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Abstract

The invariance theory in continuum mechanics is applied to analyze Reynolds stresses in high Reynolds number turbulent flows. The analysis leads to a turbulent constitutive relation that relates the Reynolds stresses to the mean velocity gradients in a more general form in which the classical isotropic eddy viscosity model is just the linear approximation of the general form. On the basis of realizability analysis, a set of model coefficients are obtained which are functions of the time scale ratios of the turbulence to the mean strain rate and the mean rotation rate. These coefficients will ensure the positivity of each component of the turbulent kinetic energy — realizability that most existing turbulence models fail to satisfy. Separated flows over backward-facing step configurations are taken as applications. The calculations are performed with a conservative finite-volume method. Grid-independent and numerical diffusion-free solutions are obtained by using differencing schemes of second-order accuracy on sufficiently fine grids. The calculated results are compared in detail with the experimental data for both mean and turbulent quantities. The comparison shows that the present proposal significantly improves the predictive capability of K-ϵ based two equation models. In addition, the proposed model is able to simulate rotational homogeneous shear flows with large rotation rates which all conventional eddy viscosity models fail to simulate.

1. Introduction

Numerical simulation of turbulence is a bottleneck in the development of computational fluid dynamics (CFD). The approach of direct numerical simulation (DNS) without any turbulence models is restricted to simple and low Reynolds number flows within the capabilities of current computers. A compromise to DNS is the large-eddy simulation (LES) approach in which the large scales of turbulence are directly computed and the small scales are modeled. LES, though applicable for high Reynolds number turbulence, is usually very expensive and has serious problems with boundary conditions. Some of these difficulties also exist in DNS. Therefore, most practical calculations at the present time are based on averaged Navier-Stokes equations with the aid of turbulence modeling.

In turbulence modeling, unknown turbulent correlations are expressed in terms of determinable flow quantities with the aid of empirical information. According to the way the Reynolds stresses (the second-order moment correlations) are treated, turbulence models may be divided into two groups: the Reynolds stress algebraic equation models and the Reynolds stress transport equation models. The former group includes the zero-, one- and two-equation models (Rodi, 1980) in which the Reynolds stresses are algebraically related to the mean flow field. The eddy viscosity K-ϵ two-equation model (Launder and Spalding, 1974) in this group is one of the most popular turbulence models used today in practical flow calculations. In the latter group, often called second-order closure models, the Reynolds stresses are determined by their own dynamical transport equations. Second-order closures are attractive because they can simulate the transport of the individual Reynolds stresses; however, it is difficult to consistently model all the higher-order turbulent correlations appearing in these second moment dynamical equations. Inappropriate modeling of higher order correlations (often due to lack of information about their underlying mechanism) could result in a serious inaccuracy and unphysical results.

In the standard K-ϵ model, all the model coefficients are constant and are determined from a set of experiments for simple flows under equilibrium or isotropic turbulence conditions (Rodi, 1980). Numerical experience over the last two decades has shown that this set of constants has a broad applicability, but this by no means signifies that they are universal. Rodi (1980) found that the K-ϵ model's ability to predict weak shear flows can be significantly improved by using C_μ as a function of the average ratio of P/ϵ (P is the production of the turbulent kinetic energy) instead of a constant. Leschziner and Rodi (1981) proposed a function for C_μ which takes into account the effect of streamline curvature and obtained improved results in the calculation of annular and twin parallel jets. Recently, Yakhot and co-workers have

developed a version of the K-ϵ model using renormalization group (RNG) methods. Their model is of the same form as the standard K-ϵ model, but all the model coefficients take different constant values. In the latest version of the RNG based K-ϵ model (Speziale and Thangam, 1992), the coefficient C_1, related to the production of dissipation term, is set to be a function of η, where η is the time scale ratio of the turbulent to mean strain rate. In applying this model to a separated flow over a backward-facing step, experimentally studied by Kim *et al.* (1978), Speziale and Thangam obtained a good prediction of the reattachment length which is an important parameter often used to assess the overall accuracy of calculations.

The standard K-ϵ model (including the RNG based one), like many others in the algebraic equation model group, uses Boussinesq's isotropic eddy-viscosity concept which assumes that the Reynolds stresses are proportional to the mean velocity gradients. The concept usually does well for the shear stresses in two-dimensional mean flows of the boundary-layer type, but not well for the normal stresses due to the erroneous isotropic nature of the concept. This suggests that linear dependence on the mean velocity gradients is insufficient and that a more general relation is needed for more complex flows. In fact, by eliminating the convection and diffusion terms in the modeled transport equations for the Reynolds stresses, Rodi (1980) developed an algebraic stress model (ASM) in which the Reynolds stresses are calculated by algebraic expressions. Owing to its anisotropic nature, the model does perform better than the isotropic K-ϵ model for certain flows; a well known example is fully-developed flow in non-circular ducts where ASM is capable of generating turbulence-driven secondary motions while the isotropic eddy viscosity K-ϵ model is not. However, ASM does not appear in a tensorial invariant form, which may limit its generality. In addition, inappropriate modeling of higher order correlations, such as pressure-strain correlations, will also cause deficiencies of the second-order closure based ASM. Moreover, special care needs to be taken to prevent the turbulent normal stresses from becoming negative (Huang and Leschziner, 1985), and the numerical implementation of ASM may even be more complicated than that of its parent second-order closure model, especially in general three dimensional flows. Recently, this numerical difficulties of ASM was first nicely resolved by Taulbee (1992).

There are other approaches to developing Reynolds stress algebraic equation models. For example, Yoshizawa (1984) derived a relation for the turbulent stresses using a two-scale direct interaction approximation. It contains both linear and quadratic terms of the mean velocity gradients. A similar relation was also derived recently by Rubinstein and Barton (1990) using Yokhot and Orszag's RNG method. An interesting point in these two methods is that the values of the model coefficients can all

be determined analytically. Speziale (1987) proposed a different expression, based on the principle of material frame-indifference, which contains the Oldroyd derivative of the mean strain rates. However, the principle of material frame-indifference is only valid in the limit of two-dimensional incompressible turbulence, hence it is not an appropriate constraint for general turbulent flows. In addition, these non-linear models are not fully realizable and have not been extensively tested.

The purpose of the present study is to develop a general and realizable Reynolds stress algebraic equation model with the method of rational mechanics. As usual, we assume that the Reynolds stresses depend on the mean velocity gradients, the turbulent velocity and length scales, then a constitutive relation for the Reynolds stresses is derived by using the invariance theory. The final form is truncated up to tensorial quadratic terms of the mean velocity gradients. Using the realizability conditions, the coefficients in the obtained relation are found to be at least functions of the time scale ratio of the turbulence to the mean strain rate. In general, they are also functions of the time scale ratio of the turbulence to the mean rotation rate.

The model validation is made on the basis of applications to the rotational homogeneous shear flows simulated by Bardina *et al.* (1983) and the two backward-facing step flows experimentally studied by Driver and Seegmiller (1985) and Kim *et al.* (1978). The latter type of flows has served as a benchmark in validating turbulence models for complex flows. Calculations are carried out with a conservative finite-volume method, and a second-order accurate and bounded differencing scheme, together with sufficiently fine grids, is used to ensure that the solution is both grid-independent and free from numerical diffusion. The calculated results are compared in detail with experimental data as well as with those obtained using standard K-ϵ model.

2. Modeling of Reynolds Stresses

Incompressible turbulent flows are governed by the following Reynolds averaged continuity and Navier-Stokes equations:

$$U_{i,i} = 0 \tag{1}$$

$$U_{i,t} + (U_j U_i - \nu U_{i,j} + \overline{u_i u_j})_{,j} = -\frac{p_{,i}}{\rho} \tag{2}$$

where U_i are the mean velocity components ($i = 1, 2, 3$), p is the mean pressure, ν and ρ are the fluid kinematic viscosity and density, $U_{i,t}$ and $U_{i,j}$ are derivatives of U_i with respect to time t and co-ordinate x_j, respectively. $\overline{u_i u_j}$ is the turbulent stress tensor which must be modeled.

The oldest and simplest proposal for modeling the turbulent stress is Boussinesq's isotropic eddy-viscosity concept that assumes an analogy between the viscous stresses in laminar flows and the turbulent stresses in turbulent flows. The general form of this concept is

$$\overline{u_i u_j} = \frac{2}{3} K \delta_{ij} - 2\nu_t S_{ij} \tag{3}$$

where ν_t is called the eddy-viscosity and S_{ij} is the mean strain rate defined by

$$S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) \tag{4}$$

Equation (3) constitutes a common basis for most turbulence models that are extensively used today.

2.1 Constitutive relation

Does a general constitutive relation exist for turbulent correlations? Lumley (1970) discussed this problem and found that such relations exist only under the situation in which the length and time scales of turbulence are much smaller than those in the mean flow field so that the effect of initial and boundary conditions on the turbulence is not significant far from the wall. In other situations such as rapidly developing mean flows or in the vicinity of walls, it is questionable whether there exists such a constitutive relation for any turbulent correlation; however, from practical point of view, we can formally derive a "constitutive" relation for any turbulent correlation to solve the closure problem. The validity of such a formally derived relation needs, of course, to be verified with the aid of experiments.

A turbulent constitutive relation, if it exists, is always of functional form. From a modeling point of view and for convenience of application, we neglect the time memory effects in the relation and consider the relationship at the present time as the first order approximation in the time expansion of the functional form. Therefore, we assume that the turbulent stress $\overline{u_i u_j}$ is a function of the mean deformation tensor $U_{i,j}$, the velocity and length scales of turbulence characterized by the turbulent kinetic energy K and its dissipation rate ϵ, i.e.,

$$\overline{u_i u_j} = F_{ij}(U_{i,j},\ K,\ \epsilon) \tag{5}$$

Note that the molecular viscosity ν is not included because we restrict our attention here only to high Reynolds number turbulent flows.

The arguments of equation (5) contain ten quantities bearing two dimensions. According to the π theorem of dimensional analysis, they may be grouped into eight independent non-dimensional quantities:

$$V_{i,j} = \frac{K}{\epsilon} U_{i,j} \tag{6}$$

By normalizing the turbulent stress as

$$\overline{v_i v_j} = \frac{\overline{u_i u_j}}{2K} \tag{7}$$

equation (5) can be written as

$$\overline{v_i v_j} = F_{ij}(V_{i,j}) \tag{8}$$

The form of the tensor valued isotropic function F_{ij} can be determined by using invariance theory (Lumley, 1978). The basic principle is that an invariant can only be a function of other invariants. In determining a set of independent invariants, we have shown in Appendices A and B (also see Shih, 1992) that only 18 independent tensors can be formed with the tensor $V_{i,j}$ and its transpose $V_{j,i}$ according to the generalized Cayley-Hamilton relations. Following Lumley (1978), let A_i and B_j be two non-dimensional arbitrary vectors, we may form the following 18 invariants:

$$\begin{aligned}
&V_{i,j}A_iB_j,\ V_{j,i}A_iB_j,\\
&V_{i,k}V_{k,j}A_iB_j,\ V_{j,k}V_{k,i}A_iB_j,\ V_{i,k}V_{j,k}A_iB_j,\ V_{k,i}V_{k,j}A_iB_j,\\
&V_{i,k}V_{j,k}^2A_iB_j,\ V_{i,k}^2V_{j,k}A_iB_j,\ V_{k,i}V_{k,j}^2A_iB_j,\ V_{k,i}^2V_{k,j}A_iB_j,\\
&V_{i,k}V_{l,k}V_{l,j}^2A_iB_j,\ V_{i,k}^2V_{j,k}^2A_iB_j,\ V_{k,i}^2V_{k,j}^2A_iB_j,\ V_{k,i}V_{k,l}V_{j,l}^2A_iB_j,\\
&V_{k,i}V_{k,l}^2V_{j,l}^2A_iB_j,\ V_{i,k}V_{l,k}^2V_{l,j}^2A_iB_j,\\
&V_{i,k}V_{l,k}V_{l,m}^2V_{j,m}^2A_iB_j,\ V_{k,i}^2V_{k,l}^2V_{m,l}V_{m,j}A_iB_j
\end{aligned} \tag{9}$$

In addition, we have other invariants:

$$\delta_{ij}A_iB_j,\ A_iA_i,\ B_iB_i,\ I,\ II,\ III,\ \cdots \tag{10}$$

where I, II, III are the three invariants of the tensor $V_{i,j}$:

$$I = V_{i,i}$$

$$II = \frac{1}{2}(V_{i,i}V_{j,j} - V_{i,j}V_{j,i}) \tag{11}$$

$$III = \frac{1}{3!}(V_{i,i}V_{j,j}V_{k,k} - 3V_{i,i}V_{j,k}V_{k,j} + 2V_{i,j}V_{j,k}V_{k,i})$$

and $\cdots$ represents the invariants of other 17 tensors, for example, $V_{i,j}V_{i,j}$.

Note that in the above list of invariants we do not include $V_{i,l}V_{l,k}V_{k,j}A_iB_j$ and higher order terms of this type because they are not independent quantities according to the Cayley-Hamilton theorem:

$$V_{i,l}V_{l,k}V_{k,j} - I \cdot V_{i,k}V_{k,j} + II \cdot V_{i,j} - III\delta_{ij} = 0 \tag{12}$$

Any other possible terms, for example, $V_{k,i}V_{k,l}V_{m,l}V_{m,j}A_iB_j$, are also not independent, hence they will not be included.

However, the invariant list can be extended by including any combination of invariants in (9) and (10), for example,

$$\begin{aligned} &V_{i,j}A_iB_j \cdot f_1(I,\ II,\ III,\ V_{i,j}V_{i,j},\ \cdots) \\ &V_{i,k}V_{k,j}A_iB_j \cdot f_2(I,\ II,\ III,\ V_{i,j}V_{i,j},\ \cdots) \end{aligned} \tag{13}$$

where f_1 and f_2 are scalar functions. Of course, these types of invariants are not independent, but they are useful in explaining why the coefficients in the final relation (16) are, in general, functions of various invariants of the tensors in question.

As a result, the invariant $\overline{v_iv_j}A_iB_j$ may be written as a function of the above invariants listed in (9), (10) and (13):

$$\begin{aligned} \overline{v_iv_j}A_iB_j = f(&V_{i,j}A_iB_j,\ V_{j,i}A_iB_j,\ V_{i,k}V_{k,j}A_iB_j,\ V_{j,k}V_{k,i}A_iB_j, \\ &V_{i,k}V_{j,k}A_iB_j,\ V_{k,i}V_{k,j}A_iB_j,\ V_{i,k}V_{j,k}^2A_iB_j, \\ &V_{i,k}^2V_{j,k}A_iB_j,\ V_{k,i}V_{k,j}^2A_iB_j,\ V_{k,i}^2V_{k,j}A_iB_j, \\ &V_{i,k}V_{l,k}V_{l,j}^2A_iB_j,\ V_{i,k}^2V_{j,k}^2A_iB_j,\ V_{k,i}^2V_{k,j}^2A_iB_j, \\ &V_{k,i}V_{k,l}V_{j,l}^2A_iB_j,\ V_{k,i}V_{k,l}^2V_{j,l}^2A_iB_j,\ V_{i,k}V_{l,k}^2V_{l,j}^2A_iB_j, \\ &V_{i,k}V_{l,k}V_{l,m}^2V_{j,m}^2A_iB_j,\ V_{k,i}^2V_{k,l}^2V_{m,l}V_{m,j}A_iB_j, \\ &\delta_{ij}A_iB_j,\ A_iA_i,\ B_iB_i,\ I,\ II,\ III,\ V_{i,j}V_{i,j},\ \cdots, \end{aligned}$$

$$V_{i,j}A_iB_jf_1,\ \cdots) \tag{14}$$

Because $\overline{v_iv_j}A_iB_j$ is bilinear in arbitrary tensors A_i and B_j, we must require that the right hand side of equation (14) be also bilinear in A_i and B_j. Therefore, equation (14) can be reduced to

$$\begin{aligned}
\overline{v_iv_j}A_iB_j &= a_1\delta_{ij}A_iB_j + a_2V_{i,j}A_iB_j + a_3V_{j,i}A_iB_j \\
&+a_4V_{i,k}V_{k,j}A_iB_j + a_5V_{j,k}V_{k,i}A_iB_j + a_6V_{i,k}V_{j,k}A_iB_j \\
&+a_7V_{k,i}V_{k,j}A_iB_j + a_8V_{i,k}V_{j,k}^2A_iB_j + a_9V_{i,k}^2V_{j,k}A_iB_j \\
&+a_{10}V_{k,i}V_{k,j}^2A_iB_j + a_{11}V_{k,i}^2V_{k,j}A_iB_j + a_{12}V_{i,k}V_{l,k}V_{l,j}^2A_iB_j \\
&+a_{13}V_{i,k}^2V_{j,k}^2A_iB_j + a_{14}V_{k,i}^2V_{k,j}^2A_iB_j + a_{15}V_{k,i}V_{k,l}V_{j,l}^2A_iB_j \\
&+a_{16}V_{k,i}V_{k,l}^2V_{j,l}^2A_iB_j + a_{17}V_{i,k}V_{l,k}^2V_{l,j}^2A_iB_j \\
&+a_{18}V_{i,k}V_{l,k}V_{l,m}^2V_{j,m}^2A_iB_j + a_{19}V_{k,i}^2V_{k,l}^2V_{m,l}V_{m,j}A_iB_j
\end{aligned} \tag{15}$$

Noting that A_i and B_j are the arbitrary vectors, we obtain

$$\begin{aligned}
\overline{v_iv_j} &= a_1\delta_{ij} + a_2V_{i,j} + a_3V_{j,i} + a_4V_{i,k}V_{k,j} + a_5V_{j,k}V_{k,i} + a_6V_{i,k}V_{j,k} \\
&+a_7V_{k,i}V_{k,j} + a_8V_{i,k}V_{j,k}^2 + a_9V_{i,k}^2V_{j,k} + a_{10}V_{k,i}V_{k,j}^2 + a_{11}V_{k,i}^2V_{k,j} \\
&+a_{12}V_{i,k}V_{l,k}V_{l,j}^2 + a_{13}V_{i,k}^2V_{j,k}^2 + a_{14}V_{k,i}^2V_{k,j}^2 + a_{15}V_{k,i}V_{k,l}V_{j,l}^2 \\
&+a_{16}V_{k,i}V_{k,l}^2V_{j,l}^2 + a_{17}V_{i,k}V_{l,k}^2V_{l,j}^2 + a_{18}V_{i,k}V_{l,k}V_{l,m}^2V_{j,m}^2 \\
&+a_{19}V_{k,i}^2V_{k,l}^2V_{m,l}V_{m,j}
\end{aligned} \tag{16}$$

where the coefficients $a_1 - a_{19}$ are, due to (13), functions of the invariants I, II, III, $V_{i,j}V_{i,j}$ $\cdots$, i.e.,

$$a_i = f_i(I,\ II,\ III,\ V_{i,j}V_{i,j},\ \cdots), \qquad i = 1, 2, \cdots, 19 \tag{17}$$

Equation (16) is a general relationship between two second rank tensors.

The normalized turbulent stresses $\overline{v_i v_j}$ have two properties

$$\overline{v_i v_j} = \overline{v_j v_i} \tag{18}$$

and

$$\overline{v_i v_i} = 1 \tag{19}$$

Using the properties of (18) and (19) in equation (16), we obtain the following relations:

$$\begin{aligned} & a_3 = a_2, \qquad a_5 = a_4, \qquad a_9 = a_8, \qquad a_{11} = a_{10} \\ & a_{12} = a_{15} = a_{16} = a_{17} = a_{18} = a_{19} = 0 \\ & a_1 = \frac{1}{3}[1 - 2a_2 I - 2a_4 D - (a_6 + a_7)\tilde{D} - 2(a_8 + a_{10})\tilde{\tilde{D}} - (a_{13} + a_{14})\bar{\tilde{D}}] \end{aligned} \tag{20}$$

where

$$D = V_{i,j}V_{j,i}\ , \qquad \tilde{D} = V_{i,j}V_{i,j}\ , \qquad \tilde{\tilde{D}} = V_{i,j}V_{i,j}^2\ , \qquad \bar{\tilde{D}} = V_{i,j}^2 V_{i,j}^2 \tag{21}$$

After introducing equations (20) into (16) and converting to the dimensional form, we obtain

$$\begin{aligned} \overline{u_i u_j} \;=\; & \frac{2}{3}K\delta_{ij} + 2a_2\frac{K^2}{\epsilon}(U_{i,j} + U_{j,i} - \frac{2}{3}U_{i,i}\delta_{i,j}) \\ & +2a_4\frac{K^3}{\epsilon^2}(U_{i,k}U_{k,j} + U_{j,k}U_{k,i} - \frac{2}{3}\Pi\delta_{ij}) \\ & +2a_6\frac{K^3}{\epsilon^2}(U_{i,k}U_{j,k} - \frac{1}{3}\tilde{\Pi}\delta_{ij}) \\ & +2a_7\frac{K^3}{\epsilon^2}(U_{k,i}U_{k,j} - \frac{1}{3}\tilde{\Pi}\delta_{ij}) \\ & +2a_8\frac{K^4}{\epsilon^3}(U_{i,k}U_{j,k}^2 + U_{i,k}^2U_{j,k} - \frac{2}{3}\tilde{\tilde{\Pi}}\delta_{ij}) \\ & +2a_{10}\frac{K^4}{\epsilon^3}(U_{k,i}U_{k,j}^2 + U_{k,j}U_{k,i}^2 - \frac{2}{3}\tilde{\tilde{\Pi}}\delta_{ij}) \\ & +2a_{13}\frac{K^4}{\epsilon^3}(U_{i,k}^2U_{j,k}^2 - \frac{1}{3}\bar{\tilde{\Pi}}\delta_{ij}) \\ & +2a_{14}\frac{K^4}{\epsilon^3}(U_{k,i}^2U_{k,j}^2 - \frac{1}{3}\bar{\tilde{\Pi}}\delta_{ij}) \end{aligned} \tag{22}$$

where

$$\Pi = U_{i,j}U_{j,i}\ , \qquad \tilde{\Pi} = U_{i,j}U_{i,j}\ , \qquad \tilde{\tilde{\Pi}} = U_{i,j}U_{i,j}^2\ , \qquad \bar{\tilde{\Pi}} = U_{i,j}^2U_{i,j}^2 \tag{23}$$

Equation (22) is the most general relationship between $\overline{u_i u_j}$ and $U_{i,j}$ under the assumption (5). Interestingly enough, the first five terms at the right-hand side of equation (22) are of the same form as those derived through both the two-scale direct interaction formalism (Yoshizawa, 1984) and the renormalization group method (Rubinstein and Barton, 1990). The fact that the three different theoretical analyses lead to a similar result indicates the rationality of equation (22).

In practice, however, a quadratic tensorial form of (22) may be sufficient, especially when $||U_{i,j}||K/\epsilon$ is less than unity (which is true if the turbulent time scale is smaller than the time scale of mean flow). Therefore, from now on, we only consider the quadratic form of equation (22).

2.2 Realizability

Realizability (Schumann, 1977, Lumley, 1978), defined as the requirement of the non-negativity of turbulent normal stresses and Schwarz' inequality between any fluctuating quantities, is a basic physical and mathematical principle that the solution of any turbulence model equation should obey. It also represents a minimal requirement to prevent a turbulence model from producing unphysical results. In the following, this principle will be applied to the constitutive relation (22) to derive constraints on its coefficients.

Consider a deformation rate tensor of the form

$$\begin{pmatrix} U_{1,1} & 0 & 0 \\ 0 & U_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{24}$$

The continuity equation (1) gives

$$U_{2,2} = -U_{1,1} \tag{25}$$

and from equation (22), the normal stress $\overline{u_1 u_1}$ can be written as

$$\frac{\overline{u_1 u_1}}{2K} = \frac{1}{3} + 2a_2 \frac{KU_{1,1}}{\epsilon} + \frac{1}{3}(2a_4 + a_6 + a_7)\left(\frac{KU_{1,1}}{\epsilon}\right)^2 \tag{26}$$

Since the time scale ratio of the turbulent to the mean strain rate is defined by

$$\eta = \frac{KS}{\epsilon} \tag{27}$$

where

$$S = (2S_{ij}S_{ij})^{1/2} \tag{28}$$

equation (26) can further be written as

$$\frac{\overline{u_1 u_1}}{2K} = \frac{1}{3} + a_2\eta + \frac{1}{12}(2a_4 + a_6 + a_7)\eta^2 \tag{29}$$

Physically, we know that $\overline{u_1 u_1}$ will decrease by a vortex stretching with an increase in $U_{1,1}$, but $\overline{u_1 u_1}$ cannot be driven to negative values. Therefore, we must require that

$$\frac{\overline{u_1 u_1}}{2K} > 0, \qquad \text{if } 0 < \eta < \infty \tag{30}$$

$$\frac{\overline{u_1 u_1}}{2K} \to 0, \qquad \text{if } \eta \to \infty \tag{31}$$

$$(\frac{\overline{u_1 u_1}}{2K})_{,\eta} \to 0, \qquad \text{if } \eta \to \infty \tag{32}$$

These are called the realizability conditions. They can be satisfied in various ways of which the simplest way is perhaps the following:

$$2a_2 = -\frac{2/3}{A_1 + \eta} \tag{33}$$

$$2a_4 = \frac{C_{\tau 1}}{f(\eta)} \tag{34}$$

$$2a_6 = \frac{C_{\tau 2}}{f(\eta)} \tag{35}$$

$$2a_7 = \frac{C_{\tau 3}}{f(\eta)} \tag{36}$$

where $f(\eta)$ is in general a polynomial of η of order higher than 2. We take its simplest form as

$$f(\eta) = A_2 + \eta^3 \tag{37}$$

$A_1, A_2, C_{\tau 1}, C_{\tau 2}$ and $C_{\tau 3}$ are adjustable constants, but they must satisfy

$$\begin{aligned} &A_1 > 0\ , \qquad A_2 > 0\ , \\ &2C_{\tau 1} + C_{\tau 2} + C_{\tau 3} > 0 \end{aligned} \tag{38}$$

Similar analysis on $\overline{u_2 u_2}$ and $\overline{u_3 u_3}$ also leads to equations (33) to (38). It should be mentioned that equations (33) to (38) also hold for a three-dimensional pure strain rate tensor

$$\begin{pmatrix} U_{1,1} & 0 & 0 \\ 0 & U_{2,2} & 0 \\ 0 & 0 & U_{3,3} \end{pmatrix} \tag{39}$$

and that any deformation rate tensor can be written in the form of (39) in the principal axes of deformation rate tensor.

It can be seen from the above analysis that realizability cannot be fully satisfied if the model coefficients are taken as constant, such as those in the standard K-ϵ model and in the recent anisotropic models of Speziale (1987), Yoshizawa (1984) and Rubinstein and Barton (1990). In fact, these models satisfy realizability only in the weak sense, that is, they only ensure the positivity of the sum of the normal Reynolds stresses.

Further constraints on the model coefficients can be obtained by considering the deformation rate tensor

$$\begin{pmatrix} 0 & U_{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{40}$$

which corresponds to a fully-developed channel flow. In this case, we have

$$\begin{aligned} \overline{u_1u_1} &= \frac{2}{3}K + \frac{\eta^2 K}{3(A_2+\eta^3)}(2C_{\tau 2} - C_{\tau 3}) \\ \overline{u_2u_2} &= \frac{2}{3}K + \frac{\eta^2 K}{3(A_2+\eta^3)}(2C_{\tau 3} - C_{\tau 2}) \\ \overline{u_3u_3} &= \frac{2}{3}K - \frac{\eta^2 K}{3(A_2+\eta^3)}(C_{\tau 2} + C_{\tau 3}) \\ \overline{u_1u_2} &= -\frac{2\eta K}{3(A_1+\eta)} \end{aligned} \tag{41}$$

where

$$\eta = \frac{KS}{\epsilon}, \qquad S = (2S_{ij}S_{ij})^{1/2} = |U_{1,2}| \tag{42}$$

Experiments indicate that

$$\begin{aligned} \overline{u_1u_1} &> \frac{2}{3}K \\ \overline{u_2u_2} &< \frac{2}{3}K \end{aligned} \tag{43}$$

which requires, from equation (41), that

$$C_{\tau 2} > 2C_{\tau 3} \tag{44}$$

2.3 Rotation effect

The parameter η represents the effect of the mean strain rate, and the effect of the mean rotation rate can be represented by ξ:

$$\xi = \frac{K\Omega}{\epsilon}, \qquad \Omega = (2\Omega^*_{ij}\Omega^*_{ij})^{1/2}, \qquad \Omega^*_{ij} = (U_{i,j} - U_{j,i})/2 + 4\epsilon_{mji}\omega_m \tag{45}$$

where ω_m represents the rotation of the frame.

In the present study, we find that it is sufficient to simply include the parameter ξ only in the coefficient a_2, i.e.,

$$2a_2 = -\frac{2/3}{A_1 + \eta + \alpha\xi} \tag{46}$$

while keeping the other coefficients the same. The dependance of the coefficients on η and ξ can be easily justified by equation (17).

2.4 Realizable algebraic equation model

By introducing equation (33)-(37) into equation (22), we obtain

$$\begin{aligned}
\overline{u_i u_j} &= \frac{2}{3}K\delta_{ij} - \nu_t(U_{i,j} + U_{j,i}) \\
&+ \frac{C_{\tau 1}}{A_2 + \eta^3}\frac{K^3}{\epsilon^2}(U_{i,k}U_{k,j} + U_{j,k}U_{k,i} - \frac{2}{3}\Pi\delta_{ij}) \\
&+ \frac{C_{\tau 2}}{A_2 + \eta^3}\frac{K^3}{\epsilon^2}(U_{i,k}U_{j,k} - \frac{1}{3}\tilde{\Pi}\delta_{ij}) \\
&+ \frac{C_{\tau 3}}{A_2 + \eta^3}\frac{K^3}{\epsilon^2}(U_{k,i}U_{k,j} - \frac{1}{3}\tilde{\Pi}\delta_{ij})
\end{aligned} \tag{47}$$

Two quantities, the turbulent kinetic energy K and its dissipation rate ϵ, remain to be determined in equation (47). To this end, the two transport equations in the standard K-ϵ model are used which read:

$$K_{,t} + [U_j K - (\nu + \frac{\nu_t}{\sigma_K})K_{,j}]_{,j} = P - \epsilon \tag{48}$$

$$\epsilon_{,t} + [U_j\epsilon - (\nu + \frac{\nu_t}{\sigma_\epsilon})\epsilon_{,j}]_{,j} = C_1\frac{\epsilon}{K}P - C_2\frac{\epsilon^2}{K} \tag{49}$$

where

$$\nu_t = C_\mu \frac{K^2}{\epsilon}, \qquad C_\mu = \frac{2/3}{A_1 + \eta + \alpha\xi} \tag{50}$$

$$P = -\overline{u_i u_j} U_{i,j} \tag{51}$$

The coefficients C_1, C_2, σ_K and σ_ϵ assume their standard values:

$$C_1 = 1.44, \quad C_2 = 1.92, \quad \sigma_K = 1, \quad \sigma_\epsilon = 1.3 \tag{52}$$

and the additional coefficients assume:

$$C_{r1} = -4, \quad C_{r2} = 13, \quad C_{r3} = -2, \quad A_1 = 1.25, \quad \alpha = 0.9, \quad A_2 = 1000. \tag{53}$$

These values have been found to work well for both test cases considered in this work.

3. Rotating Homogeneous Shear Flow

The present model is able to mimic the effect of the mean rotation rate on the turbulence. A test case is the rotating homogeneous shear flow which was studied by Bardina *et al.* (1983) using the LES method. Figure 1 is the configuration of the flow being tested. Figures 2(a – c) show the evolution of the turbulent kinetic energy K/K_0 with the nondimensional time, St, at the rotation rates of $\Omega/S =$ 0, 0.5, -0.5 respectively, where K_0 is the initial turbulent kinetic energy, S is the mean strain rate and Ω is the rotation rate of the reference frame. The calculations were performed with a fourth order Runge-Kutta scheme. The initial condition corresponding to the isotropic turbulence used in LES with $\epsilon_0/(SK_0) = 0.296$ was adopted for all the three cases. The results from both the present model and the standard K-ϵ model (hereafter referred to as s-K-ϵ) are compared with LES results in figures 2(a – c). It can be seen that at $\Omega/S = 0$ the present model cannot predict the initial nonequilibrium development of turbulent kinetic energy very well starting from an isotropic turbulence. However, it does catch up with the later "equilibrium" development and performs much better than the s-K-ϵ model which highly overpredicts the data. Figures 2(b) and 2(c) show the ability of the present model to simulate the effect of the large rotation rate on turbulence. Note that the s-K-ϵ model gives the same results as for the no rotation case because it cannot account for the effect of rotation on the evolution of turbulence.

4. Backward-Facing Step Flows

4.1 Numerical procedure

For computational convenience, the non-dimensional form of the governing equations is solved, in which

$$\begin{aligned} &< x_i >= \frac{x_i}{L_{ref}}, \quad < U_i >= \frac{U_i}{U_{ref}}, \quad < p >= \frac{p}{\rho U_{ref}^2}, \\ &< K >= \frac{K}{U_{ref}^2}, \quad < \epsilon >= \frac{\epsilon L_{ref}}{U_{ref}^3}, \quad < \nu_t >= \frac{\nu_t}{U_{ref} L_{ref}} \end{aligned} \tag{54}$$

where $< >$ refers to a non-dimensional quantity, and L_{ref}, and U_{ref} are the reference length and velocity, respectively. Accordingly, the flow Reynolds number is defined by

$$Re = \frac{L_{ref} U_{ref}}{\nu} \tag{55}$$

Hereafter, all the quantities will be of the non-dimensional form so that $< >$ will be dropped for simplicity.

In the steady-state and two dimensional cases ($x_1 = x, x_2 = y$), the transport equations (1), (2), (48) and (49) can be written in the following general form

$$[U\phi - (\frac{1}{Re} + \frac{\nu_t}{\sigma_\phi})\phi_{,x}]_{,x} + [V\phi - (\frac{1}{Re} + \frac{\nu_t}{\sigma_\phi})\phi_{,y}]_{,y} = S_\phi \tag{56}$$

where ϕ stands for 1, $U(= U_1)$, $V(= U_2)$, K and ϵ. For the momentum equations, the source term S_ϕ includes the cross-derivative diffusion and quadratic velocity gradient terms arising from equation (47). It can be seen that the non-dimensional equations are all of the same form as their dimensional counterparts, except that the kinematic molecular viscosity ν is replaced by $1/Re$.

The numerical method used to solve the system of equations (56) is a finite-volume procedure. It uses a non-staggered grid with all the dependent variables being stored at the geometric center of each control volume (Figure 3). The momentum interpolation procedure of Rhie and Chow (1983) is used to avoid spurious oscillations usually associated with the non-staggered grid, and the pressure-velocity coupling is handled with the SIMPLEC algorithm (Van Doormal and Raithby, 1984). To ensure both accuracy and stability of numerical solution, the convection terms are approximated by a second-order accurate and bounded differencing scheme (Zhu, 1991a), and all the other terms by the conventional central differencing scheme. As

a result, the discretized counterpart of equation (56) can be cast into the following linearized form

$$\sum_l A_l \phi_C = A_l \phi_l + S_C \tag{57}$$

where the coefficients A_l ($l = W, E, S, N$), which relate the principal unknown ϕ_C to its neighbours ϕ_l (Figure 3), result from the discretization of the left-hand side terms of equation (56). The convection scheme used ensures that $A_l \geq 0$ so that the resulting coefficient matrix is always diagonally dominant. The strongly implicit procedure of Stone (1968) is used to solve the system of algebraic equations. The iterative solution process is considered converged when the maximum normalized residue of all the dependent variables is less than 10^{-4}. The details of the present numerical procedure are given in Rodi *et al.* (1989) and Zhu (1991b).

4.2 Numerical results

The present model is then applied to the two backward-facing step flows experimentally studied by Kim, Kline and Johnston (1978) and Driver and Seegmiller (1985), from here on referred to as KKJ- and DS-cases, respectively. Figure 4 shows the flow configuration and the Cartesian co-ordinate system used. Table 1 gives the flow parameters for both cases; here the experimental reference free-stream velocity U_{ref} and step height H_s are taken as the reference quantities for non-dimensionalization.

Table 1. Flow parameters

case	Re	δ	L_s	L_e	H_s	H_d	U_{ref}
DS	37423	1.5	10	40	1	8	1
KKJ	44737	0.6	10	40	1	2	1

Three types of boundaries are present, i.e. inlet, outlet and solid wall. At the inlet, the experimental data are available for the streamwise mean velocity U and the turbulent normal stresses $\overline{uu}$ and $\overline{vv}$. K is calculated from these $\overline{uu}$ and $\overline{vv}$ with the assumption that

$$\overline{ww} = \frac{1}{2}(\overline{uu} + \overline{vv}) \tag{58}$$

and ϵ by

$$\epsilon = \frac{C_\mu^{3/4} K^{3/2}}{L}, \qquad L = \min(0.41\Delta y,\ 0.085\delta) \tag{59}$$

where Δy is the distance from the wall and δ is the boundary-layer thickness given in Table 1. At the outlet, the streamwise derivatives of the flow variables are set to zero. Influences of both inlet and outlet conditions on the solution are examined by changing the locations L_s and L_e, and it has been found that in both cases, the distances given in Table 1 are already sufficiently far away from the region of interest. In the earlier stage of this work, we tested several low Reynolds number K-ϵ models including those of Chien (1982), Lam and Bremhorst (1981), Launder and Sharma (1974), Shih and Lumley (1992), and Yang and Shih (1992), but none of them was found to be able to yield satisfactory solutions. Similar findings were also reported in Avva *et al.* (1990), Shuen (1992) and So and Lai (1988). Therefore in this work, we use the standard wall function approach (Launder and Spalding, 1974) to bridge the viscous sublayer near the wall.

Two sets of non-uniform computational grids are used to examine the grid dependence of the solution; they contain 110×52 (coarse) and 199×91 (fine) points for the KKJ-case and 106×56 (coarse) and 201×109 (fine) points for the DS-case. Figures 5(a) and 5(b) show the friction coefficient C_f at the bottom wall calculated with the s-K-ϵ model and the present model; also included in figure 5(a) are the experimental data for the DS-case, but no such data are available for the KKJ-case. It can be seen that the grid refinement does produce some differences for the results of the present model, more noticeable in the KKJ-case, and this is also the case for the s-K-ϵ results. This indicates that the solutions obtained on the coarse grids are not sufficiently close to the grid-independent stage. Recently, Thangam and Hur (1991) have conducted a highly-resolved calculation for the KKJ-case. They have found that quadrupling a 166×73 grid leads to only a minimal improvement. Therefore, the present results on the fine grids can be considered as grid-independent. For the DS-case, the fine grid computations with the s-K-ϵ and present model required 703 and 805 iterations, and took approximately 7.1 and 8.3 minutes of CPU time on the Cray YMP computer. In the following, only the fine grid results are presented.

The wall friction coefficient C_f is a parameter that is very sensitive to the near-wall turbulence modeling. It is C_f that the various low Reynolds number K-ϵ models tested predict much worse than those using wall functions. However, the influence of the near-wall turbulence modeling is only restricted in the near-wall regions. It is seen from figure 5(a) that both the s-K-ϵ and present model largely underpredict the negative peak of C_f, pointing to limited accuracy of the wall function approach in the recirculation region.

The computed and measured reattachment points are compared in Table 2. They are determined in the calculation from the point where C_f goes to zero. Also included in Table 2 are the result of Obi *et al.* (1989) obtained with the Reynolds stress model

(RSM) and that of Sindir (1982) with a modified algebraic stress model (ASM). The reattachment point is a critical parameter which has often been used to assess the overall performance of turbulence models as well as numerical procedures. Table 2 clearly demonstrates the significant improvement obtained with the present model. It is important to mention that this improvement is mainly due to the behavior of C_μ in the present model, and that the anisotropic behavior of the turbulent stresses only makes a marginal contribution to it.

Table 2. Comparison of reattachment points

case	experiment	s-K-ϵ	present model	RSM	ASM
DS	6.1	4.99	5.82	-	5.66
KKJ	7 $\pm$0.5	6.35	7.35	6.44	-

Figures 6(a) and 6(b) show the comparison of computed and measured static pressure coefficient C_p along the bottom wall. In both cases, the s-K-ϵ is seen to predict premature pressure rises, which is consistent with its underprediction of the reattachment lengths, while the present model captures these pressure rises quite well. The good predictions of C_p were reported in both works of Obi *et al.* (1989) and Sindir (1982), using the RMS and ASM. The results of the present model are almost comparable to those of the RSM and ASM. Again, the improved predictions of C_p are mainly attributed to the variation of C_μ.

The streamwise mean velocity U profiles are shown in figures 7(a) and 7(b) at four different cross-sections. Here, the differences between the results of the s-K-ϵ and present model are not substantial, as compared to other flow variables. The present model predicts reverse flows better than the s-K-ϵ, but results in somewhat slower recovery in the vicinity of the reattachment point. Interestingly enough, such a slow recovery also exists in the RSM prediction of Obi *et al.* (1989). Further downstream, say at x=20 in figure 7(a), the results of the two models nearly coincide with each other.

Finally, the comparisons of predicted and measured turbulent stresses $\overline{u^2}$, $\overline{v^2}$ and $\overline{uv}$ are shown in figures 8 and 9 at various x-locations. In the KKJ-case, no experimental data for the turbulent stresses are available in the recirculation region, and the reattachment point was found in the experiment to move forward and backward continuously around seven step heights downstream of the step, leaving an uncertainty of $\pm$0.5 step height for the reattachment length. This also points to some uncertainty in the measured turbulent quantities in the recovery region. On the other hand, the experimental data in the DS-case should be considered more reliable because of the smaller uncertainty of the reattachment location, indicating a smaller unsteadiness of the flow. As compared with the s-K-ϵ results in figures

8 and 9, it can be seen that the anisotropic terms increase $\overline{u^2}$ while decreasing $\overline{v^2}$, leading to significant improvements in both $\overline{u^2}$ and $\overline{v^2}$ results. On the other hand, the anisotropic terms have little impact on the turbulent shear stress $\overline{uv}$. These behaviours are clearly reflected in equations (41) which also hold qualitatively for the flows considered here. The improvement obtained by the present model in figure 8 for $\overline{uv}$ is due to the reduction in C_μ.

5. Conclusions

A constitutive relation for the turbulent stresses has been derived by using invariance theory. The relation is valid only for turbulent flows at high Reynolds numbers because the influence of the molecular viscosity has not been taken into account in the analysis. Being a second rank tensor, the general form of the turbulent stress can be expressed as a series, in terms of the mean velocity gradients, of order up to 4, while the classical eddy-viscosity representation constitutes only a first-order approximation. For practical calculations, it may suffice to use the quadratic approximation. The model coefficients are functions of the time scale ratios η and ξ, which ensure the positiveness of the turbulent normal stresses - a realizability condition that most existing turbulence models are unable to satisfy. The present model has been applied to calculate the two different flows: rotating homogeneous shear flows and backward-facing step flows. The calculated results show that all flow variables are sensitive to the variation of C_μ, and that only the turbulent normal stresses are sensitive to the terms containing nonlinear mean velocity gradients. The values of the model coefficients given in this paper seem quite appropriate for both the test cases, but the value of C_{r1} related to the cross-derivative quadratic terms has little impact on the flows considered. This indicates that C_{r1} may be further calibrated against other flows. The computed results have been compared in detail with the LES data for rotating homogeneous shear flows and the experimental data for backward-facing step flows. The comparisons show that the present model does provide significant improvement over the standard K-ϵ model, and this improvement is achieved at an insignificant penalty to the computational efficiency and algorithmic simplicity of the latter. The present model can also be expected to work well for simple inhomogeneous shear flows, as evidenced by its improved prediction in the region far downstream of the reattachment point where the flow tends to be of simple parabolic nature.

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Appendix A

Generalized Cayley-Hamilton Formulas

Rivlin (1955) showed that there are several generalized Cayley-Hamilton formulas relating matrices (product of several matrices A, B, C ...) of higher extension to matrices of lower extension. Some of them are listed here for latter use.

$$\begin{aligned} &ABC + ACB + BCA + BAC + CAB + CBA - A(trBC - trB\ trC) \\ &- B(trCA - trC\ trA) - C(trAB - trA\ trB) \\ &-(BC + CA)trA - (CA + AC)trB - (AB + BA)trC \\ &-I(trA\ trB\ trC - trA\ trBC - trB\ trCA \\ &-trC\ trAB + trABC + trCBA) = 0 \end{aligned} \tag{A.1}$$

Replacing C with A and B in Eq.(A.1), respectively, we obtain

$$\begin{aligned} ABA = &-A^2B - BA^2 + A(trAB - trA\ trB) \\ &+\frac{1}{2}B(trA^2 - trA\ trA) + (AB + BA)trA + A^2trB \\ &+I[trA^2B - trA\ trAB + \frac{1}{2}trB(trA\ trA - trA^2)] \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} BAB = &-B^2A - AB^2 + B(trBA - trB\ trA) \\ &+\frac{1}{2}A(trB^2 - trB\ trA) + (BA + AB)trB + B^2trA \\ &+I[trB^2A - trB\ trBA + \frac{1}{2}trA(trB\ trB - trB^2)] \end{aligned} \tag{A.3}$$

which indicate that the matrices ABA and BAB of extension 3 can be expressed by polynomials of matrices of extension 2 or less.

Multiplying Eq.(A.2) from the left and the right by A and Eq.(A.3) by B, and adding them correspondingly, we obtain following two relations:

$$ABA^2 + A^2BA = ABA\ trA\ +\ A^2\ trAB + A(trA^2B\ -\ trA\ trAB) \\ -B\ detA\ +\ I\ detA\ trB \tag{A.4}$$

and

$$BAB^2 + B^2AB = BAB\ trB\ +\ B^2\ trBA + B(trB^2A\ -\ trB\ trBA) \\ -A\ detB\ +\ I\ detB\ trA \tag{A.5}$$

Replacing B with B^2 in Eq.(A.4) and A with A^2 in Eq.(A.5) give

$$AB^2A^2 + A^2B^2A = AB^2A\ trA\ +\ A^2\ trAB^2 \\ +A(trA^2B^2\ -\ trA\ trAB^2) - B^2\ detA\ +\ I\ detA\ trB^2 \tag{A.6}$$

and

$$BA^2B^2 + B^2A^2B = BA^2B\ trB\ +\ B^2\ trBA^2 \\ +B(trB^2A^2\ -\ trB\ trBA^2) - A^2\ detB\ +\ I\ detB\ trA^2 \tag{A.7}$$

Replacing B with B^2 in Eq.(A.2) and A with A^2 in Eq.(A.3) yield

$$AB^2A = -A^2B^2\ -\ B^2A^2\ +\ A(trAB^2\ -\ trA\ trB^2) \\ +\frac{1}{2}\ B^2(trA^2\ -\ trA\ trA)\ +\ (AB^2 + B^2A)trA\ +\ A^2trB^2 \\ +I[trA^2B^2\ -\ trA\ trAB^2 + \frac{1}{2}trB^2(trA\ trA -\ trA^2)] \tag{A.8}$$

and

$$BA^2B = -B^2A^2\ -\ A^2B^2\ +\ B(trBA^2\ -\ trB\ trA^2) \\ +\frac{1}{2}\ A^2(trB^2\ -\ trB\ trA^2)\ +\ (BA^2 + A^2B)trB\ +\ B^2trA^2 \\ +I[trB^2A^2\ -\ trB\ trBA^2 + \frac{1}{2}trA^2(trB\ trB -\ trB^2)] \tag{A.9}$$

Eqs.(A.8) and (A.9) indicate that the matrices AB^2A and BA^2B of extension 3 can be expressed by polynomials of matrices of extension 2 or less. Therefore, the right hand sides of Eqs.(A.6) and (A.7) are also polynomials of matrices of extension of 2 or less.

Appendix B

Number of Independent Tensors Formed by Two Tensors

Let us show that the number of independent tensors formed with two general tensor A and B is 18.

Rivlin (1955) showed that any matrix product in two 3×3 matrices may be expressed as a polynomial in these matrices of extension 4 or less. Suppose we have a matrix product Π of extension 5:

$$\Pi = ABA^2B^2A \tag{B.1}$$

This can be written as

$$\Pi = ACA \tag{B.2}$$

where $C = BA^2B^2$. From Eq.(A.2), Π may be viewed as a polynomial of matrices in A and C of extension 2 or less. C itself is a matrix in A and B of extension 3 so that Π may be expressed by a polynomial in A and B of extension 4 or less. Therefore, we only need to consider the possible tensors of extension 4 or less formed by A and B.

We may show that there are only two independent tensors of extension 4. The possible tensors of extension 4 are the following 8 tensors:

$$\begin{gathered} ABA^2B^2,\ BAB^2A^2,\ A^2BAB^2,\ B^2ABA^2, \\ AB^2A^2B,\ BA^2B^2A,\ A^2B^2AB,\ B^2A^2BA. \end{gathered} \tag{B.3}$$

With Eq.(A.4), A^2BAB^2 can be expressed by $ABA^2B^2 + \cdots$. Similarly, with Eq.(A.5), $B^2ABA^2 = -BAB^2A^2 + \cdots$; with Eq.(A.7), $AB^2A^2B = -ABA^2B^2 + \cdots$; with Eq.(A.6), $BA^2B^2A = -BAB^2A^2 + \cdots$; with Eqs.(A.5) and (A.4), $A^2B^2AB = ABA^2B^2 + \cdots$; with Eqs.(A.4) and (A.5), $B^2A^2BA = BAB^2A^2 + \cdots$; where $\cdots$ represents a polynomial in A and B of extension 3 or less. As a result, only two tensors of extension 4 in Eq.(B.3) are independent, and we select them as

$$ABA^2B^2, \qquad BAB^2A^2 \tag{B.4}$$

Now we show that there are only four independent tensors of extension 3. The possible tensors of extension 3 are the following 8 tensors:

$$ABA^2,\ A^2BA,\ BAB^2,\ B^2AB,\ AB^2A^2,\ A^2B^2A,\ BA^2B^2,\ B^2A^2B. \tag{B.5}$$

Using Eqs.(A.4), (A.5), (A.6) and (A.7), we find that only four of them are independent. Let us select them as

$$ABA^2,\ BAB^2,\ AB^2A^2,\ BA^2B^2. \tag{B.6}$$

Furthermore, there are eight independent tensors of extension 2:

$$AB,\ BA,\ AB^2,\ B^2A,\ A^2B,\ BA^2,\ A^2B^2,\ B^2A^2 \tag{B.7}$$

and four independent tensors of extension 1:

$$A,\ A^2,\ B,\ B^2. \tag{B.8}$$

Therefore, we have proved that only 18 tensors can be formed independently by two general tensors.

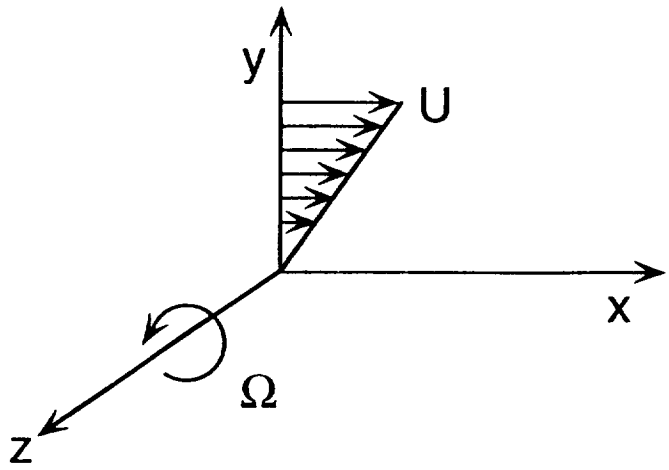

Figure 1. Rotating homogeneous shear flow

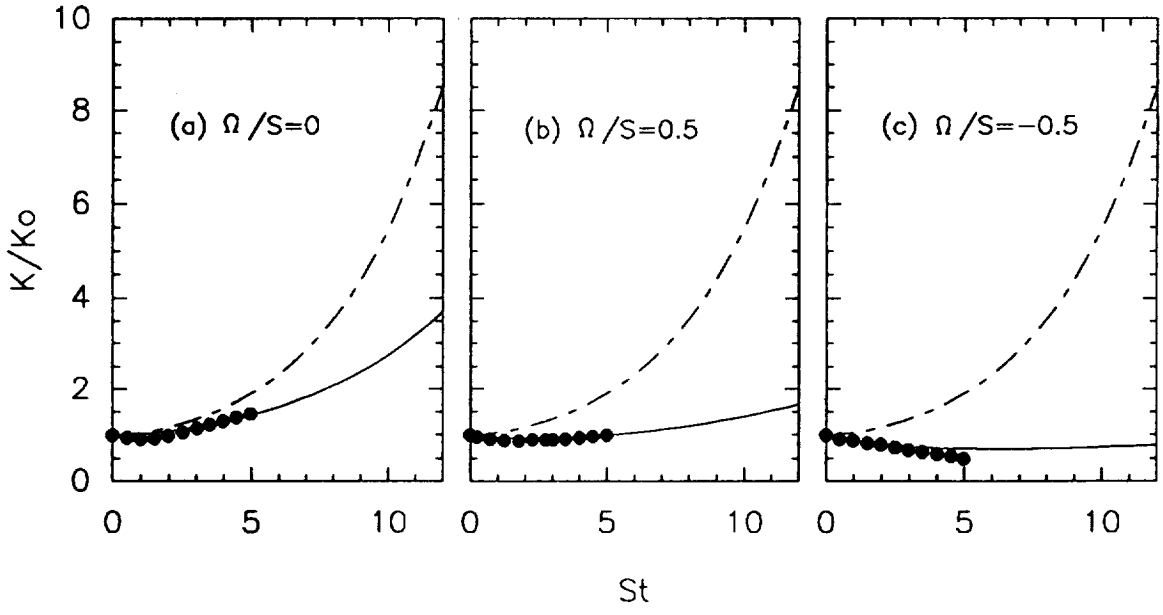

Figure 2. Evolution of turbulent kinetic energy with time.
— - —: s-K-ϵ; ——: present model; •: LES.

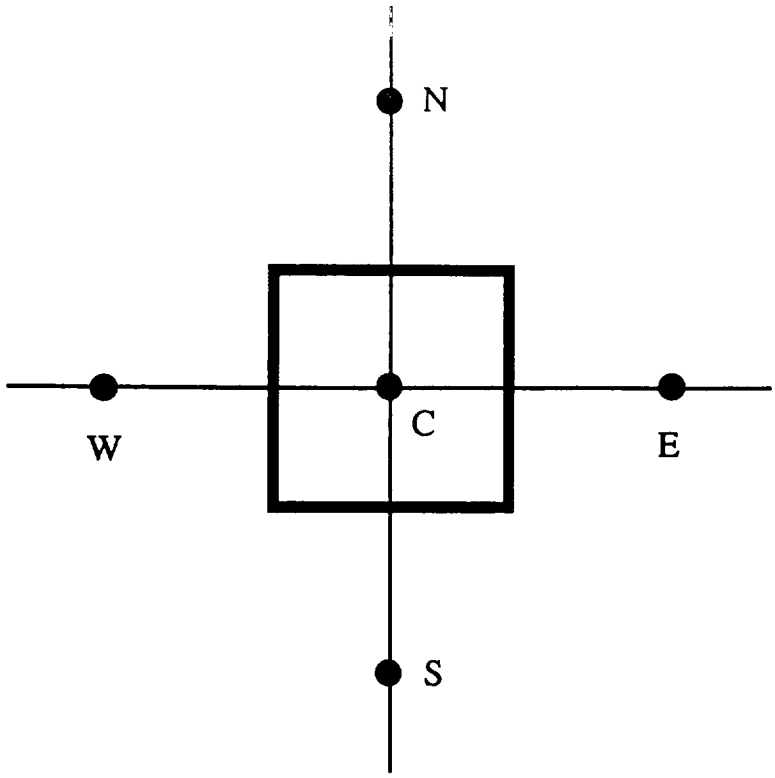

Figure 3. Typical control volume centered at C and related nodes

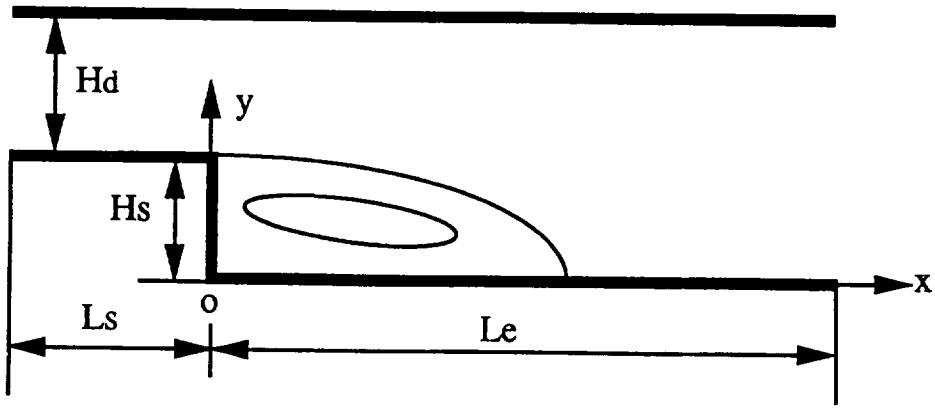

Figure 4. Backward-facing step geometry

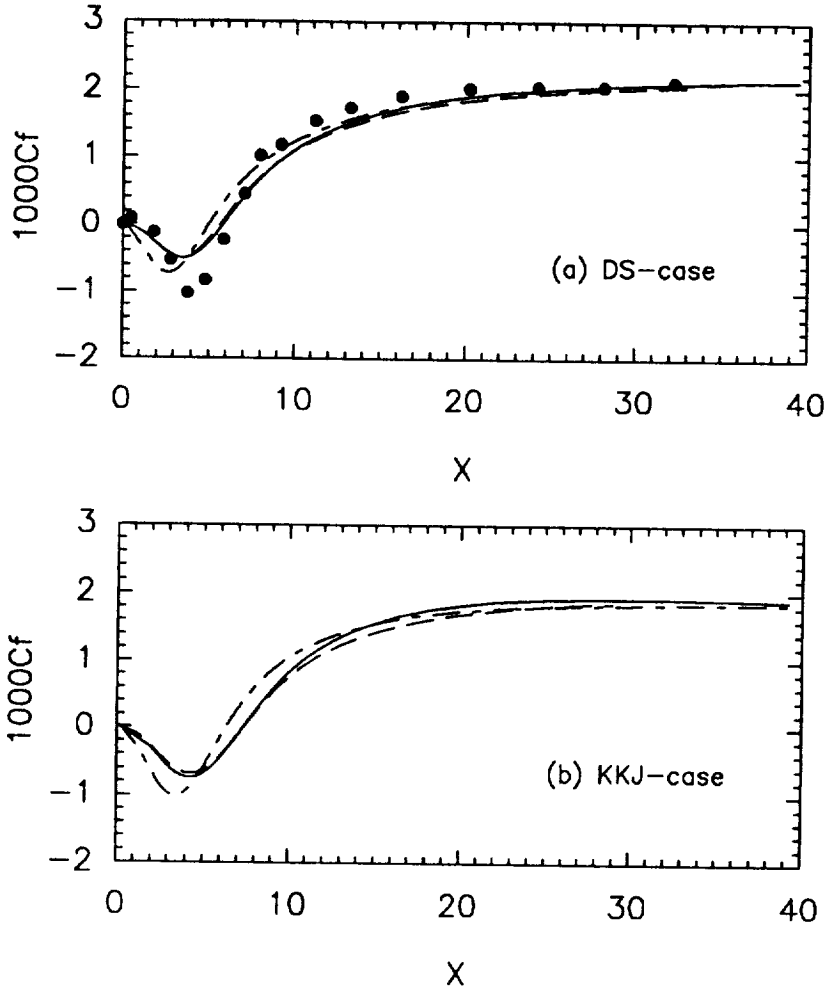

Figure 5. Friction coefficient C_f along the bottom wall.
— - —: s-K-ϵ, fine grid; ——: present model, fine grid;
— —: present model, coarse grid; •: experiment.

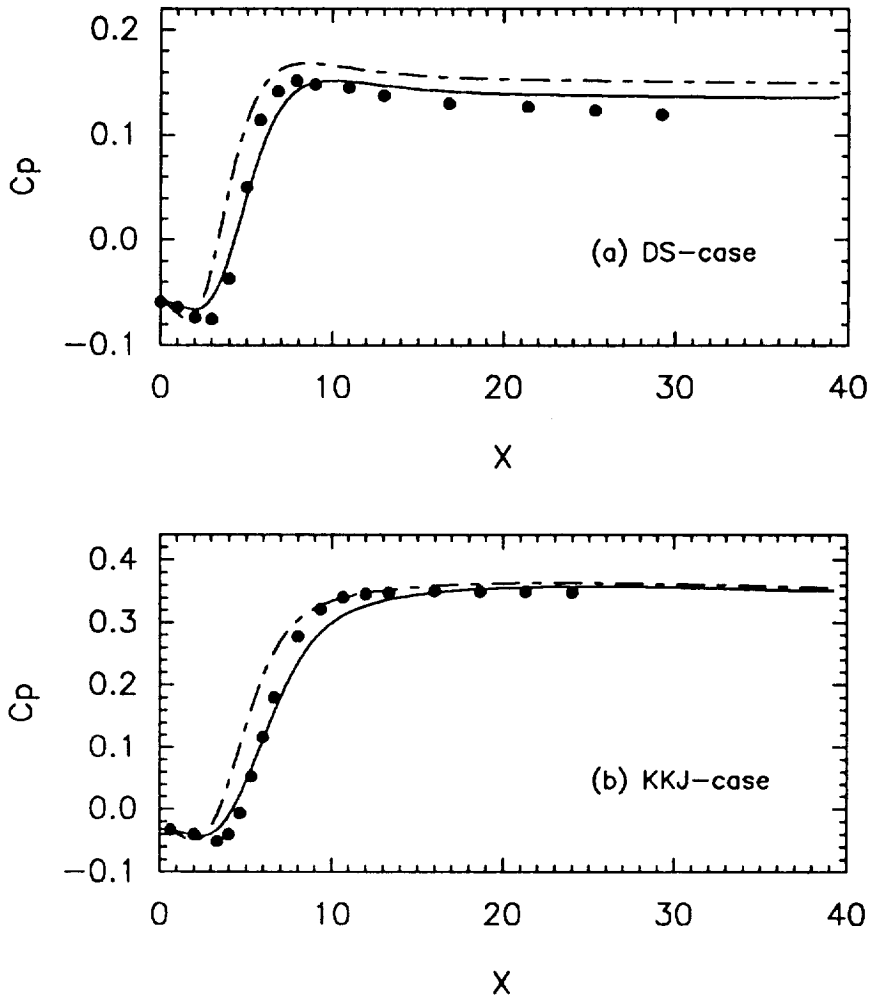

Figure 6. Static pressure coefficient C_p along the bottom wall.
— - —: s-K-ϵ; ——: present model; •: experiment

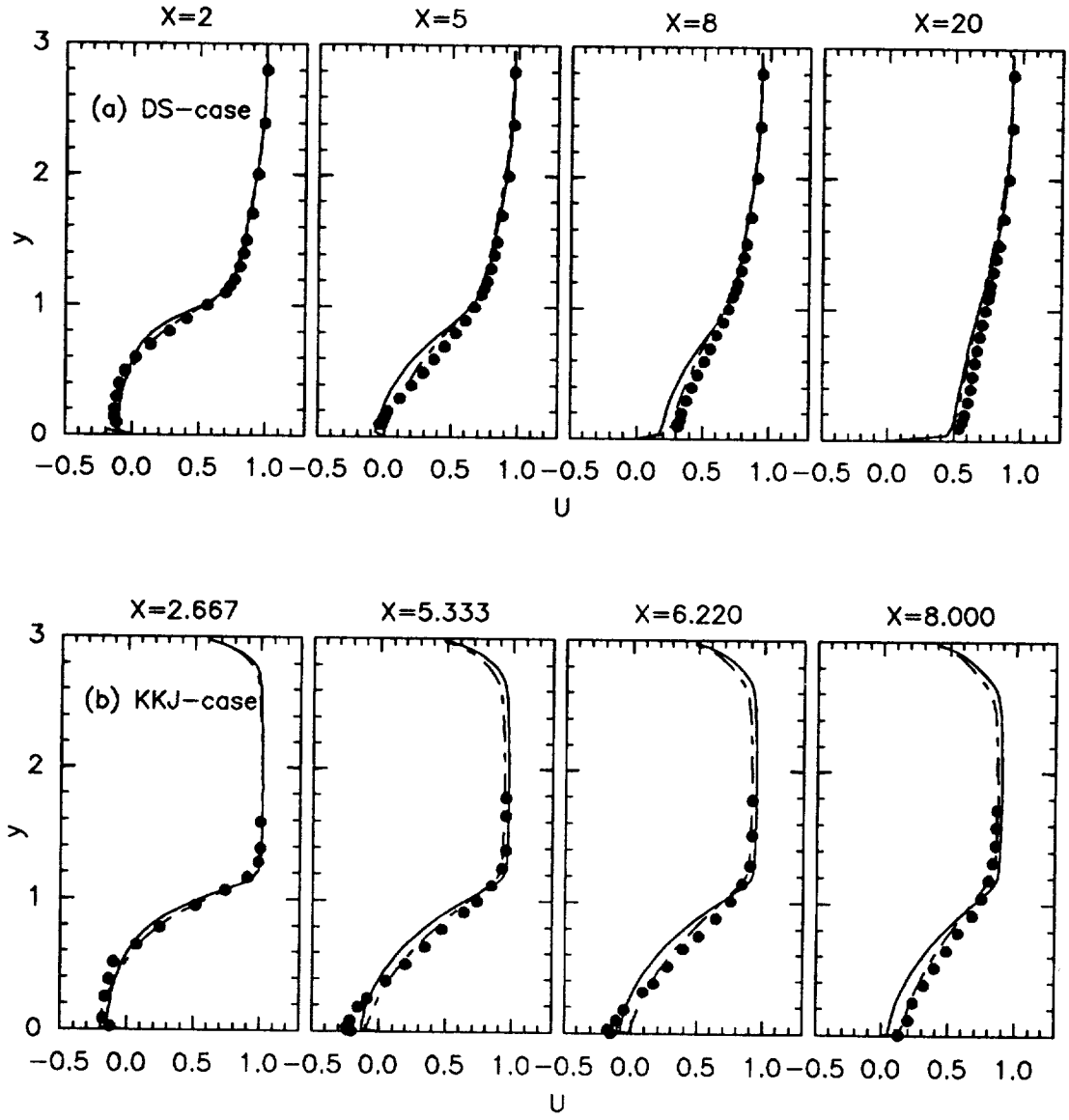

Figure 7. Streamwise mean velocity U-profiles.
— - —: s-K-ϵ; ——: present model; •: experiment

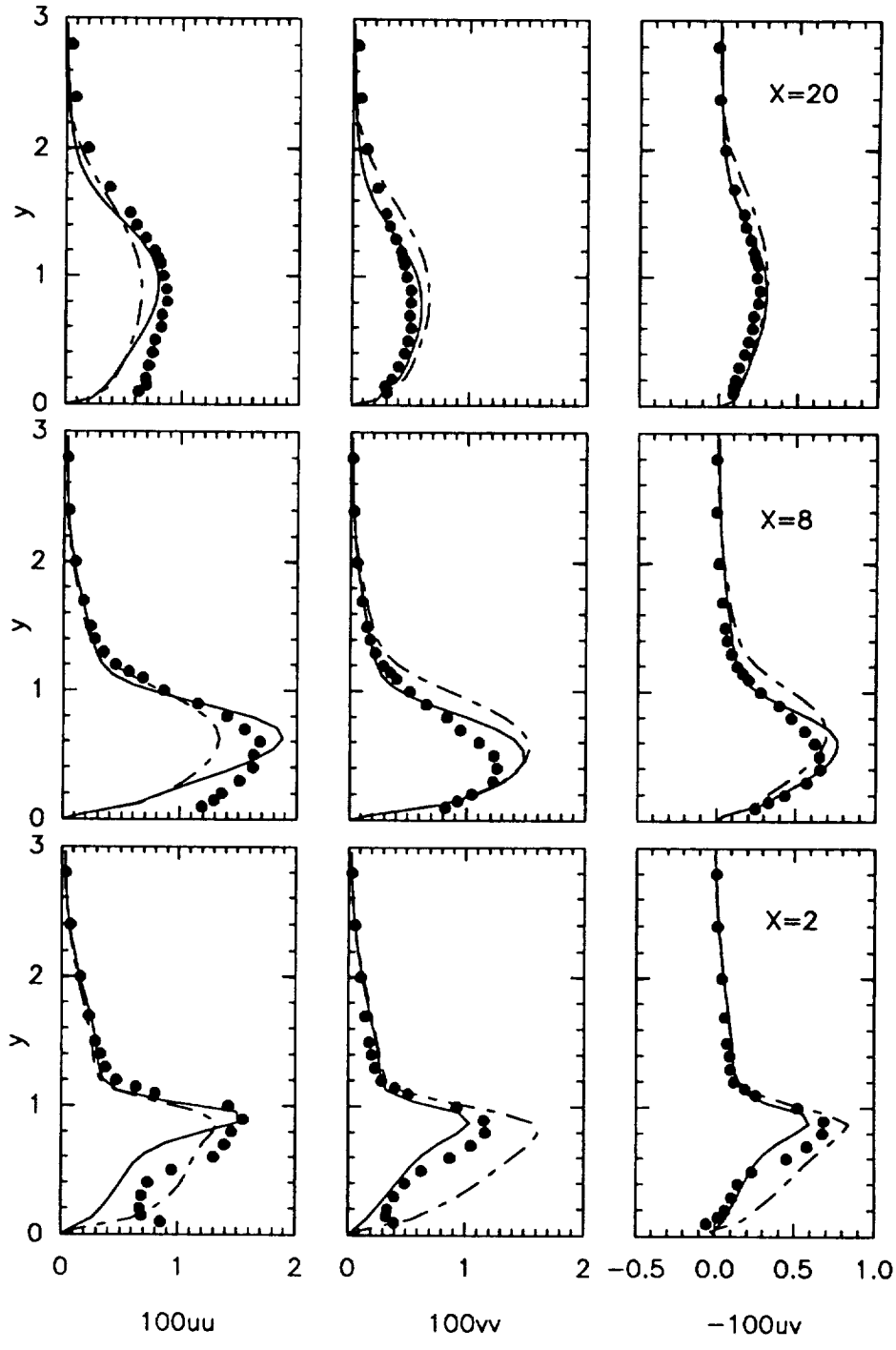

Figure 8. Turbulent stress profiles in DS-case.
— - —: s-K-ϵ; ——: present model; •: experiment

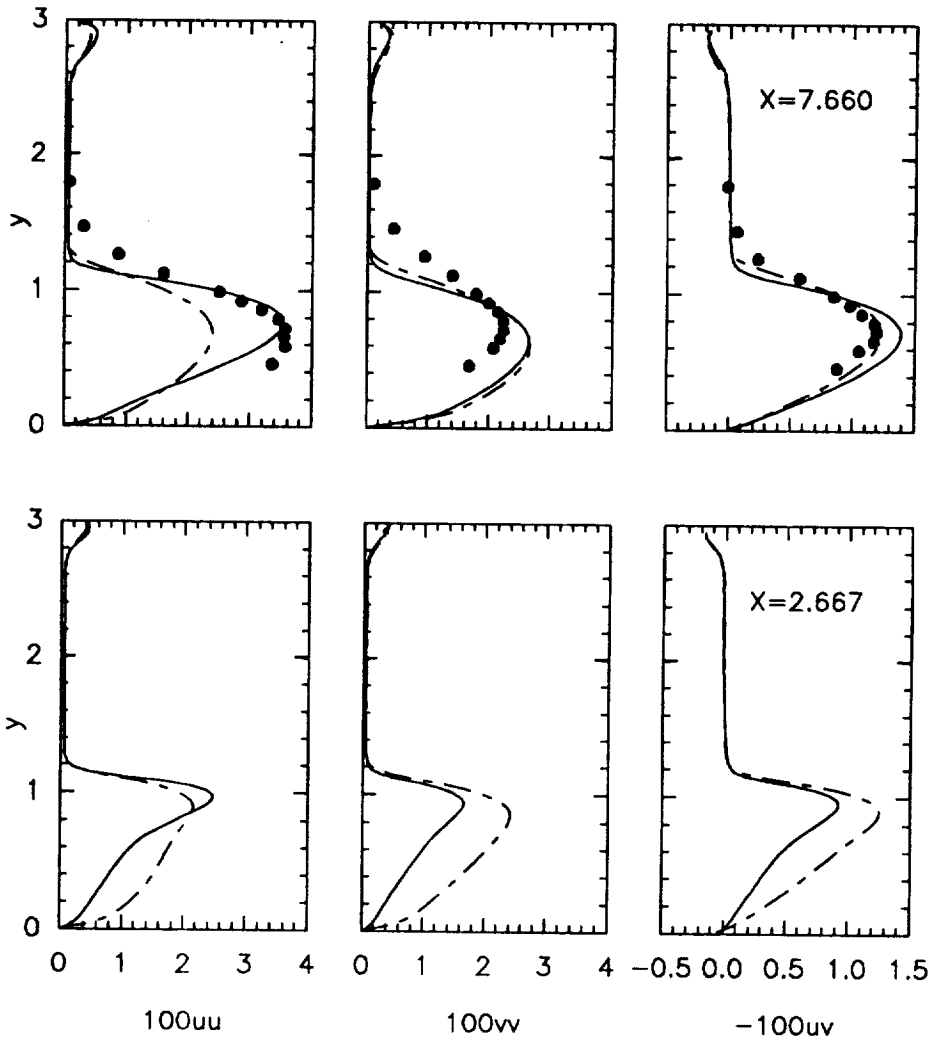

Figure 9. Turbulent stress profiles in KKJ-case.
— - —: s-K-ϵ; ——: present model; •: experiment

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13. ABSTRACT ***(Maximum 200 words)***

The invariance theory in continuum mechanics is applied to analyze Reynolds stresses in high Reynolds number turbulent flows. The analysis leads to a turbulent constitutive relation that relates the Reynolds stresses to the mean velocity gradients in a more general form in which the classical isotropic eddy viscosity model is just the linear approximation of the general form. On the basis of realizability analysis, a set of model coefficients are obtained which are fuctions of the time scale ratios of the turbulence to the mean strain rate and the mean rotation rate. The coefficients will ensure the positivity of each component of the mean rotation rate. These coefficients will ensure the positivity of each component of the turbulent kinetic energy -- realizability that most existing turbulence models fail to satisfy. Separated flows over backward-facing step configurations are taken as applications. The calculations are performed with a conservative finite-volume method. Grid-independent and numerical diffusion-free solutions are obtained by using differencing schemes of second-order accuracy on sufficiently fine grids. The calculated results are compared in detail with the experimental data for both mean and turbulent quantities. The comparison shows that the present proposal significantly improves the predictive capability of K-ε based two equation models. In addition, the proposed model is able to simulate rotational homogeneous shear flows with large rotation rates which all conventional eddy viscosity models fail to simulate.

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