/108636

P.15

NASA Technical Paper 3232

July 1992

Trajectory Fitting in Function Space With Application to Analytic Modeling of Surfaces

Raymond L. Barger

NASA

NASA
Technical
Paper
3232

1992

Trajectory Fitting in Function Space With Application to Analytic Modeling of Surfaces

Raymond L. Barger
Langley Research Center
Hampton, Virginia

National Aeronautics and
Space Administration
Office of Management
Scientific and Technical
Information Program

Abstract

A theory for representing a parameter-dependent function as a function trajectory is described, along with the theory for determining a piecewise analytic fit to the trajectory. An example is given that illustrates the application of the theory to generating a smooth surface through a discrete set of input cross-section shapes. A simple procedure for smoothing in the parameter direction is presented, along with a computed example. Application of the theory to aerodynamic surface modeling is demonstrated by applying it to a blended wing-fuselage surface.

Introduction

Analytic curves, such as cubics, parabolas, and least square polynomials, are often used to fit (precisely or approximately) an ordered set of points obtained by experiment or computation. These fitting methods can also be extended to generate space curves to fit discrete three-space vector locations (ref. 1).

One purpose of the present analysis is to demonstrate that with little theoretical difficulty, the methods can be extended to vectors in higher dimensional spaces, and even to functions. In the latter case, the resulting polynomials in function space can be used to interpolate solutions of a differential equation that depends on a parameter, where these solutions have been obtained for a relatively small set of values of the parameter. However, the primary application that is explored here is that of fitting a smooth surface through a given set of cross-section shapes. This surface is expressed in analytic form, and it is generated automatically. Thus, it does not involve time-consuming hands-on procedures, like those required for fitting surfaces with local patches.

Symbols

B	function space
B_i	polynomial coefficients, may be scalars, vectors, or functions, $i = 1, 2, \ldots$
F	function of two variables
f, g	generic functions
k	number of components of joint vector
$\mathbf{M}$	coefficient matrix (eq. (3))
m	total number of joint vectors
$\mathbf{P}_i$	joint vector, ith joint
R	real number line
t	trajectory parameter
$\mathbf{V}$	matrix whose terms are linear combinations of joint vectors or functions
$\mathbf{X}$	vector position on trajectory in $(n + 1)$-dimensional space formed as cross product of axis t with n-dimensional space
$\mathbf{x}$	position vector of point on trajectory
$\mathbf{x}'$	matrix obtained from prescribed tangent vectors at joints
x, y, z	Cartesian coordinates
λ	blending parameter, $0 \leq \lambda \leq 1$
ξ	independent variable
τ	normalized trajectory parameter (eqs. (5), (6), and (8))

Analysis

Curve Fitting in Three-Dimensional Space

In general, a parametric space curve is expressed in the form

$$\mathbf{x}(t) = \begin{Bmatrix} x(t) \\ y(t) \\ z(t) \end{Bmatrix} \tag{1}$$

where t is the trajectory parameter. The theory for fitting such a continuous curve through a given set of discrete points in space is discussed in reference 1. If the point coordinates are specified at discrete values of a parameter, then a continuous function of t can be synthesized that includes the specified points. However, if only a discrete ordered set of points is given without reference to a parameter, then it is necessary to parameterize the trajectory to define a continuous curve through the points. This parameterization is accomplished by basing the parameter t on some quantity, such as the cumulative chord length between successive points.

To distinguish the original set of discrete vector locations from other points on the continuous trajectory fit through these points, the original points are referred to as *joints*. (See fig. 1.) This terminology is consistent with that of reference 1.

Linked Cubics

If the continuous trajectory is defined by a set of cubic curves, each segment has the form

$$\mathbf{x}_i(t) = \mathbf{B}_{1,i} + \mathbf{B}_{2,i}t + \mathbf{B}_{3,i}t^2 + \mathbf{B}_{4,i}t^3 \qquad (2)$$

for the ith interval, where the coefficients $\mathbf{B}_i$ in the polynomial are vectors. The usual way of determining these coefficients is to require the trajectory to pass through the joint vectors and match specified slope vectors at the joints. If these slope vectors are determined by choosing the slopes at the initial and final points and requiring second-derivative continuity at the internal joints, the curve is called a cubic spline. Then the following matrix equation for the m tangent vectors $(\mathbf{x}'_i, i = 1, m)$ is obtained (ref. 1, ch. 5)

$$\mathbf{M}\mathbf{x}'_{k,i} = \mathbf{V}_{k,i} \qquad (3)$$

where $\mathbf{M}$ is an $m \times m$ tridiagonal matrix of scalars, and the column matrix $\mathbf{V}_{k,i}$ consists of linear combinations of the components k of the joint vectors $\mathbf{P}_i$. Although equation (3) actually represents three equations (one for each vector component), it does not require three solutions. Inverting the matrix $\mathbf{M}$ provides the solution for all components

$$\mathbf{x}'_{k,1} = \mathbf{M}^{-1}\mathbf{V}_k \qquad (4)$$

In fact, if we specify m points in an n-dimensional space, the problem does not increase in complexity, as the system is solved by inverting the same $m \times m$ matrix.

Determining the tangent vectors at the joints by requiring second-derivative continuity may result in trajectories with rapid, frequent curvature changes (wiggles) and large excursions between the joints. This effect can be reduced, with some increase in complexity, by the use of rational tension splines (ref. 2). If it is permissible to forego the second-derivative continuity condition, then the problem can be avoided by specifying the tangent vectors at the joints by a more conservative method or by using an alternative to cubic splines, such as the method of blended parabolas (ref. 1, ch. 5).

Blended-Parabola Technique

This blended-parabola technique can be described in the present notation as follows. A parabola $\mathbf{x}_j$ is fit through three consecutive joints $\mathbf{P}_{j-1}, \mathbf{P}_j$, and $\mathbf{P}_{j+1}$. Its equation is

$$\mathbf{x}_j(\tau_1) = \mathbf{B}_{1,j} + \mathbf{B}_{2,j}\tau_1 + \mathbf{B}_{3,j}\tau_1^2 \qquad (5)$$

where

$$\tau_1 = \frac{t - t_{j-1}}{t_{j+1} - t_{j-1}} \qquad (6)$$

Similarly, a parabola $\mathbf{x}_{j+1}$ is fit through the points $\mathbf{P}_j, \mathbf{P}_{j+1}$, and $\mathbf{P}_{j+2}$ with parameter

$$\tau_2 = \frac{t - t_j}{t_{j+2} - t_j} \qquad (7)$$

Between points $\mathbf{P}_j$ and $\mathbf{P}_{j+1}$ the parabolas are blended by the formula

$$\mathbf{x}_{j,j+1}(\tau) = (1 - \tau)\mathbf{x}_j[\tau_1(\tau)] + \tau\mathbf{x}_{j+1}[\tau_2(\tau)] \qquad (8)$$

where the double subscript on the left-hand side of equation (8) indicates blending of the parabolas on the jth and $j+1$st intervals, and

$$\tau = \frac{t - t_j}{t_{j+1} - t_j} \qquad (9)$$

Equation (8) is actually a cubic in τ. No blending is performed between the first two points or between the last two.

Trajectories Specified in Terms of a Parameter

Suppose a set of points in two-dimensional space is given in terms of a parameter t, which can be regarded as time as follows:

$$x = f(t) \qquad (10a)$$

$$y = g(t) \qquad (10b)$$

Then, we can construct a t-axis normal to the x-y plane and display the time history of the two-space trajectory as a curve in the three-dimensional space, as shown by the following equation:

$$\mathbf{X}(x, y, t) = \begin{Bmatrix} x(t) \\ y(t) \\ t \end{Bmatrix} \qquad (11)$$

As an example, the equation

$$\mathbf{x} = \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix} \tag{12a}$$

represents a circle in the x-y plane. However, if we take t to be time and display the curve

$$\mathbf{X} = \begin{Bmatrix} \cos t \\ \sin t \\ t \end{Bmatrix} \tag{12b}$$

in the three-dimensional (x, y, t) space, the helix now represents the time history of the circular trajectory in the physical two-dimensional (x, y) space.

Generalization of Theory

This type of distinction is crucial in the generalization of the theory, where the parameter is always considered as an additional independent variable. That is, points in an n-dimensional space are assumed to be specified in terms of a parameter t, but the trajectory is taken in the $(n+1)$-dimensional space with the parameter taken as the $(n+1)$ coordinate variable. The trajectories can be distinguished by adopting the terminology shown in the following equations:

$$\mathbf{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix} \tag{13a}$$

$$\mathbf{X}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \\ t \end{Bmatrix} = \begin{Bmatrix} x(t) \\ t \end{Bmatrix} \tag{13b}$$

It would even be possible to extend this theory to vectors representing infinite sequences ($n = \infty$) provided that the distance element series defined by

$$ds^2 = \sum dx_i^2 + dt^2 \tag{14}$$

is always convergent.

However, the purpose of the present analysis is not to pursue the development of curve fitting in n-dimensional space. Rather, the n-space vector $\mathbf{x}(t)$ is to be replaced by a function $F(\xi, t)$. (In a mathematical sense, this step is equivalent to increasing the dimension to uncountable infinity.) Thus, in the cubic equation (eq. (2)), the coefficient vectors $\mathbf{B}_i$ are replaced by functions $B(\xi)$ as follows:

$$F(\xi, t) = B_1(\xi) + B_2(\xi)t + B_3(\xi)t^2 + B_4(\xi)t^3 \tag{15}$$

The specific *points* to be fit by the curve are in the function space B, but the trajectory exists in the cross-product space $B \times R$, where R is the real line specifying the parameter t. This effect is analogous to the discussion of equations (10) to (13).

This kind of cross-product space is utilized in path-following techniques (refs. 3 and 4) for the solution of a nonlinear differential equation that depends on a parameter. In fact, one potential application of this theory for curve fitting in function space is to interpolate solutions of the equation for values of the parameter between those for which the exact solution is obtained. This application is not pursued in this paper.

In equation (15), the coefficients are determined by matching position and tangent vectors at the ends of the intervals, by the formulas (ref. 1, sec. 5–3):

$$\mathbf{B}_1(\xi) = \mathbf{P}_i(\xi) \tag{16a}$$

$$\mathbf{B}_2(\xi) = \mathbf{P}'_i(\xi) \tag{16b}$$

$$\mathbf{B}_4(\xi) = \mathbf{P}'_{i+1}(\xi) - 2\mathbf{P}_{i+1}(\xi) + 2\mathbf{B}_1(\xi) + \mathbf{B}_2(\xi) \tag{16c}$$

$$\mathbf{B}_3(\xi) = \mathbf{P}_{i+1}(\xi) - \mathbf{B}_1(\xi) - \mathbf{B}_2(\xi) - \mathbf{B}_4(\xi) \tag{16d}$$

where $\mathbf{P}(\xi)$ denotes an input (joint) shape, and $\mathbf{P}'(\xi)$ denotes the specified slope at the joint with respect to τ (not ξ) where

$$\tau = \frac{t - t_i}{t_{i+1} - t_i} \tag{17}$$

A further point regarding the computation of the coefficient functions should be noted. If cubic splines are used, so that the tangent vectors are determined by specifying second-derivative continuity with respect to t at the joints, then a matrix equation has to be inverted as in the solution of equation (3). As previously discussed for equation (3), no increase in complexity occurs as a result of replacing the vectors with functions.

Application to Analytic Surface Modeling

The problem to be considered is that of fitting a smooth surface through a given discrete set of cross-section shapes. The shapes are given as functions

$z_i(y)$ at specified locations x_i $(i = 1, 2, ..., m)$. The equations are simplified considerably by normalizing the independent variable y so that $-1 \leq y \leq 1$.

For purposes of illustration, an example was formulated with five input cross-section shapes having analytic descriptions as follows:

$$z_1(y) = 0.5 \cos \frac{\pi y}{2} \tag{18a}$$

$$z_2(y) = 0.3|1 - y| - 0.4 \tag{18b}$$

$$z_3(y) = 0.5|\sin \pi y| \tag{18c}$$

$$z_4(y) = -0.3|y| - 0.25 \tag{18d}$$

$$z_5(y) = 0.5\sqrt{1 - y^2} \tag{18e}$$

The cross-section shapes described by these formulas were disposed at equal intervals beginning at $x = 0$ and ending at $x = 6$ and computed at equally spaced values of y. These shapes are shown in figure 2. The three internal shapes (eqs. (18b) to (18d)) have discontinuous derivatives at $y = 0$.

Figure 3(a) shows a perspective view of a surface obtained by linear interpolation between the input cross-section shapes. This surface clearly has slope discontinuities in the x-direction.

Figure 3(b) shows a cubic function fit through the input cross sections, with four cross sections shown interpolated between each pair of input shapes for this particular case. For this surface, the slopes at the internal joints were not specified by requiring second-derivative continuity at the joints because that requirement led to a highly warped surface with large excursions between the joints. Consequently, a more conservative approach was taken by setting the slopes at internal joints equal to the average of the left- and right-hand derivates as follows:

$$\frac{dz_i}{dx} = \frac{1}{2}\left(\frac{z_{i+1} - z_i}{x_{i+1} - x_i} + \frac{z_i - z_{i-1}}{x_i - x_{i+1}}\right) \tag{19}$$

With the same input cross sections, the surface shown in figure 3(c) was generated by the method of blended parabolas. This method also produces a smooth fit without large excursions between the joints. With a slight increase in complexity, the method of rational tension splines (ref. 2) can also be used.

Smoothing

If the individual input cross-section shapes are defined by numerical data rather than analytic functions, the data may not be smooth, and consequently the resulting surface would be bumpy. In this case, the input shapes must be preconditioned by replacing the data with a smoothed approximation, which can be represented analytically in various ways, as for example, by smoothing splines (ref. 5). Of course, smoothing can be performed only if some deviation from the original data is permitted. Once the cross-section shapes are described analytically, whether by splines or some other formula, the coefficients $B_i(\xi)$ in equation (15) are analytic functions, and consequently, the formula $F(\xi, t)$ for the surface is analytic.

However, even with such a surface, the variation with respect to t may be too extreme for some particular application. That is, while the general shape of the surface is appropriate, the requirement that the surface passes precisely through each of the input cross sections may cause too much warping of the surface, and a smoother version of the surface is required. Such a smoothed version can be generated in various ways. One relatively simple procedure is described next.

Assume that the surface is required to coincide precisely with the two end sections but that the condition of matching the intermediate joint sections may be relaxed. If the intermediate shapes are completely ignored, the surface that satisfies the end conditions with minimum variation in the x-direction is a linear blend of the two end shapes:

$$z_\ell(y, x) = \left(1 - \frac{x}{x_5}\right) z_1(y) + \frac{x}{x_5} z_5(y) \tag{20}$$

At a location x_i of one of the joint sections, this function has the shape

$$z_{\ell i}(y) = \left(1 - \frac{x_i}{x_5}\right) z_1(y) + \frac{x_i}{x_5} z_5(y) \tag{21}$$

Now this function can be blended with the original input shape $z_i(y)$ at x_i according to the following formula:

$$z_{bi}(y) = (1 - \lambda_i) z_i(y) + \lambda_i z_{\ell i}(y) \tag{22}$$

where $0 \leq \lambda_i \leq 1$. Clearly $\lambda_i = 0$ results in no deviation from the input section, while $\lambda_i = 1$ results in total neglect of the input section shape. An appropriate value of λ_i is selected and then the function spline procedure is applied with the joint shapes $z_i(y)$ replaced with $z_{bi}(y)$.

This kind of smoothing is illustrated in figure 4, which shows the surfaces that result when the parameter λ_i is varied progressively from 0 to 1.

Aerodynamic Surface Example

Figure 5 shows the steps in transforming the upper surface of a blended wing-fuselage configuration into its topological equivalent having normalized chord lengths. Figure 5(a) is a plan view of the configuration. Figure 5(b) shows the same configuration with the leading edge unswept by subtracting the distance from the nose to the leading edge for each section, including the fuselage. Figure 5(c) depicts the surface with the chord lengths normalized to a constant value.

The function spline procedure was applied to the surface of figure 5(c) to interpolate four sections between successive joint sections. Then the transformations in figure 5 were applied in reverse order. The surface grids of the original configuration and of the resulting function spline representation are shown in figure 6 for purposes of comparison. Figure 7 shows a close-up, highly foreshortened, view in the vicinity of the wing-fuselage juncture. Figure 8 shows a similar view including both upper and lower surfaces. A smooth representation of the surface is obtained.

Concluding Remarks

A theory for representing a parameter-dependent function as a function trajectory has been presented, along with the theory for determining a piecewise analytic fit to the trajectory. An example is given that illustrates the application of the theory to generating a smooth surface through a discrete set of input cross-section shapes. A simple procedure for smoothing in the parameter direction was presented, along with a computed example. Application of the theory to aerodynamic surface modeling was demonstrated by applying it to a blended wing-fuselage surface.

NASA Langley Research Center
Hampton, VA 23665-5225
June 2, 1992

References

1. Rogers, David F.; and Adams, J. Alan: *Mathematical Elements for Computer Graphics.* McGraw-Hill, Inc., c.1976.

2. Schiess, James R.; and Kerr, Patricia A.: *Rational-Spline Approximation With Automatic Tension Adjustment.* NASA TP-2366, 1984.

3. Rabinowitz, Paul H., ed.: *Applications of Bifurcation Theory.* Academic Press, Inc., 1977.

4. Barger, Raymond L.; and Walters, Robert W.: *Some Path-Following Techniques for Solution of Nonlinear Equations and Comparison With Parametric Differentiation.* NASA TP-2654, 1986.

5. Reinsch, Christian H.: Smoothing by Spline Functions. *Numer. Math.*, vol. 10, no. 3, 1967, pp. 177 183.

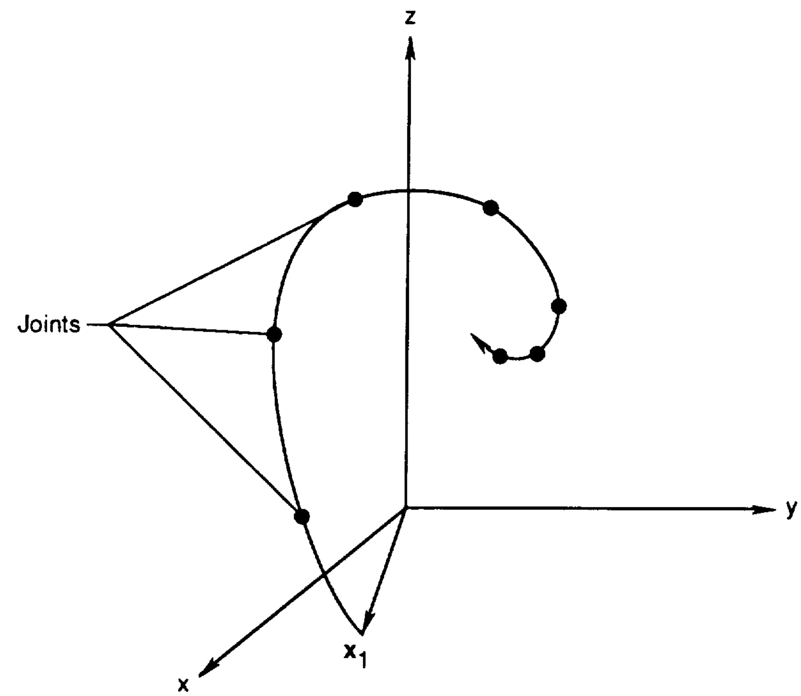


Figure 1. Smooth curve fit through discrete points in three-dimensional space. Circles denote joint locations.

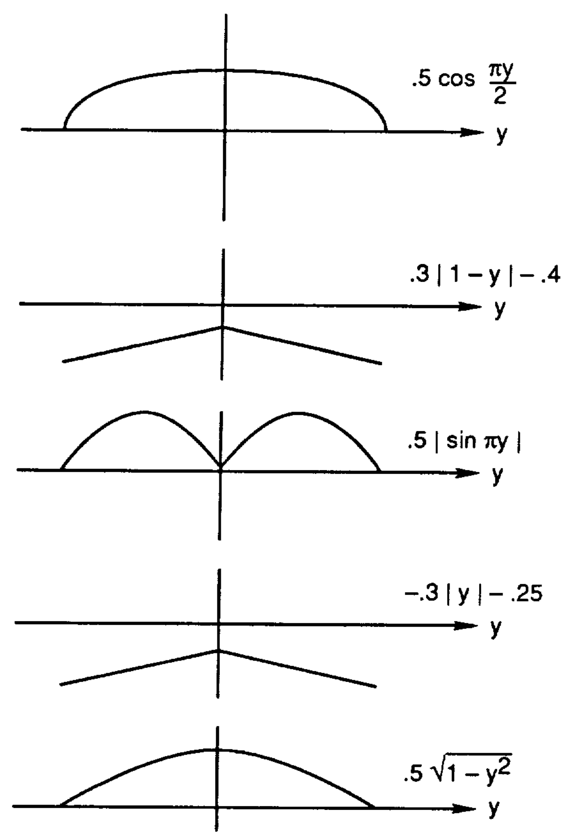
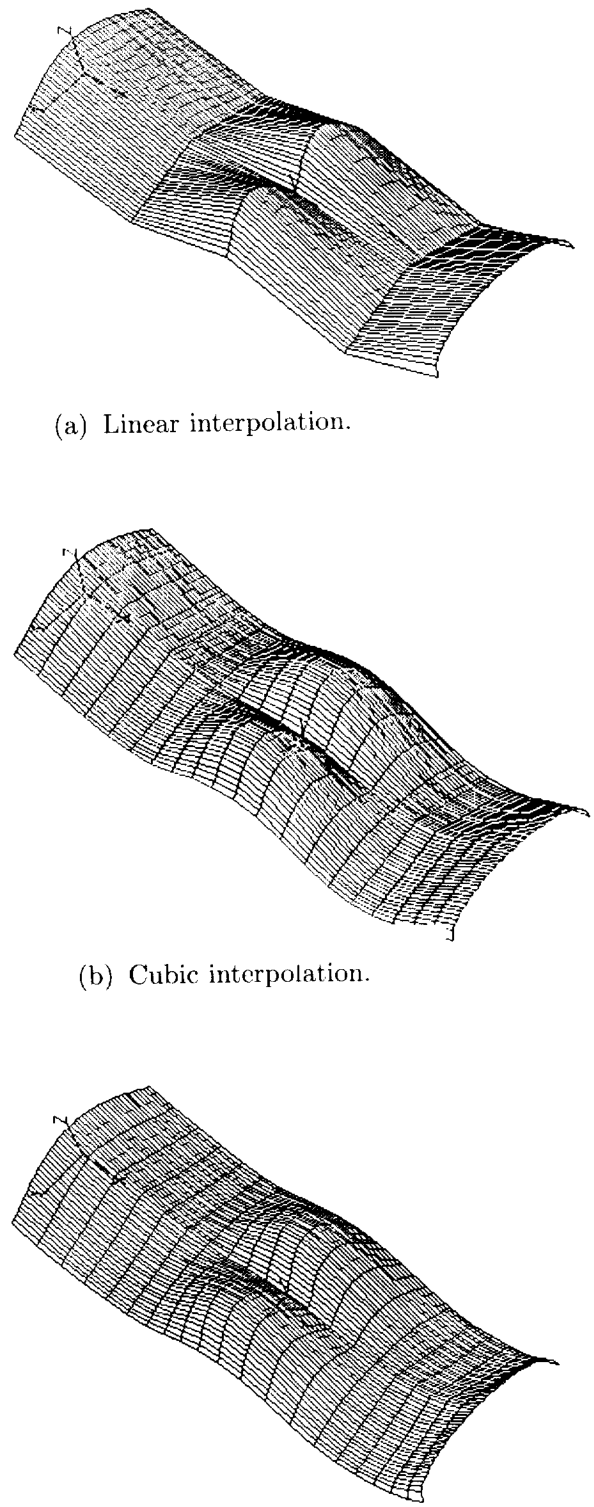


Figure 2. Cross-section shapes to be fit by surface.

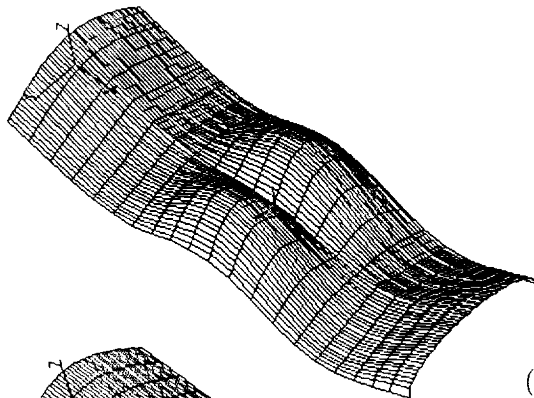


(a) Linear interpolation.

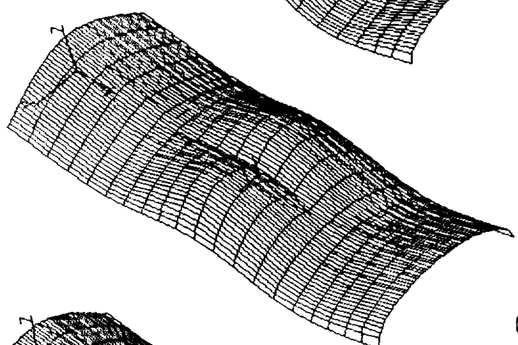
(b) Cubic interpolation.

(c) Blended-parabola interpolation.

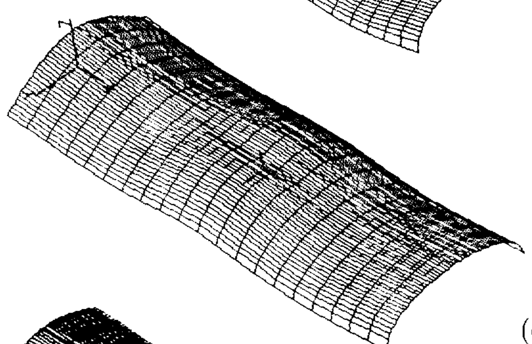
Figure 3. Surfaces fit through input cross-section shapes by various methods.



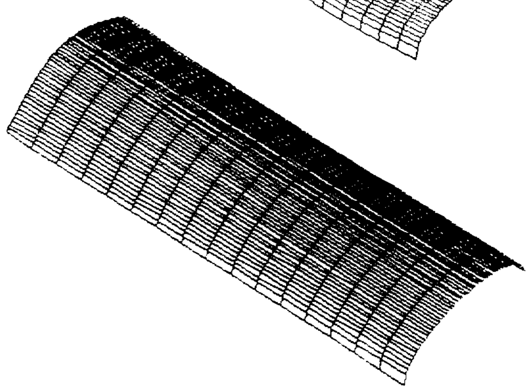
(a) $\lambda = 0.$



(b) $\lambda = 0.5.$

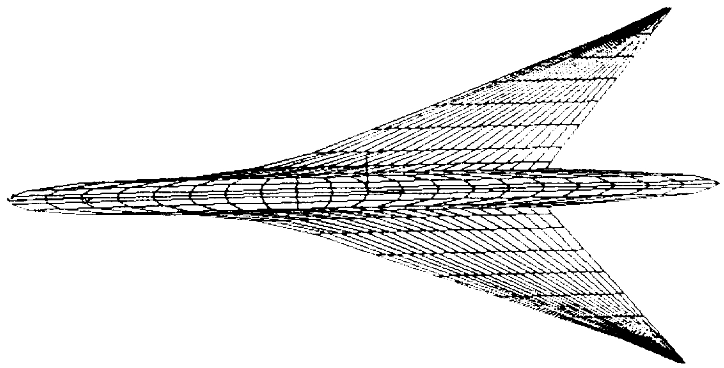


(c) $\lambda = 0.8.$

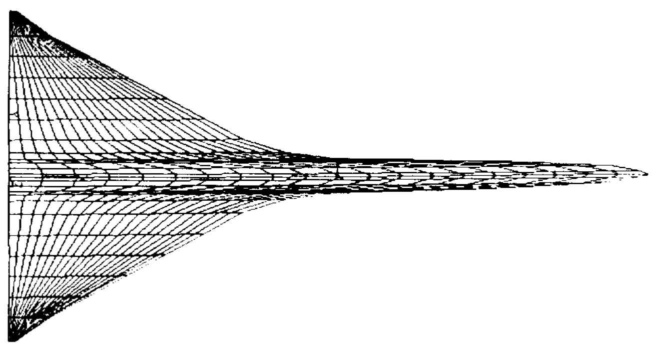


(d) $\lambda = 1.0.$

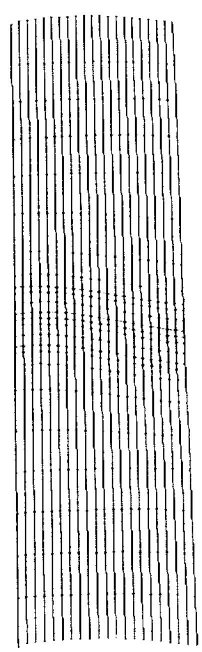
Figure 4. Progressive smoothing of surface in parameter direction.



(a) Original surface.

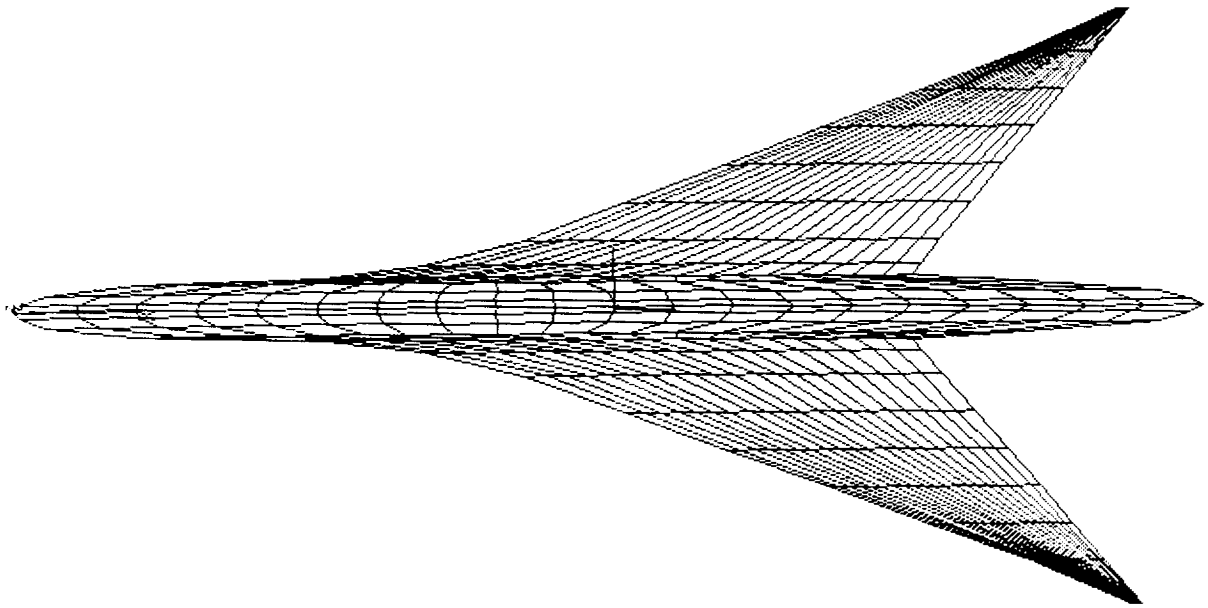


(b) Leading edge unswept.

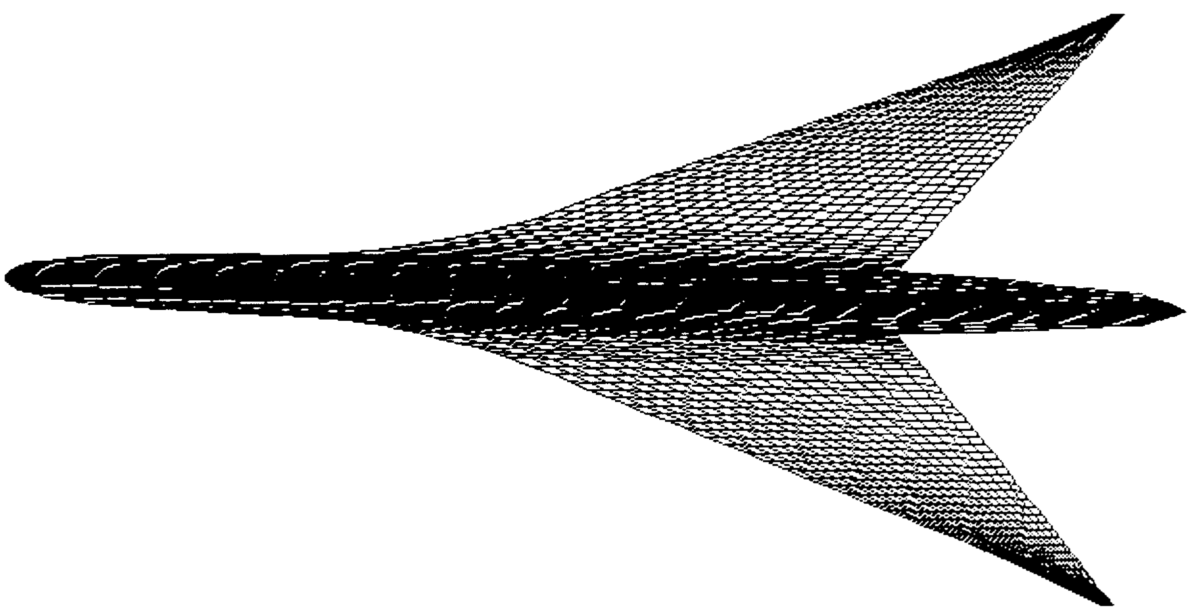


(c) Chord lengths normalized.

Figure 5. Topological transformation of configuration upper surface.

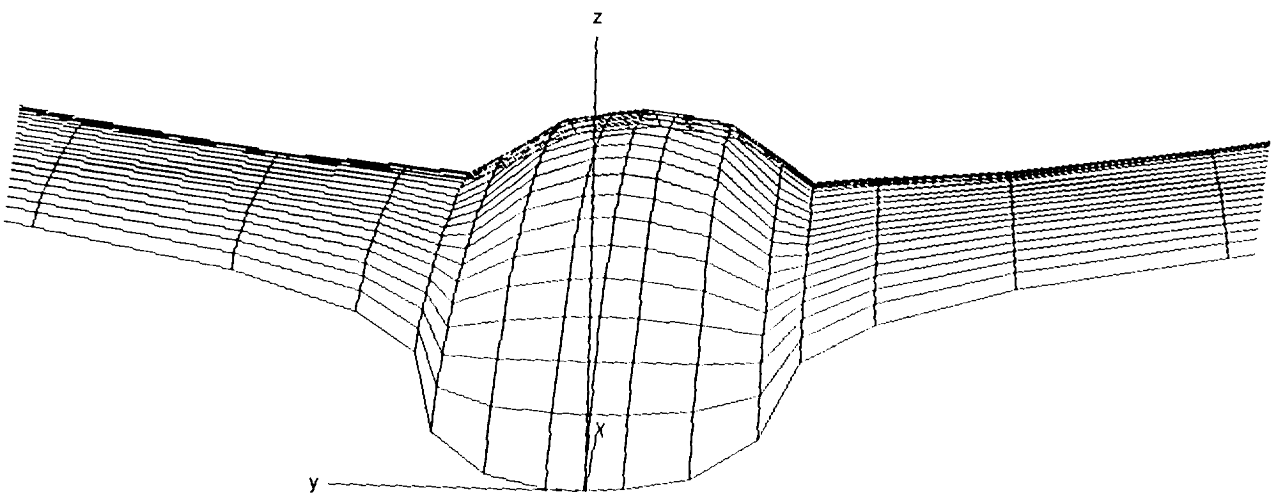


(a) Original configuration.

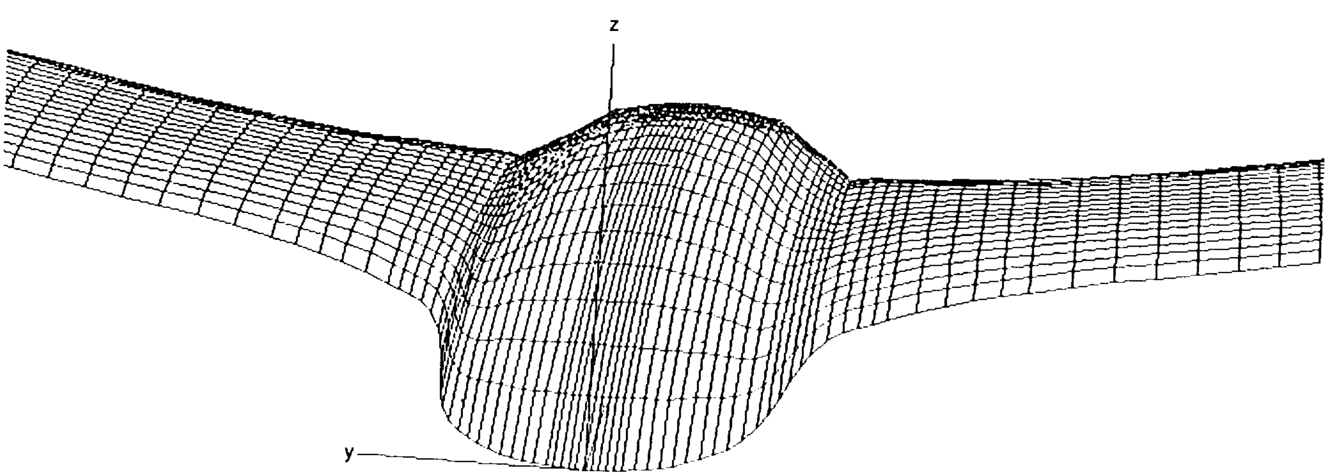


(b) Revised configuration.

Figure 6. Surface of figure 5 before and after applying shape interpolation and transformations in reverse order.

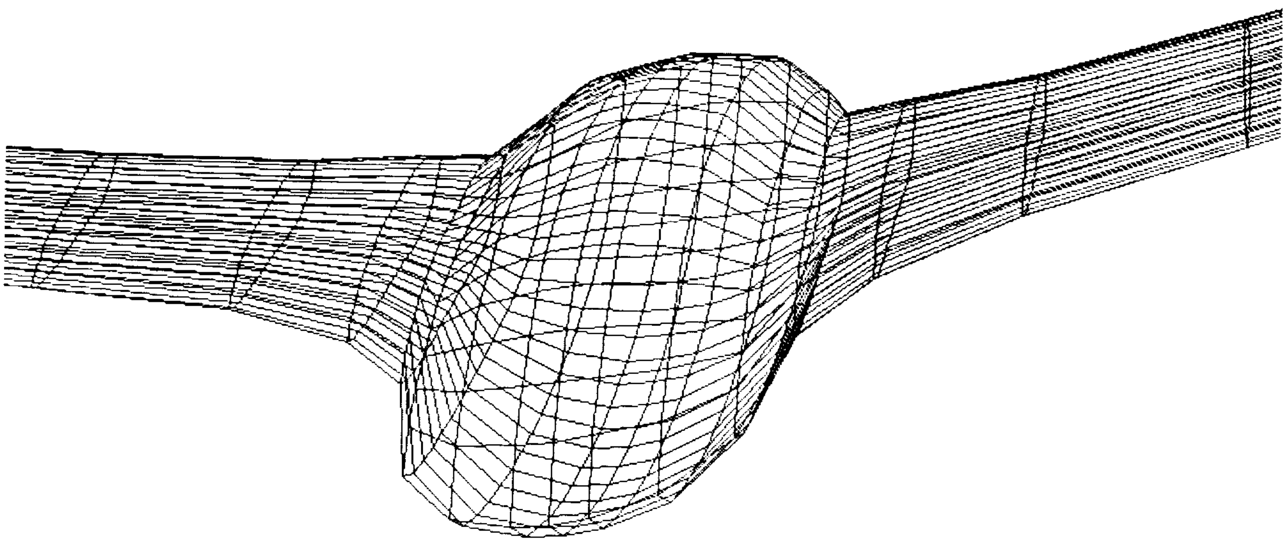


(a) Before interpolation.

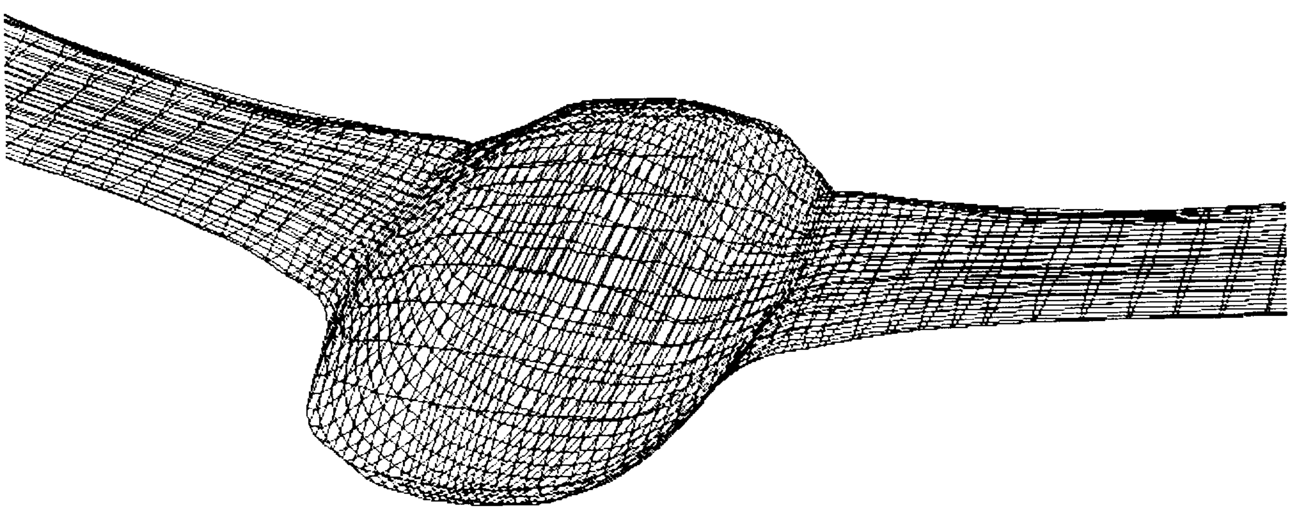


(b) After interpolation.

Figure 7. Perspective view of wing-fuselage juncture before and after interpolation.



(a) Before interpolation.



(b) After interpolation.

Figure 8. Perspective view, including lower surface of wing-fuselage juncture before and after interpolation.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
July 1992

3. REPORT TYPE AND DATES COVERED
Technical Paper

4. TITLE AND SUBTITLE
Trajectory Fitting in Function Space With Application to Analytic Modeling of Surfaces

5. FUNDING NUMBERS
WU 505-59-53-01

6. AUTHOR(S)
Raymond L. Barger

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
NASA Langley Research Center
Hampton, VA 23665-5225

8. PERFORMING ORGANIZATION REPORT NUMBER
L-17054

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
NASA TP-3232

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Unclassified Unlimited

Subject Category 02

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
A theory for representing a parameter-dependent function as a function trajectory is described, along with the theory for determining a piecewise analytic fit to the trajectory. An example is given that illustrates the application of the theory to generating a smooth surface through a discrete set of input cross-section shapes. A simple procedure for smoothing in the parameter direction is presented, along with a computed example. Application of the theory to aerodynamic surface modeling is demonstrated by applying it to a blended wing-fuselage surface.

14. SUBJECT TERMS
Geometry modeling; Analytic modeling; Surface grid generation

15. NUMBER OF PAGES
13

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified		

National Aeronautics and
Space Administration
Code JTT
Washington, D.C.
20546-0001

Official Business
Penalty for Private Use, $300

BULK RATE
POSTAGE & FEES PAID
NASA
Permit No. G-27

NASA

POSTMASTER: If Undeliverable (Section 158 Postal Manual) Do Not Return