ASTEROID ORBITAL ERROR ANALYSIS: THEORY AND APPLICATION

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We present a rigorous Bayesian theory for asteroid orbital error estimation in which the probability density of the orbital elements is derived from the noise statistics of the observations. For Gaussian noise in a linearized approximation the probability density is also Gaussian, and the errors of the orbital elements at a given epoch are fully described by the covariance matrix. The law of error propagation can then be applied to calculate past and future positional uncertainty ellipsoids (Cappellari *et al.* 1976, Yeomans *et al.* 1987, Whipple *et al.* 1991).

To our knowledge, this is the first time a Bayesian approach has been formulated for orbital element estimation. In contrast to the classical Fisherian school of statistics, the Bayesian school allows a priori information to be formally present in the final estimation. However, Bayesian estimation does give the same results as Fisherian estimation when no a priori information is assumed (Lehtinen 1988, and references therein).

BAYESIAN THEORY OF ORBIT ESTIMATION

Assume that N pairs of right ascensions and declinations $\ell = (\alpha_1, \delta_1; \ldots; \alpha_N, \delta_N)^T$ have been observed for a certain asteroid at times $\mathbf{t} = (t_1, \ldots, t_N)^T$. Let the corresponding computed sky−plane positions be described by the vector $\mathbf{L(P)}$ for the orbital elements $\mathbf{P} = (\tau, \omega, \Omega, i, e, a)^T$. The orbital elements are, respectively, the time of perihelion, argument of perihelion, longitude of ascending node, inclination, eccentricity, and semimajor axis. The astrometric observations and computed positions are related to each other through the so−called observation equation

$$\begin{aligned} \ell &= \mathbf{L(P)} + \epsilon \\ \epsilon &= (\epsilon_{\alpha_1}, \epsilon_{\delta_1}; \ldots; \epsilon_{\alpha_N}, \epsilon_{\delta_N})^T, \end{aligned} \qquad (1)$$

where ϵ describes the noise.

Following Bayesian estimation, or statistical inversion theory, the a posteriori probability density of the orbital elements is related, via Eq. (1), to the a priori and noise probability densities through

$$p_{post}(\mathbf{P}) \propto p_{pre}(\mathbf{P})\, p_{2N}(\epsilon) = p_{pre}(\mathbf{P})\, p_{2N}(\ell - \mathbf{L(P)}). \qquad (2)$$

The a priori probability density can be assumed constant, so no constraining assumptions are made about the orbital elements. Eq. (2) describes the entire solution of the orbit inversion problem in terms of probability densities.

Next, assume that the noise probability density is Gaussian with a diagonal covariance matrix; i.e., the noise is not correlated. In spite of this simplifying assumption, it is evident from Eq. (2) that the a posteriori density for orbital elements is rather complicated. In the present context, we do not study the general probability density but proceed by linearizing the sky−plane positions in the neighborhood of the ephemeris from a least−squares orbit solution $\mathbf{P_0}$. The resulting a posteriori probability density for the orbital element deviations is Gaussian and has inverse covariance matrix

$$\Sigma_{P_i P_j}^{-1} = \sum_{k=1}^{N} \frac{\cos^2 \delta_k}{\sigma_k^2} \frac{\partial \alpha}{\partial P_i}(\mathbf{P_0}, t_k) \frac{\partial \alpha}{\partial P_j}(\mathbf{P_0}, t_k) + \frac{1}{\rho_k^2} \frac{\partial \delta}{\partial P_i}(\mathbf{P_0}, t_k) \frac{\partial \delta}{\partial P_j}(\mathbf{P_0}, t_k), \qquad (3)$$

in which σ_k and ρ_k are the standard deviations in right ascension and declination, respectively. Numerical inversion of this matrix yields the covariance matrix of the orbital elements.

Instead of using a priori error estimates of the observations to obtain σ_k and ρ_k for Eq. (3), the following self–consistent method of determining the standard deviations can be used: Determine the orbit by the usual means of differential correction, in which equal weight is given to each observation; compute the rms errors σ (allowing for $\cos\delta$) and ρ, and then set $\sigma_k = \sigma$, $\rho_k = \rho$, $k = 1, \ldots, N$ in the error analysis.

A number of analytical results can be derived from the covariance matrix in a two–body orbit approximation (with minor changes, the results are valid for perturbed orbits). For example, as an observational arc (T) is lengthened, the accuracy of the semimajor axis is improved faster ($\Sigma_{aa} \propto T^{-3}$) than that of the other orbital elements ($\Sigma_{P_i P_i} \propto T^{-2}$). This is analogous to lightcurve error analysis, in which the period improves faster than the Fourier coefficients (Karttunen and Muinonen 1991). Moreover, the variances of the mean, eccentric, and true anomalies have a quadratic time dependence. It is also worth noting that the correlations among the orbital elements are relatively insensitive to the arc length. This arises from geometric restrictions of the optical groundbased observations, and could be alleviated by radar or spacecraft observations.

In the general formalism, the probability density for the range $\tilde{R}$, declination $\tilde{\delta}$, and right ascension $\tilde{\alpha}$ at time t can be obtained from the a posteriori probability density of the orbital elements from

$$p(\tilde{R}, \tilde{\delta}, \tilde{\alpha}; t) = \frac{1}{\tilde{R}^2 \cos\tilde{\delta}} \int d\mathrm{P}\, p_{post}(\mathrm{P})\, \delta_D(\tilde{R} - R(\mathrm{P}, t))\delta_D(\tilde{\delta} - \delta(\mathrm{P}, t))\delta_D(\tilde{\alpha} - \alpha(\mathrm{P}, t)), \quad (4)$$

where δ_D is Dirac's function. The factor preceding the integral derives from the spherical coordinate system. The numerical computation of this integral is straightforward, though time–consuming. Note that to compute the real uncertainties for a future observation, the probability density in Eq. (4) should still be convolved with the noise of that observation.

Using the linearized approximation for the orbital elements and linearizing the Dirac function arguments, a Gaussian probability density results for the equatorial spherical coordinates (from the integral part of Eq. (4)) with covariance matrix

$$\Lambda = \Psi^T \Sigma \Psi, \qquad \Psi = \begin{pmatrix} \frac{\partial R}{\partial \tau} & \frac{\partial \delta}{\partial \tau} & \frac{\partial \alpha}{\partial \tau} \\ \vdots & \vdots & \vdots \\ \frac{\partial R}{\partial a} & \frac{\partial \delta}{\partial a} & \frac{\partial \alpha}{\partial a} \end{pmatrix}. \quad (5)$$

Here, for a prediction at time t, the partial derivatives must be evaluated at (P_0, t). It should be noted that the linearized approach is an approximation that fails, for example, when a long time has elapsed from the observations and the uncertainties have become very large. The covariance matrix Λ gives an error ellipsoid in three–dimensional space. As is well known, the sky–plane error ellipse is usually very elongated and aligned with the line of variation. Using Eq. (5), it can be shown that, asymptotically, the range, declination, and right ascension uncertainties increase linearly with time in the two–body approximation.

MARS TROJAN 1990 MB

As an example, we study the orbit of the Mars Trojan asteroid 1990 MB using 43 observations spanning 1979 Nov. 21 through 1990 Oct. 14. A two–body orbital error analysis gives the following error estimates for the orbital elements at epoch 1991 Dec. 10.0 (B1950.0 for ω, Ω, and i):

$$\begin{aligned} \tau &= 1992 \text{ Sep. } 15.26887 \pm 0.00184 & i &= 20°28098 \pm 0°00007 \\ \omega &= 95°39992 \pm 0°00137 & e &= 0.0647663 \pm 0.0000028 \\ \Omega &= 244°44755 \pm 0°00007 & a &= 1.5235591\mathrm{AU} \pm 0.0000003\mathrm{AU} \end{aligned} \quad (6)$$

Note that the semimajor axis is very accurately known, as predicted by the error analysis. Because of the rather small eccentricity, the argument of perihelion has a larger error than the longitude of the ascending node. The error in the time of perihelion also reflects the small eccentricity. The covariance matrix reveals high correlations between semimajor axis and eccentricity, as well as between time of perihelion and argument of perihelion.

Figure 1a shows the sky–plane 1–σ uncertainty $\sqrt{\Lambda_{\alpha\alpha} \cos^2 \delta + \Lambda_{\delta\delta}}$ in position prediction for a 10–yr interval. The maxima correlate well with the minimum geocentric distance, which suggests that, for optimum ephemeris improvement, future observations should be made near the minimum distance of the object. As a demonstration, we simulated a pair of observations on 1992 Jul. 26, close to opposition. There is a three–fold improvement in the orbit. For a simulated pair of observations far from opposition on 1992 Mar. 26, there is virtually no ephemeris improvement. Figure 1b shows the present and simulated 1–σ uncertainties $\sqrt{\Lambda_{RR}}$ in range prediction. Since the true orbit is not the least–squares solution, orbital and ephemeris errors are somewhat underestimated.
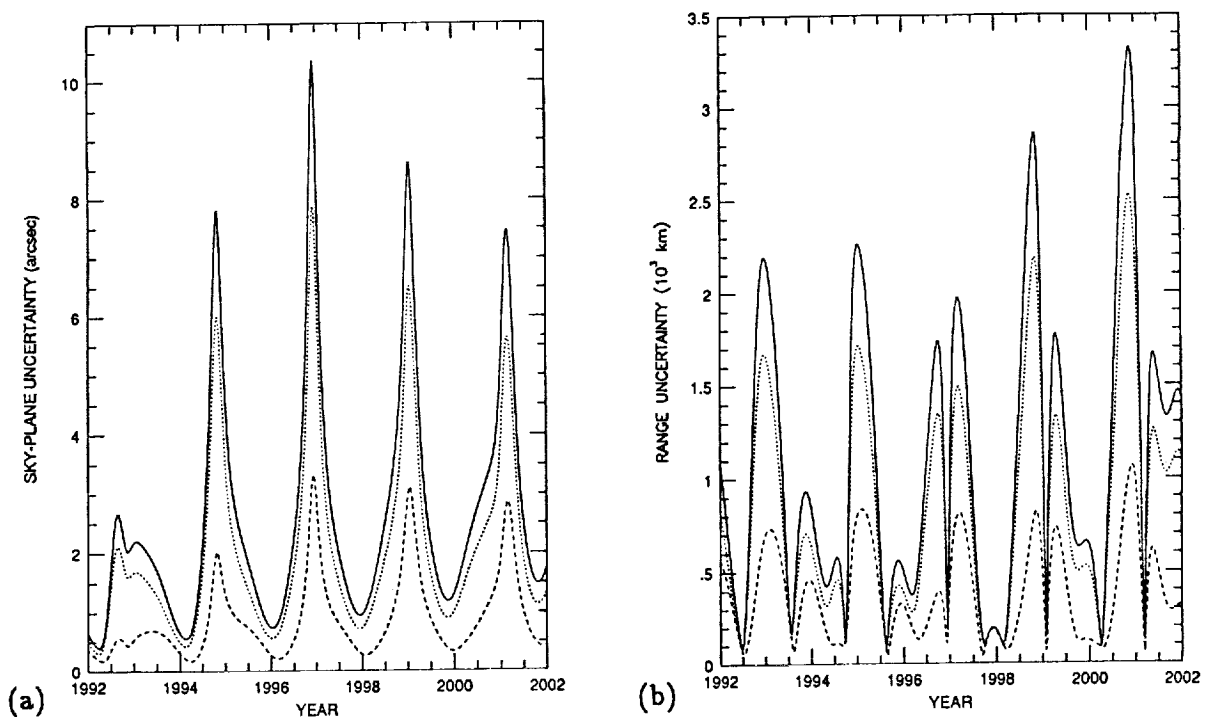

Fig. 1. Sky–plane (a) and range (b) uncertainty predictions for the Mars Trojan 1990 MB over an interval of 10 years (solid lines). The uncertainties after adding a simulated pair of observations on 1992 Jul. 26 (dashed line, close to opposition) and on 1992 Mar. 26 (dotted line, far from opposition) underscore the importance of observation date and geometry.

FUTURE WORK AND APPLICATIONS

In future work, the two–body partial derivatives should be checked against integrated derivatives. It will be important to compare the uncertainty predictions with approximate and accurate derivatives. The linearized approximation should be checked against the general Bayesian approach. Finally, we suggest that a public–domain file of covariance matrices of orbital elements, together with computer programs for error analysis and position prediction, be established. Information about the orbital uncertainties and covariance matrices could also be published in the *Minor Planet Circulars.*

We summarize selected applications of the orbital error analysis as follows:

- In orbit computation, the behavior of the covariance matrix serves as a guide for eliminating poor observations and suggests a way to automate the process.

- In the case of newly discovered asteroids, a strategy for follow–up or recovery can be devised. For example, one may decide whether an asteroid having a one–apparition orbit is recoverable using a narrow– or wide–field instrument and what the extent of the search should reasonably be (Bowell and Muinonen 1991).

- A figure of merit is associated with each possible future observation, thereby suggesting an observational strategy to optimize orbit improvement and to avoid making observations that would not contribute significantly.

- A criterion for numbering an asteroid can be established on the basis of the predicted ephemeris uncertainty over a suitable interval (Muinonen and Bowell 1991). For example, an ephemeris accuracy of 10 arcsec or better over 20 years, independent of single–night apparitions, might be required. The Mars Trojan 1990 MB would qualify for numbering after successful observations on two nights near opposition in 1992 (Figure 1).

- A criterion for determining whether an asteroid is unrecoverable can be established in a corresponding way by defining an observational lifetime for each asteroid.

- A strategy for a recovery attempt of an asteroid having a large sky–plane uncertainty can be planned with the help of orbital error analysis (Bowell and Muinonen 1991).

- Ephemeris uncertainty predictions can be used for fields observed in the past, thus aiding the identification of images on archive plates.

- Knowledge of the accuracy of orbital elements can be used to decide whether it is appropriate to calculate proper elements.

- Uncertainties in occultation ground tracks can be determined.

- Rigorous spacecraft trajectory error analysis can be undertaken.

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