) = z/(m+1)$. The velocity and entropy fluctuations associated with these energy bearing eddies, $v_H(z)$ and $s_H(z)$, are related to the mean entropy gradient, ds/dz, by

$$v_H^2 \sim \frac{gH^2}{c_p} \frac{ds}{dz} ,$$

(31)

where c_p is the specific heat at constant pressure per unit mass, and

$$s_H \sim H \frac{ds}{dz} ,$$

(32)

These relations lead to

$$F_c \sim \rho T v_H s_H \sim \rho v_H^3 .$$

(33)

Since F_c is independent of z,

$$v_H(z) = v_t \left(\frac{z_t}{z}\right)^{m/3} ,$$

(34)

where $v_t \equiv v_H(z_t)$.

In treating the convection zone as adiabatic we have been neglecting the superadiabaticity of the temperature gradient, $c_p^{-1} T ds/dz$, with respect to the adiabatic temperature gradient, g/c_p. From equation (32) it follows that the ratio of these gradients may be expressed as

$$\frac{T}{g} \frac{ds}{dz} \sim M^2 ,$$

(35)

where the Mach number of the turbulence, $M \equiv v_H/c$. Appeal to equation (33) establishes that

$$M \sim \left(\frac{F_c}{\rho c^3}\right)^{1/3} .$$

(36)

We assume that the turbulent velocities are substantially subsonic even near the top of the convection zone, that is, $M_t \ll 1$. Under these conditions we are justified in approximating the convection zone as adiabatic when calculating eigenfunctions for the normal modes.

The characteristic time scale of the energy bearing eddies is

$$\tau_H \sim \frac{H}{v_H} .$$

(37)

It is smallest at the top of the convection zone where

$$\tau_t \sim \frac{1}{M_t \omega_{ac}} \, .$$
(38)

The velocities of smaller, $h < H$, inertial range eddies are related to those of the energy bearing eddies by the Kolmogorov scaling (Tennekes and Lumley 1972),

$$\frac{v_h}{v_H} = \left(\frac{h}{H}\right)^{1/3} \, ,$$
(39)

at fixed z. The Kolmogorov spectrum applies to turbulent convection because, below the scale of the energy bearing eddies, the Reynolds stress provides greater accelerations than the buoyancy forces (Goldreich and Keeley 1977a). This implies that entropy mixes like a passive scalar contaminant in the inertial range. Thus,

$$\frac{s_h}{s_H} \sim \left(\frac{h}{H}\right)^{1/3} \, .$$
(40)

The depth dependence of the properties of eddies of fixed size h follows from equations (32), (34), (37), and (40). We find

$$v_h(z) \sim v_t \left[\frac{h z_t^m}{z^{(m+1)}}\right]^{1/3}$$

$$\tau_h(z) \sim \tau_t \left[\frac{z^{(m+1)} h^2}{z_t^{(m+3)}}\right]^{1/3} \, .$$
(41)

$$s_h(z) \sim s_t \left[\frac{h z_t^{(2m+3)}}{z^{2(m+2)}}\right]^{1/3}$$

III. MODE EXCITATION

a) Source Terms

We begin this section by adding source terms due to turbulent convection to the linear wave equation (2) for the adiabatic layer. Next, we classify the individual terms as sources of monopole, dipole, and quadrupole radiation. Then we evaluate the excitation of wave modes by these sources.

We distinguish three principal sources of wave excitation by turbulent convection. They are, the expansion and contraction of fluid due to the gain and loss of specific entropy, buoyancy force variations associated with these entropy changes, and momentum transport by the fluctuating Reynold's stress.

We derive the inhomogeneous wave equation from the linearized versions of the equations for mass and momentum conservation supplemented by the equation of state for a perfect adiabatic gas. We augment the momentum equation by the divergence of the turbulent Reynolds stress, and the adiabatic equation of state by the entropy fluctuations associated with turbulent convection. These equations now read:

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho v) = 0 \, ,$$
(42)

$$\frac{\partial(\rho v)}{\partial t} + \nabla p_1 - \rho_1 g = -\nabla \cdot (\rho v v) \equiv F \, ,$$
(43)

and

$$\frac{p_1}{p} - \frac{\Gamma \rho_1}{\rho} = \frac{s}{c_v} \, ,$$
(44)

where ρ_1, p_1, v, and s are the Eulerian density, pressure, velocity, and entropy perturbations associated with the turbulent convection and the waves it generates. The subscript 1 attached to the density and pressure perturbations denotes that only the lowest order variations of these quantities need be retained. Equation (44), the Eulerian form of the perturbed equation of state, holds because the background state is isentropic.

Eliminating ρ_1 and v from the left-hand sides of equations (42)–(44), we obtain the inhomogeneous wave equation

$$\nabla^2 Q + \frac{g}{c^2} \frac{\partial Q}{\partial z} - \frac{1}{c^2} \frac{\partial^2 Q}{\partial t^2} = \frac{S^{(1)} + S^{(2)}}{\rho} \, ,$$
(45)

where

$$S^{(1)} = -\rho \frac{\partial^2}{\partial t^2} \left(\frac{s}{c_p}\right) - g \frac{\partial}{\partial z} \left(\frac{\rho s}{c_p}\right) \, , \qquad S^{(2)} = \nabla \cdot F \, .$$
(46)

The interpretation of equation (45) is somewhat subtle. Provided we drop the final $c^{-2}\partial^2 Q/\partial t^2$ term on the left-hand side as a first approximation in the limit of subsonic turbulence, it determines the near field turbulent pressure perturbations from the turbulent velocity and entropy perturbations. The $c^{-2}\partial^2 Q/\partial t^2$ term connects the near field perturbations to the wave field perturbations. The latter may be expanded in terms of the normal modes.

The identification of sources by multipole order is a useful device in estimating wave emission by turbulent convection. It helps to separate the sources that must be retained from those that may be safely discarded. For homogeneous and isotropic turbulence the multipole expansion may be carried out in several equivalent ways. In our application the turbulence is z-dependent, and therefore inhomogeneous, and the atmosphere is stratified, and therefore anisotropic. Under these circumstances the method of choice is to identify sources according to whether they involve a change in fluid volume (monopole terms), a source of external momentum (dipole terms), or merely internal stresses (quadrupole terms).[2] Classification based on the angular dependence of the wave amplitude in the radiation zone is not useful, because the angular dependence results, in part, from the anisotropy of the medium.[3] Identification of sources by the number of their spatial derivatives also leads to ambiguity, since it differs according to the choice of dependent variable.

The first term in $S^{(1)}$ arises directly from the volume change due to an entropy change at fixed pressure. It is a monopole source. The second term in $S^{(1)}$ reflects the buoyancy force variation associated with this volume change. It involves a variation of the density of momentum supplied by the external gravitational force and is a dipole source. The double divergence of the Reynolds stress in $S^{(2)}$ reflects the redistribution of momentum by internal stresses. It is a quadrupole source.

One might suspect that the monopole and dipole terms in $S^{(1)}$ produce more acoustic radiation than the quadrupole term in $S^{(2)}$. Treating these three terms independently appears to confirm this suspicion; the monopole and dipole terms are found to excite comparably greater amounts of acoustic radiation than the quadrupole term. However, the correct solution is more subtle. As we demonstrate shortly, destructive interference causes the total monopole plus dipole acoustic emission to be of the same order as the quadrupole emission.

b) Amplitude Equation

The total enthalpy perturbation, $Q(x, t)$, is expanded in terms of the normal modes, $Q_\alpha(z)$, as

$$Q = \frac{1}{\sqrt{2\mathscr{A}}} \sum_\alpha \left[A_\alpha Q_\alpha \exp\left(-i\omega t + i k_h \cdot x\right) + A_\alpha^* Q_\alpha^* \exp\left(i\omega t - i k_h \cdot x\right) \right], \tag{47}$$

where $\mathscr{A}$ is the horizontal cross section of the atmosphere.[4] The mode amplitudes, $A_\alpha(t)$, are slowly varying functions of time, $|dA_\alpha/dt| \ll \omega |A_\alpha|$. Substituting this expansion into equation (45), multiplying both sides by $Q_\alpha^* \exp\left(i\omega t - i k_h \cdot x\right)$, and integrating over space and time, we obtain

$$A_\alpha(t) = \frac{1}{2i\omega\mathscr{A}^{1/2}} \int_{-\infty}^t dt \int d^3x \, Q_\alpha^* (S^{(1)} + S^{(2)}) \exp\left(i\omega t - i k_h \cdot x\right). \tag{48}$$

Taking $-\infty$ for the lower limit on the integral over t involves the implicit assumption that damping erases the memory of excitations from the distant past.

Next, we integrate by parts to transfer all time and space derivatives to the eigenfunctions. The contributions due to the individual source terms are discussed separately below.

The monopole plus dipole terms contribute

$$A_\alpha^{(1)}(t) = \frac{1}{2i\omega\mathscr{A}^{1/2}} \int_{-\infty}^t dt \int d^3x \, \frac{\rho s}{c_p} \left(\omega^2 Q_\alpha^* + g \frac{\partial Q_\alpha^*}{\partial z} \right) \exp\left(i\omega t - i k_h \cdot x\right). \tag{49}$$

With the aid of the homogeneous wave equation (2), we transform equation (49) to

$$A_\alpha^{(1)}(t) \approx -\frac{1}{2i\omega\mathscr{A}^{1/2}} \int_{-\infty}^t dt \int d^3x \, \frac{\rho c^2 s}{c_p} \left(\frac{\partial^2 Q_\alpha^*}{\partial z^2} - k_h^2 Q_\alpha^* \right) \exp\left(i\omega t - i k_h \cdot x\right). \tag{50}$$

The contribution due to the quadrupole term is

$$A_\alpha^{(2)}(t) \approx \frac{1}{2i\omega\mathscr{A}^{1/2}} \int_{-\infty}^t dt \int d^3x \, \rho vv : \nabla\nabla Q_\alpha^* \exp\left(i\omega t - i k_h \cdot x\right). \tag{51}$$

The normal mode eigenfunctions share the property that $k_h |Q_\alpha| \lesssim |\partial Q_\alpha/\partial z|$ near the top of the adiabatic layer. More precisely, other than the f-modes for which $\partial Q_\alpha/\partial z = -k_h Q_\alpha$, the approximate mode eigenfunctions calculated in § IIb satisfy the strict inequality. This implies that

$$A_\alpha^{(1)}(t) \approx -\frac{1}{2i\omega\mathscr{A}^{1/2}} \int_{-\infty}^t dt \int d^3x \, \frac{\rho c^2 s}{c_p} \frac{\partial^2 Q_\alpha^*}{\partial z^2} \exp\left(i\omega t - i k_h \cdot x\right), \tag{52}$$

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[2] This method preserves the ordering of source terms by the efficiency with which they generate radiation.

[3] For example, a spherically symmetric point source radiates anisotropically in a stratified atmosphere.

[4] For the moment we are treating the atmosphere as being of finite horizontal extent.

and

$$A_z^{(2)}(t) \approx \frac{1}{2i\omega\mathscr{A}^{1/2}} \int_{-\tau}^{t} dt \int d^3x \rho v_z^2 \frac{\partial^2 Q_z^*}{\partial z^2} \exp(i\omega t - ik_h \cdot x),$$ (53)

provide order of magnitude estimates for $A_z^{(1)}(t)$ and $A_z^{(2)}(t)$. However, $A_z^{(1)}(t) = 0$ for f-modes as a consequence of their incompressibility.

Now we compare the relative sizes of $A_z^{(1)}(t)$ and $A_z^{(2)}(t)$. We start with the contributions made by energy bearing eddies and go on to investigate those due to smaller, inertial range eddies.

According to equations (31)–(32), $c^2 s_H/c_p \sim v_H^2$. Thus, except for the f-modes, the entropy and the Reynolds stress sources associated with energy bearing eddies make comparable contributions to $A_z(t)$. This illustrates the destructive interference between the monopole and dipole amplitudes to which we referred earlier; for energy bearing eddies and acoustic modes with $\omega \sim v_H/H$, the monopole and dipole terms in equation (49) are each larger by a factor $\sim(c/v_H)^2$ than the combined term in equation (50). The destructive interference between monopole and dipole amplitudes is a consequence of the anisotropy of the adiabatic layer. This is expressed by the anisotropic form of equation (2) which transforms equation (49) into equation (50).

For inertial range eddies, $c^2 s_h/c_p \sim v_h^2(H/h)^{1/3}$. This suggests that, unlike energy bearing eddies, inertial range eddies might excite waves more by their entropy sources than by their Reynolds stress sources. In fact, this is not the case. From equation (50) we see that wave excitation by the entropy source depends upon the time variability of the Eulerian entropy field. Inertial range eddies contribute to this time variation in different ways. The kinetic energy in an eddy of size $h \lesssim H$ may dissipate raising the local value of s_h. Neighboring eddies of similar size having opposite signs of s_h may collide and mix their fluid thereby smoothing the spatial variation of the entropy field on scale h. An eddy of size h carrying an entropy fluctuation s_h may be advected at speeds up to v_h. Of these possibilities, the dissipation of kinetic energy into heat produces the largest entropy source. However, this source is just equal to that provided by the Reynolds stress. Thus, from here on we use equation (53) to estimate the total excitation rate of normal modes.

Destructive interference between monopole and dipole radiation fields holds the acoustic emissivity of turbulent convection at the level characteristic of free turbulence[5] for which the emissivity is dominated by acoustic quadruples. We did not appreciate this point in our earlier treatment of acoustic emission by turbulent fluids (Goldreich and Kumar 1988). There we discussed the emissivity of turbulent pseudo-convection, a surrogate for turbulent convection. Since this model has acoustic dipoles but not acoustic monopoles, its emissivity is greater than that of free turbulence.

c) Excitation Rate

Turbulent convection consists of a hierarchy of critically damped eddies. Different eddies of similar size are assumed to be uncorrelated. This assumption enables us to divide into several steps the calculation of the rate at which turbulent convection pumps energy into a wave mode.

To begin, we estimate the magnitude of the incremental amplitude, ΔA_z^h, produced by a single eddy of size h located at depth z over its lifetime $\tau_h \sim h/v_h$.

$$\Delta A_z^h \sim \frac{\rho v_h h^4}{2i\omega\mathscr{A}^{1/2}} \frac{\partial^2 Q_z^*}{\partial z^2}, \qquad \omega \lesssim \tau_h^{-1}.$$ (54)

In arriving at the above equation we have assumed that the eigenfunction does not vary dramatically over $\Delta z = h \lesssim H$. This is a good approximation for all the modes we are concerned with. At frequencies much greater than τ_h^{-1}, ΔA_z^h declines exponentially with increasing ω.

Next, we note that

$$\dot{E}_z^h \sim \frac{|\Delta A_z^h|^2}{\tau_h},$$ (55)

is the mean rate at which one eddy supplies energy to mode α.

Then, summing over eddies of all sizes and depths, we obtain

$$\dot{E}_z \sim \frac{1}{\omega^2} \int_{z_t}^{\infty} dz \rho^2 \left|\frac{\partial^2 Q_z}{\partial z^2}\right|^2 \int_0^{h_{max}} \frac{dh}{h} v_h^3 h^4,$$ (56)

where

$$h_{max}(z) \sim \frac{H(z)}{1 + [\omega\tau_H(z)]^{3/2}}.$$ (57)

In deriving equation (56) from (55), we include a factor $\mathscr{A} dz/h^3$, the number of eddies in the horizontal slice of cross-sectional area $\mathscr{A}$ between vertical depths z and $z + dz$. The appearance of dh/h in equation (56) denotes that each inertial range eddy accounts for a finite range of scale size $dh/h \sim 1$. Carrying out the integration over h yields

$$\dot{E}_z \sim \frac{\rho_t^2 H_t^8}{\omega^2\tau_t^3} \int_{z_t}^{\infty} \frac{dz}{z_t} W\left(\frac{z}{z_t}\right) \left|\frac{\partial^2 Q_z}{\partial z^2}\right|^2,$$ (58)

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[5] Free turbulence is turbulence that is not subject to external forces.

where the weight factor, W, is given by

$$W(u) = \frac{u^{m+4}}{[1 + (\omega\tau_t)^{3/2} u^{(m+3)/2}]^5} .$$ (59)

The weight factor is sharply peaked about

$$u_* \sim 1 + \frac{1}{(\omega\tau_t)^{3/(m+3)}} ,$$ (60)

where

$$W(u_*) \approx \frac{1}{(\omega\tau_t)^{3(m+4)/(m+3)}[1 + (\omega\tau_t)^{3(3m+7)/2(m+3)}]} ;$$ (61)

it decays as $u^{(m+4)}$ for $u \ll u_*$ and as $u^{-(3m+7)/2}$ for $u \gg u_*$.

The peak in W is so sharp that $\dot{E}_\alpha$ is dominated by contributions from $z \sim z_*$ for all wave modes. Physically, this means that the excitation is concentrated in the layer where the turnover time of the energy bearing eddies is most nearly equal to the mode period. This enables us to further simplify the expression for $\dot{E}_\alpha$ to

$$\dot{E}_\alpha \sim \frac{\rho_t^2 H_t^8}{\tau_t} \frac{1}{(\omega\tau_t)^{(5m+21)/(m+3)}[1 + (\omega\tau_t)^{3(3m+5)/2(m+3)}]} \left| \frac{\partial^2 Q_\alpha(z_*)}{\partial z^2} \right|^2 .$$ (62)

IV. FLUXES OF ENERGY

To evaluate the total excitation rate for each type of mode, we substitute the relevant expression for $\partial^2 Q_\alpha(z_*)/\partial z^2$ given in § IIb) into equation (62). Following that, we integrate $\dot{E}_\alpha$ over all modes of the family to determine the fraction of the convective energy flux that family receives.

The frequencies of trapped modes satisfy equation (9). The flux of energy going into modes of a given family is

$$F_\alpha = \frac{1}{\mathscr{A}} \sum_\alpha \dot{E}_\alpha = \frac{1}{2\pi} \sum_n \int dk_h\, k_h \dot{E}_\alpha ,$$ (63)

where the sum over α includes all modes in the family, the sum over n includes all dispersion ridges in the family, and $\int dk_h$ is over all modes along a ridge. The last equality follows because the spacing between adjacent k_h modes in a box of horizontal area, $\mathscr{A}$, is equal to $2\pi/\sqrt{\mathscr{A}}$. Therefore, the number of modes in $d^2 k_h$ is $\mathscr{A} d^2 k_h/(2\pi)^2 = (\mathscr{A}/2\pi)dk_h\, k_h$.

For propagating modes, ω and k_h are independently specified. The flux of energy into a family of modes is computed from

$$F_\alpha = \frac{1}{\mathscr{A}} \sum_\alpha \dot{E} = \frac{1}{2\pi} \int d\omega \int dk_h\, k_h \dot{E}_\alpha ,$$ (64)

where the double integral is over all modes in the family.

a) P-Modes

From equations (16) and (62), we obtain

$$\dot{E}_p \sim \rho_t H_t^3 v_t^3 M_t^{2(m+2)} k_h \frac{(\omega\tau_t)^{(2m^2+7m-3)/(m+3)}}{1 + (\omega\tau_t)^{3(3m+5)/2(m+3)}} .$$ (65)

At fixed k_h, $\dot{E}_p$ varies as $\omega^{(2m^2+7m-3)/(m+3)}$ for $\omega\tau_t < 1$ and as $\omega^{(4m-7)/2}$ for $\omega\tau_t > 1$. To obtain the energy input rate per mode along the n'th p-mode ridge, we eliminate k_h from equation (65) by using equation (9). This procedure yields

$$\dot{E}_p \sim \frac{\rho_t H_t^2 v_t^3 M_t^{2(m+3)}}{(n + m/2)} \frac{(\omega\tau_t)^{(2m^2+9m+3)/(m+3)}}{1 + (\omega\tau_t)^{3(3m+5)/2(m+3)}} .$$ (66)

The total flux of energy going into p-modes follows from substituting equation (66) into equation (63):

$$F_p \sim \rho_t v_t^3 M_t^{15/2} = M_t^{15/2} F_c .$$ (67)

From equation (66), we note that for $\omega\tau_t \gg 1$ the energy input rate is proportional to $(\omega\tau_t)^{(4m-3)/2}$, which increases with increasing ω for $m > \frac{3}{4}$. Since the maximum frequency for trapped p-modes is ω_{ac}, most of the energy flux goes into modes whose frequencies lie just below the acoustic cutoff, $\omega \lesssim \omega_{ac}$, and is emitted by inertial range eddies with $h \sim M_t^{3/2} H_t$ located in the top scale height of the convection zone.

b) F-Modes

The calculations for the f-modes are similar to those for the p-modes. We substitute equation (18) into equation (62) and find

$$\dot{E}_f \sim \rho_t H_t^3 v_t^3 M_t^{2(m+2)} k_h \frac{(\omega\tau_t)^{(2m^2+7m-3)/(m+3)}}{1 + (\omega\tau_t)^{3(3m+5)/2(m+3)}} ,$$ (68)

where we have set $\exp(-k_h z_*) \sim 1$ since $k_h z_* \sim M_t^2 (\omega\tau_t)^{(2m+3)/(m+3)}[1 + (\omega\tau_t)^{3/(m+3)}] \lesssim 1$ for $\omega \leq \omega_{ac}$. The rate of energy input per mode along the f-mode ridge reads

$$\dot{E}_f \sim \rho_t H_t^2 v_t^3 M_t^{2(m+3)} \frac{(\omega\tau_t)^{(2m^2+9m+3)/(m+3)}}{1 + (\omega\tau_t)^{3(3m+5)/2(m+3)}} . \tag{69}$$

The total flux of energy going into f-modes is

$$F_f \sim \rho_t v_t^3 M_t^{15/2} = M_t^{15/2} F_c . \tag{70}$$

c) Acoustic Waves

From equations (24) and (62), we obtain

$$\dot{E}_a \sim \rho_t H_t^3 v_t^2 \frac{M_t^3}{(\omega\tau_t)^{11/2}} , \tag{71}$$

after averaging over the phase ϕ_a. Substituting equation (71) into equation (64), we derive the total flux of energy carried by the acoustic waves:

$$F_a \sim \rho_t v_t^3 M_t^{15/2} = M_t^{15/2} F_c . \tag{72}$$

Most of this energy is emitted by inertial eddies of size $h \lesssim M_t^{3/2} H_t$ located in the top scale height of the convection zone. It is carried by waves with $\omega \gtrsim \omega_{ac}$ and $k_h \lesssim 1/H_t$.

d) Gravity Waves

Equations (30) and (62) yield

$$\dot{E}_g \sim \rho_t H_t^4 v_t^2 M_t k_h \frac{(\omega\tau_t)^{3(m-3)/(m+3)}}{[1 + (\omega\tau_t)^{3(7m+9)/2(m+3)}]} , \tag{73}$$

so the power input into gravity waves peaks for $\omega\tau_t \sim 1$. Equation (73) holds for k_h in the range $\omega/\omega_b < 2k_h H_t < (\omega\tau_t)^{3/(m+3)}/[1 + (\omega\tau_t)^{3/(m+3)}]$. Substituting equation (73) into equation (64), we find the total flux of energy carried by the gravity waves:

$$F_g \sim \rho_t v_t^3 M_t = M_t F_c . \tag{74}$$

Most of this energy is emitted by energy bearing eddies located in the top scale height of the convection zone. It is carried by waves with $\omega\tau_t \sim 1$ and $k_h \lesssim 1/H_t$. The vertical wave vector of these waves in the isothermal layer is $k_z \sim 1/(M_t H_t)$.

V. DISCUSSION

a) Previous Results

Our principal results are dimensional efficiencies, η, for the conversion of the convective energy flux into the energy flux in different types of wave modes; $\eta_p \sim \eta_f \sim \eta_a \sim M_t^{15/2}$, and $\eta_g \sim M_t$. It is illuminating to compare these efficiencies to those obtained in previous investigations.

The classic result for the efficiency of emission of acoustic waves by homogeneous, isotropic turbulence is that of Lighthill (1952). Translated into our notation it is $\eta_a \sim M_t^5$. Here we are thinking of the acoustic emission from a layer of turbulent fluid of thickness, H_t, embedded in an otherwise uniform atmosphere. The energy bearing eddies are characterized by size, H_t, and velocity, v_t. In this system, the acoustic emission is dominated by the energy bearing eddies, and is concentrated at $\omega \sim v_t/H_t$, $k \sim M_t H_t$. We find $\eta_p \sim \eta_a \sim M_t^{15/2}$, with the emission dominated by inertial range eddies of size $h \sim M_t^{3/2} H_t$ and concentrated at $\omega \sim c_t/H_t$, $k_h \sim 1/H_t$. There are two relevant comparisons between our results and those of Lighthill.

First, we can redo the estimate for η_a from Lighthill's treatment restricting attention to emission from inertial range eddies having $h \lesssim M_t^{3/2} H_t$. These eddies, whose lifetimes $\tau_h \lesssim \omega_{ac}^{-1}$, dominate the emission of energy into p-modes and acoustic waves in the stratified atmosphere. A simple calculation yields $\eta_a \sim M_t^{15/2}$. This result agrees with ours showing that the acoustic emission from eddies with $h \lesssim M_t^{3/2} H_t$ is not affected by stratification.

Second, we can modify our calculation of η_p so that only the emission by energy bearing eddies is included. This is accomplished by repeating the procedure described in § IVa) but now limiting the integration over frequency along the p-mode ridges to $\omega \lesssim v_t/H_t$. This exercise yields $\eta_p \sim M_t^{10}$. The factor M_t^5 by which this result differs from Lighthill's may be accounted for as follows. Both in a homogeneous atmosphere and in our stratified atmosphere, the acoustic emissivity is proportional to $|\nabla\nabla Q|^2$. However, for $\omega \sim v_t/H_t$, $|\nabla\nabla Q|^2 \sim (M_t/H_t)^4 |Q|^2$ in the homogeneous atmosphere, whereas $|\nabla\nabla Q|^2 \sim (M_t^2/H_t)^4 |Q|^2$ in the stratified atmosphere. This difference, which accounts for four factors of M_t, arises because p-modes with $\omega \sim v_t/H_t \sim M_t \omega_{ac}$ are evanescent near the top of the convection zone in the stratified atmosphere.[6] The fifth factor of M_t arises from differences in phase space mode densities. In a uniform atmosphere, the number density of modes having $\omega \sim v_t/H_t$ is approximately $(M_t/H_t)^3$. This becomes M_t^3/H_t^2 per unit area for a layer H_t thick. The corresponding area density of p-modes in the stratified atmosphere is M_t^4/H_t^2, just one power of M_t smaller.

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[6] For acoustic waves with $\omega \gtrsim \omega_{ac}$, $|\nabla\nabla Q|^2$ is of the same order in the stratified atmosphere as in a homogeneous atmosphere.

Stein (1967) investigated the emission of acoustic and gravity waves by turbulent convection in a stratified atmosphere. He paid proper attention to the roles of ω_{ac} and ω_b and to the shapes of the mode eigenfunctions. However, Stein considered an isothermal atmosphere whereas we treat a two level atmosphere with the turbulent convection confined to the lower, adiabatic layer. Finally, we relate the properties of the turbulence to the convective energy flux using the mixing length hypothesis and the Kolmogorov scaling. The differences between out model assumptions and those of Stein preclude a meaningful comparison between his results and ours.

Milkey (1970) commented on the relation between Stein's calculation of acoustic spectral emissivity, $\epsilon_a(\omega)$, and that for free turbulence.[7] He showed that the Kolmogorov spectrum implies $\epsilon_a \propto \omega^{-7/2}$ in the dual limit $\omega \gg \omega_{ac}$ and $\omega \gg 1/\tau_t$. Equation (13) in Goldreich and Kumar (1988) confirms this simple result and, written in our notation, reads

$$\epsilon_a(\omega) \sim \rho_t v_t^2 \frac{M_t^5}{(\omega \tau_t)^{7/2}} . \tag{75}$$

Our equation (71) giving $\dot{E}_a$ also leads to equation (75) since $\epsilon_a(\omega) \sim (\omega/c_t)^2 \dot{E}_a/H_t \sim (\omega \tau_t)^2 M_t^2 \dot{E}_a/H_t^3$.

b) Solar p-Modes

Libbrecht (1988) has determined $\dot{E}_p(\omega)$ from his solar p-mode observations. He finds $\dot{E}_p \propto \omega^8$ for $\omega \ll 2 \times 10^{-2} \, \text{s}^{-1}$. Equation (65) gives $\dot{E}_p \propto \omega^{(2m^2+7m-3)/(m+3)}$ for $\omega \tau_t \ll 1$, in agreement with the observational result for $m \approx 4$, the polytropic index that fits the average density profile in the hydrogen ionization zone. Our formula fails for $\omega \tau_t \gg 1$; it gives $\dot{E}_p \propto \omega^{(4m-1)/4}$, or $\dot{E}_p \propto \omega^{4.5}$ for $m = 4$, while Libbrecht finds $\dot{E}_p \propto \omega^{-5}$ for $\omega \gg 2 \times 10^{-2} \, \text{s}^{-1}$. The resolution of this discrepancy is in hand. It involves modification of the eigenfunctions in the polytropic layer for ω close to ω_{ac} by the boundary conditions imposed at the interface with the isothermal layer. These modifications, which are ignored here, will be described in a subsequent paper devoted to a detailed examination of the excitation of the solar p-modes.

Even the limited success of our theoretical calculations in matching the frequency dependence of $\dot{E}_p$ lends support to the hypothesis that the solar p-modes are stochastically excited by turbulent convection (Goldreich and Keeley 1977b).

c) General Applications

Wave emission by turbulent convection is a common process in stellar and planetary atmospheres. It is clearly implicated in the heating of stellar chromospheres and coronas. Our results provide a foundation for the theory of wave emission in stratified atmospheres. However, several additional factors need to be examined before serious applications to real systems are contemplated. Several of these are mentioned below.

Real atmospheres differ from our model atmosphere in ways that may have important practical implications. The upper part of the convective zone, where much of the wave generation occurs, may not be well approximated by an isentropic layer of constant adiabatic index. Instead, as in the Sun, it may be significantly superadiabatic and possess ionization zones through which Γ undergoes substantial variations. The model atmosphere makes an abrupt transition from an adiabatic layer to an isothermal layer. The emission of gravity waves is likely reduced by the gradual rise of ω_b with height in a real atmosphere. Moreover, radiative smoothing of temperature perturbations may damp waves and also modify their propagation by lowering the effective adiabatic index. Both effects are most likely to be relevant for the dominant gravity waves because of their low frequencies and short vertical wavelengths.

The scope of our investigation is restricted to linear waves in unmagnetized media. Wave heating depends upon the behavior of nonlinear waves. It may also involve the coupling of acoustic and gravity waves to magnetosonic and Alfvén waves. These issues remain to be addressed by future studies.

[7] Spectral emissivity is the energy emission rate, per unit volume, per unit frequency.

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Arguments about three-mode couplings that are not strongly damping lead to the conclusion that the solar p-modes are probably stochastically driven by coupling with convection. Current disagreements among authors are discussed.

[Sections 1-X omitted.]

XI. CONCLUSIONS

A careful reading of this chapter will reveal that there are many uncertainties and disagreements in the prediction of the stability or overstability of the observed solar oscillations. The simplest calculations using only radiation flow by diffusion, ignoring the error of this approximation near the solar surface, give stability of most modes. This assumes, however, that we know the opacity and its derivatives, and small errors in these values could result in predictions of overstable modes.

When the model structure in the top 1000 km or less, below the photosphere, is modified to include radiative-transport effects, and the variations of the transport are calculated by linear pulsation theory, stability seems to be enhanced.

The inclusion of the convective Cowling mechanism gives driving just below the photosphere that appears marginally to overcome damping effects for p-modes according to the Antia et al. (1982) calculations. They recently propose that low-order g-modes could also be excited by this near-surface convective Cowling mechanism.

We then note that if the p-modes are really overstable, some very effective amplitude-limiting mechanism would have to be present because the observed modes have extremely small amplitudes unlike those for classical variable stars like the Cepheids. Three-mode coupling, the most efficient mechanism that seems available for nonlinear amplitude limitation, is not good enough to suppress these p-modes. Therefore, it seems from that conclusion that the solar p-modes may not be self excited.

Finally we come to a view that the forcing of the p-modes by coupling to convection-induced acoustic noise is an adequate mechanism for their excitation. This concept is accepted by most workers, but the convective Cowling mechanism needs further consideration.

The prediction of low-degree, low-order g-mode instability needs further investigation, but unlike the p-modes, these modes may well be at a large amplitude with amplitude limitations due to nonlinear effects. The difficulty in seeing these modes at the surface may only be because they tunnel through the convection zone leaving only a small surface amplitude. Any large deep amplitudes can greatly affect neutrino production rates, if these modes actually occur in the Sun. However, this does not seem to be a problem, because

OSCILLATION MODE EXCITATION

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The excitation of the oscillation modes in the Sun is very different from that for the previously known variable stars. A review of the normal pulsation mechanisms seen in many classes of variable stars shows that they actually are operating in the Sun. However, most, but not all, studies of the solar mode excitation predict that radiative damping, and damping by convective processes also, overwhelm the driving to stabilize all radial and nonradial modes. This is in accord with the observations that frequently show measurable widths of the lines in the p-mode oscillation spectrum. These line widths indicate mode lifetimes of days for the p-modes. Most calculations predict that solar g-modes are stable, leading to the question of how they then can ever be observed. However, there is a possibility that low-degree and low-order g-modes could be just slightly unstable. Improvements to the predictions of lowest-order p-mode excitation by the inclusion of better radiative intensity formulations at the top of the convection zone and in the photosphere indicate even more mode stability. Calculations that show how convection may drive solar p-modes are presented

those g-modes that are now predicted to be possibly overstable need comparable amplitudes in the surface layers as in the deep interior to invoke the radiative and convective processes operating in the subsurface layers of the Sun. The observed small surface amplitudes suggest small deep amplitudes that would not materially affect the nuclear-burning regions near the center. These modes are probably also stochastically excited by coupling with convection, just as for the p-modes, and their large radiative damping explains their very low amplitudes. These many unresolved problems we must leave for the future.