

OCEAN TIDE MODELS
FOR SATELLITE GEODESY
AND EARTH ROTATION

(initially titled "Effects of the Oceans on Polar Motion")

Final Technical Report

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SUMMARY

Research carried out under NASA grant NAG 5-145 (including its 11 supplements) has produced a theory which predicts tides in turbulent, self-gravitating and loading oceans possessing linearized bottom friction, realistic bathymetry and continents; at coastal boundaries "no-flow" conditions are imposed. The theory is phrased in terms of spherical harmonics, which allows the tide equations to be reduced to linear matrix equations. This approach also allows an ocean-wide mass conservation constraint to be applied. Solutions have been obtained for 32 long- and short-period luni-solar tidal constituents (and the pole tide), including the tidal velocities in addition to the tide height. Calibrating the intensity of bottom friction produces reasonable phase lags for all constituents; however, tidal amplitudes compare well with those from observation and other theories only for long-period constituents.

In the most recent stage of grant research, traditional theory (Liouville equations) for determining the effects of angular momentum exchange on Earth's rotation were extended to encompass high-frequency excitations (such as short-period tides). This required incorporating a frequency-dependent response of the oceans to the rotational perturbations induced initially by those excitations, as well as frequency-dependent core decoupling. Determination of the oceanic responses, and therefore actual calculation of short-period tidal effects on Earth's rotation, would not have been possible without the use of the tide theory previously developed by the author. Such calculations were carried out for a variety of initial excitations, including the 32 tide solutions previously obtained by the author and also a number of tide models determined by other theorists and from satellite altimetry -- with tide velocities of the latter models computed by a combination of my theory and the models' tide height harmonics.

A number of graduate student research projects were also supported by grant NAG 5-145. These projects helped to improve our theoretical understanding of the dynamic pole tide in the North Sea; the equilibrium pole tide world-wide, and its effects on the Chandler wobble; and the effects of long-period luni-solar ocean tides on the length of day.

Frontispiece

CONCRETE ACHIEVEMENTS OF NASA GRANT NAG 5-145
"Ocean Tide Models for Satellite Geodesy and Earth Rotation"

OCEAN WOBBLE

Wobble of the ocean/
solid-earth system:
static ocean response

Dickman 1983, 1985a

Wobble of the ocean/
solid-earth system:
dynamic response of
turbulent oceans

Nam & Dickman, in prep.

DYNAMIC TIDES

Spherical harmonic dynamic
pole tide theory:
global oceans

Dickman 1985b

Spherical harmonic dynamic
pole tide theory:
non-global oceans

Dickman & Steinberg 1986

Modifications for oceanic
loading & self-gravitation
and mantle elasticity

Dickman 1988a

Extension of tidal theory
to turbulent oceans

Complete theory of
luni-solar tides
(turbulent, loading
& self-gravitating
non-global oceans)

Dickman 1989

Modifications for global
tide-mass conservation

Dickman 1990

Frequency dependence of
tidal admittance;
satellite-constrained
dynamic ocean tide models

Dickman, 1991

High-frequency excitation of Earth rotation;
dynamic ocean tide effects on rotation

Dickman 1992

APPLICATIONS

Stokes' Paradox and
the North Sea
pole tide

Dickman & Prellsig 1986

Re-assessment of the
static pole tide

Oceanic "tidal" re-
sponse to atmospheric
pressure variations

Dickman 1988b

Ocean tide signals
in l.o.d.

Nam & Dickman 1990

FINAL TECHNICAL REPORT

The common, underlying theme of all research projects I have carried out under support of NASA grant NAG 5-145, including its eleven supplements and extending from March 1981 through December 1991, has been the role of the oceans in problems of central importance to geodesy. The primary achievement of my grant research -- see the *frontispiece* of this proposal as well as the discussions below -- was the development of a sophisticated theory for the prediction of tides in mid-ocean; its primary applications were to extend rotational theory to include excitations of polar motion and UT1 at frequencies as high as semi-diurnal, and to accurately predict ocean tidal effects on rotation using the extended theory.

This final technical report for Grant NAG 5-145 will be considerably shorter than one might consider appropriate for a final report encompassing more than 10 years of funded research. I believe such brevity is acceptable because a very detailed and comprehensive report had been submitted two and a half years ago, at the request of incoming NASA program managers.

REVIEW OF RESEARCH ACHIEVEMENTS DURING GRANT PERIOD

I. My ocean tide theory was initially developed in the context of the pole tide, the oceans' response to the Chandler wobble, in order to quantify the contributions of the oceans to the Chandler wobble period and decay rate. *Originally*, however, my grant research began with a more novel approach to the oceanic effects on polar motion. The idea was to treat the oceans as an integral body, capable of independent rotation except for the torques exerted on it by the solid earth which partially contains it. The oceans thus impart an extra "degree of freedom" to the rotating Earth. By analogy with the oscillations of a pair of coupled springs or pendulums, the *coupled*, rotating ocean/solid-earth system should possess at least two natural wobble modes. With strong coupling (and, of course, the oceans are almost perfectly coupled to the solid earth), one wobble mode of the system will correspond to the Chandler wobble -- yielding incidentally the oceanic contribution to the Chandler period. At the time this project was proposed, it was anticipated that the frequency of the second mode would be close to the first (for very strong coupling), thereby explaining the observed modulation ("beating") of the Chandler wobble amplitude.

The theory was worked out [Dickman 1983] for the case of an elastic mantle possessing a fluid core, with fluid oceans that respond statically to the incremental centrifugal force incurred from wobble. The torques acting between the oceans and solid earth were parameterized in terms of the relative angular velocity between the two bodies, and the wobble mode frequencies,

decay rates, and relative amplitudes were determined for a wide range of coupling strengths. Unexpectedly (though explainable in hindsight), it turned out that -- because of the shape and fluidity of the oceans -- the second wobble mode was necessarily *retrograde* (clockwise wobble motion); at any coupling strength it would not lead to a second Chandler frequency, since that wobble is *prograde*. Surprisingly, at high but physically reasonable coupling strengths, the second mode exhibited many characteristics of the controversial "Markowitz wobble", a low-amplitude ultra-long period apparent polar motion [see, e.g., Dickman 1981] which many researchers claim to be pure noise. Thus, the results from the first grant project appeared to indicate that an Earth with oceans possesses two natural wobble modes rather than just one.

The theory developed for that project, however, was based on an assumed equilibrium behavior of the fluid oceans; it was argued that this approximation would not be far wrong because the wobble periods are so long (in excess of a year). Such arguments were not universally accepted; it is not, after all, intuitively obvious how a wobble of the oceanic "body" could survive fluid dynamic motions [cf. Wahr 1984, Dickman 1985a].

The extent of fluid dynamic disruption of ocean/solid-earth wobble at physically reasonable coupling strengths is related to the extent to which the oceanic response to wobble departs from static; that is, it is related to the extent to which the pole tide is non-equilibrium. Observations of the pole tide for the past century have, almost without exception, indicated it to be significantly enhanced -- at least in a number of shallow seas, and probably in the deep ocean as well -- and capable of significant contributions to the Chandler wobble period and decay rate [see Naito 1979, Dickman 1979, Lambeck 1980]. The expected low amplitude of the pole tide (~ 0.5 cm for typical wobble amplitudes) and relatively high noise level of most tide data, however, cast doubt on the accuracy of the data and on the statistical reliability of even the best analyses. [But in time series analysis, long durations of data (and repetitions of the periodicity in question) always allow a low-amplitude signal to be picked out. It turns out (see below) that most likely the data analyses were not wrong, but misinterpreted: they were revealing coastal features of the pole tide, unrelated to its global character.] Clearly, it was time for a fluid dynamic theory of the pole tide.

II. Prior to my beginning the development of a dynamic pole tide theory, work by others [see, e.g., Munk & MacDonald 1960, Smith & Dahlen 1981, Carton & Wahr 1983 (based on a 1981 paper)] had suggested that dynamic effects were likely to be very small. Such conclusions had major implications for the extent of low-frequency mantle anelasticity, for core viscosity, and even for the evolution of the lunar orbit; they would also resolve long-standing questions about the nature of the Chandler wobble. Those conclusions, however, depended on approximations either to the equations or to the methods of solution; if the pole tide data analyses were indeed correct, it appeared that an *exact* fluid dynamic theory would be necessary.

I chose a spherical harmonic approach for two reasons: spherical harmonics are appropriate for a global description of tidal quantities; and, the effects of the pole tide are primarily the result of oceanic mass redistribution, i.e. changes in Earth's inertia generated by the pole tide, and the oceanic *products of inertia* could be easily represented as a single spherical harmonic [see Dickman 1985b]. Thus, the tide height T and

north-south, west-east tide current velocities u_θ , u_λ were expressed as complex variables, periodic in time with frequency σ ,

$$\begin{aligned} T &= \text{Re}\{\underline{T}\} & u_\theta &= \text{Re}\{\underline{u}_\theta\} & u_\lambda &= \text{Re}\{\underline{u}_\lambda\} \\ \underline{T} &= \hat{\underline{T}} \exp(i\sigma t) & \underline{u}_\theta &= \hat{\underline{u}}_\theta \exp(i\sigma t) & \underline{u}_\lambda &= \hat{\underline{u}}_\lambda \exp(i\sigma t); \end{aligned}$$

the time-independent portions of these variables were expanded in complex spherical harmonics,

$$\hat{\underline{T}} = \sum_{\ell,n} \underline{T}_\ell^n Y_\ell^n \quad \hat{\underline{u}}_\theta = \sum_{\ell,n} \underline{u}_\ell^n Y_\ell^n \quad \hat{\underline{u}}_\lambda = \sum_{\ell,n} \underline{v}_\ell^n Y_\ell^n$$

where Y_ℓ^n is the harmonic function (of colatitude θ and longitude λ) of degree ℓ and order n . The sets of unknown coefficients

$$\vec{T} = \{\underline{T}_\ell^n\} \quad \vec{U} = \{\underline{u}_\ell^n\} \quad \vec{V} = \{\underline{v}_\ell^n\}$$

are the quantities our theory must determine.

The frequency σ of the tide in this case would be that of the Chandler wobble, which causes the pole tide. However, the period and decay rate of the Chandler wobble are partially the result of pole tide effects (the 14-month Chandler period would be ~1 month shorter, and the wobble energy would be dissipated more slowly, if the Earth were oceanless). This "feedback" situation implies that the pole tide characteristics -- which depend on σ -- cannot be determined independently of σ itself. For this reason, and because no approximations would be made in the development, I began the project with a simple ocean model: non-turbulent, global oceans of uniform depth possessing bottom friction with constant drag coefficient.

The tide-governing equations, including two horizontal "momentum" equations and one equation of pointwise mass conservation ("continuity"), will be presented in full later in this report. For the relatively simple ocean model employed at this stage of work, the momentum equations could be solved analytically for the tide velocities in terms of the tide height, then substituted into continuity to yield a single differential equation for the tide height $\hat{\underline{T}}$. With $\hat{\underline{T}}$ expanded in spherical harmonics, it became necessary to formulate a general expression for derivatives of harmonic functions; after I developed the required relations [see Dickman 1985b for details] it became apparent that the theory would depend on the computation of triple harmonic product integrals, viz.

$$A_{\ell\rho j}^{nq\Delta} = \int (Y_\ell^n)^* Y_\rho^q Y_j^\Delta ds$$

(these integrals are over the unit sphere). Eventually it was possible to reduce the tide-governing equations to

$$(1) \quad \underline{\underline{B}} \cdot \vec{T} = \underline{\underline{M}}_p \vec{b},$$

where the polar motion forcing the tide is of amplitude $\underline{\underline{M}}_p$. For the ocean model considered at this stage the components of equation (1) are relatively

uncomplicated; for example, only the tide height coefficients $I_{2\ell}^{-1}$ are non-zero and need be determined [see Dickman 1985b for details and properties of $\underline{\beta}$ and $\underline{\delta}$].

Joint solution of equation (1) and the Liouville equation, the latter describing conservation of angular momentum, allowed the pole tide (\dot{T}) in global oceans, and its effects on polar motion ($\underline{\sigma}$), to be determined self-consistently. The computed solutions included the effects of both the tidal inertia (associated with changes in sea-level) and tidal relative momentum (associated with tidal currents). Overall, it was found that the pole tide in global oceans is very close to equilibrium; however, the slight dynamic effects on period and decay time implied by these results could well be amplified by other factors, such as non-globality or self-gravitation, in real oceans. In short, the ocean model used in this stage of my research was simply too preliminary to settle the question of pole tide effects on wobble.

The tidal theory was extended to include more realistic ocean models in later stages of the grant research. Such work is described in the sections following the next.

III. One region where the pole tide has been consistently observed with greatly enhanced amplitudes -- up to 10 times equilibrium -- is the North and Baltic Seas area. Such enhancements, in those seas alone, would be sufficient to dissipate up to half of the Chandler wobble energy [Dickman 1979; see also Dickman 1986]. Given the apparently nearly static behavior of the pole tide globally, it was important that such shallow-sea observations be verified theoretically.

A fluid dynamic model of the North Sea pole tide was indeed proposed in the mid-1970's [Wunsch 1974], and it appeared able to explain the observations -- in particular, its combination of depth-dependent bottom friction and shallowing to the south generated an "eastward intensification" of the tide height similar to what had been observed along the North Sea coast. However, Wunsch [1975] later declared, with little elaboration, that his results were erroneous.

In the earliest years of the grant period, I had supervised a graduate student (J. Preisig) who re-evaluated the analysis of Wunsch in light of his later *erratum*. Our review of Wunsch [1974 and 1975] had revealed that his solutions exhibited a mild eastward intensification; but they failed to satisfy one of his tidal equations and either did not satisfy fundamental boundary conditions or else depended on invalid approximations. We found that a slightly different model of the North Sea bathymetry avoided such difficulties. Our better-quality solution would later allow me to discover the essential problem with North Sea pole tide dynamics.

I began my grant research for this project by re-deriving the North Sea tide equations, paying special attention to the various approximations invoked in earlier works. Obtaining (or searching for) solutions to these equations required an extensive effort. All solutions [see Dickman & Preisig 1986] shared several noteworthy features: 1) they all exhibited an eastward intensification along the south shore of the Sea -- more so than Wunsch's solutions, and in good agreement with observation; 2) they all dissipated far more Chandler wobble energy than is physically possible (with the most intense

currents occurring in the northern areas); and 3) they all were non-unique solutions. The implication was that the North Sea tide equations themselves were deficient.

The non-uniqueness and extreme intensity of the solutions bear a resemblance to the elements of *Stokes' Paradox*; details may be found in Dickman & Preisig [1986]. Like Stokes' Paradox, these features suggested that the inclusion of *inertial* terms in the North Sea tide equations would produce a more reasonable solution. The inertial terms would either be time derivatives ($\partial \vec{u}/\partial t$ for the momentum equations, $\partial T/\partial t$ for continuity), which are normally a part of "Laplace" tide equations but were omitted by Wunsch [1974]; or -- more likely -- advective terms ($\vec{u} \cdot \nabla \vec{u}$ in the momentum equations), which are normally excluded from tide equations because they are non-linear, but which might be important in shallow, partially confined seas.

A rigorous fluid dynamic theory of the pole tide, including inertial terms, has yet to be worked out for the North Sea; thus, the observations of markedly amplified pole tide heights in the North and Baltic Seas have yet to be theoretically substantiated. In light of the work described in the next section -- which demonstrates that the dissipation of wobble energy by the deep-ocean pole tide is negligible -- the North-Baltic Seas observations alone stand in the way of attributing the bulk of Chandler wobble dissipation to mantle anelasticity. Without solution of this problem, our knowledge of mantle rheology at low frequencies remains speculative.

During this time I had just finished supervising another graduate student (D. Steinberg). He had (among other things) analyzed tide data at a number of ports -- mid-ocean, coastal, and shallow-sea; comparing the results with those from carefully constructed equilibrium pole tide time series, and accounting for the high noise level in the tide data, he showed that the pole tide in mid-Pacific (Hawaii) and mid-Atlantic (Bermuda) was statistically significantly greater than static, perhaps 1.5 to 2 times greater. The calculations of the static pole tide time series, incidentally, were based on an extension I developed of earlier theory using suggestions by Dahlen [1976]; that development led me to appreciate the distinction between "untruncated" (or *global*) and "truncated" (i.e. *restricted* to the oceans) tidal quantities [see Dickman & Steinberg 1986 for details]. That distinction would play a major role in my subsequent work.

The potential effects on the Chandler wobble of a global pole tide with the characteristics suggested by our careful analysis -- see Dickman & Steinberg [1986] -- were severe enough, and their consequences for mantle and core properties were important enough, that further theoretical work on pole tide dynamics in realistic oceans was warranted.

IV. The importance of accounting for sea-floor topography and land barriers (i.e. continents) in tidal theory is matched only by its difficulty. The extension of my global dynamic pole tide theory to the case of non-global oceans faced several obstacles. *First*, a variable ocean depth h makes the tide equations more complicated; with $h = h(\theta, \lambda)$ expanded in spherical harmonics ($h = \sum \underline{h}_\alpha Y_\alpha$), for example, the development would involve quadruple rather than triple products of harmonic functions. *Second*, an adequate

spherical harmonic description of oceanic bathymetry had been unavailable, and would need to be constructed. *Third*, the existence of oceanic boundaries requires that the tidal currents satisfy no-flow conditions at the boundaries. *Fourth*, the non-globality of the oceans (i.e. the presence of continents) changes the character of the polar motion, imparting an ellipticity to the (otherwise circular) Chandler pole path. This last difficulty implied that the polar motion, of complex amplitude $\underline{m} = m_x + im_y$, will contain prograde and retrograde components:

$$(2) \quad \underline{m} = \underline{M}_P e^{i\sigma t} + \underline{M}_R e^{-i\sigma^* t}.$$

Since the potential which forces the pole tide depends on \underline{m} , equation (2) implies that the tidal potential will be more complicated as well.

These problems were treated as follows [see Dickman 1988a for details].

1. Relations which allow products of harmonic functions (e.g. $Y_1^{\pm 1} Y_j^\Delta$, $Y_2^{\pm 1} Y_j^\Delta$, and so on) to be reduced to *sums* of harmonic functions were laboriously developed, then selectively applied to reduce quadruple products to triple products, for which the integrals ($A_{lpj}^{nq\Delta}$, see earlier) are easily computed. It was then possible to re-write the tide-governing equation as a single non-differential matrix equation of the form

$$(3) \quad \underline{B} \cdot \vec{T} = \underline{M}_P \vec{b}_P + \underline{M}_R^* \vec{b}_R$$

where now the collection of unknown tide height coefficients $\vec{T} = \{\underline{T}_l^n\}$ involves all degree and order harmonics. The elements of the matrix \underline{B} and vectors \vec{b}_P , \vec{b}_R , which are significantly more complicated than in the case of global oceans (—it now took ~ 5 sheets of notepaper to write them out symbolically), depend on the tidal frequency σ , bottom friction P , and the oceanic bathymetry coefficients \underline{h}_j^Δ .

2. At the time this research was performed, published values of \underline{h}_j^Δ were available only through degree and order 8. Given the other obstacles to this research, I decided to postpone construction of a more complete set of bathymetry coefficients (see next section). From the work leading to Dickman & Steinberg [1986], we had obtained two extensive sets of \underline{O}_j^Δ , the coefficients of the ocean function (see below), so that an accurate description of the continent - ocean distribution was already in hand. Thus, the results at this stage of my work would represent tides in non-global but flat-bottom (i.e. uniform depth) oceans.

3. The constraint that continents are barriers to tidal currents -- that is, there is no flow of tidewater across continental boundaries, is

$$\vec{u} \cdot \hat{n} = 0 \quad \text{at coastlines}$$

where \hat{n} is the unit normal to the coast and $\vec{u} = (u_\theta, u_\lambda)$ is the horizontal tide velocity. I re-wrote this "pointwise" constraint as the more suitable, global constraint

$$(4) \quad \vec{u} \cdot \vec{\nabla}(h_0 O) = 0 \quad \text{everywhere}$$

where h_0 is a constant (for scaling the boundary conditions) and O is the ocean function, defined as being zero at land locations and unity over the

oceans. With O expanded in spherical harmonics, and u_θ , u_λ expressed in terms of the tide height T , it was possible to reduce (4) to a single non-differential matrix equation,

$$(5) \quad \underline{B}' \cdot \vec{T} = \underline{M}_P \vec{b}'_P + \underline{M}_R^* \vec{b}'_R .$$

The pole tide in non-global oceans, constrained at coastlines, is determined as the joint solution to equations (3) and (5). Eventually, the approach I adopted was to use an equal number of tide equations (comprising (3)) and boundary condition equations (comprising (5)), thus altogether a greater number of equations than unknowns: an "overdetermined" situation; the actual matrix inversion procedure to obtain \vec{T} would yield a least-squares solution -- given the factor h_0 , a weighted least-squares solution.

4. Non-global oceans modify both the wobble frequency (σ) and the proportion of retrograde to prograde wobble motion, $\underline{Z} \equiv \underline{M}^*/\underline{M}_P$. I found a way to rewrite the Liouville equations in this case so that σ^R and \underline{Z} could be determined, once \underline{M}_P had been prescribed and a tide solution obtained; see Dickman [1988a] for details.

Thus, the self-consistent dynamic pole tide in non-global oceans would be found as the joint solution to Liouville equations and equations (3) & (5). The complexity of the problem, and need to search for joint solutions, meant that a supercomputer (like the CYBER 205 or IBM 3090 vector facility) was required. Early results suggested that non-globality had a tremendous effect on the Chandler wobble [see Dickman 1988a for details]. However, a number of theoretical tests eventually led me to realize that my solutions generate a non-zero tide on land, as an automatic artifact of analytical tide equations; that only the oceanic portion OT of the solution should be employed (e.g. in the Liouville equations) to determine tidal effects on rotation; and that the land portions of the solution could safely be ignored.

The "truncated" harmonics of the solution, i.e. the coefficients of OT , were easily computed from the solution for \vec{T} . The results indicate that pole tide dynamical effects on wobble are small. For example, tide dynamics add 5/3 days more to the Chandler period than a static pole tide would, but dissipate a negligibly small amount of wobble energy.

During this stage of research I also modified the non-global theory to account for the response of the elastic mantle to the tidal force, and to account (at this point somewhat approximately) for self-gravitation and loading by the tide layer. Table 3 in Dickman [1988a] (not presented here) summarizes the pole tide effects on wobble including these factors, for a range of bottom friction strengths. Although loading and self-gravitation increase the height of the ocean tide, the corresponding depression of the sea-floor reduces the solid earth's contribution to the Chandler period. The net effect, as shown in that table, is even less dissipation of wobble energy by the pole tide, and a lengthening of the Chandler period by just over 1 day more than that from a static tide.

At this point in my research, my tide theory was based on an ocean model which lacked turbulence, realistic bathymetric variations, and exact loading/self-gravitation. The long period of the Chandler wobble implied that none of these deficiencies would be likely to change our conclusions significantly [cf. Dickman 1989]. For the study of luni-solar ocean tides (almost all of which are much shorter period than the pole tide), these factors would of

course need to be included for accurate modeling.

The question of how the oceans respond to atmospheric pressure variations has been a recurrent one in wobble and l.o.d. research. With the recently recognized importance of atmospheric angular momentum in Earth rotation [e.g., Eubanks et al. 1985, Dickey & Eubanks 1986], an answer to that question becomes even more crucial: essentially, the extent to which the sea surface depresses under barometric high pressures (-- the ideal static response, the "inverted barometer", represents a perfect "isostatic" compensation) determines how much atmospheric pressure excitation of wobble or l.o.d. is canceled out.

My tidal theory could be used to resolve this question. An atmospheric pressure fluctuation of amplitude p_A is equivalent to a tidal potential perturbation of amplitude $-p_A/\rho_w$ [Défant 1961]. However, my theory had to be generalized because the barometric forcing could possess any spatial structure (and, for that matter, any time-dependence), not just the degree-2 spherical harmonic structure of a tidal forcing. Thus, we write

$$p_A = \text{Re}\{p_A\} \qquad p_A = (\sum_{\ell} a_{\ell}^n Y_{\ell}^n) e^{i\sigma t}$$

to describe the atmospheric forcing at frequency σ .

Letting $\vec{a} = \{a_{\ell}^n\}$ denote the "vector" atmospheric forcing, I found that the equations governing the "tide height" or oceanic response, $\vec{T} = \{T_{\ell}^n\}$, could be written

$$(6) \qquad \underline{\underline{B}} \cdot \vec{T} = -\frac{1}{\rho_w g} \underline{\underline{C}}_{\underline{\underline{B}}} \cdot \vec{a} .$$

The matrix $\underline{\underline{B}}$ would have been the same as in equation (3), except that at this point I had taken the opportunity to account for loading and self-gravitation exactly -- a generalization of my earlier tidal theory (atmospheric loading & self-gravitation were also included now). The "no-flow" oceanic boundary conditions now took the form

$$(7) \qquad \underline{\underline{B}}' \cdot \vec{T} = -\frac{1}{\rho_w g} \underline{\underline{C}}_{\underline{\underline{B}}'} \cdot \vec{a}$$

with similar comments about $\underline{\underline{B}}'$ in relation to equation (5). See Dickman [1988b] for details regarding $\underline{\underline{B}}$, $\underline{\underline{C}}_{\underline{\underline{B}}}$, $\underline{\underline{B}}'$, and $\underline{\underline{C}}_{\underline{\underline{B}}}'$.

Solutions to (6) and (7) were computed for atmospheric forcing described by a single harmonic of degree L , order N at period $2\pi/\sigma$. To evaluate the range of oceanic responses, I considered $2\pi/\sigma = 5$ days, 50 days, and 433.2 days; for each frequency I tried $L=2, N=0$; $L=2, N=1$; $L=5, N=5$; and $L=10, N=10$ (note that $L=2$ corresponds to global, $L=5$ to "basin"-scale, and $L=10$ to regional forcing). For flat ocean basins I found that the actual, dynamic response does not differ from the ideal, inverted barometer response by more than 10% - 15%, especially at forcing periods exceeding a week (my work did reveal a slight, 9-day resonance, however). The effects of bathymetry and turbulence, both of which could be very influential at moderate or short periods, were not considered at this stage.

V. The addition of turbulence to my ocean model required extensive re-working of the spherical harmonic theory. The mathematical complexity of the turbulent drag forces prevented the three starting equations (2 momentum, 1 continuity) from being combined into a single equation reducible to matrix form; instead, each equation had to be written directly as a matrix equation. This in turn resulted in the appearance of modified harmonic product integrals,

$$\int Y_{\ell}^N Y_{\ell}^N \sin\theta \, ds$$

which are similar to $A_{\ell j p}^{n \Delta q}$ but which had never been solved before. Once I had worked these out, it was possible to write the momentum equations symbolically as

$$\underline{\underline{M}} \cdot \underline{\underline{U}} + \underline{\underline{C}} \cdot \underline{\underline{V}} = \underline{\underline{\Delta}}(\underline{\underline{M}}_P, \underline{\underline{M}}_R, \underline{\underline{T}})$$

$$\underline{\underline{M}} \cdot \underline{\underline{U}} + \underline{\underline{C}} \cdot \underline{\underline{V}} = \underline{\underline{T}}(\underline{\underline{M}}_P, \underline{\underline{M}}_R, \underline{\underline{T}}) ,$$

relating the unknown tide velocities to the tide height. With these solved for $\underline{\underline{U}}$ and $\underline{\underline{V}}$ in terms of $\underline{\underline{T}}$, they could be substituted into the continuity equation, yielding a matrix equation for $\underline{\underline{T}}$ analogous to equation (3); when substituted into the boundary constraint, (4), they would yield a second matrix equation for $\underline{\underline{T}}$, analogous to (5). Details of all this may be inferred from Dickman [1989].

The preceding relation between tide velocities and tide height is, to my knowledge, the first of its kind to be developed. When applied to luni-solar tides, for example, it would allow tidal velocities to be determined from the various tide height models published by other researchers (whose theories yield T but not u_{θ} , u_{λ}). It would also allow tidal velocities to be inferred from SLR or satellite altimetry observations of tide heights.

The development of the turbulent theory also required that a modified ocean depth

$$\tilde{h} = h \sin\theta$$

be expanded in spherical harmonics. At this point in my research I computed the values of \tilde{h}_{Δ} from those of h_{Δ} theoretically, for the case of flat ocean basins (where $h_{\Delta} = h_0 \cdot \mathcal{O}(\theta, \lambda)$). The results for the pole tide in turbulent, non-global, self-gravitating & loading oceans possessing bottom friction but no bathymetry were -- not surprisingly -- very similar to the non-turbulent results (except when I employed unrealistically large turbulent viscosity coefficients).

Around this time I began to construct my own set of spherical harmonic coefficients for \mathcal{O} , h , and \tilde{h} directly from actual bathymetry data. The data was kindly provided by J. Marsh at NASA/GSFC, in the form of the DBDB5 5' x 5' digital bathymetry tape.

VI. The preceding research had answered questions about pole tide dynamics in fairly realistic oceans, and about pole tide effects on wobble; the work also had implications for dissipation of wobble energy by mantle anelasticity and core viscosity. At this point I was in possession of a pretty good pole tide model, and I set about applying it to the case of luni-solar ocean tides.

The luni-solar tidal potential is of degree 2 order s ($|s| = 0, 1, \text{ or } 2$ for long-period, diurnal, or semi-diurnal tides), and has frequency σ . We wrote it as

$$(8) \quad U = \text{Re}\{\underline{U}\}, \quad \underline{U} = \frac{M_0}{2} Y_2^s e^{i\sigma t}$$

where M_0 is the complex forcing amplitude.

The momentum equations appropriate for the tide in self-gravitating and loading turbulent oceans are

$$(9) \quad \frac{\partial}{\partial t} u_\theta - f u_\lambda = -\frac{g}{a} \frac{\partial}{\partial \theta} \left[\text{Re} \left\{ \sum_{\ell, n} \sum_{\rho, q} L_{\ell\rho}^{nq} Y_{\ell\rho}^n \exp(i\sigma t) \right\} - \alpha U/g \right] - P u_\theta$$

$$+ A \left[\nabla_H^2 u_\theta - \frac{u_\theta}{a^2 \sin^2 \theta} - \frac{2 \cos \theta}{a^2 \sin^2 \theta} \frac{\partial u_\lambda}{\partial \lambda} \right]$$

$$\frac{\partial}{\partial t} u_\lambda + f u_\theta = -\frac{g}{a \sin \theta} \frac{\partial}{\partial \lambda} \left[\text{Re} \left\{ \sum_{\ell, n} \sum_{\rho, q} L_{\ell\rho}^{nq} Y_{\ell\rho}^n \exp(i\sigma t) \right\} - \alpha U/g \right] - P u_\lambda$$

$$+ A \left[\nabla_H^2 u_\lambda - \frac{u_\lambda}{a^2 \sin^2 \theta} + \frac{2 \cos \theta}{a^2 \sin^2 \theta} \frac{\partial u_\theta}{\partial \lambda} \right];$$

A , the horizontal eddy viscosity coefficient, governs the strength of the turbulent drag forces. These equations account for the effects of Earth rotation (f is the Coriolis parameter), the yielding of the solid earth to the tide-generating potential ($\alpha = 1+k-h$ is the usual Love-number combination), and tidal loading & self-gravitation;² the matrix \underline{L} is defined as

$$(10) \quad L_{\ell\rho}^{nq} = \delta_{\ell\rho} \delta_{nq} - \frac{3\rho_w}{\bar{\rho}} \frac{1+k'_\ell-h'_\ell}{2\ell+1} \left[A_{\ell j\rho-j}^{n\sigma q_0} \right]$$

where k'_ℓ, h'_ℓ are load Love numbers of degree ℓ , and $\bar{\rho}$ is mean Earth density.

Following the same procedures worked out for the turbulent pole tide, I reduced the momentum equations to matrix form and solved for the unknown tide velocity coefficients:

$$(11) \quad \begin{aligned} \vec{V} &= \underline{\underline{E}} \cdot \left[\underline{\underline{L}} \cdot \vec{T} \right] + \alpha \underline{\underline{M}}_0 \vec{\lambda} \\ \vec{U} &= \underline{\underline{F}} \cdot \left[\underline{\underline{L}} \cdot \vec{T} \right] + \alpha \underline{\underline{M}}_0 \vec{\psi} \end{aligned}$$

where $\underline{\underline{E}}$, $\underline{\underline{F}}$, $\vec{\lambda}$, and $\vec{\psi}$ are defined in Dickman [1989]. As mentioned earlier, in situations where tide heights only have been determined, such as satellite ranging or altimetric observations, or finite-difference tide theories, equations (11) make it possible to estimate the associated tidal currents. Such estimates were unobtainable prior to this work.

The dynamic description of the tides is completed by specifying continuity, i.e. pointwise conservation of mass,

$$(12) \quad \frac{\partial}{\partial t} T = - \frac{1}{a \sin^2 \theta} \left\{ \sin \theta \frac{\partial}{\partial \theta} \left[\tilde{h} u_\theta \right] + \frac{\partial}{\partial \lambda} \left[\tilde{h} u_\lambda \right] \right\} .$$

With (12) reduced to matrix form and the expressions for the tide velocities substituted in, I obtained a single, non-differential matrix equation for the tide height,

$$(13) \quad \underline{\underline{B}} \cdot \vec{T} = \underline{\underline{M}}_0 \vec{b} .$$

When the velocity expressions were substituted into a matrix version of the global boundary constraint, (4), the result was

$$(14) \quad h_0 \underline{\underline{B}}' \cdot \vec{T} = h_0 \underline{\underline{M}}_0 \vec{b}' .$$

$\underline{\underline{B}}$, $\underline{\underline{b}}$, $\underline{\underline{B}}'$, and $\underline{\underline{b}}'$, which depend on σ , P , A , and \tilde{h}_j^Δ , are detailed in Dickman [1989].

With the coefficients of \mathcal{O} , h , and \tilde{h} for the actual oceans constructed directly, the solutions to (13) and (14) describe the lunar and solar tides in realistic oceans. In Dickman [1989] I obtained such solutions for 5 long-period tides. Comparison of the amplitudes and phases of the $\underline{\underline{I}}_0$ coefficient -- the primary coefficient of all long-period tides -- from my² theory with those from other theories and observations revealed an overall agreement of my amplitudes with the others (but see the discussion in Dickman [1989] for some interesting exceptions); however, the phase lags predicted by my theory were consistently lower than those found by others; this is discussed later. At this time, I also predicted the effects of my long-period ocean tides on l.o.d. and wobble; in all cases the effects were found to be small -- but marginally detectable in recent (and, of course, future) high-quality space-geodetic rotation data.

VII. One of the major limitations of my luni-solar tidal predictions was an ambiguity resulting from the failure of T_0 to vanish. T_0 relates to the total amount of tide water world-wide, $\rho_w \int T ds$, and should be zero since tidal forces only re-distribute the oceanic mass (but do not add to it). Calculations of T and OT are ambiguous depending on whether T_0 is included in their spherical harmonic expansions or excluded (in Dickman [1989] I had found the *preferred* solutions to be those which excluded T_0).

In fact, other tide theories [e.g. Schwiderski 1983] deliberately generate non-zero T_0 , by allowing their oceans to periodically "leak" water, in order to model shallow-sea dissipative processes (Parke [1982] additionally arranges the leakiness so that the world-wide net leak is zero). A glance at the tables of results in Dickman [1989] demonstrates how serious the ambiguity can be: with the T_0 produced by my theory, for example, tidal effects on wobble and l.o.d. are uncertain by up to 100% in amplitude and phase, depending on how the T_0 is treated. The role of T_0 was not evaluated by other tide researchers, perhaps because their models are not spherical harmonic; their analyses thus failed to recognize such ambiguity.

As discussed in Dickman [1990], I considered a number of alternatives for achieving a reduced (or zero) T_0 . I judged the optimal approach to be the addition of an oceanic mass-conservation constraint

$$\int_{\text{ocean}} T ds = 0$$

or, equivalently,

$$[OT]_0^0 = 0,$$

implemented via Lagrange multipliers. Thus, my best estimate of the tide height in realistic oceans was the least-squares solution of

$$(15) \quad \begin{pmatrix} \underline{\underline{B}} \\ h_0 \underline{\underline{B'}} \end{pmatrix} \cdot \vec{T} = \underline{\underline{M}}_0 \begin{pmatrix} \vec{b} \\ h_0 \vec{b'} \end{pmatrix} \quad \text{and} \quad \Lambda_1 [OT]_0^0 = 0$$

where Λ_1 is an undetermined Lagrange multiplier.

Such mass-constrained tide solutions are the first of their kind. These solutions are more realistic, and less ambiguous, than other luni-solar tide models. For the five long-period tides I had studied, my predicted amplitudes differed noticeably with the constraint. After adjusting the boundary condition weighting factor (h_0) according to other criteria [see Dickman 1990], it turned out that the tide amplitudes were very similar to the *preferred* values in Dickman [1989] (and thus still in general agreement with other estimates). Additionally, the phases showed a slight improvement.

During this time I also supervised a student (Y.-S. Nam) whose master's project involved tidal effects on the length of day. He constructed a relatively long span of high-quality l.o.d. data using the IRIS program's UT1 observations; corrected it for atmospheric wind and pressure effects; and estimated the 9-day, fortnightly, and monthly tide effects on l.o.d. (the analysis was actually phrased in terms of the "zonal response coefficient" [Agnew & Farrell 1978]). Using my luni-solar tide program, he computed theoretical ocean tide effects at those periods and compared the theoretical predictions with observation.

His work was carefully researched and he accounted *fully* for an expected core-mantle decoupling. Any statistically significant discrepancies between theory and observation would be attributed to factors not included in our model, such as mantle anelasticity. We hoped that, with the differing oceanic effects subtracted out, the frequency dependence of those factors could be elucidated. Because the effects are small, comparable to the noise level present in that VLBI data, it was difficult to draw strong conclusions; our results, which are described in Nam & Dickman [1990], suggest that the anelastic contribution to the fortnightly tidal effect on l.o.d. cannot exceed 1.7%, while partial coupling between core & mantle at monthly periods cannot be ruled out.

VIII. It was clear even from my pole tide investigation [see Dickman 1988a] that the tidal response of the oceans at various frequencies depends on how close those frequencies are to the natural resonances of the ocean basins. For luni-solar tides, which span three very different frequency bands (semi-diurnal, diurnal, and long-period), the dependence on oceanic resonances should be even more significant [see Platzman 1984]. The oceanic response to tidal forcing must unavoidably be considered strongly frequency-dependent.

Knowledge of that frequency dependence would be potentially important for the determination of satellite orbits and, simultaneously, satellite determination of ocean tides. For example, prior to 1990, the most comprehensive estimation of ocean tide constituents from satellite geodesy yielded 616 harmonic coefficients of 32 major and minor tides [Christodoulidis et al. 1988]; however, the bulk of those coefficients -- corresponding to minor or "sideband" tides -- were merely interpolated from the solutions for the major tide coefficients. Christodoulidis et al. [1988] based that interpolation on a linear model (i.e. a linear frequency dependence) of tidal admittance (tidal admittances are the ratios of actual to static tide height amplitude coefficients); although such a model represents an improvement over earlier models of constant admittance (in which a minor tide amplitude was taken to be in the same proportion to its forcing potential as the major tide in that band), it may not account very well for the effects of resonances in or near the tidal bands. A better model of the frequency dependence within the tidal bands would improve both the satellite orbit determination and the ocean tide inferences.

In an effort to elucidate the frequency dependence of tidal admittance, and consequently gain insight into the character of natural oceanic resonances, I computed the actual oceanic responses to a pre-set forcing, at the frequencies of 32 major and minor luni-solar tides. In addition to providing characteristics of tidal constituents (in many cases, unavailable prior to this work) of immediate use in satellite geodesy, these responses could be used to assess sideband tide models previously employed, and also to infer the existence of oceanic eigenfrequencies.

In early stages of this research, I took the opportunity to improve the resolution of my spherical harmonic tide model. Using the original (DBDB5 tape) world-wide topography data, I constructed spherical harmonic coefficients of h , \dot{h} , and \ddot{h} through degree and order 48. This took some effort, because the computations involved numbers larger than mainframes can store, and because the computations had to be performed using extended precision. Subsequent numerical tests (and a further look at equation (13))

led me to the realization that my programs do not recognize high-resolution bathymetry unless high-degree tide height coefficients are also obtained. That is, the theory automatically and correctly ignores \underline{h}_j , \underline{h}_j^Δ , and O_j^Δ of harmonic degree $j \geq 2(L+1)$ if I solve only for \underline{T}_ℓ^n of degree $j \leq L$. Thus, to assess the effects of high- or even moderate-resolution bathymetry I would have to run larger programs (on the IBM 3090 vector facility I estimated that 2-minute, 8-Megabyte programs would have to give way to 35-minute, 12-Megabyte programs or 4-hour, 32-Megabyte programs in order to include bathymetry exceeding degree 12 or degree 24, respectively).

Following these modifications I discovered that, with the increased resolution, the spherical harmonic solutions had become sensitive to the intensity of bottom friction (and, marginally, to the extent of lateral eddy dissipation). Further experiments revealed that, with a magnitude of $P = 2.5 \times 10^{-4} \text{ sec}^{-1}$ for long-period tides and $1.5 \times 10^{-4} \text{ sec}^{-1}$ for diurnal and semi-diurnal tides, it was possible to produce phase lags for the degree 2 tide height components which were consistent with tide observations and other tide theories. *This represented a major improvement in my tide models.* However, calibrating the intensity of friction in my model ocean did not significantly improve the abnormally low amplitudes predicted for the short-period constituents.

The results of the study are presented in Dickman [1991]. One major conclusion is that the linear sideband models used in the most recent satellite geodesy analyses are definitely better than constant-admittance models, but typically still fail to account for ~ 40% of the sideband admittance variations. This in turn suggests that theoretically constrained satellite analyses might yield much-improved ocean tide models and satellite orbit tracks.

The tidal solutions in Dickman [1991] were also evaluated with regard to a number of "diagnostics" indicating anomalous oceanic behavior. From those clues we concluded that a search for oceanic resonances should focus on the O1 - Q1 and OO1 - J1 frequencies in the diurnal tide band; the M2 - 2N2 frequencies, and frequencies beyond 2N2, in the semi-diurnal tide band; and the MSm - Mm frequencies, and frequencies beyond M9, in the long-period tide band. See Dickman [1991] for details.

At this time, I also explored the use of external constraints -- for example, from satellite observations of ocean tides, or from finite-difference predictions of tide coefficients -- in my spherical harmonic ocean tide theory. Initially, it was hoped that, like the "hydrodynamical interpolation" technique used in numerical tide theories, the incorporation of actual data would lead to improved short-period tide amplitudes; and, because spherical harmonic constraints are global, the results should be less affected by coastal anomalies than are the finite-difference theories.

The constraints, implemented via Lagrange multipliers [cf. Menke 1984] according to

$$(16) \quad \begin{pmatrix} \underline{B} \\ h_0 \underline{B}' \end{pmatrix} \cdot \vec{T} = \underline{M}_0 \begin{pmatrix} \vec{b} \\ h_0 \vec{b}' \end{pmatrix} \quad \text{and} \quad \Lambda_1 [\underline{OT}]_0^0 = 0 \quad \text{and} \quad \Lambda_2 [\underline{OT}]_L^S = \text{SPECIFIED}$$

where Λ_1 and Λ_2 are the undetermined (Lagrange) multipliers for global mass conservation and the specified harmonic constraint, respectively, quickly

revealed another use: they could be employed to evaluate the fluid dynamic validity of tide observations or theoretical predictions. For instance, satellite determination of the principal lunar monthly tide height coefficient ($[OT]_2^0$) by Christodoulidis et al. [1988] yielded an amplitude of 0.36 ± 0.50 cm and phase lead of $4.1 \pm 76.6^\circ$. The surprisingly small amplitude (static $[OT]_2^0$ is 1.17 cm, and at such a long period the tide should not be so far below equilibrium) as well as the large uncertainties and the phase lead suggested that this determination was questionable. To evaluate the satellite result, solutions were obtained with $[OT]_2^0$ constrained to assume the range of reported values, also more reasonable values. As described in Dickman [1991], the experiments did indeed confirm that the satellite value was not realistic.

Another example considered was the prediction by Schwiderski of the primary semi-annual tide height coefficient, $[OT]_2^0 = (1.24$ cm amplitude, 48.3° phase lag); the excessively large amplitude ($\sim 20\%$ bigger than static) and phase lag of this tide, considering its long period, suggest that his prediction is inaccurate -- perhaps contaminated by seasonal meteorological effects on the incorporated data. Diagnostics were determined for the tidal solution constrained according to Schwiderski's value and constrained by alternative values. Again, the investigation appeared to confirm that the coefficient was fluid dynamically questionable.

Finally, by using simultaneous multiple constraints it was possible to test whether the high- and low-degree harmonics of some tide model are mutually consistent. As detailed in Dickman [1991], short-period degree-4 harmonics as large as were reported by satellite and numerical tide models -- those admittances were an order of magnitude larger than those in my tide model -- may well be dynamically incompatible with their degree-2 admittances.

During the year of that grant project, a graduate student (Y.-S. Nam) began to research the problem of wobble of the coupled ocean/solid-earth system [Dickman 1983 -- the focus of the first year of this grant]. He was to generalize my original work to allow for a dynamic response of the oceans during wobble; the dynamic response would be determined using my tide programs. Under my supervision, he finally mastered the theory enough to begin generalizing it; and completed a first set of computations. The results were encouraging but needed checking; however, after writing a summary of his efforts, he left the university to undertake doctoral studies elsewhere.

IX. The goal of my final project under grant NAG 5-145 was the determination of tidal effects on Earth's rotation and on the lunar orbit. The tides in question were the 32 long- and short-period constituents I had previously obtained as the solutions to my spherical harmonic ocean tide model. The effects on Earth's rotation derive from the *change in the oceans' inertia tensor*, i.e. the redistribution of oceanic mass associated with the rise and fall of the sea surface, and from the *relative angular momentum* produced by the tidal currents (see later discussion for details). The effects appeared to be calculable from standard Liouville equations (-- but, again, see later) and after a brief effort I had produced a detailed list of the effects of tidal inertia and relative momentum on both polar motion and the length of day.

To determine the effects of ocean tides on the lunar orbit, I began to review the theory presented in Christodoulidis et al. [1988], with frequent reference also to Kaula [1966]. In the midst of developing and de-bugging my own programs for calculating these effects, communication with some of the Christodoulidis et al. co-authors (particularly S. Klosko and R. Williamson) led to their giving me a copy of the programs used in their work (it also led to the discovery, by me, of some errors in their programs...). By the start of the Spring 1991 AGU Meeting, I had also produced a list of ocean tide effects on the lunar orbit. These results, which are summarized in a manuscript currently *in press* as part of an IUGG Monograph (following the IUGG Meetings in Vienna), demonstrate that the bulk of tidal friction is produced by just a few tidal constituents (M2, O1, N2, and also Mf and Q1). I found that my tidal theory failed to predict the observed amounts of Earth's secular deceleration and changes in the lunar orbit, due to the theory's abnormally small amplitudes for M2 and N2. However, the ability of any tide model to predict these effects in agreement with observation does not establish the general validity of that tide model -- or conversely -- since its prediction depends on so few constituents.

At the Spring 1991 AGU Meeting, conversations with several people led me to reconsider all of the preceding work. Discussions in particular with D. McCarthy, R. Gross, and M. Eubanks emphasized that theoretical quantities such as the Earth's angular velocity vector (and thus wobble and changes in the length of day) are not actually what is observed by current geodetic techniques, not even approximately so at high frequencies; for that matter, the very meaning of changes in the length of 'day' is unclear for excitations which possess semi-diurnal or diurnal periods. After the Meeting, as I attempted to understand this new viewpoint, which is especially important at high frequencies, I began to re-evaluate the Liouville equations themselves.

As described in my latest manuscript, "Dynamic ocean tide effects on Earth's rotation", which was recently accepted for publication by the *Geophysical Journal International*, the fundamental problems with the 'traditionally formulated' Liouville equations are its failure to incorporate frequency-dependent fluid core decoupling (the correct formulation can be found in Smith & Dahlen [1981]!) and its failure to correctly model the response of the oceans to the excited rotational perturbations. In modifying the Liouville equations at this time, I took the opportunity to incorporate frequency-dependent Love numbers [Wahr 1981] and load Love numbers [Wahr & Sasao 1981] into both the Liouville theory and the spherical harmonic tidal theory. This in turn required that I carefully distinguish between prograde and retrograde phenomena, since the Love number frequency dependence stems from the *retrograde* nearly diurnal core resonance.

Ultimately, then, I re-defined the tidal potential as

$$U = \operatorname{Re}\{ U \} \qquad U = M_0 Y_2^{-s} e^{-i\sigma t}$$

for the tidal constituent of frequency σ , where M_0 is the forcing amplitude and Y_2^n is the fully normalized complex spherical harmonic function of degree l and order n . This definition allows us to adhere to a rule that $\sigma > 0$ corresponds to prograde and $\sigma < 0$ to retrograde phenomena. The generalization of my tidal theory to include forcing by harmonics of negative order is easily accomplished: the only modification is that in the matrixed momentum equations, eq. (6b) of Dickman [1989],

$$\alpha \mathbf{M}_0 \vec{J} \quad \text{and} \quad \alpha \mathbf{M}_0 \vec{J}'$$

are replaced with

$$\underline{\eta} \cdot \vec{M}' \quad \text{and} \quad \underline{\nu} \cdot \vec{M}',$$

respectively, where \vec{M}' is the collection of $\{ M_l^n \}$ and

$$M_l^n = \begin{cases} 0 & \text{for all } (l, n) \text{ except } (2, -s) \\ -\alpha M_0 / g & \text{for } l = 2, n = -s \end{cases}$$

The original Liouville equations are [e.g., Lambeck 1980]

$$\dot{\mathbf{m}} - i\sigma_r \mathbf{m} = -i \frac{\Omega}{A} \left\{ \mathbf{c} - \frac{i}{\Omega} \dot{\mathbf{c}} + \frac{1}{\Omega} \mathbf{l} - \frac{i}{\Omega^2} \dot{\mathbf{l}} + \frac{i}{\Omega^2} \boldsymbol{\tau} \right\} = -i\sigma_r \underline{\Psi}$$

$$m_3 = -\frac{1}{C} \left\{ c_{33} + \frac{1}{\Omega} \ell_3 - \frac{1}{\Omega} \int \tau_3 dt \right\} = \psi_3$$

where $\vec{\omega} = \Omega \hat{z} + \Omega_m \vec{m}$ is the Earth's angular velocity vector and $\vec{\psi}$ is the dimensionless excitation vector. The first equation allows us to determine the polar motion $\mathbf{m} = m_1 + im_2$ produced by any internal or external process; the second equation similarly yields the resulting axial perturbation m_3 in rotation, thus the change in the length of day $\Delta LOD = -86400m_3$ sec and the change in rotational angle $\Delta \tau_1 = -\int m_3 dt$. Excitation is possible from externally applied torques $\vec{\tau}$, from internal changes $\vec{\ell}$ in angular momentum associated with currents relative to the rotating coordinate frame, or from perturbations \mathbf{c} in the Earth's inertia tensor. In these equations $c_{11} \ll A$, C and $\ell_1 \ll \Omega C$; that is, the perturbations in inertia are small compared to the average principal moments of inertia of the Earth ($A =$ equatorial, $C =$ polar), and the relative momentum perturbations are much less than the bulk angular momentum of the Earth. Finally, the equatorial components of the Liouville equation have been combined into a single complex equation, with $\mathbf{c} = c_{13} + ic_{23}$ and $\mathbf{l} = l_1 + il_2$, and the corresponding excitation function $\underline{\Psi} = \psi_1 + i\psi_2$ is multiplied by the scale factor $\sigma_r = \Omega(C-A)/A$.

The mantle, core, and oceans all respond to perturbations $\Omega_m \vec{m}$ in Earth's rotation. In order to determine the net effect of any rotational excitation, all of these responses must be characterized. In this report I will present their traditional formulations first and then, in stages, my modifications. I will go into a fair amount of detail since the manuscript has not yet appeared in print.

The mantle's response is the most straightforward: if kE denotes the degree-2 Love number of the oceanless earth, then the changes in the inertia tensor associated with the solid earth response are [Munk & MacDonald 1960]

$$\mathbf{c} = \frac{kE}{ks} (C-A) \mathbf{m} \quad c_{33} = \frac{4}{3} \frac{kE}{ks} (C-A) m_3$$

where the 'secular' Love number (often called the 'fluid Earth' Love number) is $ks = 3G(C-A)/(\Omega^2 a^5)$, G is the gravitational constant and a is Earth's mean radius. (In the traditional view, the oceans' response is often included in $kE \dots$).

To the extent that the fluid core is inviscid, non-magnetic, and axially symmetric, with polar moment of inertia C_c , it will be decoupled from axial perturbations in mantle rotation [Yoder et al. 1981, Wahr et al. 1981, Merriam 1982]; as measured from the rotating mantle, it therefore contributes a relative angular momentum

$$l_3 = -C_c \cdot \Omega m_3 .$$

At decade-scale periods, of course, geomagnetic torques will succeed in coupling core to mantle [Rochester 1960].

Because the core boundary has the shape of an oblate spheroid, with flattening f_c , a wobble of the mantle engenders a slight response of the core fluid [Hough 1895]. As summarized in Wahr [1986], the relative angular momentum contributed by the core in this case is

$$l = -\Omega \frac{\sigma}{\sigma + (1+f_c)\Omega} A_c \dot{m} = +\Omega \frac{i}{\sigma + (1+f_c)\Omega} A_c \dot{m} ,$$

where A_c is the core's equatorial moment of inertia (the first of these results C_c may be obtained from Smith & Dahlen [1981] by combining their equations 6, 7, 10, and 11 in Appendix B with 4.11b). When σ is small compared to Ω , this relation yields l proportional to \dot{m} with no significant frequency dependence; however, in our work the frequency-dependent denominator in this expression cannot be treated as constant.

Mantle elasticity allows the core-mantle boundary to deform slightly in response to the core flow [see Sasao et al. 1977, also Sasao & Wahr 1981]; the resulting products of inertia can be written

$$c = -\hat{\xi} \frac{\sigma}{\sigma + (1+f_c)\Omega} A_c \dot{m} = +\hat{\xi} \frac{i}{\sigma + (1+f_c)\Omega} A_c \dot{m}$$

where $\xi = 2.252 \times 10^{-4}$ and $\hat{\xi} \equiv (A/A_c)\xi$.

Combining the preceding mantle and core effects, the Liouville equations can be written

$$\left[1 - \frac{\hat{\kappa}}{\kappa} \frac{A_c}{A} + \frac{C-A}{A} \frac{kE}{ks} \right] \dot{m} - i\sigma_r \left[1 - \frac{kE}{ks} \right] m = -i\sigma_r \underline{\Psi}$$

$$\left[1 - \frac{C_c}{C} + \frac{4}{3} \frac{kE}{ks} \frac{C-A}{C} \right] m_3 = \psi_3$$

where

$$\hat{\kappa} \equiv \frac{1 + \hat{\xi}}{1 + f_c \frac{\Omega}{\sigma + \Omega}} \equiv \frac{\sigma + \Omega}{\sigma + \Omega(1+f_c)} (1 + \hat{\xi})$$

might be viewed as a frequency-dependent "Hough number" [cf. Dickman 1983] and where now $\underline{\Psi}$ and ψ_3 denote additional excitations beyond those associated with the response of the oceanless earth to rotational perturbations.

Incidentally, the free wobble frequency σ_e for an Earth without oceans is easily found from these equations by setting $\underline{\psi} = 0$; since $\sigma_e \ll \Omega$, $\hat{k} \approx (1 + \hat{\xi})[1 + f_c]^{-1} \approx 1 + \hat{\xi} - f_c$ and we have

$$\sigma_e = \sigma_r \left(1 - \frac{kE}{kS}\right) \frac{A}{A_H + (f_c - \hat{\xi})A_C + (C-A) \frac{kE}{kS}}$$

As discussed in my manuscript, the oceanic effect on wobble period -- determined using my most recent spherical harmonic tide model (including the features discussed presently) -- implies that $2\pi/\sigma_e \approx 404.4$ sidereal days, if the Chandler period is 434.4 sidereal days [Wilson & Vicente 1980] (-- thus, an estimate of 30.0 sidereal days for the lengthening of the Chandler period by the oceans); this in turn leads to $kE \approx 0.31189$.

The free core nutation causes solid-earth Love numbers and load Love numbers to be strongly frequency-dependent at short periods [see, e.g., Melchior 1983], if the forcing potential is of harmonic degree 2 and order $|s| = 1$. Following Wahr [1981] and Lambeck [1988, Table 11.6] we first write

$$kE(\sigma) = 0.298 + \frac{\sigma - \sigma_{01}}{\sigma - \sigma_{FCN}} (1.23 \times 10^{-3}) \quad (s = -1)$$

where σ_{01} is the frequency of the O1 tide, chosen as reference, and σ_{FCN} is the free core nutation frequency; $\sigma_{FCN} = -2\pi(1.0021714)$ and from Lambeck [1988], $\sigma_{01} = -2\pi(0.926998)$ rad/sidereal day. Wahr [1981] also found

$$\begin{aligned} kE &= 0.299 & (s = 0) \\ kE &= 0.302 & (s = -2); \end{aligned}$$

these differ from the non-resonant portion of the $s=-1$ kE as a result of mantle ellipticity and Earth's prograde rotation.

The oceanic response to polar motion and changes in l.o.d. will later require that we consider various types of forcing potentials; to keep track of the mantle's response to all of those forcings we denote the Love number by kE_0 , kE_1 , and kE_2 when the forcing potential is of order $|s| = 0, 1, \text{ and } 2$. Additionally, as a result of mantle anelasticity, all Love number values must -- as noted above -- become equal to 0.312 at long periods, in order to produce the observed Chandler wobble period. Including all these considerations, we write the frequency-dependent potential Love number as

$$kE_0(\sigma) = \begin{cases} 0.299 & s=0, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ 0.312 & s=0, \quad 2\pi/|\sigma| > 4 \text{ days} \end{cases}$$

$$kE_1(\sigma) = \begin{cases} 0.298 + \frac{\sigma - \sigma_{01}}{\sigma - \sigma_{FCN}} (1.23 \times 10^{-3}) & s=-1, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ 0.312 & s=-1, \quad 2\pi/|\sigma| > 4 \text{ days} \end{cases}$$

$$kE_2(\sigma) = \begin{cases} 0.302 & s=-2, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ 0.312 & s=-2, \quad 2\pi/|\sigma| > 4 \text{ days} \end{cases}$$

with 4 days chosen arbitrarily as the demarcation between elastic and measurably anelastic responses of the mantle. We also define

$$kE_P(\sigma) = kE_1(\sigma) \quad \sigma \geq 0$$

$$kE_R(\sigma) = kE_1(\sigma) \quad \sigma < 0$$

and

$$\hat{k}_P(\sigma) = \hat{k}(\sigma) \quad \sigma \geq 0$$

$$\hat{k}_R(\sigma) = \hat{k}(\sigma) \quad \sigma < 0 \quad ;$$

these definitions allow us to keep track of prograde versus retrograde forcing.

Finally, quantifying the oceanic effects on rotation, as we will describe shortly, requires us to use various load Love numbers. As discussed in Wahr & Sasao [1981], the only load Love numbers possessing a core-resonance frequency dependence are those associated with the solid-earth response to loading by degree 2, order $|s| = 1$ loads. From Wahr & Sasao, but noting also that Love numbers and load Love numbers should be equally dispersive, we write

$$k'_2(\sigma) = \begin{cases} -0.3083 & s=0, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ -0.3092 - \frac{\Omega}{\sigma - \sigma_{FCN}} (1.45 \times 10^{-4}) & s=-1, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ -0.3083 & s=-2, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ (-0.3092)(0.312/0.298) & \text{all } s, \quad 2\pi/|\sigma| > 4 \text{ days} \end{cases}$$

where the long-period value for k'_2 has the same scale factor, (0.312/0.298), as the long-period kE (and, as with kE , the scale factor is applied to the $s=-1$ value of k'_2). The constant terms in these expressions have been altered slightly (by about 1/4%) from Wahr & Sasao, to account for the slight differences between earth models 1066A and 1066B [see also Dahlen 1976].

It is useful to define $k'_{202} = k'_{202}(\sigma)$ and $k'_{21} = k'_{21}(\sigma)$ as the values of $k'_2(\sigma)$ corresponding to loads of harmonic order $s=0$ or 2 and $|s|=1$, respectively; and to distinguish

$$k'_P(\sigma) = k'_{21}(\sigma) \quad \sigma \geq 0 \quad ; \quad k'_R(\sigma) = k'_{21}(\sigma) \quad \sigma < 0 \quad .$$

Completing the description of Love number frequency dependence, it

follows from the same references as were used for k_E and k'_2 that we can write

$$h_E(\sigma) = \begin{cases} 0.606 & s=0, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ 0.603 + \frac{\sigma - \sigma_{01}}{\sigma - \sigma_{FCN}} (2.46 \times 10^{-3}) & s=-1, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ 0.609 & s=-2, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ (0.603)(0.312/0.298) & \text{all } s, \quad 2\pi/|\sigma| > 4 \text{ days} \end{cases}$$

and

$$h'_2(\sigma) = \begin{cases} -0.9994 & s=0, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ -1.0048 - \frac{\Omega}{\sigma - \sigma_{FCN}} (2.88 \times 10^{-4}) & s=-1, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ -0.9994 & s=-2, \quad 2\pi/|\sigma| \leq 4 \text{ days} \\ (-1.0048)(0.312/0.298) & \text{all } s, \quad 2\pi/|\sigma| > 4 \text{ days} \end{cases}$$

Incorporation of frequency-dependent Love numbers into our rotational and tidal theories requires that these expressions must be substituted, where appropriate, into the Liouville equations (for the rotational theory), and into the α factor and \underline{L} matrix in equations (9) and (10) (for the tidal theory).

Now for the oceans' response. The complexities incurred because of the oceans can best be understood by considering the rotational perturbations in detail. The polar motion generated by a periodic excitation may be elliptical, including retrograde as well as prograde components, so for complete generality we write

$$\underline{m} = \underline{M}_P e^{i\sigma t} + \underline{M}_R e^{-i\sigma^* t},$$

henceforth also allowing for the possibility of a complex forcing frequency, σ , i.e. periodic motion ($Re\{\sigma\}$) plus damping ($Im\{\sigma\}$). In terms of these variables, the change W in centrifugal potential accompanying wobble can be expressed as

$$W = Re\left\{-\Omega^2 a^2 \sqrt{\frac{8\pi}{15}} \left[\underline{M}_P \underline{Y}_2^{-1}\right] e^{i\sigma t}\right\} + Re\left\{-\Omega^2 a^2 \sqrt{\frac{8\pi}{15}} \left[\underline{M}_R \underline{Y}_2^{-1}\right] e^{-i\sigma^* t}\right\}$$

[W is presented in slightly different (but equivalent) form in Dickman 1988].

When an excitation generates a change in the length of day, the accompanying change W' in centrifugal potential also acts as a tidal potential; in this case the potential can be written

$$W' = \text{Re} \left\{ -\Omega^2 a^2 \frac{4}{3} \sqrt{\frac{4\pi}{5}} M'_p Y_2^0 e^{i\sigma t} \right\}$$

[see Lambeck 1988] where we set

$$m_3 = M'_p \exp(i\sigma t) + \text{conjugate} .$$

The response of realistic (zonally asymmetric) oceans to W or W' will include spherical harmonic components of all degrees and orders, exciting further changes in all components of rotation; with a dynamic response, these further excitations will derive from relative momentum as well as inertia perturbations. The initial tidal excitations also include momentum and inertia contributions. From our spherical harmonic solution for tide height, the changes in the oceanic inertia tensor can be calculated in a straightforward manner:

$$c = -\frac{1}{2} \sqrt{\frac{8\pi}{15}} \rho_w a^4 (1+k'_2(\sigma)) \left\{ \mathbb{I}_2^{-1} e^{i\sigma t} - [\mathbb{I}_2^1]^* e^{-i\sigma^* t} \right\}$$

$$c_{33} = -\frac{2}{3} \sqrt{\frac{4\pi}{5}} \rho_w a^4 \text{Re} \left\{ e^{i\sigma t} \left[\left(1 + \frac{\alpha_3}{\alpha_1} k'_2(\sigma)\right) \mathbb{I}_2^0 - \sqrt{5} \left(1 + \frac{\alpha_3}{\alpha_1} k'_0\right) \mathbb{I}_0^0 \right] \right\}$$

[cf. Dickman 1989], where ρ_w is oceanic density and the tide height coefficients correspond to the tide height 'truncated' to zero over the continents [see Dickman 1988]. The present formula for c_{33} corrects a typographical error in Dickman [1989 Appendix 2] mistakenly involving $1/\sqrt{7}$. In these formulae we have also incorporated the change in mantle inertia produced by ocean loading, via the load Love numbers k'_0 , k'_2 [Lambeck 1980, 1988], now treated as frequency-dependent. In the present work we also include a second consequence of core fluidity [Merriam 1980], which is that axial de-coupling of the core modifies the load Love numbers; in the notation of Nam & Dickman [1990], the modification factor is $\alpha_3/\alpha_1 \approx 0.7919$.

Defining

$$\gamma_p(\sigma) \equiv -\frac{1}{2} \sqrt{\frac{8\pi}{15}} \rho_w a^4 (1+k'_p(\sigma)) , \quad \gamma_R(\sigma) \equiv -\frac{1}{2} \sqrt{\frac{8\pi}{15}} \rho_w a^4 (1+k'_R(\sigma)) ,$$

$$\gamma'(\sigma) \equiv -\frac{2}{3} \sqrt{\frac{4\pi}{5}} \rho_w a^4 \left(1 + \frac{\alpha_3}{\alpha_1} k'_{202}(\sigma)\right) ,$$

and

$$\kappa_{02}(\sigma) \equiv \sqrt{5} \left(1 + \frac{\alpha_3}{\alpha_1} k'_0\right) / \left(1 + \frac{\alpha_3}{\alpha_1} k'_{202}(\sigma)\right) ,$$

we can write

$$(17a) \quad c = \gamma_p(\sigma) \mathbb{I}_2^{-1} e^{i\sigma t} - \gamma_R(\sigma) [\mathbb{I}_2^1]^* e^{-i\sigma^* t} \equiv c_p e^{i\sigma t} + c_R e^{-i\sigma^* t}$$

and

$$(17b) \quad c_{33} = \gamma'(\sigma) \text{Re} \left\{ [\mathbb{I}_2^0 - \kappa_{02}(\sigma) \mathbb{I}_0^0] e^{i\sigma t} \right\} .$$

From the solution for the tidal currents, the equatorial relative angular momentum of the oceanic response to W or W' can be calculated according to

$$(18a) \quad l = l_P e^{i\sigma t} + l_R e^{-i\sigma^* t};$$

here

$$l_P = \frac{1}{2} \rho_W a^3 \sum_j h_j^\Delta \sum \left[i \underline{u}_l^n \tilde{Q}_{lj}^{n\Delta} - \underline{v}_l^n C_j^\Delta \tilde{Q}_{l(j+1)}^{n\Delta} - \underline{v}_l^n \hat{C}_j^\Delta \tilde{Q}_{l(j-1)}^{n\Delta} \right]$$

and the expression for l is the same as that for l_P but with the tide velocity coefficients $\underline{u}_l^n, \underline{v}_l^n$ replaced by their complex conjugates and $\tilde{Q}^{n\Delta}$ replaced by $(-1)^{\tilde{Q}^{n\Delta}}$. $\underline{u}_l^n, \underline{v}_l^n, C_j^\Delta, \hat{C}_j^\Delta$, and $\tilde{Q}_{lj}^{n\Delta}$ are defined in Dickman [1988 and 1989]. The formula for l presented here differs from that listed in Dickman [1989]; the latter was actually the "motion" term $(1 - ic)$ of $\underline{\Psi}$ [Lambeck 1980]. For consistency with the core and mantle effects, where l and c terms are explicitly treated in our Liouville equations, the angular momentum term here is restricted solely to l .

For the axial component of the relative angular momentum, we find

$$(18b) \quad l_3 = \rho_W a^3 \text{Re} \left\{ e^{i\sigma t} \sum_j \underline{v}_j^\Delta \left[\tilde{h}_j^\Delta \right]^* \right\}$$

as in Dickman [1989]. Note that l_3 involves harmonic coefficients of \tilde{h} rather than h .

A traditional approach to the Liouville equations would include the oceanic responses to W or W' through a standard Love-number formalism, for example writing c as directly proportional to m (or to m and to m^*) with the proportionality constant an effective Love number. In the course of this project I was able to demonstrate, using my spherical harmonic tidal theory, that such an approach is not valid for a dynamic response of the oceans. After much effort, I discovered a relation between c and \dot{m} which would remain valid under all conditions, including non-global oceans and other situations where the resulting polar motion may be elliptical and may be "cross-coupled" [Dahlen 1976] to the axial spin. This general relation can be written

$$(19a) \quad c = E\dot{m} + i\hat{E}\dot{m} + F\dot{m}^* + i\hat{F}\dot{m}^* + Gm_3 + i\hat{G}m_3$$

where

$$E = \frac{\underline{\sigma}^* \gamma_P(\sigma) [{}_{P-2}T^{-1}] + \underline{\sigma} \gamma_R(\sigma) [{}_{R-2}T^{-1}]}{\underline{\sigma}^* + \underline{\sigma}} \quad \hat{E} = \frac{\gamma_R(\sigma) [{}_{R-2}T^{-1}] - \gamma_P(\sigma) [{}_{P-2}T^{-1}]}{\underline{\sigma}^* + \underline{\sigma}}$$

$$F = -\frac{\underline{\sigma}^* \gamma_P(\sigma) [{}_{R-2}T^1]^* + \underline{\sigma} \gamma_R(\sigma) [{}_{P-2}T^1]^*}{\underline{\sigma}^* + \underline{\sigma}} \quad \hat{F} = \frac{\gamma_P(\sigma) [{}_{R-2}T^1]^* - \gamma_R(\sigma) [{}_{P-2}T^1]^*}{\underline{\sigma}^* + \underline{\sigma}}$$

$$G = \frac{\underline{\sigma}^* \gamma_P(\sigma) [({}_{P-2}T')^{-1}] - \underline{\sigma} \gamma_R(\sigma) [({}_{P-2}T')^1]^*}{\underline{\sigma}^* + \underline{\sigma}}$$

and

$$\hat{G} = \frac{-\gamma_P(\sigma)[(\underline{I}'_P)_2^{-1}] - \gamma_R(\sigma)[(\underline{I}'_R)_2^1]^*}{\underline{\sigma}^* + \underline{\sigma}} ;$$

in these expressions the prograde and retrograde tidal responses to the potential W are given by

$$\vec{T} = M_P(\vec{T}) \quad \text{and} \quad \vec{T} = M_R(\vec{T}) ,$$

whereas the tidal response to the potential W' is given by

$$\vec{T} = M'_P(\vec{T}') ,$$

and all tide height harmonic components in these formulae are 'truncated'.

In a similar fashion, it is possible to write the relative angular momentum l , and the axial perturbations to inertia and momentum c_{33} , l_3 produced by W and/or W' , in terms of the rotational variables; the results are

$$(19b) \quad l = E_L m + i\hat{E}_L \dot{m} + F_L m^* + i\hat{F}_L \dot{m}^* + G_L m_3 + i\hat{G}_L \dot{m}_3$$

$$(19c) \quad c_{33} = E'_m + i\hat{E}'_m \dot{m} + F'_m m^* + i\hat{F}'_m \dot{m}^* + G'_m m_3 + i\hat{G}'_m \dot{m}_3$$

$$(19d) \quad l_3 = E'_L m + i\hat{E}'_L \dot{m} + F'_L m^* + i\hat{F}'_L \dot{m}^* + G'_L m_3 + i\hat{G}'_L \dot{m}_3 .$$

The various E , F , and G coefficients, which depend on $\underline{\sigma}$, load Love numbers (in the case of c or c_{33}), the parameters of the ocean model, and the tidal coefficients generated by W or W' , are similar in form to those presented for c .

The reader should note that determination of the E , F , and G coefficients requires a tidal theory from which solutions can be obtained to both tesseral and zonal forcing potentials (W and W'), at each tidal frequency. These coefficients cannot be estimated from known tidal solutions since they represent oceanic responses to potentials which are 'structurally' different than the original tidal forcing. And, tidal theories based on "hydrodynamic interpolation" are incapable of estimating the coefficients since there is no tidal data to which these responses can be tied.

Finally, with periodic excitations,

$$\underline{\psi} = \underline{\psi}_P e^{i\sigma t} + \underline{\psi}_R e^{-i\sigma^* t} , \quad \psi_3 = \underline{\psi}'_P e^{i\sigma t} + (\underline{\psi}'_P)^* e^{-i\sigma^* t} ,$$

and equations (19) describing the oceanic responses to $\vec{\psi}$, the Liouville equations can be reduced to

$$(20) \quad \begin{aligned} D_{11} M_P + D_{12} M_R^* + D_{13} M'_P &= -\sigma \underline{\psi}_{\Gamma-P} \\ D_{21} M_P + D_{22} M_R^* + D_{23} M'_P &= -\sigma \underline{\psi}_{\Gamma-R}^* \\ D_{31} M_P + D_{32} M_R^* + D_{33} M'_P &= \underline{\psi}'_P . \end{aligned}$$

These equations may be termed "broad-band" Liouville equations, because of their applicability at high as well as low frequencies. The D coefficients

are found according to

$$\begin{aligned}
 D_{11} &= -\sigma_r \left(1 - \frac{kT_P(\sigma)}{ks} \right) + \sigma \left(1 - \hat{k}_P \frac{A_C}{A} + \frac{\sigma_r}{\Omega} \frac{kT_P(\sigma)}{ks} \right) - \sigma \left(1 + \frac{\sigma}{\Omega} \right) \frac{\hat{E}_L + \Omega \hat{E}}{A} \\
 D_{22} &= -\sigma_r \left(1 - \frac{kT_R(\sigma)}{ks} \right)^* - \sigma \left(1 - \hat{k}_R \frac{A_C}{A} + \frac{\sigma_r}{\Omega} \frac{kT_R(\sigma)}{ks} \right)^* + \sigma \left(1 - \frac{\sigma}{\Omega} \right) \left(\frac{\hat{E}_L + \Omega \hat{E}}{A} \right)^* \\
 D_{33} &= 1 - \frac{C_C}{C} + \frac{4}{3} \frac{C-A}{C} \frac{kE_0(\sigma)}{ks} + \frac{G' + \frac{1}{\Omega} G'_L}{C} - \frac{\hat{G}' + \frac{1}{\Omega} \hat{G}'_L}{C} \\
 D_{12} &= \left(\frac{\Omega + \sigma}{A} \right) \left[\left(F + \frac{1}{\Omega} F_L \right) - \sigma \left(\hat{F} + \frac{1}{\Omega} \hat{F}_L \right) \right] \\
 D_{21} &= \left(\frac{\Omega - \sigma}{A} \right) \left[\left(F + \frac{1}{\Omega} F_L \right)^* + \sigma \left(\hat{F} + \frac{1}{\Omega} \hat{F}_L \right)^* \right] \\
 D_{13} &= \left(\frac{\Omega + \sigma}{A} \right) \left[\left(G + \frac{1}{\Omega} G_L \right) - \sigma \left(\hat{G} + \frac{1}{\Omega} \hat{G}_L \right) \right] \\
 D_{23} &= \left(\frac{\Omega - \sigma}{A} \right) \left[\left(G + \frac{1}{\Omega} G_L \right)^* + \sigma \left(\hat{G} + \frac{1}{\Omega} \hat{G}_L \right)^* \right] \\
 D_{31} &= \frac{1}{C} \left[\left(E' + \frac{1}{\Omega} E'_L \right) - \sigma \left(\hat{E}' + \frac{1}{\Omega} \hat{E}'_L \right) \right] \\
 D_{32} &= \frac{1}{C} \left[\left(F' + \frac{1}{\Omega} F'_L \right) - \sigma \left(\hat{F}' + \frac{1}{\Omega} \hat{F}'_L \right) \right]
 \end{aligned}$$

where

$$kT_P(\sigma) = kE_P(\sigma) + ks \frac{E + \frac{1}{\Omega} E_L}{A} \frac{\Omega}{\sigma_r} \quad \text{and} \quad kT_R(\sigma) = kE_R(\sigma) + ks \frac{E + \frac{1}{\Omega} E_L}{A} \frac{\Omega}{\sigma_r}$$

are, as a result of the oceanic responses, complex-valued as well as frequency-dependent whole-Earth Love numbers.

Once the D coefficients have been calculated, tidal effects on rotation can be determined by using the tidal model and (17), (18) to construct the tidal excitation and then inverting equations (20) for the unknown rotational perturbations M_P , M_R , and M'_R . That is, when solved jointly, the broad-band equations (20) yield the polar motion m and axial perturbation m_3 produced by the initial tidal excitation $\underline{\Psi}$ or ψ_3 acting at frequency σ on an Earth whose mantle, core, and oceans all feed back to modify the polar motion or change in l.o.d..

As indicated earlier, results have been obtained for all 32 of the tidal constituents modeled in Dickman [1991]. These results were also re-phrased, following the work by Gross [1991], in terms of the quantities directly observed in modern rotational geodesy: the so-called 'nutation' of the body axis (celestial ephemeris pole) relative to space; and the angular spin increment (change in UT1). The results suggest that diurnal changes in UT1 produced by ocean tides may be marginally detectable in modern high-accuracy, high-frequency space-geodetic data. Diurnal effects on polar motion may also be measurable. The detectability of semi-diurnal tidal changes in UT1 is less certain, because of the great disparity [cf. Brosche et al. 1989] in the inertia and relative momentum excitations predicted by semi-diurnal tide models.

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