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THE OPTIMAL FIBER VOLUME FRACTION AND FIBER-MATRIX PROPERTY COMPATIBILITY IN FIBER REINFORCED COMPOSITES

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Abstract

Although the question of minimum or critical fiber volume fraction beyond which a composite can then be strengthened due to addition of fibers has been dealt with by several investigators for both continuous and short fiber composites, a study of maximum or optimal fiber volume fraction at which the composite reaches its highest strength has not been reported yet. The present analysis has investigated this issue for short fiber case based on the well-known shear lag (the elastic stress transfer) theory as the first step. Using the relationships obtained, the minimum spacing between fibers is determined upon which the maximum fiber volume fraction can be calculated, depending on the fiber packing forms within the composites. The effects on the value of this maximum fiber volume fraction due to such factors as fiber and matrix properties, fiber aspect ratio and fiber packing forms are discussed. Furthermore, combined with the previous analysis on the minimum fiber volume fraction, this maximum fiber volume fraction can be used to examine the property compatibility of fiber and matrix in forming a composite. This is deemed to be useful for composite design. Finally some examples are provided to illustrate the results[1–14].

1   INTRODUCTION

Adding fibers to strengthen materials is a technique which has been used since ancient times. It is applied mainly to materials which are much weaker in tension than in compression so that by adding fibers into them the superior tensile property of fibers can be fully utilized and stronger new materials are obtained.

Yet it is understandable that if very few fibers are added to a matrix, the material is weakened rather than strengthened. Therefore there must be a minimum critical fiber volume fraction V_{min}, only exceeding that with which the fiber reinforcing function can be realized. There have been several studies which addressed this problem and derived the specific values of V_{min} for various cases and under different conditions [1,8,11]. On the other hand, however, as the fiber amount in the system is being increased, the tensile strength of the

composite will increase to a certain point where, upon further increasing of fiber amount, the bonding between the fibers and the matrix will start to deteriorate as the fibers become too close to each other. As a result, for a short-fiber composite, the tensile strength of the composite will decline due to the bond failure of the system caused by the excessive fibers. Therefore there will be a maximum value of fiber volume fraction as well, that being the upper limit of fiber amount allowable in the system for reinforcement.

There have been no reported studies on this issue as far as the present author is aware. This may be due mainly to the fact that, in most cases, the fiber amount which can be incorporated into a matrix system is limited by the processing technology [4] so that technically it may be difficult for the fiber volume fraction to reach this maximum allowable value. Hence the maximum fiber volume problem may not be as significant as the minimum one for practical applications. Nevertheless, study on this issue is still desirable partly due to its theoretical significance, and more importantly, because the investigation of this issue as presented in this article cannot only provide the maximum value of fiber volume fraction, but also determines the property compatibility of various fiber and matrix materials for a composite so as to guide the design procedures in achieving the optimum composite strength and full material usage.

The present study deals with this problem based on a shear strength criterion between fibers within the composite. The effects of fiber length and fiber misalignment are also investigated.

2 THE MINIMUM ALLOWABLE SPACING BETWEEN FIBERS IN A SHORT FIBER COMPOSITE

It has been a well known mechanism that when a fiber composite is under a uniaxial tension, the axial displacements in the fiber and in the matrix will be different because of the differences in tensile properties of these two components. As a result, shear strains will be created on all planes parallel to the axes of the fibers. The shear strain and the resulting shear stress are the primary means by which load is transferred to fibers (for a short fiber composite), or distributed between and supported by the two components of composites. It is through this interaction between fibers and matrix that a fiber reinforcing function is realized. There have been several theories trying to explain this fiber-matrix interaction. The first one was entirely based on the elastic mechanism by Cox[3] in 1952, and is now referred to as the shear lag theory, and another similar version was later proposed by Rosen [12]. Since then, a number of new theories were suggested such as the slip theory [7] to account for matrix plasticity at the fiber surface near the fiber ends, applicable to well-bonded reinforced metals, and the theory of frictional sliding [7] to reinforced polymers and ceramics. However for the present study, the model of the elastic stress transfer will be used as the main theoretical basis. It will be shown that, although this theory basically only explains the behavior of composites at low stress, it still provides adequate information in determining the maximum fiber volume fraction for design purposes. Furthermore the analysis will surely be helpful in the attempt to look into the case of the inelastic interaction as well.

Assumptions made in this analysis include :

1. Since the elastic model is used here, conclusions from the present analysis are valid only if the original assumptions associated with this model hold.

2. The composite consists of many short fibers each with constant length l, circular cross-section area A_f of uniform radius r and tensile modulus E_f.

3. All fibers are distributed uniformly along the length of the composite so that the fiber area fractions on all the cross sections of the composite are identical.

4. Both fibers and matrix behave elastically, and the interface transfers the stress between fibers and matrix without yielding or slipping.

5. Fiber ends are all normal so that the shape effect of fiber end on the stress transfer [6] is excluded in this analysis.

6. Furthermore, the fiber-fiber interaction within the composite and the effect of matrix property change as a result of the fiber interfering with dislocation motion in the matrix are also ignored.

We take the mean fiber center to center spacing normal to their length to be $2R$ (see Figure 1). Assume the composite as a whole is subject to a strain ϵ_c which will cause a strain ϵ_f in a fiber. If P is the load in the fiber at a distance x from the fiber end, then according to Cox [3], the distribution of tensile stress in this arbitrary fiber is

$$\sigma = \frac{P}{A_f} = E_f \epsilon_f [1 - \frac{\cosh \beta(\frac{l}{2} - x)}{\cosh \beta \frac{l}{2}}] \tag{1}$$

where

$$\beta = \frac{1}{r} \sqrt{\frac{G_m}{E_f}(\frac{2}{\ln(R/r)})} \tag{2}$$

and G_m is the shear modulus of the matrix. Note that $\sigma = 0$ at $x = 0$, and l.

The maximum stress occurs at the middle of position $x = l/2$ where

$$\sigma_{max} = E_f \epsilon_f [1 - \frac{1}{\cosh \beta \frac{l}{2}}] \tag{3}$$

It can be seen from Equation 3 that, in order to fully make use of the tensile strength of the fibers, i.e. to make $\sigma_{max} = \sigma_{bf}$, the fiber-fiber spacing R is the key factor for given fiber strain, and fiber and matrix properties.

If τ is the shear stress in the direction of the fiber axis, on planes parallel to this axis, then at the fiber surface we have

$$\frac{dP}{dx} = -2\pi r \tau \tag{4}$$

Equation 1 and 4 give the expression for the shear stress distribution

$$\tau = E_f \epsilon_f \sqrt{\frac{G_m}{2 E_f \ln(R/r)}} \frac{\sinh \beta(\frac{l}{2} - x)}{\cosh \beta \frac{l}{2}} \tag{5}$$

The maximum value of τ occurs at the fiber ends, i.e. at $x = 0$ and l,

$$\tau_{max} = E_f \epsilon_f \sqrt{\frac{G_m}{2E_f \ln(R/r)}} \tanh \beta \frac{l}{2} \qquad (6)$$

and it is zero at the middle of the fiber. Both of these stress distributions are shown in Figure 2. The ratio of the maximum value of shear stress to the maximum tensile stress in the fiber is

$$\frac{\tau_{max}}{\sigma_{max}} = \sqrt{\frac{G_m}{2E_f \ln(R/r)}} \coth \beta \frac{l}{4} \qquad (7)$$

This ratio is of great importance as it represents fiber and matrix properties as well as fiber spacing R within the matrix. The validity of equations 1 - 7 has been verified by several experimental studies [6,7].

In reinforcing the composite to its maximum tensile strength, the tensile strength of fibers has to be utilized to the fullest. In other words, a stress equal to the tensile breaking stress of the fibers σ_{bf} must be reached at the middle of a fiber, i.e. $\sigma_{max} = \sigma_{bf}$. So the above equation can be rearranged as

$$\tau_{max} = \sigma_{bf} \sqrt{\frac{G_m}{2E_f \ln(R/r)}} \coth \beta \frac{l}{4} \qquad (8)$$

This equation gives the relationship between fiber spacing R (or the spacing ratio R/r) and the maximum value of shear stress. When the spacing between fibers R decreases, the value of τ_{max} will increase as shown by Figure 3. The minimum spacing R_{min} is thus determined when τ_{max} has been increased to such a large value that it reaches the shear strength of the matrix adjacent to the interface or the shear strength of the fiber/matrix interface, whichever is less, designated as τ_s. Because of the elastic assumption where the matrix can not deform plastically, this will cause either the fiber /matrix interface or the matrix to fail in shear.

Furthermore replacing τ_{max} by τ_s and rearranging Equation 8 give the final relationship between the minimum spacing ratio R_{min}/r and the strength ratio $\frac{\sigma_{bf}}{\tau_s}$, the fiber aspect ratio $\frac{l}{r}$ as well as the modulus ratio $\frac{G_m}{E_f}$

$$\ln(R_{min}/r) = (\frac{\sigma_{bf}}{\tau_s} \coth \beta \frac{l}{4})^2 \frac{G_m}{2E_f} = \frac{\sigma_{bf}}{\tau_s} \coth[\frac{1}{4}\frac{l}{r}\sqrt{\frac{G_m}{E_f}\frac{2}{\ln(R_{min}/r)}}]^2 \frac{G_m}{2E_f} \qquad (9)$$

This is a transcendental equation for R_{min}/r, and its solution can only be calculated numerically.

However if the fiber length is relatively long so that $\coth \beta \frac{l}{4} \rightarrow 1$, we have an explicit relationship between the fiber spacing ratio and the fiber-matrix properties

$$\ln(R_{min}/r) = (\frac{\sigma_f}{\tau_s})^2 \frac{G_m}{2E_f} \qquad (10)$$

or

$$R_{min}/r = e^{(\frac{\sigma_f}{\tau_s})^2 \frac{G_m}{2E_f}} \qquad (11)$$

In this analysis, the effect of stress transfer across the fiber ends is neglected which will cause an extra load on both the fiber and the matrix in this region. However this effect is considered insignificant [11] as long as the fiber aspect ratio $l/r > 10$. Also the influence of stress concentration across the fiber ends, which will lead to a greater shear stress [8] and will affect the slip behavior of the fiber ends, is ignored.

In addition, in the present analysis, fiber and the matrix are assumed to be completely elastic. This is of course an ideal case, and only valid in practice to brittle materials. For some cases where plastic deformation of the matrix does exist, the conclusion drawn from this study will be a conservative one and certain modification may be needed, since the plastic deformation of the matrix will alleviate the shear stress. However a different model of the spacing/stress relationship is desirable for a matrix which is significantly plastic and flows under loading, or for structures where the effect of frictional sliding between fiber and matrix during the stress transfer is not negligible.

3 THE MAXIMUM FIBER VOLUME FRACTION IN COMPOSITES

As indicated above since there is a minimum spacing R_{min}/r between fibers within a composite below which the structure will collapse due to shear failure, correspondingly this minimum spacing will define an upper limit of fiber amount which is allowed to be incorporated into a given matrix. The composite will reach its highest strength at this maximum fiber volume fraction V_{max}, as there will be a maximum amount of fibers in the composite and each is fully utilized. In other words, this maximum fiber volume fraction is also the optimal value for maximizing the composite tensile strength. Obviously the specific value of V_{max} is dependent on the forms of fiber arrangement within a composite as well. Moreover, for the short fiber case where fiber ends don't meet, the maximum fiber volume fraction also varies with the distance between fiber ends. Let us assume this distance between the ends of two fibers is $2\delta_f$ as shown in Figure 4(a).

The following are the two cases most often encountered in a composite [11].

3.1 Hexagonally Packed Fibers

The fiber arrangement of this type is schematically shown in Figure 1-(c). Suppose there are totally N fibers within the composite. According to the definition of fiber volume fraction of a composite, we have

$$V_f = \frac{V_{fiber}}{V_{total}} \tag{12}$$

Considering the hexagonal area enclosed by the dotted line in Figure 1-(c), the maximum fiber volume fraction in this case is

$$V_{fmh} = \frac{3\pi r^2 l}{\frac{3\sqrt{3}}{2}(2R_{min})^2(l + 2\delta_f)} = \frac{\pi}{2\sqrt{3}}(\frac{r}{R_{min}})^2(\frac{1}{1 + 2\delta_f/l}) \tag{13}$$

When all fibers are packed so densely that they are actually contacting each other in full with fiber ends also connected, that is

$$R_{min} = r, \qquad \delta_f = 0 \tag{14}$$

the maximum fiber volume fraction becomes

$$V_{fmh} = \frac{\pi}{2\sqrt{3}} \tag{15}$$

Also for the case when the fiber length is so long as $l >> \delta_f$ that the fiber end effect can be neglected

$$V_{fmh} = \frac{\pi}{2\sqrt{3}}(\frac{r}{R_{min}})^2 \tag{16}$$

Because of this direct relationship between the maximum fiber volume fraction and the minimum fiber spacing, it is equivalent in the later analysis to refer to either of them.

3.2 Square-Packed Fibers

The fiber arrangement in this case is shown in Figure 1-(d) and we will have

$$V_{fms} = \frac{\pi r^2 l}{(2R_{min})^2(l + 2\delta_f)} = \frac{\pi}{4}(\frac{r}{R_{min}})^2(\frac{1}{1 + 2\delta_f/l}) \tag{17}$$

For the longer fiber case we have

$$V_{fms} = \frac{\pi}{4}(\frac{r}{R_{min}})^2 \tag{18}$$

In the extreme case when fibers are closely packed to each other so that $R_{min} = r$, there will be

$$V_{fms} = \frac{\pi}{4} \tag{19}$$

In either of two packing forms, the value of maximum fiber volume fraction monotonically increases as the fiber spacing decreases. The relationship of or difference between the maximum fiber volume fractions of these two packing forms is given by

$$\frac{V_{fms}}{V_{fmh}} = \frac{\sqrt{3}}{2} \tag{20}$$

That is, the maximum possible fiber volume fraction for square-packed fibers is less than that of a hexagonally Packed case. Again because of this direct relationship between the two fiber packing forms, for briefness, only the Square-Packed form is used in the following analysis.

Note that when there is fiber misalignment existing in the composite, the fiber arrangement may not be as regular as the two examples shown here. Consequently the value of the actual maximum fiber volume fraction may be lower than the present results.

4   THE MINIMUM FIBER VOLUME FRACTION IN COMPOSITES

In the next paragraph, ϵ_{bf}, ϵ_{bm} and ϵ_{ym} represent the fiber breaking strain, the matrix breaking strain and the matrix yield strain. Although there may be three cases [11]

1. $\epsilon_{bf} < \epsilon_{ym}$,

2. $\epsilon_{ym} < \epsilon_{bf} < \epsilon_{bm}$,

3. $\epsilon_{bm} < \epsilon_{bf}$.

existing in composite, each of which will lead to different failure behavior of the composite, usually the breaking strain of the fiber ϵ_{bf} is less than the yield strain of the matrix ϵ_{ym} so that only the first case is considered here. The treatments of two other cases, however, are in principle the same.

If the variations of fiber tensile properties are ignored, according to the Law of Mixture, the breaking strength of the composite σ_{bc} is of the contributions from both fibers and the matrix, and can be expressed as [1]

$$\sigma_{bc} = \sigma_{fm}(1 - V_f) + \eta_l \eta_0 \sigma_{bf} V_f, \qquad V_f > V_{min} \tag{21}$$

where σ_{bf} is the breaking strength of the fiber, and σ_{fm} is the stress on the matrix at the breaking tensile strain of the fiber. The factors η_l and η_0 account for the effects of limited fiber length for the short fiber case, and of fiber misalignment, and are often called the length efficiency and fiber orientation efficiency factors. V_{min} is the minimum value of the fiber volume fraction which must be exceeded if the strength of the composite is to be given by the Law of Mixture. The value of V_{min} can be determined analytically, according to Kelly [8],as shown below.

If the amount of fiber added into the composite is very small, it will actually weaken the composite so that the strength of the composite becomes

$$\sigma_{bc} = \sigma_{bm}(1 - V_f) \tag{22}$$

where σ_{bm} is the breaking strength of the matrix.

Inserting this relation into Equation 21 gives:

$$\sigma_{bm}(1 - V_f) = \sigma_{fm}(1 - V_f) + \eta_l \eta_0 \sigma_{bf} V_f \tag{23}$$

The minimum fiber volume fraction can be derived from this equation,i.e.

$$V_{min} = \frac{\sigma_{bm} - \sigma_{fm}}{\eta_l \eta_0 \sigma_{bf} + \sigma_{bm} - \sigma_{fm}} \tag{24}$$

For a continuous fiber composite where all fibers are aligned in the loading direction, there will be

$$\eta_l = 1, \eta_0 = 1$$

and

$$V_{min} = \frac{\sigma_{bm} - \sigma_{fm}}{\sigma_{bf} + \sigma_{bm} - \sigma_{fm}} \tag{25}$$

5 DETERMINATION OF THE FIBER LENGTH EFFICIENCY FACTOR

It has been claimed [9] that in the post-cracking stage the combined efficiency factors due to both length and orientation can not be simply calculated as the product of the length efficiency factor and the orientation efficiency factor because the orientation efficiency factor is also a function of the fiber length in the case of short fibers. For the elastic model of pre-cracking stage as in the present case however, these two factors can be considered to be independent of each other and therefore can be determined separately.

The length efficiency factor, specifying the effect of a definite length of fibers in a short fiber composite, has two expressions, depending on the stress interaction mechanisms [9]. For the inelastic case, the most common version of this fiber length efficiency factor is expressed in terms of critical fiber length [1,8, 9]. For the elastic case, it can be easily determined based on the tensile stress distribution. From Equation 1, the average tensile stress over the length of this short fiber can be calculated as

$$\bar{\sigma}_{fl} = E_f \epsilon_f [1 - \frac{\tanh \beta \frac{l}{2}}{\beta \frac{l}{2}}] \tag{26}$$

While for continuous fibers, there is

$$\bar{\sigma}_{fl} = \sigma_{fl} = E_f \epsilon_f \tag{27}$$

Therefore the fiber length efficiency factor can be defined as

$$\eta_l = 1 - \frac{\tanh \beta \frac{l}{2}}{\beta \frac{l}{2}} \tag{28}$$

When $l \to \infty$, $\eta_l = 1$.

This expression shows that, compared to the continuous fibers, the tensile stress on a short fiber is discounted by a factor η_l due to limited fiber length.

It is easy to prove that when $l \to 0$, $\eta_l = 0$.

6 DETERMINATION OF FIBER ORIENTATION EFFICIENCY FACTOR

In most published studies where the effect of fiber misalignment was considered, fibers were assumed either all aligned in the same direction at a fixed angle with respect to the axis of the composite loading direction[1,8,11], i.e., there is no variation or spread existing in

fiber orientations, or distributed totally in random [3, 9]. Although it is usually desirable to orient the fibers to enhance stiffness and strength properties, in short-fiber composites, it is normally very difficult, if not impossible, to achieve perfect alignment or completely random distribution of short fibers. The orientation distributions of fibers in a composite are determined by the processing conditions. Partial fiber alignment is typical in injection and transfer moulded composites while planar partial random orientation is typical in sheet moulding compounds [2]. Therefore for most cases, variation of fiber orientation distribution has to be included in the study. There have been several reports [2,5,13,14] dealing with fiber orientation spreading. A more explicit form of the expression of this fiber orientation efficiency factor is obtained in the present study.

6.1 Form of Fiber Orientation Density Function

Since it is impractical to deal with fibers of different orientations individually, a statistical approach is usually a better, or the only, alternative. To do this, a known form of the function to describe the fiber orientation probability density is the premise.

Two cases of the fiber orientation distribution are of practical importance [2]. In the case of injection moulded objects, fiber orientation distribution is independent of the base angle ϕ if the direction of flow is along the $x_3(z)$ axis. In sheet moulding compounds it is reasonable to assume that the short fibers all lie within a plane and the problem is reduced to a two-dimensional one. In either case, by properly arranging the coordinate system, the fiber orientation density function can be expressed as

$$\Omega(\theta) = \begin{cases} \frac{1}{\alpha} & 0 \le \theta \le \alpha \\ 0 & \alpha \le \theta < \pi/2 \end{cases}$$

where θ is the polar angle of a fiber with respect to the x_3 axis (the loading direction), and α is the limit of θ.

6.2 Relationship Between Strains of Composite, Matrix and Fiber

Assume the composite as a whole is subject to a strain ϵ_c which will cause strain ϵ_f in the fiber and ϵ_m in the matrix. It has been widely accepted that the elastic stress transfer mechanism is dominant at the pre-cracking stage and therefore the longitudinal displacements of the fiber and matrix interface are considered geometrically compatible. In other words, the matrix strain will be the same as the composite strain before cracking. The fiber strain however is dependent on the fiber orientation with respect to the loading direction.

There are several approaches in finding the relationship between the composite strain and fiber strain, such as the tensor transformation method[10] and the affine deformation model [5]. However a few simple differentiation operations as shown below can also derive the same result.

Let us consider a cylinder of matrix material with height H and radius R_c. Inside the matrix there is a fiber with length l and orientation θ (see Figure 4-(b)). We have a relationship between the three variables

$$l^2 = R_c^2 + H^2 \tag{29}$$

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Differentiating both sides gives

$$2ldl = 2R_c dR_c + 2HdH \tag{30}$$

It can be further expressed as

$$\frac{dl}{l} = \frac{R_c^2}{l^2}\frac{dR_c}{R_c} + \frac{H^2}{l^2}\frac{dH}{H} \tag{31}$$

Bringing

$$\frac{dH}{H} = \epsilon_c \tag{32}$$

and

$$\frac{dR_c}{R_c} = -\nu_m \epsilon_c \tag{33}$$

into it yields

$$\epsilon_f = \frac{dl}{l} = \epsilon_c(\cos^2\theta - \nu_m \sin^2\theta) \tag{34}$$

where ν_m represents the matrix Poisson's ratio. Note that, similar to previous analyses [5, 8], the effect of the fiber Poisson's ratio has been excluded in equation 33. Equation 34 has been found to be consistent with both the experimental data and the results based on other more sophisticated analytical analysis [5]. However, since the change of the fiber orientation during composite deformation is neglected in the above analysis, it is preferable to apply equation 34 to the small strain case.

6.3   The Result of Fiber Orientation Efficiency Factor

Once we have the relationship between fiber strain and the overall composite strain, the average strain on an arbitrary fiber due to its misalignment can then be calculated as

$$\bar{\epsilon}_f = \int_0^\alpha \epsilon_c(\cos^2\theta - \nu_m \sin^2\theta)\Omega(\theta)d\theta \tag{35}$$

Bringing the distribution function into the above equation gives

$$\bar{\epsilon}_f = \frac{\epsilon_c}{4\alpha}[2\alpha(1-\nu_m) + (1+\nu_m)\sin 2\alpha] \tag{36}$$

The overall average tensile stress on this fiber thus becomes

$$\bar{\sigma}_f = \eta_l E_f \bar{\epsilon}_f = \eta_l E_f \frac{\epsilon_c}{4\alpha}[2\alpha(1-\nu_m) + (1+\nu_m)\sin 2\alpha] \tag{37}$$

Furthermore because of fiber misalignment, the contribution of this fiber toward the composite strength will be discounted according to the equation in [9]

$$[\sigma_{fc}] = [T][\sigma_f] \tag{38}$$

where $[T]$ is the transformation matrix

$$[T] = \begin{bmatrix} \cos^2 \alpha & \sin^2 \alpha & -2\sin\alpha\cos\alpha \\ \sin^2 \alpha & \cos^2 \alpha & 2\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & -\sin\alpha\cos\alpha & \cos^2 \alpha - \sin^2 \alpha \end{bmatrix} \qquad (39)$$

and $[\sigma_f]$ and $[\sigma_{fc}]$ are the actual fiber stress tensor and the fiber stress tensor in the orthogonal directions with respect to loading direction. For the present uniaxially loading case, the above equation reduces to

$$\sigma_{fc} = \bar{\sigma}_f \cos^2 \theta - 2\bar{\tau}_f \sin\theta\cos\theta \qquad (40)$$

It can be easily proven from Equation 5 that

$$\bar{\tau}_f = 0 \qquad (41)$$

So we have

$$\sigma_{fc} = \bar{\sigma}_f \cos^2 \theta \qquad (42)$$

The average value of this stress with respect to fiber orientation is

$$\bar{\sigma}_{fc} = \int_0^\alpha \bar{\sigma}_f \cos^2 \theta \, \Omega(\theta) d\theta = \eta_l E_f \epsilon_c \frac{1}{16\alpha^2}[2\alpha(1 - \nu_m) + (1 + \nu_m)\sin 2\alpha](2\alpha + \sin 2\alpha) \qquad (43)$$

The fiber orientation efficiency factor is thus derived as

$$\eta_\theta = \frac{1}{16\alpha^2}[2\alpha(1 - \nu_m) + (1 + \nu_m)\sin 2\alpha](2\alpha + \sin 2\alpha) \qquad (44)$$

It can be proved that when $\alpha \to 0$, $\eta_\theta = 1$. The minimum value of $\eta_\theta = \frac{(1-\nu_m)}{4}$ is achieved when $\alpha \to \pi/2$

7 THE PROPERTY COMPATIBILITY OF FIBER AND MATRIX IN COMPOSITES

Now that we have determined the maximum allowable fiber volume fraction and the minimum necessary fiber volume fraction, we can use these two values to examine the fiber-matrix property compatibility.

Obviously, for composite design with any possible combinations of fiber and matrix, the criterion

$$V_{max} \geq V_{min} \qquad (45)$$

has to be satisfied. As these two values are determined by the properties of the fiber and matrix as well as the spacing between fibers, Equation 45 actually provides the inter-relationships between all these parameters in a composite, and can hence be used to study the fiber-matrix property compatibility and to select proper materials for a composite.

The easier way of using this equation is to study the boundaries enclosed by the maximum and the minimum fiber volume fraction curves for a given property. Some examples will be shown in the next section.

8 CALCULATION AND DISCUSSION

First of all, since we have had all the equations describing the relationships between the composite structural parameters and the fiber and matrix properties, a parametric study becomes possible to show the effects of these properties on a composite structure. The data used for calculation are listed in Table 1. For generality, the ratios of fiber matrix properties are used wherever possible. When the effect of a specific parameter in Table 1 is investigated over the given range, other parameters will take the typical values provided. The results are illustrated in Figures 5 - 12.

Table I. The Fiber Matrix Properties Used for Calculation

Item	Range	Typical
Strength Ratio $\frac{\sigma_{bf}}{\tau_s}$	1.5 - 5.0 [7]	2.4
Modulus Ratio $\frac{G_m}{E_f}$	0.02 - 0.3 [7]	0.03
Fiber Aspect Ratio $\frac{l}{r}$	5 - 200 [assumed]	60
Fiber Breaking Strength σ_{bf}	4 - 20 Gpa [9]	8 Gpa
Matrix Stress Difference $\sigma_{bm} - \sigma_{fm}$	0.2 - 4.0 Gpa [assumed]	2 Gpa
Fiber Orientation Range α	$0 - \frac{\pi}{3}$	$\frac{\pi}{6}$
Matrix Poisson's Ratio ν_m	0.2 - 0.5 [assumed]	0.3
Fiber End spacing Length ratio $\frac{\delta}{l}$	0.0008 - 0.05 [assumed]	0.003

Figure 5 shows the effect of the Strength Ratio $\frac{\sigma_{bf}}{\tau_s}$ on the values of V_{max}. As the strength ratio increases, meaning stronger fibers are used, or a weaker bonding shear strength between the matrix and fibers, V_{max} is decreasing, a greater spacing between fibers is required in order to maintain a stable structure. Note that fiber length does not have significant effect on the result.

The effect of the modulus ratio $\frac{G_m}{E_f}$ on the V_{max} value is illustrated in Figure 6. It is also a monotonically decreasing relationship. This means that a matrix with higher shear modulus or a less tough fiber will result in a smaller V_{max} value, or allow greater spacing between fibers. In other words, fewer fibers will be needed in the structure. Again there is no noticeable difference for different fiber length cases.

Figure 7 and 8 show the relationships between the fiber length efficiency factor η_l and the fiber aspect ratio $\frac{l}{r}$, and between the fiber orientation range α and the fiber orientation efficiency factor η_θ respectively. As shown in the results, increasing the fiber aspect ratio (a longer or thinner fiber) will raise the fiber length efficiency factor, and a wider spread (a greater α value) of fiber orientation will lower the fiber orientation efficiency factor.

Figures 9 and 10 on the other hand indicate the effects of the matrix Poisson's ratio ν_m and the fiber orientation range α on the value of minimum fiber volume fraction V_{min}. When α becomes larger, the value of V_{min} will increase as shown in Figure 10, indicating that the fiber reinforcing function is hampered due to fiber misalignment so that more fibers are needed. A similar trend is found between ν_m and V_{min} in Figure 9 except that the relationship appears to be linear.

The curves in Figures 11 and 12 can be used to test the property compatibility between the fibers and matrix. First of all, Figure 11 shows the effects of the modulus ratio on the values of V_{max} (the same curve as the short fiber case in Figure 6) and V_{min}. Unlike V_{max},

V_{min} decreases very slightly when $\frac{G_m}{E_f}$ is increasing. Based on Equation 45, only those fiber and matrix types whose $\frac{G_m}{E_f}$ values are greater than the critical $\frac{G_m}{E_f}$ value are compatible for being selected to form a properly functioning composite. Likewise in Figure 12, first, the effects of $\frac{l}{r}$ on both volume fraction values V_{max} and V_{min} can be seen, showing different trends but both gradually approaching its own asymptote as $\frac{l}{r}$ increases. On the other hand, as shown in the figure, there is a critical $\frac{l}{r}$ value above which a feasible structure can then be made.

9 CONCLUSIONS

The stress transfer between matrix and fibers in a composite is not only determined by the intrinsic properties of fiber and matrix, but also affected by the geometric parameters of fiber arrangement within the matrix such as the spacing between fibers and the orientation of fibers.Consequently the shear strength of the interface between fibers and the matrix can be used as a criterion to determine the spacing between fibers in a composite.

For a composite made of given fiber and matrix materials, there is an optimal spacing between fibers at which the fiber tensile strength will be fully exploited. Moreover this optimal spacing is also the minimum allowable spacing between fibers below which the structure will start to disintegrate under loading before the fiber tensile failure. This minimum spacing then defines a maximum fiber volume fraction allowable for a composite.

The maximum fiber volume fraction combined with the minimum fiber volume fraction studied previously can be used for composite design. Both volume fractions are found dependent on such parameters as fiber modulus E_f, fiber tensile strength σ_{bf}, fiber aspect ratio $\frac{l}{r}$ and fiber orientation range α, the matrix properties as Poisson's Ratio ν_m, shear modulus G_m, and the bonding shear strength τ_s between fiber and matrix, as shown in this study. Consequently, these two values of fiber volume fraction V_{max} and V_{min} can be applied to define the boundaries in determining the property compatibility of various combinations of fiber and matrix types for a particular application so as to optimize the result of composite design.

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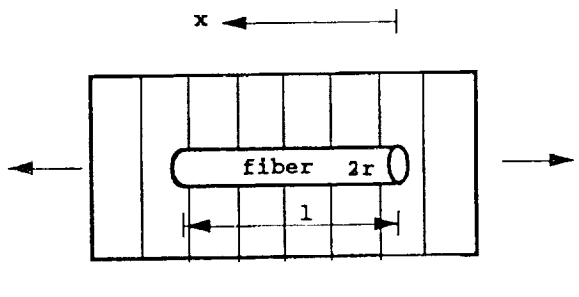
(a) Before loading
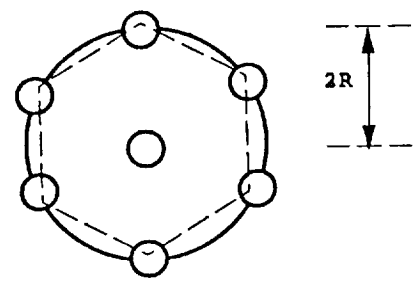
(c) Fiber with nearest hexagonally packed neighbours
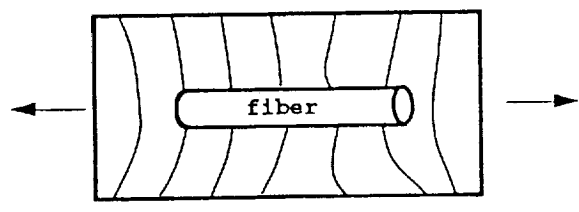
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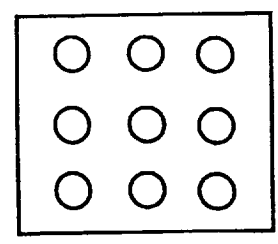
(d) Fiber with nearest square-packed neighbours

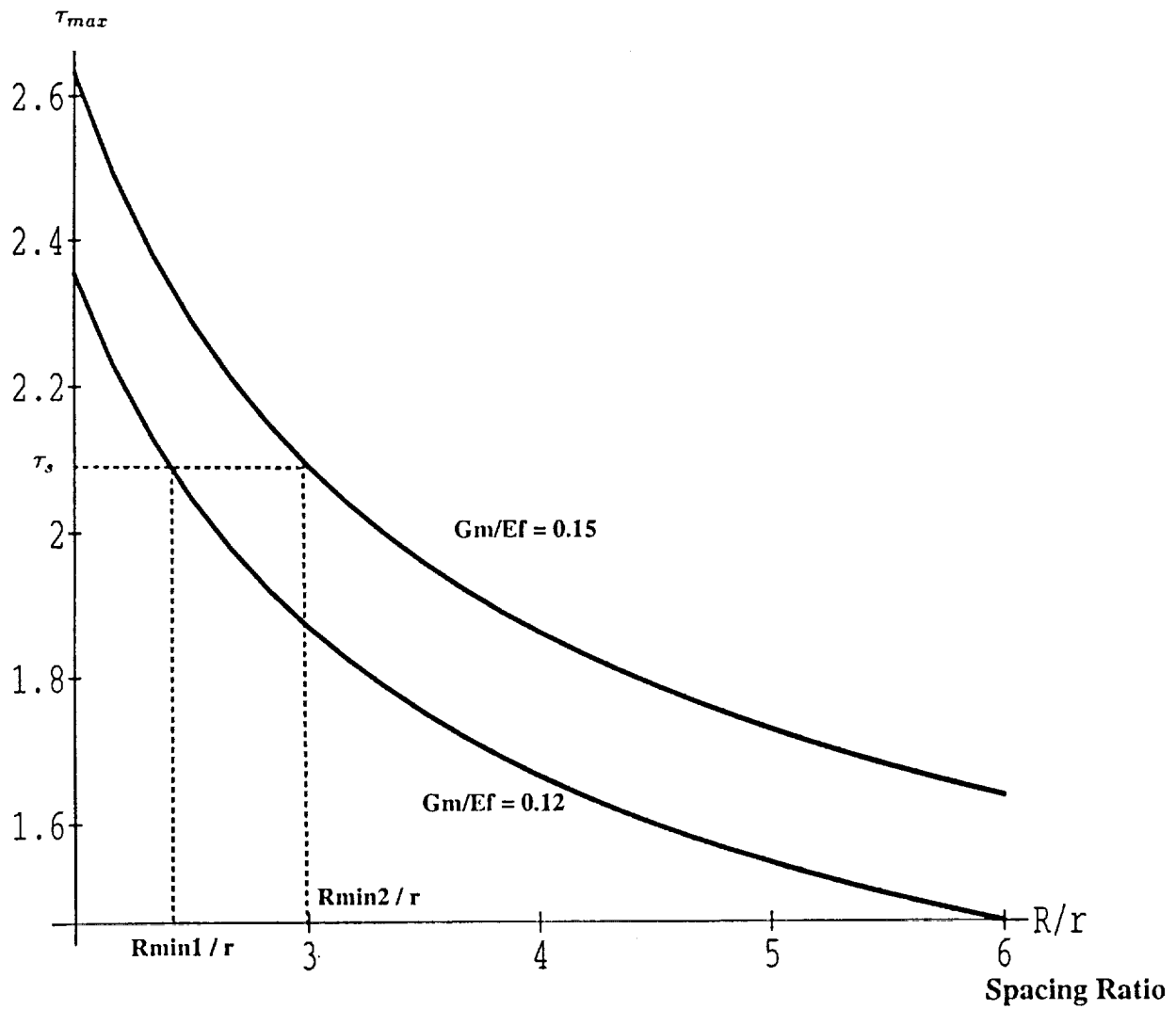Figure 1   Fiber embedded in a matrix and the fiber packing forms
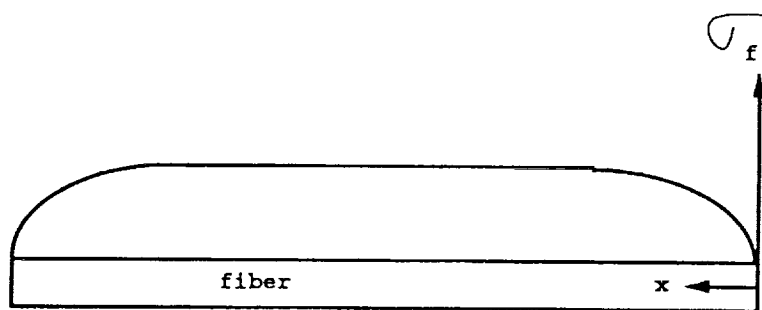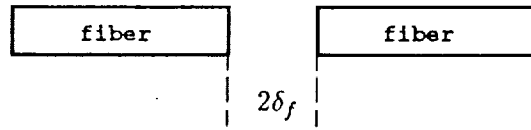
(a) The tensile stress distribution
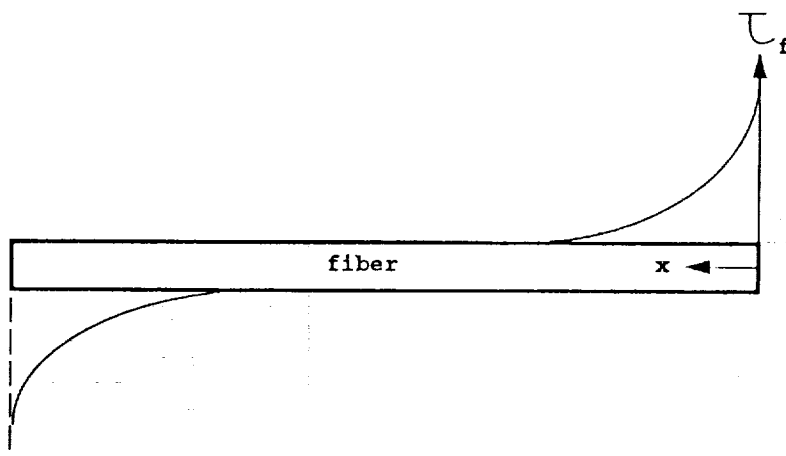
(b) The shear stress distribution
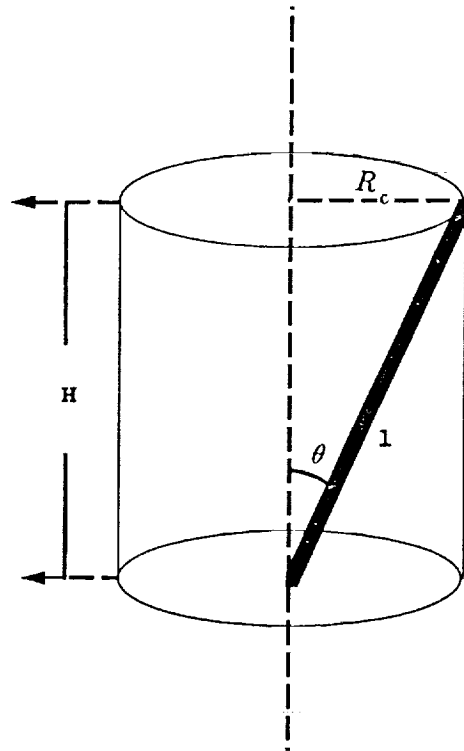
Figure 2   Elastic Stress Distribution in the Fiber

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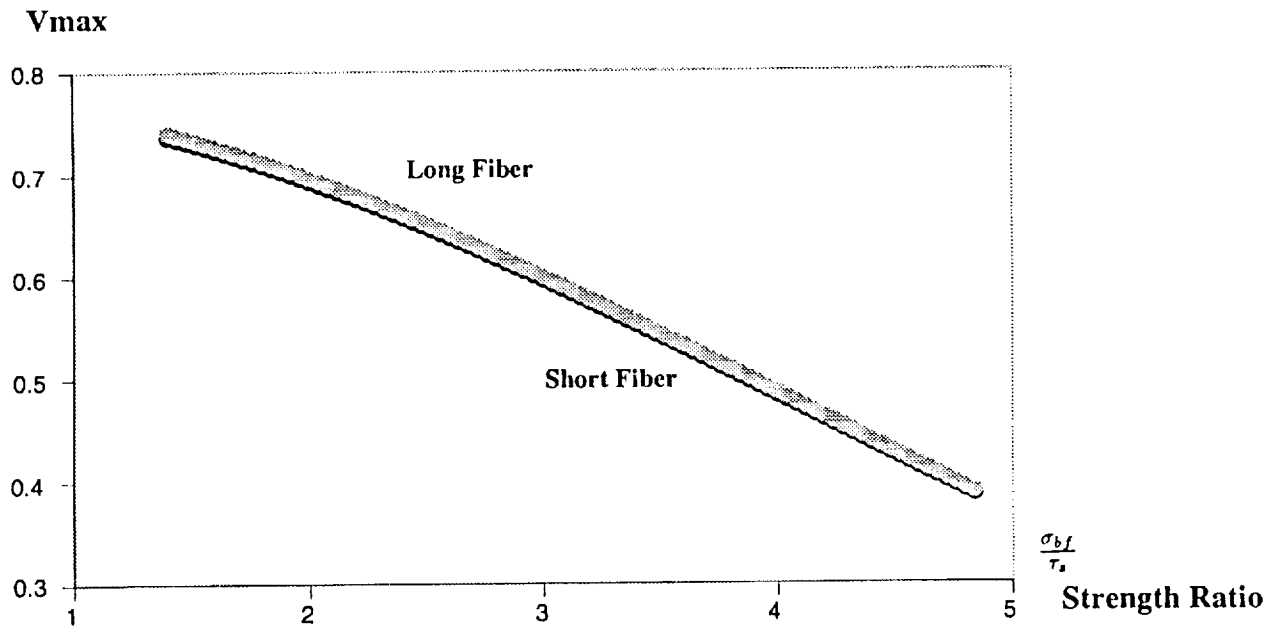
Figure 3   Maximum Shear Stress vs. Spacing Ratio
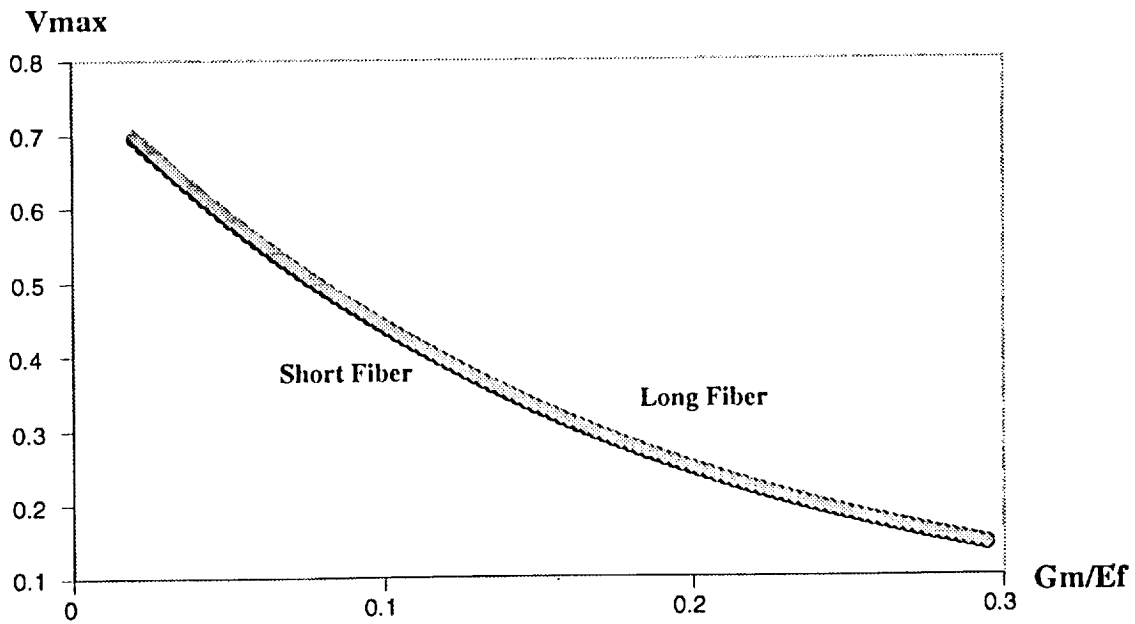
(a) Distance between fiber ends

(b) A fiber in the matrix

Figure 4   Geometric Relationships between Fiber and Matrix

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Figure 5   Maximum Volume Fraction vs. Strength Ratio

The y-axis is labeled "Vmax" with values 0.3, 0.4, 0.5, 0.6, 0.7, 0.8. The x-axis is labeled "Strength Ratio" with the notation $\frac{\sigma_{bf}}{\tau_s}$, values 1, 2, 3, 4, 5. Curves are labeled "Long Fiber" and "Short Fiber".

Figure   6   Fiber Volume Fraction vs. Gm/Ef

The y-axis is labeled "Vmax" with values 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8. The x-axis is labeled "Gm/Ef" with values 0, 0.1, 0.2, 0.3. Curves are labeled "Short Fiber" and "Long Fiber".
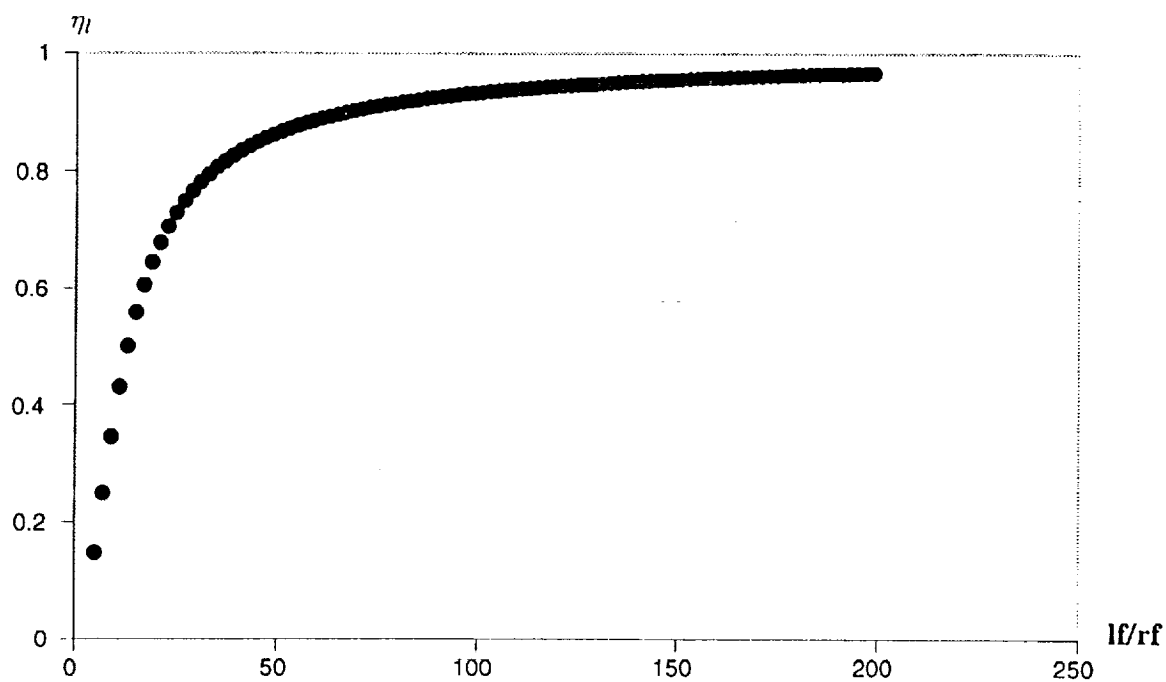
Figure 7   Length Efficiency Factor vs. lf/rf
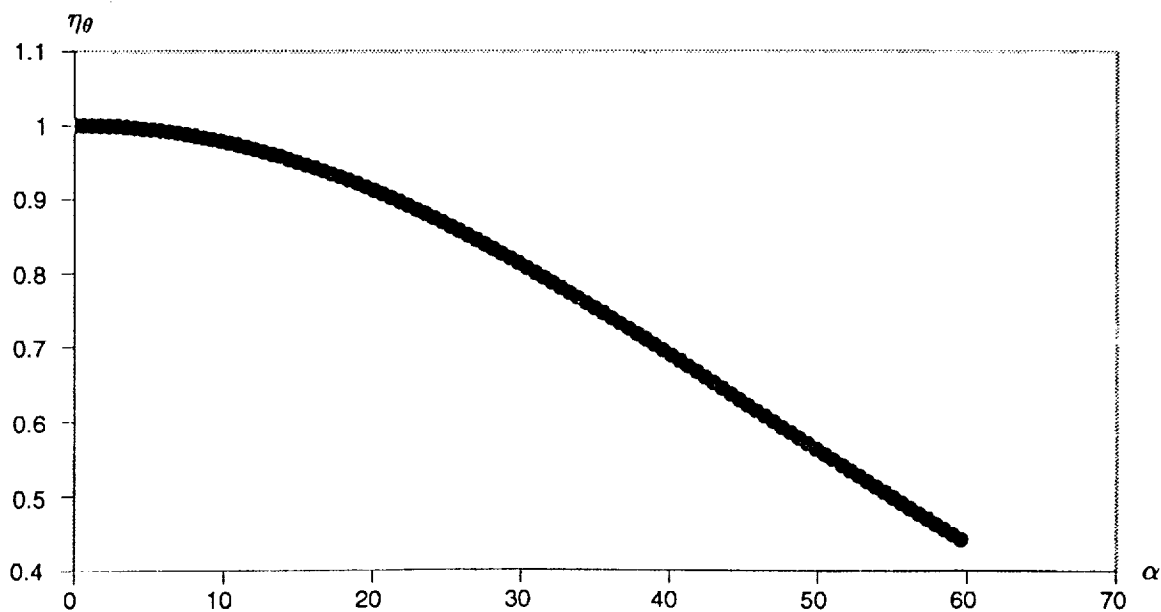
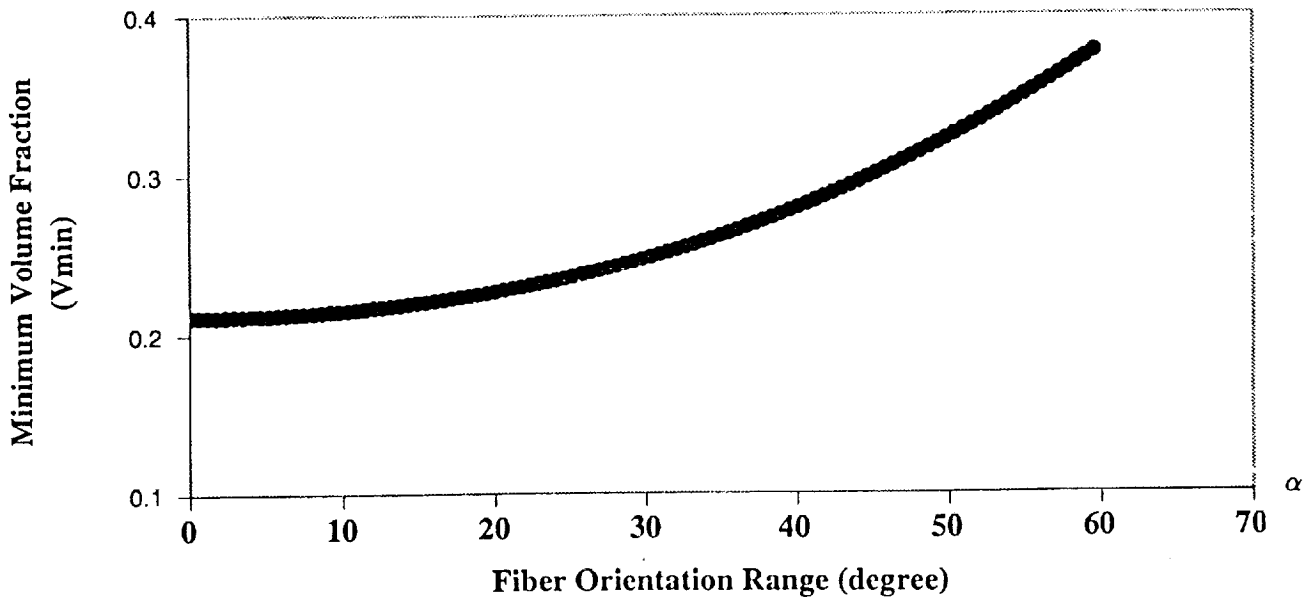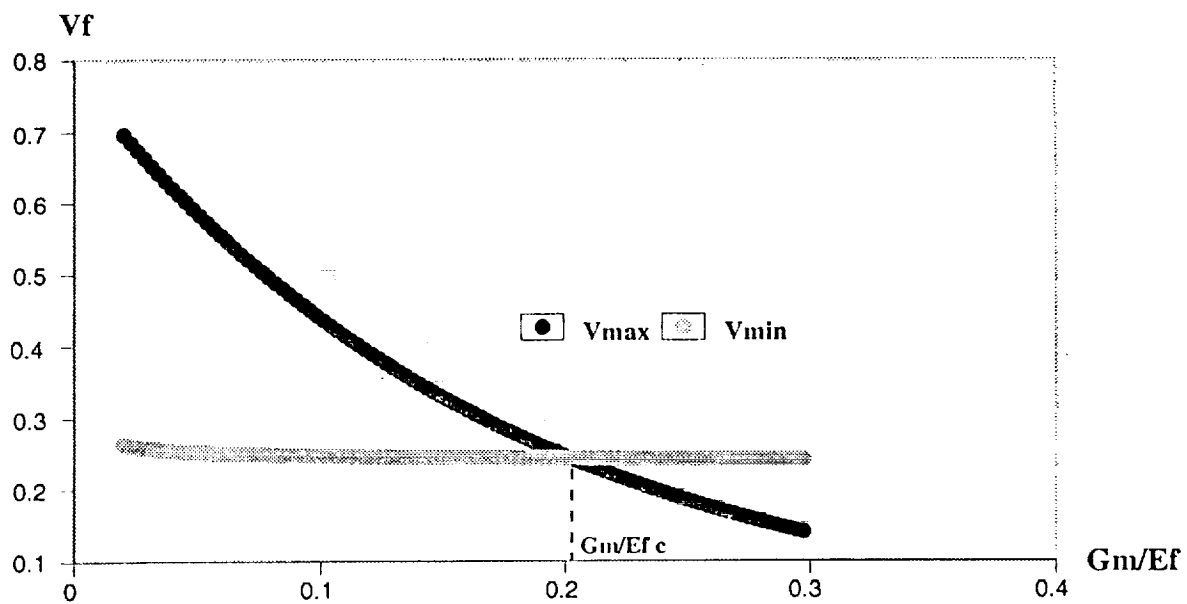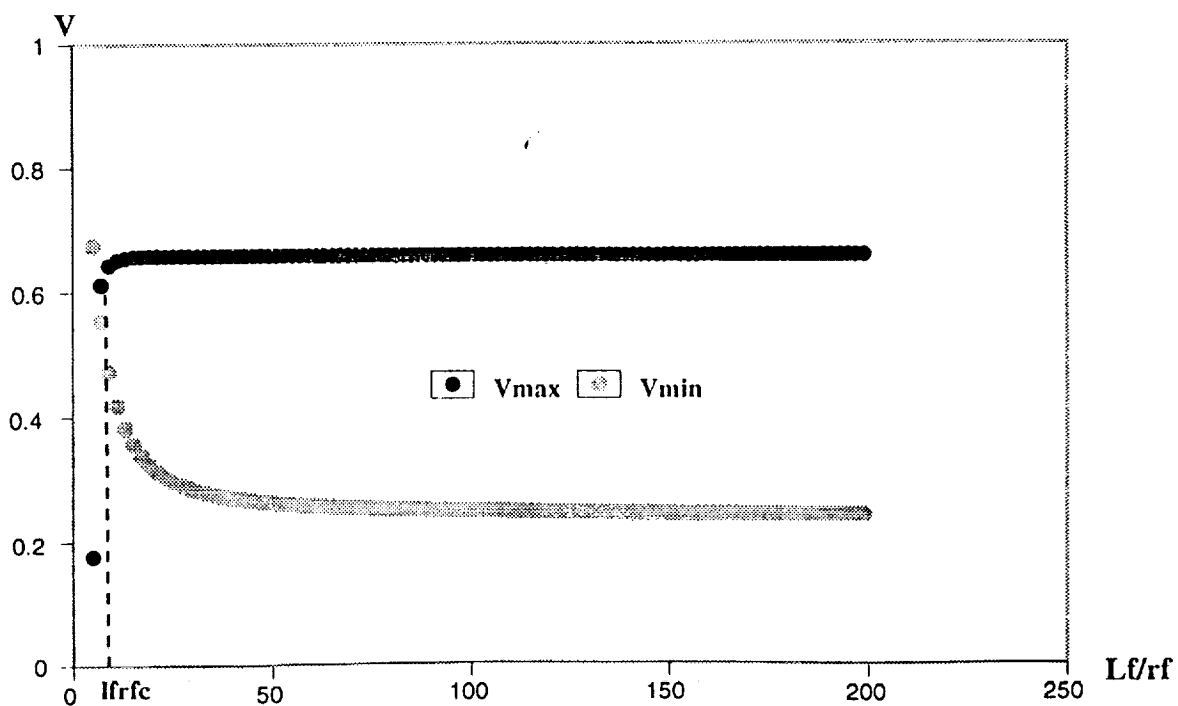**Figure 8 Relationship between Orientation Range and Efficiency Factor**
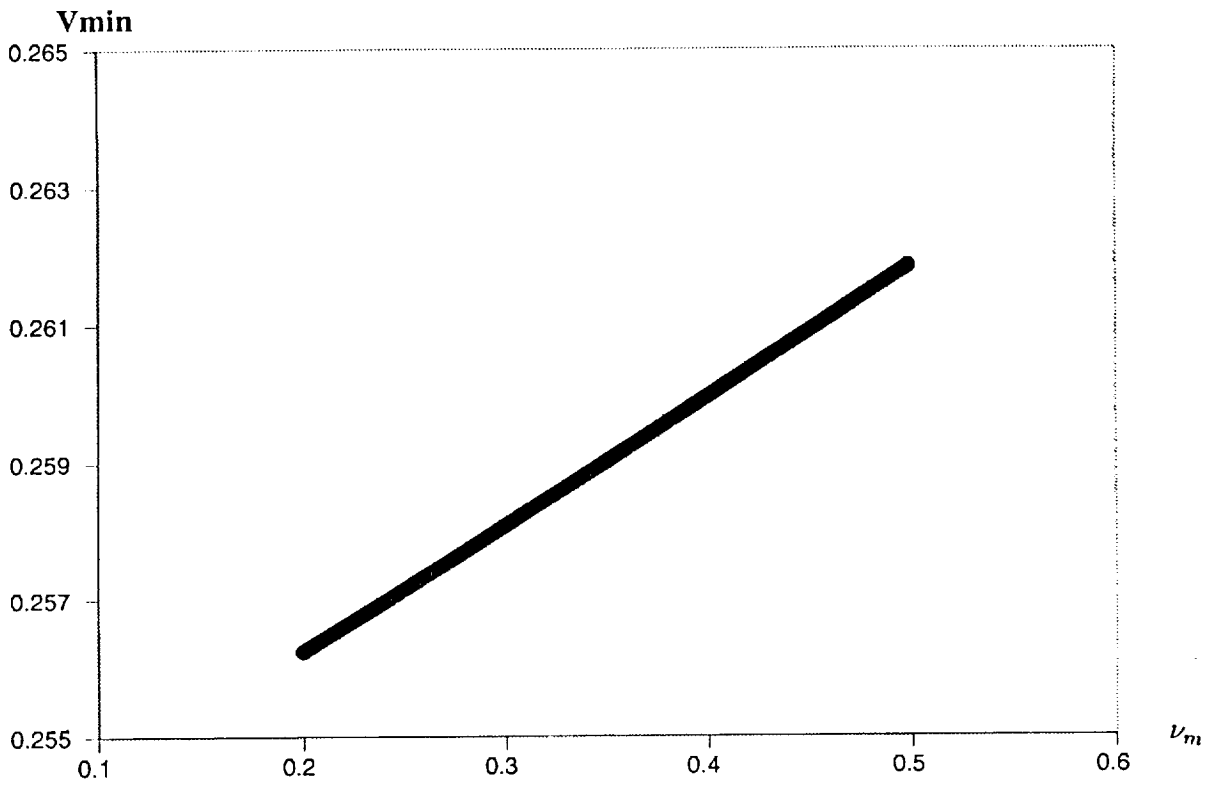
Figure 9   Matrix Poisson's Ratio vs. Vfmin

Figure 10   Vmin vs. Fiber Orientation

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Figure 11 Effects of the Modulus Ratio on Volume Fractions and the The Property Compatibility Boundary Defined

Figure 12 Critical Fiber Aspect Ratio