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Solution of Two-Body Relativistic Bound State Equations with Confining plus Coulomb Interactions

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Abstract

Studies of meson spectroscopy have often employed a non-relativistic Coulomb plus Linear Confining potential in position space. However because the quarks in mesons move at an appreciable fraction of the speed of light, it is necessary to use a relativistic treatment of the bound state problem. Such a treatment is most easily carried out in momentum space. However the position space Linear and Coulomb potentials lead to singular kernels in momentum space. Using a subtraction procedure we show how to remove these singularities *exactly* and thereby solve the Schrödinger equation in momentum space for *all* partial waves. Furthermore, we generalize the Linear and Coulomb potentials to relativistic kernels in 4-dimensional momentum space. Again we use a subtraction procedure to remove the relativistic singularities exactly for all partial waves. This enables us to solve 3-dimensional reductions of the Bethe-Salpeter equation. We solve six such equations for Coulomb plus Confining interactions for all partial waves.

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1. INTRODUCTION

Meson spectroscopy[1] has been one of the most interesting and fundamental subjects in elementary particle physics for the last two decades. It has provided one of the basic testing grounds for our understanding of both the symmetries and the dynamics of the strong interaction between quarks, mediated by gluons. Future studies are also of great interest, particularly as they may provide evidence of constituent glue. Given the important role of meson spectroscopy it is vital that our theoretical descriptions of these relativistic $q\bar{q}$ systems be as accurate and consistent as possible. Thus one would ideally like to be able to connect the theoretical description of mesons to the fundamental theory of the strong interactions, namely Quantum Chromodynamics (QCD). However the non-abelian nature of QCD leads to strong self-interactions between the gluons resulting in field equations that are highly nonlinear and are unable to be solved by standard diagrammatic methods except in the perturbative regime. In the region of large distances , lattice gauge calculations[2], which provide the most direct link to QCD, have led to the conclusion that in the static quark limit the force between quarks can be very well described with a Linearly rising plus Coulomb potential. Nonrelativistic models which use such a potential have been very successful in accounting for both the masses and decays of mesons, particularly those containing heavy quarks.

However the pure non-relativistic model calculations have limitations. Firstly for systems containing one light quark the use of pure nonrelativistic formalism is obviously unjustified. Secondly the nonrelativistic formalism has intrinsic problems such as the incorrect dependence of the meson mass on the quark mass, i.e. the mesons with light quarks can become heavier than the mesons with heavier quarks[3,4]. Also the nonrelativistic Linear potential does not lead to Linear Regge trajectories[3]. None of these problems occur in semirelativistic treatments where the relativistic expression for the energy is used. Clearly then one must also introduce relativistic effects. Such studies have been made and good descriptions of the entire meson family have been obtained[5]. However, if one incorporates relativity into a position space calculation then many different relativistic

effects must be put in "by hand" leading to a significant number of adjustable parameters[5]. A much more satisfactory approach can be made by doing calculations in momentum space where relativistic effects can be handled in a much more economic way.

Such calculations immediately present two difficulties. Firstly, because one would like to retain a manifestly covariant approach it is natural to transform the Linear plus Coulomb potentials to momentum space. The problem is however, that both potentials lead to singular kernels. Secondly, because many mesons of interest contain quarks of comparable mass, one should ideally solve the two-body Bethe-Salpeter (BS) equation[6] and certainly not consider only the one-body Klein-Gordon or Dirac equations. Although the best way to do meson physics in the two-body framework would be to solve the Bethe-Salpeter(BS) equation, it is more practical to solve a three-dimensional[7-11] reduction of it. However there exist in principle infinitely many possible three-dimensional reductions[7-11] of the BS equation and generally there is no reason to prefer one reduction to another, although in some cases the physical problem itself might suggest a particular reduction scheme. Therefore for the general $q\bar{q}$ problem it is useful to carry out a systematic study of the various reductions of the BS equation.

In this paper we present a complete study of how to solve relativistic two-body bound state equations in momentum space with kernels which are a generalization of Coulomb plus Linear potentials. A method for treating linear and Coulomb potentials in momentum space for the nonrelativistic case was presented by Spence and Vary[18] but their method is not easily generalizable for the relativistic case if one retains retardation in the interaction. We present a systematic treatment of how to deal with the momentum space singularities for both Coulomb and Confining interactions for all partial waves and for both the non-relativstic Schrödinger equation and for six different 3-dimensional reductions[7-11] of the Bethe-Salpeter equation. The only parameters that our method permits are the quark masses and the Coulomb and Confining couplings. Our study is a comprehensive treatment of relativistic effects but with a very restrictive parameter set and should thus eventually provide a definitive description of the entire meson spectrum. *The main purpose of the present paper is to present the theoretical subtraction techniques necessary to solve two-body relativistic bound state equations in momentum space.*

2. SCHRÖDINGER EQUATION AND POTENTIALS IN MOMENTUM SPAC

The nonrelativistic power law potential in r-space can be written as

$$V^N(r) = \lambda_N \lim_{\eta \to 0} r^N e^{-\eta r} \tag{2.1}$$

Where λ_N is the strength of the potential and η is the screening parameter. The index N indicates the type of potential under consideration, i.e $N = -1$ corresponds to Coulomb potential ($\lambda_N = \lambda_C$) and $N = 1$ corresponds to a Linear potential ($\lambda_N = \lambda_L$). In the present paper we shall be considering only these two types of potential. For the bound state problem of two particles with masses m_1 and m_2 interacting via $V^N(\mathbf{q})$ the Schrödinger equation in momentum space is

$$\frac{\mathbf{p}^2}{2\mu}\phi(\mathbf{p}) + \int V^N(\mathbf{q})\phi(\mathbf{p}')d\mathbf{p}' = E\phi(\mathbf{p}) \tag{2.2}$$

where μ is the reduced mass. The momentum space potential is given by the Fourier transform of Eq.(2.1) namely

$$V^N(\mathbf{q}) = \frac{\lambda_N}{2\pi^2} \lim_{\eta \to 0}(-1)^{N+1} \frac{\partial^{N+1}}{\partial \eta^{N+1}}[\frac{1}{\mathbf{q}^2 + \eta^2}] \tag{2.3}$$

Where $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. The Schrödinger equation for the l^{th} partialwave is given by

$$\frac{p^2}{2\mu}\phi_{nl}(p) + \int_0^\infty V_l^N(p', p)\phi_{nl}(p')p'^2 dp' = E_{nl}\phi_{nl}(p) \tag{2.4}$$

where $p = |\mathbf{p}|$, n is the principal quantum number and l is the orbital quantum number. The partial wave components of the potential is readily obtained as

$$\begin{aligned} V_l^N(p', p) &= 2\pi \int_{-1}^{1} V^N(\mathbf{q})P_l(x)dx \\ &= \frac{\lambda_N}{\pi} \lim_{\eta \to 0}(-1)^{N+1} \frac{\partial^{N+1}}{\partial \eta^{N+1}} \frac{Q_l(y)}{pp'} \end{aligned} \tag{2.5}$$

where $x = cos\theta_{\mathbf{pp}'}$ and y is defined as

$$y = \frac{p^2 + p'^2 + \eta^2}{2pp'} \tag{2.6}$$

The exact $\lim_{\eta\to 0}$ will be taken shortly. Special cases of interest are the Coulomb case ($N=-1$) and the Linear potential ($N=1$) and they are readily obtained from Eq(2.5) as

$$V_l^C(p',p) = \frac{\lambda_C}{\pi} \lim_{\eta\to 0} \frac{Q_l(y)}{pp'} \tag{2.7}$$

and

$$V_l^L(p',p) = \frac{\lambda_L}{\pi} \lim_{\eta\to 0} \frac{\partial^2}{\partial\eta^2} \frac{Q_l(y)}{pp'} \tag{2.8a}$$

$$= \frac{\lambda_L}{\pi} \lim_{\eta\to 0} \left[\frac{Q'_l(y)}{(pp')^2} + \frac{\eta^2}{(pp')^3} Q''_l(y) \right] \tag{2.8b}$$

Here $Q_l(y)$ are the Legendre polynomials of the second kind and their first and second derivatives are taken with respect to y, i.e.

$$Q'_l(y) = \frac{\partial Q_l(y)}{\partial y} \tag{2.9a}$$

$$Q''_l(y) = \frac{\partial^2 Q_l(y)}{\partial y^2} \tag{2.9b}$$

We note here that these potentials (at $\eta = 0$) have singularities when $p' = p$ which corresponds to $y = 1$. In order to see the singularity structure explicitly we rewrite $Q_l(y)$ in terms of $Q_0(y)$ as

$$Q_l(y) = P_l(y)Q_0(y) - w_{l-1}(y) \tag{2.10a}$$

where

$$w_{l-1}(y) = \sum_{m=1}^{l} \frac{1}{m} P_{l-m}(y)P_{m-1}(y) \tag{2.10b}$$

Note also that

$$Q_0(y) = 1/2\ ln|(y+1)/(y-1)| = 1/2\ ln\left[\frac{(p'+p)^2+\eta^2}{(p'-p)^2+\eta^2}\right] \tag{2.11a}$$

$$Q'_0(y) = \frac{1}{1-y^2} = pp'\left[\frac{-1}{(p'-p)^2+\eta^2} + \frac{1}{(p'+p)^2+\eta^2}\right] \tag{2.11b}$$

and

$$\frac{\eta^2}{pp'}Q''_0(y) = \eta^2(p^2 + p'^2 + \eta^2)\left[\frac{-1}{(p'-p)^2+\eta^2} + \frac{1}{(p'+p)^2+\eta^2}\right]^2. \qquad (2.11c)$$

In the expression for $Q_l(y)$ the only term that is singular (at $\eta = 0$) is $Q_0(y)$. Therefore the Coulomb potential has a logarithmic singularity from $Q_0(y)$ and the Linear potential has higher order singularities from $Q'_0(y)$ and $Q''_0(y)$. We note that the singularity structure of these potentials are the same for all partial waves.

As mentioned above the potentials we are interested in have singularities at $p' = p$ and in the following sections we will show how to take care of these singularities in the momentum space Schrödinger equation. There are two useful integrals which will be used repeatedly in the following sections. They are

$$\int_0^\infty \frac{Q_0(y, \eta = 0)}{p'} dp' = \frac{\pi^2}{2} \qquad (2.12)$$

$$\int_0^\infty [\frac{\eta^2}{pp'}Q''_0(y) + Q'_0(y)]dp' = 0 \qquad (2.13)$$

2.1 NON-RELATIVISTIC COULOMB PROBLEM

In this subsection we will present a subtraction method which will treat the Coulomb singularity properly. For the pure Coulomb problem in momentum space the exact analytic bound state solutions were found by Fock[12], but our aim is to solve the Schrödinger equation and later relativistic equations for a combined Linear plus Coulomb interaction. Thus we need to be able to implement a numerical subtraction procedure in momentum space. Apart from the rearrangement of terms this method is identical to the one developed in references 13 and 14, but we reproduce it here for completeness. With the potential given in Eq.(2.7) and using the expression (2.10a) for $Q_l(y)$ the Schrödinger equation (2.4), with $\eta = 0$ becomes

$$\frac{p^2}{2\mu}\phi_{nl}(p) + \frac{\lambda_C}{\pi p}\int_0^\infty P_l(y)\frac{Q_0(y)}{p'}\phi_{nl}(p')p'^2dp' - \frac{\lambda_C}{\pi p}\int_0^\infty w_{l-1}(y)\phi_{nl}(p')p'dp' = E_{nl}\phi_{nl}(p) \tag{2.14}$$

Since $w_{l-1}(y)$ contains no singularity the second integral needs no special treatment. In order to remove the singularity arising from the $Q_0(y)$ term we subtract and add a term from the integrand fo the first integral of Eq(2.13). The added term is propotional to the integral of Eq(2.12) and we obtain a singularity free equation

$$\frac{p^2}{2\mu}\phi_{nl}(p) + \frac{\lambda_C}{\pi p}\int_0^\infty P_l(y)\frac{Q_0(y)}{p'}\left[p'^2\phi_{nl}(p') - \frac{p^2\phi_{nl}(p)}{P_l(y)}\right]dp'$$
$$+ \frac{\lambda_C}{\pi p}\left[\frac{\pi^2}{2}p^2\phi_{nl}(p)\right] - \frac{\lambda_C}{\pi p}\int_0^\infty w_{l-1}(y)\phi_{nl}(p')p'dp' = E_{nl}\phi_{nl}(p) \tag{2.15}$$

Note that at the singular point we have $p = p'$, $y = 1$ and $P_l(1) = 1$. Therefore the terms in the square brackets cancel exactly and removes the singularity arising from $Q_0(y)$. The numerical solution of this equation is discussed in the section on Numerical Methods (Sec. 2.3).

2.2 NON-RELATIVISTIC CONFINING PROBLEM

In the case of the Linear potential there are singularities arising from $Q_0(y)$, $Q'_0(y)$ and $Q''_0(y)$. We are interested in solving the Schrödinger equation in the limit $\eta = 0$. For the sake of clarity we will first consider the $l = 0$ case. For $l = 0$ the potential is

$$V_0^L(p',p) = \frac{\lambda_L}{\pi}\lim_{\eta\to 0}\Big(\frac{\eta^2}{(pp')^3}Q''_0(y) + \frac{Q'_0(y)}{(pp')^2}\Big) \tag{2.16}$$

Therefore the s-wave Schrödinger equation is

$$\frac{p^2}{2\mu}\phi_{n0}(p) + \frac{\lambda_L}{\pi p^2}\lim_{\eta\to 0}\int_0^\infty\Big(\frac{\eta^2}{pp'}Q''_0(y) + Q'_0(y)\Big)\phi_{n0}(p')dp' = E_{n0}\phi_{n0}(p) \tag{2.17}$$

Now by adding and subtracting a term in the integral we obtain

$$\frac{p^2}{2\mu}\phi_{n0}(p) + \frac{\lambda_L}{\pi p^2}\lim_{\eta\to 0}\int_0^\infty (\frac{\eta^2}{pp'}Q''_0(y) + Q'_0(y))\Big(\phi_{n0}(p') - \phi_{n0}(p)\Big)dp'$$

$$+ \frac{\lambda_L}{\pi p^2}\phi_{n0}(p)\lim_{\eta\to 0}\int_0^\infty (\frac{\eta^2}{pp'}Q''_0(y) + Q'_0(y)) = E_{n0}\phi_{n0}(p) \qquad (2.18)$$

From Eq(2.13) we see that the last integral is identically zero. Now we can take the $\eta = 0$ limit explicitly[15,16] and we finally get

$$\frac{p^2}{2\mu}\phi_{n0}(p) + \frac{\lambda_L}{\pi p^2}\int_0^\infty Q'_0(y)\Big(\phi_{n0}(p') - \phi_{n0}(p)\Big)dp' = E_{n0}\phi_{n0}(p) \qquad (2.19)$$

In the above equation $Q'_0(y)$ has a double pole singularity at $p = p'$ (see Eq2.11b). By Taylor series expansion of $\phi_{n0}(p')$ around the point $p' = p$ we can see that only a principal value singularity is left, which can be treated by conventional means.(see section on numerical methods and reference 17. Next we consider the case for general l. After removing the terms which can be shown to vanish when the $\eta = 0$ limit is taken the form of the potential is

$$V_l^L(p',p) = \frac{\lambda_L}{\pi}\lim_{\eta\to 0}\left[P_l(y)\left(\frac{\eta^2}{(pp')^3}Q''_0(y) + \frac{Q'_0(y)}{(pp')^2}\right) + \frac{P'_l(y)Q_0(y) - w'_{l-1}(y)}{(pp')^2}\right] \qquad (2.20)$$

Substituting this into the Schrödinger equation (2.4) we have

$$\frac{p^2}{2\mu}\phi_{nl}(p) + \frac{\lambda_L}{\pi p^2}\lim_{\eta\to 0}\int_0^\infty \Big[P_l(y)\Big(\frac{\eta^2}{pp'}Q''_0(y) + Q'_0(y)\Big)$$

$$+ P'_l(y)Q_0(y) - w'_{l-1}(y)\Big]\phi_{nl}(p')dp' = E_{nl}\phi_{nl}(p) \qquad (2.21)$$

In order to remove the singularities now we must perform two separate subtractions. The first subtraction is for the singularities coming from $Q'_0(y)$ and $Q''_0(y)$ and the second subtraction is for $Q_0(y)$. For the singularities arising from Q' and Q'', by using Eq.(2.13) we can make a subtraction without having to add anything back and for the singularity

arising from Q_0, by using Eq.(2.12) we subtract and add a term as for the Coulomb case. In addition, we have shown previously[15] how to take the explicit $\eta = 0$ limit. Thus our singularity free equation, in the exact $\eta = 0$ limit is

$$\frac{p^2}{2\mu}\phi_{nl}(p) + \frac{\lambda_L}{\pi p^2}\int_0^\infty P_l(y)Q'_0(y)\left(\phi_{nl}(p') - \frac{\phi_{nl}(p)}{P_l(y)}\right)dp'$$
$$+ \frac{\lambda_L}{\pi p^2}\int_0^\infty P'_l(y)\frac{Q_0(y)}{p'}\left(p'\phi_{nl}(p') - \frac{l(l+1)}{2}\frac{p\phi_{nl}(p)}{P'_l(y)}\right)dp'$$
$$+ \frac{\lambda_L}{\pi p^2}\frac{l(l+1)}{2}\left[\frac{\pi^2}{2}p\phi_{nl}(p)\right]$$
$$- \frac{\lambda_L}{\pi p^2}\int_0^\infty w'_{l-1}(y)\phi_{nl}(p')dp' = E_{nl}\phi_{nl}(p) \qquad (2.22)$$

At the singular point ($p' = p$; $y = 1$) we have $P_l(y = 1) = 1$ and the bracketed term in the first integral vanishes. Therefore, as in the $l = 0$ case we are left a pricipal value singularity. In the second integral at the singular point $P'_l(y = 1) = l(l + 1)/2$ and the term in the bracket again vanishes and kills the logarithmic singularity arising from $Q_0(y)$. Note that for $l = 0$ Eq(2.22) reduces to Eq(2.19). Now we are in a position to solve Eq.(2.15) for the pure Coulomb or Eq.(2.22) for the pure Linear potential for all partial waves. It is also obvious how to treat the combined Coulomb plus Linear potentials together.

2.3 NUMERICAL METHODS

Consider first the Coulomb equation (2.15). An important point to note is that at the singular point $p' = p$ the term in the square brackets of the first integral goes to zero *faster* than the logarithmic singularity in $Q_0(y)$, and therefore the integrand of the first term is *identically zero* at $p' = p$. By using Gaussian quadrature one can easily write the whole equation (2.15) as a matrix equation with $\phi_{nl}(p)$ as the eigenvectors and E_{nl} as the eigenvalues. Because the kernel of the first integral is zero when $p' = p$, the diagonal term of

the corresponding matrix will vanish; i.e. the matrix coefficients of the matrix eigenvectors will vanish at $p' = p$. However there remain non-zero terms multiplying $\phi_{nl}(p)$. These can be used as non-zero diagonal coefficient terms for the eigenvectors. The result is that one can obtain eigenvalues *and eigenvectors* directly from one's matrix equation. As we shall see later this is no longer possible for the Linear potential. These techniques for the pure Coulomb case are also very well explained in references 13 and 14.

However for the Linear potential this method does not work. (For the sake of simplicity let us discuss the $l = 0$ Linear potential equation only. The methods are identical for the higher l equation.) The reason that the above method does not work in Eq.(2.19) is because $Q'_0(y)$ has a double pole singularity and even after the subtraction, a principal value singularity is left. ***Thus the integral must be evaluated explicitly.*** However to do this we must know what the functions $\phi_{n0}(p)$ are before we solve the problem ! The way around this dilemma is to expand $\phi_{n0}(p)$ in a suitable set of basis functions:

$$\phi_{n0}(p) = \sum_i^M C_i g_i(p) \tag{2.23}$$

Inserting this expansion in Eq(2.19), multiplying by $p^2 g_j(p)$ and integrating over p, we obtain:

$$\sum_i^M C_i [\int_0^\infty \frac{p^4}{2\mu} g_j(p) g_i(p) dp + \frac{\lambda_L}{\pi} \int_0^\infty \int_0^\infty Q'_0(y) g_j(p) \Big(g_i(p') - g_i(p) \Big) dp' dp]$$
$$= E_{n0} \sum_i^M C_i \int_0^\infty p^2 g_j(p) g_i(p) dp \tag{2.24}$$

which is just the matrix equation:

$$\sum_i A_{ji} C_i = E_{n0} \sum_i G_{ji} C_i \tag{2.25}$$

which is symmetric under interchange of i and j (equivalent to symmetry under interchange of p and p') thus ensuring that the eigenvalues are all real.

The double integral still contains a principal value singularity. In order to treat this, the integral over p' is performed by integrating from 0 to $2p$ and then $2p$ to ∞ with

the singularity at the midpoint of the first region, which is carried out using Gaussian quadrature with an even number of points. This type of integration yields the Cauchy principal value automatically[17]. When we solve (2.25) we get M eigenvalues E_{10} to E_{M0} and a corresponding set of M eigenvectors C_1 to C_M. Thus Eq.(2.25) is solved for the energies E_{n0} and the coefficients C_i, which yield the wave function when substituted back into Eq.(2.23) Convergence is obtained by increasing the number of basis functions M and integration points. In order to obtain the wave function in coordinate space, one simply takes the Fourier transform $g_i(r)$ of the basis functions $g_i(p)$ and uses the same set of coefficients C_i but now multiplying $g_i(r)$ to obtain the coordinate space wave function. (Thus it is very convenient to pick $g_i(p)$ so that they have a simple Fourier transform.) For the masses and couplings considered in this paper a convenient set of functions $g_i(p)$ is

$$g_i(p) = exp[-p^2 i^2/M] \tag{2.26}$$

where M is the maximum number of functions used in the expansion Eq(2.23). Note however that for different masses and couplings[15], a different set of basis functions is necessary to achieve rapid convergence.

When solving the general Coulomb plus Linear problem one cannot take advantage of the simplicity of the Coulomb numerical procedure[13,14] by itself. One must employ the basis function expansion method described above. The basis functions appropriate to the Linear potential alone also turn out to be suitable for the general Linear plus Coulomb problem for the masses and couplings of this paper.

2.4 NON-RELATIVISTIC RESULTS

We have carried out many different tests of our methods. Firstly, for the pure Coulomb case we solved the problem with the method of references 13 and 14, which does not require any basis function expansion. We compared to the exact Coulomb energies and found that we could easily generate over 20 eigenvalues very accurately. Secondly, as an additional

check we also solved the pure Coulomb case using an appropriate set of basis expansion functions and were able to obtain about 10 eigenvalues quite accurately. Thirdly, the pure Linear problem was solved for $l = 0$ (see reference 15 for details) and compared to the exact results. (For the $l = 0$, pure Linear potential case, the exact eigenvalues can be obtained in terms of the roots of the Airy function). The calculated eigenfunctions also agreed with the exact results. Fourthly, the combined Coulomb plus Linear problem was solved with the expansion functions in Eq.(2.26) for $l = 0, 1, 2, 3$ and compared to a coordinate space calculation. (The coordinate space code integrates the Schrödinger equation out from the origin at $r = 0$ and in from large r, and matches the logarithmic derivatives at the classical turning point). Fifthly, the combined Coulomb plus Linear results were also compared to those listed in reference 18 and also with a coordinate space code. Excellent agreement was obtained.

In summary, we have very thoroughly tested our methods for Coulomb plus Linear potentials for many partial waves against results from exact calculations, coordinate space codes and the results of other authors for both eigenvalues and wave functions.

3. RELATIVISTIC TWO-BODY EQUATIONS AND INTERACTION KERNELS

In traditional nuclear physics, the deuteron is the only two particle bound state system. It has been studied in both the nonrelativistic framework and also in numerous relativistic frameworks. Compared to the deuteron the $q\bar{q}$ system is a very rich system and its spectra provides an ideal testing ground in which a systematic study of the 3-dimensional relativistic equations can be made.

The Bethe-Salpeter(BS) equation for the bound state problem in the center of mass frame is given by

$$\Psi(\mathbf{p}, P_0) = \frac{i}{(2\pi)^4} \int V(p, p') G(p', P_0) \Psi(\mathbf{p}, P_0) d^4p' \tag{3.1}$$

As mentioned above, there are infinitely many 3-dimensional reductions of the BS

equation. In this section we are going to work with six particular reductions which we believe to be a fair representative sample of the most commonly used 3-dimensional reductions of the BS equation. In order to reduce Eq(3.1) in to a 3-dimensional equation, we replace the propagator G by a 3-dimensional propagator g which has the same elastic cut. A systematic study of these 3-dimensional relativistic equations for the problem of scattering of scalar particles has been performed in reference 11. As stated in the introduction, in this paper we will make a similar study of the bound state of two particles interacting via a confining interaction. Some results have already been previously discussed[16]. The choice of the 3-dimensional propagator can be categorized into two types in general. One which renders the interaction to be instantaneous and one which does not. In this paper we study six 3-dimensional reductions, three of each type. Minimal Relativity (MR) equation[8?], Kadyshevsky (K) equation[9], and Gross (G) equation[7] all of which retain the retardation in the interaction. The equations with instantaneous interaction(no retardation) are the Blankenbecler-Sugar (BBS)equation[8], Kadyshevsky (K0) (without retardation)[9] and Thompson (T) equation[10]. All six equations can be generically written as (compare to Eq.2.2)

$$D_i\phi(\mathbf{p}) = -\int d\mathbf{p}' V_i(p,p')\phi(\mathbf{p}') \tag{3.2}$$

where $\phi(\mathbf{p})$ is a Schrödinger like wave function. We will neglect the couplings to the negative energy channels since the subtraction method is the same for the coupled channel case. The D_i are given in table I and the index i can be MR, K, G, BBS, K0 and T. Note that for MR, K and G equations the interaction V_i has retardation and for the other three equations it does not. We will choose to use MR, BBS, K and K0 equations to study the bound states of two scalar particles interacting via a confining interaction and G and T equations to study the bound state of spinor quarks[7].

The confining interaction to be used in these relativistic equations is a straightforward generalization of the Linearly rising potential discussed in section 2. We simply replace the three vector $\mathbf{q}$ of Eq(2.2) by a four-vector q. Now q^2 is given by

$$q^2 = (\mathbf{p}-\mathbf{p}')^2 - (E_{\mathbf{p}} - E_{\mathbf{p}'})^2 \tag{3.3}$$

In this generalization the form of the Coulomb type interaction and the confining interaction remain the same as in the nonrelativistic case but now $\mathbf{q}^2$ is replaced by q^2 and the partial wave components of these interactions will be given by Eq(2.5) but for the equations that include retardation (MR, K, G) the variable y is now replaced by $\bar{y}$ (instead of Eq.2.6 with $\eta = 0$) where

$$\bar{y} = \frac{p'^2 + p^2 - (E_p - E_{p'})^2}{2pp'} \tag{3.4}$$

Equations without retardation (BBS, K0, T) retain the original form of y in Eq.(2.6) with $\eta = 0$. Here p and p' are only the magnitude of the three vectors. Again we note that these relativistic interactions will introduce singularities as in the nonrelativistic case at $q^2 = 0$ or at $\bar{y} = 1$. Note also that although the variables are different the singularity structures are similar to the nonrelativistic case; i.e. the Coulomb interaction will have a logarithmic singularity and the confining interacting has higher order singularities. For the equations without retardation the interaction $V_i(p, p')$ is instantaneous and it is exactly the same as the nonrelativistic case. For the instantaneous interaction, relativistic effects come in to the equation only through the kinematics; i.e. only through the operator D_i. The singularities in this interaction can be handled exactly the same way as in the nonrelativistic case.

In the following subsections we will discuss how the singularities in the relativistic confining and Coulomb interactions can be treated properly.

3.1 RELATIVISTIC COULOMB PROBLEM

The relativistic generalization of the Coulomb interaction in the partial wave form is given by

$$V_l^C(p', p) = \frac{\lambda_C}{\pi} \lim_{\eta \to 0} \frac{Q_l(\bar{y})}{pp'} \tag{3.5}$$

and by using the expression for Q_l Eq(3.2) becomes

$$D_i\phi_{nl}(p) + \frac{\lambda_C}{\pi p}\int_0^\infty P_l(\bar{y})\frac{Q_0(\bar{y})}{p'}\phi_{nl}(p'){p'}^2dp' - \frac{\lambda_C}{\pi p}\int_0^\infty w_{l-1}(\bar{y})\phi_{nl}(p')p'dp' = 0 \qquad (3.6)$$

for the MR, K and G equations only. For the instantaneous equations BBS, K0, T instead of the above Eq.(3.6), we have simply the Schrödinger equation (2.14) but with the operator D_i replacing the Schrödinger propagator. Note that the only singularity in equation (3.6) arises from $Q_0(\bar{y})$. We want to handle this singularity in a similar fasion as in the nonrelativistic case; i.e. by adding and subtracting a term. But we must also be able to handle the added term analytically or numerically. Unfortunately because of the presense of retardation we cannot just subtract a ϕ_{nl} and use Eq.(2.12) as in the nonrelativistic case. In order to take advantage of Eq(2.12), we subtract a term propotional to the nonrelativistic interaction and obtain (compare to Eq. 2.15)

$$D_i\phi_{nl}(p) + \frac{\lambda_C}{\pi p}\int_0^\infty \Big[Q_0(\bar{y})\phi_{nl}(p')p' - Q_0(y)\frac{p^2\phi_{nl}(p)}{p'P_l(\bar{y})}\Big]P_l(\bar{y})dp'$$
$$+ \frac{\lambda_C}{\pi p}[p^2\phi_{nl}(p)\frac{\pi^2}{2}] - \frac{\lambda_C}{\pi p}\int_0^\infty w_{l-1}(\bar{y})\phi_{nl}(p')p'dp' = 0 \qquad (3.7)$$

for the MR, K and G equations only. Again for the instantaneous equations BBS, K0, T instead of the above Eq.(3.7), we have simply the Schrödinger equation (2.15) but with the operator D_i replacing the Schrödinger propagator. Note that we again have at the singular point $p' = p$, $\bar{y} = 1$ and $P_l(\bar{y} = 1) = 1$ and by Taylor expanding $Q_0(\bar{y})$ around $p' = p$, one can show that the term in the square brackets vanishes at the singular point.

3.2 RELATIVISTIC CONFINING PROBLEM

In the case of the relativistic confining interaction, the functional structure of the in teraction is again the same as the nonrelativistic case but y replaced by $\bar{y}$. We therefore use

the same type of subtraction used in the relativistic Coulomb case. That is, we subtract and add a term propotional to the nonrelativistic confining interaction. We obtain (compare to Eq. 2.22)

$$D_i\phi_{nl}(p) + \frac{\lambda_L}{\pi p^2}\int_0^\infty \Big[Q'_0(\bar{y})\phi_{nl}(p') - (\frac{E_p}{m})^2 Q'_0(y)\frac{\phi_{nl}(p)}{P_l(\bar{y})}\Big]P_l(\bar{y})dp'$$

$$\frac{\lambda_L}{\pi p^2}\int_0^\infty \Big[Q_0(\bar{y})\phi_{nl}(p') - \frac{p}{p'}\frac{l(l+1)}{2P'_l(\bar{y})}Q_0(y)\phi_{nl}(p)\Big]P'_l(\bar{y})dp'$$

$$-\frac{\lambda_L}{\pi p^2}\int_0^\infty W'_{l-1}(\bar{y})\phi_{nl}(p')dp' + \frac{\lambda_L}{\pi p^2}\frac{l(l+1)}{2}p\frac{\pi^2}{2}\phi_{nl}(p) = 0 \qquad (3.8)$$

for the MR, K and G equations only. Once more for the instantaneous equations BBS, K0, T instead of the above Eq.(3.8), we have the Schrödinger equation (2.22) but with the operator D_i replacing the Schrödinger propagator. The factor E_p^2/m^2 in the subtracted term of the first integral in Eq(3.8) is necessary in order to cancel the singularity arising from $Q'_0(\bar{y})$ exactly at the singular point. This can easily be seen by Taylor expanding $Q'_0(\bar{y})$ at $p' = p$. Equation (3.8) is now ready to be solved for various choices of D_i when there is retardation in the interaction. For cases without retardation V_i is identical to the nonrelativistic problem and the subtraction technique developed in the nonrelativistic section can be used.

3.3 RELATIVISTIC RESULTS AND CONCLUSIONS

The main purpose of the present paper is to present the theoretical subtraction techniques necessary to solve two-body relativistic bound state equations in momentum space. Therefore equations (3.7) and (3.8) are our major results.

Nevertheless for the sake of illustration we shall present some numerical solutions for the pure confining problem with equation (3.8) written in terms of a single channel. Such results will at least allow us to see whether our theoretical methods give reasonable results.

The usefulness of these relativistic equations depends on the extent to which they reproduce global properties of the spectrum characterized by the dependence of the energy E_{nl} on the principal quantum number n. This dependence is most easily revealed by studying the ratio E_{nl}/E_{1l}. E_{nl} is related to the total energy W_{nl} through $E_{nl} = W_{nl} - 2m$. Tables 2, 3 and 4 contain the results for the ratio E_{nl}/E_{1l} for the equations listed above for a reasonable choice of mass and coupling parameters. l values range from 0 to 2.

There are three observations to make from these tables. First all of the energy ratios are reasonably close to the non-relativistic results for heavy quark masses. Second the difference between the relativistic results and the non-relativistic results gets bigger for smaller quark mass. Third, the higher radial excitations show more pronounced relativistic corrections, which is consistent with the virial theorem[3] for a positive power law potential which requires larger kinetic energies for orbits with greater average radii. These results lead us to conclude that our theoretical methods are valid and give us confidence that the methods developed herein will be suitable when a full coupled channels calculation is performed and compared to experimental data.

In conclusion we have presented the theoretical subtraction techniques necessary to solve two-body relativistic bound state equations in momentum space with Coulomb plus Confining interactions. Future work will be devoted to including spinors and coupling to the negative energy channels in all six equations so that detailed comparisons to experiment can be carried out.

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Table 1

D_i operators for relativistic equations

G and T equations are describing pseudoscalar mesons with spinor quarks

The other four relativistic equations are for scalar quarks

i	Name	D_i	Retardation
MR	Minimal Relativity	$4E_k(E_k{}^2 - W^2/4)$	Yes
BBS	Blanckenbecler Sugar	*same as MR*	No
K	Kadyshevsky	$2E_k{}^2(E_k - W/2)$	Yes
K0	Kadyshevsky	*same as K*	No
G	Gross	$2E_k - W$	Yes
T	Thompson	*same as G*	No

Table 2

Energy ratios $\frac{E_{n+1}}{E_1}$ for pure Confining interaction with $l = 0$.

G and T equations are for spinor quarks with $k = 0.2GeV^2$. The other four relativistic equations are for scalar quarks with $k = 0.2GeV^4$. The nonrelativistic(NR) equation is with $k = 0.2GeV^2$. All masses are in units of GeV.

n	MR	BBS	K	K0	G	T	NR	Mass
1	1.73	1.71	1.74	1.72	1.79	1.72	1.75	1.5
2	2.31	2.27	2.34	2.30	2.47	2.30	2.36	1.5
3	2.81	2.75	2.87	2.80	3.09	2.80	2.90	1.5
1	1.58	1.50	1.68	1.54	1.90	1.67	1.75	0.5
2	2.00	1.82	2.21	1.89	2.73	2.18	2.36	0.5
3	2.35	2.08	2.65	2.16	3.52	2.62	2.90	0.5
1	1.51	1.41	1.66	1.44	1.98	1.63	1.75	0.3
2	1.87	1.65	2.13	1.69	2.92	2.11	2.36	0.3
3	2.17	1.84	2.52	1.89	3.84	2.51	2.90	0.3

Table 3

Energy ratios $\frac{E_{n+1}}{E_1}$ for pure Confining interaction with $l = 1$.

Notation and units are the same as Table 2.

n	MR	BBS	K	K0	G	T	NR	Mass
1	1.44	1.43	1.45	1.44	1.49	1.43	1.45	1.5
2	1.82	1.79	1.85	1.81	1.92	1.80	1.85	1.5
3	2.16	2.11	2.20	2.15	2.33	2.13	2.20	1.5
1	1.38	1.31	1.47	1.35	1.56	1.39	1.45	0.5
2	1.67	1.54	1.84	1.59	2.09	1.71	1.85	0.5
3	1.92	1.72	2.17	1.79	2.60	2.00	2.20	0.5
1	1.37	1.27	1.52	1.30	1.61	1.36	1.45	0.3
2	1.64	1.45	1.90	1.50	2.21	1.67	1.85	0.3
3	1.87	1.60	2.22	1.65	2.80	1.93	2.20	0.3

Table 4

Energy ratios $\frac{E_{n+1}}{E_1}$ for pure Confining interaction with $l = 2$.

Notation and units are the same as Table 2.

n	MR	BBS	K	K0	G	T	NR	Mass
1	1.32	1.31	1.33	1.31	1.35	1.31	1.33	1.5
2	1.59	1.57	1.62	1.59	1.68	1.58	1.62	1.5
3	1.84	1.81	1.88	1.84	1.99	1.82	1.89	1.5
1	1.29	1.23	1.37	1.26	1.41	1.27	1.33	0.5
2	1.52	1.40	1.67	1.45	1.80	1.51	1.62	0.5
3	1.71	1.55	1.94	1.61	2.19	1.72	1.89	0.5
1	1.30	1.20	1.44	1.23	1.45	1.25	1.33	0.3
2	1.52	1.35	1.78	1.39	1.89	1.47	1.62	0.3
3	1.72	1.47	2.06	1.52	2.33	1.67	1.89	0.3

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