NASA-CR-190745

EVALUATION OF THE INCOMPLETE AIRY FUNCTIONS AND THEIR APPLICATION TO HIGH FREQUENCY SCATTERING AND DIFFRACTION Semiannual Report (Ohio State Univ.) 57 p

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Numerical Evaluation of the Incomplete Airy Functions and Their Application to High Frequency Scattering and Diffraction

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Semi-Annual Report 725311-1
Grant No. NSG-1492
September 1992

National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23665

NOTICES

50272-101

REPORT DOCUMENTATION PAGE	1. REPORT NO.	2.	3. Recipient's Accession No.
4. Title and Subtitle Numerical Evaluation of the Incomplete Airy Functions and Their Application to High Frequency Scattering and Diffraction			5. Report Date September 1992 6.
7. Author(s) E.D. Constantinides and R.J. Marhefka			8. Performing Org. Rept. No. 725311-1
9. Performing Organization Name and Address The Ohio State University ElectroScience Laboratory 1320 Kinnear Road Columbus, OH 43212			10. Project/Task/Work Unit No. 11. Contract(C) or Grant(G) No. (C) (G) NSG-1498
12. Sponsoring Organization Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665			13. Report Type/Period Covered Semi-Annual Report 14.

15. Supplementary Notes

16. Abstract (Limit: 200 words)

The incomplete Airy integrals serve as canonical functions for the uniform ray optical solutions to several high-frequency scattering and diffraction problems that involve a class of integrals characterized by two stationary points that are arbitrarily close to one another or to an integration endpoint. Integrals of such analytical properties describe transition region phenomena associated with composite shadow boundaries. An efficient and accurate method for computing the incomplete Airy functions would make the solutions to such problems useful for engineering purposes. In this report, a convergent series solution form for the incomplete Airy functions is derived. Asymptotic expansions involving several terms are also developed and serve as large argument approximations. The combination of the series solution form with the asymptotic formulae provides for an efficient and accurate computation of the incomplete Airy functions. Validation of accuracy is accomplished using direct numerical integration data.

17. Document Analysis a. Descriptors

ASYMPTOTIC
DIFFRACTION
NUMERICAL
SCATTERING
UTD

b. Identifiers/Open-Ended Terms

c. COSATI Field/Group

18. Availability Statement	19. Security Class (This Report)	21. No. of Pages
A. Approved for public release; Distribution is unlimited.	Unclassified	56
	20. Security Class (This Page) Unclassified	22. Price

(See ANSI-Z39.18) *See Instructions on Reverse* OPTIONAL FORM 272 (4-77)
Department of Commerce

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Chapter 1

Introduction

A common method of analysis for high-frequency electromagnetic and acoustic scattering and diffraction problems involves the use of radiation integrals as well as plane wave integral representations for the fields, with the asymptotic approximations to the various scattering mechanisms found from the critical point contributions of the integrand. The incomplete Airy integrals [1] serve as canonical functions for the uniform asymptotic approximation of a class of integrals characterized by two stationary phase points that are arbitrarily close to one another or to an integration endpoint [2]. Integrals of such analytical properties describe transition region phenomena associated with composite shadow boundaries resulting from the confluence of two stationary points and an endpoint in the integration interval. A typical problem of particular interest, where transition region phenomena of this type exist, involves the scattering from smoothly indented boundaries containing an edge as illustrated in Figures 1.1 and 1.2. When the reflection shadow boundary (RSB) is not in the immediate vicinity of the smooth caustic, the conventional UTD[1] edge diffraction coefficient [3] which involves the Fresnel integral as a canonical function can be used to effectively describe the edge diffracted field behavior in the neighborhood of the reflection shadow boundary. Furthermore, the ordinary Airy integrals and their derivatives are the appropriate canonical functions for the description of

[1] Uniform Geometrical Theory of Diffraction

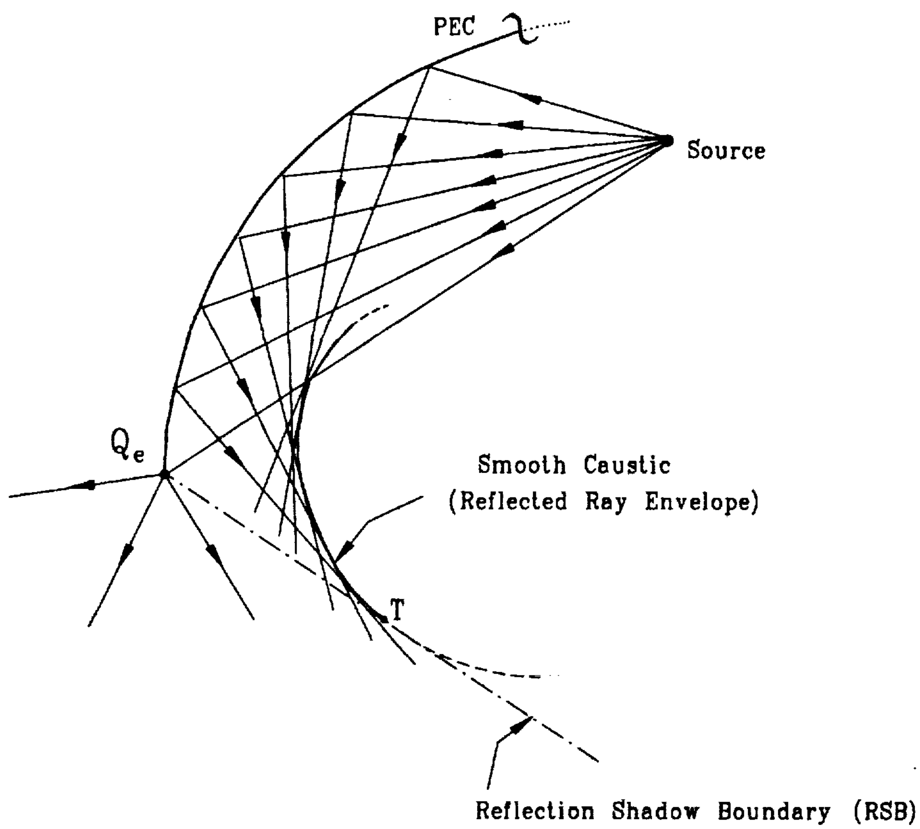

Figure 1.1: Scattering and diffraction from a concave boundary containing an edge.

the high-frequency fields in the neighborhood of the smooth caustic [4]. However, when there is a confluence of both reflected and caustic type shadow boundaries as shown in Figures 1.1 and 1.2, neither the Fresnel integral nor the ordinary Airy integral adequately describe the transition region phenomena, and they must be appropriately replaced by the incomplete Airy function.

The formulation of uniform ray optical solutions for problems involving composite shadow boundary transition phenomena, that are useful for engineering purposes, requires an efficient and accurate method for computing the incomplete Airy functions. In the original work of Levey and Felsen [1], several other diffraction problems whose solutions involve the incomplete Airy functions are also described, and some general and asymptotic characteristics of these functions are examined. However, the asymptotic formulae given in [1] are valid when the saddle points are sufficiently

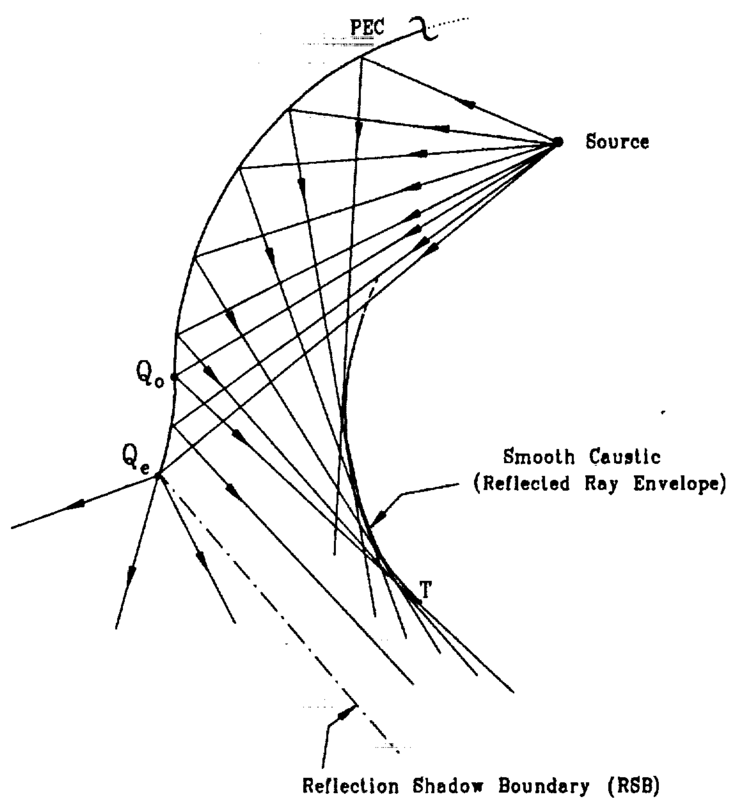

Figure 1.2: Scattering and diffraction from a concave-convex boundary containing an edge.

far apart or far removed from the integration endpoint. When the two saddle points move close together or coalesce to form a caustic, the asymptotic formulae break down and an alternative computation method for the incomplete Airy functions is needed.

In this report, a convergent series solution for the incomplete Airy functions is rigorously derived, and asymptotic expansions involving several terms are also developed for large argument approximations. The higher order terms in the asymptotic expansions greatly improve the accuracy of the asymptotic formulae, and also improve computational efficiency by limiting the region of the argument space where the more time consuming series solution form should be used. Thus, the combination of the series solution form with the more accurate asymptotic formulae provides for an efficient and accurate computation of the incomplete Airy functions for the entire range of their arguments. These results would allow the formulation of uniform ray optical solutions for high-frequency scattering and diffraction problems that are useful for engineering purposes. A method for uniformly evaluating certain stationary phase integrals using the incomplete Airy integral as a canonical function is briefly discussed in Appendix A. The details of a systematic uniform asymptotic analysis for particular applications that involve transition region phenomena describable by the incomplete Airy functions will be reported a separately.

The outline of this report is as follows: In Chapter 2, some general properties of the incomplete Airy functions are reviewed, and a convergent series solution form is rigorously derived using the parabolic differential equation. The asymptotic formulae for large argument approximations are derived in Chapter 3 using the integral forms of the incomplete Airy functions. In Chapter 4, some indicative numerical results are presented and discussed, with their accuracy demonstrated via comparison with data obtained from direct numerical integration of the integral forms. The regions of validity for each formula used in the computations are also shown in this chapter, and some error assessments for the asymptotic results are provided. Finally, the main results and accomplishments of this work are summarized in Chapter 5.

Chapter 2

Derivation of the Series Solution Form

In this chapter, a convergent series solution form for the incomplete Airy functions is derived. Before proceeding with the derivation, however, some of their general properties are briefly reviewed.

The incomplete Airy functions are functions of two variables and they satisfy the parabolic partial differential equation applied by Fock [5] to the study of fields near the surface of a smooth convex scattering surface; i.e.,

$$\left[\frac{\partial^2}{\partial\beta^2} - \beta - j\frac{\partial}{\partial\xi}\right] g_i(\beta,\xi) \equiv 0 \;; \quad i = 0,1,2. \tag{2.1}$$

In integral form, the solutions of (2.1) are given by:

$$g_i(\beta,\xi) \equiv \int_{\xi_i}^{\infty \exp(j\psi_i)} e^{j(\beta z + z^3/3)}\, dz \;; \quad i = 0,1,2 \tag{2.2}$$

where the upper limit lies within one of the three sectors of the complex z-plane in which the integral converges; i.e.,

$$2i\frac{\pi}{3} \le \psi_i \le (2i+1)\frac{\pi}{3} \;; \quad i = 0,1,2. \tag{2.3}$$

The contours of integration for the incomplete Airy functions are shown in Figure 2.1.

In this report, we only examine $g_0(\beta,\xi)$ since the other two functions, namely g_1 and g_2, can be obtained from g_0 and the well known complete Airy functions [6];

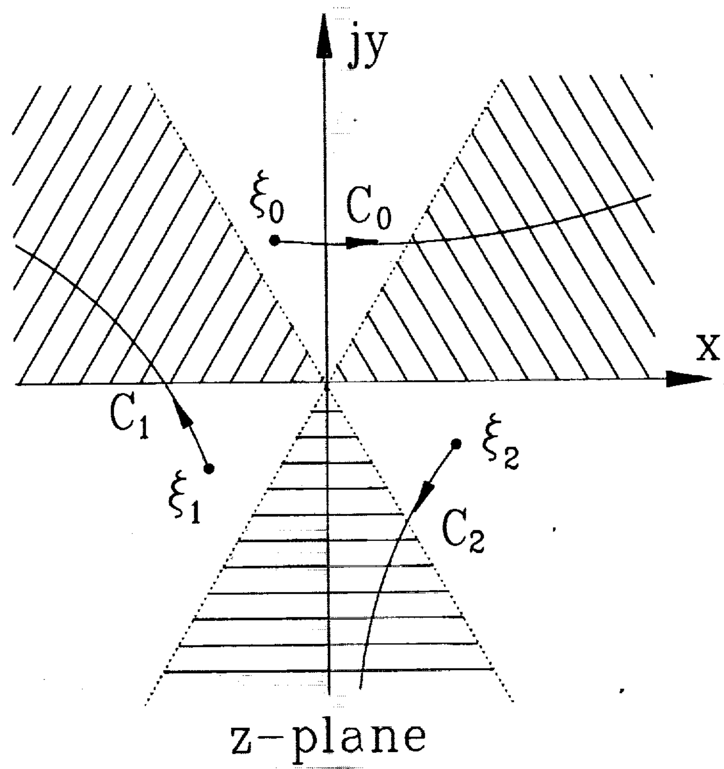

Figure 2.1: Contours of integration for the incomplete Airy functions.

i.e.,

$$g_1(\beta,\xi) = g_0(\beta,\xi) - 2\pi \mathrm{Ai}(\beta)\,, \tag{2.4}$$

and

$$g_2(\beta,\xi) = g_0(\beta,\xi) - \pi[\mathrm{Ai}(\beta) + j\mathrm{Bi}(\beta)]\,, \tag{2.5}$$

where

$$\mathrm{Ai}(\beta) = \frac{1}{2\pi}\int_{L_1} e^{j(\beta z + z^3/3)}\,dz\,, \tag{2.6}$$

and

$$\mathrm{Bi}(\beta) = \frac{j}{2\pi}\int_{L_2+L_3} e^{j(\beta z + z^3/3)}\,dz\,. \tag{2.7}$$

The contours of integration for the complete Airy functions are shown in Figure 2.2. Also, the quantities β and ξ will be taken as real since in most practical applications real β and ξ are of primary interest. However, this is not a requirement for the analysis that follows and the resulting formulae are valid for arbitrary values of β and ξ. In addition, ξ will be restricted to positive values since for negative values of ξ, g_0 may be obtained using the expression:

$$g_0(\beta,-|\xi|) = 2\pi\mathrm{Ai}(\beta) - g_0^*(\beta,|\xi|)\,, \tag{2.8}$$

with $(*)$ denoting the complex conjugate operation.

In order to obtain a series solution form for $g_0(\beta,\xi)$, we begin with the parabolic differential equation and assume two independent solutions of the form:

$$y_1(\beta,\xi) = \sum_{n=0}^{\infty} a_n(\xi)\beta^n\,, \tag{2.9}$$

$$\text{and}\quad y_2(\beta,\xi) = \sum_{n=0}^{\infty} b_n(\xi)\beta^n\,. \tag{2.10}$$

Substituting (2.9) and (2.10) in (2.1) we obtain the following expressions:

$$\sum_{n=2}^{\infty} n(n-1)a_n(\xi)\beta^{n-2} - \sum_{n=0}^{\infty} a_n(\xi)\beta^{n+1} - j\sum_{n=0}^{\infty} a_n'(\xi)\beta^n \equiv 0\,, \tag{2.11}$$

$$\text{and}\quad \sum_{n=2}^{\infty} n(n-1)b_n(\xi)\beta^{n-2} - \sum_{n=0}^{\infty} b_n(\xi)\beta^{n+1} - j\sum_{n=0}^{\infty} b_n'(\xi)\beta^n \equiv 0\,. \tag{2.12}$$

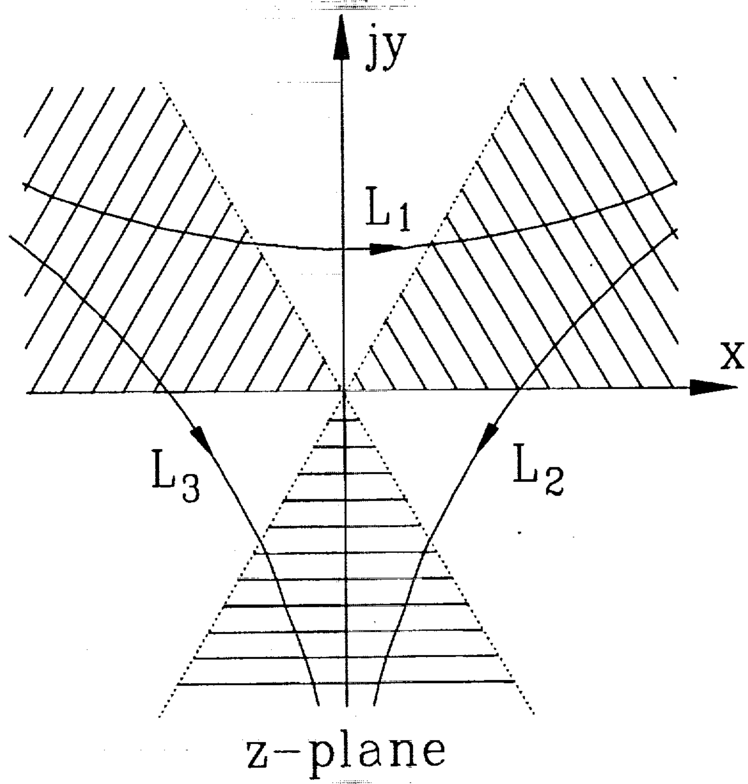

Figure 2.2: Contours of integration for the complete Airy functions.

For Equations (2.11) and (2.12) to be satisfied, the sum of coefficients of like powers of β must be zero for any value of ξ. To obtain the first independent solution, y_1, we let $a_1(\xi) \equiv 0$, and setting the sum of coefficients of like powers in (2.11) equal to zero we obtain:

$$a_1(\xi) = 0, \ \forall \xi, \tag{2.13}$$

$$a_2(\xi) = \frac{j}{2}a_0'(\xi), \tag{2.14}$$

$$\text{and} \quad a_n(\xi) = \frac{a_{n-3}(\xi) + ja_{n-2}'(\xi)}{n(n-1)}, \quad n \geq 3. \tag{2.15}$$

Similarly, for the second independent solution, y_2, we let $b_0(\xi) \equiv 0$ and setting the sum of coefficients of like powers in (2.12) equal to zero we obtain:

$$b_0(\xi) = 0, \ \forall \xi, \tag{2.16}$$

$$b_2(\xi) = 0, \ \forall \xi, \tag{2.17}$$

$$\text{and} \quad b_n(\xi) = \frac{b_{n-3}(\xi) + jb_{n-2}'(\xi)}{n(n-1)}, \quad n \geq 3. \tag{2.18}$$

Thus, y_1 and y_2 are given by:

$$y_1(\beta,\xi) = a_0(\xi) + \sum_{n=2}^{\infty} a_n(\xi)\beta^n, \tag{2.19}$$

$$\text{and} \quad y_2(\beta,\xi) = b_1(\xi)\beta + \sum_{n=2}^{\infty} b_n(\xi)\beta^n, \tag{2.20}$$

where $a_n(\xi)$ and $b_n(\xi)$ for $n \geq 2$ can be expressed in terms of $a_0(\xi)$, $b_1(\xi)$ and their derivatives, respectively, via Equations (2.13)–(2.18). Now, $g_0(\beta,\xi)$ must be equal to the sum of the two independent solutions; i.e.,

$$g_0(\beta,\xi) = a_0(\xi) + b_1(\xi)\beta + \sum_{n=2}^{\infty}[a_n(\xi) + b_n(\xi)]\beta^n, \tag{2.21}$$

$$\text{and} \quad \frac{\partial}{\partial\beta}g_0(\beta,\xi) = b_1(\xi) + \sum_{n=2}^{\infty} n[a_n(\xi) + b_n(\xi)]\beta^{n-1}. \tag{2.22}$$

Although the coefficients $a_n(\xi)$ and $b_n(\xi)$ can be combined into a single coefficient, keeping them separate greatly simplifies their evaluation in a computer code.

Finally, it remains to find expressions for $a_0(\xi)$ and $b_1(\xi)$ in order to complete the solution. This can be accomplished by applying the proper boundary conditions at $\beta = 0$ using the integral form of $g_0(\beta, \xi)$ given in Equation (2.2); i.e.,

$$a_0(\xi) = g_0(\beta, \xi)|_{\beta=0} = \int_{\xi}^{\infty} e^{jz^3/3}\, dz\,, \tag{2.23}$$

and

$$b_1(\xi) = \frac{\partial}{\partial \beta} g_0(\beta, \xi)|_{\beta=0} = \int_{\xi}^{\infty} jz e^{jz^3/3}\, dz\,. \tag{2.24}$$

The functions $a_0(\xi)$ and $b_1(\xi)$ can be expressed in terms of the incomplete Gamma function [7], and their computation is straight forward. Details are provided in Appendix B.

When $\xi \to -\infty$, our solution should reduce to the series solution form for the complete Airy function, $\mathrm{Ai}(\beta)$ [6]. In this case we have:

$$a_0(\xi \to -\infty) = 2\pi \mathrm{Ai}(0) = 2.23070703\,, \tag{2.25}$$

$$b_1(\xi \to -\infty) = 2\pi \mathrm{Ai}'(0) = -1.62621025\,, \tag{2.26}$$

and using Equations (2.13)–(2.20) we obtain:

$$\begin{aligned} g_0(\beta, \xi \to -\infty) &= 2\pi \mathrm{Ai}(0) \left(1 + \frac{1}{3!}\beta^3 + \frac{1 \cdot 4}{6!}\beta^6 + \frac{1 \cdot 4 \cdot 7}{9!}\beta^9 + \cdots\right) \\ &+ 2\pi \mathrm{Ai}'(0) \left(\beta + \frac{2}{4!}\beta^4 + \frac{2 \cdot 5}{7!}\beta^7 + \frac{2 \cdot 5 \cdot 8}{10!}\beta^{10} + \cdots\right) \\ &= 2\pi \mathrm{Ai}(\beta)\,, \end{aligned} \tag{2.27}$$

which is a necessary condition for the validity of our result.

Chapter 3

Derivation of Asymptotic Formulae

In this chapter, a pair of asymptotic formulae for the incomplete Airy function, $g_0(\beta,\xi)$, involving several terms are derived and serve as large argument forms. We begin by introducing the large parameter Ω in the integral form of $g_0(\beta,\xi)$, and examine the integral:

$$I_0(\sigma,\gamma;\Omega) = \int_{\gamma}^{\infty} e^{j\Omega(\sigma z + z^3/3)}\, dz\,. \tag{3.1}$$

Our objective is to obtain asymptotic expansions for I_0 as $\Omega \to \infty$ for various dispositions of the saddle points and endpoint. Then, the asymptotic formulae for $g_0(\beta,\xi)$ may be obtained using the expression:

$$g_0(\beta,\xi) = \Omega^{1/3} I_0(\sigma = \beta\Omega^{-2/3}, \gamma = \xi\Omega^{-1/3}; \Omega)\,. \tag{3.2}$$

Since we are interested in real β and ξ with $\xi \geq 0$, the analysis that follows will be restricted to real σ and γ, with $\gamma \geq 0$.

3.1 Asymptotic Formula for $\xi \gg |\beta|^{\frac{1}{2}}$

This case corresponds to the endpoint being far removed from the possibly neighboring saddle points of the integrand in Equation (3.1), or $\gamma \gg |\sigma|^{\frac{1}{2}}$. Although the sign of σ is irrelevant in this case, the saddle points $z_{1,2} = \pm(-\sigma)^{1/2}$ are taken as real for simplicity. Also, the original integration path P_0 in Figure 3.1 is deformed into the steepest descent path, P_{sdp}, leading away from the endpoint at $z = \gamma$ and into the sector $0 \leq \arg(z) \leq \pi/3$ of the complex z-plane. The asymptotic evaluation of I_0 is then performed using repeated integration by parts that yields:

$$
\begin{aligned}
I_0(\sigma,\gamma;\Omega) \sim\; & e^{j\Omega(\sigma\gamma+\gamma^3/3)}\left\{\frac{1}{\Omega}\left[\frac{j}{\sigma+\gamma^2}\right]+\frac{1}{\Omega^2}\left[\frac{2\gamma}{(\sigma+\gamma^2)^3}\right]\right. \\
& +\frac{1}{\Omega^3}\left[\frac{2j}{(\sigma+\gamma^2)^4}-\frac{12j\gamma^2}{(\sigma+\gamma^2)^5}\right]+\frac{1}{\Omega^4}\left[\frac{40\gamma}{(\sigma+\gamma^2)^6}-\frac{120\gamma^3}{(\sigma+\gamma^2)^7}\right] \\
& \left.+\frac{1}{\Omega^5}\left[\frac{-40j}{(\sigma+\gamma^2)^7}-\frac{840j\gamma^2}{(\sigma+\gamma^2)^8}+\frac{1680j\gamma^4}{(\sigma+\gamma^2)^9}\right]\right\}+O(\Omega^{-6}). \qquad (3.3)
\end{aligned}
$$

Now, using Equations (3.2) and (3.3), the asymptotic formula for $g_0(\beta,\xi)$ when $\xi \gg |\beta|^{1/2}$ is given by:

$$
\begin{aligned}
g_0(\beta,\xi) \simeq\; & e^{j(\beta\xi+\xi^3/3)}\left[\frac{j}{\beta+\xi^2}+\frac{2\xi}{(\beta+\xi^2)^3}+\frac{2j}{(\beta+\xi^2)^4}-\frac{12j}{(\beta+\xi^2)^5}\right. \\
& \left.+\frac{40\xi}{(\beta+\xi^2)^6}-\frac{120\xi^3-40j}{(\beta+\xi^2)^7}-\frac{840j\xi^2}{(\beta+\xi^2)^8}+\frac{1680j\xi^4}{(\beta+\xi^2)^9}\right]. \qquad (3.4)
\end{aligned}
$$

The derivative of $g_0(\beta,\xi)$ is given by:

$$
\begin{aligned}
\frac{\partial}{\partial\beta}g_0(\beta,\xi) \simeq\; & e^{j(\beta\xi+\xi^3/3)}\left[\frac{-\xi}{\beta+\xi^2}-\frac{j}{(\beta+\xi^2)^2}+\frac{2j\xi^2}{(\beta+\xi^2)^3}-\frac{8\xi}{(\beta+\xi^2)^4}\right. \\
& -\frac{8j-12\xi}{(\beta+\xi^2)^5}+\frac{60j+40j\xi^2}{(\beta+\xi^2)^6}-\frac{280\xi+120j\xi^3}{(\beta+\xi^2)^7} \\
& \left.+\frac{1680\xi^3-280j}{(\beta+\xi^2)^8}+\frac{6720j\xi^2-1680\xi^4}{(\beta+\xi^2)^9}-\frac{15120j\xi^4}{(\beta+\xi^2)^{10}}\right]. \qquad (3.5)
\end{aligned}
$$

3.2 Asymptotic Formula for $\beta \ll -1$

This case corresponds to real and widely separated saddle points ($\sigma \ll -1$) in the integrand of Equation (3.1), with the endpoint γ arbitrarily close to the saddle point $z_1 = (-\sigma)^{1/2}$, as shown in Figure 3.2. For an asymptotic evaluation of Equation (3.1) that holds uniformly as the endpoint γ approaches the saddle point z_1, we make the following transformation [1]:

$$
q(z) = \sigma z + z^3/3 = q(z_1) + s^2 = -\frac{2}{3}(-\sigma)^{3/2} + s^2 = \tau(s), \qquad (3.6)
$$

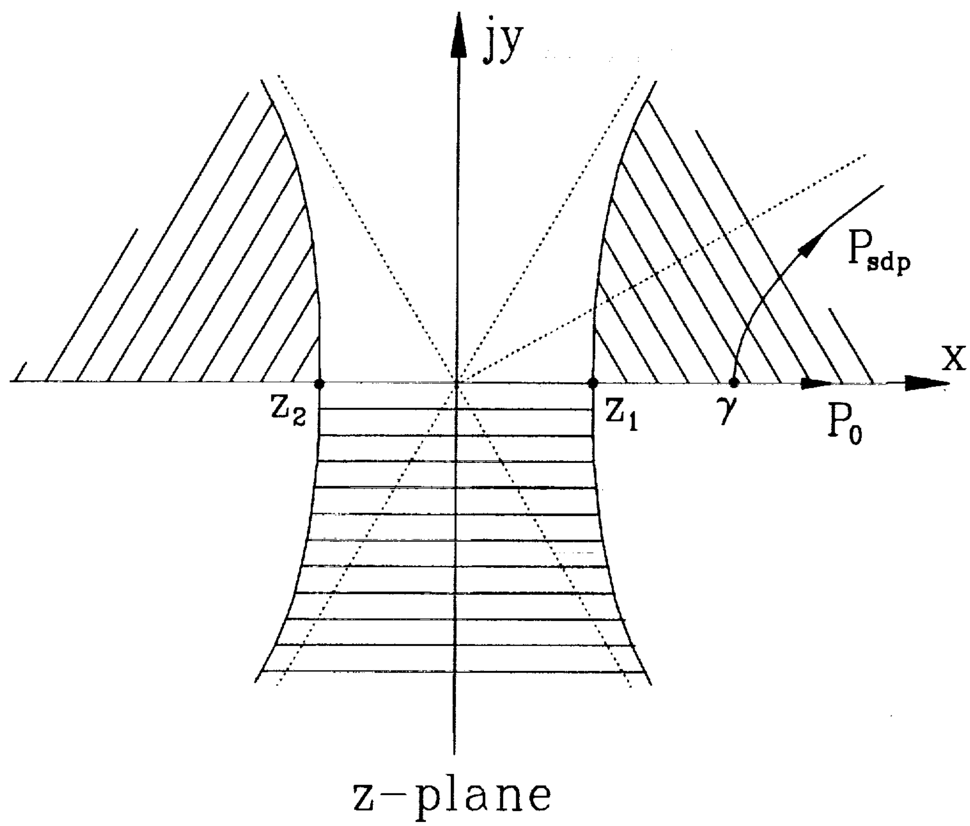

Figure 3.1: Contours of integration for the incomplete Airy integral, $\gamma \gg (-\sigma)^{1/2}$.

with arg(s) restricted so that $I_0(\sigma,\gamma;\Omega)$ converges as $s \to \infty$. Hence, employing (3.6) in Equation (3.1) we have:

$$I_0(\sigma,\gamma;\Omega) = e^{-j\frac{2}{3}\Omega(-\sigma)^{3/2}} \int_\zeta^\infty f(s) e^{j\Omega s^2}\, ds\,, \tag{3.7}$$

with the upper limit taken in the sector $0 \le \arg(s) \le \pi/2$ of the complex s-plane. The quantities ζ and $f(s)$ are given by:

$$\zeta = \pm\left[\sigma\gamma + \gamma^3/3 + \frac{2}{3}(-\sigma)^{3/2}\right]^{1/2} \quad ; \quad \gamma \gtrless (-\sigma)^{1/2}\,, \tag{3.8}$$

and

$$f(s) = \frac{dz}{ds} = \frac{\tau'(s)}{q'(z)} = \frac{2s}{\sigma + z^2}\,. \tag{3.9}$$

Equation (3.7) can be written as follows [8]:

$$I_0(\sigma,\gamma;\Omega) = e^{-j\frac{2}{3}\Omega(-\sigma)^{3/2}} \left\{ f(0)\int_\zeta^\infty e^{j\Omega s^2}\, ds + \frac{1}{2j\Omega}\int_\zeta^\infty \left[\frac{f(s)-f(0)}{s}\right] \frac{d}{ds} e^{j\Omega s^2}\, ds \right\}, \tag{3.10}$$

and using integration by parts in the second integral we get:

$$\begin{aligned} I_0(\sigma,\gamma;\Omega) \;&=\; e^{-j\frac{2}{3}\Omega(-\sigma)^{3/2}} \left\{ f(0)\int_\zeta^\infty e^{j\Omega s^2}\, ds - \frac{1}{2j\Omega}\left[\frac{f(\zeta)-f(0)}{\zeta}\right] e^{j\Omega\zeta^2} \right. \\ &\left. -\frac{1}{2j\Omega}\int_\zeta^\infty g(s) e^{j\Omega s^2}\, ds \right\}, \end{aligned} \tag{3.11}$$

where

$$g(s) = \frac{s f'(s) - f(s) + f(0)}{s^2}\,. \tag{3.12}$$

In a similar way, the second integral in Equation (3.11) can be written as follows:

$$\begin{aligned} \int_\zeta^\infty g(s) e^{j\Omega s^2}\, ds \;&=\; g(0)\int_\zeta^\infty e^{j\Omega s^2}\, ds - \frac{1}{2j\Omega}\left[\frac{g(\zeta)-g(0)}{\zeta}\right] e^{j\Omega\zeta^2} \\ &-\frac{1}{2j\Omega}\int_\zeta^\infty h(s) e^{j\Omega s^2}\, ds\,, \end{aligned} \tag{3.13}$$

where

$$h(s) = \frac{s g'(s) - g(s) + g(0)}{s^2}\,. \tag{3.14}$$

Thus, using (3.13), Equation (3.11) becomes:

$$\begin{aligned} I_0(\sigma,\gamma;\Omega) &= e^{-j\frac{2}{3}\Omega(-\sigma)^{3/2}}\left[f(0)-\frac{1}{2j\Omega}g(0)\right]\int_\zeta^\infty e^{j\Omega s^2}\,ds \\ &+e^{j\Omega(\sigma\gamma+\gamma^3/3)}\left\{-\frac{1}{2j\Omega}\left[\frac{f(\zeta)-f(0)}{\zeta}\right]+\frac{1}{(2j\Omega)^2}\left[\frac{g(\zeta)-g(0)}{\zeta}\right]\right\} \\ &+e^{-j\frac{2}{3}\Omega(-\sigma)^{3/2}}\left[\frac{1}{(2j\Omega)^2}\int_\zeta^\infty h(s)e^{j\Omega s^2}\,ds\right]. \end{aligned} \tag{3.15}$$

The same procedure is repeated once more for the last integral in Equation (3.15); i.e.,

$$\begin{aligned} \int_\zeta^\infty h(s)e^{j\Omega s^2}\,ds &\sim \left[h(0)-\frac{1}{2j\Omega}k(0)\right]\int_\zeta^\infty e^{j\Omega s^2}\,ds \\ &+e^{j\Omega\zeta^2}\left\{-\frac{1}{2j\Omega}\left[\frac{h(\zeta)-h(0)}{\zeta}\right]+\frac{1}{(2j\Omega)^2}\left[\frac{k(\zeta)-k(0)}{\zeta}\right]\right\}+O(\Omega^{-3}), \end{aligned} \tag{3.16}$$

where

$$k(s)=\frac{sh'(s)-h(s)+h(0)}{s^2}, \tag{3.17}$$

and finally combining Equations (3.15) and (3.16), the asymptotic expansion of $I_0(\sigma,\gamma;\Omega)$ when $\sigma \ll -1$ is given by:

$$\begin{aligned} I_0(\sigma,\gamma;\Omega) &\sim e^{-j\frac{2}{3}(-\sigma)^{3/2}}\left[f(0)-\frac{1}{2j\Omega}g(0)+\frac{1}{(2j\Omega)^2}h(0)-\frac{1}{(2j\Omega)^3}k(0)\right]\int_\zeta^\infty e^{j\Omega s^2}\,ds \\ &+e^{j\Omega(\sigma\gamma+\gamma^3/3)}\left\{-\frac{1}{2j\Omega}\left[\frac{f(\zeta)-f(0)}{\zeta}\right]+\frac{1}{(2j\Omega)^2}\left[\frac{g(\zeta)-g(0)}{\zeta}\right]\right. \\ &\left.-\frac{1}{(2j\Omega)^3}\left[\frac{h(\zeta)-h(0)}{\zeta}\right]+\frac{1}{(2j\Omega)^4}\left[\frac{k(\zeta)-k(0)}{\zeta}\right]\right\}+O(\Omega^{-5}). \end{aligned} \tag{3.18}$$

In order to express the functions f, g, h, and k in Equation (3.18) in terms of σ and γ, we need to derive an expression for $f(s)$ and its derivatives when $s \approx 0$. This is done using a procedure introduced by Erdélyi [9] that yields:

$$f(s) = \frac{1}{(-\sigma)^{1/4}}\sum_{n=1}^{\infty}\frac{\Gamma(3n/2-1)(-1)^{n-1}}{(n-1)!\Gamma(n/2)[3(-\sigma)^{3/4}]^{n-1}}s^{n-1}, \tag{3.19}$$

and

$$\frac{d^k}{ds^k}f(s) = \frac{1}{(-\sigma)^{1/4}}\sum_{n=k+1}^{\infty}\frac{\Gamma(3n/2-1)(-1)^{n-1}}{(n-k-1)!\Gamma(n/2)[3(-\sigma)^{3/4}]^{n-1}}s^{n-k-1} \tag{3.20}$$

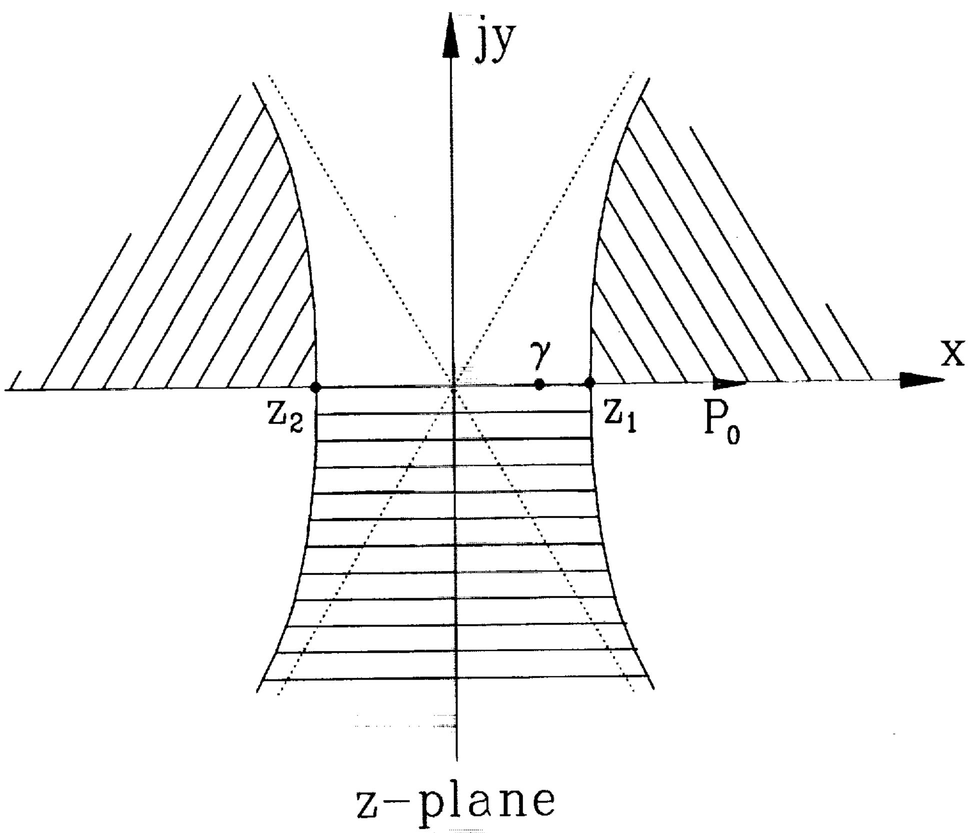

Figure 3.2: Contour of integration for the incomplete Airy integral, $\sigma \ll -1$ and $\gamma \approx (-\sigma)^{1/2}$.

Hence, using Equations (3.9), (3.12), (3.14), (3.17), (3.19), and (3.20) we have:

$$f(\zeta) = \frac{2\zeta}{\sigma+\gamma^2}, \tag{3.21}$$

$$g(\zeta) = \frac{-8\gamma\zeta}{(\sigma+\gamma^2)^3} + \frac{1}{\zeta^2(-\sigma)^{1/4}}, \tag{3.22}$$

$$h(\zeta) = \frac{-16\zeta}{(\sigma+\gamma^2)^4} + \frac{96\gamma^2}{(\sigma+\gamma^2)^5} - \frac{3}{\zeta^4(-\sigma)^{1/4}} + \frac{5}{24\zeta^2(-\sigma)^{7/4}}, \tag{3.23}$$

$$k(\zeta) = \frac{640\gamma}{(\sigma+\gamma^2)^6} - \frac{1920\gamma^3}{(\sigma+\gamma^2)^7} + \frac{15}{\zeta^6(-\sigma)^{1/4}} - \frac{5}{8\zeta^4(-\sigma)^{7/4}} + \frac{77}{3456\zeta^3(-\sigma)^{13/4}}, \tag{3.24}$$

$$f(0) = \frac{1}{(-\sigma)^{1/4}}, \tag{3.25}$$

$$f'(0) = \frac{-1}{3(-\sigma)}, \tag{3.26}$$

$$g(0) = \frac{f''(0)}{2} = \frac{5}{24(-\sigma)^{7/4}}, \tag{3.27}$$

$$g'(0) = \frac{f'''(0)}{3} = \frac{-8}{27(-\sigma)^{5/2}}, \tag{3.28}$$

$$h(0) = \frac{f^{(4)}(0)}{8} = \frac{77}{3456(-\sigma)^{13/4}}, \tag{3.29}$$

$$h'(0) = \frac{f^{(5)}(0)}{15} = \frac{-56}{81(-\sigma)^4}, \tag{3.30}$$

$$k(0) = \frac{f^{(6)}(0)}{48} = \frac{12155}{82944(-\sigma)^{19/4}}, \quad \text{and} \tag{3.31}$$

$$k'(0) = \frac{f^{(7)}(0)}{105} = \frac{-640}{243(-\sigma)^{11/2}}. \tag{3.32}$$

Also, the Fresnel integral in Equation (3.18) may be expressed in terms of the Fresnel transition function $F(x)$ [3]; i.e.,

$$\int_\zeta^\infty e^{j\Omega s^2}\,ds = \Omega^{1/2}\left[\sqrt{\pi}\,e^{j\pi/4}\,U(-\eta) - \frac{F^*(\eta^2)}{2j\eta}e^{j\eta^2}\right] = \Omega^{1/2}\Lambda(\eta), \tag{3.33}$$

where $U(x)$ is the Heaveside unit step function, and $\eta = \Omega^{1/2}\zeta$. The general properties of the Fresnel transition function and details on its computation are provided in Appendix C. Now, using Equations (3.21)–(3.33), the asymptotic expansion of

$I_0(\sigma, \gamma; \Omega)$ when $\sigma \ll -1$ is given by:

$$I_0(\sigma,\gamma;\Omega) \sim \frac{e^{-j\frac{2}{3}\Omega(-\sigma)^{3/2}}}{\Omega^{1/3}(-\sigma)^{1/4}} S(\sigma;\Omega)\,\Lambda(\eta) + e^{j\Omega(\sigma\gamma+\gamma^3/3)} E(\sigma,\gamma,\eta;\Omega) + O(\Omega^{-5})\,, \quad (3.34)$$

where

$$S(\sigma;\Omega) = 1 - \frac{s_1}{2j\Omega(-\sigma)^{3/2}} + \frac{s_2}{(2j\Omega)^2(-\sigma)^3} - \frac{s_3}{(2j\Omega)^3(-\sigma)^{9/2}}\,, \quad (3.35)$$

$$\begin{aligned} E(\sigma,\gamma,\eta;\Omega) &= -\frac{1}{2j\Omega}\left[\frac{2}{\sigma+\gamma^2} - \frac{\Omega^{1/2}}{\eta(-\sigma)^{1/4}}\right] \\ &+\frac{1}{(2j\Omega)^2}\left[\frac{-8\gamma}{(\sigma+\gamma^2)^3} + \frac{\Omega^{3/2}}{\eta^3(-\sigma)^{1/4}} - \frac{\Omega^{1/2}s_1}{\eta(-\sigma)^{7/4}}\right] \\ &-\frac{1}{(2j\Omega)^3}\left[\frac{-16}{(\sigma+\gamma^2)^4} + \frac{96\gamma^2}{(\sigma+\gamma^2)^5} - \frac{3\Omega^{5/2}}{\eta^5(-\sigma)^{1/4})} + \frac{\Omega^{3/2}s_1}{\eta^3(-\sigma)^{7/4}} - \frac{\Omega^{1/2}s_2}{\eta(-\sigma)^{13/4}}\right] \\ &+\frac{1}{(2j\Omega)^4}\left[\frac{640\gamma}{(\sigma+\gamma^2)^6} - \frac{1920\gamma^3}{(\sigma+\gamma^2)^7} + \frac{15\Omega^{7/2}}{\eta^7(-\sigma)^{1/4})} - \frac{3\Omega^{5/2}s_1}{\eta^5(-\sigma)^{7/4}} + \frac{\Omega^{3/2}s_2}{\eta^3(-\sigma)^{13/4}}\right. \\ &\left.-\frac{\Omega^{1/2}s_3}{\eta(-\sigma)^{19/4}}\right]\,, \end{aligned} \quad (3.36)$$

$s_1 = 0.20833333$, $s_2 = 0.33420139$, and $s_3 = 1.02581260$.

The asymptotic formulae for $g_0(\beta,\xi)$ and its derivative when $\beta \ll -1$ are then obtained using Equations (3.2) and (3.34)–(3.36); i.e.,

$$\begin{aligned} g_0(\beta,\xi) &\simeq \frac{e^{-j\frac{2}{3}(-\beta)^{3/2}}}{(-\beta)^{1/4}} S(\beta)\,\Lambda(\eta) + e^{j(\beta\xi+\xi^3/3)} E(\beta,\xi,\eta)\,, & (3.37) \\ \frac{\partial}{\partial\beta} g_0(\beta,\xi) &\simeq \frac{e^{-j\frac{2}{3}(-\beta)^{3/2}}}{(-\beta)^{1/4}}\left\{\left[j(-\beta)^{1/2} + \frac{1}{4(-\beta)}\right] S(\beta)\Lambda(\eta) + \Lambda(\eta)\frac{d}{d\beta}S(\beta)\right. \\ &\left. +S(\beta)\frac{d}{d\beta}\Lambda(\eta)\right\} + e^{j(\beta\xi+\xi^3/3)}\left[j\xi E(\beta,\xi,\eta) + \frac{\partial}{\partial\beta}E(\beta,\xi,\eta)\right] & (3.38) \end{aligned}$$

where

$$\begin{aligned} S(\beta) &= 1 - \frac{s_1}{2j(-\beta)^{3/2}} + \frac{s_2}{(2j)^2(-\beta)^3} - \frac{s_3}{(2j)^3(-\beta)^{9/2}}\,, & (3.39) \\ \frac{d}{d\beta}S(\beta) &= -\frac{1.5s_1}{2j(-\beta)^{5/2}} + \frac{3s_2}{(2j)^2(-\beta)^4} - \frac{4.5s_3}{(2j)^3(-\beta)^{11/2}}\,, & (3.40) \end{aligned}$$

$$E(\beta,\xi,\eta) = -\frac{1}{2j}\left[\frac{2}{\beta+\xi^2} - \frac{1}{\eta(-\beta)^{1/4}}\right]$$
$$+\frac{1}{(2j)^2}\left[\frac{-8\xi}{(\beta+\xi^2)^3} + \frac{1}{\eta^3(-\beta)^{1/4}} - \frac{s_1}{\eta(-\beta)^{7/4}}\right]$$
$$-\frac{1}{(2j)^3}\left[\frac{-16}{(\beta+\xi^2)^4} + \frac{96\xi^2}{(\beta+\xi^2)^5} - \frac{3}{\eta^5(-\beta)^{1/4}} + \frac{s_1}{\eta^3(-\beta)^{7/4}} - \frac{s_2}{\eta(-\beta)^{13/4}}\right]$$
$$+\frac{1}{(2j)^4}\left[\frac{640\xi}{(\beta+\xi^2)^6} - \frac{1920\xi^3}{(\beta+\xi^2)^7} + \frac{15}{\eta^7(-\beta)^{1/4}} - \frac{3s_1}{\eta^5(-\beta)^{7/4}} + \frac{s_2}{\eta^3(-\beta)^{13/4}}\right.$$
$$\left.-\frac{s_3}{\eta(-\beta)^{19/4}}\right], \tag{3.41}$$

$$\frac{\partial}{\partial\beta}E(\beta,\xi,\eta) = -\frac{1}{2j}\left[\frac{-2}{(\beta+\xi^2)^2} - \frac{1}{4\eta(-\beta)^{5/4}} + \frac{\xi-(-\beta)^{1/2}}{2\eta^3(-\beta)^{1/4}}\right]$$
$$+\frac{1}{(2j)^2}\left\{\frac{24\xi}{(\beta+\xi^2)^4} + \frac{1}{4\eta^3(-\beta)^{5/4}} - \frac{3[\xi-(-\beta)^{1/2}]}{2\eta^5(-\beta)^{1/4}} - \frac{7s_1}{4\eta(-\beta)^{11/4}}\right.$$
$$\left.+\frac{s_1[\xi-(-\beta)^{1/2}]}{2\eta^3(-\beta)^{7/4}}\right\} - \frac{1}{(2j)^3}\left\{\frac{64}{(\beta+\xi^2)^5} - \frac{480\xi^2}{(\beta+\xi^2)^6} - \frac{3}{4\eta^5(-\beta)^{5/4}}\right.$$
$$+\frac{15[\xi-(-\beta)^{1/2}]}{2\eta^7(-\beta)^{1/4}} + \frac{7s_1}{4\eta^3(-\beta)^{11/4}} - \frac{3s_1[\xi-(-\beta)^{1/2}]}{2\eta^5(-\beta)^{7/4}} - \frac{13s_2}{4\eta(-\beta)^{17/4}}$$
$$\left.+\frac{s_2[\xi-(-\beta)^{1/2}]}{2\eta^3(-\beta)^{13/4}}\right\} + \frac{1}{(2j)^4}\left\{\frac{-3840\xi}{(\beta+\xi^2)^7} + \frac{13440\xi^3}{(\beta+\xi^2)^8} + \frac{15}{4\eta^7(-\beta)^{5/4}}\right.$$
$$-\frac{105[\xi-(-\beta)^{1/2}]}{2\eta^9(-\beta)^{1/4}} - \frac{21s_1}{4\eta^5(-\beta)^{11/4}} + \frac{15s_1[\xi-(-\beta)^{1/2}]}{2\eta^7(-\beta)^{7/4}} + \frac{13s_2}{4\eta^3(-\beta)^{17/4}}$$
$$\left.-\frac{3s_2[\xi-(-\beta)^{1/2}]}{2\eta^5(-\beta)^{13/4}} - \frac{19s_3}{4\eta(-\beta)^{23/4}} + \frac{s_3[\xi-(-\beta)^{1/2}]}{2\eta^3(-\beta)^{19/4}}\right\}, \tag{3.42}$$

$$\frac{d}{d\beta}\Lambda(\eta) = -\frac{\partial\eta}{\partial\beta}e^{j\eta^2} = -\left[\frac{\xi-(-\beta)^{1/2}}{2\eta}\right]e^{j\eta^2}, \tag{3.43}$$

and

$$\eta = \pm\left[\beta\xi + \xi^3/3 + \frac{2}{3}(-\beta)^{3/2}\right]^{1/2}; \quad \xi \gtrless (-\beta)^{1/2}. \tag{3.44}$$

It can be easily shown using the large argument form of the Fresnel transition function (see Appendix C) that when $\xi \gg (-\beta)^{1/2}$ or $\eta \gg 1$, Equation (3.37) appropriately reduces to the first six terms of Equation (3.4). Also, Equations (3.41) and (3.42) remain finite near the caustic when $\xi \to (-\beta)^{1/2}$ and $\eta \to 0$, however, they become numerically unstable and an alternative formulation should be used. Applying L'Hospital's rule in Equation (3.18), and then using Equations (3.2) and (3.19)–(3.32) we have:

$$\begin{aligned} E(\beta,\xi,\eta\approx 0) \;\simeq\; & \frac{f_0}{2j(-\beta)}\left[1-\frac{f_1\eta}{(-\beta)^{3/4}}+\frac{f_2\eta^2}{(-\beta)^{3/2}}\right] \\ & -\frac{g_0}{(2j)^2(-\beta)^{5/2}}\left[1-\frac{g_1\eta}{(-\beta)^{3/4}}+\frac{g_2\eta^2}{(-\beta)^{3/2}}\right] \\ & +\frac{h_0}{(2j)^3(-\beta)^4}\left[1-\frac{h_1\eta}{(-\beta)^{3/4}}+\frac{h_2\eta^2}{(-\beta)^{3/2}}\right] \\ & -\frac{k_0}{(2j)^4(-\beta)^{11/2}}\left[1-\frac{k_1\eta}{(-\beta)^{3/4}}+\frac{k_2\eta^2}{(-\beta)^{3/2}}\right], \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} \frac{\partial}{\partial\beta}E(\beta,\xi,\eta\approx 0)\simeq & \frac{f_0}{2j(-\beta)^2}\left\{1-\frac{7f_1\eta}{4(-\beta)^{3/4}}-\frac{f_1}{2}+\frac{5f_2\eta^2}{2(-\beta)^{3/2}}+\frac{f_2[\xi-(-\beta)^{1/2}]}{(-\beta)^{1/2}}\right\} \\ & -\frac{g_0}{(2j)^2(-\beta)^{7/2}}\left\{1-\frac{13g_1\eta}{4(-\beta)^{3/4}}-\frac{g_1}{2}+\frac{4g_2\eta^2}{(-\beta)^{3/2}}+\frac{g_2[\xi-(-\beta)^{1/2}]}{(-\beta)^{1/2}}\right\} \\ & +\frac{h_0}{(2j)^3(-\beta)^5}\left\{1-\frac{19h_1\eta}{4(-\beta)^{3/4}}-\frac{h_1}{2}+\frac{11h_2\eta^2}{2(-\beta)^{3/2}}+\frac{h_2[\xi-(-\beta)^{1/2}]}{(-\beta)^{1/2}}\right\} \\ & -\frac{k_0}{(2j)^4(-\beta)^{13/2}}\left\{1-\frac{25k_1\eta}{4(-\beta)^{3/4}}-\frac{k_1}{2}+\frac{7k_2\eta^2}{(-\beta)^{3/2}}+\frac{k_2[\xi-(-\beta)^{1/2}]}{(-\beta)^{1/2}}\right\}, \end{aligned} \tag{3.46}$$

where

$f_0 = 1/3,\ f_1 = 1.25,\ f_2 = 4/3,$

$g_0 = 0.29629630,\ g_1 = 3.00781250,\ g_2 = 5.83333333,$

$h_0 = 0.69135802,\ h_1 = 4.74804688,\ h_2 = 13.3333333,$

$k_0 = 2.63374486,\ k_1 = 6.48405151,\ k_2 = 23.8333333.$

Also, $\Lambda(\eta)$ and its derivative near the caustic are given by:

$$\Lambda(\eta\approx 0) \;\simeq\; \frac{\sqrt{\pi}}{2}e^{j\pi/4}-\eta-\frac{j\eta^3}{3}, \tag{3.47}$$

$$\text{and}\quad \frac{d}{d\beta}\Lambda(\eta\approx 0) \;=\; \frac{-e^{j\eta^2}}{2(-\beta)^{1/4}\left[1+\frac{\xi-(-\beta)^{1/2}}{3(-\beta)^{1/2}}\right]^{1/2}}. \tag{3.48}$$

Chapter 4

Numerical Results and Discussion

For an efficient and accurate computation of the incomplete Airy function $g_0(\beta, \xi)$ and its derivative, the argument space is divided into three regions as shown in Figure 4.1. Three different sets of formulae are used, one for each region in Figure 4.1, and a fourth set of formulae that is used in the immediate vicinity of the caustic ($\beta + \xi^2 \approx 0$ or $\eta \approx 0$). In region I, Equations (3.4) and (3.5) are used, in region II, Equations (3.37)–(3.44) are used, and in region III, the series solution is used given by Equations (2.21) and (2.22). In the immediate vicinity of the caustic and specifically when $\eta < 0.1$, Equations (3.37)–(3.40) and (3.44)–(3.48) are used.

An empirical expression for the number of terms used in the series solution is given by:

$$N(\beta, \xi) = 8|\beta| + 4, \quad \text{when } \xi < 2.0, \tag{4.1}$$

$$= 8|\beta| + 4 + 3|\beta|(\xi - 2.0), \quad \text{when } \xi \geq 2.0, \tag{4.2}$$

and results in a truncation error of less than 10^{-6}.

Figure 4.2 shows the percent amplitude error of the asymptotic result relative to the series solution along the boundary between regions I and III. The results are plotted vs. the parameter β, with $\xi = (12 - 2\beta)^{1/2}$. The asymptotic result shows excellent agreement with the series solution, exhibiting a maximum error of 0.12%. Figure 4.3 shows the percent amplitude error of the asymptotic result relative to the series solution along the boundary between regions II and III. The results for this

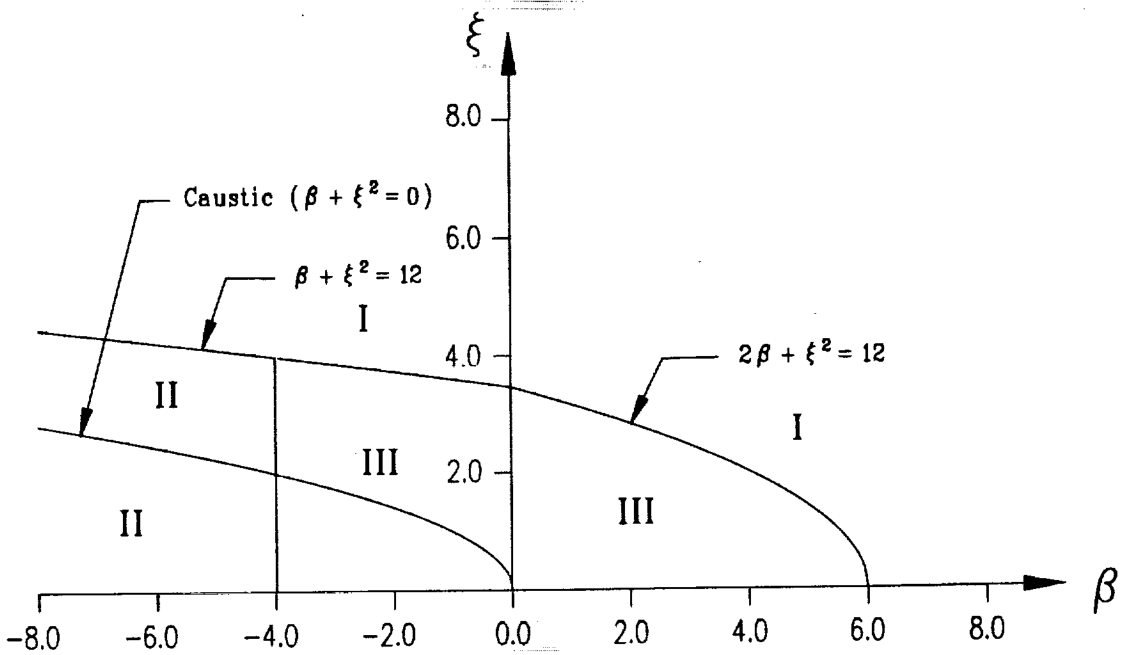

Figure 4.1: Three different sets of formulae are used for the computation of the incomplete Airy function, one for each region in the figure, and a fourth set that is used in the immediate vicinity of the caustic.

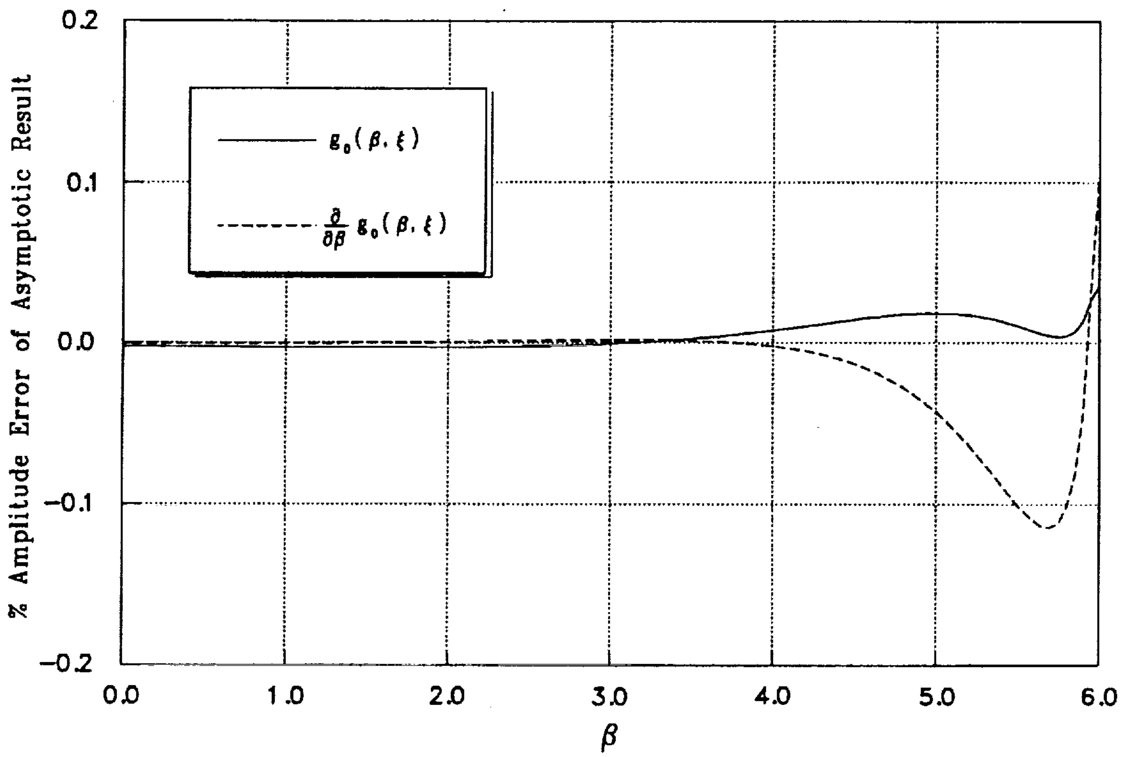

Figure 4.2: Percent amplitude error of the asymptotic result for the incomplete Airy function (solid line) and its derivative (broken line) along the boundary between regions I and III. Results are plotted vs. the parameter β with $\xi = (12 - 2\beta)^{1/2}$.

case are plotted vs. the parameter ξ, with $\beta = -4$. Again the asymptotic result shows excellent agreement with the series solution, exhibiting a maximum error of only 0.075%.

Figures 4.4 and 4.5 show plots of the incomplete Airy function $g_0(\beta, \xi)$ and its derivative vs. the parameter β for $\xi = -3$ and 2, respectively. Figures 4.6 and 4.7 show plots of the incomplete Airy function $g_0(\beta, \xi)$ and its derivative vs. the parameter ξ for $\beta = -5$ and 0, respectively. The marks on the contours represent direct numerical integration data, and show excellent agreement with the results obtained using the formulae derived in this report.

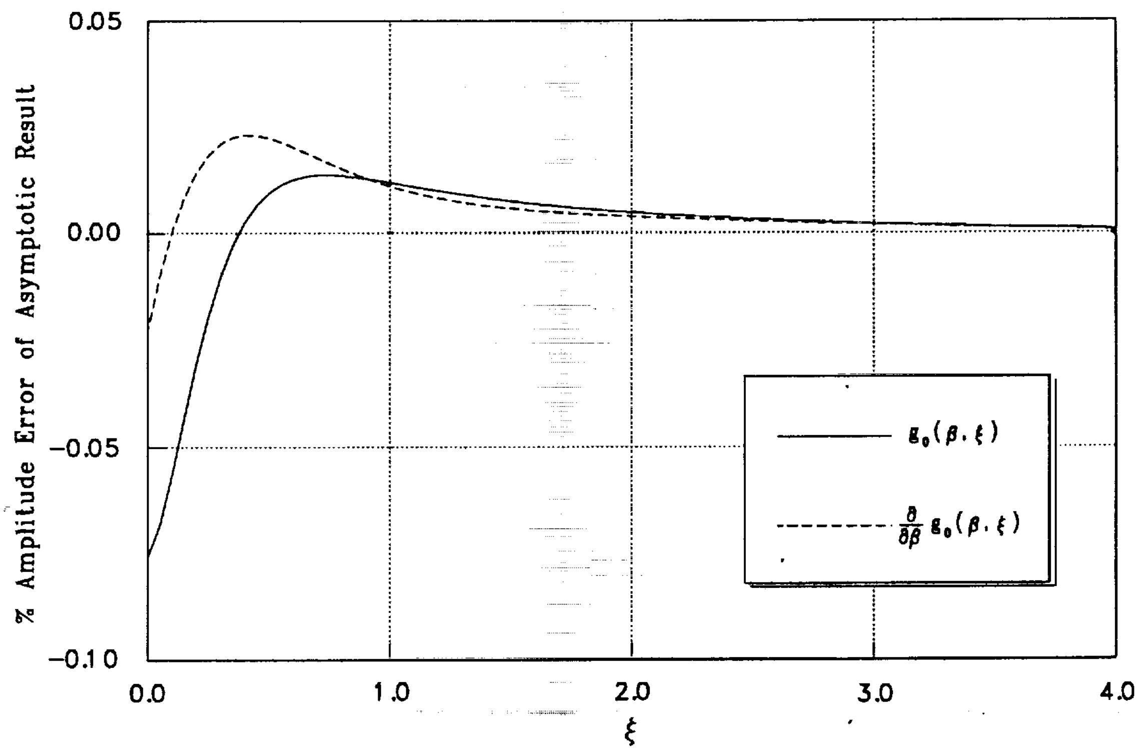

Figure 4.3: Percent amplitude error of the asymptotic result for the incomplete Airy function (solid line) and its derivative (broken line) along the boundary between regions II and III. Results are plotted vs. the parameter ξ with $\beta = -4.0$.

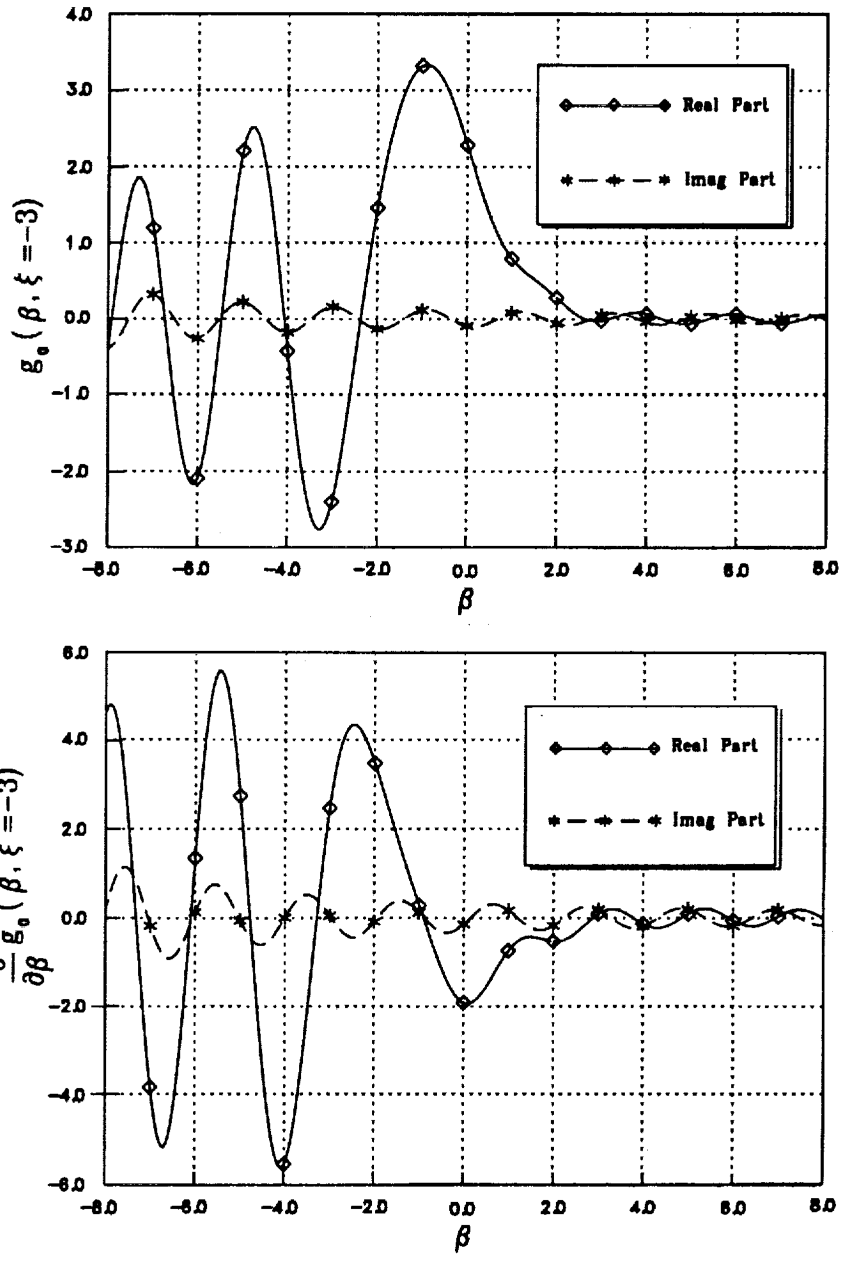

Figure 4.4: Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter β with $\xi = -3.0$. The marks on the contours represent direct numerical integration data.

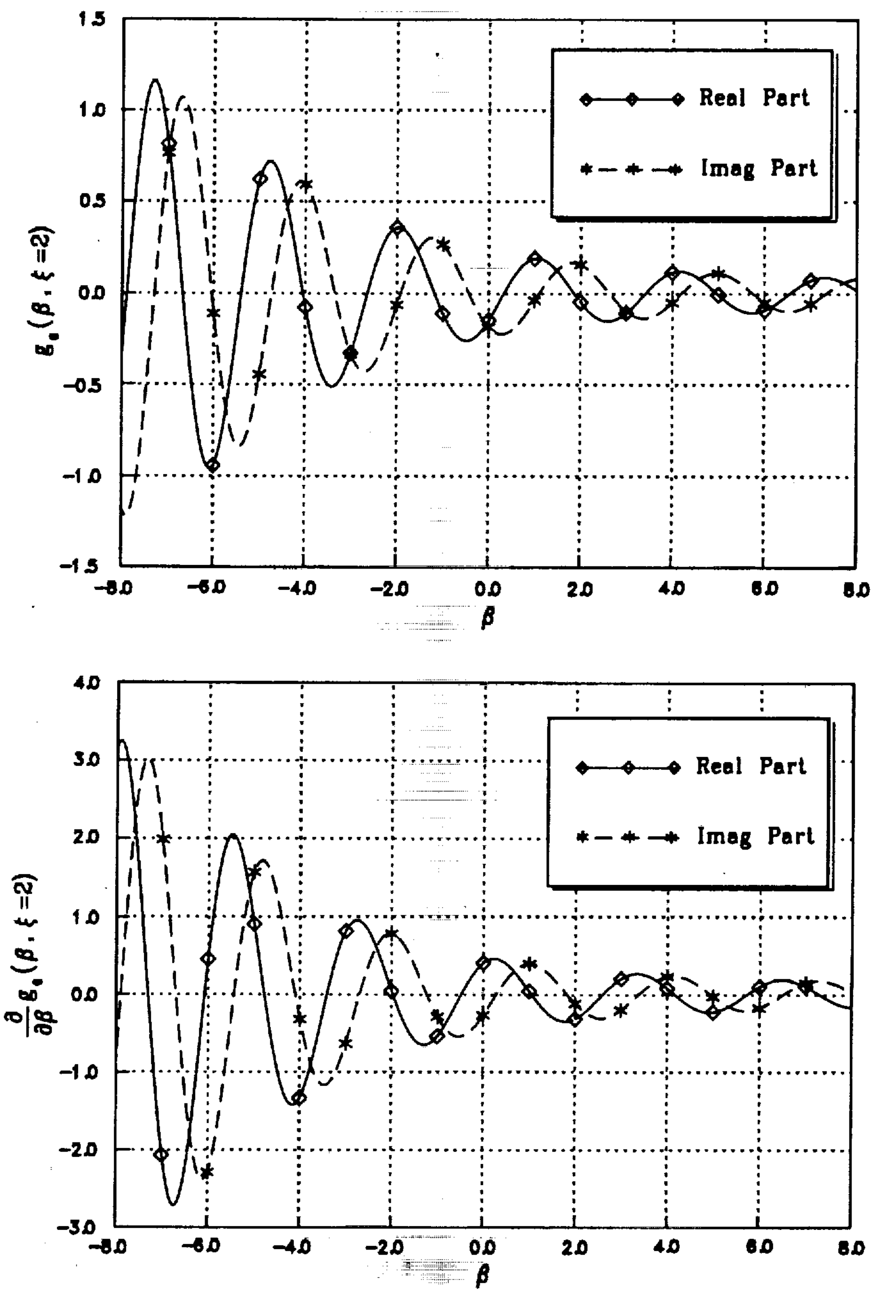

Figure 4.5: Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter β with $\xi = 2.0$. The marks on the contours represent direct numerical integration data.

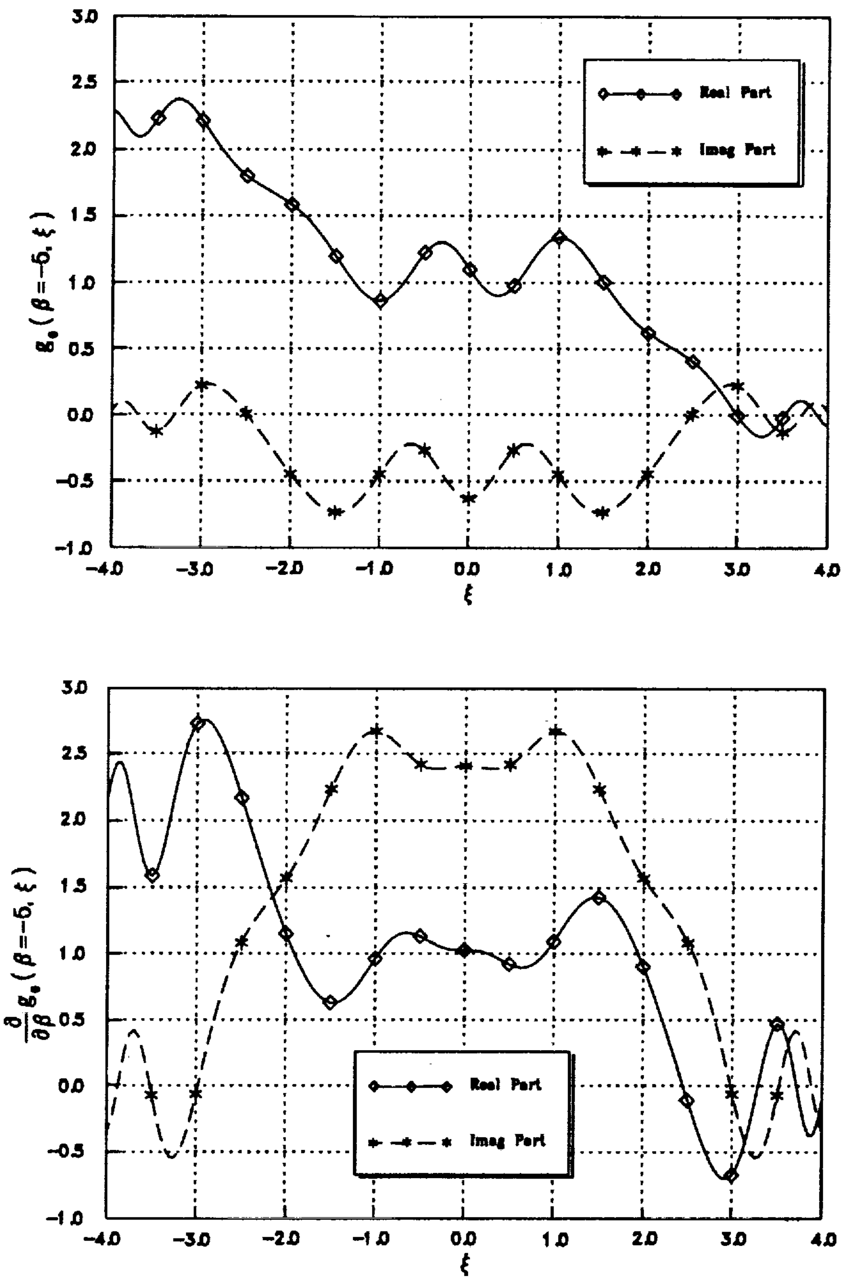

Figure 4.6: Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter ξ with $\beta = -5.0$. The marks on the contours represent direct numerical integration data.

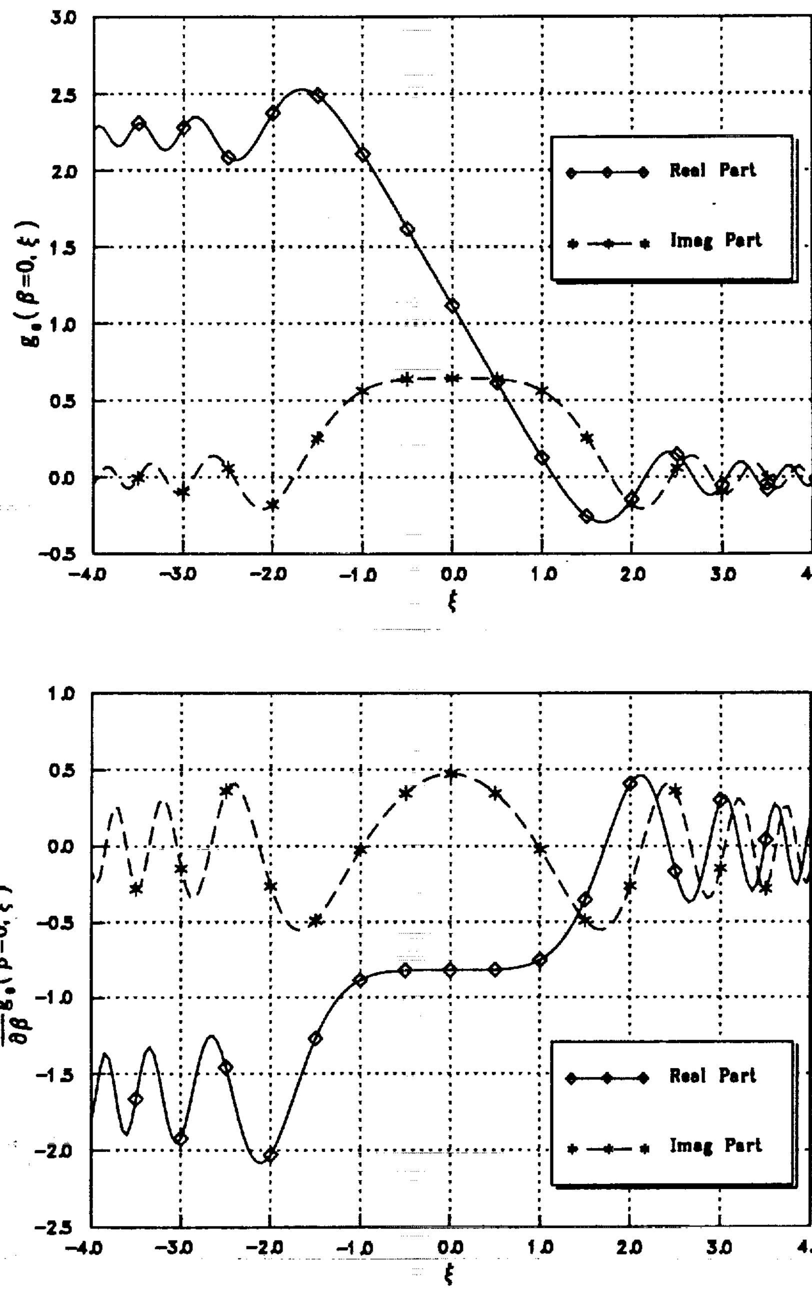

Figure 4.7: Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter ξ with $\beta = 0.0$. The marks on the contours represent direct numerical integration data.

Chapter 5

Summary and Conclusions

In this report, a convergent series solution form for the incomplete Airy functions has been derived, and asymptotic expansions involving several terms have been developed for use in large argument approximations. It has been demonstrated that the combination of the series solution with the asymptotic formulae results in an efficient and accurate means for computing the incomplete Airy functions for the entire range of their arguments.

A necessary requirement for the applicability of uniform asymptotic solutions to practical engineering problems is the efficient and accurate computation of the canonical functions involved. The results of this report would allow the formulation of useful uniform asymptotic solutions for several high-frequency scattering and diffraction problems in which the incomplete Airy integrals serve as canonical functions for the description of high-frequency field behavior in the vicinity of composite shadow boundaries. Furthermore, the methods used in this report may provide useful insight to the computation of other multivariable canonical functions occurring in high-frequency scattering and diffraction theory.

A FORTRAN code for the computation of the incomplete Airy functions based on the formulae derived in this report is available from the authors. A complete code listing appears in Appendix D.

Appendix A

Uniform Asymptotic Evaluation of Certain Stationary Phase Integrals

Let's consider the uniform asymptotic evaluation of a stationary phase integral of the form

$$I(a,b;k) = \int_a^\infty f(s)e^{jk\phi(s,b)}\,ds \tag{A.1}$$

where the phase function $\phi(s,b)$ possesses two stationary phase points $s_{1,2}(b)$ satisfying $\phi'(s_{1,2},b) = 0$ with no restrictions placed on their location relative to the integration endpoint a. The amplitude function $f(s)$ is assumed to be a slowly varying and analytic function of s. The integral in (A.1) may be transformed into a canonical form using the following transformation:

$$\phi(s,b) = \tau(z,\beta) = \alpha + \beta z + z^3/3 \tag{A.2}$$

where

$$\alpha = \tau(0,\beta) = \phi(s_p,b); \quad \phi''(s_p) = 0 \tag{A.3}$$

$$\beta = \phi'(s_p)\left[\frac{2}{\phi'''(s_p)}\right]^{\frac{1}{3}} \tag{A.4}$$

or alternatively

$$\alpha = \tau(z_{1,2},\beta) + \frac{2}{3}(-\beta)^{3/2}, \tag{A.5}$$

$$\beta = -\left\{\frac{3}{2}[\tau(0,\beta) - \tau(z_{1,2},\beta)]\right\}^{2/3}, \qquad (A.6)$$

with $z_{1,2} = \pm(-\beta)^{1/2}$. The proper branch for β depends on the sign of $\phi'(s_p)$ and $\phi'''(s_p)$. Thus, using (A.2) the integral in (A.1) becomes:

$$I(\xi_a,\beta;k) = \int_{\xi_a}^{\text{“}\infty\text{”}} G(z)e^{jk\tau(z,\beta)}\,dz \qquad (A.7)$$

where

$$\xi_a = (a - s_p)\left[\frac{\phi'''(s_p)}{2}\right]^{\frac{1}{3}} = [\phi'(a) - \phi'(s_p)]^{1/2}\left[\frac{2}{\phi'''(s_p)}\right]^{\frac{1}{6}}, \qquad (A.8)$$

$$G(z) = f(s)\frac{dz}{ds}, \quad \text{and} \qquad (A.9)$$

$$\frac{dz}{ds} = \left[\frac{2}{\phi'''(s_p)}\right]^{\frac{1}{3}}. \qquad (A.10)$$

The proper branches for ξ_a and $\frac{dz}{ds}$ depend on the sign of $\phi'''(s_p)$. Next, we employ the Chester et. al. expansion [10] for the amplitude function in (A.7), i.e.,

$$G(z) = \sum_{m=0}^{\infty}[a_m(z^2+\beta)^m + b_m z(z^2+\beta)^m] \qquad (A.11)$$

and since only the leading terms in the asymptotic expansion of (A.7) will be retained, Equation (A.11) may be rewritten as follows:

$$G(z) = a_0 + zb_0 + (z^2+\beta)g(z) \qquad (A.12)$$

where

$$a_0 = \frac{1}{2}[G(z_1 + G(z_2)], \qquad (A.13)$$

$$b_0 = \frac{1}{2z_1}[G(z_1 - G(z_2)], \qquad (A.14)$$

$$g(z) = \sum_{m=1}^{\infty}[a_m(z^2+\beta)^{m-1} + b_m z(z^2+\beta)^{m-1}], \quad \text{and} \qquad (A.15)$$

$$G(z_{1,2}) = f(s_{1,2})\left.\frac{ds}{dz}\right|_{z=z_{1,2}} = f(s_{1,2})\left[\frac{2\beta^{1/2}}{\phi''(s_{1,2})}\right]^{\frac{1}{2}}. \qquad (A.16)$$

Inserting (A.12) into (A.7) yields

$$\begin{aligned} I(\xi_a,\beta;k) &= e^{jk\alpha}\left[a_0\int_{\xi_a}^{\text{“}\infty\text{”}} e^{jk(\beta z+z^3/3)}\,dz + b_0\int_{\xi_a}^{\text{“}\infty\text{”}} z e^{jk(\beta z+z^3/3)}\,dz\right.\\ &\quad \left. -\ \frac{1}{jk}g(\xi_a)e^{jk(\beta\xi_a+\xi_a^3/3)} - \frac{1}{jk}\int_{\xi_a}^{\text{“}\infty\text{”}} g'(z)e^{jk(\beta z+z^3/3)}\,dz\right]. \qquad \text{(A.17)} \end{aligned}$$

The last two terms in the right side of (A.17) are obtained by an integration by parts of the integrand involving $g(z)$. The desired uniform asymptotic approximation for $k \to \infty$ is given by the first three terms of (A.17), and may be expressed in the form

$$\begin{aligned} I(\xi_a,\beta;k) &\sim e^{jk\alpha}\left\{k^{-1/3}a_0\overline{\text{Ai}}(k^{2/3}\beta,k^{1/3}\xi_a) - jk^{-2/3}b_0\frac{\partial}{\partial\beta}\overline{\text{Ai}}(k^{2/3}\beta,k^{1/3}\xi_a)\right.\\ &\quad \left. -\ \frac{1}{jk}\left[\frac{G(\xi_a)-a_0-\xi_a b_0}{\xi_a^2+\beta}\right]e^{jk(\beta\xi_a+\xi_a^3/3)}\right\} \qquad \text{(A.18)} \end{aligned}$$

where $\overline{\text{Ai}}(\sigma,\zeta)$ is the incomplete Airy integral defined by

$$\overline{\text{Ai}}(\sigma,\zeta) \triangleq \int_{\zeta}^{\text{“}\infty\text{”}} e^{j(\sigma z+z^3/3)}\,dz \qquad \text{(A.19)}$$

and the upper limit terminates in one of the three sectors in the complex z-plane where the integral converges. For example, in the case when the upper limit terminates in sector $0 < \arg z < \pi/3$ we have that $\overline{\text{Ai}}(\sigma,\zeta) \equiv g_0(\sigma,\zeta)$.

Appendix B

Computation of $a_0(\xi)$, $b_1(\xi)$ and their Derivatives

The functions $a_0(\xi)$ and $b_1(\xi)$ needed in the series solution form of the incomplete Airy functions can be expressed in terms of the incomplete Gamma function; i.e.,

$$a_0(\xi) = e^{j\pi/6}\, 3^{-2/3}\, \Gamma(1/3, -j\xi^3/3)\,, \tag{B.1}$$

$$b_1(\xi) = -e^{-j\pi/6}\, 3^{-1/3}\, \Gamma(2/3, -j\xi^3/3)\,, \tag{B.2}$$

where

$$\Gamma(x,y) = \int_y^\infty t^{x-1}\, e^{-t}\, dt\,, \quad \Re(x) > 0\,. \tag{B.3}$$

Using the series solution form of the incomplete Gamma function [7], $a_0(\xi)$ and $b_1(\xi)$ are computed using the expressions:

$$a_0(\xi) \simeq e^{j\pi/6}\, 3^{-2/3}\, \Gamma(1/3) - \xi \sum_{n=0}^{N(\xi)} \frac{(j\xi^3/3)^n}{(3n+1)n!}\,, \tag{B.4}$$

$$\text{and} \quad b_1(\xi) \simeq -e^{-j\pi/6}\, 3^{-1/3}\, \Gamma(2/3) - j\xi^2 \sum_{n=0}^{N(\xi)} \frac{(j\xi^3/3)^n}{(3n+2)n!}\,. \tag{B.5}$$

The number of terms in the series, $N(\xi)$, is given by the empirical formula:

$$N(\xi) = 2 + 4\xi^2\,, \tag{B.6}$$

and results in a truncation error of less than 10^{-6}.

The derivatives of $a_0(\xi)$ and $b_1(\xi)$ are given by:

$$
\begin{aligned}
a_0^{(k)}(\xi) &= j d_k a_0^{(k-3)}(\xi) + e_k a_0^{(k-2)}(\xi) + j\xi^2 a_0^{(k-1)}(\xi)\,, \;\; n \geq 3\,, &\text{(B.7)}\\
b_1^{(k)}(\xi) &= j(k-1) a_0^{(k-1)}(\xi) + j\xi a_0^{(k)}(\xi)\,, \;\; n \geq 1\,, &\text{(B.8)}
\end{aligned}
$$

where

$$
\begin{aligned}
a_0^{(1)}(\xi) &= -e^{j\xi^3/3}\,, &\text{(B.9)}\\
\text{and} \;\; a_0^{(2)}(\xi) &= j\xi^2 a_0^{(1)}(\xi)\,. &\text{(B.10)}
\end{aligned}
$$

The constants d_k and e_k are obtained using the recursive relationships:

$$
\begin{aligned}
d_k &= d_{k-1} + e_{k-1}\,, &\text{(B.11)}\\
\text{and} \;\; e_k &= e_{k-1} + 2\,, \;\; k \geq 3\,, &\text{(B.12)}
\end{aligned}
$$

with $d_2 = e_2 = 0$.

Appendix C

Computation of the Fresnel Transition Function

The Fresnel transition function [3] is defined as follows:

$$F(x) \triangleq 2j\sqrt{x}\, e^{jx} \int_{\sqrt{x}}^{\infty} e^{-j\tau^2}\, d\tau\,, \tag{C.1}$$

where $\sqrt{x} = |\sqrt{x}|$ if $x \geq 0$, and $\sqrt{x} = j|\sqrt{x}|$ if $x < 0$. Also,

$$F(-|x|) = F^*(|x|)\,. \tag{C.2}$$

When $x < 6.0$, $F(x)$ is computed using its series solution form; i.e.,

$$F(x) \simeq \sqrt{\pi x}\, e^{j(x+\pi/4)}\, \mathrm{sign}(x) - 2jxe^{jx} \sum_{n=0}^{N(x)} \frac{(-jx)^n}{(2n+1)n!}\,. \tag{C.3}$$

The number of terms in the series, $N(x)$, is given by the empirical formula:

$$N(x) = 10\sqrt{x}\,, \tag{C.4}$$

and results in a truncation error of less than 10^{-6}. The large argument form of the Fresnel transition function ($x \geq 6.0$) is given by:

$$F(x) \sim \sum_{m=0}^{5} \frac{(-1)^m 2^m \left(\frac{1}{2}\right)_m}{(2jx)^m}\,, \tag{C.5}$$

where

$$\left(\alpha + \frac{1}{2}\right)_0 = 1, \text{ and } 2^m \left(\alpha + \frac{1}{2}\right)_m = (2\alpha+1)(2\alpha+3)\cdots(2\alpha+2k-1)\,. \tag{C.6}$$

Appendix D

Code Listing

```
C-----------------------------------------------------------------------
      OPTIONS/EXTEND_SOURCE
      SUBROUTINE INCAIRY(B,X,G0,G0P,G1,G1P,G2,G2P)
      IMPLICIT NONE
C!!!
C!!!  This subroutine computes the incomplete Airy functions and their
C!!!  first derivatives with respect to the parameter B. Both B and X
C!!!  are considered real. Double precision arithmetic is used.
C!!!
C!!!  G0  (B,X) = INT(X,INFTY,FS)
C!!!  G0P (B,X) = INT(X,INFTY,CJ*S*FS)
C!!!
C!!!  where  FS = EXP[CJ*(B*S+S**3/3)]
C!!!
C!!!  G1(B,X) = G0(B,X)-TPI*AI(B)
C!!!  G2(B,X) = G0(B,X)-PI*[AI(B)+CJ*BI(B)]
C!!!
C!!!  AI(B) and BI(B) are the complete Airy functions.
C!!!
C!!!  Version 1.1 4-12-1992
C!!!
C!!!  Author: E. D. Constantinides
C!!!          The Ohio State University
C!!!          ElectroScience Laboratory
C!!!          1320 Kinnear Road
C!!!          Columbus, OH 43212
C!!!
      REAL*8      B,X,XP,X2,BXX,PI/3.141592653589793/,
     &            TPI/6.283185307179586/
```

```
      COMPLEX*16 G0,G0P,G1,G1P,G2,G2P,CJ/(0.0D0,1.0D0)/
      COMPLEX    BZ,AI,AIP,BI,BIP
      EXTERNAL   INC_AIRY_SS,INC_AIRY_FF,INC_AIRY_IP,AIBI
      COMMON     CJ,PI,TPI
C!!!
      XP=DABS(X)
      X2=X*X
      BXX=B+X2
      IF (B.GT.0.0D0) BXX=2.0D0*B+X2
      IF (BXX.GT.12.0D0) THEN
        CALL INC_AIRY_IP(B,XP,G0,G0P)
      ELSEIF (B.LT.-4.0D0) THEN
        CALL INC_AIRY_FF(B,XP,G0,G0P)
      ELSE
        CALL INC_AIRY_SS(B,XP,G0,G0P)
      ENDIF
      BZ=CMPLX(B)
      CALL AIBI(BZ,AI,AIP,BI,BIP)
      IF (X.LT.0.0D0) THEN
        G0=TPI*AI-DCONJG(G0)
        G0P=TPI*AIP-DCONJG(G0P)
      ENDIF
      G1=G0-TPI*AI
      G1P=G0P-TPI*AIP
      G2=G0-PI*(AI+CJ*BI)
      G2P=G0P-PI*(AIP+CJ*BIP)
C!!!
      END
C--------------------------------------------------------------------
      OPTIONS/EXTEND_SOURCE
      SUBROUTINE INC_AIRY_IP(BS,XS,G0,G0P)
      IMPLICIT NONE
C!!!
C!!!  This subroutine computes the incomplete airy function
C!!!  using integration by parts when BS+XS**2>>1.0
C!!!
      REAL*8     BS,XS,BXS,BXS1,BXS2,BXS3,BXS4,BXS5,BXS6,BXS7,
     &           BXS8,BXS9,BXS10,BXS11,PI,TPI
      COMPLEX*16 CJ,CT,G0,G0P
      COMMON     CJ,PI,TPI
C!!!
      BXS=BS+XS*XS
```

```
      BXS1=1.0D0/BXS
      BXS2=BXS1*BXS1
      BXS3=BXS2*BXS1
      BXS4=BXS3*BXS1
      BXS5=BXS4*BXS1
      BXS6=BXS5*BXS1
      BXS7=BXS6*BXS1
      BXS8=BXS7*BXS1
      BXS9=BXS8*BXS1
      BXS10=BXS9*BXS1
      BXS11=BXS10*BXS1
      CT=CDEXP(CJ*(BS*XS+XS*XS*XS/3.0D0))
      IF (XS.EQ.0.0D0) GOTO 10
      G0=CJ*BXS1+2.0D0*XS*BXS3+2.0D0*CJ*BXS4-12.0D0*CJ*XS*XS*BXS5
     &    +40.0D0*XS*BXS6+40.0D0*CJ*BXS7-120.0D0*XS*XS*XS*BXS7
     &    -840.0D0*CJ*XS*XS*BXS8+1680.0D0*CJ*XS*XS*XS*XS*BXS9
     &    +2240.0D0*XS*BXS9
      G0=CT*G0
      G0P=-CJ*BXS2-6.0D0*XS*BXS4-8.0D0*CJ*BXS5+60.0D0*CJ*XS*XS*BXS6
     &     -240.0D0*XS*BXS7-280.0D0*CJ*BXS8+840.0D0*XS*XS*XS*BXS8
     &     +6720.0D0*CJ*XS*XS*BXS9-15120.0D0*CJ*XS*XS*XS*XS*BXS10
     &     -20160.0D0*XS*BXS10
      G0P=CJ*XS*G0+CT*G0P
      RETURN
C!!!
10    G0=CJ*BXS1+2.0D0*CJ*BXS4+40.0D0*CJ*BXS7
      G0P=-CJ*BXS2-8.0D0*CJ*BXS5-280.0D0*CJ*BXS8-22400.0D0*CJ*BXS11
      RETURN
C!!!
      END
C------------------------------------------------------------------
      OPTIONS/EXTEND_SOURCE
      SUBROUTINE INC_AIRY_FF(BS,XS,G0,G0P)
      IMPLICIT NONE
C!!!
C!!!  This subroutine computes the incomplete airy function
C!!!  using the Fresnel integral when BS<<0.0
C!!!
      REAL*8      BS,XS,BSP,Z,ZZ,BXS,BX,PI,TPI,B1,B2,B3,
     &            G1,G2,G3,Z3,Z5,Z7,Z9,BXS1,BXS2,BXS3,BXS4,BXS5,BXS6,
     &            BXS7,BXS8,BS14,BX1,BSQP,BSS,F1,F2,F3,F4,F11,F12,F21,
     &            F22,F31,F32,F41,F42,DZXS,DZBS
```

```
      COMPLEX*16 CJ,CT,G0,GOP,FCTZ,CZZ,G01,G02,G03,
     &           EPC,EPCP,SPC,SPCP,CT1,GZ,FZ,HZ,KZ,FFCT,GPZ,FPZ,HPZ,
     &           KPZ,FCTZP
      EXTERNAL   FTRANSD
      COMMON     CJ,PI,TPI
C!!!
      DATA       G1,G2,G3   /0.2083333333333333,0.3342013888888889,
     &                       1.025812596450617/
      DATA       F1,F11,F12 /0.3333333333333333,1.25D0,1.333333333333333/
      DATA       F2,F21,F22 /0.2962962962962963,3.0078125D0,5.833333333333333/
      DATA       F3,F31,F32 /0.6913580246913580,4.748046875D0,
     &                       13.33333333333333/
      DATA       F4,F41,F42 /2.633744855967078,6.484051513671875,
     &                       23.83333333333333/
C!!!
      BSP=DABS(BS)
      BSQP=DSQRT(BSP)
      BS14=DSQRT(BSQP)
      BSS=BSP*BSQP
      B1=1.0D0/BSS
      B2=B1/BSS
      B3=B2/BSS
      BX=BS*XS+XS*XS*XS/3.0D0
      BX1=-2.0D0*BSS/3.0D0
      ZZ=DABS(BX+2.0D0*BSS/3.0D0)
      Z=DSQRT(ZZ)
      IF (XS.LT.BSQP) Z=-Z
      Z3=Z*ZZ
      Z5=Z3*ZZ
      Z7=Z5*ZZ
      Z9=Z7*ZZ
      CT=CDEXP(CJ*BX)
      CT1=CDEXP(CJ*BX1)
      G01=0.5D0*CJ*G1*B1
      G02=-0.25D0*G2*B2
      G03=-0.125D0*CJ*G3*B3
      SPC=1.0D0+G01+G02+G03
      SPCP=(1.5D0*G01+3.0D0*G02+4.5D0*G03)/BSP
      BXS=BS+XS*XS
      DZXS=XS-BSQP
      CZZ=CDEXP(CJ*ZZ)
      IF (ABS(BXS).LT.0.1D0) GOTO 10
```

```
C!!!
      BXS1=1.0D0/BXS
      BXS2=BXS1*BXS1
      BXS3=BXS2*BXS1
      BXS4=BXS3*BXS1
      BXS5=BXS4*BXS1
      BXS6=BXS5*BXS1
      BXS7=BXS6*BXS1
      BXS8=BXS7*BXS1
      CALL FTRANSD(ZZ,FFCT)
      FCTZ=0.5D0*CJ*CONJG(FFCT)*CZZ/Z
      FCTZP=-0.5D0*CZZ*DZXS/Z
      IF (Z.LT.0.0D0) FCTZ=FCTZ+CDSQRT(CJ*PI)
      GZ=2.0D0*BXS1-1.0D0/Z/BS14
      GPZ=-2.0D0*BXS2-0.25D0/Z/BSP/BS14+0.5D0*DZXS/Z3/BS14
      FZ=-8.0D0*XS*BXS3+1.0D0/Z3/BS14-G1*B1/Z/BS14
      FPZ=24.0D0*XS*BXS4+0.25D0/Z3/BS14/BSP-1.75D0*G1*B1/Z/BS14/BSP
     &   -1.5D0*DZXS/Z5/BS14+0.5D0*G1*B1*DZXS/Z3/BS14
      HZ=-16.0D0*BXS4+96.0D0*XS*XS*BXS5-3.0D0/Z5/BS14+G1*B1/Z3/BS14
     &   -G2*B2/Z/BS14
      HPZ=64.0D0*BXS5-480.0D0*XS*XS*BXS6-0.75D0/Z5/BS14/BSP+
     &    1.75D0*G1*B1/Z3/BS14/BSP-3.25D0*G2*B2/Z/BS14/BSP+
     &    7.5D0*DZXS/Z7/BS14-1.5D0*DZXS*G1*B1/Z5/BS14+
     &    0.5D0*DZXS*G2*B2/Z3/BS14
      KZ=640.0D0*XS*BXS6-1920.0D0*XS*XS*XS*BXS7+15.0D0/Z7/BS14-
     &   3.0D0*G1*B1/Z5/BS14+G2*B2/Z3/BS14-G3*B3/Z/BS14
      KPZ=-3840.0D0*XS*BXS7+13440.0D0*XS*XS*XS*BXS8+3.75D0/Z7/BS14/BSP
     &    -5.25D0*G1*B1/Z5/BS14/BSP+3.25D0*G2*B2/Z3/BS14/BSP
     &    -4.75*G3*B3/Z/BS14/BSP-52.5D0*DZXS/Z9/BS14
     &    +7.5D0*DZXS*G1*B1/Z7/BS14-1.5D0*DZXS*G2*B2/Z5/BS14
     &    +0.5D0*DZXS*G3*B3/Z3/BS14
      EPC=CT*(0.5D0*CJ*GZ-0.25D0*FZ-0.125D0*CJ*HZ+0.0625D0*KZ)
      EPCP=CT*(0.5D0*CJ*GPZ-0.25D0*FPZ-0.125D0*CJ*HPZ+0.0625D0*KPZ)
      G0=CT1*SPC*FCTZ/BS14+EPC
      G0P=(CJ*BS14+0.25D0/BS14/BSP)*CT1*SPC*FCTZ+CT1*SPCP*FCTZ/BS14
     &    +CT1*SPC*FCTZP/BS14+CJ*XS*EPC+EPCP
      RETURN
C!!!
C!!!  Small argument Form
C!!!
10    FCTZ=0.5D0*CDSQRT(CJ*PI)-Z-CJ*ZZ*Z/3.0D0
      DZBS=0.5D0/BS14/DSQRT(1.0D0+DZXS/3.0D0/BSQP)
```

```
      FCTZP=-DZBS*CZZ
      GZ=1.0D0-F11*Z/BSQP/BS14+F12*ZZ/BSS
      GPZ=-0.75D0*F11*Z/BSQP/BS14/BSP+1.5D0*F12*ZZ/BSS/BSP
     &    -F11*DZBS/BSQP/BS14+2.0D0*Z*F12*DZBS/BSS
      FZ=1.0D0-F21*Z/BSQP/BS14+F22*ZZ/BSS
      FPZ=-0.75D0*F21*Z/BSQP/BS14/BSP+1.5D0*F22*ZZ/BSS/BSP
     &    -F21*DZBS/BSQP/BS14+2.0D0*Z*F22*DZBS/BSS
      HZ=1.0D0-F31*Z/BSQP/BS14+F32*ZZ/BSS
      HPZ=-0.75D0*F31*Z/BSQP/BS14/BSP+1.5D0*F32*ZZ/BSS/BSP
     &    -F31*DZBS/BSQP/BS14+2.0D0*Z*F32*DZBS/BSS
      KZ=1.0D0-F41*Z/BSQP/BS14+F42*ZZ/BSS
      KPZ=-0.75D0*F41*Z/BSQP/BS14/BSP+1.5D0*F42*ZZ/BSS/BSP
     &    -F41*DZBS/BSQP/BS14+2.0D0*Z*F42*DZBS/BSS
      EPC=CT*(-0.5*CJ*F1*GZ/BSP+0.25D0*F2*FZ/BSP/BSS
     &    +0.125D0*F3*CJ*HZ/BSP/BSS/BSS
     &    -0.0625D0*F4*KZ/BSP/BSS/BSS/BSS)
      EPCP=CT*(-0.5*CJ*F1*GZ/BSP/BSP-0.5*CJ*F1*GPZ/BSP
     &    +2.5D0*0.25D0*F2*FZ/BSP/BSS/BSP+0.25D0*F2*FPZ/BSP/BSS
     &    +4.0D0*0.125D0*F3*CJ*HZ/BSP/BSS/BSS/BSP
     &    +0.125D0*F3*CJ*HPZ/BSP/BSS/BSS
     &    -5.5D0*0.0625D0*F4*KZ/BSP/BSS/BSS/BSS/BSP
     &    -0.0625D0*F4*KPZ/BSP/BSS/BSS/BSS)
      G0=CT1*SPC*FCTZ/BS14+EPC
      G0P=(CJ*BS14+0.25D0/BS14/BSP)*CT1*SPC*FCTZ
     &    +CT1*SPCP*FCTZ/BS14+CT1*SPC*FCTZP/BS14+CJ*XS*EPC+EPCP
C!!!
      RETURN
      END
C-------------------------------------------------------------------------
      OPTIONS/EXTEND_SOURCE
      SUBROUTINE FTRANSD(XF,FFCT)
      IMPLICIT NONE
C!!!
C!!!  This routine evaluates the Fresnel Transition function F(X)
C!!!
      REAL*8      X,XF,PI4,PI,TPI
      INTEGER     N,NT
      COMPLEX*16 CJ,A0,AN,FFCT,FFCTS
      COMMON      CJ,PI,TPI
C!!!
      X=DABS(XF)
      IF (X.GT.6.0D0) GOTO 1
```

```
C!!!
C!!!  Small argument (series) form
C!!!
      IF (X.EQ.0.0D0) THEN
        FFCT=(0.0D0,0.0D0)
        RETURN
      ENDIF
      PI4=PI/4.0D0
      FFCT=DSQRT(PI*X)*CDEXP(CJ*(X+PI4))
      A0=(1.0D0,0.0D0)
      FFCTS=A0
      NT=10*DSQRT(X)
      DO N=1,NT
        AN=-CJ*X*A0/DBLE(N)
        FFCTS=FFCTS+AN/DBLE(2*N+1)
        A0=AN
      END DO
      FFCT=FFCT-2.0D0*CJ*X*CDEXP(CJ*X)*FFCTS
      GOTO 20
C!!!
C!!!  Large argument form
C!!!
1     A0=(1.0D0,0.0D0)
      FFCT=A0
      DO N=1,8
        AN=0.5D0*CJ*DBLE(2*N-1)*A0/X
        FFCT=FFCT+AN
        A0=AN
      END DO
20    IF (XF.GE.0.0) RETURN
      FFCT=DCONJG(FFCT)
C!!!
      RETURN
      END
C------------------------------------------------------------------------
      OPTIONS/EXTEND_SOURCE
      SUBROUTINE INC_AIRY_SS(BS,XS,G0,G0P)
      IMPLICIT NONE
C!!!
C!!!  This subroutine computes the Incomplete Airy function using a
C!!!  convergent series solution with error less than 1.0E-6.
C!!!
```

```
      REAL*8      BS,BSP,BS2,BS3,XS,XS2,XS3,CO,C1,PI,TPI
      INTEGER     M,N,MM,NT,MT
      COMPLEX*16  GO,GOP,A(66,33),B(66,33),AOP(33),A1P(33),
     &            CA(66),CB(66),CJ
      EXTERNAL    AOA1XS
      COMMON      CJ,PI,TPI
C!!!
C!!!  Initialize the coefficient values.
C!!!
      BSP=DABS(BS)
      XS2=XS*XS
      XS3=XS2*XS
      CALL AOA1XS(XS,AOP(1),A1P(1))
      AOP(2)=-CDEXP(CJ*XS3/3.0DO)
      AOP(3)=CJ*XS2*AOP(2)
      AOP(4)=2.0DO*CJ*XS*AOP(2)+CJ*XS2*AOP(3)
      A1P(2)=CJ*XS*AOP(2)
      DO N=3,4
        A1P(N)=CJ*DBLE(N-2)*AOP(N-1)+CJ*XS*AOP(N)
      END DO
      BS2=BS*BS
      BS3=BS2*BS
      GO=AOP(1)+A1P(1)*BS+0.5DO*CJ*AOP(2)*BS2
      GOP=A1P(1)+CJ*AOP(2)*BS
      IF (BS.EQ.0.0DO) RETURN
      MT=8*BSP+4
      IF (XS.GT.2.0DO) MT=MT+3*BSP*(XS-2)
      NT=(MT+1)/2
      CO=0.0DO
      C1=2.0DO
      DO M=1,3
        DO N=1,NT
          A(M,N)=(0.0DO,0.0DO)
          B(M,N)=(0.0DO,0.0DO)
        END DO
      END DO
C!!!
      A(1,1)=(1.0DO,0.0DO)
      B(2,1)=(1.0DO,0.0DO)*BS
      A(3,2)=(0.0DO,0.5DO)*BS2
C!!!
C!!!  Compute the series
```

```
C!!!
      DO M=4,MT
        MM=(M+1)/2
        A(M,1)=A(M-3,1)*BS3/DBLE(M-1)/DBLE(M-2)
        B(M,1)=B(M-3,1)*BS3/DBLE(M-1)/DBLE(M-2)
        DO N=2,MM
          A(M,N)=(A(M-3,N)*BS3+CJ*A(M-2,N-1)*BS2)/DBLE(M-1)/DBLE(M-2)
          B(M,N)=(B(M-3,N)*BS3+CJ*B(M-2,N-1)*BS2)/DBLE(M-1)/DBLE(M-2)
        END DO
        DO N=MM+1,NT
          A(M,N)=(0.0D0,0.0D0)
          B(M,N)=(0.0D0,0.0D0)
        END DO
        IF ((MM.NE.(M/2)).AND.(MM.GE.5)) THEN
          C0=C0+C1
          C1=C1+2.0D0
          A0P(MM)=CJ*XS2*A0P(MM-1)+C1*CJ*XS*A0P(MM-2)+C0*CJ*A0P(MM-3)
          A1P(MM)=CJ*DBLE(MM-2)*A0P(MM-1)+CJ*XS*A0P(MM)
        ENDIF
        CA(M)=(0.0D0,0.0D0)
        CB(M)=(0.0D0,0.0D0)
        DO N=1,MM
          CA(M)=CA(M)+A(M,N)*A0P(N)
          CB(M)=CB(M)+B(M,N)*A1P(N)
        END DO
        G0=G0+(CA(M)+CB(M))
        G0P=G0P+DBLE(M-1)*(CA(M)+CB(M))/BS
      END DO
C!!!
      RETURN
      END
C-----------------------------------------------------------------------
      OPTIONS/EXTEND_SOURCE
      SUBROUTINE A0A1XS(XS,A0,A1)
      IMPLICIT NONE
C!!!
      REAL*8      XS,G13/2.6789385347D0/,G23/1.3541179394D0/,PI,TPI,
     &            XS1,XS2,AI0/0.3550280539D0/,AIP0/-0.2588194038D0/
      COMPLEX*16 A0,A1,CJ,X3,B(100),C13,C23
      INTEGER     I,NT
      COMMON      CJ,PI,TPI
C!!!
```

```
      DATA       C13,C23/(0.4163415888278022,0.2403749283845681),
     &                   (-0.6004684775880014,0.3466806371753174)/
C!!!
      XS1=DABS(XS)
      X3=CJ*XS1*XS1*XS1/3.0D0
      XS2=XS1*XS1
C!!!
C!!!  Series Solution
C!!!
      A0=C13*G13
      A1=C23*G23
      IF (XS1.EQ.0.0D0) RETURN
      B(1)=X3
      A0=A0-XS1-0.25D0*XS1*B(1)
      A1=A1-0.5D0*CJ*XS2-0.2*CJ*XS2*B(1)
      NT=2+4*XS2
      DO I=2,NT
       B(I)=B(I-1)*X3/DBLE(I)
       A0=A0-XS1*B(I)/DBLE(3*I+1)
       A1=A1-CJ*XS2*B(I)/DBLE(3*I+2)
      END DO
C!!!
10    IF (XS.LT.0.0D0) THEN
        A0=2.0D0*PI*AI0-DCONJG(A0)
        A1=2.0D0*PI*AIP0-DCONJG(A1)
      ENDIF
C!!!
      RETURN
      END
C-----------------------------------------------------------------------
      SUBROUTINE AIBI(Z,AI,AIP,BI,BIP)
C!!!
C!!!  This routine calculates the Airy functions AI(Z),BI(Z),
C!!!  and their derivatives AIP(Z),BIP(Z).
C!!!  Ref. Abramowitz and Stegun, Handbook of Mathematical Functions.
C!!!  For CABS(Z) .LE. 6.0 ,a Taylor Series is used.
C!!!  ARG(Z) may take any value. See eqs. (10.4.2) to (10.4.5) .
C!!!
      COMPLEX Z,AI,AIP,BI,BIP
C!!!
      IF(CABS(Z).GT.6.)GO TO 12
      CALL AIBI1(Z,AI,AIP,BI,BIP)
```

```
      RETURN
12    CALL AIBI2(Z,AI,AIP,BI,BIP)
C!!!
      RETURN
      END
C-------------------------------------------------------------------
      SUBROUTINE AIBI1(Z,AI,AIP,BI,BIP)
C!!!
      COMPLEX    Z,AI,AIP,BI,BIP
      COMPLEX*16 F,G,FP,GP
      DOUBLE PRECISION CC1,CC2
C!!!
      DATA S3,CC1,CC2/1.732050808,.355028053887817,.258819403792807 /
C!!!
      CALL FZGZ(Z,F,G,FP,GP)
      AI=CC1*F-CC2*G
      AIP=CC1*FP-CC2*GP
      BI=S3*(CC1*F+CC2*G)
      BIP=S3*(CC1*FP+CC2*GP)
C!!!
      RETURN
      END
C-------------------------------------------------------------------
      SUBROUTINE FZGZ(Z,F,G,FP,GP)
C!!!
C!!!  The auxiliary functions F(Z),G(Z),FP(Z),GP(Z) are computed as in
C!!!  "Tables of the Modified Hankel functions of order one-third and
C!!!  of their derivatives", Computation Lab, Harvard Univ. Press,1945.
C!!!
      COMPLEX*16 F,G,FP,GP,Z3,Z3M,ZD
      COMPLEX    Z
      REAL*8     AM,BM,CM,DM,AO,BO,CO,DO
      REAL       ZMBD(5)
      INTEGER    MAX(5)
C!!!
      DATA       ZMBD /6.1, 5.6, 4.8, 4.1, 3.2 /
      DATA       MAX /22, 19, 16, 14, 11 /
C!!!
      ZD=0.DO
      ZD=Z
      AO=1.DO
      BO=1.DO
```

```
      C0=0.5D0
      D0=1.D0
      Z3=(ZD**3)/200
      Z3M=Z3
      ZMAG=CABS(Z)
      DO 3 M=1,5
3     IF(ZMAG .LE. ZMBD(M))MADMAX=MAX(M)
      F=A0
      G=B0
      FP=C0
      GP=D0
      DO 10 M=1,MADMAX
      TM=FLOAT(3*M)
      AM=200.D0*A0/TM/(TM-1)
      BM=200.D0*B0/TM/(TM+1)
      CM=200.D0*C0/TM/(TM+2)
      DM=200.D0*D0/TM/(TM-2)
      F=F+AM*Z3M
      G=G+BM*Z3M
      FP=FP+CM*Z3M
      GP=GP+DM*Z3M
          Z3M=Z3M*Z3
      A0=AM
      B0=BM
      C0=CM
      D0=DM
10    CONTINUE
      G=ZD*G
      FP=(ZD**2)*FP
C!!!
      RETURN
      END
C-----------------------------------------------------------------------
      SUBROUTINE AIBI2(XX,AI,AIP,BI,BIP)
C!!!
C!!!  This Routine calculates the Airy functions AI(XX),BI(XX),
C!!!  and their derivatives AIP(XX),BIP(XX).
C!!!  Ref. Abramowitz and Stegun, Handbook of Mathematical Functions.
C!!!
      COMPLEX Z,AI,AIP,BI,BIP,XX
      COMPLEX Z25,ZTB,ZT,ZT2,ZT3,ZT4,ZT5
      COMPLEX CT1,A2L2,EIPI3,EIPI6,C,S
```

```
C!!!
      DATA    RTPI,TWORPI,RTOP,POF
     %        /1.772453851,3.544907702,.797884561,.785398164 /
      DATA    A2L2,EIPI6,EIPI3
     %        /(0.,.346573590),(.866025404,.5),(.5,.866025404) /
      DATA    C1/.069444444/,C2/.037133487/,C3/.037993059/,
     %        C4/.057649190/,C5/.116099064/
      DATA    D1/-.097222222/,D2/-.043885030/,D3/-.042462830/,
     %        D4/-.062662163/,D5/-.124105896/
C!!!
      ZTB=(2./3.)*XX**1.5
      ARG=ATAN2(AIMAG(XX),REAL(XX))
      IF(ABS(ARG).GE.2.1) GO TO 100
C!!!
C!!! EQN. (10.4.59),(10.4.61)
C!!!
      Z25=XX**.25
      ZT=ZTB
      ZT2=ZT*ZT
      ZT3=ZT2*ZT
      ZT4=ZT2*ZT2
      ZT5=ZT3*ZT2
      CT1=CEXP(-ZT)/TWORPI
      AI=CT1/Z25*(1-C1/ZT+C2/ZT2-C3/ZT3+C4/ZT4-C5/ZT5)
      AIP=-CT1*Z25*(1-D1/ZT+D2/ZT2-D3/ZT3+D4/ZT4-D5/ZT5)
      IF(ARG.LT.0.)GO TO 20
      ZT=(0.,-1.)*ZTB
C!!!
C!!! EQN. (10.4.65),(10.4.68) WITH UPPER SIGNS.
C!!!
      Z=XX/EIPI3
      CT1=ZT+POF-A2L2
      BI=EIPI6
      BIP=1./EIPI6
      GO TO 30
20    ZT=(0.,1.)*ZTB
C!!!
C!!! EQN. (10.4.65),(10.4.68) WITH LOWER SIGNS.
C!!!
      Z=XX*EIPI3
      CT1=ZT+POF+A2L2
      BI=1./EIPI6
```

```
      BIP=EIPI6
 30   S=CSIN(CT1)
      C=CCOS(CT1)
      Z25=Z**.25
      ZT2=ZT*ZT
      ZT3=ZT2*ZT
      ZT4=ZT2*ZT2
      ZT5=ZT3*ZT2
      BI=BI*RTOP/Z25*(S*(1-C2/ZT2+C4/ZT4)-C*(C1/ZT-C3/ZT3+C5/ZT5))
      BIP=BIP*RTOP*Z25*(C*(1-D2/ZT2+D4/ZT4)+S*(D1/ZT-D3/ZT3+D5/ZT5))
      RETURN
100   ZT=(0.,1.)*ZTB
C!!!
C!!!  EQN. (10.4.60),(10.4.62),(10.4.64),(10.4.67)
C!!!
      IF(ARG.LT.0.)ZT=-ZT
      Z=-XX
      Z25=Z**.25
      ZT2=ZT*ZT
      ZT3=ZT2*ZT
      ZT4=ZT2*ZT2
      ZT5=ZT3*ZT2
      CT1=ZT+POF
      S=CSIN(CT1)
      C=CCOS(CT1)
      AI=1./RTPI/Z25*(S*(1-C2/ZT2+C4/ZT4)-C*(C1/ZT-C3/ZT3+C5/ZT5))
      AIP=-Z25/RTPI*(C*(1-D2/ZT2+D4/ZT4)+S*(D1/ZT-D3/ZT3+D5/ZT5))
      BI=1./RTPI/Z25*(C*(1-C2/ZT2+C4/ZT4)+S*(C1/ZT-C3/ZT3+C5/ZT5))
      BIP=Z25/RTPI*(S*(1-D2/ZT2+D4/ZT4)-C*(D1/ZT-D3/ZT3+D5/ZT5))
C!!!
      RETURN
      END
C-----------------------------------------------------------------------
```

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