

POLYNOMIAL APPROXIMATIONS OF POINCARÉ MAPS FOR  
HAMILTONIAN SYSTEMS

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**Abstract** Different methods are proposed and tested for transforming a non-linear differential system, and more particularly a Hamiltonian one, into a map without integrating the whole orbit as in the well-known Poincaré return map technique. We construct piecewise polynomial maps by coarse-graining the phase-space surface of section into parallelograms and using either only values of the Poincaré maps at the vertices or also the gradient information at the nearest neighbours to define a polynomial approximation within each cell. The numerical experiments are in good agreement with both the real symplectic and Poincaré maps.

**A synthetic approach**

Poincaré maps are now of common use for studying the qualitative behaviour of differential equations (see Hénon, 1981). Moreover, in order to study stability problems, many authors have sought explicit algebraic mappings which approximate, at least qualitatively, the Poincaré maps obtained from the original Newton equations. Froeschlé and Petit (1990, paper I) have reviewed some of these mappings and showed that they are reliable only as long as one remains within the domain of validity of the approximations made in order to isolate either — in the case of deterministic mappings — an integrable part and some instantaneous perturbations, or — for stochastic mappings — a source of endogeneous/exogeneous stochasticity (see Froeschlé and Rickman, 1988). All these mappings are *ad hoc* and reliable only in some region of the phase space and for some specific purpose. In paper I we built a mapping valid everywhere in the phase space, following an idea already used by Varosi et al. (1987) but in the framework of non-Hamiltonian systems (i.e., systems where attractors do exist). The method consists of coarse-graining the phase-space surface of section and then interpolating the value of the image of a point. Linear interpolation requires a rather fine graining of the phase space, hence it is necessary to compute a lot of points on the grid. However, Taylor expansions of order 3 and 5 can provide very good results as long as symmetrical interpolation formulae are applied, for which it is necessary to use an extended grid. Since there are cases where one cannot cross a given limit, asymmetrical interpolation formulae have been tested, but their accuracy was found to be inferior. Therefore Petit and Froeschlé (1991, paper II) have developed another type of interpolation, where the information, including that on the gradients, is stored to the same level of accuracy only for the nearest-neighbouring vertices. Thus, not only images of vertices are computed, but also tangential mappings at each vertex.

There are in any case two key parameters: the number of bins in each direction  $N = (\text{total number of cells})^{1/D}$ , where  $D (= 2 \text{ and } 3 \text{ in papers I and II})$  is the dimension of the surface of section, and  $M$  the order of the Taylor expansion. In order to explore the validity of the synthetic approach, we have applied our method in two cases:

- 1) An algebraic area-preserving mapping for which the computation of orbits is very fast. This allows one to follow a large number of orbits and to carry out enough iterations for a meaningful comparison.
- 2) A special case of the restricted three-body problem, already studied by Duncan et al. (1989).

In the former case the well-known standard mapping has been used (Froeschlé, 1970; Lichtenberg and Leiberman, 1983):

$$\begin{aligned}x^{(n+1)} &= x^{(n)} + a \sin(x^{(n)} + y^{(n)}), \\y^{(n+1)} &= x^{(n)} + y^{(n)}.\end{aligned} \quad (\text{mod } 2\pi)$$

Fig. 1 shows orbits of the standard mapping for  $a = -1.3$ . Indeed such a mapping exhibits all the well-known typical features of problems with two degrees of freedom, such as invariant curves, "islands", and stochastic zones where the points wander in a chaotic way. Figs. 1b and 1c are magnifications of the

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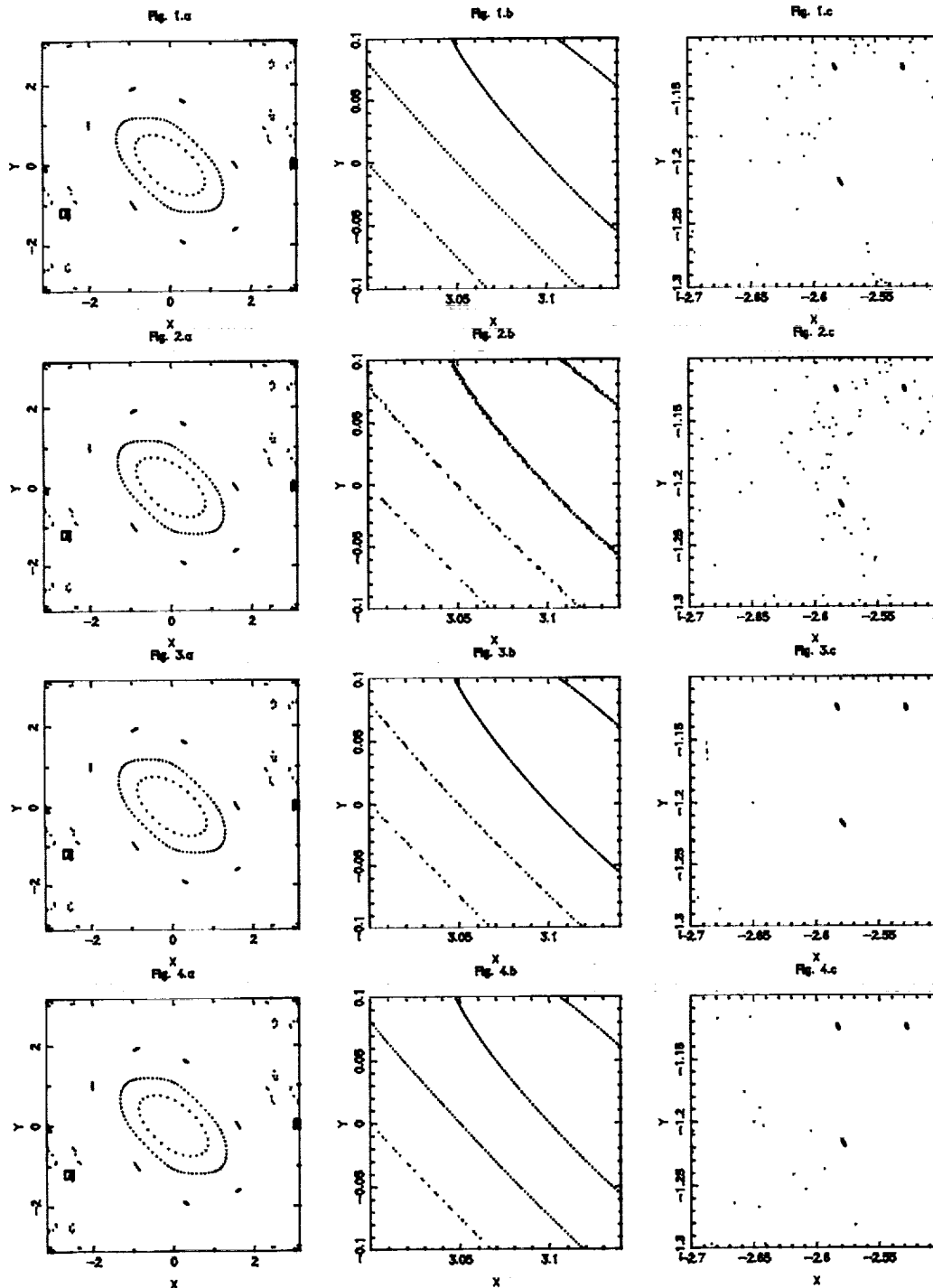


Fig. 1: (a) Plots of the standard mapping for  $a = -1.3$ . (b) and (c) are enlargements of the small boxes shown in (a), respectively at the right border and near the left border.

Figs. 2a-c: Same as in Figs. 1a-c but using a Taylor approximation of order 3 with a regular grid and decentered formulae.

Figs. 3a-c: The same as in Figs. 1a-c but using a Taylor approximation of order 3 with a non-regular grid and decentered formulae.

Figs. 4a-c: The same as in Figs. 1a-c but using a Taylor approximation of order 3 with a regular grid and the gradient method.

small boxes indicated in Fig. 1a. At this magnification level, details like second-order islands become evident and the approximation levels of the synthetic maps are easily visualized. Figs. 2a, 2b and 2c correspond to the same orbits and the same magnifications as Figs. 1a-c, but using the Taylor interpolation mapping of order  $M = 3$  with decentered formulae on the edges of the mapping; here the grid was regular and characterized by  $N = 40$ . The results are very similar to those of the original map, except for the box close to the frontier. Figs. 3a-c correspond to the same formulae, but with cells having half the size of the previous ones close to the edges. Apparently, this does not improve drastically the quality of the mapping in the frontier box. On the other hand, a definite improvement is obtained using the gradient formulae (for more details about these formulae, see papers I and II).

Fig. 5.a

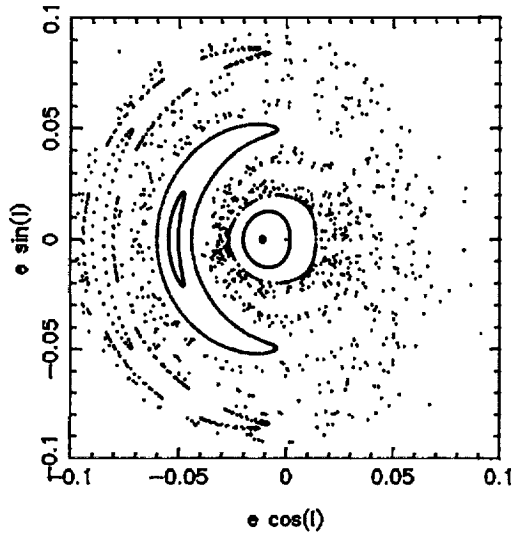


Fig. 5.c

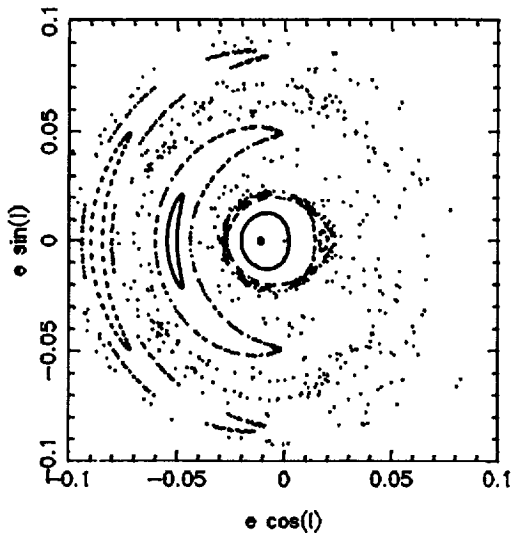


Fig. 5.b

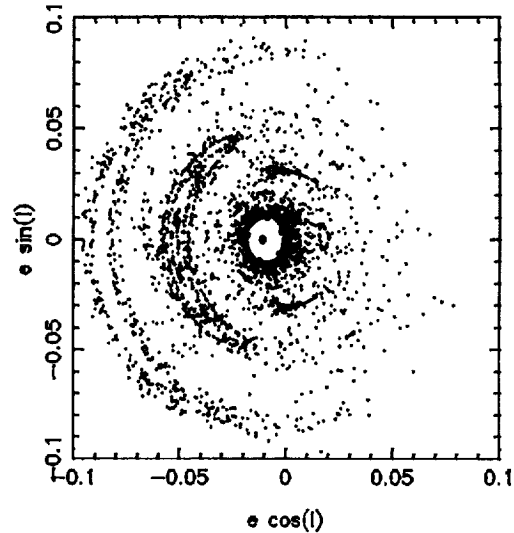


Fig. 5.d

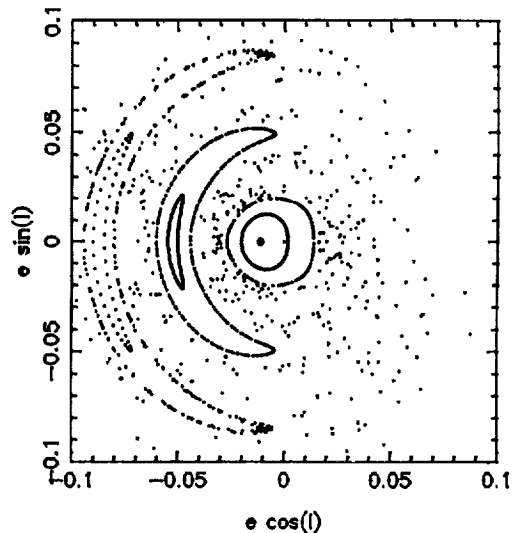


Fig. 5a: Trajectories of the Poincaré map of the restricted three-body problem, in the plane giving (as polar coordinates) eccentricity and mean longitude at conjunction with the planet, for the Neptune-to-Sun mass ratio  $m/M_0 = 5.178 \cdot 10^{-5}$  and a Jacobi constant of 3.0080694.

Figs. 5b-d: The same as Fig. 5a but for the synthetic maps T1, T3 and T5, respectively.

Let us switch to the tests of the method on a special case of the restricted three-body problem, for which Duncan *et al.* have developed a special mapping. Fig. 5a shows orbits of the Poincaré map taking as surface of section the plane defined by the eccentricity and the mean longitude as polar coordinates, at the times when the particle is in conjunction with the planet (i.e., in the rotating frame, when  $y = 0$  and  $\dot{y} > 0$ ). Figs. 5b-d show plots of the corresponding orbits for the Taylor synthetic mappings of order 1, 3 and 5, using a grid with  $N = 100$ . While the linear mapping T1 displays only poor qualitative similarities with the Poincaré map, the map T3 correctly reproduces the locations, shapes and sizes of the zones containing regular orbits. Of course, the T5 map is even better. It should be noted that in order to obtain the same accuracy, we had to use a smaller grid size (i.e., more points) than for the standard map, since the functions which have to be interpolated are less regular.

### Conclusions

Synthetic maps appear to be valuable tools for celestial mechanics. We have presented here only some partial results. For instance another important development concerns problems with more than two degrees of freedom, for which the situation is less straightforward than described above. The number of operations required for the Taylor approximation increases drastically with the dimensions of the surface of section. Of course a lower-order map can be used by decreasing the grid size, but a further difficulty lies in the task of storing and recalling the values of the computed images at the vertices. This is the reason why we have used a hash function when dealing with problems with three degrees of freedom (see paper II).

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