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Bivariate Extreme Value Distributions

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I. INTRODUCTION

A successful engineering design is usually a compromise between the capacity of the elements in the design and the actual operating conditions. These elements as well as the operating conditions are usually subject to such random effects that they become random variables. Hence, the properties of the elements involved and the external conditions are not known with complete certainty. Two extreme results of a classical engineering design are those related to under- or over-design which led to failure of the design or to a large waste of resources. It is essential to guarantee that no catastrophic limiting states occur during the operation or lifetime of a design. Thus, the extreme values are the ones affecting the failure of any given system. Consequently, the problem of the distribution of extreme value is of great interest.

The pioneering work of E. J. Gumbel which was published in his book "Statistics of Extremes" in 1958 had a large impact on the engineering community. Since that time many new advances and developments have taken place in the area of extreme value theory. These new developments are in engineering applications as well as in areas of probability, reliability, stochastic processes, multivariate structures, and statistical decision theory. The engineering applications involve a broad range of topics such as breaking strength of material [3], extreme river stages [3,4], wind shear [1], wind energy [8], aircraft gust loads [12], etc. However, with few exceptions, most of these applications have been restricted to the univariate cases.

An engineering application, which is of particular interest to us, is the "ascent structural wind loads" for the National Space Transportation System (NSTS). The

response of the space vehicle to loads induced by the winds, in particular, the fluctuation in the structural loads due to wind profile variability, has been and continues to be the subject for intensive investigations. An aerospace vehicle must withstand the design ascent structural load or be within any flight constraints to safely reach orbit. For the NSTS an assessment for flight readiness is made for each mission. This assessment for ascent loads and performance parameters is stated in terms of launch probability, which is derived from analyses of certain probability models.

The univariate extreme forms are the Gumbel type, which is unbounded, the Weibull (of largest extremes) and the Fréchet, which have lower bounds. Smith et al. [13,14] used the bivariate Gumbel "m – case" distribution in the analyses of extreme loads for the Space Transportation System (STS). However, some of the load variables had a physical lower bound of zero, and hence application of the bivariate Gumbel was not strictly appropriate. Therefore, the need for practical models for the bivariate Weibull and Fréchet was clearly identified for this specific application. This is the motivation for this report. It is envisioned that in many other engineering fields, in which the univariate models of extremes are used, applications of the bivariate models will evolve after dissemination of the tools provided in this report.

Extreme value distributions were obtained as limiting distributions of greatest (or smallest) values in random samples of increasing size. Since $\min\limits_{1 \le i \le n} X_i = -\max\limits_{1 \le i \le n}(-X_i)$, minima results can be obtained from the corresponding maxima, and vice versa.

Let (X_i, Y_i), $i = 1, 2, \cdots, n$, be a sequence of i.i.d. random pairs each having the same continuous joint cumulative distribution

$$F(x, y) = P\{X_i \le x, Y_i \le y\}.$$

We consider the joint distribution of $X_{max} := \max(X_1, X_2, \cdots, X_n)$, and $Y_{max} := \max(Y_1, Y_2, \cdots, Y_n)$.

To obtain a nondegenerate limiting distribution, it is necessary to reduce the actual greatest value by applying a linear transformation whose coefficients depend on the sample size. That is, we find linear transformations

$$X_{(n)} = a_n X_{max} + b_n \ , \ \ a_n > 0 \ , \ \text{and} \ \ Y_{(n)} = c_n Y_{max} + d_n \ , \ c_n > 0.$$

such that each of $X_{(n)}$ and $Y_{(n)}$ has a limiting distribution which is one of three types of extreme value distributions (see section II below). The limiting joint distribution of $X_{(n)}, Y_{(n)}$ is a bivariate extreme value distribution. The joint cumulative distribution function of X_{max} and Y_{max} is $[F(x,y)]^n$. Denoting the bivariate extreme value cumulative distribution function by $F_{(\infty)}(x,y)$, we then have

$$F_{(\infty)}(x,y) = \lim_{n \to \infty} [F(a_n x + b_n, c_n y + d_n)]^n, \tag{1}$$

For notational convenience we shall write $F(x,y)$ for the bivariate extreme distribution, if no ambiguity is likely to arise.

Equation (1) is sometimes referred to as the "stability relation" or "stability postulate." It is a natural extension of a similar equation obtained by Fréchet, and also used by Fisher and Tippett, in the univariate case.

II. UNIVARIATE EXTREME VALUE DISTRIBUTIONS

The univariate extreme value distributions are among the three following families;

(1) Gumbel: $P\{X \leq x\} = \exp\left[-e^{-x}\right]$,

(2) Fréchet: $P\{X \leq x\} = \begin{cases} 0 & , \ x < 0 \\ \exp[-x^{-\theta}], & x \geq 0, \end{cases}$

(3) Weibull: $P\{X \leq x\} = \begin{cases} \exp\left[-(-x)^{\theta}\right], & x \leq 0 \\ 0 & , \ x > 0, \end{cases}$

where $X = \frac{x - \mu}{\sigma}$ is a reduced variate, x (bold x) is the original variate, μ is the location parameter, σ is the dispersion parameter, and $\theta > 0$ is a parameter. The corresponding distributions of $(-X)$ are also called extreme value distributions.

Of these three families of distributions, Gumbel is by far the most commonly referred to in discussions of extreme value distributions, see [3].

The Weibull distribution above (sometimes called "reversed" Weibull) is an asymptotic distribution of smallest extremes. It is used in the modeling of failure (or survival), and those applications connected with reliability analysis.

The mirror image of the Weibull distribution, the distribution of (-X), is called Fisher-Tippet type III distribution of largest extremes,

(3)′ $\quad P\{-X \geq -x\} = 1 - P\{-X \leq -x\} = 1 - \exp\left(-x\right)^{\theta}, \ x \geq 0.$

The distribution in (3)′ has been used to model maximum temperatures, maximum wind speeds [7], and maximum earthquake magnitudes.

The Fréchet distribution is an asymptotic distribution of largest extremes derived by Fréchet under the condition that the initial variates be nonnegative. It has been widely used as a model for floods and maximum rainfall [2,7].

II.1. REMARK: We note that the Fréchet and Weibull distributions, with parameter θ, can be transformed into Gumbel distribution (with $\sigma=\frac{1}{\theta}$, and $\mu=0$) by the transformations Z=lnX, and Z=$-\ln(-$X$)$, respectively. Therefore, we can obtain the Fréchet or the Weibull distributions, with parameter $\theta=\frac{1}{\sigma}>0$, from Gumbel type with $\mu=0$ and $\sigma=\frac{1}{\theta}$, by using the transformations z=e^x or z=$-e^{-x}$, respectively.

III. STRUCTURE OF BIVARIATE EXTREME VALUE DISTRIBUTIONS

Gumbel and Mustafi [5] have described two general forms for the bivariate extreme value distributions in terms of the marginal distributions;

(1.) The a-case:

$$F(x,y) = F_1(x) \, F_2(y) \, \exp[-a(\tfrac{1}{\ln F_1(x)} + \tfrac{1}{\ln F_2(y)})^{-1}], \qquad (1)$$

where $0 \le a \le 1$, X and Y are reduced variates. The parameter a indicates the association between the two extremes, and a=0 corresponds to the case where X and Y are independent.

(2.) The m-case:

$$F(x,y) = \exp[-\{\ln F_1(x))^m + (-\ln F_2(y))^m\}^{\frac{1}{m}}], \qquad (2)$$

where $1 \le m \le \infty$, X and Y are reduced variates. The parameter m indicates the association between the two extremes, and m=1 gives the independence case.

We shall consider bivariate extreme value distributions when the margins $F_1(x)$ and $F_2(y)$ belong to the same family of univariate extreme value distributions.

A. Gumbel Margins

J. Tiago de Oliveira [15,16] proved that a bivariate extreme value distribution with Gumbel margins can be defined by a cumulative distribution function of the form

$$F(x,y) = \exp[-(e^{-x} + e^{-y}) \, g(y-x)], \qquad (3)$$

where X and Y are reduced variates.

In this case a density function exists if the "dependence" function g satisfies the following conditions:

(i) $\lim_{t \to \infty} g(t) = \lim_{t \to -\infty} g(t) = 1$,

(ii) $\frac{d}{dt}[(1 + e^{-t}) \, g(t)] \le 0$,

(iii) $\frac{d}{dt}\left[(1 + e^t)\, g(t)\right] \geq 0,$ and

(iv) $(1 + e^{-t})\, g''(t) + (1 - e^{-t})\, g'(t) \geq 0.$

A.1. Remarks: (i) To obtain the a–case equation (1) from equation (3) above, we take

$$g(t) = g_1(t) = 1 - \tfrac{a}{4}\operatorname{sech}^2(\tfrac{t}{2}),\ \ 0 \leq a \leq 1. \tag{4}$$

(ii) To obtain the m–case equation (2) from equation (3), we take

$$g(t) = g_2(t) = \frac{(e^{mt}+1)^{\frac{1}{m}}}{e^t+1},\ \ 1 \leq m \leq \infty. \tag{5}$$

We will now verify these remarks.

(i) The a–case:

$$F(x,y) = \exp\left[-(e^{-x}+e^{-y})\left(1 - \frac{a}{4\operatorname{ch}^2\frac{(y-x)}{2}}\right)\right]$$

$$= \exp\left[-(e^{-x}+e^{-y})\left(1 - \tfrac{a}{4}\left(\frac{2}{e^{\frac{y-x}{2}}+e^{-\frac{(y-x)}{2}}}\right)^2\right)\right]$$

$$= \exp\left[-(e^{-x}+e^{-y}) + a\,\frac{(e^{-x}+e^{-y})}{\left(e^{\frac{y-x}{2}}+e^{-\frac{(y-x)}{2}}\right)^2}\right]$$

$$= \exp\left[-(e^{-x}+e^{-y}) + a\,\frac{e^{-\frac{(y+x)}{2}}}{\left(e^{\frac{y-x}{2}}+e^{-\frac{(y-x)}{2}}\right)}\right]$$

$$= \exp\left[-(e^{-x}+e^{-y}) + a\,\frac{1}{e^y + e^x}\right].$$

Therefore,

$$F(x,y;a) = \exp\left[-(e^{-x}+e^{-y}) + a\,(e^x + e^y)^{-1}\right],\ \ 0 \leq a \leq 1. \tag{6}$$

(ii) The m–case:

$$F(x,y) = \exp\left[-(e^{-x}+e^{-y})\,\frac{(e^{m(y-x)}+1)^{\frac{1}{m}}}{e^{y-x}+1}\right]$$

$$= \exp\left[-(e^{-x}+e^{-y})\left(\frac{e^{-mx}+e^{-my}}{e^{-my}}\right)^{\frac{1}{m}} \Big/ \left(\frac{e^{-x}+e^{-y}}{e^{-y}}\right)\right].$$

Therefore,

$$F(x,y;m) = \exp\left[-(e^{-mx} + e^{-my})^{\frac{1}{m}}\right], \quad 1 \leq m \leq \infty. \tag{7}$$

A.2. Rotation of the Axes by 45°

We may want to take advantage of the symmetry around the diagonal $y=x$ by rotating the coordinates 45°.

$$\left.\begin{aligned}
x &= x' \cos 45° - y' \sin 45° = \frac{\sqrt{2}}{2}(x' - y'). \\
y &= x' \sin 45° + y' \cos 45° = \frac{\sqrt{2}}{2}(x' + y').
\end{aligned}\right\} \tag{8}$$

(i) The a-case:

$$F(x',y') = \exp\left[-\left(e^{-\frac{\sqrt{2}}{2}(x'-y')} + e^{-\frac{\sqrt{2}}{2}(x'+y')}\right) + a\left(e^{\frac{\sqrt{2}}{2}(x'-y')} + e^{\frac{\sqrt{2}}{2}(x'+y')}\right)^{-1}\right]$$

$$= \exp\left[-\left(e^{-\frac{\sqrt{2}}{2}x'}\left(e^{\frac{\sqrt{2}}{2}y'} + e^{-\frac{\sqrt{2}}{2}y'}\right)\right)\left(1 - \frac{a}{4}\frac{1}{\left(\frac{e^{-\frac{\sqrt{2}}{2}y'} + e^{\frac{\sqrt{2}}{2}y'}}{2}\right)^2}\right)\right]$$

$$= \exp\left[-2\,e^{-\frac{\sqrt{2}}{2}x'}\,\mathrm{ch}\left(\tfrac{\sqrt{2}}{2}y'\right)\left(1 - \tfrac{a}{4}\,\mathrm{sech}^2\left(-\tfrac{\sqrt{2}}{2}y'\right)\right)\right], \quad 0 \leq a \leq 1.$$

Thus,

$$F(x',y') = \exp\left[-2\,e^{-\frac{\sqrt{2}}{2}x'}\,\mathrm{ch}\left(\tfrac{\sqrt{2}}{2}y'\right)g_1(\sqrt{2}\,y')\right], \tag{9}$$

where $g_1(t) = 1 - \tfrac{a}{4}\,\mathrm{sech}^2(\tfrac{t}{2})$, $0 \leq a \leq 1$, is the dependence function in A.1.

(ii) The m-case:

$$F(x',y') = \exp\left[-\left(e^{-\frac{m\sqrt{2}}{2}(x'-y')} + e^{-\frac{m\sqrt{2}}{2}(x'+y')}\right)^{\frac{1}{m}}\right]$$

$$= \exp\left[-e^{\frac{\sqrt{2}}{2}x'}\left(e^{\frac{m\sqrt{2}}{2}y'} + e^{-\frac{m\sqrt{2}}{2}y'}\right)^{\frac{1}{m}}\right]$$

$$= \exp\left[-e^{-\frac{\sqrt{2}}{2}x'}\left(\frac{e^{m\sqrt{2}y'}+1}{e^{m\frac{\sqrt{2}}{2}y'}}\right)^{\frac{1}{m}}\right]$$

$$= \exp\left[-e^{-\frac{\sqrt{2}}{2}x'}(e^{\sqrt{2}y'}+1)\frac{\left(\frac{e^{m\sqrt{2}y'}+1}{e^{m\frac{\sqrt{2}}{2}y'}}\right)^{\frac{1}{m}}}{(e^{\sqrt{2}y'}+1)}\right]$$

$$= \exp\left[-e^{\frac{\sqrt{2}}{2}x'}\left(e^{\frac{\sqrt{2}}{2}y'} + e^{\frac{-\sqrt{2}}{2}y'}\right)\frac{(e^{m\sqrt{2}y'}+1)^{\frac{1}{m}}}{(e^{\sqrt{2}y'}+1)}\right]$$

$$= \exp\left[-2\,e^{\frac{\sqrt{2}}{2}x'}\,ch(\tfrac{\sqrt{2}}{2}y')\,\frac{(e^{m\sqrt{2}y'}+1)^{\frac{1}{m}}}{(e^{\sqrt{2}y'}+1)}\right], \quad 1 \le m \le \infty.$$

Thus,

$$F(x',y') = \exp\left[-2\,e^{-\frac{\sqrt{2}}{2}x'}\,ch(\tfrac{\sqrt{2}}{2}y')\,g_2(\sqrt{2}\,y')\right], \tag{10}$$

where $g_2(t) = \dfrac{(e^{mt}+1)^{\frac{1}{m}}}{e^t+1}$, $1 \le m \le \infty$, is the dependence function in A.1.

A.3. The Joint Density functions

(i) The a – case:

$$f(x,y;a) = F(x,y;a)\,e^{-(x+y)}\,[1 - a(e^{2x}+e^{2y})(e^x+e^y)^{-2} + 2\,a\,e^{2(x+y)}\,(e^x+e^y)^{-3}$$
$$+ a^2\,e^{2(x+y)}\,(e^x+e^y)^{-4}] \tag{11}$$

where $F(x,y;a)$ is the distribution function given by (6), and $0 \le a \le 1$.

(ii) The m – case:

$$f(x,y;m) = F(x,y;m)\,e^{-m(x+y)}\,(e^{-mx}+e^{-my})^{-2+\frac{1}{m}}\,[m-1 + (e^{-mx}+e^{-my})^{\frac{1}{m}}], \tag{12}$$

where $F(x,y;m)$ is the distribution function given by (7), and $1 \le m \le \infty$.

A.4. The Bivariate Mode

(i) The a – case:

The value $(\tilde{x}, \tilde{y})$ which maximizes the density $f(x,y;a)$, given by (11), is called the modal location. This value lies on the line $y = x$ (or $y'= 0$), and hence to obtain it we write the density in terms of x' and y', the $45°$–rotated axes. From (8), we have,

$$x + y = \sqrt{2}\,x' \,, \quad e^{2x}+e^{2y} = 2\,e^{\sqrt{2}\,x'}\left(\frac{e^{-\sqrt{2}y'}+e^{\sqrt{2}y'}}{2}\right) = 2\,e^{\sqrt{2}x'}ch(\sqrt{2}\,y') \,, \text{ and}$$

$$e^x + e^y = 2\, e^{\frac{\sqrt{2}}{2}x'} \mathrm{ch}(\tfrac{\sqrt{2}}{2}\, y').$$

Therefore (11) becomes

$$f(x',y';a) = F(x',y';a)\, e^{-\sqrt{2}x'}\Big[1 - \frac{a\, 2\, e^{\sqrt{2}x'}\mathrm{ch}(\sqrt{2}\, y')}{(2\, e^{\frac{\sqrt{2}}{2}x'}\mathrm{ch}(\frac{\sqrt{2}}{2}\, y'))^2} + \frac{2\, a\, e^{\sqrt{2}x'}}{(2\, e^{\frac{\sqrt{2}}{2}x'}\mathrm{ch}(\frac{\sqrt{2}}{2}\, y'))^3}$$

$$+ \frac{a^2\, e^{2\sqrt{2}x'}}{(2\, e^{\frac{\sqrt{2}}{2}x'}\mathrm{ch}(\frac{\sqrt{2}}{2}\, y'))^4}\Big].$$

$$f(x',y';a) = F(x',y';a)\, e^{-\sqrt{2}x'}\Big[1 - \tfrac{a}{2}\,\mathrm{ch}(\sqrt{2}\, y')\,\mathrm{sech}^2(\tfrac{\sqrt{2}}{2}y') + \tfrac{a}{4}\, e^{\frac{\sqrt{2}}{2}x'}\mathrm{sech}^3(\tfrac{\sqrt{2}}{2}\, y')$$

$$+ \tfrac{a^2}{16}\,\mathrm{sech}^4(\tfrac{\sqrt{2}}{2}\, y')\Big].$$

Thus,

$$f(x',0;a) = F(x',0;a)\, e^{-\sqrt{2}x'}\Big[1 - \tfrac{a}{2} + \tfrac{a^2}{16} + \tfrac{a}{4}\, e^{\frac{\sqrt{2}}{2}x'}\Big],$$

where $F(x',0;a) = \exp\big[-2(1 - \tfrac{a}{4})e^{-\frac{\sqrt{2}}{2}x'}\big].$

Therefore,

$$f(x',0;a) = \exp\big[-2(1 - \tfrac{a}{4})e^{-\frac{\sqrt{2}}{2}x'}\big]\big((1-\tfrac{a}{4})^2 e^{-\sqrt{2}x'} + \tfrac{a}{4}\, e^{-\frac{\sqrt{2}}{2}x'}\big), \qquad (13)$$

and

$$\frac{\partial}{\partial x'}f(x',0;a) = \exp\big[-2(1 - \tfrac{a}{4})e^{-\frac{\sqrt{2}}{2}x'}\big]\,(1-\tfrac{a}{4})\sqrt{2}\, e^{-\sqrt{2}x'}\big((1-\tfrac{a}{4})^2 e^{-\sqrt{2}x'} + \tfrac{a}{4}\, e^{-\frac{\sqrt{2}}{2}x'}\big)$$

$$+ \exp\big[-2(1 - \tfrac{a}{4})e^{-\frac{\sqrt{2}}{2}x'}\big]\big(-\sqrt{2}(1-\tfrac{a}{4})^2 e^{-\sqrt{2}x'} - \tfrac{a\sqrt{2}}{8}\, e^{-\frac{\sqrt{2}}{2}x'}\big).$$

Setting $\frac{\partial}{\partial x'}f(x',0;a) = 0$, we obtain

$$(1-\tfrac{a}{4})\, e^{-\sqrt{2}x'}\big((1-\tfrac{a}{4})^2\, e^{-\sqrt{2}x'} + \tfrac{a}{4}\, e^{-\frac{\sqrt{2}}{2}x'}\big) - (1-\tfrac{a}{4})^2\, e^{-\sqrt{2}x'} - \tfrac{a}{8}\, e^{-\frac{\sqrt{2}}{2}x'} = 0$$

$$(1-\tfrac{a}{4})\big((1-\tfrac{a}{4})^2\, e^{-\sqrt{2}x'} + \tfrac{a}{4}\, e^{-\frac{\sqrt{2}}{2}x'}\big) - (1-\tfrac{a}{4})^2\, e^{-\frac{\sqrt{2}}{2}x'} - \tfrac{a}{8} = 0$$

$$(1-\tfrac{a}{4})^3 e^{-\sqrt{2}x'} - (1-\tfrac{a}{4})(1-\tfrac{a}{2})e^{-\frac{\sqrt{2}}{2}x'} - \tfrac{a}{8} = 0 .$$

Therefore,

$$e^{-\frac{\sqrt{2}}{2}x'} = \frac{(1 - \tfrac{a}{4})(1 - \tfrac{a}{2}) \pm \sqrt{(1 - \tfrac{a}{4})^2(1 - \tfrac{a}{2})^2 - 4(1 - \tfrac{a}{4})^3(-\tfrac{a}{8})}}{2(1 - \tfrac{a}{4})^3}$$

$$= \frac{(1 - \frac{a}{4})(1 - \frac{a}{2}) \pm \sqrt{(1 - \frac{a}{4})^2(1 - a + \frac{a^2}{4} + \frac{a}{2} - \frac{a^2}{8})}}{2(1 - \frac{a}{4})^3}$$

$$= \frac{(1 - \frac{a}{2}) \pm \sqrt{1 - \frac{a}{2} + \frac{a^2}{8}}}{2(1 - \frac{a}{4})^2} \quad . \text{ Since } e^{-\frac{\sqrt{2}}{2}x'} \text{ has to be positive, we see that,}$$

$$e^{-\frac{\sqrt{2}}{2}x'} = \frac{(1 - \frac{a}{2}) + \sqrt{1 - \frac{a}{2} + \frac{a^2}{8}}}{2(1 - \frac{a}{4})^2} \quad .$$

Therefore, modal location is $\tilde{y}' = 0$, and

$$\tilde{x}' = -\sqrt{2} \left[\ln \left((1 - \frac{a}{2}) + \sqrt{1 - \frac{a}{2} + \frac{a^2}{8}} \right) - \ln \left(2(1 - \frac{a}{4})^2 \right) \right]. \tag{14}$$

The corresponding modal value is obtained by substituting (14) into (13):

$$f(\tilde{x}', 0; a) = \exp\left[-\frac{(1 - \frac{a}{2}) + \sqrt{1 - \frac{a}{2} + \frac{a^2}{8}}}{(1 - \frac{a}{4})} \right] \left\{ (1 - \frac{a}{4})^2 \left(\frac{(1 - \frac{a}{2}) + \sqrt{1 - \frac{a}{2} + \frac{a^2}{8}}}{2(1 - \frac{a}{4})^2} \right)^2 + \right.$$

$$\left. + \frac{a}{4} \frac{(1 - \frac{a}{2}) + \sqrt{1 - \frac{a}{2} + \frac{a^2}{8}}}{2(1 - \frac{a}{4})^2} \right\}$$

$$= \exp\left[-\frac{(1 - \frac{a}{2}) + \sqrt{1 - \frac{a}{2} + \frac{a^2}{8}}}{(1 - \frac{a}{4})} \right] \left\{ \left(\frac{(1 - \frac{a}{2}) + \sqrt{1 - \frac{a}{2} + \frac{a^2}{8}}}{2(1 - \frac{a}{4})} \right)^2 + \frac{a}{4} \frac{(1 - \frac{a}{2}) + \sqrt{1 - \frac{a}{2} + \frac{a^2}{8}}}{2(1 - \frac{a}{4})^2} \right\}$$

Thus

$$f(\tilde{x}', 0; a) = \frac{1}{2} \exp\left[-\frac{2(2 - a) + \sqrt{16 - 8a + 2a^2}}{(4 - a)} \right] \left(1 + \frac{1}{4 - a} \sqrt{(4 - a)^2 + a^2} \right). \tag{15}$$

(ii) The m − case:

As in the a − case, by using (8), we see that

$$e^{-mx} + e^{-my} = e^{-\frac{m\sqrt{2}}{2}(x' - y')} + e^{-\frac{m\sqrt{2}}{2}(x' + y')}$$

$$= e^{-\frac{m\sqrt{2}}{2}x'}\left(e^{\frac{m\sqrt{2}}{2}y'} + e^{-\frac{m\sqrt{2}}{2}y'}\right)$$

$$= 2\,e^{-\frac{m\sqrt{2}}{2}x'}\mathrm{ch}(\tfrac{m\sqrt{2}}{2}y').$$

Therefore, (12) becomes

$$f(x',y';m)=F(x',y';m)e^{-m\sqrt{2}x'}\left(2e^{-\frac{m\sqrt{2}}{2}x'}\mathrm{ch}(\tfrac{m\sqrt{2}}{2}y')\right)^{-2+\frac{1}{m}}[m-1$$

$$+ 2\,e^{-\frac{\sqrt{2}}{2}x'}(\mathrm{ch}(\tfrac{m\sqrt{2}}{2}y'))^{\frac{1}{m}}].\tag{16}$$

Thus,

$$f(x',0;m) = F(x',0;m)e^{-m\sqrt{2}x'}\left(2e^{-\frac{m\sqrt{2}}{2}x'}\right)^{-2+\frac{1}{m}}(m-1+2\,e^{-\frac{\sqrt{2}}{2}x'}),$$

where $F(x',0;m)=\exp[-2^{\frac{1}{m}}\,e^{-\frac{\sqrt{2}}{2}x'}]$, or

$$f(x',0;m) = \exp[-2^{\frac{1}{m}}\,e^{-\frac{\sqrt{2}}{2}x'}]\,e^{-m\sqrt{2}x'}\left(2e^{-\frac{m\sqrt{2}}{2}x'}\right)^{-2+\frac{1}{m}}[m-1+2\,e^{-\frac{\sqrt{2}}{2}x'}],$$

$$f(x',0;m) = \exp[-2^{\frac{1}{m}}\,e^{-\frac{\sqrt{2}}{2}x'}]\,2^{-2+\frac{1}{m}}\left((m-1)\,e^{-\frac{\sqrt{2}}{2}x'}+2^{\frac{1}{m}}\,e^{-\sqrt{2}x'}\right).\tag{17}$$

Now, differentiating w.r.t. x' and setting the derivative to zero, we obtain

$$\frac{\partial}{\partial x'}f(x',0;m) = \exp[-2^{\frac{1}{m}}\,e^{-\frac{\sqrt{2}}{2}x'}]\left(-2^{\frac{1}{m}}\,(\tfrac{-\sqrt{2}}{2})e^{-\frac{\sqrt{2}}{2}x'}\right)\left((m-1)\,e^{-\frac{\sqrt{2}}{2}x'}+2^{\frac{1}{m}}\,e^{-\sqrt{2}x'}\right)$$

$$+ \exp[-2^{\frac{1}{m}}\,e^{-\frac{\sqrt{2}}{2}x'}]\left(-\tfrac{\sqrt{2}}{2}(m-1)\,e^{-\frac{\sqrt{2}}{2}x'}+2^{\frac{1}{m}}\,(-\sqrt{2})e^{-\sqrt{2}x'}\right)=0,$$

which gives,

$$2^{\frac{1}{m}}e^{-\frac{\sqrt{2}}{2}x'}\left((m-1)\,e^{-\frac{\sqrt{2}}{2}x'}+2^{\frac{1}{m}}\,e^{-\sqrt{2}x'}\right) - \left((m-1)\,e^{-\frac{\sqrt{2}}{2}x'}+2^{\frac{1}{m}}\,e^{-\sqrt{2}x'}\right)=0$$

$$2^{\frac{1}{m}}(m-3)\,e^{-\sqrt{2}x'}+2^{\frac{2}{m}}\,e^{-\frac{3}{2}\sqrt{2}x'}-(m-1)\,e^{-\frac{\sqrt{2}}{2}x'}=0$$

$$2^{\frac{2}{m}}\,e^{-\sqrt{2}x'}+2^{\frac{1}{m}}(m-3)\,e^{-\frac{\sqrt{2}}{2}x'}-(m-1)=0.\ \text{Thus,}$$

$$e^{-\frac{\sqrt{2}}{2}x'} = \frac{-2^{\frac{1}{m}}(m-3) \pm \sqrt{2^{\frac{2}{m}}(m-3)^2 + 4(2)^{\frac{2}{m}}(m-1)}}{2(2)^{\frac{2}{m}}}$$

$$= \frac{-(m-3) \pm \sqrt{(m-3)^2 + 4(m-1)}}{2(2)^{\frac{1}{m}}}, \quad m \geq 1.$$

It follows that,

$$e^{-\frac{\sqrt{2}}{2}x'} = \frac{-(m-3) + \sqrt{(m-3)^2 + 4(m-1)}}{(2)^{\frac{1}{m}+1}}$$

$$= \frac{-(m-3) + \sqrt{(m-1)^2 + 4}}{(2)^{\frac{1}{m}+1}}, \text{ and}$$

$$-\frac{\sqrt{2}}{2}x' = \ln\left[-(m-3) + \sqrt{(m-1)^2 + 4}\right] - (\tfrac{1}{m}+1)\ln 2.$$

Thus the modal location is $\widetilde{y}' = 0$, and

$$\widetilde{x}' = \sqrt{2}\left\{(\tfrac{1}{m}+1)\ln 2 - \ln\left[-(m-3) + \sqrt{(m-1)^2 + 4}\right]\right\}. \tag{18}$$

Substituting (18) into (17) we obtain the modal value

$$f(\widetilde{x}',0;m) = \exp\left[\frac{-(m-3) + \sqrt{(m-1)^2 + 4}}{2}\right]\left(\frac{-(m-3) + \sqrt{(m-1)^2 + 4}}{8}\right)$$

$$\times \left((m-1) + \left(\frac{-(m-3) + \sqrt{(m-1)^2 + 4}}{2}\right)\right)$$

$$= \exp\left[\frac{-(m-3) + \sqrt{(m-1)^2 + 4}}{2}\right]\left(\frac{-(m-3) + \sqrt{(m-1)^2 + 4}}{8}\right)\left(\frac{(m+1) + \sqrt{(m-1)^2 + 4}}{2}\right). \tag{19}$$

B. Fréchet Margins

Using the transformation $z = e^x$ or $\ln z = x$, (we use Gumbel type with $\mu=0$ and $\sigma=\frac{1}{\theta}$, see Remark II.1.) in equations (3), (6), (7), (9), (10), (11), and (12), we obtain the following (after changing the variables back to x and y):

B.1. Bivariate Extreme Distributions (with Fréchet margins)

The equation that corresponds to equation (3) is

$$F(x,y) = \exp[-(x^{-\theta}+ y^{-\theta})\, g(\theta(\ln y - \ln x))]$$

$$= \exp[-(x^{-\theta}+ y^{-\theta})\, g(\theta \ln \tfrac{y}{x})], \quad \theta=\tfrac{1}{\sigma} > 0,\ x \geq 0,\ y \geq 0, \tag{20}$$

and $F(x,y) = 0$, for $x < 0,\ y < 0$,

where X and Y are the original variates (with $\mu = 0$).

The function g is the dependence function given in Remarks A.1. The choice of g, as g_1 or g_2 , will yield (respectively):

(i) The a – case:

$$F(x,y;a) = \begin{cases} 0, & \text{if } x < 0,\ y < 0, \\ \exp[-(x^{-\theta}+ y^{-\theta}) + a(x^{\theta}+ y^{\theta})^{-1}], & \text{if } x \geq 0,\ y \geq 0, \end{cases} \tag{21}$$

where $\theta=\tfrac{1}{\sigma} > 0$ is a parameter and $0 \leq a \leq 1$.

(ii) The m – case:

$$F(x,y;m) = \begin{cases} 0, & \text{if } x < 0,\ y < 0, \\ \exp[-(x^{-m\theta}+ y^{-m\theta})^{\frac{1}{m}}], & \text{if } x \geq 0,\ y \geq 0, \end{cases} \tag{22}$$

where $\theta=\tfrac{1}{\sigma} > 0$ is a parameter and $1 \leq m \leq \infty$.

B.2. Rotation of the Axes by 45°

(i) The a – case:

$$F(x',y';a) = \begin{cases} 0, & \text{if } x' < 0, y' < 0, \\ \exp[-2(x')^{-\frac{\sqrt{2}}{2}\theta}ch(\frac{\sqrt{2}}{2}\theta \ln y')(1-\frac{a}{4}\,sech^2(\frac{\sqrt{2}}{2}\theta \ln y'))], & \text{if } x' \geq 0, y' \geq 0, \end{cases} \quad (23)$$

where $\theta > 0$, and $0 \leq a \leq 1$.

(ii) The m – case:

$$F(x',y';m) = \begin{cases} 0, & \text{if } x' < 0, y' < 0, \\ \exp[-2(x')^{-\frac{\sqrt{2}}{2}\theta}\, ch(\frac{\sqrt{2}}{2}\theta \ln y')\, \dfrac{((y')^{m\theta\sqrt{2}}+1)^{\frac{1}{m}}}{((y')^{\theta\sqrt{2}}+1)}], & \text{if } x' \geq 0, y' \geq 0, \end{cases} \quad (24)$$

where $\theta > 0$, and $1 \leq m \leq \infty$.

Note that the two cases above (equations (23) and (24)) can be written as

$$F(x',y') = \begin{cases} 0, & \text{if } x' < 0, y' < 0, \\ \exp[-2(x')^{-\frac{\sqrt{2}}{2}\theta}\, ch(\frac{\sqrt{2}}{2}\theta \ln y')\, g(\ln(y')^{\sqrt{2}\theta})], & \text{if } x' \geq 0, y' \geq 0, \end{cases} \quad (25)$$

where $\theta > 0$, and g is as given in Remarks A.1.

B.3. The Joint Density Functions

(i) The a – case:

$$f(x,y;a) = F(x,y;a)\,(xy)^{-\theta}[1 - a(x^{2\theta} + y^{2\theta})(x^\theta + y^\theta)^{-2} + 2a(xy)^{2\theta}(x^\theta+y^\theta)^{-3}$$
$$+ a^2\,(xy)^{2\theta}(x^\theta + y^\theta)^{-4}], \quad (26)$$

where F(x,y;a) is the distribution function given by (21), $\theta > 0$, $0 \leq a \leq 1$, $x \geq 0$, $y \geq 0$, and f(x,y,;a) = 0, otherwise.

(ii) The m – case:

$$f(x,y;m) = F(x,y;m)(xy)^{-m\theta}(x^{-m\theta} + y^{-m\theta})^{-2+\frac{1}{m}}[\,m-1 + (x^{-m\theta} + y^{-m\theta})^{\frac{1}{m}}\,] \quad (27)$$

where $\theta > 0$, $1 \leq m \leq \infty$, $x \geq 0$, $y \geq 0$, and f(x,y;m) = 0, otherwise.

B.4. The Bivariate Mode

(i) The a-case:

To write the density function (26) in terms of x′ and y′, the 45°-rotated axes, using relations (8), we have $xy = \frac{1}{2}(x'^2 - y'^2)$, $x^{2\theta} + y^{2\theta} = (\frac{\sqrt{2}}{2})^{2\theta} [(x'-y')^{2\theta} + (x'+y')^{2\theta}]$, and $x^\theta + y^\theta = (\frac{\sqrt{2}}{2})^\theta [(x'-y')^\theta + (x'+y')^\theta]$.

Now, the Fréchet density function (26) takes the form

$$f(x',y';a) = F(x',y';a)(\frac{x'^2 - y'^2}{2})^{-\theta} [1 - a((x'-y')^{2\theta} + (x'+y')^{2\theta})((x'-y')^\theta + (x'+y')^\theta)^{-2}$$

$$+ 2a(\frac{x'^2 - y'^2}{2})^{2\theta} (\frac{\sqrt{2}}{2})^{-3\theta} ((x'-y')^\theta + (x'+y')^\theta)^{-3}$$

$$+ a^2 (\frac{x'^2 - y'^2}{2})^{2\theta} (\frac{\sqrt{2}}{2})^{-4\theta} ((x'-y')^\theta + (x'+y')^\theta)^{-4}],$$

$$= F(x',y';a)2^\theta (x'^2 - y'^2)^{-\theta} [1 - a((x'-y')^{2\theta} + (x'+y')^{2\theta})((x'-y')^\theta + (x'+y')^\theta)^{-2}$$

$$+ 2^{1-\frac{\theta}{2}} a(x'^2 - y'^2)^{2\theta} ((x'-y')^\theta + (x'+y')^\theta)^{-3}$$

$$+ a^2 (x'^2 - y'^2)^{2\theta} ((x'-y')^\theta + (x'+y')^\theta)^{-4}], \tag{28}$$

where $F(x',y';a)$ is as given (23), $\theta > 0$, $0 \le a \le 1$, $x' \ge 0$, $y' \ge 0$, and $f(x',y';a) = 0$, otherwise.

The modal location for the Fréchet can be obtained from the corresponding value for the Gumbel, equation(14), as follows $\widetilde{y_F} = e^{\widetilde{y'}} = e^0 = 1$, and

$$\widetilde{x_F} = \sqrt{2} \exp\{ \frac{1}{\sigma} [\ln\left(2(1-\frac{a}{4})^2\right) - \ln\left((1-\frac{a}{2})+\sqrt{1 - \frac{a}{2} + \frac{a^2}{8}}\right)]\}$$

$$= \sqrt{2} [\left(2(1-\frac{a}{4})^2\right)^\theta \left((1-\frac{a}{2}) + \sqrt{1 - \frac{a}{2} + \frac{a^2}{8}}\right)^{-\theta}], \tag{29}$$

where $\theta = \frac{1}{\sigma}$.

The modal value is, then, obtained by substituting $\widetilde{y_F}$ and $\widetilde{x_F}$, given above, into equation (28).

(ii) The m-case:

As in the a-case, the density (27), can be written as

$$f(x',y';m) = F(x',y';m)(x'^2 - y'^2)^{-m\theta} (2)^{\frac{\theta}{2}}((x' - y')^{-m\theta} + (x'+y')^{-m\theta})^{-2+\frac{1}{m}}$$

$$\times (m - 1 + (2)^{\frac{\theta}{2}} ((x' - y')^{-m\theta} + (x'+y')^{-m\theta})^{\frac{1}{m}}), \tag{30}$$

where $F(x',y';m)$ is given by (24), $\theta > 0$, $1 \le m \le \infty$, $x' \ge 0$, $y' \ge 0$, and

$f(x',y';m) = 0$, otherwise.

The modal location is obtained as in the a-case; $\tilde{y_F} = 1$, and

$$\tilde{x_F} = \sqrt{2} \exp[\tfrac{1}{\sigma} \{(\tfrac{1}{m} + 1) \ln 2 - \ln(- (m - 3) + \sqrt{ (m - 1)^2 + 4 })\}]$$

$$= \sqrt{2} (2)^{\theta(\frac{1}{m}+1)} (- (m - 3) + \sqrt{ (m - 1)^2 + 4 })^{-\theta}, \tag{31}$$

where $\theta = \tfrac{1}{\sigma}$.

C. Weibull Margins

For notational convenience we present here bivariate extreme distributions of smallest values, the bivariate distributions of largest extremes can be easily obtained from it, see equation (3)' of section II and the discussion therein. Using the transformation $z = -e^{-x}$ or $\ln(-z) = -x$, we obtain:

C.1. Bivariate Extreme Distributions (with Weibull margins)

$$F(x,y) = \begin{cases} \exp[-((-x)^\theta + (-y)^\theta) g(\theta \ln \tfrac{x}{y})], & x \le 0, y \le 0, \\ 0, & x > 0, y > 0, \end{cases} \tag{32}$$

where $\theta = \tfrac{1}{\sigma} > 0$, and X,Y are the original variates (with $\mu = 0$).

The dependence function g is as given in Remarks A.1.

(i) The a – case:

$$F(x,y;a) = \begin{cases} \exp[-((-x)^\theta + (-y)^\theta) + a((-x)^{-\theta} + (-y)^{-\theta})^{-1}], & x \le 0, y \le 0, \\ 0, & x > 0, y > 0, \end{cases} \tag{33}$$

where $\theta = \tfrac{1}{\sigma} > 0$ is a parameter and $0 \le a \le 1$.

(ii) The m – case:

$$F(x,y;m) = \begin{cases} \exp[-((-x)^{m\theta} + (-y)^{m\theta})^{\frac{1}{m}}], & x \leq 0, \ y \leq 0, \\ 0, & x > 0, \ y > 0, \end{cases} \qquad (34)$$

where $\theta = \frac{1}{\sigma} > 0$ is a parameter and $1 \leq m \leq \infty$.

C.2. Rotation of the Axes by 45°

(i) The a – case:

$$F(x',y';a) = \begin{cases} \exp[-2(-x')^{-\frac{\sqrt{2}}{2}\theta}\mathrm{ch}(\frac{\sqrt{2}}{2}\theta \ln(-y'))(1 - \frac{a}{4}\mathrm{sech}^2(\frac{\sqrt{2}}{2}\theta \ln(-y')))], & \\ & x' \leq 0, \ y' \leq 0, \\ 0, & x' > 0, \ y' > 0, \end{cases} \qquad (35)$$

where $\theta > 0$, and $0 \leq a \leq 1$.

(ii) The m – case:

$$F(x',y';m) = \begin{cases} \exp[-2(-x')^{-\frac{\sqrt{2}}{2}\theta}\mathrm{ch}(\frac{\sqrt{2}}{2}\theta \ln(-y'))\frac{((-y')^{m\theta\sqrt{2}}+1)^{\frac{1}{m}}}{((-y')^{\theta\sqrt{2}}+1)}], & x' \leq 0, \ y' \leq 0, \\ 0, & x' > 0, \ y' > 0, \end{cases} \qquad (36)$$

where $\theta > 0$, and $1 \leq m \leq \infty$.

Again, we notice that both the cases above can be written as

$$F(x',y') = \begin{cases} \exp[-2(-x')^{-\frac{\sqrt{2}}{2}\theta}\mathrm{ch}(\frac{\sqrt{2}}{2}\theta \ln(-y'))\ g(\ln(-y')^{\sqrt{2}\theta})], & x' \leq 0, \ y' \leq 0, \\ 0, & x' > 0, \ y' > 0, \end{cases} \qquad (37)$$

where $\theta > 0$, and g is as given in Remarks A.1.

C.3. The Joint Density Functions

(i) The a – case:

$$f(x,y;a) = F(x,y;a)\ (xy)^{\theta}\ [1 - a((-x)^{-2\theta} + (-y)^{-2\theta})\ ((-x)^{-\theta} + (-y)^{-\theta})^{-2}$$

$$+ 2a\ (xy)^{-2\theta}((-x)^{-\theta} + (-y)^{-\theta})^{-3} + a^2(xy)^{-2\theta}((-x)^{-\theta} + (-y)^{-\theta})^{-4}], \qquad (38)$$

where $F(x,y;a)$ is the distribution function given by (33), $\theta > 0$, $0 \leq a \leq 1$, $x \leq 0$, $y \leq 0$,

and $f(x,y;a) = 0$, otherwise.

(ii) The m–case:

$$f(x,y;m) = F(x,y;m) \ (xy)^{m\theta}((-x)^{m\theta} + (-y)^{m\theta})^{-2 + \frac{1}{m}}$$

$$\times \ [m - 1 + ((-x)^{m\theta} + (-y)^{m\theta})^{\frac{1}{m}}], \tag{39}$$

where $F(x,y;m)$ is the distribution function given by (34), $\theta > 0$, $1 \le m \le \infty$, $x \le 0$, $y \le 0$, and $f(x,y;m) = 0$, otherwise.

C.4. The Bivariate Mode

The bivariate mode for the Weibull is similarly obtained as that of the Fréchet.

(i) The a-case:

The Weibull density function (38) takes the form

$$f(x',y';a) = F(x',y';a)2^\theta \ (x'^2 - y'^2)^{-\theta} \ [1 - a((x'-y')^{2\theta} + (x'+y')^{2\theta})((x'-y')^\theta + (x'+y')^\theta)^{-2}$$

$$+ \ 2^{1-\frac{\theta}{2}} \ (\text{-}1)^\theta \ a(x'^2 - y'^2)^{2\theta} \ ((x'-y')^\theta + (x'+y')^\theta)^{-3}$$

$$+ \ a^2 \ (x'^2 - y'^2)^{2\theta} \ ((x'-y')^\theta + (x'+y')^\theta)^{-4}], \tag{40}$$

where $F(x',y';a)$ is as given in (35), $\theta > 0$, $0 \le a \le 1$, $x' \le 0$, $y' \le 0$, and $f(x',y';a) = 0$, otherwise.

The modal location for the Weibull can be obtained from the corresponding value for the Gumbel, equation(14), as follows $\tilde{y_{\text{W}}} = -e^{-\tilde{y'}} = -e^0 = -1$, and

$$\tilde{x_{\text{W}}} = -\sqrt{2} \ \exp\{ \tfrac{1}{\sigma} \ [\ln \left(2(1-\tfrac{a}{4})^2\right) - \ln\!\left((1-\tfrac{a}{2}) + \sqrt{1 - \tfrac{a}{2} + \tfrac{a^2}{8}}\right)]\}$$

$$= -\sqrt{2} \ [\left(2(1-\tfrac{a}{4})^2\right)^{-\theta} \left((1-\tfrac{a}{2}) + \sqrt{1 - \tfrac{a}{2} + \tfrac{a^2}{8}} \ \right)^\theta], \tag{41}$$

where $\theta = \tfrac{1}{\sigma}$.

The modal value is, then, obtained by substituting $\tilde{y_{\text{W}}}$ and $\tilde{x_{\text{W}}}$, given above, into equation (40).

(ii) The m-case:

As in the a-case, the density (39), can be written as

$$f(x',y';m) = F(x',y';m)(x'^2 - y'^2)^{-m\theta} (2)^{\frac{\theta}{2}} (-1)^{\theta}((x'-y')^{-m\theta} + (x'+y')^{-m\theta})^{-2+\frac{1}{m}}$$

$$\times (m - 1 + (2)^{\frac{\theta}{2}} (-1)^{\theta}((x'-y')^{-m\theta} + (x'+y')^{-m\theta})^{\frac{1}{m}}) , \qquad (42)$$

where $F(x',y';m)$ is given by (36), $\theta > 0$, $1 \leq m \leq \infty$, $x' \leq 0$, $y' \leq 0$, and $f(x',y';m) = 0$, otherwise.

The modal location is obtained as in the a-case; $\tilde{y}_W = -1$, and

$$\tilde{x}_W = -\sqrt{2} \exp[\frac{-1}{\theta} \{(\frac{1}{m} + 1) \ln 2 - \ln(-(m-3) + \sqrt{(m-1)^2 + 4})\}]$$

$$= -\sqrt{2} (2)^{-\theta(\frac{1}{m}+1)} (-(m-3) + \sqrt{(m-1)^2 + 4})^{\theta}, \qquad (43)$$

where $\theta = \frac{1}{\sigma}$.

The modal value is, then, obtained by substituting $\tilde{y}_W$ and $\tilde{x}_W$, given above, into equation (42).

IV. PROPERTIES OF BIVARIATE EXTREME DISTRIBUTIONS

Bivariate extreme value distributions have several properties (for example, see Gumbel [3], and Tiago de Oliveira [19]); we only discuss a few here.

(1.) For any bivariate extreme value distribution the following holds

$$F_{(\infty)}(x,y) \geq F_{1(\infty)}(x) \, F_{2(\infty)}(y) \, ,$$

where $F_{1(\infty)}(x)$, and $F_{2(\infty)}(y)$ are the (marginal) univariate extreme distribution functions.

(2.) If $F_{(\infty)}(x,y)$ and $G_{(\infty)}(x,y)$ are two bivariate extreme distributions, then so is their weighted geometric mean

$$(\, F_{(\infty)}(x,y))^\lambda \, (G_{(\infty)}(x,y))^{1-\lambda} \, , \quad 0 \leq \lambda \leq 1.$$

(3.) Fréchet bounds. Let $F(x,y)$ be any bivariate distribution function with marginals $F_1(x)$ and $F_2(y)$. Then

$$\max \, (0, \, F_1(x)+F_2(y) \, -1) \leq F(x,y) \leq \min \, (F_1(x) \, , \, F_2(y)).$$

(4.) Each of $F(\cdot,\cdot;a)$ and $f(\cdot,\cdot;a)$, the bivariate extreme distribution and density functions, given by equations (6) and (7) respectively, is an increasing function of the parameter a , see Gumbel and Mustafi [5].

(5.) The Gumbel type distribution, $F_{1(\infty)}(x) = \exp(-e^{-x})$, has expected value γ (=0.5772$\cdots$), the Euler's constant, and variance $\frac{\pi^2}{6}$.

(6.) Since $\lim_{m\to\infty} (e^{-mx} + e^{-my})^{\frac{1}{m}} = \max(e^{-x} , e^{-y})$, it follows that

$$\lim_{m\to\infty} F(x,y;m) = \min[\exp(-e^{-x}), \exp(-e^{-y})].$$

(7.) The conditional distributions: We shall use the formula

$$F_{X\,|\,Y}(x_0|Y=y) = \frac{\frac{\partial}{\partial y}F(x,y)}{\frac{d}{dy}F_Y(y)} \qquad (1)$$

to drive the conditional distributions.

A. Gumbel margins

(i) The a–case: We have

$$F(x,y;a) = \exp[-(e^{-x}+e^{-y})+a\,(e^x+e^y)^{-1}], \quad 0 \le a \le 1,$$

thus

$$\frac{\partial F(x,y;a)}{\partial y} = \exp[-(e^{-x}+e^{-y})+a\,(e^x+e^y)^{-1}]\,(e^{-y}-a\,e^y(e^x+e^y)^{-2}).$$

Also,

$$\frac{d}{dy}\,F_y(y) = \frac{d}{dy}\,[\exp(-e^{-y})] = \exp(-e^{-y})\,(e^{-y}).$$

Using equation (1) above, we obtain

$$\begin{aligned}
F_{X\,|\,Y}(x_0|Y=y) &= \exp[-e^{-x_0}+a(e^{x_0}+e^y)^{-1}]\,(1-a\,e^{2y}(e^{x_0}+e^y)^{-2}) \\
&= \left(1-a\left(\frac{e^y}{e^{x_0}+e^y}\right)^2\right)\exp[-e^{-x_0}+a(e^{x_0}+e^y)^{-1}]
\end{aligned} \qquad (2)$$

where $0 \le a \le 1$, X and Y are reduced variates.

(ii) The m–case: We have

$$F(x,y;m) = \exp[-(e^{-mx}+e^{-my})^{\frac{1}{m}}], \quad 1 \le m \le \infty,$$

$$\begin{aligned}
\frac{\partial F(x,y;m)}{\partial y} &= \exp[-(e^{-mx}+e^{-my})^{\frac{1}{m}}]\,(-\tfrac{1}{m}(e^{-mx}+e^{-my})^{\frac{1}{m}-1}(-m)\,e^{-my}) \\
&= \exp[-(e^{-mx}+e^{-my})^{\frac{1}{m}}]\,(e^{-mx}+e^{-my})^{\frac{1}{m}-1}\,e^{-my}.
\end{aligned}$$

Also,

$$\frac{d}{dy}\,F_y(y) = \exp[-e^{-y}]\,(e^{-y}).$$

Equation (1) yields:

$$F_{X\,|\,Y}(x_0|Y=y) = (e^{-mx_0}+e^{-my})^{\frac{1}{m}-1}\exp[-(e^{-mx_0}+e^{-my})^{\frac{1}{m}}-(m-1)y+e^{-y}], \quad (3)$$

where $1 \le m \le \infty$ and X,Y are reduced variates.

If we let $z = e^{-mx_0}+e^{-my}$, equation (3) takes the form

$$F_{X\,|\,Y}(x_0|Y=y) = z^{\frac{1}{m}-1}\exp[-x^{\frac{1}{m}}-(m-1)y+e^{-y}]. \qquad (4)$$

Equation (3) can also be written as

$$F_{X \mid Y}(x_0 \mid Y=y) = \frac{(e^{-mx_0} + e^{-my})^{\frac{1}{m}-1}}{e^{(m-1)y}} \exp[-(e^{-mx_0} + e^{-my})^{\frac{1}{m}} + e^{-y}]$$

$$= \left(\frac{e^{-mx_0} + e^{-my}}{e^{-my}}\right)^{\frac{1}{m}-1} \exp[-(e^{-mx_0} + e^{-my})^{\frac{1}{m}} + e^{-y}] \tag{5}$$

where $1 \leq m \leq \infty$, X and Y are reduced variates.

Now, using the transformations given in section II, we can obtain the corresponding formulas for the Fréchet and Weibull margin cases.

B. Fréchet margins

(i) The a – case:

$$F_{X \mid Y}(x_0 \mid Y=y) = \left(1 - a\left(\frac{y^\theta}{x_0^\theta + y^\theta}\right)^2\right) \exp[-x_0^{-\theta} + a(x_0^\theta + y^\theta)^{-1}], \tag{6}$$

where $0 \leq a \leq 1$, $\theta = \frac{1}{\sigma} > 0$, and $x_0 \geq 0$, $y \geq 0$ are values of the variates X and Y.

(ii) The m – case:

$$F_{X \mid Y}(x_0 \mid Y=y) = \left(\frac{x_0^{-m\theta} + y^{-m\theta}}{y^{-m\theta}}\right)^{\frac{1}{m}-1} \exp[-(x_0^{-m\theta} + y^{-m\theta})^{\frac{1}{m}} + y^{-\theta}], \tag{7}$$

where $1 \leq m \leq \infty$, $\theta = \frac{1}{\sigma} > 0$, and $x_0 \geq 0$, $y \geq 0$ are values of the variates X and Y.

C. Weibull margins

(i) The a – case:

$$F_{X \mid Y}(x_0 \mid Y=y) = \left(1 - a\left(\frac{(-y)^{-\theta}}{(-x_0)^{-\theta} + (-y)^{-\theta}}\right)^2\right) \exp[-(-x_0)^\theta + a((-x_0)^\theta + (-y)^{-\theta})^{-1}], \tag{8}$$

where $0 \leq a \leq 1$, $\theta = \frac{1}{\sigma} > 0$, and $x_0 \leq 0$, $y \leq 0$ are values of the variates X and Y.

(ii) The m – case:

$$F_{X \mid Y}(x_0 \mid Y=y) = \left(\frac{(-x_0)^{m\theta} + (-y)^{-m\theta}}{(-y)^{m\theta}}\right)^{\frac{1}{m}-1} \exp[-((-x_0)^{m\theta} + (-y)^{m\theta})^{\frac{1}{m}} + (-y)^\theta], \tag{9}$$

where $1 \leq m \leq \infty$, $\theta = \frac{1}{\sigma} > 0$, and $x_0 \leq 0$, $y \leq 0$ are values of the variates X and Y.

V. PARAMETER ESTIMATION

There are five parameters for each type (Gumbel, Fréchet, and Weibull). These are the location and dispersion (shape or scale) parameters μ and σ for each of the two variates and the association parameter a or m as the case may be.

A. The location and dispersion parameters

The marginal reduced variates are

$$Z_1 = \frac{1}{\sigma_1}(X - \mu_1) \text{ and } Z_2 = \frac{1}{\sigma_2}(Y - \mu_2).$$

The estimates $\hat{\mu}_i$, i=1,2, are the modal values, and the values of the shape parameters $\hat{\sigma}_i$, i=1,2, can be calculated from the separate marginal distributions using any of the following methods:

1. The method of maximum likelihood.
2. The method of moments.
3. Best linear unbiased estimates (Lieblein's method).
4. Best linear invariant estimators.

The method of maximum likelihood yields a system of equations whose exact solutions are difficult to obtain. However, Kimball developed a simpler procedure to get approximate solutions. This is described by Kimball in some sections of Gumbel's book, pp. 229-235 of [3].

The method of moments, as proposed by Gumbel [3], for the variate X gives (and similar corresponding results for Y)

$\hat{\sigma}_1 Z_1 = X - \mu_1$, where X_1 is a reduced variate.

Thus,

$\hat{\sigma}_1 \gamma = \overline{X} - \hat{\mu}_1$ or $\hat{\mu}_1 = \overline{X} - \gamma \hat{\sigma}_1$,

where $\gamma(=0.5772)$ is Euler's constant.

Also,

$$\hat{\sigma}_1^2 \, \frac{\pi^2}{6} = \text{var}(x) = s_x^2 \quad \text{or} \quad \hat{\sigma}_1 = \frac{\sqrt{6}}{\pi} \, s_x \ ,$$

where $S_x^2 = \sum_{i=1}^{N} \frac{(X_i - \bar{X})^2}{N}$.

If the sample size N is small γ and $\frac{\pi}{\sqrt{6}}$ are replaced by the corresponding population parameters $\bar{\mu}_N$ and σ_N , which are tabulated as a function of the sample size N. This is referred to as "Gumbel modified moment method." In this case, we have

$$\hat{\mu}_1 = \bar{X} - \hat{\sigma}_1 \, \hat{\mu}_N \quad \text{and} \quad \hat{\sigma}_1 = \frac{S_x}{\sigma_N} \ .$$

Lieblein proposed to estimate a parameter θ by $\hat{\theta}$, a linear combination of order statistics in which weights are determined to yield minimum variance unbiased estimators. In the 4th method of best linear invariant estimators, the condition of unbiasedness was relaxed from Lieblein's method.

Gumbel's moment method is the most simple of all four methods.

B. The a and m parameters

There are three possible methods that can be used to estimate the parameters a and m. The first method is based on the contents of the quadrants (the quadrant method). The second method is based on relating the parameters to a correlation coefficient. The third method (the strip method) is based on the distribution of $|x - y|$.

We shall discuss these methods briefly, but first we need the following relations.

$$
\begin{aligned}
F(x,x;a) &= \exp\left[-2\,e^{-x} + \tfrac{a}{2}\,e^{-x}\right] \\
&= \exp\left[-\left(2 - \tfrac{a}{2}\right)e^{-x}\right] \\
&= \exp\left[-e^{-(x - \ln(2 - \frac{a}{2}))}\right] \\
&= F(x - \ln(2 - \tfrac{a}{2})) \\
&= [F(x)]^{2 - \frac{a}{2}} \ ,
\end{aligned}
\qquad (1)
$$

where $F(x)$ is the first asymptotic (univariate) distribution $(F(x) = \exp(-e^{-x}))$ and $0 \le a \le 1$.

$$F(x,x;m) = \exp(-2^{\frac{1}{m}} e^{-x})$$
$$= [F(x)]^{2^{\frac{1}{m}}}, \qquad (2)$$

where $F(x)$ is the first asymptotic distribution and $1 \le m \le \infty$.

To obtain the median $\breve{x}$ we set

$$F(\breve{x}) = \exp(-e^{-\breve{x}}) = \tfrac{1}{2}$$

$$\Rightarrow e^{-\breve{x}} = -\ln(\tfrac{1}{2}) \quad \text{or} \quad \breve{x} = -\ln(-\ln\tfrac{1}{2})$$

Thus $\quad \breve{x} = \breve{y} = 0.3665129$.

B.1. The Quadrant method (see Gumbel and Mustafi [5])

(i) The a – parameter

The population medians of the marginal distribution divide the domain of variation of x and y into 4 quadrants as shown. The number of sample points, Q_1 and Q_3, contained in quadrants 1 and 3 are obtained. The "critical content" is defined by $\overline{Q}_{1,3} = \frac{Q_1 + Q_3}{2N}$, where N is the total number of sample points.
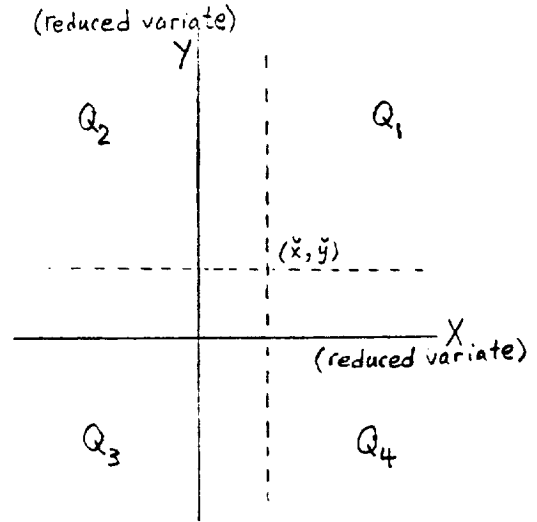
Note that with increasing a, i.e. increasing association, the distribution tends to concentrate along the diagonal. Therefore, the maximum likelihood estimate of a is the value $\hat{a}$ such that, (from equation (1)).

$$F(x,x;a) = \left(e^{-e^{-x}}\right)^{2-\frac{\hat{a}}{2}} = \overline{Q}_{1,3} \quad \text{when } x = \breve{x}.$$

Since

$$F(\check{x},\check{x};a) = (\tfrac{1}{2})^{2-\frac{\hat{a}}{2}}$$

$$= \tfrac{1}{4}\,(\tfrac{1}{2})^{-\frac{\hat{a}}{2}}$$

$$= \tfrac{1}{4}\,(2)^{\frac{\hat{a}}{2}}\,,$$

it follows that

$$4\,\overline{Q}_{1,3}= 2^{\frac{\hat{a}}{2}}$$

or

$$\ln 4 + \ln \overline{Q}_{1,3}= \tfrac{\hat{a}}{2}\ln 2$$

$$\frac{2\ln 4 + 2\ln \overline{Q}_{1,3}}{\ln 2} = \hat{a}$$

$$4 + \frac{2\ln \overline{Q}_{1,3}}{\ln 2} = \hat{a}\,. \tag{3}$$

Because of the fact that $0 \le a \le 1$, we see that

$$F(\check{x},\check{x};0) = (\tfrac{1}{2})^{2-\frac{0}{2}} = \tfrac{1}{4}\,,\ \text{and}$$

$$F(\check{x},\check{x};1) = (\tfrac{1}{2})^{\frac{3}{2}} = 0.35355.$$

Therefore, if $\overline{Q}_{1,3}$ lies outside the interval $[0.25,\ 0.35355]$, the quadrant method cannot

be used to estimate the parameter a.

(ii) The m – parameter

From equation (2), we have $(a+x=\check{x})$

$$F(\check{x},\check{x};m) = (\tfrac{1}{2})^{2^{\frac{1}{m}}}.$$

Since $1 \le m \le \infty$, we see that

$$F(\check{x},\check{x};1) = (\tfrac{1}{2})^2 = \tfrac{1}{4},\ \text{and}$$

$$F(\check{x},\check{x};\infty) = (\tfrac{1}{2})^{2^0} = \tfrac{1}{2}\,.$$

Therefore, if $\overline{Q}_{1,3}$ is outside the interval $[\tfrac{1}{4},\tfrac{1}{2}]$ the quadrant method cannot be used

to estimate m. Otherwise, as in the a – case,

$$\overline{Q}_{1,3} = (\tfrac{1}{2})^{2^{\frac{1}{\hat{m}}}}$$

$$\ln \overline{Q}_{1,3} = -2^{\frac{1}{\hat{m}}} \ln 2$$

$$\ln(-\ln \overline{Q}_{1,3}) = \tfrac{1}{\hat{m}} \ln 2 + \ln \ln 2$$

$$\tfrac{1}{\hat{m}} = \frac{\ln(-\ln \overline{Q}_{1,3}) - \ln(\ln 2)}{\ln 2} \; . \tag{4}$$

Note that if $\overline{Q}_{1,3}$ is such that $0.25 \leq \overline{Q}_{1,3} \leq 0.35355$, both a and m can be estimated by the quadrant method.

B.2. (i) Estimating the parameter a using the grade correlation coefficient

The grade correlation coefficient was introduced by Hoeffding [6]. Let t be the difference sign correlation coefficient, k′ be the (Spearman) rank correlation coefficient, and k be the grade correlation coefficient. If Z_1, Z_2, ..., Z_n, $n \geq 2$, is a random sample from a bivariate population F (see Konijn[9]), where $X_i = (X_i, Y_i)$, then

$$\frac{(n-1)(n+1)}{3n^2} k' = \frac{1}{n^3} \sum_{i,j,k} \text{sgn}(X_i - X_j)\, \text{sgn}(Y_i - Y_k)$$

$$= \frac{4}{n^3} \sum_{i} (R_{(i)} - \tfrac{n+1}{2})(S_{(i)} - \tfrac{n+1}{2}); \tag{5}$$

$$t = \frac{1}{n(n-1)} \sum_{i \neq j} \text{sgn}(X_i - X_j)\, \text{sgn}(Y_i - Y_j); \tag{6}$$

and

$$k = \begin{cases} \frac{3}{n(n-1)(n-2)} \sum_{\substack{i \neq j \\ j \neq k \\ i \neq k}} \text{sgn}(X_i - X_j)\, \text{sgn}(Y_i - Y_k), & \text{for } n > 2 \\ 0, & \text{for } n=2, \end{cases} \tag{7}$$

where $R_{(i)}$ is the rank of X_i, $S_{(i)}$ is the rank of Y_i, and $\text{sgn}(u) = \begin{cases} -1 & \text{if } u < 0 \\ 0 & \text{if } u=0 \\ +1 & \text{if } u > 0 \end{cases}$.

We also have

$$k' = \frac{(n-2)k + 3t}{n+1} \tag{8}$$

The grade correlation coefficient k can be expressed as, see Konijn [9],

$$\frac{k+3}{12} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x,y)\, dF_1(x)\, dF_2(y), \tag{9}$$

where $F(x,y)$ is the bivariate probability distribution function, and $F_1(x)$, $F_2(y)$ are the marginal distributions.

For the Gumbel a – case, the right side of (9) becomes

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-x-y-2\,e^{-x}-2\,e^{-y} + a(e^x + e^y)^{-1}]\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-x-y-2\,e^{-x}(1+\tfrac{e^{-y}}{e^{-x}}) + a\,e^{-x}(1+\tfrac{e^y}{e^x})^{-1}]\, dx\, dy. \tag{10}$$

Let $e^{y-x} = e^t$ or $y-x=t$, $y=t+x$, and (10) becomes

$$I = \int_{-\infty}^{\infty} e^{-t} \int_{-\infty}^{\infty} \exp[-2x - e^{-x}(2(1+e^{-t}) - a(1+e^t)^{-1})]\, dx\, dt.$$

The inner integral is

$$\int_{-\infty}^{\infty} e^{-2x-Ae^{-x}}\, dx, \quad \text{where} \quad A = 2(1+e^{-t}) - a(1+e^t)^{-1}$$

$$= \tfrac{1}{A} \int_{-\infty}^{\infty} e^{-x}\, d(e^{-Ae^{-x}})$$

$$= \tfrac{1}{A} \left[e^{-x}\, e^{-Ae^{-x}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x}\, e^{-Ae^{-x}}\, dx \right]$$

$$= \tfrac{1}{A} \left[0 + \tfrac{1}{A} \int_{-\infty}^{\infty} d(e^{-Ae^{-x}}) \right] = \tfrac{1}{A^2}.$$

Therefore,

$$I = \int_{-\infty}^{\infty} \frac{e^{-t}}{\{2(1+e^{-t}) - a(1+e^t)^{-1}\}^2}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{e^{-t}(1+e^t)^2}{\{2(1+e^{-t})(1+e^t) - a\}^2}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{e^{-t}(1+e^t)^2}{\{2(2+e^t+e^{-t}) - a\}^2}\, dt.$$

Let $1+e^t = z$, $e^t dt = dz$, then

$$I = \int_{1}^{\infty} \frac{e^{-2t}\, z^2\, dz}{\{2(2\,e^t+e^{2t}+1-\tfrac{a}{2}\,e^t)/e^t\}^2}$$

$$= \int_{1}^{\infty} \frac{z^2}{\{2(1+e^t)^2 - \tfrac{a}{2}(z-1)\}^2}$$

$$= \tfrac{1}{4} \int_{1}^{\infty} \frac{z^2\, dz}{(z^2 - \tfrac{a}{2}\,z + \tfrac{a}{2})^2}.$$

Thus equation (9) can be written as

$$1 + \tfrac{k}{3} = \int_1^\infty \frac{z^2 \, dz}{(z^2 - \tfrac{a}{2} z + \tfrac{a}{2})^2} := I .$$

(11)

$$I = \begin{cases} \tfrac{a}{q}(1 + \tfrac{\pi}{\sqrt{q}} - \tfrac{2}{\sqrt{q}} \tan^{-1} \tfrac{2 - \tfrac{a}{2}}{\sqrt{q}}), & \text{if } q > 0, \\ 1, & \text{if } q = 0, \end{cases}$$

where $q = a(2 - \tfrac{a}{4})$, $0 < a \leq 1$.

Therefore,

$$1 + \tfrac{k}{3} = \begin{cases} \tfrac{a}{q}(1 + \tfrac{\pi}{\sqrt{q}} - \tfrac{2}{\sqrt{q}} \tan^{-1} \tfrac{2 - \tfrac{a}{2}}{\sqrt{q}}), & \text{if } q > 0, \\ 1, & \text{otherwise.} \end{cases}$$

(12)

A table is given in Gumbel and Mustafi [5], which shows the grade correlation coefficient k as a function of a (k is almost a linear function of a). Hence a can be estimated from the sample grade correlation coefficient using (12).

B.2. (ii) Estimating the parameter m using the standard deviation of the difference of the reduced variates

The quadrant method was based on the value of the critical content $\overline{Q}_{1,3}$. The conditions $0 \leq a, \tfrac{1}{m} \leq 1$ on the parameters impose conditions on $\overline{Q}_{1,3}$, and hence some restrictions on the quadrant method. Another method is available to estimate the parameter m.

The difference of the two reduced variates for the bivariate extreme value distribution is a logistic distribution (this does not hold for the two dependent reduced variates in the a - case).

Let $t = Y - X$, X and Y are reduced variates, then the probability distribution of t is
$F(t) = (1 + e^{-mt})^{-1}$.

The t variate has zero mean and standard deviation

$$\sigma_t = \frac{\pi}{\sqrt{3}m} .$$

(13)

This leads to the following estimate of m

$$\hat{m} = \frac{\pi}{\sqrt{3}S_t} ,$$ (14)

where S_t is the sample value of σ_t (obtained from the observed reduced differences).

Also, it is known that the correlation coefficient ρ between the reduced variates X and Y is related to m by

$$m = (1-p)^{-\frac{1}{2}} .$$ (15)

This gives another estimate of m

$$\hat{m} = (1-\hat{p})^{-\frac{1}{2}} .$$ (16)

B.3. The "strip estimate"

Posner et al. [11] proposed a method of estimating a or m based on the distribution of $|X-Y|$. Recall that in general the bivariate extreme value distribution, see equation (III.3), is given by

$$F(x,y) = \exp[-(e^{-x} + e^{-y}) g(y-x)],$$

where g is the dependence function.

In this case they proved that

$$P \{a \leq X - Y \leq b\} = h(b) - h(a) ,$$ (17)

where

$$h(t) = (1 + e^{-t})^{-1} + \frac{g'(t)}{g(t)}$$ (18)

In particular, if g is given by

$$g(t) = 1 - \frac{a}{4} \operatorname{sech}^2(\frac{t}{2}) := 1 - \frac{a}{4} u(t) ,$$

then

$$P\{|X-Y| \leq \delta\} = \frac{e^{\delta}-1}{e^{\delta}+1} + \frac{2g'(\delta)}{g(\delta)} := P(\delta) .$$

If $\nu_N(\delta)$ is the number of times $|X-Y| \leq \delta$ in a sample of size N, then $\frac{\nu_N(\delta)}{N}$ is an estimator of the probability $P(\delta)$ above. Thus an estimator â of a is obtained by solving the equation

$$\frac{\nu_N(\delta)}{N} = \frac{e^\delta - 1}{e^\delta + 1} + \frac{2g'(\delta)}{g(\delta)}$$

$$= \frac{e^\delta - 1}{e^\delta + 1} - \frac{2(\frac{a}{4}) \, u'(\delta)}{1 + (\frac{a}{4}) \, u(\delta)} , \tag{19}$$

where

$$u(t) = \operatorname{sech}^2\left(\tfrac{t}{2}\right), \text{ and}$$

$$u'(t) = -\operatorname{sech}^2\left(\tfrac{t}{2}\right) \tanh\left(\tfrac{t}{2}\right).$$

For the m – case:

$$g(t) = \frac{(e^{mt} + 1)^{\frac{1}{m}}}{e^t + 1} , \quad 1 \le m \le \infty.$$

$$g'(t) = \frac{(e^t + 1)(e^{mt} + 1)^{\frac{1}{m} - 1} e^{mt} - (e^{mt} + 1)^{\frac{1}{m}} e^t}{(e^t + 1)^2} .$$

$$\frac{g'(t)}{g(t)} = \frac{e^{mt}}{e^{mt} + 1} - \frac{e^{mt}}{e^{mt} + 1} .$$

Therefore, an estimator $\hat{m}$ of a is obtained by solving the equation

$$\frac{\nu_N(\delta)}{N} = \frac{e^\delta - 1}{e^\delta + 1} + 2\left(\frac{e^{\hat{m}\delta}}{e^{\hat{m}\delta} + 1} - \frac{e^\delta}{e^\delta + 1}\right)$$

$$= \frac{2\,e^{\hat{m}\delta}}{e^{\hat{m}\delta} + 1} - 1$$

$$= \frac{e^{\hat{m}\delta} - 1}{e^{\hat{m}\delta} + 1} . \tag{20}$$

VI. HYPOTHESES TESTING AND GOODNESS-OF-FIT TEST

Tests of hypotheses on parameters can be done by relating the acceptance region to the confidence region. Confidence interval estimation is available in the literature, see for example [19], pp. 49-80.

A goodness-of-fit test for the smallest extreme is Gumbel distribution against the alternative, which is Weibull extreme distribution; this was discussed by Montfort [10].

Another test of the hypotheses that the largest extreme is Gumbel against the alternative that it is either Weibull or Fréchet is due to Tiago de Oliveira [17,18].

As pointed out in B.2.(ii), if $t=Y-Y$, where X and Y are dependent reduced variates of the Gumbel type, then t has a logistic distribution in the m−case but not the a−case. This fact could be used to discriminate between the choice of the a−case and the m−case.

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In certain engineering applications, such as those occurring in the analyses of ascent structural loads for the Space Transportation System (STS), some of the load variables have a lower bound of zero. Thus, the need for practical models of bivariate extreme value probability distribution functions with lower limits was identified. We discuss the Gumbel models and present practical forms of bivariate extreme probability distributions of Weibull and Fréchet types with two parameters. Bivariate extreme value probability distribution functions can be expressed in terms of the marginal extremel distributions and a "dependence" function subject to certain analytical conditions. Properties of such bivariate extreme distributions, sums and difference of paired extremals, as well as the corresponding forms of conditional distributions, are discussed. Practical estimation techniques are also given.

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