PASSIVITY/LYAPUNOV BASED CONTROLLER DESIGN FOR TRAJECTORY TRACKING OF FLEXIBLE JOINT MANIPULATORS

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Abstract

A passivity and Lyapunov based approach for the control design for the trajectory tracking problem of flexible joint robots is presented. The basic structure of the proposed controller is the sum of a model-based feedforward and a model-independent feedback. Feedforward selection and solution is analyzed for a general model for flexible joints, and for more specific and practical model structures. Passivity theory is used to design a motor state-based controller in order to input-output stabilize the error system formed by the feedforward. Observability conditions for asymptotic stability are stated and verified. In order to accommodate for modeling uncertainties and to allow for the implementation of a simplified feedforward compensation, the stability of the system is analyzed in presence of approximations in the feedforward by using a Lyapunov based robustness analysis. It is shown that under certain conditions, e.g. the desired trajectory is varying slowly enough, stability is maintained for various approximations of a canonical feedforward.

1 Introduction

Joint flexibility is well recognized for its adverse effect on stability and performance of robotic manipulators [23, 28]. The main implications of joint flexibility are that the number of degrees of freedom is larger than the number of inputs, and that the system is not passive from the torque input to the link velocity as for rigid robots such that most of the control schemes designed for rigid robots are inappropriate for the control of flexible joint robots. Different approaches have been considered to solve the problem of controlling robots with joint elasticity including singular perturbation techniques, exact linearization and passivity based design. Singular perturbation techniques and exact linearization generally require linear spring assumption and exact knowledge of the system parameters, and are characterized by their computational complexity and their lack of robustness to parameter uncertainty (for a summary, see [27]). Furthermore, exact linearization requires zero gyroscopic force, and the feedforward compensation for linearization and the feedback stabilization are intertwined and errors in the feedforward may affect the closed loop stability in an adverse way.

The concept of passivity is traditionally defined as an input/output condition describing a class of physical systems that do not generate energy [18]. This property has been used in the feedback stabilization for fully actuated rigid robots [29], satellites [6], and flexible joint robots [4, 31]. The passivity property for flexible joint robots (motor torque and motor velocity form a passive pair) was recognized in [4] and was used in a proportional-derivative (PD) type controller design. The method requires inherent damping in both links and motors. Similar results without requiring the inherent damping have recently appeared in [31]. The PD controller has been generalized to a general passive controller in [22]. This method also requires inherent damping and furthermore, linear spring assumption. Moreover, a frequency analysis that yields a non-causal solution is used to find the feedforward. The design of general passive controllers without the requirement of inherent damping and of the linear spring assumption, and that uses causal feedforward was presented in [15, 24]. This approach allows to consider both the set point and tracking problems, and does not require a large elastic coupling stiffness as singular perturbation techniques does.

The basic structure of the proposed controller is the sum of a model-based feedforward and a model-independent feedback [15, 24]. The feedforward design in the proposed scheme is very similar to the exact linearization approach, both essentially solve an inverse plant problem, but requires much less model information in the set point control case, and the additive separation between the feedback and feedforward implies that errors in the feedforward do not lead to instability. However, the closed loop performance cannot be arbitrarily assigned and the feedforward for the tracking problem may be complex. We address the issue of computing a causal feedforward for a nonlinear spring.

Passivity based controllers have been recognized for their robustness to parameter uncertainties [1, 31]. This robustness may allow to simplify the control law for implementation as opposed to exact linearization techniques for which simplifications may compromise stability. We will analyze the stability of the system under feedforward approximations.

This paper addresses the following trajectory tracking problem for flexible joint robots:

> Given the desired output trajectory described by $\theta_d(t)$ and its derivatives, $t \geq t_0$, design a feedback control law $u(t)$ so that $\theta(t)$ tracks $\theta_d(t)$ in some sense while assuring internal stability of the system.

In Section 2, background information on passivity, and useful definitions and theorems are provided. In Section 3, the model of the class of systems that is considered and some of its properties are presented. The proposed approach for controller design which exploits the passivity property of flexible joint robots is exposed in Section 4. In Section 5, various possible forms of feedforward for the trajectory tracking problem are presented and the issue of solving the feedforward is addressed. In Section 6, the controller design is carried out for particular choices of feedforward, referred to as the canonical feedforward compensation schemes, under the assumption that certain conditions are satisfied. Zero-state detectability of the canonical feedforward compensation schemes is analyzed in Section 7. In Section 8, the stability of different possible forms of feedforward is analyzed by using a Lyapunov based robustness analysis. Conclusions are drawn and future work is summarized in Section 9.

2 Passivity and definitions

The notion of passivity of an input-output system, motivated by the dissipation of energy across resistors in an electrical circuit, has been widely used in order to analyze stability of a general class of interconnected nonlinear systems, e.g. [32, 34]. Passivity was also studied for state-space representations of nonlinear systems, allowing a more geometric interpretation of notions such as available, stored, and dissipated energy in terms of Lyapunov functions [38, 39, 40]. This point of view has been specifically developed in [10, 11], and leads to Lyapunov-theoretic counterparts to many stability results developed within an input-output perspective, as well as to a nonlinear form of the Kalman-Yacubovitch-Popov lemma. The great interest of considering passive (strictly passive) systems lies in the fact that passivity is invariant under feedback connections and that passive systems are always stable, and under additional assumptions asymptotically stable [17, 18, 38, 41].

In [18], necessary and sufficient conditions for a nonlinear autonomous system with the control entering linearly to be passive are given. It is well known that passive linear systems are necessarily minimum phase and, conversely, a minimum phase plant with relative degree zero or one can be rendered passive via a static state feedback. A similar relationship for nonlinear systems has recently been published [5]. This allows to extend the class of systems to which the stability results for passive systems applies.

We now present definitions and theorems related to passivity and to the stabilization of passive systems. The information presented here was mainly drawn from [5, 10, 11, 18, 33, 38, 39, 41, 42, 43]. Other papers of interest that treat positive realness, which is a subcase of passivity, are [2, 12, 37].

Definition 2.1 [33] *A function $f : \mathbf{R}_+ \rightarrow \mathbf{R}^n$ is an element of the vector space $L_p(\mathbf{R}_+, \mathbf{R}^n)$ if*

$$\int_{-\infty}^{\infty} |f(t)|^p \, dt < \infty \qquad \text{for } p \in [1, \infty)$$

Define the input and output signal spaces, $\mathcal{U}_e$, $\mathcal{Y}_e$, respectively, as the extended spaces $L_{2e}(\mathbf{R}_+, \mathbf{R}^n)$, i.e. the causal extension of $L_2(\mathbf{R}_+, \mathbf{R}^n)$:

$$L_{2e}(\mathbf{R}_+, \mathbf{R}^n) \triangleq \{f : \mathbf{R}_+ \rightarrow \mathbf{R}^n \mid P_T f \in L_2(\mathbf{R}_+, \mathbf{R}^n), \forall T \in \mathbf{R}\}$$

where P_T is the projection operator that truncates f at T:

$$(P_T f)(t) \triangleq \begin{cases} f(t) & \text{for } t \leq T \\ 0 & \text{otherwise} \end{cases}$$

Define the truncated inner product by

$$\langle u(\cdot), v(\cdot) \rangle_T \triangleq \langle P_T u(\cdot), P_T v(\cdot) \rangle_2 = \int_0^{\infty} (P_T u(t))^T P_T v(t) \, dt$$

By a dynamical system, we mean an I/O mapping $H : \mathcal{U}_e \rightarrow \mathcal{Y}_e$. The following definitions are given for $\mathcal{U}_e$, $\mathcal{Y}_e$ being extended L_p-spaces, but we will use uniquely L_2-spaces in the remaining of the report.

Definition 2.2 [33] *For $p \in [1, \infty)$, the function $\|\cdot\|_p : L_p[0, \infty) \rightarrow [0, \infty)$ is defined by*

$$\|f(\cdot)\|_p = \left[\int_0^{\infty} |f(t)|^p \, dt \right]^{\frac{1}{p}}$$

The function $\|\cdot\|_\infty : L_\infty[0, \infty) \rightarrow [0, \infty)$ is defined by

$$\|f(\cdot)\|_\infty = ess \sup_{t \in [0, \infty)} |f(t)|$$

Definition 2.3 [33] *A dynamical system H is L_p-stable if*

(i) *$y \in L_p$ whenever $u \in L_p$, and*

(ii) *There exist finite constants k, b such that*

$$\|y\|_p \leq k \|u\|_p + b \qquad \forall u \in L_p$$

Definition 2.4 [33] *A dynamical system H is L_∞-stable (Bounded Input–Bounded Output-stable, BIBO stable) if*

(i) *$y \in L_p$ whenever $u \in L_p$, and*

(ii) *There exist finite constants k, b such that*

$$\|y\|_\infty \leq k \|u\|_\infty + b \qquad \forall u \in L_p$$

Different definitions for passivity, which characterizes systems that do not generate energy, have been used [43]. We will adopt one of these definitions, but first will define the more general concept of dissipativity [10, 11].

Definition 2.5 [11] *A dynamical system H is dissipative with respect to the triplet (Q, S, R) if*

$$w(u, y) = \langle y, Qy\rangle_T + 2\,\langle y, Su\rangle_T + \langle u, Ru\rangle_T \geq 0$$

for all $T \geq 0$ and $u \in \mathcal{U}_e$, where $w(u, y)$ is defined as the supply rate, and Q, S and R are memoryless bounded operators with Q and R self-adjoint.

Definition 2.6 [43] *A dynamical system H storing no energy at $t = 0$ is passive if for all $T \in \mathrm{R}_+$ and all admissible pairs $\{u(\cdot), y(\cdot)\} \in \{\mathcal{U}_e, \mathcal{Y}_e\}$,*

$$\langle y, u\rangle_T \geq 0$$

The quantity $\langle y, u\rangle_T$ is sometimes defined as the input energy of the system [18]. This nomenclature will be used here.

Definition 2.7 [38] *A dynamical system H storing no energy at $t = 0$ is strictly passive if for all $T \in \mathrm{R}_+$ and all admissible pairs $\{u(\cdot), y(\cdot)\} \in \{\mathcal{U}_e, \mathcal{Y}_e\}$, $H - \varepsilon I$ is passive for some real constant $\varepsilon > 0$, i.e. if*

$$\langle y, u\rangle_T - \varepsilon\,\langle u, u\rangle_T \geq 0$$

Hence, a finite-gain I/O stable system is dissipative with respect to $(-I, 0, k^2 I)$, while a passive system is dissipative with respect to $(0, \frac{1}{2}I, 0)$ and a strictly-passive system is dissipative with respect to $(0, \frac{1}{2}I, -\varepsilon)$ [11]. This definition of strict-passivity corresponds to the definition of U-strong-passivity (USP) in [10]. Two additional forms of strong passivity as defined in [10] are Y-strong-passivity (YSP) which corresponds to systems that are dissipative with respect to $(-\varepsilon, \frac{1}{2}I, 0)$ for some real constant $\varepsilon > 0$, and Very-strong-passivity (VSP) which corresponds to systems that are dissipative with respect to $(-\varepsilon_1, \frac{1}{2}I, -\varepsilon_2)$ for some real constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

In [38, 39], different energy quantities are defined. The required energy E_r is the energy needed to excite a system to a given set of initial conditions. The available energy E_a is the maximum energy that can be extracted from a system. The cycle energy E_c is the minimum energy it takes to cycle a system between the equilibrium and a given state. Under mild conditions, these quantities are well defined for a passive system :

Lemma 2.1 [38] *Consider a realization of a passive system and assume that the state space is reachable. Then E_a, E_r and E_c exist (i.e. $E_a, E_r, E_c < \infty$) and are nonnegative. Moreover, $0 \leq E_a$, $E_c \leq E_r$.*

In principle, these quantities can be used to construct Lyapunov functions as illustrated by the following theorem.

Theorem 2.1 [39] *The set of possible storage functions of a dissipative dynamical system forms a convex set. Hence, $\alpha E_a + (1-\alpha)E_r$, $0 \leq \alpha \leq 1$, is a possible storage function for a dissipative dynamical system whose state space is reachable from the point in space where the storage function attains its minimum.*

An important characteristic of the systems studied here is that they satisfy the following conservation of energy equation [17, 18]:

$$\text{Input energy} = \text{Final energy - Initial energy + Dissipated energy}$$

where, for passive systems, the dissipated energy is always nonnegative.

In [10], a necessary and sufficient condition for a nonlinear system with the control input entering linearly to be dissipative is given. The test involves the construction of some functions, including a function representing the generalized energy of the system. This function is positive for dissipative systems such that this condition can be used as an aid for the construction of a Lyapunov function for the system.

Input/output and state space stability and stabilization of passive systems have been extensively studied , and we now present some of these results, but first provide two more definitions.

Definition 2.8 *A dynamical system H is said to be zero–state detectable if $u(t) \equiv 0$ and $y(t) \equiv 0$ imply that the state $x(t) \equiv 0$.*

For linear systems, this corresponds to observability.

We introduce here a weaker form of zero–state detectability.

Definition 2.9 *A dynamical system H is said to be weakly zero–state detectable if $u(t) \equiv 0$ and $y(t) \equiv 0$ imply that the state $x(t) \rightarrow 0$ asymptotically.*

There is no consensus actually in the literature about the nomenclature regarding zero–state detectability. Hence, in [5], Definition 2.8 corresponds to observability, and Definition 2.9 to zero–state detectability.

Theorem 2.2 [10] *Consider a dynamical system H which is dissipative with respect to the triplet (Q, S, R) and zero–detectable. Then, the system with zero input is Lyapunov stable if $Q \leq 0$ and asymptotically stable if $Q < 0$.*

Hence, passive and strictly–passive (USP) systems are stable while YSP and VSP systems are asymptotically stable.

The same conclusions hold for weakly zero–detectable systems.

An important characteristic of passivity is that it is invariant under feedback connections [39, 41]. Hence, a feedback system consisting of a passive dynamical system in both the feedforward and feedback loop is itself passive and thus stable. Moreover, the sum of the stored energies in the forward loop and in the feedback loop is a Lyapunov function for the closed loop system. Hence, the Lyapunov function used to show passivity of the two system components may be used as part of the overall Lyapunov function used to show the stability of the system. This procedure to show stability is formalized in [41].

We now state a simplified version of a very important theorem in the stabilization of passive systems, the *Passivity Theorem*, which provides sufficient conditions to determine stability of the interconnection of systems.

Theorem 2.3 *Passivity Theorem* [9]: *The system formed by the negative feedback connection of a passive dynamical system and of a strictly passive dynamical system with finite gain is L_2-stable.*

Hence, any passive system can be rendered L_2–stable by closing a strictly passive loop with finite gain. I/O stability infers internal state space asymptotic stability if the closed-loop system is stabilizable and zero–state detectable (if these properties hold globally, the internal stability is also global). Hence, under observability (zero–state detectability) and reachability conditions, the interconnection of a passive and a finite gain strictly passive system is asymptotically stable [38, 41]. Also, it is sometimes possible to show via a Lyapunov type argument that $y(t) \rightarrow 0$ asymptotically if $u(t) \equiv 0$. Then the zero–state detectability alone guarantees internal asymptotic stability.

It is well known that passive linear systems are necessarily minimum phase and, conversely, a minimum phase plant with relative degree zero or one can be rendered passive via a static state feedback (see [5] for a particular form of this statement). A similar relationship for nonlinear systems has recently been published in [5]. It is shown that a nonlinear system can be rendered passive via static feedback, i.e. it is feedback equivalent to a passive feedback, if and only if the system is weakly minimum phase and the relative degree is one (see Appendix B for the formal statement). An equivalent statement for this is that the system is asymptotically stabilizable by state feedback [5]. Hence, the class of systems to which stabilization by passivity approach applies can be extended by using this result.

3 Modeling of flexible joint manipulators

The dynamical equation of motion for flexibly jointed manipulators with rigid links can be written as:

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + D(\dot{\theta}) + r(\theta) = Bu \tag{3.1}$$

$$r(\theta) = g(\theta) + k(\theta) \tag{3.2}$$

where $\theta \in \mathbf{R}^n$ is the displacement vector, $u \in \mathbf{R}^m$ is the input force vector, M is the mass–inertia matrix, D is the viscous damping and Coulomb friction, C corresponds to the centrifugal and Coriolis forces, g is the gravity force, and k represents the spring coupling force. Note that in model (3.1), it is assumed that no external force is exerted on the manipulator, and that joint and link friction is position independent. We will consider manipulators with all its joints exhibiting flexibility and with each link being actuated, i.e. $\theta = \begin{bmatrix} \theta_\ell^T & \theta_m^T \end{bmatrix}^T$ where $\theta_\ell, \theta_m \in \mathbf{R}^m$ are respectively the link and motor displacement vectors, and $n = (2 \cdot m)$. These assumptions are not essential but will allow to obtain a more concise presentation.

We define the output of interest as

$$y = h(\theta) \tag{3.3}$$

Usually y is a function uniquely of the link state.

3.1 Properties of the system

We now state some properties of the system that will be useful in this paper.

$$C(\theta,\dot{\theta})\dot{\theta} \triangleq \left\{ \dot{M}(\theta,\dot{\theta}) - \frac{1}{2}\left[\frac{\partial\left(M(\theta)\dot{\theta}\right)}{\partial\theta}\right]^T \right\} \dot{\theta}$$

$\dot{M}$ is used to denote the derivative of $M(\theta)$ along the solution, i.e.

$$\dot{M}(\theta,\dot{\theta}) \triangleq \sum_{i=1}^{n} \frac{\partial M(\theta)}{\partial\theta_i}\dot{\theta}_i \tag{3.4}$$

While $C(\theta,\dot{\theta})\dot{\theta}$ is unique, $C(\theta,\dot{\theta})$ is not. However, there is a close relationship between $C(\theta,\dot{\theta})$ and $M(\theta)$: for any choice of $C(\theta,\dot{\theta})$ and for any $z \in \mathbf{R}^n$,

$$z\left(\frac{1}{2}\dot{M}(\theta,z) - C(\theta,z)\right)z = 0$$

Two frequently used representations for C will be considered.

Representation of C using Christoffel's symbols : In this representation,

$$\begin{aligned} C_{kj} &\triangleq \sum_{i=1}^{n} c_{ijk}(\theta)\dot{\theta}_i \\ &= \sum_{i=1}^{n} \frac{1}{2}\left[\frac{\partial M_{kj}}{\partial\theta_i} + \frac{\partial M_{ki}}{\partial\theta_j} - \frac{\partial M_{ij}}{\partial\theta_k}\right]\dot{\theta}_i \end{aligned}$$

For this representation, and this representation only, $\left(\frac{1}{2}\dot{M}(\theta,z) - C(\theta,z)\right)$ is skew–symmetric [21].

Representation of C using M_D–notation : First define a matrix M_D that depends on two vector arguments:

$$M_D(\theta, v) \triangleq \sum_{i=1}^{n} \frac{\partial M(\theta)}{\partial \theta_i} v e_i^T \tag{3.5}$$

where e_i is the i^{th} unit vector in $\mathbf{R}^n$. M_D and $\dot{M}$ are related as follows:

$$M_D(\theta, v)w = \dot{M}(\theta, w)v \tag{3.6}$$

M_D also satisfies

$$M_D^T(\theta, v)w = M_D^T(\theta, w)v \tag{3.7}$$

$C(\theta, \dot{\theta})$ can be expressed succinctly as

$$C(\theta, \dot{\theta}) \triangleq M_D(\theta, \dot{\theta}) - \frac{1}{2} M_D^T(\theta, \dot{\theta}) \tag{3.8}$$

The Coriolis and centrifugal coefficient matrix C will be represented by C to denote an arbitrary choice of representation unless noted otherwise, by C_C to denote the representation by the Christoffel's symbols, and by C_D to denote the representation by the M_D–notation.

Also, it is always possible to find a vector function $\bar{r}(\theta)$ such that $r(\theta)$ can be factored as

$$r(\theta) = \bar{r}(\theta) + R\theta \tag{3.9}$$

for any user defined constant matrix $R \in \mathbf{R}^{n \times n}$. R will be represented as

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

where $R_{11}, R_{12}, R_{21}, R_{22} \in \mathbf{R}^{m \times m}$.

We will also represent r as

$$r(\theta) = \begin{bmatrix} r_1(\theta) \\ r_2(\theta) \end{bmatrix}$$

where $r_1, r_2 \in \mathbf{R}^{m \times 1}$. Similarly,

$$\bar{r}(\theta) = \begin{bmatrix} \bar{r}_1(\theta) \\ \bar{r}_2(\theta) \end{bmatrix}$$

where $\bar{r}_1, \bar{r}_2 \in \mathbf{R}^{m \times 1}$.

The friction term D can also be represented by the sum of a linear component D_0 and of a nonlinear term D_1 as follows

$$D(\dot{\theta}) = D_0\dot{\theta} + D_1(\dot{\theta})$$

3.2 Structure of the model

It is well recognized that the control of flexible joint robots is highly dependent upon the structure of the model, which depends on various factors including the kinematic arrangement of the links and the way motors are mounted [7, 14, 20, 27, 35]. The following standard modeling assumptions have been used in the literature [7, 20, 26, 27, 30] :

- Linear spring assumption.

- Motor inertia is symmetric about motor axis of rotation (M and gravity forces only depend on θ_ℓ).
- Kinetic energy of each motor is due mainly to its own rotation, or motion of motor is a pure rotation with respect to an inertial frame (Neglect gyroscopic effects; $M_{12} = M_{12}^T = 0$).

None of these assumptions will be used *a priori*. However, we present here the structure of the model that is obtained under some standard assumptions and for specific manipulator structures.

3.2.1 General model

For a general flexible joint manipulator, equation (3.1) can be expanded as [8, 20]:

$$\begin{bmatrix} M_{11}(\theta_\ell,\theta_m) & M_{12}(\theta_\ell,\theta_m) \\ M_{12}^T(\theta_\ell,\theta_m) & M_{22}(\theta_m) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_\ell \\ \ddot{\theta}_m \end{bmatrix} + \begin{bmatrix} C_{11}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m) & C_{12}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m) \\ C_{21}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m) & C_{22}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m) \end{bmatrix} \begin{bmatrix} \dot{\theta}_\ell \\ \dot{\theta}_m \end{bmatrix} + \begin{bmatrix} D_\ell(\dot{\theta}_\ell) \\ D_m(\dot{\theta}_m) \end{bmatrix} + \begin{bmatrix} r_1(\theta_\ell,\theta_m) \\ r_2(\theta_\ell,\theta_m) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} u \tag{3.10}$$

$$\begin{bmatrix} r_1(\theta_\ell,\theta_m) \\ r_2(\theta_\ell,\theta_m) \end{bmatrix} = \begin{bmatrix} g_\ell(\theta_\ell,\theta_m) \\ g_m(\theta_\ell,\theta_m) \end{bmatrix} + \begin{bmatrix} Nk_1(\theta_\ell,\theta_m) \\ -k_1(\theta_\ell,\theta_m) \end{bmatrix} \tag{3.11}$$

where $\theta_\ell, \theta_m \in \mathbf{R}^m$ are the link and motor displacement vectors respectively with elements numbered from base to tip, link m being the last link; $u \in \mathbf{R}^m$ is the input force vector; M_{11} is the mass–inertia matrix of the links including the mass of the motors mounted on the links as if they were rigidly attached to the links; M_{22} is the motor inertia matrix; M_{12} represents the coupling or interaction matrix that gives the dynamic coupling between the motor and link accelerations; M_{12}^T is the counterpart that gives the interaction between the link and motor accelerations; D_ℓ and D_m are respectively the torques due to link and motor viscous and Coulomb friction (the effect of friction in the transmission element is neglected); C_{11} and C_{12} represent the coefficients of the torque acting on the link shafts due to centrifugal and Coriolis forces; C_{12}^T and C_{22} represent the coefficients of the torque acting on the motor shafts due to centrifugal and Coriolis forces; g_ℓ and g_m are the gravity forces acting respectively upon the link and motor shafts; k_1 represents the spring coupling forces; N is the matrix of gear ratios; and $0, I \in \mathbf{R}^{m \times m}$ are respectively the zero and the identity matrices.

3.2.2 M_{12} nonsingular

This condition may arise when the stator of motor i, $i = 1, \cdots, m$, is mounted on the link it drives (rotor elastically coupled to link $i-1$), or on a link higher in the chain, i.e. on a link j_i, $j_i > i$, for certain kinematic configurations such as planar robots with the axis of rotation of the motors and links all coplanar. The system is then represented by the same equations as the general model, i.e. (3.10, 3.11).

Subcase : Symmetric motor inertia

If the inertia of the motors is symmetric about their axis of rotation, then (3.10,3.11) become

$$\begin{bmatrix} M_{11}(\theta_\ell) & M_{12}(\theta_\ell) \\ M_{12}^T(\theta_\ell) & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_\ell \\ \ddot{\theta}_m \end{bmatrix} + \begin{bmatrix} C_{11}(\theta_\ell,\dot{\theta}_\ell,\dot{\theta}_m) & C_{12}(\theta_\ell,\dot{\theta}_\ell) \\ C_{21}(\theta_\ell,\dot{\theta}_\ell) & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_\ell \\ \dot{\theta}_m \end{bmatrix} + \begin{bmatrix} D_\ell(\dot{\theta}_\ell) \\ D_m(\dot{\theta}_m) \end{bmatrix} + \begin{bmatrix} r_1(\theta_\ell,\theta_m) \\ r_2(\theta_\ell,\theta_m) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} u \tag{3.12}$$

$$\begin{bmatrix} \tau_1(\theta_\ell, \theta_m) \\ \tau_2(\theta_\ell, \theta_m) \end{bmatrix} = \begin{bmatrix} g_\ell(\theta_\ell) \\ 0 \end{bmatrix} + \begin{bmatrix} N k_1(\theta_\ell, \theta_m) \\ -k_1(\theta_\ell, \theta_m) \end{bmatrix} \tag{3.13}$$

3.2.3 M_{12} strictly upper triangular

The matrix M_{12} may be singular but non–zero when the energy of at least one motor is due to its own rotation. Also, a common configuration yielding this condition is when the stator of motor i is mounted on link $i-1$ while its rotor is elastically attached to link i for $i = 1, \cdots, m$ (the stator of motor 1 is mounted in the inertial frame). In a more general case, the stator of motor i is mounted on link j_i while its rotor is elastically attached to link i for $j_i < i$ and $i = 1, \cdots, m$ (This leads to $M_{12} = 0$ if all the motors are mounted in the inertial frame). It has been independently pointed out in [19] and [31] (the former is for the exact case) that for that case, M_{12} is strictly upper triangular.

We now present the structure of the model with symmetric motors for M_{12} strictly upper triangular where M_{12} is state dependent or constant.

Subcase 1 : Symmetric motor inertia and M_{12} strictly upper triangular

In this case, the system dynamics are described by (3.12, 3.13) with the Coriolis and centrifugal matrix having the following structure:

$$C(\theta_\ell, \dot\theta_\ell, \dot\theta_m) = \begin{bmatrix} C_{11}^A(\theta_\ell, \dot\theta_\ell) + C_{11}^B(\theta_\ell, \dot\theta_m) & C_{12}^A(\theta_\ell, \dot\theta_\ell) \\ C_{21}^A(\theta_\ell, \dot\theta_\ell) & 0 \end{bmatrix}$$

where C_{11}^A is the Coriolis and centrifugal term for the rigid robot and the structure of the other terms is described below (see Appendix A for the complete set of equations).

It was pointed out in [31] that M_{12} has the following structure:

$$M_{12}(\theta_\ell) = \begin{bmatrix} 0 & m_{1,2}(\theta_{\ell,1}) & m_{1,3}(\theta_{\ell,1}, \theta_{\ell,2}) & \cdots & m_{1,m}(\theta_{\ell,1}, \cdots, \theta_{\ell,m-1}) \\ 0 & 0 & m_{2,3}(\theta_{\ell,2}) & \cdots & m_{2,m}(\theta_{\ell,2}, \cdots, \theta_{\ell,m-1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $m_{i,j}$ are scalar functions. This particular structure for M_{12} yields that (Appendix A)

- C_{11}^B is skew–symmetric with row m equal to zero and row i independent of any $\dot\theta_{m,j}$, $j \le i$, $i = 1, \cdots, m-1$,
- C_{12}^A is strictly upper triangular,
- C_{21}^A is strictly lower triangular.

Subcase 2 : Symmetric motor inertia, and M_{12} constant and strictly upper triangular

Matrix M_{12} may be constant due to the configuration of the manipulator and due to the position (orientation in particular) of the motors on the links. This is the case when the axis of rotation of each motor is coplanar with the axis of rotation of the link that it drives, and that the axis of rotation of the links are mutually perpendicular (possible for up to three links, unless some of the motors are mounted in the inertial frame yielding weaker constraints and allowing for a larger number of links).

In this case, the system dynamics are described by

$$\begin{bmatrix} M_{11}(\theta_\ell) & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \begin{bmatrix} \ddot\theta_\ell \\ \ddot\theta_m \end{bmatrix} + \begin{bmatrix} C_{11}(\theta_\ell, \dot\theta_\ell) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_\ell \\ \dot\theta_m \end{bmatrix} + \begin{bmatrix} D_\ell(\dot\theta_\ell) \\ D_m(\dot\theta_m) \end{bmatrix} +$$

$$\begin{bmatrix} r_1(\theta_\ell,\theta_m) \\ r_2(\theta_\ell,\theta_m) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} u \quad (3.14)$$

$$\begin{bmatrix} r_1(\theta_\ell,\theta_m) \\ r_2(\theta_\ell,\theta_m) \end{bmatrix} = \begin{bmatrix} g_\ell(\theta_\ell) \\ 0 \end{bmatrix} + \begin{bmatrix} Nk_1(\theta_\ell,\theta_m) \\ -k_1(\theta_\ell,\theta_m) \end{bmatrix} \quad (3.15)$$

where C_{11} is the Coriolis and centrifugal term for the rigid robot.

3.2.4 $M_{12} = 0$

There is no gyroscopic coupling when the energy of each motor is due to its own rotation, e.g. the motors are mounted in an inertial frame, or all the motor axis are perpendicular to the link axis [20].

In this case, the dynamics of the system are described by

$$\begin{bmatrix} M_{11}(\theta_\ell,\theta_m) & 0 \\ 0 & M_{22}(\theta_m) \end{bmatrix} \begin{bmatrix} \ddot\theta_\ell \\ \ddot\theta_m \end{bmatrix} + \begin{bmatrix} C_{11}(\theta_\ell,\theta_m,\dot\theta_\ell) & C_{12}(\theta_\ell,\theta_m,\dot\theta_\ell) \\ C_{21}(\theta_\ell,\theta_m,\dot\theta_\ell) & C_{22}(\theta_m,\theta_m) \end{bmatrix} \begin{bmatrix} \dot\theta_\ell \\ \dot\theta_m \end{bmatrix} + \begin{bmatrix} D_\ell(\dot\theta_\ell) \\ D_m(\dot\theta_m) \end{bmatrix} + \begin{bmatrix} r_1(\theta_\ell,\theta_m) \\ r_2(\theta_\ell,\theta_m) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} u \quad (3.16)$$

$$\begin{bmatrix} r_1(\theta_\ell,\theta_m) \\ r_2(\theta_\ell,\theta_m) \end{bmatrix} = \begin{bmatrix} g_\ell(\theta_\ell,\theta_m) \\ g_m(\theta_\ell,\theta_m) \end{bmatrix} + \begin{bmatrix} Nk_1(\theta_\ell,\theta_m) \\ -k_1(\theta_\ell,\theta_m) \end{bmatrix} \quad (3.17)$$

Subcase : Symmetric motor inertia

If the inertia of the motors is symmetric about their axis of rotation, (3.16,3.17) become

$$\begin{bmatrix} M_{11}(\theta_\ell) & 0 \\ 0 & M_{22} \end{bmatrix} \begin{bmatrix} \ddot\theta_\ell \\ \ddot\theta_m \end{bmatrix} + \begin{bmatrix} C_{11}(\theta_\ell,\dot\theta_\ell) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_\ell \\ \dot\theta_m \end{bmatrix} + \begin{bmatrix} D_\ell(\dot\theta_\ell) \\ D_m(\dot\theta_m) \end{bmatrix} + \begin{bmatrix} r_1(\theta_\ell,\theta_m) \\ r_2(\theta_\ell,\theta_m) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} u \quad (3.18)$$

$$\begin{bmatrix} r_1(\theta_\ell,\theta_m) \\ r_2(\theta_\ell,\theta_m) \end{bmatrix} = \begin{bmatrix} g_\ell(\theta_\ell) \\ 0 \end{bmatrix} + \begin{bmatrix} Nk_1(\theta_\ell,\theta_m) \\ -k_1(\theta_\ell,\theta_m) \end{bmatrix} \quad (3.19)$$

This model has the same structure as used by Spong [27] and many other researchers.

3.3 Notation and useful bounds

We define the position error and velocity error as

$$\Delta\theta = \theta - \theta_d, \qquad \Delta\dot\theta = \dot\theta - \dot\theta_d$$

and so on for higher derivatives.

Also, $\theta^{(i)}$ denotes the i^{th} derivative of θ, and a function $f \in C^k$ if it is continuous and differentiable k times. Furthermore, the i^{th} row of a matrix or the i^{th} element of a vector will be noted by the subscript "$,i$" (without the comma for a matrix or vector without subscript), and a particular element of a matrix will be denoted by its coordinates in parenthesis as a subscript.

The following notation for various bounds will be used later in this paper:

$$\alpha_M = \inf_\theta \sigma_{min}\{M(\theta)\} \qquad \alpha_p = \sigma_{min}\left\{R + BK_pB^T\right\}$$

$$\alpha_v = \sigma_{min}\left\{D_0 + BK_vB^T\right\} \qquad \gamma_d = \sup_{t\geq 0}\left\|\dot{\theta}_d(t)\right\|$$

$$\gamma_{dd} = \sup_{t\geq 0}\left\|\ddot{\theta}_d(t)\right\| \qquad \gamma_D = \sup_\theta \sum_{i=1}^{n}\left\|\frac{\partial M(\theta)}{\partial\theta_i}\right\|$$

$$\gamma_{Dd} = \sup_\theta \sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\frac{\partial^2 M(\theta)}{\partial\theta_i\partial\theta_j}\right\| \qquad \gamma_{D_0} = \|D_0\|$$

$$\gamma_e = \sup_{t\geq 0}\|\Delta\theta(t)\| \qquad \gamma_{ed} = \sup_{t\geq 0}\left\|\Delta\dot{\theta}(t)\right\|$$

$$\gamma_M = \sup_\theta \|M(\theta)\| \qquad \gamma_p = \left\|R + BK_pB^T\right\|$$

$$\gamma_R = \sup_\theta \|\nabla_\theta \bar{r}(\theta)\| \qquad \gamma_v = \left\|D_0 + BK_vB^T\right\|$$

where $K_p, K_v \in \mathbf{R}^{n\times n}$ are controller gains.

The following results will also be used in the paper.

Due to the positive definitiveness of the mass matrix M in (3.1), and of M_{11} and M_{22}, we can always solve for $\ddot{\theta}_\ell$ and $\ddot{\theta}_m$ from (3.10). In (3.10), dropping the dependence on the state in the equation, define

$$\begin{bmatrix}\rho_1\\ \rho_2\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}\begin{bmatrix}\dot{\theta}_\ell\\ \dot{\theta}_m\end{bmatrix} + \begin{bmatrix}D_\ell\\ D_m\end{bmatrix} + \begin{bmatrix}r_1\\ r_2\end{bmatrix}$$

Then, we have, for the general case,

$$\ddot{\theta}_\ell = \left[M_{11} - M_{12}M_{22}^{-1}M_{12}^T\right]^{-1}\left[M_{12}M_{22}^{-1}(\rho_2 - u) - \rho_1\right] \tag{3.20}$$

$$\ddot{\theta}_m = \left[M_{22} - M_{12}^TM_{11}^{-1}M_{12}\right]^{-1}\left[M_{12}^TM_{11}^{-1}\rho_1 - \rho_2 + u\right] \tag{3.21}$$

Also, the following relations hold:

$$C_A(\theta,\dot{\theta},\dot{\theta}_d) \triangleq C_D(\theta,\dot{\theta})\dot{\theta}_d - \frac{1}{2}M_D(\theta,\dot{\theta})\dot{\theta}_d + \frac{1}{2}M_D(\theta,\dot{\theta}_d)\dot{\theta} \tag{3.22}$$

$$= \frac{1}{2}M_D(\theta,\dot{\theta})\dot{\theta}_d - \frac{1}{2}M_D^T(\theta,\dot{\theta})\dot{\theta}_d + \frac{1}{2}M_D(\theta,\dot{\theta}_d)\dot{\theta} \tag{3.23}$$

$$= \frac{1}{2}C_D(\theta,\dot{\theta})\dot{\theta}_d + \frac{1}{2}C_D(\theta,\dot{\theta}_d)\dot{\theta} \tag{3.24}$$

Proof : Equation (3.23): direct substitution of (3.8) in (3.22).

Equation (3.24): add and subtract $\frac{1}{4}M_D^T(\theta,\dot{\theta})\dot{\theta}_d$ to (3.22), and use (3.7) and (3.8). ■

$$J \triangleq \Delta\dot{\theta}^T\left[C_D(\theta,\dot{\theta})\Delta\dot{\theta} + \frac{1}{2}M_D(\theta,\dot{\theta})\dot{\theta}_d - \frac{1}{2}M_D(\theta,\dot{\theta}_d)\dot{\theta} - \frac{1}{2}\dot{M}(\theta,\dot{\theta})\Delta\dot{\theta}\right] = 0 \tag{3.25}$$

Proof : By direct substitution,

$$J = \Delta\dot{\theta}^T\left[M_D(\theta,\dot{\theta})\Delta\dot{\theta} - \frac{1}{2}M_D^T(\theta,\dot{\theta})\Delta\dot{\theta} + \frac{1}{2}M_D(\theta,\dot{\theta})\dot{\theta}_d - \frac{1}{2}M_D(\theta,\dot{\theta}_d)\dot{\theta} - \frac{1}{2}M_D(\theta,\Delta\dot{\theta})\dot{\theta}\right]$$

Add and subtract $\frac{1}{2}M_D(\theta,\dot\theta)\dot\theta$ to this equation and use the linearity property of M_D in its second argument to obtain

$$\begin{aligned} J &= \Delta\dot\theta^T\left[M_D(\theta,\dot\theta)\Delta\dot\theta - \frac{1}{2}M_D^T(\theta,\dot\theta)\Delta\dot\theta - \frac{1}{2}M_D(\theta,\dot\theta)\Delta\dot\theta\right] \\ &= \Delta\dot\theta^T\left[\frac{1}{2}M_D(\theta,\dot\theta)\Delta\dot\theta - \frac{1}{2}M_D^T(\theta,\dot\theta)\Delta\dot\theta\right] \\ &= 0 \end{aligned}$$

due to the skew-symmetry of the term in brackets. ∎

$$\|M(\theta_d) - M(\theta)\| \le \gamma_D \|\Delta\theta\| \tag{3.26}$$

Proof : γ_D corresponds to the supremum of the norm of the gradient of M, leading directly to this identity. ∎

$$J_n \triangleq \left\|C_D(\theta,\dot\theta)\Delta\dot\theta + \frac{1}{2}M_D(\theta,\dot\theta)\dot\theta_d - \frac{1}{2}M_D(\theta,\dot\theta_d)\dot\theta - \dot M(\theta,\dot\theta)\Delta\dot\theta\right\| \le \frac{3}{2}\gamma_D\left\|\Delta\dot\theta\right\|^2 + \frac{3}{2}\gamma_D\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\| \tag{3.27}$$

Proof : Add and subtract $\frac{1}{2}M_D(\theta,\dot\theta)\dot\theta$ (inside the norm operation) and use (3.6) to write

$$\begin{aligned} J_n &= \frac{1}{2}\left\|M_D(\theta,\dot\theta)\Delta\dot\theta - M_D^T(\theta,\dot\theta)\Delta\dot\theta - M_D(\theta,\Delta\dot\theta)\dot\theta\right\| \\ &\le \frac{1}{2}\left\|M_D(\theta,\dot\theta)\right\|\left\|\Delta\dot\theta\right\| + \frac{1}{2}\left\|M_D^T(\theta,\dot\theta)\right\|\left\|\Delta\dot\theta\right\| + \frac{1}{2}\left\|M_D(\theta,\Delta\dot\theta)\right\|\left\|\dot\theta\right\| \end{aligned}$$

and use (3.5) to write

$$\begin{aligned} J_n &\le \frac{3}{2}\gamma_D\left\|\dot\theta\right\|\left\|\Delta\dot\theta\right\| \\ &= \frac{3}{2}\gamma_D\left\|\Delta\dot\theta + \dot\theta_d\right\|\left\|\Delta\dot\theta\right\| \\ &\le \frac{3}{2}\gamma_D\left\|\Delta\dot\theta\right\|^2 + \frac{3}{2}\gamma_D\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\| \end{aligned}$$

which concludes the proof. ∎

$$\left\|C_A(\theta,\dot\theta,\dot\theta_d)\right\| \le \frac{3}{2}\gamma_D\left\|\dot\theta_d\right\|^2 + \frac{3}{2}\gamma_D\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\| \tag{3.28}$$

Proof : Use (3.23) to write

$$\begin{aligned} \left\|C_A(\theta,\dot\theta,\dot\theta_d)\right\| &\le \frac{1}{2}\left\|M_D(\theta,\dot\theta)\dot\theta_d\right\| + \frac{1}{2}\left\|M_D^T(\theta,\dot\theta)\dot\theta_d\right\| + \frac{1}{2}\left\|M_D(\theta,\dot\theta_d)\dot\theta\right\| \\ &\le \frac{3}{2}\gamma_D\left\|\dot\theta\right\|\left\|\dot\theta_d\right\| \end{aligned}$$

which leads to the conclusion by using $\left\|\dot\theta\right\| \le \left\|\dot\theta_d\right\| + \left\|\Delta\dot\theta\right\|$. ∎

$$\left\|\frac{1}{2}M_D(\theta,\dot\theta)\dot\theta_d - \frac{1}{2}M_D(\theta,\dot\theta_d)\dot\theta\right\| \le \gamma_D\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\| \tag{3.29}$$

Proof : Add and subtract $\frac{1}{2}M_D(\theta,\dot\theta_d)\dot\theta_d$ (inside the norm operation) to write

$$\begin{aligned}\left\|\frac{1}{2}M_D(\theta,\dot\theta)\dot\theta_d-\frac{1}{2}M_D(\theta,\dot\theta_d)\dot\theta\right\| &= \left\|\frac{1}{2}M_D(\theta,\Delta\dot\theta)\dot\theta_d-\frac{1}{2}M_D(\theta,\dot\theta_d)\Delta\dot\theta\right\| \\ &\le \gamma_D\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\|\end{aligned}$$

which ends the proof. ■

$$J_m \triangleq \left\|C_D(\theta_d,\dot\theta_d)\dot\theta_d - C_A(\theta,\dot\theta,\dot\theta_d)\right\| \le \frac{3}{2}\gamma_{Dd}\left\|\dot\theta_d\right\|^2\|\Delta\theta\| + \frac{3}{2}\gamma_D\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\| \tag{3.30}$$

Proof : Use (3.22) to write

$$J_m = \left\|C_D(\theta_d,\dot\theta_d)\dot\theta_d - C_D(\theta,\dot\theta)\dot\theta_d + \frac{1}{2}M_D(\theta,\dot\theta)\dot\theta_d - \frac{1}{2}M_D(\theta,\dot\theta_d)\dot\theta\right\|$$

Add and subtract $C_D(\theta,\dot\theta_d)\dot\theta_d$ and $\frac{1}{2}M_D^T(\theta,\dot\theta_d)\dot\theta_d$ (inside the norm operation) to obtain

$$\begin{aligned}J_m &= \left\|C_D(\theta_d,\dot\theta_d)\dot\theta_d - C_D(\theta,\dot\theta_d)\dot\theta_d - C_D(\theta,\Delta\dot\theta)\dot\theta_d + \frac{1}{2}M_D(\theta,\Delta\dot\theta)\dot\theta_d - \frac{1}{2}M_D(\theta,\dot\theta_d)\Delta\dot\theta\right\| \\ &\le \left\|C_D(\theta_d,\dot\theta_d)\dot\theta_d - C_D(\theta,\dot\theta_d)\dot\theta_d\right\| + \left\|-C_D(\theta,\Delta\dot\theta)\dot\theta_d + \frac{1}{2}M_D(\theta,\Delta\dot\theta)\dot\theta_d - \frac{1}{2}M_D(\theta,\dot\theta_d)\Delta\dot\theta\right\| \\ &\le \left\|M_D(\theta_d,\dot\theta_d)\dot\theta_d - M_D(\theta,\dot\theta_d)\dot\theta_d\right\| + \frac{1}{2}\left\|M_D^T(\theta_d,\dot\theta_d)\dot\theta_d - M_D^T(\theta,\dot\theta_d)\dot\theta_d\right\| + \\ &\quad \left\|-\frac{1}{2}M_D(\theta,\Delta\dot\theta)\dot\theta_d + \frac{1}{2}M_D^T(\theta,\Delta\dot\theta)\dot\theta_d - \frac{1}{2}M_D(\theta,\dot\theta_d)\Delta\dot\theta\right\| \\ &\le \frac{3}{2}\gamma_{Dd}\left\|\dot\theta_d\right\|^2\|\Delta\theta\| + \frac{3}{2}\gamma_D\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\|\end{aligned}$$

Where we have used the definition of M_D (3.5), and in particular, the supremum of the bound of the gradient of M_D with respect to θ to obtain the bound in γ_{Dd}. ■

For C represented using the Christoffel's symbols, the relations listed below hold.

There exists a unique and finite real constant γ_C that satisfies, for any θ and any finite w,

$$\|C_C(\theta,w)\| \le \gamma_C\|w\| \tag{3.31}$$

Proof : The evaluation of this bound, and in particular of a tight bound γ_C is not straightforward and is not provided here. ■

There exists some constant γ_{Cd} so that

$$\left\|C_C(\theta_d,\dot\theta) - C_C(\theta,\dot\theta)\right\| \le \gamma_{Cd}\|\Delta\theta\|\left\|\dot\theta\right\| \tag{3.32}$$

Proof : This bound is a function of the supremum of the norm of the gradient of C_C but is difficult to find analytically so that no proof is provided. ■

Also, there exists some constant γ_{C_2} so that

$$\left\|C_C(\theta,\dot\theta)\Delta\dot\theta - \dot M(\theta,\dot\theta)\Delta\dot\theta\right\| \le \gamma_{C_2}\left\|\Delta\dot\theta\right\|^2 + \gamma_{C_2}\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\| \tag{3.33}$$

Proof : We may write

$$\begin{aligned}\left\|C_C(\theta,\dot\theta)\Delta\dot\theta - \dot M(\theta,\dot\theta)\Delta\dot\theta\right\| &\le \left\|C_C(\theta,\dot\theta)\Delta\dot\theta\right\| + \left\|\dot M(\theta,\dot\theta)\Delta\dot\theta\right\| \\ &\le \gamma_C\left\|\Delta\dot\theta\right\|\left\|\dot\theta\right\| + \gamma_D\left\|\Delta\dot\theta\right\|\left\|\dot\theta\right\| \\ &\le (\gamma_C+\gamma_D)\left(\left\|\Delta\dot\theta\right\|^2 + \left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\|\right)\end{aligned}$$

where we have used (3.4) and (3.31). We obtain the desired conclusion by choosing $\gamma_{C_2} = \gamma_C + \gamma_D$, which gives a loose evaluation of the bound. ∎

$$\left\|C_C(\theta_d, \dot\theta_d) - C_C(\theta, \dot\theta)\right\| \le \gamma_{Cd} \left\|\Delta\dot\theta\right\| \left\|\dot\theta_d\right\| + \gamma_C \left\|\Delta\dot\theta\right\| \tag{3.34}$$

Proof : We may write

$$\begin{aligned}
\left\|C_C(\theta_d, \dot\theta_d) - C_C(\theta, \dot\theta)\right\| &= \left\|C_C(\theta_d, \dot\theta_d) - C_C(\theta, \dot\theta_d) - C_C(\theta, \Delta\dot\theta)\right\| \\
&\le \left\|C_C(\theta_d, \dot\theta_d) - C_C(\theta, \dot\theta_d)\right\| + \left\|C_C(\theta, \Delta\dot\theta)\right\| \\
&\le \gamma_{Cd} \left\|\Delta\theta\right\| \left\|\dot\theta_d\right\| + \gamma_C \left\|\Delta\dot\theta\right\|
\end{aligned}$$

where (3.32) has been used in the last step. ∎

Finally,

$$\left\|\bar r(\theta) - \bar r(\theta_d)\right\| \le \gamma_R \left\|\Delta\theta\right\| \tag{3.35}$$

Proof : γ_R corresponds to the supremum of the norm of the gradient of $\bar r$, leading directly to this identity. ∎

4 Controller design approach

The synthesis of the material presented in Section 2 leads to a control design approach based on passivity for a large class of systems. Assuming that we can form an error system and render the error system passive by static feedback, we can then L_2-stabilize the error system by using any finite gain strictly passive controller. Then, if furthermore the error system with static feedback is at least weakly zero-state detectable, then internal asymptotic stability is also guaranteed.

Flexible joint manipulators are members of the class of systems to which such an approach can be applied since such systems are feedback equivalent to passive systems for a proper choice of outputs, and in particular for any arm configuration if the motor velocities are used as outputs as shown in Appendix B.

Problem statement Consider the dynamical equation of the system (3.1,3.2) rewritten here for completeness:

$$\begin{aligned}
M(\theta)\ddot\theta + C(\theta, \dot\theta)\dot\theta + D(\dot\theta) + r(\theta) &= Bu \\
r(\theta) &= g(\theta) + k(\theta) \\
B &= \begin{bmatrix} 0_{m\times m} & I_{m\times m} \end{bmatrix}^T
\end{aligned}$$

Find u so that the output of interest $y = h(\theta_\ell) \to y_d$ asymptotically.

We now present a design procedure that involves essentially two steps defining the basic structure of the controller, i.e. (i) define a model–based feedforward in order to form an error system; (ii) add a model–independent stabilizing feedback to the feedforward.

1. **Feedforward Design:** Decompose the control input as

$$u = u_o + u_{ff}$$

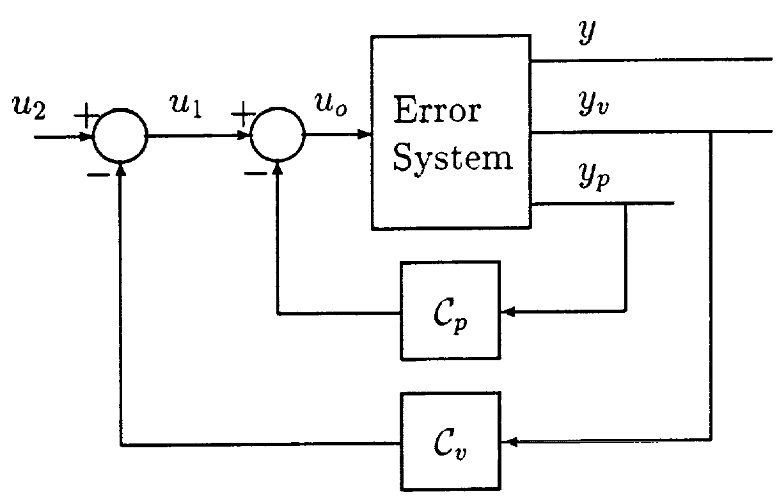

Figure 1: Structure of the proposed controller

where u_{ff} is the feedforward input and u_o the stabilizing input. The feedforward is chosen such that the output of interest y tracks the desired trajectory under matched initial conditions. The choice of feedforward must also allow to form an error system that is passive or can be rendered passive between a particular input/output pair (u_1, y_v) via static feedback from some output y_p (Fig. 1). The results presented in [5] can be used to guide the choice of the output y_v (Appendix B).

2. **Feedback Stabilization:** Find a static feedback (inner loop $\mathcal{C}_p$) to obtain passivity and zero–state detectability for the pair (u_1, y_v). Then choose a finite gain strictly passive feedback (outer loop $\mathcal{C}_v$) between the passive input/output pair. By the *Passivity Theorem* [9], this guarantees L_2–stability of the system from u_2 to y_v. Use an energy based Lyapunov analysis to show that output $y_v \to 0$ asymptotically. From zero–state detectability, the zero error state is also asymptotically stable, and the output of interest $y \to y_d$ asymptotically.

NOTE : Three outputs are defined in the design procedure. In the case of flexible joint robots, the output of interest y is generally defined in terms of θ_ℓ, while $\dot{\theta}_m$ (or $\Delta\dot{\theta}_m$) has been used as the passive output y_v, and θ_m (or $\Delta\theta_m$) has been used as y_p in [16, 24] to render the system zero-state detectable from u_1 to y_v.

NOTE : Weak zero–state detectability is also sufficient to guarantee asymptotic stability given $y_v \to 0$ asymptotically.

This approach has interesting features, namely :

- $\mathcal{C}_v$ can be any BIBO strictly passive feedback and can be tuned for performance enhancement. Note that BIBO stability is implied by strict passivity for finite dimensional linear time invariant systems.
- Only y_p and y_v are needed for stabilization, i.e. only motor position and velocity for flexible joint robots.
- Error in the feedforward does not cause instability, i.e. small errors in the feedforward only lead to a weaker form of stability (see Section 8).
- Applicable to both position set point stabilization [16] and tracking control.

The design approach offers a certain degree of flexibility in the sense that certain choices must be made. In particular, different forms of feedforward can be used yielding different characteristics regarding their solution (solubility, required model information, required signal measurements, complexity) (Section 5), and the type of stability and tracking accuracy that are obtained (Section 8). Another choice that has to be made is which controller in the class of strictly passive controllers should be used in order to obtain the desired performance. Among this class of controllers, we will use the classical PD (Proportional–Derivative) controller to carry out the design in Section 6.

In the following section, we present various possible forms of feedforward for the trajectory tracking problem and address the issue of solving the feedforward. We then carry out the controller design in Section 6 for particular choices of feedforward under the assumption that the condition of (weak) zero–state detectability is met, assumption that is verified in Section 7. The design approach guarantees asymptotic stability only if all the design conditions are met, which is strongly dependent on the feedforward signal that is used. A Lyapunov stability analysis is carried out in Section 8 to analyze the stability of different forms of feedforward, in particular approximations of the feedforward based on the inverse plant.

5 Feedforward: selection and solution

In this section, we present a series of possible forms of feedforward for the problem of trajectory tracking and address the issue of solving for the feedforward. In particular, conditions for solubility including required measurement and modeling information are stated.

5.1 Selection of feedforward

Assume that, given $y_d = h(\theta_{\ell_d})$, we can solve for θ_d. There are many possible feedforward compensation based on the inverse dynamics of the system (**FF1** below) and its approximations that can be used, e.g.

FF1. $Bu_{ff} = M(\theta_d)\ddot{\theta}_d + C(\theta_d, \dot{\theta}_d)\dot{\theta}_d + D(\dot{\theta}_d) + r(\theta_d)$

FF2. $Bu_{ff} = M(\theta_d)\ddot{\theta}_d + C(\theta_d, \dot{\theta}_d)\dot{\theta}_d + D(\dot{\theta}_d) + \bar{r}(\theta) + R\theta_d$

FF3. $Bu_{ff} = M(\theta)\ddot{\theta}_d + C(\theta, \dot{\theta})\dot{\theta}_d + D(\dot{\theta}_d) + r(\theta_d)$

FF4. $Bu_{ff} = M(\theta)\ddot{\theta}_d + C(\theta, \dot{\theta})\dot{\theta}_d + D(\dot{\theta}_d) + \bar{r}(\theta) + R\theta_d$

FF5. $Bu_{ff} = M(\theta)\ddot{\theta}_d + C_D(\theta, \dot{\theta})\dot{\theta}_d - \frac{1}{2}M_D(\theta, \dot{\theta})\dot{\theta}_d + \frac{1}{2}M_D(\theta, \dot{\theta}_d)\dot{\theta} + D(\dot{\theta}_d) + r(\theta_d)$

FF6. $Bu_{ff} = M(\theta)\ddot{\theta}_d + C_D(\theta, \dot{\theta})\dot{\theta}_d - \frac{1}{2}M_D(\theta, \dot{\theta})\dot{\theta}_d + \frac{1}{2}M_D(\theta, \dot{\theta}_d)\dot{\theta} + D(\dot{\theta}_d) + \bar{r}(\theta) + R\theta_d$

FF7. $Bu_{ff} = r(\theta_d)$

FF8. $Bu_{ff} = \bar{r}(\theta) + R\theta_d$

Note that feedforward **FF7** is the same as used for the set point control, e.g. [16, 31].

There are multiple considerations in the choice of feedforward, namely, the measurements and model information required to solve for the feedforward and for its implementation, the complexity of the solution, and the performance it allows to obtain regarding stability and tracking. We now analyze the solution of the different forms of feedforward and will analyze their performance in Section 8.

The solution of the feedforward is a stable inversion problem. We will use a procedure for the solution that involves essentially three steps defined as follows

1. From the last m equations of the feedforward equation (**FF1–FF8**), define what signals are required to compute the feedforward.

2. Given the desired output (θ_{ℓ_d} and its higher derivatives, i.e. link reference trajectory provided by the user) and the signals available by measurement, determine the desired internal state (θ_{m_d} and its higher derivatives as required by the feedforward, i.e. desired motor reference trajectory) by using the first m equations of the feedforward, and possibly additional information such as the original system equation. The result that we are seeking at this step is a causal and bounded desired internal state yielding a bounded feedforward signal provided that the desired output state is also bounded, i.e. it should be a stable and causal solution.

3. Evaluate the feedforward.

Proposition 5.1 *Consider the properties of the system (Section 3.1) and assume that $D(x)$ and $r(y)$ are bounded functions of their argument. Then, if the desired trajectory (θ_d and its higher derivatives) is bounded, u_{ff} is bounded in*

- **FF1** *and* **FF7**,
- **FF2** *and* **FF8** *if the following assumption is satisfied*

  Assumption 5.1 *$\|\Delta\theta\|$ is uniformly bounded in time by a finite constant γ_e.*

- **FF3** *and* **FF5** *if the following assumption is satisfied*

  Assumption 5.2 *$\left\|\Delta\dot{\theta}\right\|$ is uniformly bounded in time by a finite constant γ_{ed}.*

- **FF4** *and* **FF6** *if both Assumption 5.1 and 5.2 are satisfied.*

Proof : Consider **FF4** with C represented using the M_D-notation. Premultiply the equation of **FF4** by B^T and use (3.9) to obtain

$$\begin{aligned}
u_{ff} &= B^T\left[M(\theta)\ddot{\theta}_d + C_D(\theta,\dot{\theta})\dot{\theta}_d + D(\dot{\theta}_d) + r(\theta) - R\Delta\theta\right] \\
\|u_{ff}\| &\leq \left\|B^T\right\|\left[\|M(\theta)\|\left\|\ddot{\theta}_d\right\| + \left\|C_D(\theta,\dot{\theta})\right\|\left\|\dot{\theta}_d\right\| + \left\|D(\dot{\theta}_d)\right\| + \|r(\theta)\| + \|R\|\,\|\Delta\theta\|\right] \\
&\leq \left\|B^T\right\|\left[\gamma_M\left\|\ddot{\theta}_d\right\| + \frac{3}{2}\gamma_D\left\|\dot{\theta}\right\|\left\|\dot{\theta}_d\right\| + \left\|D(\dot{\theta}_d)\right\| + \|r(\Delta\theta+\theta_d)\| + \|R\|\,\|\Delta\theta\|\right] \\
&\leq \left\|B^T\right\|\left[\gamma_M\left\|\ddot{\theta}_d\right\| + \frac{3}{2}\gamma_D\left\|\dot{\theta}_d\right\|^2 + \frac{3}{2}\gamma_D\left\|\Delta\dot{\theta}\right\|\left\|\dot{\theta}_d\right\| + \left\|D(\dot{\theta}_d)\right\| + \|r(\Delta\theta+\theta_d)\| + \|R\|\,\|\Delta\theta\|\right]
\end{aligned}$$

which leads to the conclusion of the proposition since $\left\|B^T\right\|$ and $\|R\|$ are also bounded. The same procedure is used for the other forms of feedforward and for the C matrix represented using the Christoffel's symbols. ■

The following assumptions will be particularly useful for the solution of the feedforward:

Assumption 5.3 *r_1 is continuous and differentiable, and the gradient of $r_1(\theta_\ell,\theta_m)$ with respect to θ_m, $\nabla_{\theta_m} r_1(\theta_{\ell_d},\theta_m)$, is invertible in some open set in θ_m.*

Assumption 5.4 *$D(\dot{\theta})$ is continuously differentiable as many times as this function needs to be differentiated to solve for the feedforward.*

We will also assume that the reference output trajectory is bounded.

5.2 Solution of FF1

Expand **FF1** to obtain, for the general model,

$$0 = M_{11}(\theta_{\ell_d},\theta_{m_d})\ddot{\theta}_{\ell_d} + M_{12}(\theta_{\ell_d},\theta_{m_d})\ddot{\theta}_{m_d} + C_{11}(\theta_{\ell_d},\theta_{m_d},\dot{\theta}_{\ell_d},\dot{\theta}_{m_d})\dot{\theta}_{\ell_d} + C_{12}(\theta_{\ell_d},\theta_{m_d},\dot{\theta}_{\ell_d},\dot{\theta}_{m_d})\dot{\theta}_{m_d} + D_\ell(\dot{\theta}_{\ell_d}) + r_1(\theta_{\ell_d},\theta_{m_d}) \quad (5.1)$$

$$u_{ff} = M_{12}^T(\theta_{\ell_d},\theta_{m_d})\ddot{\theta}_{\ell_d} + M_{22}(\theta_{m_d})\ddot{\theta}_{m_d} + C_{21}(\theta_{\ell_d},\theta_{m_d},\dot{\theta}_{\ell_d},\dot{\theta}_{m_d})\dot{\theta}_{\ell_d} + C_{22}(\theta_{m_d},\dot{\theta}_{m_d})\dot{\theta}_{m_d} + D_m(\dot{\theta}_{m_d}) + r_2(\theta_{\ell_d},\theta_{m_d}) \quad (5.2)$$

To obtain a closed loop form for the feedforward control input u_{ff} from (5.2) for $(\theta_{\ell_d},\dot{\theta}_{\ell_d},\ddot{\theta}_{\ell_d})$ given , we must first define $(\theta_{m_d},\dot{\theta}_{m_d},\ddot{\theta}_{m_d})$.

The problem of solving for $(\theta_{m_d},\dot{\theta}_{m_d},\ddot{\theta}_{m_d})$ resumes to solving (5.1) for these variables given the input $\ddot{\theta}_{\ell_d}$ and $\theta_{\ell_d},\dot{\theta}_{\ell_d}$. This problem may have multiple solutions as in the case where M_{12} is nonsingular. In that particular case, the user is free of choosing the initial values for $\theta_{m_d},\dot{\theta}_{m_d}$. An important characteristic that is desired for the solution is its stability in the sense that the desired motor state should stabilize in order to obtain internal stability not only of the error state, but also of the actual state. This allows, among others, to reduce the stress on the equipment.

Zero–state detectability or weakly zero–state detectability properties of (5.1), with $y = (\theta_{\ell_d},\dot{\theta}_{\ell_d},\ddot{\theta}_{\ell_d})$, and $x = (\theta_{m_d},\dot{\theta}_{m_d},\ddot{\theta}_{m_d})$ in Definitions 2.8 and 2.9, allows to characterize partially the solution for $(\theta_{m_d},\dot{\theta}_{m_d},\ddot{\theta}_{m_d})$ either locally or globally.

Fact 5.1 *Zero–state detectability of (5.1) constitutes a necessary condition for uniqueness of the solution for* $(\theta_{m_d},\dot{\theta}_{m_d},\ddot{\theta}_{m_d})$.

Fact 5.2 *Weak zero–state detectability of (5.1) constitutes a necessary condition for the solutions for* $(\theta_{m_d},\dot{\theta}_{m_d},\ddot{\theta}_{m_d})$ *to be asymptotically stable.*

A more global condition is obtained if we require that θ_{ℓ_d} constant implies $\theta_{m_d} = \theta_c$ (zero–state detectability of any shifted state), or implies $(\theta_{m_d},\dot{\theta}_{m_d},\ddot{\theta}_{m_d}) \to (\theta_c,0,0)$ (weak zero–state detectability of any shifted state), where $\theta_c \in \mathbf{R}^m$ is constant and uniquely defined for each θ_{ℓ_d}.

Example 5.1 One case for which (5.1) is zero–state detectable and **FF1** has a unique solution is the following. First, assume that the inertia of the motors is symmetric about their axis of rotation, and that M_{12} is strictly upper triangular (Section 3.2.3). Also assume that the i^{th} element of $r^1(\theta_\ell,\theta_m)$ depends only on $(\theta_{\ell,i},\theta_{m,i})$. Under these assumptions, expand **FF1** to obtain

$$0 = M_{11}(\theta_{\ell_d})\ddot{\theta}_{\ell_d} + M_{12}(\theta_{\ell_d})\ddot{\theta}_{m_d} + C_{11}^A(\theta_{\ell_d},\dot{\theta}_{\ell_d})\dot{\theta}_{\ell_d} + C_{11}^B(\theta_{\ell_d},\dot{\theta}_{m_d})\dot{\theta}_{\ell_d} + C_{12}^A(\theta_{\ell_d},\dot{\theta}_{\ell_d})\dot{\theta}_{m_d} + D_\ell(\dot{\theta}_{\ell_d}) + r_1(\theta_{\ell_d},\theta_{m_d}) \quad (5.3)$$

$$u_{ff} = M_{12}^T(\theta_{\ell_d})\ddot{\theta}_{\ell_d} + M_{22}\ddot{\theta}_{m_d} + C_{21}^A(\theta_{\ell_d},\dot{\theta}_{\ell_d})\dot{\theta}_{\ell_d} + D_m(\dot{\theta}_{m_d}) + r_2(\theta_{\ell_d},\theta_{m_d}) \quad (5.4)$$

Also suppose that Assumption 5.3 holds. This implies that, by the Implicit Function Theorem [3], there exists a locally unique solution θ_{m_d} to (5.3) for any given $\dot{\theta}_{m_d}$, $\ddot{\theta}_{m_d}$, and θ_{ℓ_d} and its higher derivatives. However, $\dot{\theta}_{m_d}$ and $\ddot{\theta}_{m_d}$ are not known *a priori*.

In order to solve for θ_{m_d} and its derivatives, we take advantage of the fact that M_{12} and C_{12}^A are strictly upper triangular and that row i of C_{11}^B does not depend on any $\dot{\theta}_{m,j}$ for $j \leq i$ and $i = 1,\cdots,m$. This allows to solve iteratively for the elements of θ_{m_d} from its m^{th} element to the first.

Given θ_{ℓ_d} and its higher derivatives, use the last row of (5.3) to solve algebraically for $\theta_{m_d,m}$. Then suppose that Assumption 5.4 holds and evaluate $\dot{\theta}_{m_d,m}$ and $\ddot{\theta}_{m_d,m}$ by taking the time derivative of $\theta_{m_d,m}$. Now, given $\dot{\theta}_{m_d,m}$ and $\ddot{\theta}_{m_d,m}$, use the second last row of (5.3) to solve algebraically for $\theta_{m_d,m-1}$, and continue the process down to $\theta_{m_d,1}$. Note that the desired trajectory has to be very smooth since the solution of the last term of the iteration, i.e. $\ddot{\theta}_{m_d,1}$, requires $\theta_{l_d,i}^{2(i+1)}$, $i = 1, m$, and that the characteristic of the equivalent spring of joint i be differentiable $2i$ times in both its arguments for $i = 1, m$.

Noting that all the coefficients of the equations use to solve for the feedforward are bounded, we conclude that the desired internal state is bounded given a bounded desired output trajectory.

We then obtain the feedforward input by substituting the variables in (5.4).

Hence, the solution of u_{ff} for this case requires invertibility (at least local) of r_1 and of $\nabla_{\theta_{m_d}} r_1$ along with knowledge of the full model, a very smooth input trajectory and differentiability to a high order of r_1. However, no measurements are required. □

5.3 Solution of FF2

Expand **FF2** to obtain, for the general model,

$$\begin{aligned} 0 &= M_{11}(\theta_{\ell_d}, \theta_{m_d})\ddot{\theta}_{\ell_d} + M_{12}(\theta_{\ell_d}, \theta_{m_d})\ddot{\theta}_{m_d} + C_{11}(\theta_{\ell_d}, \theta_{m_d}, \dot{\theta}_{\ell_d}, \dot{\theta}_{m_d})\dot{\theta}_{\ell_d} + \\ &\quad C_{12}(\theta_{\ell_d}, \theta_{m_d}, \dot{\theta}_{\ell_d}, \dot{\theta}_{m_d})\dot{\theta}_{m_d} + D_\ell(\dot{\theta}_{\ell_d}) + \bar{r}_1(\theta_\ell, \theta_m) + R_{11}\theta_{\ell_d} + R_{12}\theta_{m_d} \qquad (5.5) \\ u_{ff} &= M_{12}^T(\theta_{\ell_d}, \theta_{m_d})\ddot{\theta}_{\ell_d} + M_{22}(\theta_{m_d})\ddot{\theta}_{m_d} + C_{21}(\theta_{\ell_d}, \theta_{m_d}, \dot{\theta}_{\ell_d}, \dot{\theta}_{m_d})\dot{\theta}_{\ell_d} + \\ &\quad C_{22}(\theta_{m_d}, \dot{\theta}_{m_d})\dot{\theta}_{m_d} + D_m(\dot{\theta}_{m_d}) + \bar{r}_2(\theta_\ell, \theta_m) + R_{21}\theta_{\ell_d} + R_{22}\theta_{m_d} \qquad (5.6) \end{aligned}$$

To obtain a closed loop form for the feedforward control input u_{ff} from (5.6) for $(\theta_{\ell_d}, \dot{\theta}_{\ell_d}, \ddot{\theta}_{\ell_d})$ given, we must first define $(\theta_{m_d}, \dot{\theta}_{m_d}, \ddot{\theta}_{m_d})$.

The problem of solving for $(\theta_{m_d}, \dot{\theta}_{m_d}, \ddot{\theta}_{m_d})$ is very similar to the one encountered with **FF1** except that (5.5) depends on the actual state (θ_ℓ, θ_m). Hence, the feedforward and the control input are intertwined. This renders the task of characterizing the solutions more difficult: we must know the controller in order to ascertain stability of the feedforward input. However, we may assume that the closed loop system is stable and that the error signals are bounded in order to pursue the analysis.

Example 5.2 Consider Example 5.1 with the additional assumption that R_{12} is chosen nonsingular and upper triangular (or diagonal). Expanding **FF2**, we obtain

$$\begin{aligned} 0 &= M_{11}(\theta_{\ell_d})\ddot{\theta}_{\ell_d} + M_{12}(\theta_{\ell_d})\ddot{\theta}_{m_d} + C_{11}^A(\theta_{\ell_d}, \dot{\theta}_{\ell_d})\dot{\theta}_{\ell_d} + C_{11}^B(\theta_{\ell_d}, \dot{\theta}_{m_d})\dot{\theta}_{\ell_d} + C_{12}^A(\theta_{\ell_d}, \dot{\theta}_{\ell_d})\dot{\theta}_{m_d} + \\ &\quad D_\ell(\dot{\theta}_{\ell_d}) + \bar{r}_1(\theta_\ell, \theta_m) + R_{11}\theta_{\ell_d} + R_{12}\theta_{m_d} \qquad (5.7) \\ u_{ff} &= M_{12}^T(\theta_{\ell_d})\ddot{\theta}_{\ell_d} + M_{22}\ddot{\theta}_{m_d} + C_{21}^A(\theta_{\ell_d}, \dot{\theta}_{\ell_d})\dot{\theta}_{\ell_d} + D_m(\dot{\theta}_{m_d}) + \bar{r}_2(\theta_\ell, \theta_m) + R_{21}\theta_{\ell_d} + R_{22}\theta_{m_d} \qquad (5.8) \end{aligned}$$

In order to solve for θ_{m_d} and its derivatives, we take advantage of the structure of M_{12}, C_{11}^B, C_{12}^A and R_{12} to solve iteratively for the elements of θ_{m_d} from its m^{th} element to the first.

Given θ_{ℓ_d} and its higher derivatives, and assuming that θ_ℓ and θ_m are available for measurement, use the last row of (5.7) to solve algebraically for the m^{th} element of θ_{m_d}:

$$\begin{aligned} \theta_{m_d,m} &= -R_{12,(m,m)}^{-1}\Big[M_{11,m}(\theta_{\ell_d})\ddot{\theta}_{\ell_d} + C_{11,m}^A(\theta_{\ell_d}, \dot{\theta}_{\ell_d})\dot{\theta}_{\ell_d} + \\ &\qquad D_{\ell,m}(\dot{\theta}_{\ell_d}) + \bar{r}_{1,m}(\theta_{\ell,m}, \theta_{m,m}) + R_{11,(m,m)}\theta_{\ell_d,m}\Big] \qquad (5.9) \end{aligned}$$

We solve for $\dot{\theta}_{m_d,m}$ by taking the first time derivative of (5.9) under the assumption that $\dot{\theta}_\ell$ and $\dot{\theta}_m$ are available for measurement and that Assumption 5.4 holds. Now, taking the second time derivative of (5.9) in order to find $\ddot{\theta}_{m_d,m}$, we obtain, for $\mathcal{X}$ properly defined:

$$\begin{aligned} \ddot{\theta}_{m_d,m} &= -R_{12,(m,m)}^{-1}\frac{d^2}{dt^2}\Big[M_{11,m}(\theta_{\ell_d})\ddot{\theta}_{\ell_d} + C_{11,m}^A(\theta_{\ell_d}, \dot{\theta}_{\ell_d})\dot{\theta}_{\ell_d} + \\ &\qquad D_{\ell,m}(\dot{\theta}_{\ell_d}) + \bar{r}_{1,m}(\theta_{\ell,m}, \theta_{m,m}) + R_{11,(m,m)}\theta_{\ell_d,m}\Big] \\ &= \mathcal{X}\left(\theta_{\ell_d}, \dot{\theta}_{\ell_d}, \ddot{\theta}_{\ell_d}, \theta_{\ell_d}^{(3)}, \theta_{\ell_d}^{(4)}, \theta_{\ell,m}, \dot{\theta}_{\ell,m}, \theta_{m,m}, \dot{\theta}_{m,m}\right) \\ &\quad - R_{12,(m,m)}^{-1}\Big[\nabla_{\theta_{\ell,m}}\bar{r}_{1,m}(\theta_{\ell,m}, \theta_{m,m})\ddot{\theta}_{\ell,m} + \nabla_{\theta_{m,m}}\bar{r}_{1,m}(\theta_{\ell,m}, \theta_{m,m})\ddot{\theta}_{m,m}\Big] \end{aligned}$$

Since $\ddot{\theta}_\ell$ and $\ddot{\theta}_m$ are usually not available for measurement, we want to remove the dependence on these variables. For this, we use the dynamical equations of the system (3.20, 3.21). However, due to the gyroscopic couplings, $\ddot{\theta}_{m,m}$ may depend directly on all the inputs, i.e. in general we must know all the feedforward inputs in order to solve for this signal. We conclude that, in general, this approach does not lead to a *practical* closed loop form for the solution in this case. □

NOTE : Under the additional assumption that M_{12} is constant in the previous example, the same problems are encountered in the solution of the feedforward.

Example 5.3 Continue the previous example (Example 5.2) with the assumption that there is no gyroscopic coupling, i.e. $M_{12} = 0$ (Section 3.2.4). Expanding **FF2**, we obtain

$$\begin{aligned} 0 &= M_{11}(\theta_{\ell_d})\ddot{\theta}_{\ell_d} + C_{11}(\theta_{\ell_d},\dot{\theta}_{\ell_d})\dot{\theta}_{\ell_d} + D_\ell(\dot{\theta}_{\ell_d}) + \bar{r}_1(\theta_\ell,\theta_m) + R_{11}\theta_{\ell_d} + R_{12}\theta_{m_d} \quad (5.10)\\ u_{ff} &= M_{22}\ddot{\theta}_{m_d} + D_m(\dot{\theta}_{m_d}) + \bar{r}_2(\theta_\ell,\theta_m) + R_{21}\theta_{\ell_d} + R_{22}\theta_{m_d} \quad (5.11)\end{aligned}$$

Given θ_{ℓ_d} and its higher derivatives, that R_{12} is nonsingular but otherwise arbitrary, and assuming that θ_ℓ and θ_m are available for measurement, we have, from (5.10),

$$\theta_{m_d} = -R_{12}{}^{-1}\left[M_{11}(\theta_{\ell_d})\ddot{\theta}_{\ell_d} + C_{11}(\theta_{\ell_d},\dot{\theta}_{\ell_d})\dot{\theta}_{\ell_d} + D_\ell(\dot{\theta}_{\ell_d}) + \bar{r}_1(\theta_\ell,\theta_m) + R_{11}\theta_{\ell_d}\right] \quad (5.12)$$

and θ_{m_d} is bounded if Assumption 5.1 is satisfied.

We solve for $\dot{\theta}_{m_d}$ by taking the first time derivative of (5.12) under the assumption that $\dot{\theta}_\ell$ and $\dot{\theta}_m$ are available for measurement and that Assumption 5.4 holds. Now, taking the second time derivative of (5.12) in order to find $\ddot{\theta}_{m_d}$, we obtain, for $\mathcal{X}$ properly defined:

$$\begin{aligned}\ddot{\theta}_{m_d} &= -R_{12}{}^{-1}\frac{d^2}{dt^2}\left[M_{11}(\theta_{\ell_d})\ddot{\theta}_{\ell_d} + C_{11}(\theta_{\ell_d},\dot{\theta}_{\ell_d})\dot{\theta}_{\ell_d} + D_\ell(\dot{\theta}_{\ell_d}) + \bar{r}_1(\theta_\ell,\theta_m) + R_{11}\theta_{\ell_d}\right]\\ &= \mathcal{X}\left(\theta_{\ell_d},\dot{\theta}_{\ell_d},\ddot{\theta}_{\ell_d},\theta_{\ell_d}^{(3)},\theta_{\ell_d}^{(4)},\theta_\ell,\dot{\theta}_\ell,\theta_m,\dot{\theta}_m\right)\\ &\quad -R_{12}{}^{-1}\left[\nabla_{\theta_\ell}\bar{r}_1(\theta_\ell,\theta_m)\ddot{\theta}_\ell + \nabla_{\theta_m}\bar{r}_1(\theta_\ell,\theta_m)\ddot{\theta}_m\right] \quad (5.13)\end{aligned}$$

We now use the dynamical equation of the system to write θ_ℓ and θ_m as a function of know signals. From (3.20, 3.21),

$$\begin{aligned}\ddot{\theta}_\ell &= M_{11}(\theta_\ell)^{-1}\left[C_{11}(\theta_\ell,\dot{\theta}_\ell)\dot{\theta}_\ell + D_\ell(\dot{\theta}_\ell) + r_1(\theta_\ell,\theta_m)\right] \quad (5.14)\\ \ddot{\theta}_m &= -M_{22}^{-1}\left[D_m(\dot{\theta}_m) + r_2(\theta_\ell,\theta_m)\right] + M_{22}^{-1}u \quad (5.15)\end{aligned}$$

Substitute (5.14,5.15) in (5.13) to obtain, for $\mathcal{X}_2$ properly defined,

$$\ddot{\theta}_{m_d} = \mathcal{X}_2\left(\theta_{\ell_d},\dot{\theta}_{\ell_d},\ddot{\theta}_{\ell_d},\theta_{\ell_d}^{(3)},\theta_{\ell_d}^{(4)},\theta_\ell,\dot{\theta}_\ell,\theta_m,\dot{\theta}_m\right) - R_{12}{}^{-1}\nabla_{\theta_m}\bar{r}_1(\theta_\ell,\theta_m)M_{22}^{-1}u \quad (5.16)$$

which is bounded if Assumptions 5.1 and 5.2 are satisfied and if $\Delta\ddot{\theta}$ remains bounded.

Use (5.11) and (5.16) to write

$$\begin{aligned}u_{ff} &= \left[D_m(\dot{\theta}_{m_d}) + \bar{r}_2(\theta_\ell,\theta_m) + R_{21}\theta_{\ell_d} + R_{22}\theta_{m_d}\right] +\\ &\quad M_{22}\left\{\mathcal{X}_2\left(\theta_{\ell_d},\dot{\theta}_{\ell_d},\ddot{\theta}_{\ell_d},\theta_{\ell_d}^{(3)},\theta_{\ell_d}^{(4)},\theta_\ell,\dot{\theta}_\ell,\theta_m,\dot{\theta}_m\right) - R_{12}{}^{-1}\nabla_{\theta_m}\bar{r}_1(\theta_\ell,\theta_m)M_{22}^{-1}\left[(u-u_{ff})+u_{ff}\right]\right\}\\ &= M_{22}\left[I + R_{12}{}^{-1}\nabla_{\theta_m}\bar{r}_1(\theta_\ell,\theta_m)\right]^{-1}\left\{M_{22}^{-1}\left[D_m(\dot{\theta}_{m_d}) + \bar{r}_2(\theta_\ell,\theta_m) + R_{21}\theta_{\ell_d} + R_{22}\theta_{m_d}\right] +\right.\\ &\quad \left.\left[\mathcal{X}_2\left(\theta_{\ell_d},\dot{\theta}_{\ell_d},\ddot{\theta}_{\ell_d},\theta_{\ell_d}^{(3)},\theta_{\ell_d}^{(4)},\theta_\ell,\dot{\theta}_\ell,\theta_m,\dot{\theta}_m\right) - R_{12}{}^{-1}\nabla_{\theta_m}\bar{r}_1(\theta_\ell,\theta_m)M_{22}^{-1}(u-u_{ff})\right]\right\}\end{aligned}$$

where the term $(u - u_{ff})$ represents the contribution of the stabilizing controller, and under the assumption that $\left[I + R_{12}{}^{-1}\nabla_{\theta_m}\bar{r}_1(\theta_\ell,\theta_m)\right]$ is invertible for all (θ_ℓ,θ_m), which is always possible by a proper choice of R_{12}.

Hence, the solution of u_{ff} for this case requires invertibility of R_{12} and of $\left[I + R_{12}{}^{-1}\nabla_{\theta_m}\bar{r}_1(\theta_\ell, \theta_m)\right]$ along with knowledge of the full model, measurement of the full state, differentiability of r_1 twice in both its arguments, and knowledge of the stabilizing controller signal. This last requirement also means that the feedforward and the feedback are intertwined. □

5.4 Solution of FF3

Expand **FF3** to obtain, for the general model,

$$\begin{aligned} 0 &= M_{11}(\theta_\ell, \theta_m)\ddot{\theta}_{\ell_d} + M_{12}(\theta_\ell, \theta_m)\ddot{\theta}_{m_d} + C_{11}(\theta_\ell, \theta_m, \dot{\theta}_\ell, \dot{\theta}_m)\dot{\theta}_{\ell_d} + C_{12}(\theta_\ell, \theta_m, \dot{\theta}_\ell, \dot{\theta}_m)\dot{\theta}_{m_d} + \\ &\quad D_\ell(\dot{\theta}_{\ell_d}) + r_1(\theta_{\ell_d}, \theta_{m_d}) \qquad (5.17) \\ u_{ff} &= M_{12}^T(\theta_\ell, \theta_m)\ddot{\theta}_{\ell_d} + M_{22}(\theta_m)\ddot{\theta}_{m_d} + C_{21}(\theta_\ell, \theta_m, \dot{\theta}_\ell, \dot{\theta}_m)\dot{\theta}_{\ell_d} + C_{22}(\theta_m, \dot{\theta}_m)\dot{\theta}_{m_d} + \\ &\quad D_m(\dot{\theta}_{m_d}) + r_2(\theta_{\ell_d}, \theta_{m_d}) \qquad (5.18) \end{aligned}$$

To obtain a closed loop form for the feedforward control input u_{ff} from (5.18) for $(\theta_{\ell_d}, \dot{\theta}_{\ell_d}, \ddot{\theta}_{\ell_d})$ given , we must first define $(\theta_{m_d}, \dot{\theta}_{m_d}, \ddot{\theta}_{m_d})$.

The problem of solving for $(\theta_{m_d}, \dot{\theta}_{m_d}, \ddot{\theta}_{m_d})$ is very similar to the one encountered with **FF1** except that (5.17) depends on the actual state $(\theta_\ell, \dot{\theta}_\ell, \theta_m, \dot{\theta}_m)$. Hence, the feedforward and the control input are intertwined. This renders the task of characterizing the solutions more difficult as noted in the discussion on the solution of **FF2** (Section 5.3).

If we try to find the closed loop solution under the same assumptions as in Example 5.2, in particular M_{12} strictly upper triangular, we encounter the same problems as with **FF2**. We will thus consider the system in Example 5.3 to demonstrate the solution of this feedforward.

Example 5.4 Assume that the inertia of the motors is symmetric about their axis of rotation, and that $M_{12} = 0$ (Section 3.2.4). Also assume that the i^{th} element of $r^1(\theta_\ell, \theta_m)$ depends only on $(\theta_{\ell,i}, \theta_{m,i})$. Under these assumptions, expand **FF3** to obtain

$$\begin{aligned} 0 &= M_{11}(\theta_\ell)\ddot{\theta}_{\ell_d} + C_{11}(\theta_\ell, \dot{\theta}_\ell)\dot{\theta}_{\ell_d} + D_\ell(\dot{\theta}_{\ell_d}) + r_1(\theta_{\ell_d}, \theta_{m_d}) \qquad (5.19) \\ u_{ff} &= M_{22}\ddot{\theta}_{m_d} + D_m(\dot{\theta}_{m_d}) + r_2(\theta_{\ell_d}, \theta_{m_d}) \qquad (5.20) \end{aligned}$$

Also suppose that Assumption 5.3 holds. This implies that, by the Implicit Function Theorem [3], there exists a locally unique solution θ_{m_d} to (5.19) for any given θ_ℓ, $\dot{\theta}_\ell$, θ_{ℓ_d} and $\ddot{\theta}_{\ell_d}$.

To solve for $\dot{\theta}_{m_d}$ and $\ddot{\theta}_{m_d}$, assume that r_1 is differentiable twice in both its arguments and that Assumption 5.4 holds. Take the first time derivative of (5.19) to obtain:

$$\begin{aligned} 0 &= \dot{M}_{11}(\theta_\ell, \dot{\theta}_\ell)\ddot{\theta}_{\ell_d} + M_{11}(\theta_\ell, \dot{\theta}_\ell)\theta_{\ell_d}^{(3)} + \dot{C}_{11}(\theta_\ell, \dot{\theta}_\ell, \ddot{\theta}_\ell)\dot{\theta}_{\ell_d} + C_{11}(\theta_\ell, \dot{\theta}_\ell, \ddot{\theta}_\ell)\ddot{\theta}_{\ell_d} + \\ &\quad \dot{D}_\ell(\dot{\theta}_{\ell_d}, \ddot{\theta}_{\ell_d}) + \nabla_{\theta_{\ell_d}} r_1(\theta_{\ell_d}, \theta_{m_d})\dot{\theta}_{\ell_d} + \nabla_{\theta_{m_d}} r_1(\theta_{\ell_d}, \theta_{m_d})\dot{\theta}_{m_d} \qquad (5.21) \end{aligned}$$

We then solve for $\ddot{\theta}_\ell$ by using the system equations in order to remove dependence of the feedforward on this variable (see (3.20)):

$$\ddot{\theta}_\ell = -M_{11}(\theta_\ell)^{-1}\left[C_{11}(\theta_\ell, \dot{\theta}_\ell)\dot{\theta}_\ell + D_\ell(\dot{\theta}_\ell) + r_1(\theta_\ell, \theta_m)\right] \qquad (5.22)$$

By substituting (5.22) into (5.21), we may write

$$0 = \mathcal{X}(\theta_\ell, \dot{\theta}_\ell, \theta_m, \theta_{\ell_d}, \dot{\theta}_{\ell_d}, \ddot{\theta}_{\ell_d}, \theta_{\ell_d}^{(3)}, \theta_{m_d}) + \nabla_{\theta_{m_d}} r_1(\theta_{\ell_d}, \theta_{m_d})\dot{\theta}_{m_d} \qquad (5.23)$$

such that we may solve for $\dot{\theta}_{m_d}$ by Assumptions 5.3 and 5.4 provided that θ_{m_d} has been previously evaluated and that θ_ℓ, $\dot{\theta}_\ell$ and θ_m are available by measurement.

We proceed similarly to evaluate $\ddot{\theta}_{m_d}$ by taking the first time derivative of (5.23), removing the dependence on $\ddot{\theta}_\ell$ by use of (5.22), and by assuming that $\dot{\theta}_{m_d}$ has been previously evaluated and that $\dot{\theta}_m$ is also available by measurement. We note that the fourth time derivative of θ_{ℓ_d} is required.

The internal state remains bounded if Assumptions 5.1 and 5.2 are satisfied.

We then obtain the feedforward input by substituting the variables into (5.20).

Hence, solution of u_{ff} requires invertibility (at least local) of r_1, that r_1 is twice differentiable in both its arguments, along with knowledge of the full model and full state measurement. □

5.5 Solution of FF4

Expand **FF4** to obtain, for the general model,

$$\begin{aligned} 0 &= M_{11}(\theta_\ell,\theta_m)\ddot{\theta}_{\ell_d} + M_{12}(\theta_\ell,\theta_m)\ddot{\theta}_{m_d} + C_{11}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m)\dot{\theta}_{\ell_d} + C_{12}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m)\dot{\theta}_{m_d} + \\ &\quad D_\ell(\dot{\theta}_{\ell_d}) + \bar{r}_1(\theta_\ell,\theta_m) + R_{11}\theta_{\ell_d} + R_{12}\theta_{m_d} \qquad (5.24) \\ u_{ff} &= M_{12}^T(\theta_\ell,\theta_m)\ddot{\theta}_{\ell_d} + M_{22}(\theta_m)\ddot{\theta}_{m_d} + C_{21}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m)\dot{\theta}_{\ell_d} + C_{22}(\theta_m,\dot{\theta}_m)\dot{\theta}_{m_d} + \\ &\quad D_m(\dot{\theta}_{m_d}) + \bar{r}_2(\theta_\ell,\theta_m) + R_{21}\theta_{\ell_d} + R_{22}\theta_{m_d} \qquad (5.25) \end{aligned}$$

To obtain a closed loop form for the feedforward control input u_{ff} from (5.25) for $(\theta_{\ell_d},\dot{\theta}_{\ell_d},\ddot{\theta}_{\ell_d})$ given, we must first define $(\theta_{m_d},\dot{\theta}_{m_d},\ddot{\theta}_{m_d})$.

The problem of solving for $(\theta_{m_d},\dot{\theta}_{m_d},\ddot{\theta}_{m_d})$ is very similar to the one encountered with **FF2** except that (5.5) depends on the actual state $(\theta_\ell,\dot{\theta}_\ell,\theta_m,\dot{\theta}_m)$.

Example 5.5 Consider the system used in Example 5.3. Expand **FF4** to obtain

$$\begin{aligned} 0 &= M_{11}(\theta_\ell)\ddot{\theta}_{\ell_d} + C_{11}(\theta_\ell,\dot{\theta}_\ell)\dot{\theta}_{\ell_d} + D_\ell(\dot{\theta}_{\ell_d}) + \bar{r}_1(\theta_\ell,\theta_m) + R_{11}\theta_{\ell_d} + R_{12}\theta_{m_d} \qquad (5.26) \\ u_{ff} &= M_{22}\ddot{\theta}_{m_d} + D_m(\dot{\theta}_{m_d}) + \bar{r}_2(\theta_\ell,\theta_m) + R_{21}\theta_{\ell_d} + R_{22}\theta_{m_d} \qquad (5.27) \end{aligned}$$

Given θ_{ℓ_d} and its higher derivatives, that R_{12} is nonsingular but otherwise arbitrary, and assuming that θ_ℓ, $\dot{\theta}_\ell$ and θ_m are available for measurement, we have, from (5.26),

$$\theta_{m_d} = -R_{12}{}^{-1}\left[M_{11}(\theta_\ell)\ddot{\theta}_{\ell_d} + C_{11}(\theta_\ell,\dot{\theta}_\ell)\dot{\theta}_{\ell_d} + D_\ell(\dot{\theta}_{\ell_d}) + \bar{r}_1(\theta_\ell,\theta_m) + R_{11}\theta_{\ell_d}\right] \qquad (5.28)$$

Suppose that Assumption 5.4 is satisfied. We solve for $\dot{\theta}_{m_d}$ by taking the first time derivative of (5.28) under the assumption that $\dot{\theta}_m$ is available for measurement, and by removing the dependence of this equation on $\ddot{\theta}_\ell$ by using the system equations as done in Example 5.4. Now, taking the second time derivative of (5.28) in order to find $\ddot{\theta}_{m_d}$, and substituting $\ddot{\theta}_\ell$, $\theta_\ell^{(3)}$ by functions of measurable state only, we may write, for $\mathcal{X}$ properly defined:

$$\begin{aligned} \ddot{\theta}_{m_d} &= -R_{12}{}^{-1}\frac{d^2}{dt^2}\left[M_{11}(\theta_\ell)\ddot{\theta}_{\ell_d} + C_{11}(\theta_\ell,\dot{\theta}_\ell)\dot{\theta}_{\ell_d} + D_\ell(\dot{\theta}_{\ell_d}) + \bar{r}_1(\theta_\ell,\theta_m) + R_{11}\theta_{\ell_d}\right] \\ &= \mathcal{X}\left(\dot{\theta}_{\ell_d},\ddot{\theta}_{\ell_d},\theta_{\ell_d}^{(3)},\theta_{\ell_d}^{(4)},\theta_\ell,\dot{\theta}_\ell,\theta_m,\dot{\theta}_m\right) \\ &\quad -R_{12}{}^{-1}\left[\nabla_{\theta_\ell}\bar{r}_1(\theta_\ell,\theta_m)\ddot{\theta}_\ell + \nabla_{\theta_m}\bar{r}_1(\theta_\ell,\theta_m)\ddot{\theta}_m\right] \end{aligned}$$

i.e. $\ddot{\theta}_{m_d}$ is of the same form as in (5.13) so that we pursue the solution as in Example 5.3, and the same conditions for solubility are obtained as for **FF2**. Furthermore, the signals remain bounded if Assumptions 5.1 and 5.2 are satisfied. □

5.6 Solution of FF5, FF6

The solution of **FF5** and **FF6** are respectively very similar to those of **FF3** and **FF4** such that the reader is referred to Sections 5.4 and 5.5 for information about their solution.

5.7 Solution of FF7

Expand **FF7** to obtain

$$0 = r_1(\theta_{\ell_d}, \theta_{m_d}) \tag{5.29}$$

$$u_{ff} = r_2(\theta_{\ell_d}, \theta_{m_d}) \tag{5.30}$$

To compute the feedforward control input u_{ff} from (5.30) for a given θ_{ℓ_d}, θ_{m_d} must first be defined.

Suppose that Assumption 5.3 holds. Then by the Implicit Function Theorem [3], there exists a locally unique solution θ_{m_d} to (5.29). The solution can then be substituted in (5.30) to obtain u_{ff}.

Hence, solution of u_{ff} requires invertibility (at least local) of $\nabla_{\theta_m} r_1(\theta_\ell, \theta_m)$ along with knowledge of r, i.e. gravity load and spring characteristics. No other restrictions are imposed on the system and no measurements are required.

If higher order derivatives of θ_{m_d} are required for the controller, we may proceed as follows. In order to obtain the i^{th} derivative of θ_{m_d} further assume that r_1 is differentiable i times in both its arguments, and that all higher derivatives of θ_{ℓ_d} up to the i^{th} are available. Take the first time derivative of (5.29) to get

$$0 = \nabla_{\theta_{\ell_d}} r_1(\theta_{\ell_d}, \theta_{m_d})\dot{\theta}_{\ell_d} + \nabla_{\theta_{m_d}} r_1(\theta_{\ell_d}, \theta_{m_d})\dot{\theta}_{m_d} \tag{5.31}$$

$$\dot{\theta}_{m_d} = -\left[\nabla_{\theta_{m_d}} r_1(\theta_{\ell_d}, \theta_{m_d})\right]^{-1} \nabla_{\theta_{\ell_d}} r_1(\theta_{\ell_d}, \theta_{m_d})\dot{\theta}_{\ell_d}$$

given that θ_{m_d} has been previously evaluated.

Taking the time derivative of (5.31), and given that θ_{m_d} and $\dot{\theta}_{m_d}$ have been previously solved, $\ddot{\theta}_{m_d}$ is found in the same fashion. Hence, we can solve iteratively for the higher derivatives, and boundedness of the signals is guaranteed for any bounded desired output trajectory.

Example 5.6 A common case for which we can solve for the feedforward is as follows. Assume that the inertia of the motors is symmetric about their axis of rotation (Section 3.2.2), and that k_1 is defined as follows

$$(k_1)_{ij}(\theta_\ell, \theta_m) = \begin{cases} f_i(N_i\theta_{\ell i} - \theta_{m i}) & i = j \\ 0 & i \neq j \end{cases} \qquad i, j = 1, \cdots, n \tag{5.32}$$

where N_i and f_i are respectively the gear ratio and the elasticity characteristic of the ith joint. If all the f_i's are monotonically increasing, continuous, differentiable, and their range is $\mathbf{R}$, then k_1 is globally invertible.

Equations (5.29, 5.30) may be written as

$$0 = g_\ell(\theta_{\ell_d}) + Nk_1(\theta_{\ell_d}, \theta_{m_d}) \tag{5.33}$$

$$u_{ff} = -k_1(\theta_{\ell_d}, \theta_{m_d}) \tag{5.34}$$

and a unique solution, θ_{m_d}, to (5.33) can be found for any θ_{ℓ_d} and is given by

$$\theta_{m_d} = N\theta_{\ell_d} - k_1^{-1}\left(-N^{-1}g_\ell(\theta_{\ell_d})\right) \tag{5.35}$$

We obtain u_{ff} by substituting (5.35) into (5.34) which gives, after simplification:

$$u_{ff} = N^{-1}g_\ell(\theta_{\ell_d})$$

□

Note that for the previous example without the motor symmetry assumption, a local result may be obtained under proper assumptions.

5.8 Solution of FF8

Expand **FF8** to obtain

$$0 = \bar{r}_1(\theta_\ell, \theta_m) + R_{11}\theta_{\ell_d} + R_{12}\theta_{m_d} \quad (5.36)$$

$$u_{ff} = \bar{r}_2(\theta_\ell, \theta_m) + R_{21}\theta_{\ell_d} + R_{22}\theta_{m_d} \quad (5.37)$$

To compute the feedforward control input u_{ff} from (5.37) for a given θ_{ℓ_d}, we assume that θ_ℓ and θ_m are available for measurement and the reference position θ_{m_d} must be defined.

Assume that R is chosen with R_{12} nonsingular, we obtain θ_{m_d} from (5.36):

$$\theta_{m_d} = -R_{12}{}^{-1}\left[\bar{r}_1(\theta_\ell, \theta_m) + R_{11}\theta_{\ell_d}\right] \quad (5.38)$$

We obtain u_{ff} by substituting (5.38) into (5.37):

$$u_{ff} = \bar{r}_2(\theta_\ell, \theta_m) + R_{21}\theta_{\ell_d} - R_{22}R_{12}{}^{-1}\left[\bar{r}_1(\theta_\ell, \theta_m) + R_{11}\theta_{\ell_d}\right]$$

$$u_{ff} = \bar{r}_2(\theta_\ell, \theta_m) - R_{22}R_{12}{}^{-1}\bar{r}_1(\theta_\ell, \theta_m) + \left[R_{21} - R_{22}R_{12}{}^{-1}R_{11}\right]\theta_{\ell_d} \quad (5.39)$$

and all the signals are bounded if Assumption 5.1 is satisfied.

We see from (5.39) that if R is singular, then θ_{ℓ_d} does not affect u_{ff} directly in certain directions. However, there seems to be no reasons at this point to restrict R to be nonsingular.

Hence, solution of u_{ff} requires invertibility of R_{12}, which is achieved by proper design, exact knowledge of r, i.e. gravity load and spring characteristics and measurement of motor and link position. Also, solution of the feedforward imposes no restriction on the structure of the system and on the system parameter values.

However, the controller may require higher order derivatives of θ_{m_d}, in particular $\dot{\theta}_{m_d}$ to implement a PD motor loop. Given $\dot{\theta}_{\ell_d}$, $\dot{\theta}_{m_d}$ is obtained by taking the time derivative of (5.38) assuming that $\bar{r}_1$ is differentiable in both θ_ℓ and θ_m and that measurement of motor and link velocity are available:

$$\dot{\theta}_{m_d} = -R_{12}{}^{-1}\left(\nabla_\theta \bar{r}_1(\theta_\ell, \theta_m)\dot{\theta} + R_{11}\dot{\theta}_{\ell_d}\right)$$

where $\nabla_x \bar{r}_1(x)$ is the gradient of $\bar{r}_1$ with respect to x. Note that the condition of differentiability of $\bar{r}_1$ implies a similar condition on r_1, i.e. on the spring characteristic itself. For boundedness of $\dot{\theta}_{m_d}$, Assumption 5.2 is required.

6 Controller design for tracking

In this section, we carry out the controller design, i.e. error system stabilization, for the error system formed by using using **FF4** and **FF6** defined in Section 5.1 in order to demonstrate the application of the procedure established in Section 4 and to present some stability results and design requirements. We recall that the procedure allows to ascertain asymptotic stability for the system by using any controller in the class of strictly passive and BIBO stable controllers if we can show that $y_v \to 0$ and if the zero–state detectability condition is met.

6.1 Stabilization of error system formed by FF4

For **FF4** with C represented using Christoffel's symbols (Section 3.1), define

$$u = u_o + u_{ff}$$

to obtain the following error equation for the system :

$$M(\theta)\Delta\ddot{\theta} + C_C(\theta, \dot{\theta})\Delta\dot{\theta} + D(\dot{\theta}) - D(\dot{\theta}_d) + R\Delta\theta = Bu_o \quad (6.1)$$

6.1.1 Static feedback

In Appendix B (Section B.2), we show that the use of the motor position as part of the output may allow to obtain stable zero dynamics for flexible joint robots, implying at least weak zero–state detectability, and also to render the system passive if, for example, the motor velocity is also part of the output (see case 1 of Section B.2). This motivates the following design.

Consider a static feedback of the motor error state ($C_p(y_p) = K_p B^T \Delta\theta$):

$$u_o = u_1 - K_p B^T \Delta\theta$$

The closed loop error equation becomes:

$$M(\theta)\Delta\ddot{\theta} + C_C(\theta,\dot{\theta})\Delta\dot{\theta} + D(\dot{\theta}) - D(\dot{\theta}_d) + \left[R + BK_pB^T\right]\Delta\theta = Bu_1 \tag{6.2}$$

Proposition 6.1 *The pair* $(u_1, B^T\Delta\dot{\theta})$ *is a passive pair for the error system if the two following assumptions are satisfied:*

Assumption 6.1 *R and* K_p *are chosen such that* $\left[R + BK_pB^T\right] > 0$.

Assumption 6.2 *There is no negative damping in the system.*

Proof : In order to show passivity, consider the following energy function based on the total energy of the error system

$$V = \frac{1}{2}\Delta\theta^T\left[R + BK_pB^T\right]\Delta\theta + \frac{1}{2}\Delta\dot{\theta}^T M(\theta)\Delta\dot{\theta} \tag{6.3}$$

Using (6.2,6.3) and the fact that $(\frac{1}{2}\dot{M} - C_C)$ is skew–symmetric, we obtain

$$\dot{V} = \Delta\dot{\theta}^T Bu_1 - \Delta\dot{\theta}^T\left[D(\dot{\theta}) - D(\dot{\theta}_d)\right] \tag{6.4}$$

Evaluate the time integral of (6.4) :

$$V(t) - V(t_0) = \int_{t_0}^{t}\Delta\dot{\theta}^T Bu_1\,dt - \int_{t_0}^{t}\Delta\dot{\theta}^T\left[D(\dot{\theta}) - D(\dot{\theta}_d)\right]dt \tag{6.5}$$

$$\int_{t_0}^{t}\Delta\dot{\theta}^T Bu_1\,dt = V(t) - V(t_0) + \int_{t_0}^{t}\Delta\dot{\theta}^T\left[D(\dot{\theta}) - D(\dot{\theta}_d)\right]dt \tag{6.6}$$

Assumption 6.1 implies that V is positive definite, and Assumption 6.2 implies that

$$\Delta\dot{\theta}^T\left[D(\dot{\theta}) - D(\dot{\theta}_d)\right] \geq 0,\ \forall\dot{\theta},\dot{\theta}_d$$

such that (6.6) satisfies the condition for passivity (Definition 2.6). ■

Note that the position feedback is not required in general to guarantee passivity, e.g. the spring coupling matrix for manipulators with linear flexible joints may be used to define R which then takes the form [31]

$$R = \begin{bmatrix} K_eN^2 & -K_eN \\ -K_eN & K_e \end{bmatrix}$$

and is then positive semi–definite (K_e is positive definite). However, the position feedback is useful to guarantee (weak) zero–state detectability (Section 7).

Also note that the system remains passive in the absence of friction, i.e. if $D(\dot{\theta}) = 0$.

Moreover, assumption 6.2 could be relaxed by employing the factorization

$$D(\dot{\theta}) = D_1(\dot{\theta}) + D_0\dot{\theta} \tag{6.7}$$

and replacing $D(\dot{\theta}_d)$ in the feedforward by $D_1(\dot{\theta}) + D_0\dot{\theta}_d$, $D_0 \geq 0$, assuming that the feedforward is solvable.

Then, $\left[D(\dot{\theta}) - D(\dot{\theta}_d)\right]$ is replaced by $D_0\Delta\dot{\theta}$ in (6.1, 6.2, 6.4, 6.5, 6.6) such that due to the positive semi–definitiveness of D_0, only Assumption 6.1 is required to guarantee passivity.

6.1.2 Strictly passive feedback

Choose

$$u_1 = u_2 - \mathcal{C}_v(B^T\Delta\dot{\theta}) \tag{6.8}$$

where $\mathcal{C}_v$ is strictly passive and BIBO stable. Then, by the Passivity Theorem (Section 2), the map from u_2 to $B^T\Delta\dot{\theta}$ is L_2 stable.

The closed loop error system is given by

$$M(\theta)\Delta\ddot{\theta} + C_C(\theta,\dot{\theta})\Delta\dot{\theta} + D(\dot{\theta}) - D(\dot{\theta}_d) + \left[R + BK_pB^T\right]\Delta\theta + B\mathcal{C}_v(B^T\Delta\dot{\theta}) = Bu_2 \tag{6.9}$$

and, using the same energy function V as before (6.3),

$$\dot{V} = \Delta\dot{\theta}^T Bu_2 - \Delta\dot{\theta}^T\left[D(\dot{\theta}) - D(\dot{\theta}_d)\right] - \left[B^T\Delta\dot{\theta}\right]^T \mathcal{C}_v(B^T\Delta\dot{\theta}) \tag{6.10}$$

Proposition 6.2 *If $u_2 = 0$, we can conclude that $(\Delta\theta, \Delta\dot{\theta})$ converges to the largest invariant set in $\{(\Delta\theta,\Delta\dot{\theta}) : B^T\Delta\dot{\theta} = 0\}$ for $(\dot{\theta}_d, \ddot{\theta}_d)$ uniformly bounded.*

Proof : Recall that the strict passivity of $\mathcal{C}_v$ means

$$\int_{t_0}^{T} w^T\mathcal{C}_v(w)\,dt \geq -\gamma^2 + \eta\int_{t_0}^{T}\|w\|^2\,dt \tag{6.11}$$

for any $T \geq t_0$, $w \in L_{2_e}$, where $\eta > 0$ is a constant, and the constant γ depends on the initial condition of the internal state of $\mathcal{C}_v$. Using (6.11) and integrating (6.10), we have

$$\begin{aligned} V\left(\Delta\theta(T),\Delta\dot{\theta}(T)\right) - V\left(\Delta\theta(t_0),\Delta\dot{\theta}(t_0)\right) &\leq \gamma^2 - \eta\int_{t_0}^{T}\left\|B^T\Delta\dot{\theta}(t)\right\|^2 dt \\ &\quad - \int_{t_0}^{T}\Delta\dot{\theta}^T(t)\left[D(\dot{\theta}(t)) - D(\dot{\theta}_d(t))\right] dt \end{aligned}$$

From this inequality, we conclude that $V(\Delta\theta,\Delta\dot{\theta})$ is uniformly bounded. Also, positive definitiveness of V implies that $\Delta\theta$ and $\Delta\dot{\theta}$ are uniformly bounded. By (3.31), the Coriolis and centrifugal force term in the error equation (6.9) is bounded above in norm under the assumption that $\dot{\theta}_d$ is uniformly bounded. Hence, given the BIBO stability of $\mathcal{C}_v$, we conclude from (6.9) that $\Delta\ddot{\theta}$ is uniformly bounded. Uniform boundedness of $\ddot{\theta}_d$ must also be assumed in order to guarantee uniform boundedness of the feedforward. Uniform boundedness of $\Delta\ddot{\theta}$ implies continuity of $\Delta\dot{\theta}$, which, along with the fact that $B^T\Delta\dot{\theta}(t) \in L_2$, lead to the conclusion that $B^T\Delta\dot{\theta}(t) \to 0$ asymptotically by Barbalat's lemma [25]. ∎

Furthermore, assume that $\theta_d \in C^k$, $k \geq 2$, and that $D \in C^{k-2}$ (no Coulomb friction or exact compensation of the discontinuous terms). Also consider the fact that $M(\theta)$ and $C_C(\theta, v)$ are functions of *sin* and *cos*, and are uniformly bounded as well as all their higher time derivatives if v and its higher derivatives are uniformly bounded. Then, by taking successive time derivatives of (6.9) and from ($\Delta\theta = \theta - \theta_d$), we find that all higher derivatives of $\Delta\dot{\theta}$ up to its $(k-1)^{th}$ derivative are uniformly bounded, and all higher derivatives of $B^T\Delta\dot{\theta}$ up to its $(k-2)th$ derivative tend to zero asymptotically. This last result will be useful in the demonstration of zero–state detectability where the convergence to zero of the higher derivatives of $B^T\Delta\dot{\theta}$ is needed.

Proposition 6.3 *The zero error state of the system with strictly passive BIBO stable feedback $\mathcal{C}_v$ and for $u_2 = 0$ is (locally) asymptotically stable under the following assumption*

Assumption 6.3 *$B^T\Delta\dot{\theta}$ is (locally) zero–state detectable in $(\Delta\theta, \Delta\dot{\theta})$ with respect to the following equation:*

$$M(\theta)\Delta\ddot{\theta} + C_C(\theta,\dot{\theta})\Delta\dot{\theta} + D(\dot{\theta}) - D(\dot{\theta}_d) + \left[R + BK_pB^T\right]\Delta\theta = 0$$

If the detectability is global, then so is the asymptotic stability.

NOTE : conditions for (local) zero–state detectability will be given in Section 7.

Proof : Proposition 6.2 and the BIBO stability of $\mathcal{C}_v$ imply that

$$u_1 = \mathcal{C}_v(B^T\Delta\dot{\theta}) \rightarrow 0$$

Substitute this result in (6.2) to get that for $B^T\Delta\dot{\theta} \rightarrow 0$ asymptotically,

$$M(\theta)\Delta\ddot{\theta} + C_C(\theta,\dot{\theta})\Delta\dot{\theta} + D(\dot{\theta}) - D(\dot{\theta}_d) + \left[R + BK_pB^T\right]\Delta\theta \rightarrow 0$$

If $B^T\Delta\dot{\theta}$ is (locally) zero–state detectable in $(\Delta\theta, \Delta\dot{\theta})$ with respect to this equation, then, $(\Delta\theta, \Delta\dot{\theta}) \rightarrow 0$ and the error state of the system is (locally) asymptotic stable. ■

6.2 Stabilization of error system formed by FF6

The design procedure and requirements for the use of **FF6** (C is represented using the M_D–notation (Section 3.1)) are the same as for **FF4** in Section 6.1. However, given the new error equation with static position error feedback

$$\begin{aligned} M(\theta)\Delta\ddot{\theta} + C_D(\theta,\dot{\theta})\Delta\dot{\theta} + \frac{1}{2}M_D(\theta,\dot{\theta})\dot{\theta}_d - \frac{1}{2}M_D(\theta,\dot{\theta}_d)\dot{\theta} + D(\dot{\theta}) - D(\dot{\theta}_d) + \\ \left[R + BK_pB^T\right]\Delta\theta &= Bu_1 \qquad (6.12) \end{aligned}$$

property (3.25) is used to evaluate the time derivative of V (6.3) to be (6.4), and to be (6.10) after the strictly passive feedback is added.

Also, identity (3.30) is used to bound the Coriolis and centrifugal force term in the new error equation for Proposition 6.2 to hold.

7 Zero–state detectability

In this section, conditions for zero–state detectability are defined for feedforward **FF4** with C represented using the Christoffel's symbols and for **FF6** in which C is represented using the M_D–notation. In essence, we want to show that in (6.2), (6.12), if the passive output $B^T\Delta\dot{\theta}$ and the input u_1 are zero then the internal state of the system is zero, or tends to zero for weak zero–state detectability (Section 2).

7.1 General result with link damping

For **FF4**, consider the error equation (6.2), and the scalar function V (6.3) and its derivative $\dot{V}$ (6.4). We want to use V to demonstrate that the link state error tends to zero asymptotically. Suppose that

Assumption 6.1 holds. After setting u_1, and $B^T\dot{\theta}$ to zero, we get, for the general model,

$$V = \frac{1}{2}\Delta\theta^T\left[R + BK_pB^T\right]\Delta\theta + \frac{1}{2}\Delta\dot{\theta}_\ell^T M_{11}(\theta_\ell, \theta_m)\Delta\dot{\theta}_\ell$$

We note that V is positive definite in $\Delta\dot{\theta}_\ell$. In order to have V positive definite in $\Delta\theta_\ell$ with $\Delta\theta_\ell = 0$ at its minimum, choose R such that $R_{12} = -R_{21}{}^T$ (recall that $R_{11} > 0$ to satisfy Assumption 6.1). We also have

$$\dot{V} = -\Delta\dot{\theta}^T\left[D(\dot{\theta}) - D(\dot{\theta}_d)\right]$$

Hence, if there is positive link damping, $\dot{V}$ is negative semi-definite and $\Delta\dot{\theta}_\ell \to 0$. Substituting this result along with the assumptions on u_1 and $B^T\Delta\dot{\theta}$ in (6.2), we obtain

$$\left[R + BK_pB^T\right]\Delta\theta = 0$$

which leads to the conclusion that $\Delta\theta \to 0$ by Assumption 6.1, implying that the internal state converges to zero.

Hence, for any arm configuration and in presence of positive damping at the links, the system is weakly zero–state detectable (globally) if Assumption 6.1 is satisfied and if we choose R such that $R_{12} = -R_{21}{}^T$.

The same conclusions are reached for **FF6** by using the same procedure.

7.2 Results without link damping

In order to assess (weak) zero–state detectability, or zero–state detectability, in the absence of link damping, we use a different approach than in Section 7.1. This will allow us to obtain a general conclusion along with some positive results for some general cases.

NOTE : As seen in Section 7.1, $(u_1, B^T\Delta\dot{\theta}, \Delta\dot{\theta}_\ell) = (0,0,0)$ implies that $\Delta\theta_\ell$ is zero, and hence the internal state is also zero.

Consider the error equation of the system (feedforward **FF4**) with static feedback only (6.2). For the general model (3.10), after setting $u_1 = 0$, and $\Delta\dot{\theta}_m$ and its higher derivatives to zero, (6.2) simplifies to

$$M_{11}(\theta_\ell, \theta_m)\Delta\ddot{\theta}_\ell + C_{11}(\theta_\ell, \dot{\theta}_\ell, \theta_m, \dot{\theta}_m)\Delta\dot{\theta}_\ell + D_\ell(\dot{\theta}_\ell) - D_\ell(\dot{\theta}_{\ell_d}) + R_{11}\Delta\theta_\ell + R_{12}\Delta\theta_m = 0 \quad (7.1)$$

$$M_{12}^T(\theta_\ell, \theta_m)\Delta\ddot{\theta}_\ell + C_{21}(\theta_\ell, \dot{\theta}_\ell, \theta_m, \dot{\theta}_m)\Delta\dot{\theta}_\ell + R_{21}\Delta\theta_\ell + R_{22}\Delta\theta_m + K_p\Delta\theta_m = 0 \quad (7.2)$$

Then take the time derivative of (7.2) to get, at $\Delta\dot{\theta}_m = 0$,

$$\begin{aligned} M_{12}^T(\theta_\ell, \theta_m)\Delta\theta_\ell^{(3)} + \dot{M}_{12}^T(\theta_\ell, \dot{\theta}_\ell, \theta_m, \dot{\theta}_m)\Delta\ddot{\theta}_\ell + C_{21}(\theta_\ell, \dot{\theta}_\ell, \theta_m, \dot{\theta}_m)\Delta\ddot{\theta}_\ell & \\ + \dot{C}_{21}(\theta_\ell, \dot{\theta}_\ell, \ddot{\theta}_\ell, \theta_m, \dot{\theta}_m, \ddot{\theta}_m)\Delta\dot{\theta}_\ell + R_{21}\Delta\dot{\theta}_\ell &= 0 \end{aligned} \quad (7.3)$$

For **FF6** we may write similar equations.

RESULT : If (7.3) is asymptotically stable ($\Delta\dot{\theta}_\ell \to 0$) then the system is at least weakly zero–state detectable and a similar conclusion is obtained for the equivalent equation obtained with **FF6**.

We now consider three general cases for which we seek conclusions on zero–state detectability.

Case 1: $M_{12} = 0$

For $M_{12} = 0$ and under the additional assumption that the motors are symmetric or that M_{11} is independent of the motor position, (7.3) becomes

$$R_{21}\Delta\dot{\theta}_\ell = 0 \quad (7.4)$$

such that, for R_{21} arbitrary but nonsingular, $\Delta\dot{\theta}_\ell$ and its higher derivatives equal zero and, as noted previously the internal state is also zero, i.e. we obtain global zero–state detectability.

NOTE : If M_{11} depends on the motor position, then we may still be able to conclude zero–state detectability depending on the structure of C_{21}.

The same procedure may be used for **FF6** and leads to similar conclusions.

Case 2: M_{12} is singular but non–zero

We will particularly consider the case when the motors are symmetric and M_{12} is strictly upper triangular as an example (Section 3.2.3).

For **FF4**, (7.3) becomes

$$M_{12}^T(\theta_\ell)\Delta\theta_\ell^{(3)} + \dot{M}_{12}^T(\theta_\ell,\dot{\theta}_\ell)\Delta\ddot{\theta}_\ell + C_{21}(\theta_\ell,\dot{\theta}_\ell)\Delta\ddot{\theta}_\ell + \dot{C}_{21}(\theta_\ell,\dot{\theta}_\ell,\ddot{\theta}_\ell)\Delta\dot{\theta}_\ell + R_{21}\Delta\dot{\theta}_\ell = 0 \qquad (7.5)$$

where M_{12}^T, C_{21} and their time derivatives are strictly lower triangular (Appendix A). Then, choosing R_{21} to be lower triangular (or diagonal) and nonsingular, it follows from (7.5) that $\Delta\dot{\theta}_\ell = 0$ (recursively solve for $\Delta\dot{\theta}_{\ell_i}$, $i = 1,\cdots,m$), also implying $\Delta\ddot{\theta}_\ell = 0$.

As seen before, $\Delta\dot{\theta} = \Delta\ddot{\theta} = 0$ implies that the internal state is zero, and the system is globally zero–state detectable.

However, without the symmetric motor inertia assumption, C_{21} may not be strictly lower triangular such that zero–state detectability may not be shown in the same fashion. However, depending on the new structure of C_{12}, we may possibly show weak zero–state detectability.

For **FF6**, the derivative of the motor equation with $(u_1, B^T\Delta\dot{\theta}, \Delta\dot{\theta}_\ell) = (0,0,0)$ is

$$\begin{aligned} M_{12}^T(\theta_\ell)\Delta\theta_\ell^{(3)} + \dot{M}_{12}^T(\theta_\ell,\dot{\theta}_\ell)\Delta\ddot{\theta}_\ell + C_{21}(\theta_\ell,\dot{\theta}_\ell)\Delta\ddot{\theta}_\ell + \dot{C}_{21}(\theta_\ell,\dot{\theta}_\ell,\ddot{\theta}_\ell)\Delta\dot{\theta}_\ell - \frac{1}{2}M_{D,21}(\theta_\ell,\dot{\theta}_\ell)\Delta\ddot{\theta}_\ell & \\ -\frac{1}{2}\dot{M}_{D,21}(\theta_\ell,\dot{\theta}_\ell,\ddot{\theta}_\ell)\Delta\dot{\theta}_\ell + \frac{1}{2}M_{D,21}(\theta_\ell,\Delta\dot{\theta}_\ell)\ddot{\theta}_\ell + \frac{1}{2}\dot{M}_{D,21}(\theta_\ell,\dot{\theta}_\ell,\Delta\dot{\theta}_\ell,\Delta\ddot{\theta}_\ell)\dot{\theta}_\ell + R_{21}\Delta\dot{\theta}_\ell &= 0 \quad (7.6) \end{aligned}$$

where M_{12}^T, $C_{21}, M_{D,21}$ and their time derivatives are strictly lower triangular (Appendix A). Also, row i of $M_{D,21}(\theta_\ell,\Delta\dot{\theta}_\ell)$ depends only on $\Delta\dot{\theta}_{\ell,j}$, $j < i$ for $i = 1,\cdots,m$. Then, choosing R_{21} to be lower triangular (or diagonal) and nonsingular, it follows from (7.6) that $\Delta\dot{\theta}_\ell = 0$ (recursively solve for $\Delta\dot{\theta}_{\ell_i}$, $i = 1,2,...m$), also implying $\Delta\ddot{\theta}_\ell = 0$.

As seen before, $\Delta\dot{\theta} = \Delta\ddot{\theta} = 0$ implies that the internal state is zero, and the system is globally zero–state detectable.

Case 3: M_{12} is nonsingular

At this point, we have no results on the zero–state detectability of the system if M_{12} is nonsingular.

8 Stability analysis under feedforward approximation

The design approach presented in Sections 4 and 6 allows to obtain a complete class of controller guaranteeing uniform asymptotic stability for systems meeting all the design conditions. Such a case for flexible joint manipulator is when motors are symmetric and there is no gyroscopic coupling guaranteeing solubility of **FF4** and of **FF6**, and zero–state detectability under some additional design assumptions (Sections 5, 6 and 7). But if for some feedforward we cannot verify (weak) zero–state detectability, we cannot draw any conclusions about the system stability at this point. This may append in particular if an approximation of the exact inverse system based feedforward is used.

If for some canonical feedforward all conditions are met except possibly the solvability condition (no causal and bounded solution), then we can draw some information from the stability properties of the system under the canonical feedforward to analyze stability under other forms of feedforward. To do so, we use the identity that uniform asymptotic stability implies local exponential stability. We then consider the variations in the feedforward as perturbations and find the bounds on these perturbations allowing to maintain stability for the feedforward under analysis, i.e. we perform a robustness analysis with respect to the variation of the feedforward (see [36] for the application of this method to rigid robots).

We will use this approach to analyze the stability of the different feedforward forms by using **FF4** and **FF6** as canonical feedforward compensations. However, we first introduce the formal procedure.

8.1 Procedure

Assume that with some canonical feedforward the zero error equilibrium point is locally exponentially stable. Then, locally to $(\Delta\theta, \Delta\dot{\theta}) = (0,0)$, there exists a scalar function V_1, $V_1 = 0$ for $(\Delta\theta, \Delta\dot{\theta}) = (0,0)$, and constants $(\alpha\,,\beta\,,\gamma > 0)$ such that

$$V_1 \geq \alpha\,\|x\|^2 \tag{8.1}$$

$$\dot{V}_1 \leq -\gamma\,\|x\|^2 \tag{8.2}$$

$$\|\nabla_x V_1\| \leq \beta\,\|x\| \tag{8.3}$$

where $x = \begin{bmatrix} \Delta\theta^T & \Delta\dot{\theta}^T \end{bmatrix}$.

Proof : To see how these conditions lead to exponential stability, note that (8.3) along with $V_1 = 0$ for $(\Delta\theta, \Delta\dot{\theta}) = (0,0)$ imply that for some constant $\kappa > 0$,

$$V_1 \leq \kappa\,\|x\|^2 \tag{8.4}$$

where we may choose $\kappa = \frac{\beta}{2}$: write

$$V_1 = \int_0^x \nabla_x V_1(\xi)\,d\xi \leq \left|\int_0^x \nabla_x V_1(\xi)\,d\xi\right| \leq \int_0^{\|x\|} \|\nabla_x V_1(\xi)\|\;d\xi \leq \frac{\beta}{2}\,\|x\|^2$$

Combine this result with (8.2) to obtain

$$\dot{V}_1 \;\leq\; -\sigma V_1 \tag{8.5}$$

where $\sigma = \frac{\gamma}{\kappa}$. Now write (8.5) as

$$\begin{aligned} \dot{V}_1 + \sigma V_1 \;&\leq\; 0 \\ e^{-\sigma(t-t_0)}\frac{d\left(e^{\sigma(t-t_0)}V_1\right)}{dt} \;&\leq\; 0 \end{aligned}$$

Integrate this last equation to get

$$V_1(t, \Delta\theta(t), \Delta\dot{\theta}(t)) \leq e^{-\sigma(t-t_0)}\left[V_1(t_0, \Delta\theta(t_0), \Delta\dot{\theta}(t_0))\right]$$

i.e. local exponential stability with rate of convergence larger or equal to σ is obtained. ∎

Write the different forms of feedforward, u_{ff}, in terms of the canonical feedforward, u_{ff_c}, and of a variation term, e, as

$$Bu_{ff} = Bu_{ff_c} + e$$

Now consider a scalar function $V_2 = V_1$. The time derivative of V_2 along the solution of the system with the various forms of feedforward is

$$\dot{V}_2 = \dot{V}_1 + \left[\nabla_{\Delta\dot{\theta}} V_1\right]^T M^{-1}(v)e \tag{8.6}$$

where $M(v)$ is the multiplier of $\Delta\ddot{\theta}$ in the error equation of the system with the canonical feedforward, e.g. $M(\theta)$ in (6.9) for feedforward **FF4**. Write

$$\begin{aligned} \dot{V}_2 &\leq \dot{V}_1 + \left\| [\nabla_{\Delta\dot{\theta}} V_1]^T M^{-1}(v) e \right\| \\ &\leq \dot{V}_1 + \frac{1}{\alpha_M} \left\| \nabla_{\Delta\dot{\theta}} V_1 \right\| \|e\| \end{aligned} \tag{8.7}$$

Now use (8.2,8.3) and note that $\|\nabla_{\Delta\dot{\theta}} V_1\| \leq \|\nabla_x V_1\| \leq \beta \|x\|$, to write

$$\dot{V}_2 \leq -\gamma \|x\|^2 + \zeta \|x\| \|e\| \tag{8.8}$$

where $\zeta = \frac{\beta}{\alpha_M}$. Then, if the norm of the variation of the feedforward satisfies one of the two bounds described below, we can assert stability of the system under the new feedforward.

<u>Case 1</u> : Assume that for some constant $\chi > 0$,

$$\|e\| \leq \chi \|x\| \tag{8.9}$$

Then, using (8.8), we obtain

$$\dot{V}_2 \leq -(\gamma - \zeta\chi) \|x\|^2$$

Hence, exponential stability with guaranteed rate of convergence $\sigma_1 > 0$ is obtained if the variation of the feedforward is small enough in the sense that

$$\chi < \frac{\gamma}{\zeta} \tag{8.10}$$

and using (8.4), σ_1 is given by

$$\sigma_1 = \frac{\gamma - \zeta\chi}{\kappa} \tag{8.11}$$

<u>Case 2</u> : Assume that for some constants $(\chi, \rho) > 0$,

$$\|e\| \leq \chi \|x\| + \rho \tag{8.12}$$

Then, using (8.8), we obtain

$$\dot{V}_2 \leq -(\gamma - \zeta\chi) \|x\|^2 + \zeta\rho \|x\| \tag{8.13}$$

Write

$$\zeta\rho \|x\| = -\zeta\rho \left[\left(\epsilon^{-1} \|x\| - \frac{1}{2}\epsilon \right)^2 - \epsilon^{-2} \|x\|^2 - \frac{\epsilon^2}{4} \right] \tag{8.14}$$

Substitute (8.14) into (8.13) to get

$$\dot{V}_2 \leq -\left(\gamma - \zeta\chi - \zeta\rho\epsilon^{-2} \right) \|x\|^2 + \frac{\zeta\rho\epsilon^2}{4} \tag{8.15}$$

which, using (8.4), is written as

$$\dot{V}_2 \leq -\sigma_2 V_2 + \rho_2 \tag{8.16}$$

where

$$\sigma_2 = \frac{(\gamma - \zeta\chi - \zeta\rho\epsilon^{-2})}{\kappa} \tag{8.17}$$

$$\rho_2 = \frac{\zeta\rho\epsilon^2}{4} \tag{8.18}$$

Now write

$$\dot{V}_2 + \sigma_2 V_2 \leq \rho_2$$
$$e^{-\sigma_2(t-t_0)} \frac{d\left(e^{\sigma_2(t-t_0)}V_2\right)}{dt} \leq \rho_2$$

Integrate this last equation to get

$$V_2(t, \Delta\theta(t), \Delta\dot{\theta}(t)) \leq e^{-\sigma_2(t-t_0)} \left[V_2(t_0, \Delta\theta(t_0), \Delta\dot{\theta}(t_0)) - \frac{\rho_2}{\sigma_2}\right] + \frac{\rho_2}{\sigma_2}$$

Assume that the variation of the feedforward is small enough in the sense that (8.10) holds such that $\sigma_2 > 0$ for $\epsilon^2 > \epsilon_m^2$ where

$$\epsilon_m^2 = \frac{\zeta\rho}{\gamma - \zeta\chi}$$

Define the supremum of V_2 over t as

$$\overline{V}_2 = \max\left\{V_2(t_0, \Delta\theta(t_0), \Delta\dot{\theta}(t_0)), \frac{\rho_2}{\sigma_2}\right\}$$

Since V_2 is positive definite, this implies that the state remains bounded, i.e. we obtain Lagrange stability. Also, convergence to the set $\left\{\Delta\theta(t), \Delta\theta(t) : V_2 \leq \frac{\rho_2}{\sigma_2}\right\}$ with rate of at least σ_2 is guaranteed. Note that choice of a large ϵ guarantees a fast convergence to a large set, while choice of a smaller $\epsilon > \epsilon_m > 0$ does not guarantee such a fast convergence but guarantees convergence to a smaller set, i.e. the state converges closer to zero.

Following closely this approach tends to give conservative bounds on $||e||$. In practice, one may evaluate explicitly $\dot{V}_2$ using (8.6) and use directly V_2 and $\dot{V}_2$ to find relationship (8.16) and hence obtain less conservative bounds.

8.2 Exponential stability of canonical feedforward

Uniform asymptotic stability of the system under the canonical feedforward with any strictly passive BIBO stabilizing controller implies local exponential stability. However, it is difficult to explicitly construct V_1 in (8.1,8.2,8.3) in general. One case for which we can explicitly construct V_1 is if there is damping at the links, case that we will consider here.

We do not provide a demonstration for a general strictly passive controller C_v but for a simple case, i.e. $C_v(B^T\Delta\dot{\theta}) = K_v B^T \Delta\dot{\theta}$ (the global controller structure is feedforward with proportional–derivative (PD) controller). Also, for the demonstration, we will assume that only a linear damping term, $D_0\Delta\dot{\theta}$, appears in the error equation (see (6.7)).

Proposition 8.1 *Consider a flexible joint manipulator with linear damping at the links $D_{l_0} > 0$. Assume that feedforward* **FF4** *with C represented using Christoffel's symbols or feedforward* **FF6** *is used. Also, close a static state error $C_p(y_p) = K_p B^T \Delta\theta$ and a strictly passive BIBO controller loop $C_v(B^T\Delta\dot{\theta}) = K_v B^T \Delta\dot{\theta}$ where the gains are chosen to satisfy Assumption 6.1 and the following assumption:*

Assumption 8.1 *K_v are chosen such that* $\left[D_0 + BK_vB^T\right] > 0$.

Then, for $\dot{\theta}_d$ sufficiently small in norm, the zero error equilibrium point is locally (to $\Delta\theta = 0$) exponentially stable.

Proof : Consider the following energy based Lyapunov function candidate where the last term is used to cancel extra terms introduced by the cross term in $\Delta\theta$ and $\Delta\dot{\theta}$:

$$V_1 = \frac{1}{2}\Delta\theta^T\left[R + BK_pB^T\right]\Delta\theta + c\Delta\theta^T M(\theta)\Delta\dot{\theta} + \frac{1}{2}\Delta\dot{\theta}^T M(\theta)\Delta\dot{\theta} + \frac{c}{2}\Delta\theta^T\left[D_0 + BK_vB^T\right]\Delta\theta \quad (8.19)$$

Evaluate the time derivative of V_1 along the solution of the system given by its error equation ((6.9) for **FF4**, and (6.12) plus the stabilizing loop for **FF6**) to get, using the properties listed in Sections 3.1 and 3.3,

$$\begin{aligned}\dot{V}_1 &= -c\Delta\theta^T\left[R + BK_pB^T\right]\Delta\theta + c\Delta\theta^T\left[\dot{M}(\theta,\dot{\theta}) - C_C(\theta,\dot{\theta})\right]\Delta\dot{\theta} \\ &\quad -\Delta\dot{\theta}^T\left[D_0 + BK_vB^T - cM(\theta)\right]\Delta\dot{\theta} + \left(c\Delta\theta^T + \Delta\dot{\theta}^T\right)Bu_2 \end{aligned} \tag{8.20}$$

for **FF4** where used the skew–symmetry property of $\left(\frac{1}{2}\dot{M}(\theta,\dot{\theta}) - C(\theta,\dot{\theta})\right)$ and we assume $u_2 = 0$, and

$$\begin{aligned}\dot{V}_1 &= -c\Delta\theta^T\left[R + BK_pB^T\right]\Delta\theta \\ &\quad +c\Delta\theta^T\left[\dot{M}(\theta,\dot{\theta})\Delta\dot{\theta} - C_D(\theta,\dot{\theta})\Delta\dot{\theta} - \frac{1}{2}M_D(\theta,\dot{\theta})\dot{\theta}_d + \frac{1}{2}M_D(\theta,\dot{\theta}_d)\dot{\theta}\right] \\ &\quad -\Delta\dot{\theta}^T\left[D_0 + BK_vB^T - cM(\theta)\right]\Delta\dot{\theta} + \left(c\Delta\theta^T + \Delta\dot{\theta}^T\right)Bu_2 \end{aligned} \tag{8.21}$$

for **FF6** where we used (3.25) and also assume $u_2 = 0$.

We now proceed to show that V_1 and $\dot{V}_1$ satisfy (8.1,8.2,8.3,8.4).

Lower bound of V_1 (8.1) : From (8.19), we have

$$V_1 \geq \frac{1}{2}(\alpha_p + c\alpha_v)\|\Delta\theta\|^2 - c\gamma_M\|\Delta\theta\|\left\|\Delta\dot{\theta}\right\| + \frac{1}{2}\alpha_M\left\|\Delta\dot{\theta}\right\|^2 \tag{8.22}$$

for $c \geq 0$. Write the cross term as

$$-c\gamma_M\|\Delta\theta\|\left\|\Delta\dot{\theta}\right\| = \frac{c\gamma_M}{2}\left[\left(\epsilon_1\|\Delta\theta\| - \epsilon_1^{-1}\left\|\Delta\dot{\theta}\right\|\right)^2 - \epsilon_1^2\|\Delta\theta\|^2 - \epsilon_1^{-2}\left\|\Delta\dot{\theta}\right\|^2\right] \tag{8.23}$$

such that, using (8.23) in (8.22), V_1 satisfies

$$V_1 \geq \alpha_1\|\Delta\theta\|^2 + \alpha_2\left\|\Delta\dot{\theta}\right\|^2 \geq \|x\|^2$$

where

$$\alpha_1 = \frac{1}{2}\left(\alpha_p + c\alpha_v - c\gamma_M\epsilon_1^2\right) \qquad \alpha_2 = \frac{1}{2}\left(\alpha_M - c\gamma_M\epsilon_1^{-2}\right)$$
$$\alpha = min\{\alpha_1, \alpha_2\}$$

Proof : of the last inequality. Write, for the $2-norm$, (we assume α_1 and α_2 nonnegative)

$$\alpha_1\|\Delta\theta\|^2 + \alpha_2\left\|\Delta\dot{\theta}\right\|^2 = \left\|\begin{bmatrix}\sqrt{\alpha_1}\Delta\theta \\ \sqrt{\alpha_2}\Delta\dot{\theta}\end{bmatrix}\right\|^2 \geq min\{\alpha_1,\alpha_2\}\|x\|^2$$

which concludes the proof. ∎

To guarantee positive definitiveness of V_1, we impose $\alpha_1 > 0$, $\alpha_2 > 0$ yielding

$$\frac{c\gamma_M}{\alpha_M} < \epsilon_1^2 < \frac{\alpha_p + c\alpha_v}{c\gamma_M}$$

which, for a nontrivial solution requires

$$\frac{c\gamma_M}{\alpha_M} < \frac{\alpha_p + c\alpha_v}{c\gamma_M}$$

This is satisfied for c small enough, i.e. $0 \leq c < c_v$ where

$$c_v = \frac{\alpha_M \alpha_v}{2\gamma_M^2} + \left(\frac{\alpha_M^2 \alpha_v^2}{4\gamma_M^4} + \frac{\alpha_M \alpha_p}{\gamma_M^2} \right)^{\frac{1}{2}} \tag{8.24}$$

Upper bound of V_1 (8.4) : From (8.19), we have

$$V_1 \leq \frac{1}{2}(\gamma_p + c\gamma_v) \|\Delta\theta\|^2 + c\gamma_M \|\Delta\theta\| \left\|\Delta\dot{\theta}\right\| + \frac{1}{2}\gamma_M \left\|\Delta\dot{\theta}\right\|^2 \tag{8.25}$$

Write

$$c\gamma_M \|\Delta\theta\| \left\|\Delta\dot{\theta}\right\| = \frac{c\gamma_M}{2} \left[-\left(\epsilon_2 \|\Delta\theta\| - \epsilon_2^{-1} \left\|\Delta\dot{\theta}\right\| \right)^2 + \epsilon_2^2 \|\Delta\theta\|^2 + \epsilon_2^{-2} \left\|\Delta\dot{\theta}\right\|^2 \right] \tag{8.26}$$

such that, using (8.26) in (8.25), V_1 satisfies

$$V_1 \leq \kappa_1 \|\Delta\theta\|^2 + \kappa_2 \left\|\Delta\dot{\theta}\right\|^2 \leq \kappa \|x\|^2$$

where

$$\kappa_1 = \frac{1}{2}\left(\gamma_p + c\gamma_v + c\gamma_M \epsilon_2^2\right) > 0 \qquad \kappa_2 = \frac{1}{2}\left(\gamma_M + c\gamma_M \epsilon_2^{-2}\right) > 0$$

$$\kappa = max\{\kappa_1, \kappa_2\}$$

Proof : of the last inequality. Write, for the $2-norm$,

$$\kappa_1 \|\Delta\theta\|^2 + \kappa_2 \left\|\Delta\dot{\theta}\right\|^2 = \left\| \begin{bmatrix} \sqrt{\kappa_1}\Delta\theta \\ \sqrt{\kappa_2}\Delta\dot{\theta} \end{bmatrix} \right\|^2 \leq max\{\kappa_1, \kappa_2\} \|x\|^2$$

which concludes the proof. ∎

Hence, V_1 is upper bounded for any bounded c.

Upper bound of $\nabla_{\Delta\dot{\theta}} V_1$ (8.3) : V_1 being lower and upper bounded by quadratic functions, and continuous, the norm of the gradient of V_1 with respect to its argument is also bounded. Evaluate explicitly the gradient of V_1 with respect to $\Delta\dot{\theta}$ from (8.19) :

$$\nabla_{\Delta\dot{\theta}} V_1(\Delta\theta, \Delta\dot{\theta}) = cM(\theta)\Delta\theta + M(\theta)\Delta\dot{\theta} \tag{8.27}$$

such that

$$\left\|\nabla_{\Delta\dot{\theta}} V_1\right\| \leq \beta_1 \|\Delta\theta\| + \beta_2 \left\|\Delta\dot{\theta}\right\| \leq \beta \|x\|$$

where

$$\beta_1 = c\gamma_M > 0 \qquad \beta_2 = \gamma_M > 0$$

$$\beta = \beta_1 + \beta_2$$

Proof : of the last inequality. Write, for β_1 and β_2 nonnegative,

$$\beta_1 \|\Delta\theta\| + \beta_2 \left\|\Delta\dot{\theta}\right\| \leq \beta_1 \|x\| + \beta_2 \|x\|$$

which leads directly to the bound in β. ∎

Upper bound of $\dot{V}_1$ (8.2) : Use (8.20) and property (3.33) to obtain, for **FF4**,

$$\dot{V}_1 \leq -c\alpha_p \|\Delta\theta\|^2 + c\gamma_{C_2} \|\Delta\theta\| \left\|\Delta\dot{\theta}\right\|^2 + c\gamma_{C_2} \|\Delta\theta\| \left\|\Delta\dot{\theta}\right\| \left\|\dot{\theta}_d\right\| - [\alpha_v - c\gamma_M] \left\|\Delta\dot{\theta}\right\|^2 \tag{8.28}$$

and use (8.21) and property (3.27) to obtain, for **FF6**,

$$\dot{V}_1 \leq -c\alpha_p \|\Delta\theta\|^2 + c\frac{3}{2}\gamma_D \|\Delta\theta\| \left\|\Delta\dot{\theta}\right\|^2 + c\frac{3}{2}\gamma_D \|\Delta\theta\| \left\|\Delta\dot{\theta}\right\| \left\|\dot{\theta}_d\right\| - [\alpha_v - c\gamma_M] \left\|\Delta\dot{\theta}\right\|^2 \tag{8.29}$$

Identify γ_I with γ_{C_2} in (8.28) and γ_I with $\frac{3}{2}\gamma_D$ in (8.29) to write, for both **FF4** and **FF6**,

$$\dot{V}_1 \leq -c\alpha_p \|\Delta\theta\|^2 + c\gamma_I \|\Delta\theta\| \left\|\Delta\dot{\theta}\right\|^2 + c\gamma_I \|\Delta\theta\| \left\|\Delta\dot{\theta}\right\| \left\|\dot{\theta}_d\right\| - [\alpha_v - c\gamma_M] \left\|\Delta\dot{\theta}\right\|^2 \tag{8.30}$$

We will use this last equation to find conditions for the negative definitiveness of $\dot{V}_1$.

For negative definitiveness of $\dot{V}_1$, we first need

$$c\alpha_p > 0 \qquad \text{and} \qquad \alpha_v - c\gamma_M > 0$$

Due to Assumption 6.1, this is satisfied for $0 < c < c_2$ where

$$c_2 = \frac{\alpha_v}{\gamma_M} \tag{8.31}$$

Then, use

$$c\gamma_I \left\|\dot{\theta}_d\right\| \|\Delta\theta\| \left\|\Delta\dot{\theta}\right\| = c\frac{\gamma_I}{2} \left\|\dot{\theta}_d\right\| \left[-\left(\epsilon_3 \|\Delta\theta\| - \epsilon_3^{-1} \left\|\Delta\dot{\theta}\right\|\right)^2 + \epsilon_3^2 \|\Delta\theta\|^2 + \epsilon_3^{-2} \left\|\Delta\dot{\theta}\right\|^2\right]$$

to get

$$\dot{V}_1 \leq -c\left[\alpha_p - \frac{\gamma_I}{2}\epsilon_3^2 \left\|\dot{\theta}_d\right\|\right] \|\Delta\theta\|^2 + c\gamma_I \|\Delta\theta\| \left\|\Delta\dot{\theta}\right\|^2 - \left[\alpha_v - c\gamma_M - c\frac{\gamma_I}{2}\epsilon_3^{-2} \left\|\dot{\theta}_d\right\|\right] \left\|\Delta\dot{\theta}\right\|^2$$

Assume that $\|\Delta\theta\|$ is uniformly bounded above by γ_e such that

$$\alpha_v - c\gamma_M - c\gamma_I\gamma_e > 0$$

and that $\left\|\dot{\theta}_d\right\|$ is uniformly bounded above by γ_d. Then

$$\dot{V}_1 \leq -\gamma_1 \|\Delta\theta\|^2 - \gamma_2 \left\|\Delta\dot{\theta}\right\|^2 \leq -\gamma \|x\|^2$$

where

$$\gamma_1 = c\left[\alpha_p - \frac{\gamma_I}{2}\epsilon_3^2\gamma_d\right] \qquad \gamma_2 = \alpha_v - c\gamma_M - c\gamma_I\gamma_e - c\frac{\gamma_I}{2}\epsilon_3^{-2}\gamma_d$$

$$\gamma = min\{\gamma_1, \gamma_2\}$$

where we assumed γ_1 and γ_2 positive. This last condition is satisfied and $\dot{V}_1$ is negative definite if

$$\frac{c\frac{\gamma_I}{2}\gamma_d}{\alpha_v - c\gamma_M - c\gamma_I\gamma_e} < \epsilon_3^2 < \frac{\alpha_p}{\frac{\gamma_I}{2}\gamma_d}$$

For a non trivial solution, we need

$$\frac{c\frac{\gamma_I}{2}\gamma_d}{\alpha_v - c\gamma_M - c\gamma_I\gamma_e} < \frac{\alpha_p}{\frac{\gamma_I}{2}\gamma_d}$$

Hence, γ_d must satisfy

$$\gamma_d < \frac{2}{\gamma_I}\left[\frac{\alpha_p(\alpha_v - c\gamma_M - c\gamma_I\gamma_e)}{c}\right]^{\frac{1}{2}} \tag{8.32}$$

Summary : Conditions (8.1,8.2,8.3) are satisfied if $\Delta\theta$ and $\dot{\theta}_d$ are small enough in norm and satisfy relation (8.32) for

$$0 < c < \min\{c_v, c_2\}$$

where c_v and c_2 are respectively given by (8.24) and (8.31). ■

We may ascertain a rate of convergence larger or equal to $\sigma = \frac{\gamma}{\kappa}$. Note that a better evaluation of the convergence rate is given by $\sigma = \min\left\{\frac{\gamma_1}{\kappa_1}, \frac{\gamma_2}{\kappa_2}\right\}$.

Also note that (8.32) indicates that an increased supremum of the norm of $\dot{\theta}_d$ reduces the allowable bound γ_e such that the region of convergence decreases.

Moreover, we obtain similar qualitative results for both **FF4** and **FF6** and the effective difference would show up in the evaluation of the bounds through γ_I.

8.3 Stability for various forms of feedforward

We now conclude on the stability with the various forms of feedforward by using the results of the previous section on the exponential stability of **FF4** and **FF6**.

The variation terms e for the different forms of feedforward with respect to the canonical feedforward depending on the canonical feedforward used, we will consider separately the analysis based on the results for **FF4** (representation of C using Christoffel's symbols) and for **FF6** (representation of C using M_D-notation).

NOTE : In order to obtain a better bound on the variation of $\dot{V}_2$ than in (8.7), explicitly evaluate (8.6) using (8.27) or equivalently by identifying Bu_2 with e in (8.20) :

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + \left(c\Delta\theta^T + \Delta\dot{\theta}^T\right) e \\
&\leq \dot{V}_1 + \left(c\|\Delta\theta\| + \left\|\Delta\dot{\theta}\right\|\right)\|e\| \\
&\leq \dot{V}_1 + \beta'\|x\|\|e\|
\end{aligned}$$

where we may use $\beta' = (c+1)$. We then identify ζ with β' in (8.8).

8.3.1 FF4 as the canonical feedforward

The variation terms e for the different forms of feedforward with respect to the canonical feedforward **FF4** with C represented using Christoffel's symbols are given by

FF1. $e = [M(\theta_d) - M(\theta)]\ddot{\theta}_d + \left[C_C(\theta_d, \dot{\theta}_d) - C_C(\theta, \dot{\theta})\right]\dot{\theta}_d + \bar{r}(\theta_d) - \bar{r}(\theta)$

FF2. $e = [M(\theta_d) - M(\theta)]\ddot{\theta}_d + \left[C_C(\theta_d, \dot{\theta}_d) - C_C(\theta, \dot{\theta})\right]\dot{\theta}_d$

FF3. $e = \bar{r}(\theta_d) - \bar{r}(\theta)$

FF7. $e = -M(\theta)\ddot{\theta}_d - C_C(\theta, \dot{\theta})\dot{\theta}_d - D_0\dot{\theta}_d + \bar{r}(\theta_d) - \bar{r}(\theta)$

FF8. $e = -M(\theta)\ddot{\theta}_d - C_C(\theta, \dot{\theta})\dot{\theta}_d - D_0\dot{\theta}_d$

We did not consider **FF5** and **FF6** because C is not represented by the same notation as **FF4**.

The norm of the variation e can be bounded above by

$$\textbf{FF1.}\quad \|e\| \le \left\|[M(\theta_d)-M(\theta)]\,\ddot\theta_d\right\| + \left\|\left[C_C(\theta_d,\dot\theta_d)-C_C(\theta,\dot\theta)\right]\dot\theta_d\right\| + \|\bar r(\theta_d)-\bar r(\theta)\|$$
$$\le \gamma_D\left\|\ddot\theta_d\right\|\|\Delta\theta\| + \gamma_{Cd}\left\|\dot\theta_d\right\|^2\|\Delta\theta\| + \gamma_C\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\| + \gamma_R\|\Delta\theta\|$$
$$= (\gamma_D\gamma_{dd}+\gamma_{Cd}\gamma_d^2+\gamma_R)\|\Delta\theta\| + \gamma_C\gamma_d\left\|\Delta\dot\theta\right\|$$

where we used (3.26, 3.34, 3.35).

$$\textbf{FF2.}\quad \|e\| \le \left\|[M(\theta_d)-M(\theta)]\,\ddot\theta_d\right\| + \left\|\left[C_C(\theta_d,\dot\theta_d)-C_C(\theta,\dot\theta)\right]\dot\theta_d\right\|$$
$$\le \gamma_D\left\|\ddot\theta_d\right\|\|\Delta\theta\| + \gamma_{Cd}\left\|\dot\theta_d\right\|^2\|\Delta\theta\| + \gamma_C\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\|$$
$$= (\gamma_D\gamma_{dd}+\gamma_{Cd}\gamma_d^2)\|\Delta\theta\| + \gamma_C\gamma_d\left\|\Delta\dot\theta\right\|$$

where we used (3.26, 3.34).

$$\textbf{FF3.}\quad \|e\| \le \gamma_R\|\Delta\theta\|$$

where we used (3.35).

$$\textbf{FF7.}\quad \|e\| \le \left\|M(\theta)\ddot\theta_d\right\| + \left\|C_C(\theta,\dot\theta)\dot\theta_d\right\| + \left\|D_0\dot\theta_d\right\| + \|\bar r(\theta_d)-\bar r(\theta)\|$$
$$\le \gamma_M\left\|\ddot\theta_d\right\| + \gamma_C\left\|\dot\theta_d\right\|^2 + \gamma_C\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\| + \gamma_{D_0}\left\|\dot\theta_d\right\| + \gamma_R\|\Delta\theta\|$$
$$= \gamma_R\|\Delta\theta\| + \gamma_C\gamma_d\left\|\Delta\dot\theta\right\| + (\gamma_M\gamma_{dd}+\gamma_C\gamma_d^2+\gamma_{D_0}\gamma_d)$$

where we used (3.31, 3.35).

$$\textbf{FF8.}\quad \|e\| \le \left\|M(\theta)\ddot\theta_d\right\| + \left\|C_C(\theta,\dot\theta)\dot\theta_d\right\| + \left\|D_0\dot\theta_d\right\|$$
$$\le \gamma_M\left\|\ddot\theta_d\right\| + \gamma_C\left\|\dot\theta_d\right\|^2 + \gamma_C\left\|\dot\theta_d\right\|\left\|\Delta\dot\theta\right\| + \gamma_{D_0}\left\|\dot\theta_d\right\|$$
$$= \gamma_C\gamma_d\left\|\Delta\dot\theta\right\| + (\gamma_M\gamma_{dd}+\gamma_C\gamma_d^2+\gamma_{D_0}\gamma_d)$$

where we used (3.31).

The variation of the feedforward signal e for **FF1**, **FF2** and **FF3** may be written in the form of (8.9), while for **FF7** and **FF8** it may be written in the form of (8.12).

The conclusions on the stability and performance for the different forms of feedforward are summarized in Table 1. Achievable type of stability, and qualitative conditions on design parameters for stability, for fast rate of convergence and for smallness of the convergence set (for Lagrange stability) are listed in the table. We recall that to guarantee stability, we require that σ_1 in (8.11) or σ_2 in (8.17) is positive depending on the form of the bound on the error signal and that a faster rate of convergence is obtained as these variables increase. For the Lagrange stability, convergence to a smaller set about the zero error state is guaranteed as ρ_2 decreases in (8.18).

8.3.2 FF6 as the canonical feedforward

The variation terms e for the different forms of feedforward with respect to the canonical feedforward **FF6** are given by

$$\textbf{FF1.}\ e = [M(\theta_d)-M(\theta)]\,\ddot\theta_d + \left[C_D(\theta_d,\dot\theta_d)-C_D(\theta,\dot\theta)+\tfrac12 M_D(\theta,\dot\theta)\right]\dot\theta_d - \tfrac12 M_D(\theta,\dot\theta_d)\dot\theta + \bar r(\theta_d)-\bar r(\theta)$$

$$\textbf{FF2.}\ e = [M(\theta_d)-M(\theta)]\,\ddot\theta_d + \left[C_D(\theta_d,\dot\theta_d)-C_D(\theta,\dot\theta)+\tfrac12 M_D(\theta,\dot\theta)\right]\dot\theta_d - \tfrac12 M_D(\theta,\dot\theta_d)\dot\theta$$

$$\textbf{FF3.}\ e = \tfrac12 M_D(\theta,\dot\theta)\dot\theta_d - \tfrac12 M_D(\theta,\dot\theta_d)\dot\theta + \bar r(\theta_d)-\bar r(\theta)$$

	Condition for stability: small enough	Type of stability	Condition for fast rate of convergence: small	Condition for convergence to small set: small
FF1	$(\gamma_d, \gamma_{dd}, \gamma_R)$	local exponential	$(\gamma_d, \gamma_{dd}, \gamma_R)$	—
FF2	(γ_d, γ_{dd})	local exponential	(γ_d, γ_{dd})	—
FF3	(γ_R)	local exponential	(γ_R)	—
FF4	—	local exponential	—	—
FF7	(γ_d, γ_R)	Lagrange	$(\gamma_d, \gamma_{dd}, \gamma_R)$	(γ_d, γ_{dd})
FF8	(γ_d)	Lagrange	(γ_d, γ_{dd})	(γ_d, γ_{dd})

Table 1: Stability and performance analysis, **FF4** as canonical feedforward

FF4. $e = \frac{1}{2}M_D(\theta, \dot{\theta})\dot{\theta}_d - \frac{1}{2}M_D(\theta, \dot{\theta}_d)\dot{\theta}$

FF5. $e = \bar{r}(\theta_d) - \bar{r}(\theta)$

FF7. $e = -M(\theta)\ddot{\theta}_d - \left[C_D(\theta, \dot{\theta}) - \frac{1}{2}M_D(\theta, \dot{\theta})\right]\dot{\theta}_d - \frac{1}{2}M_D(\theta, \dot{\theta}_d)\dot{\theta} - D_0\dot{\theta}_d + \bar{r}(\theta_d) - \bar{r}(\theta)$

FF8. $e = -M(\theta)\ddot{\theta}_d - \left[C_D(\theta, \dot{\theta}) - \frac{1}{2}M_D(\theta, \dot{\theta})\right]\dot{\theta}_d - \frac{1}{2}M_D(\theta, \dot{\theta}_d)\dot{\theta} - D_0\dot{\theta}_d$

The norm of the variation e can be bounded above by

FF1.
$$\begin{aligned} \|e\| &\leq \left\|[M(\theta_d) - M(\theta)]\ddot{\theta}_d\right\| + \left\|\left[C_D(\theta_d, \dot{\theta}_d) - C_D(\theta, \dot{\theta}) + \tfrac{1}{2}M_D(\theta, \dot{\theta})\right]\dot{\theta}_d - \tfrac{1}{2}M_D(\theta, \dot{\theta}_d)\dot{\theta}\right\| + \\ &\quad \|\bar{r}(\theta_d) - \bar{r}(\theta)\| \\ &\leq \gamma_D\left\|\ddot{\theta}_d\right\|\|\Delta\theta\| + \tfrac{3}{2}\gamma_{Dd}\left\|\dot{\theta}_d\right\|^2\|\Delta\theta\| + \tfrac{3}{2}\gamma_D\left\|\dot{\theta}_d\right\|\left\|\Delta\dot{\theta}\right\| + \gamma_R\|\Delta\theta\| \\ &= \left(\gamma_D\gamma_{dd} + \tfrac{3}{2}\gamma_{Dd}\gamma_d^2 + \gamma_R\right)\|\Delta\theta\| + \tfrac{3}{2}\gamma_D\gamma_d\left\|\Delta\dot{\theta}\right\| \end{aligned}$$

where we used (3.26, 3.30, 3.35).

FF2.
$$\begin{aligned} \|e\| &\leq \left\|[M(\theta_d) - M(\theta)]\ddot{\theta}_d\right\| + \left\|\left[C_D(\theta_d, \dot{\theta}_d) - C_D(\theta, \dot{\theta}) + \tfrac{1}{2}M_D(\theta, \dot{\theta})\right]\dot{\theta}_d - \tfrac{1}{2}M_D(\theta, \dot{\theta}_d)\dot{\theta}\right\| \\ &\leq \gamma_D\left\|\ddot{\theta}_d\right\|\|\Delta\theta\| + \tfrac{3}{2}\gamma_{Dd}\left\|\dot{\theta}_d\right\|^2\|\Delta\theta\| + \tfrac{3}{2}\gamma_D\left\|\dot{\theta}_d\right\|\left\|\Delta\dot{\theta}\right\| \\ &= \left(\gamma_D\gamma_{dd} + \tfrac{3}{2}\gamma_{Dd}\gamma_d^2\right)\|\Delta\theta\| + \tfrac{3}{2}\gamma_D\gamma_d\left\|\Delta\dot{\theta}\right\| \end{aligned}$$

where we used (3.26, 3.30).

FF3.
$$\begin{aligned} \|e\| &\leq \left\|\tfrac{1}{2}M_D(\theta, \dot{\theta})\dot{\theta}_d - \tfrac{1}{2}M_D(\theta, \dot{\theta}_d)\dot{\theta}\right\| + \|\bar{r}(\theta_d) - \bar{r}(\theta)\| \\ &\leq \gamma_D\left\|\dot{\theta}_d\right\|\left\|\Delta\dot{\theta}\right\| + \gamma_R\|\Delta\theta\| \\ &= \gamma_R\|\Delta\theta\| + \gamma_D\gamma_d\left\|\Delta\dot{\theta}\right\| \end{aligned}$$

where we used (3.29, 3.35).

FF4.
$$\begin{aligned} \|e\| &\leq \left\|\tfrac{1}{2}M_D(\theta, \dot{\theta})\dot{\theta}_d - \tfrac{1}{2}M_D(\theta, \dot{\theta}_d)\dot{\theta}\right\| \\ &\leq \gamma_D\left\|\dot{\theta}_d\right\|\left\|\Delta\dot{\theta}\right\| \\ &= \gamma_D\gamma_d\left\|\Delta\dot{\theta}\right\| \end{aligned}$$

where we used (3.29).

FF5. $\|e\| \leq \gamma_R\|\Delta\theta\|$

where we used (3.35).

FF7.
$$\begin{aligned}\|e\| &\le \left\|M(\theta)\ddot{\theta}_d\right\| + \left\|\left[C_D(\theta,\dot{\theta}) - \tfrac{1}{2}M_D(\theta,\dot{\theta})\right]\dot{\theta}_d + \tfrac{1}{2}M_D(\theta,\dot{\theta}_d)\dot{\theta}\right\| + \left\|D_0\dot{\theta}_d\right\| + \|\bar{r}(\theta_d) - \bar{r}(\theta)\| \\ &\le \gamma_M\left\|\ddot{\theta}_d\right\| + \tfrac{3}{2}\gamma_D\left\|\dot{\theta}_d\right\|^2 + \tfrac{3}{2}\gamma_D\left\|\dot{\theta}_d\right\|\left\|\Delta\dot{\theta}\right\| + \gamma_{D_0}\left\|\dot{\theta}_d\right\| + \gamma_R\|\Delta\theta\| \\ &= \gamma_R\|\Delta\theta\| + \tfrac{3}{2}\gamma_D\gamma_d\left\|\Delta\dot{\theta}\right\| + \left(\gamma_M\gamma_{dd} + \tfrac{3}{2}\gamma_D\gamma_d^2 + \gamma_{D_0}\gamma_d\right)\end{aligned}$$

where we used (3.28, 3.35).

FF8.
$$\begin{aligned}\|e\| &\le \left\|M(\theta)\ddot{\theta}_d\right\| + \left\|\left[C_D(\theta,\dot{\theta}) - \tfrac{1}{2}M_D(\theta,\dot{\theta})\right]\dot{\theta}_d + \tfrac{1}{2}M_D(\theta,\dot{\theta}_d)\dot{\theta}\right\| + \left\|D_0\dot{\theta}_d\right\| \\ &\le \gamma_M\left\|\ddot{\theta}_d\right\| + \tfrac{3}{2}\gamma_D\left\|\dot{\theta}_d\right\|^2 + \tfrac{3}{2}\gamma_D\left\|\dot{\theta}_d\right\|\left\|\Delta\dot{\theta}\right\| + \gamma_{D_0}\left\|\dot{\theta}_d\right\| \\ &= \tfrac{3}{2}\gamma_D\gamma_d\left\|\Delta\dot{\theta}\right\| + \left(\gamma_M\gamma_{dd} + \tfrac{3}{2}\gamma_D\gamma_d^2 + \gamma_{D_0}\gamma_d\right)\end{aligned}$$

where we used (3.28).

The variation of the feedforward signal e for **FF1**, **FF2**, **FF3**, **FF4** and **FF5** may be written in the form of (8.9), while for **FF7** and **FF8** it may be written in the form of (8.12).

The conclusions on the stability and performance for the different forms of feedforward are summarized in Table 2. Achievable type of stability, and qualitative conditions on design parameters for stability, for fast rate of convergence and for smallness of the convergence set (for Lagrange stability) are listed in the table.

	Condition for stability: small enough	Type of stability	Condition for fast rate of convergence: small	Condition for convergence to small set: small
FF1	$(\gamma_d, \gamma_{dd}, \gamma_R)$	local exponential	$(\gamma_d, \gamma_{dd}, \gamma_R)$	——
FF2	(γ_d, γ_{dd})	local exponential	(γ_d, γ_{dd})	——
FF3	(γ_d, γ_R)	local exponential	(γ_d, γ_R)	——
FF4	(γ_d)	local exponential	(γ_d)	——
FF5	(γ_R)	local exponential	(γ_R)	——
FF6	——	local exponential	——	——
FF7	(γ_d, γ_R)	Lagrange	$(\gamma_d, \gamma_{dd}, \gamma_R)$	(γ_d, γ_{dd})
FF8	(γ_d)	Lagrange	(γ_d, γ_{dd})	(γ_d, γ_{dd})

Table 2: Stability and performance analysis, **FF6** as canonical feedforward

8.3.3 Summary

We see from Tables 1 and 2 that the conditions for stability and fast rate of convergence with the different forms of feedforward involve the upper bound on three parameters that can be affected by design :

- Bounds (γ_d, γ_{dd}) are made small by using a slow desired output trajectory (θ_{ℓ_d} and its higher derivatives). These bounds are also affected by the system parameters and by the parameters of the compensators through the solution of θ_{m_d} and its higher derivatives (see Section 5).
- The bound γ_R is made small by choosing R giving the best fit of the gradient of the spring characteristic in the sense that $\sup_\theta \{\|\nabla_\theta r(\theta) - R\|\}$ is minimized for R in the set of matrices meeting the design requirements (Sections 5, 6 and 7). For a linear spring characteristic, this leads to

$\gamma_R = 0$ if we may choose $[r(\theta) - R\theta] = 0$. Then, odd and even numbered feedforward forms become equivalent.

9 Conclusion

We have presented a controller design approach based on passivity and Lyapunov stability theory for the tracking of flexible joint manipulators. The overall design procedure may be viewed as consisting of two main steps described below.

Firstly, compensators are designed by taking advantage of the inherent passivity properties of flexible joint manipulators. The procedure for the design of the passivity based controller involves essentially the formation of a passive and zero–state detectable error system by use of feedforward compensation and static state feedback, and the asymptotic stabilization of the system by using any strictly passive controller with finite gain. However, the feedforward compensation, which is based on plant inversion, may not have a causal solution, or it may be very difficult to solve for a stable inverse of the system due to its nonlinear nature, such that implementation of the controller is then compromised. Also, certain terms of the dynamical equation may have a negligible effect on the overall dynamics of the system, e.g. Coriolis and centrifugal effects at low velocities, such that we might want to neglect these terms in the feedforward.

In the second step, we use stability results from the passivity based design, i.e. uniform asymptotic stability of the system implies local exponential stability, to analyze the stability of the system with various feedforward compensation. To do so, we conduct a Lyapunov based robustness analysis with respect to approximations in the feedforward compensation. This analysis allows to ascertain local asymptotic stability or Lagrange stability of the system under certain conditions involving bounds on the parameters of the system, and on other parameters that are affected by design such as the rapidity of the desired output trajectory. In order to obtain quantitative results, i.e. obtain numerical values for the bounds, we must explicitly construct a Lyapunov function (local) for the system, which may impose additional constraints on the system (we required the presence of damping at the links in our analysis).

This design approach may be applied to both rigid robots and flexible robots. A particular case where this approach may be particularly useful is for the controller design of flexible joint robots with small, unmodeled or badly known gyroscopic effects, so that inclusion of these effects in the feedforward compensator is not practical.

Future work will tackle both theoretical an practical problems encountered in the control of flexible joint robots, and of flexible structures in general. Plans include

- Analysis of the active use of the link state measurement in the design framework presented here in order to improve the performance of the system.
- Design a nonlinear observer to estimate the link state based on the motor state measurement, since measurement of the link state is not available in general.
- Design a saturation–driven trajectory generator in order to maintain stability and performance of the system in the presence of hard constraints, in particular of input torque saturation.
- Pursue the validation of the results.

A Centrifugal and Coriolis matrix for M_{12} strictly upper triangular and symmetric motor inertia

In this section, we present the equations for the Coriolis and centrifugal matrix for flexible joint robots with symmetric motor inertia, and with the matrix of gyroscopic couplings M_{12} strictly upper triangular and having the following structure

$$M_{12}(\theta_\ell) = \begin{bmatrix} 0 & m_{1,2}(\theta_{\ell,1}) & m_{1,3}(\theta_{\ell,1},\theta_{\ell,2}) & \cdots & m_{1,m}(\theta_{\ell,1},\cdots,\theta_{\ell,m-1}) \\ 0 & 0 & m_{2,3}(\theta_{\ell,2}) & \cdots & m_{2,m}(\theta_{\ell,2},\cdots,\theta_{\ell,m-1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $m_{i,j}$ are scalar functions.

We recall that the mass matrix of the system can be represented as follows

$$M(\theta_\ell) = \begin{bmatrix} M_{11}(\theta_\ell) & M_{12}(\theta_\ell) \\ M_{12}^T(\theta_\ell) & M_{22} \end{bmatrix}$$

and note that we will use $\theta = \begin{bmatrix} \theta_\ell{}^T & \theta_m{}^T \end{bmatrix}$ with $\theta_\ell, \theta_m \in \mathbf{R}^m$.

We will see that we may factor the Coriolis and centrifugal matrix as

$$C(\theta_\ell,\theta_m,\dot\theta_\ell,\dot\theta_m) = \begin{bmatrix} C_{11}^A(\theta_\ell,\dot\theta_\ell)+C_{11}^B(\theta_\ell,\dot\theta_m) & C_{12}^A(\theta_\ell,\dot\theta_\ell) \\ C_{21}^A(\theta_\ell,\dot\theta_\ell) & 0 \end{bmatrix}$$

where the individual terms of this matrix are defined below.

A.1 Equations for C represented using Christoffel's symbols

We recall that the coefficient of the Coriolis and centrifugal matrix are given by (see Section 3.1)

$$C_{kj} = \sum_{i=1}^{n} \frac{1}{2}\left[\frac{\partial M_{kj}}{\partial\theta_i} + \frac{\partial M_{ki}}{\partial\theta_j} - \frac{\partial M_{ij}}{\partial\theta_k}\right]\dot\theta_i \tag{A.1}$$

First, restrict $(1 \le k \le m)$, $(1 \le j \le m)$ in (A.1). After simplification due to the dependence of M_{11}, M_{12}, M_{12}^T uniquely on θ_ℓ and due to the fact that M_{22} is constant, we can rewrite the equation as

$$C_{11,(k,j)} = C_{11}^A(\theta_\ell,\dot\theta_\ell)+C_{11}^B(\theta_\ell,\dot\theta_m)$$

with

$$C_{11}^A(\theta_\ell,\dot\theta_\ell) = \sum_{i=1}^{m}\frac{1}{2}\left[\frac{\partial M_{11,(k,j)}(\theta_\ell)}{\partial\theta_{\ell,i}} + \frac{\partial M_{11,(k,i)}(\theta_\ell)}{\partial\theta_{\ell,j}} - \frac{\partial M_{11,(i,j)}(\theta_\ell)}{\partial\theta_{\ell,k}}\right]\dot\theta_{\ell,i}$$

$$C_{11}^B(\theta_\ell,\dot\theta_m) = \sum_{i=1}^{m}\frac{1}{2}\left[\frac{\partial M_{12,(k,i)}(\theta_\ell)}{\partial\theta_{\ell,j}} - \frac{\partial M_{12,(i,j)}^T(\theta_\ell)}{\partial\theta_{\ell,k}}\right]\dot\theta_{m,i}$$

where we recognize $C_{11}^A(\theta_\ell,\dot\theta_\ell)$ as the Coriolis and centrifugal term of the rigid robot, and $C_{11}^B(\theta_\ell,\dot\theta_m)$ is skew–symmetric and has the following structure

$$C_{11}^B(\theta_\ell,\dot\theta_m) = \begin{bmatrix} 0 & c_{12}(\theta_{\ell,1},\cdots,\theta_{\ell,m-1};\dot\theta_{m,3},\cdots,\dot\theta_{m,m}) \\ -c_{12}(\theta_{\ell,1},\cdots,\theta_{\ell,m-1};\dot\theta_{m,3},\cdots,\dot\theta_{m,m}) & 0 \\ \vdots & \vdots \\ -c_{1(m-2)}(\theta_{\ell,1},\cdots,\theta_{\ell,m-1};\dot\theta_{m,m-1},\dot\theta_{m,m}) & -c_{2(m-2)}(\theta_{\ell,2},\cdots,\theta_{\ell,m-1};\dot\theta_{m,m-1},\dot\theta_{m,m}) \\ -c_{1(m-1)}(\theta_{\ell,1},\cdots,\theta_{\ell,m-1};\dot\theta_{m,m}) & -c_{2(m-1)}(\theta_{\ell,2},\cdots,\theta_{\ell,m-1};\dot\theta_{m,m}) \\ 0 & 0 \end{bmatrix}$$

$$
\begin{array}{cccc}
c_{13}(\theta_{\ell,1},\cdots,\theta_{\ell,m-1};\dot{\theta}_{m,4},\cdots,\dot{\theta}_{m,m}) & \cdots & c_{1(m-1)}(\theta_{\ell,1},\cdots,\theta_{\ell,m-1};\dot{\theta}_{m,m}) & 0 \\
c_{23}(\theta_{\ell,2},\cdots,\theta_{\ell,m-1};\dot{\theta}_{m,4},\cdots,\dot{\theta}_{m,m}) & \cdots & c_{2(m-1)}(\theta_{\ell,2},\cdots,\theta_{\ell,m-1};\dot{\theta}_{m,m}) & 0 \\
\vdots & \vdots & \vdots & \vdots \\
-c_{3(m-2)}(\theta_{\ell,3},\cdots,\theta_{\ell,m-1};\dot{\theta}_{m,m-1},\dot{\theta}_{m,m}) & \cdots & c_{(m-2)(m-1)}(\theta_{\ell,m-2},\theta_{\ell,m-1};\dot{\theta}_{m,m}) & 0 \\
-c_{3(m-1)}(\theta_{\ell,3},\cdots,\theta_{\ell,m-1};\dot{\theta}_{m,m}) & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0
\end{array}
\right]
$$

where c_{ij} are properly defined scalar functions. We note that for $i = 1, \cdots, m$, row i of C_{11}^B does not depend on any $\dot{\theta}_{m,j}$, $j \leq i$.

Now, restrict $(1 \leq k \leq m)$, $(m < j \leq 2m)$ in (A.1). After simplification, and defining $p = j - m$, we can rewrite the equation as

$$
\begin{aligned}
C_{12,(k,p)} &= C_{12,(k,p)}^A \\
&= \sum_{i=1}^{m} \frac{1}{2} \left[\frac{\partial M_{12,(k,p)}(\theta_\ell)}{\partial \theta_{\ell,i}} - \frac{\partial M_{12(i,p)}(\theta_\ell)}{\partial \theta_{\ell,k}} \right] \dot{\theta}_{\ell,i}
\end{aligned}
$$

We note that for $k \geq p$, the elements of the matrix are zero such that this matrix is strictly upper triangular.

Now, restrict $(m < k \leq 2m)$, $(1 \leq j \leq m)$ in (A.1). After simplification, and defining $p = k - m$, we can rewrite the equation as

$$
\begin{aligned}
C_{21,(p,j)} &= C_{21,(p,j)}^A \\
&= \sum_{i=1}^{m} \frac{1}{2} \left[\frac{\partial M_{12(p,j)}^T(\theta_\ell)}{\partial \theta_{\ell,i}} + \frac{\partial M_{12(p,i)}^T(\theta_\ell)}{\partial \theta_{\ell,j}} \right] \dot{\theta}_{\ell,i}
\end{aligned}
$$

We note that for $j \geq p$, the elements of the matrix are zero such that this matrix is strictly lower triangular.

Now, restrict $(m < k \leq 2m)$, $(m < j \leq 2m)$ in (A.1). After simplification, we obtain

$$
C_{22} = 0
$$

A.2 Equations for C represented using M_D–notation

We recall that the coefficient of the Coriolis and centrifugal matrix are given by (see Section 3.1)

$$
C(\theta,\dot{\theta}) = M_D(\theta,\dot{\theta}) - \frac{1}{2} M_D^T(\theta,\dot{\theta}) \tag{A.2}
$$

$$
M_D(\theta, v) = \sum_{i=1}^{n} \frac{\partial M(\theta)}{\partial \theta_i} v e_i^T \tag{A.3}
$$

where e_i is the i^{th} unit vector in $\mathbf{R}^n$.

Using (A.3), we obtain

$$
M_D(\theta_\ell,\dot{\theta}_\ell,\dot{\theta}_m) = \left[\begin{array}{cc} M_{D,11}^A(\theta_\ell,\dot{\theta}_\ell) + M_{D,11}^B(\theta_\ell,\dot{\theta}_m) & 0 \\ M_{D,21}^A(\theta_\ell,\dot{\theta}_\ell) & 0 \end{array} \right]
$$

where

$$
M_{D,11}^A(\theta_\ell,\dot{\theta}_\ell) = \sum_{i=1}^{m} \frac{\partial M_{11}(\theta_\ell)}{\partial \theta_{\ell,i}} \dot{\theta}_\ell e m_i^T
$$

$$
\begin{aligned}
M^B_{D,11}(\theta_\ell, \dot\theta_m) &= \sum_{i=1}^{m} \frac{\partial M_{12}(\theta_\ell)}{\partial \theta_{\ell,i}} \dot\theta_m em_i^T \\
M^A_{D,21}(\theta_\ell, \dot\theta_\ell) &= \sum_{i=1}^{m} \frac{\partial M_{12}^T(\theta_\ell)}{\partial \theta_{\ell,i}} \dot\theta_\ell em_i^T
\end{aligned}
$$

where em_i is the i^{th} unit vector in $\mathbf{R}^m$.

Furthermore, using (A.2), we have

$$
\begin{aligned}
C^A_{11}(\theta_\ell, \dot\theta_\ell) &= M^A_{D,11}(\theta_\ell, \dot\theta_\ell) - \frac{1}{2} {M^A_{D,11}}^T(\theta_\ell, \dot\theta_\ell) \\
C^B_{11}(\theta_\ell, \dot\theta_m) &= M^B_{D,11}(\theta_\ell, \dot\theta_m) - \frac{1}{2} {M^B_{D,11}}^T(\theta_\ell, \dot\theta_m) \\
C^A_{12}(\theta_\ell, \dot\theta_\ell) &= -\frac{1}{2} {M^A_{D,21}}^T(\theta_\ell, \dot\theta_\ell) \\
C^A_{21}(\theta_\ell, \dot\theta_\ell) &= M^A_{D,21}(\theta_\ell, \dot\theta_\ell)
\end{aligned}
$$

We note that $C^A_{11}(\theta_\ell, \dot\theta_\ell)$ is the Coriolis and centrifugal term for the rigid robot. Also,

$$
M^B_{D,11}(\theta_\ell, \dot\theta_m) = \begin{bmatrix}
md_{11}(\theta_{\ell,1}, \cdots, \theta_{\ell,m-1}; \dot\theta_{m,2}, \cdots, \dot\theta_{m,m}) & md_{12}(\theta_{\ell,1}, \cdots, \theta_{\ell,m-1}; \dot\theta_{m,3}, \cdots, \dot\theta_{m,m}) & md_{13}(\theta_{\ell,1}, \cdots, \theta_{\ell,m-1}; \dot\theta_{m,4}, \cdots, \dot\theta_{m,m}) & \cdots & md_{1(m-1)}(\theta_{\ell,1}, \cdots, \theta_{\ell,m-1}; \dot\theta_{m,m}) & 0 \\
0 & md_{22}(\theta_{\ell,2}, \cdots, \theta_{\ell,m-1}; \dot\theta_{m,3}, \cdots, \dot\theta_{m,m}) & md_{23}(\theta_{\ell,2}, \cdots, \theta_{\ell,m-1}; \dot\theta_{m,4}, \cdots, \dot\theta_{m,m}) & \cdots & md_{2(m-1)}(\theta_{\ell,2}, \cdots, \theta_{\ell,m-1}; \dot\theta_{m,m}) & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & md_{(m-1)(m-1)}(\theta_{\ell,m-1}; \dot\theta_{m,m}) & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
$$

where md_{ij} are properly defined scalar functions, so that for $i = 1, \cdots, m$, row i of $C^B_{11}(\theta_\ell, \dot\theta_m)$ does not depend on any $\dot\theta_{m,j}$, $j \leq i$.

We also note that $M^A_{D,21}(\theta_\ell, \dot\theta_\ell)$ is strictly lower triangular such that $C^A_{12}(\theta_\ell, \dot\theta_\ell)$ is strictly upper triangular, and $C^A_{21}(\theta_\ell, \dot\theta_\ell)$ is strictly lower triangular.

NOTE : We note that the two representations for C considered here yield a Coriolis and centrifugal matrix with the same structure and characteristics.

B Feedback equivalence of flexible joint robots to passive systems

Here, we want to analyze the property of feedback equivalence of flexible joint robots to passive systems. To achieve this, we first present some definitions and results extracted from [5] and then apply the theorems to flexible joint robots.

B.1 Feedback equivalence of nonlinear systems to passive systems

Consider a nonlinear system Σ described by equations of the form

$$
\begin{aligned}
\dot x &= f(x) + g(x)u && \text{(B.1)} \\
y &= h(x) && \text{(B.2)}
\end{aligned}
$$

with state space $X = \mathbf{R}^q$, set of input values $U = \mathbf{R}^m$ and set of output values $Y = \mathbf{R}^m$. The set $\mathcal{U}$ of admissible inputs consists of all U-valued piecewise continuous functions defined on $\mathbf{R}$. f and the m columns of g are smooth (i.e. C^∞) vector fields and h is a smooth mapping. We suppose that the vector field f has at least one equilibrium; without loss of generality, we can assume $f(0) = 0$ and $h(0) = 0$.

Definition B.1 *By regular static (i.e. memoryless) state feedback, we mean a feedback of the form*

$$u = \alpha(x) + \beta(x)v \tag{B.3}$$

where $\alpha(x)$ and $\beta(x)$ are smooth functions defined either locally near $x = 0$ or globally, and $\beta(x)$ is invertible for all x.

Definition B.2 *The system represented by (B.1, B.2) is feedback equivalent to a passive system if there exists a regular static state feedback (B.3) such that the closed loop system*

$$\begin{aligned} \dot{x} &= [f(x) + g(x)\alpha(x)] + g(x)\beta(x)v \\ y &= h(x) \end{aligned}$$

is passive.

Definition B.3 *A system of the form (B.1) is said to have relative degree $\{1, \cdots, 1\}$ at $x = 0$ if the matrix $L_g h(0)$ is nonsingular.*

In Definition B.3, we use the following notation [13]:

$$L_g h(x) \triangleq \sum_{i=1}^{q} \frac{\partial h(x)}{\partial x_i} g_i(x)$$

where $g_i(x) = E_i g(x)$ with $E_i \in \mathbf{R}^{m \times m}$ has element i, i equal to one, and all its other elements equal zero.

The relative degree is also equal to the smallest order of time derivative of the output in which the input appears explicitly.

Definition B.4 *The distribution Δ spanned by the vector fields $g_1(x), \cdots, g_m(x)$ is involutive if, for $g_i, g_j \in \Delta$, the Lie bracket*

$$[g_i, g_j] \triangleq \frac{\partial g_j}{\partial x} g_i - \frac{\partial g_i}{\partial x} g_j \in \Delta \quad \textit{for } i, j = 1, \cdots, m$$

This is equivalent to say that

$$\textit{rank}\, \{g_1(x), \cdots, g_m(x)\} = \textit{rank}\, \{g_1(x), \cdots, g_m(x), [g_i, g_j]\} \quad \textit{for } i, j = 1, \cdots, m$$

If system (B.1,B.2) has relative degree $\{1, \cdots, 1\}$ at $x = 0$ and the distribution Δ spanned by the vector fields $g_1(x), \cdots, g_m(x)$ is involutive, it is possible to find $q - m$ real-valued functions $z_1(x), \cdots, z_{q-m}(x)$, locally defined near $x = 0$ and vanishing at $x = 0$, which, together with the m components of the output map, qualify as a new set of local coordinates. In the new set of coordinates (z, y) the system is represented in its normal form

$$\dot{z} = q(z, y) \tag{B.4}$$
$$\dot{y} = b(z, y) + a(z, y)u \tag{B.5}$$

where the matrix $a(z, y)$ is nonsingular for all z, y near $(0, 0)$.

The zero dynamics of a system describe those internal dynamics which are consistent with the external constraint $y = 0$. If a system has relative degree $\{1, \cdots, 1\}$ at $x = 0$, its zero dynamics exist in a neighborhood X^0 of $x = 0$, evolve on the smooth $(q - m)$-dimensional submanifold

$$Z^* = \left\{x \in X^0 : h(x) = 0\right\}$$

and are described by a differential equation of the form

$$\dot{x} = f^*(x) \qquad x \in Z^*$$

in which $f^*(x)$ (the zero dynamics vector fields) denotes the restriction to Z^* of the vector field

$$f^*(x) = f(x) + g(x)u^*(x)$$

with

$$u^*(x) = -\left[L_g h(x)\right]^{-1} L_f h(x)$$

In the normal form (B.4,B.5) the zero dynamics are characterized by the equation

$$\dot{z} = q(z, 0)$$

In order to have a globally defined normal form, the following conditions must be satisfied:

H1 : the matrix $L_g h(x)$ is nonsingular for each $x \in X$,

H2 : the vector fields $\tilde{g}_1(x) \cdots \tilde{g}_m(x)$ (defined below) are complete,

H3 : the distribution spanned by $g_1(x) \cdots g_m(x)$ is involutive,

where

$$[\tilde{g}_1(x) \cdots \tilde{g}_m(x)] = g(x)\left[L_g h(x)\right]^{-1}$$

Note that by complete, we mean that the integral curves are defined for all $t \geq 0$ for any initial conditions.

Definition B.5 *A nonnegative function $V : X \to \mathbf{R}$ is said to be proper if for each $a > 0$, the set $V^{-1}([0, a]) = \{x \in X : 0 \leq V(x) \leq a\}$ is compact.*

Definition B.6 *Suppose $L_g h(0)$ is nonsingular. Then Σ is said to be:*

i) *minimum phase if $z = 0$ is an asymptotically stable equilibrium of $q(z, 0)$,*

ii) *weakly minimum phase if there exists a C^k, $k \geq 2$, function $W^*(z)$, defined near $z = 0$ with $W^*(0) = 0$, which is positive definite and such that $L_{q(z,0)} W^*(z) \leq 0$ for all z near $z = 0$.*

Suppose **H1**, **H2** *and* **H3** *hold. Then Σ is said to be:*

i) *globally minimum phase if $z = 0$ is a globally asymptotically stable equilibrium of $q(z, 0)$,*

ii) *globally weakly minimum phase if there exists a C^k, $k \geq 2$, function $W^*(z)$, defined for all z with $W^*(0) = 0$, which is positive definite and proper such that $L_{q(z,0)} W^*(z) \leq 0$ for all z.*

Definition B.7 *A point x^o is a regular point for a system Σ of the form (B.1) if rank $\{L_g h(x)\}$ is constant in a neighborhood of x^o.*

Also, we assume that $rank\{g(0)\} = rank\{dh(0)\} = m$.

We can now state the main theorems relating feedback equivalence of nonlinear systems to passive systems.

Theorem B.1 [5] *Suppose $x = 0$ is a regular point for Σ. Then Σ is locally feedback equivalent to a passive system with a C^2 storage function V, which is positive definite, if and only if Σ has relative degree $\{1, \cdots, 1\}$ at $x = 0$ and is weakly minimum phase.*

Theorem B.2 [5] *Assume* **H1**, **H2** *and* **H3**. *Then Σ is globally feedback equivalent to a passive (respectively, strictly passive) system with a C^2 storage function V, which is positive definite, if and only if Σ is globally weakly minimum phase (respectively, globally minimum phase).*

B.2 Flexible joint manipulators

We now find the conditions under which a manipulator with flexible joints is feedback equivalent to a passive system, i.e. under which Theorem B.1 or B.2 is satisfied.

First consider the general model of a manipulators with flexible joints (3.10) repeated here for completeness

$$\begin{bmatrix} M_{11}(\theta_\ell,\theta_m) & M_{12}(\theta_\ell,\theta_m) \\ M_{12}^T(\theta_\ell,\theta_m) & M_{22}(\theta_m) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_\ell \\ \ddot{\theta}_m \end{bmatrix} + \begin{bmatrix} C_{11}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m) & C_{12}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m) \\ C_{21}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m) & C_{22}(\theta_\ell,\theta_m,\dot{\theta}_\ell,\dot{\theta}_m) \end{bmatrix} \begin{bmatrix} \dot{\theta}_\ell \\ \dot{\theta}_m \end{bmatrix} + \begin{bmatrix} D_\ell(\dot{\theta}_\ell) \\ D_m(\dot{\theta}_m) \end{bmatrix} + \begin{bmatrix} r_1(\theta_\ell,\theta_m) \\ r_2(\theta_\ell,\theta_m) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} u \tag{B.6}$$

First, write (B.6)in the form of (B.1,B.2), i.e. define

$$\begin{aligned} x &= \left[x_1^T x_2^T x_3^T x_4^T\right]^T \\ x_1 &= \dot{\theta}_\ell \qquad x_2 = \dot{\theta}_m \qquad x_3 = \theta_\ell \qquad x_4 = \theta_m \\ \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_\ell \\ D_m \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \end{aligned}$$

where the state dependence has been dropped, and use the fact that M_{11} is nonsingular (in fact so is M_{22}) to write the inverse of the mass matrix as

$$M^{-1} = \begin{bmatrix} M_{11}^{-1} + M_{11}^{-1}M_{12}\Delta_2^{-1}M_{12}^TM_{11}^{-1} & -M_{11}^{-1}M_{12}\Delta_2^{-1} \\ -\Delta_2^{-1}M_{12}^TM_{11}^{-1} & \Delta_2^{-1} \end{bmatrix}$$

where

$$\Delta_2 = M_{22} - M_{12}^TM_{11}^{-1}M_{12}$$

We may then write, after some simple manipulations,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\left[M_{11}^{-1} + M_{11}^{-1}M_{12}\Delta_2^{-1}M_{12}^TM_{11}^{-1}\right]\rho_1 + M_{11}^{-1}M_{12}\Delta_2^{-1}\rho_2 \\ \Delta_2^{-1}M_{12}^TM_{11}^{-1}\rho_1 - \Delta_2^{-1}\rho_2 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -M_{11}^{-1}M_{12}\Delta_2^{-1} \\ \Delta_2^{-1} \\ 0_{m\times m} \\ 0_{m\times m} \end{bmatrix} u$$

Consider the case where the output is a linear combination of the state and is given by

$$y = h(x) = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix} x$$

where $C_1, C_2, C_3, C_4 \in \mathbf{R}^{m\times m}$ are constant.

We readily verify that $rank\{g(0)\} = m$ for any arm configuration since Δ_2 is always nonsingular. We will impose the constraint on $h(x)$ that $rank\{dh(0)\} = m$, i.e. the vector $\begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix}$ has *rank* m.

First, find conditions on the output matrix such that the system has relative degree $\{1,\cdots,1\}$ at $x = 0$. Write

$$\begin{aligned} L_g h(x) &= -\frac{\partial h(x)}{\partial x_1} M_{11}^{-1} M_{12} \Delta_2^{-1} + \frac{\partial h(x)}{\partial x_2} \Delta_2^{-1} \\ &= \left[-C_1 M_{11}^{-1} M_{12} + C_2\right] \Delta_2^{-1} \end{aligned}$$

Fact B.1 *Since the matrix Δ_2 is nonsingular for any state value, the system has relative degree $\{1,\cdots,1\}$ at $x = 0$ if and only if the matrix $\Phi = \left[-C_1 M_{11}^{-1} M_{12} + C_2\right]$ is nonsingular for $x = 0$.*

NOTES :

- For any arm configuration, the presence or absence of the motor and link positions in the output does not affect the conclusion of Fact B.1.
- If there is no gyroscopic coupling, i.e. $M_{12} = 0$, then C_2 must be nonsingular, i.e. the motor velocities must appear at the output, and C_1 is arbitrary.
- If M_{12} is nonsingular for any state value, choosing the output as the link velocities with C_1 nonsingular is sufficient to guarantee that Φ is nonsingular since M_{11} is nonsingular for any state value.
- If M_{12} is not zero at $x = 0$, then $\Phi(0)$ is singular only for specific combinations of C_1 and C_2.

The point $x = 0$ is regular if $rank\{\Phi\} = m$ in a neighborhood of $x = 0$. This is guaranteed by the continuity of M_{11} and M_{12} if Φ is nonsingular at $x = 0$. This holds globally (condition **H1**) for proper choices of C_1 and C_2 for any arm configuration due to the boundedness of M_{11} and M_{12}, i.e. if we choose C_1 and C_2 such that

$$\sup_x \sigma_{max}\left\{C_1 M_{11}(x_3,x_4)^{-1} M_{12}(x_3,x_4)\right\} < \sigma_{min}\{C_2\}$$

This holds in particular if $C_1 = 0$ or $M_{12} = 0$, and C_2 is nonsingular.

We now proceed to show the (weak) minimum phase property. The zero dynamics of the system are described by the following equation :

$$\begin{aligned} \dot{x} &= f(x) - g(x)\left[L_g h(x)\right]^{-1} L_f h(x) \qquad x \in Z^* \\ &= f(x) - g(x)\Delta_2 \Phi^{-1} \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix} f(x) \qquad x \in Z^* \end{aligned}$$

assuming that Φ is nonsingular in the neighborhood of $x = 0$. After some manipulations, this leads to

$$\dot{x} = \begin{bmatrix} -\left[M_{11}^{-1} + M_{11}^{-1} M_{12} \Phi^{-1} C_1 M_{11}^{-1}\right]\rho_1 + M_{11}^{-1} M_{12} \Phi^{-1}\left[C_3 x_1 + C_4 x_2\right] \\ \Phi^{-1} C_1 M_{11}^{-1} \rho_1 - \Phi^{-1}\left[C_3 x_1 + C_4 x_2\right] \\ x_1 \\ x_2 \end{bmatrix} \qquad x \in Z^*$$

We consider two cases here while keeping in mind that we must satisfy Φ nonsingular. We define the output as a function of the motor state only for the first case, and of the link state only in the second case.

Case 1 : $C_1 = C_3 = 0$, C_2 nonsingular, C_4 nonzero.

The dynamics on the zero manifold are described by

$$\dot{x} = \begin{bmatrix} -M_{11}^{-1}\rho_1 + M_{11}^{-1}M_{12}C_2^{-1}C_4x_2 \\ -C_2^{-1}C_4x_2 \\ x_1 \\ x_2 \end{bmatrix} \qquad x \in Z^* = \left\{x \in X^0 : C_2x_2 + C_4x_4 = 0\right\}$$

Setting the output to zero, we get

$$\begin{aligned} C_2x_2 + C_4x_4 &= 0 \\ x_2 &= -C_2^{-1}C_4x_4 \end{aligned}$$

Hence, if $C_2^{-1}C_4 > 0$ then $x_4 \to 0$ exponentially; if $C_4 = 0$ then x_4 is a constant; in general, $x_4 \to$ null space of $C_2^{-1}C_4 \geq 0$.

NOTE : With the output of the system defined as the output of a exponentially stable first order exosystem with the motor position as its input, zero system output implies that the motor state goes to zero. If the exosystem is just stable, then we can only guarantee that the motor position will go to a constant.

Given that the motor position x_4 converges exponentially to a constant x_{4f} and the motor velocity x_2 to zero, the question of stability of the zero-dynamics resumes to analyze the stability of

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} -M_{11}^{-1}\rho_1 \\ x_1 \end{bmatrix} \qquad x_2 = 0;\ x_4 = x_{4f} \\ &= \begin{bmatrix} -M_{11}^{-1}(x_3, x_{4f})\left[C_{11}(x_3, x_{4f}, x_1, 0)x_1 + D_\ell(x_1) + r_1(x_3, x_{4f})\right] \\ x_1 \end{bmatrix} \end{aligned}$$

This equation has the same stability properties as a rigid robot with a constant forcing term in x_{4f}. Hence, consider the following scalar function

$$\begin{aligned} V(x_1, x_3) &= \frac{1}{2}x_1^T M_{11}(x_3, x_{4f})x_1 + U(x_1, x_3) \\ U(x_1, x_3) &= \int_0^{x_3} r_1(\xi, x_{4f})\, d\xi \end{aligned} \tag{B.7}$$

$U(x_1, x_3)$, and thus $V(x_1, x_3)$, is positive definite if the equivalent torsional spring is strong enough with respect to gravity, i.e. if

$$N\nabla_{x_3}k_1(x_3, x_{4f}) > -\nabla_{x_3}g_\ell(x_3, x_{4f}) \qquad \forall x_3,\ x_{4f} \in \mathbf{R}^m$$

for r_1 defined as in (3.11) with $k_1(x_3, x_4)$ monotonically increasing in x_3, and $k_1(0,0) = g_\ell(0,0) = 0$.

Represent the centrifugal and Coriolis matrix using Christoffel's symbols such that $(\frac{1}{2}\dot{M} - C)$ is skew symmetric, which also imply that $(\frac{1}{2}\dot{M}_{11} - C_{11})$ is skew symmetric. Then, take the time derivative of (B.7) to obtain

$$\dot{V}(x_1, x_3) = -x_1^T D_\ell(x_1)$$

such that, for positive or at least non negative damping, $\dot{V} \leq 0$. Hence, we conclude that the zero dynamics of the system are stable, and that the system is at least weakly minimum phase (Definition B.6).

NOTE : If there is positive link damping then we conclude that $x_1 \to 0$ as well as its higher derivatives, such that the system equation becomes

$$0 = r_1(x_3, x_{4f})$$

which defines the steady state value(s) of x_3, i.e. the system is stable and weakly minimum phase.

With or without damping, if C_4 is nonsingular such that $x_{4f} = 0$, then the zero state is asymptotically stable such that the system is minimum phase (Definition B.6).

In conclusion, the conditions of Theorem B.1 are satisfied, i.e. the system is locally feedback equivalent to a passive system, for the output containing only the motor state, i.e. if

$$y = C_2\dot{\theta}_m + C_4\theta_m$$

- if and only if
  - $rank\{C_2\} = m$ such that $rank\{dh(0)\} = m$, and such that the system has relative degree $\{1, \cdots, 1\}$ and is regular at any point in the state space,
  - $C_2^{-1}C_4 \geq 0$, i.e. stable exosystem,
- and if the spring is stiff enough compared to gravity and the link damping is at least positive semi-definite such that the system is weakly minimum phase.

Case 2 : $C_2 = C_4 = 0$, C_1 nonsingular, C_3 nonzero.

We must first assume that M_{12} is nonsingular to guarantee that Φ is invertible.

The dynamics on the zero manifold are described by

$$\dot{x} = \begin{bmatrix} -C_1^{-1}C_3x_1 \\ -M_{12}^{-1}\rho_1 + M_{12}^{-1}M_{11}C_1^{-1}C_3x_1 \\ x_1 \\ x_2 \end{bmatrix} \qquad x \in Z^* = \left\{x \in X^0 : C_1x_1 + C_3x_3 = 0\right\}$$

Setting the output to zero, we get

$$\begin{aligned} C_1x_1 + C_3x_3 &= 0 \\ x_1 &= -C_1^{-1}C_3x_3 \end{aligned}$$

Hence, if $C_1^{-1}C_3 > 0$ then $x_3 \to 0$ exponentially; if $C_3 = 0$ then x_3 is a constant; in general, $x_3 \to$ null space of $C_1^{-1}C_3 \geq 0$.

NOTE : With the output of the system defined as the output of a exponentially stable first order exosystem with the link position as its input, zero system output implies that the link state goes to zero. If the exosystem is just stable, then we can only guarantee that the link position will go to a constant.

Given that the link position x_3 converges exponentially to a constant x_{3f} and the link velocity x_1 to zero, the question of stability of the zero-dynamics resumes to analyze the stability of

$$\begin{aligned} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} -M_{12}^{-1}\rho_1 \\ x_2 \end{bmatrix} \qquad\qquad x_1 = 0;\ x_3 = x_{3f} \\ &= \begin{bmatrix} -M_{12}^{-1}(x_{3f}, x_4)\left[C_{12}(x_{3f}, x_4, 0, x_2)x_2 + r_1(x_{3f}, x_4)\right] \\ x_2 \end{bmatrix} \end{aligned}$$

Hence, the stability of the zero dynamics of the system depends on the stability of the gyroscopic coupling subsystem (note that M_{12} may be positive definite, negative definite or indefinite (and also zero)). If this subsystem is stable, then so are the zero dynamics. With the assumption that r_1 is defined as previously and that $k_1(x_3, x_4)$ is monotonically decreasing with x_4, this imposes the condition that $M_{12}(x_{3f}, x_4) > 0\ \forall x_{3f},\ x_4 \in \mathbf{R}^m$. Also note that if the motors have symmetric inertia, C_{12} is independent of x_2 and is thus zero in this case such that we can get Lyapunov stability only (which is still sufficient to guarantee the system to be weakly minimum phase).

In conclusion, the conditions of Theorem B.1 are satisfied, i.e. the system is locally feedback equivalent to a passive system, for the output containing only the link state, i.e. if

$$y = C_1\dot{\theta}_\ell + C_3\theta_\ell$$

if and only if

- M_{12} is nonsingular,
- $rank\{C_1\} = m$ such that $rank\{dh(0)\} = m$, and such that the system has relative degree $\{1, \cdots, 1\}$ and is regular at any point in the state space,
- $C_1^{-1}C_3 \geq 0$, i.e. stable exosystem,
- the subsystem representing gyroscopic coupling is stable.

We have established some conditions to obtain feedback equivalence to a passive system by using uniquely the motor or the link state. The same method may be used to analyze other choices of output, but this analysis is not carried out at this point due to the difficulty in verifying the stability of the reduced system describing the zero–dynamics.

In case 1 previously presented, the minimum phase or weak minimum phase properties are in fact global. No such conclusion can be drawn for case 2 under the actual assumptions. Also, the two previous cases also satisfy condition **H1**. Hence, only conditions **H2** and **H3** have to be verified to obtain global results using Theorem B.2.

To verify **H2**, evaluate $\tilde{g}(x)$:

$$\tilde{g}(x) = \begin{bmatrix} -M_{11}^{-1}M_{12}\Phi^{-1} \\ \Phi^{-1} \\ 0 \\ 0 \end{bmatrix}$$

such that the integral curves are defined for all $t \geq 0$ and for all initial conditions provided that Φ is nonsingular over the complete state space.

To verify **H3**, note that the Lie bracket $[g_i, g_j] = 0$, $\forall i, j = 1, \cdots, m$ so that (B.4) is verified, i.e. **H3** is satisfied without additional conditions.

Hence, in order to conclude on global equivalence, the only additional condition to verify with respect to the local case is the global asymptotic or global stability of the zero dynamics, e.g. conditions regarding damping in case 1 previously analyzed.

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