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Progress for the Third Reporting Period

During the third reporting period our efforts were focused on a reformulation of the optimal control problem involving active state-variable inequality constraints. In the reformulated problem the optimization is carried out not with respect to all controllers, but only with respect to asymptotic controllers leading to the state constraint boundary.

Intimately connected with the traditional formulation is the fact that when the reduced solution for such problems lies on a state constraint boundary, the corresponding boundary layer transitions are of finite time in the stretched time scale. Thus, it has been impossible so far to apply the classical asymptotic boundary layer theory to such problems¹. Moreover, the traditional formulation leads to optimal controllers that are one-sided, that is, they break down when a disturbance throws the system on the prohibited side of the state constraint boundary.

In our second reporting period we tried to remedy these problems by using two successive transformations². The first (Valentine's) transformation converts the state-variable inequality constraint into an equality constraint but at the expense of introducing a singular arc along the reduced solution. The second (Goh's) transformation then converts to a regular optimization problem. Although this procedure appeared to supply a mechanism for the construction of linearized boundary layer corrections (in Goh's variables), the resulting solutions (in Valentine's variables) consisted of impulse like controls and step-function-like fast states in the (linearized) boundary layer. This meant that the resulting (linearized) boundary layers (in Valentine's variables) supplied no additional information over that supplied by the reduced solution. Closer examination of the whole procedure in a general setting revealed that this behavior was rather not surprising and intimately connected with the singular arc. Specifically, any attempt to obtain a linearized boundary layer correction about a singular arc results in a (trivial) zero-time boundary layer which happens to be just a step function.

The difficulties encountered during the second reporting period led us in the first half of the third reporting period (see Appendix A) to give up on Goh's transformation and try to invent an alternative near-optimal procedure for treating such problems. The idea that we came up with was to use a Valentine-like transformation to transform to new variables, such that, in the transformed system no trajectory violated the constraint, and, the reduced solution that rides the state constraint boundary corresponded to a singular arc at infinity. Linearized boundary layer expansions about such a singular arc were, just as expected, trivial step functions. By expanding however about a finite arc (that was nonsingular) we were able to obtain a nonlinear, two-sided feedback control law that tracked the reduced solution asymptotically. This corresponded to allowing -in the boundary layer only- two-sided perturbations of a prescribed size beyond the state constraint boundary. Our procedure was nearly-optimal in the same sense as feedback linearization, that is, because we were tracking a nearly-optimal reduced solution that usually tends to dominate the performance index if the final time is large compared to the characteristic times of the boundary layers. However, our procedure did not imply any optimality in the boundary layer and did not result in any uniformly valid approximation of the exact solution, because as the finite arc approached the singular arc at infinity the boundary layer degenerated again into a step function. For more details on this effort the reader is referred to Appendix A. Although the effort proved unsatisfactory, it made us look into the problem from a new perspective, which eventually, as we explain in the remainder of this report, did lead to some fruitful results.

Our futile attempts during the first two reporting periods and the first half of the third reporting period to combine the classical asymptotic boundary layer theory with the traditional formulation of optimal control problems involving active state-variable inequality constraints led us in the second half of the third reporting period to a complete reconsideration of the main issues. Above all was the realization that the way in which these problems were formulated resulted in finite-time, one-sided feedback control laws that were useless from a practical point of view. We suspected that the reason we were not getting the right answers was that we were not asking the right questions. We gradually realized that what we really had to insist upon and retain was the asymptoticity and the two-sidedness of our feedback control laws, not their optimality, defined according to these traditional standards. After all, these two features are both necessary (but not sufficient) to make a control law attractive from a practical point of view. Note that such a conflict never arises when a problem does not involve any active state-variable inequality constraints. In such a case the classical asymptotic boundary layer theory leads to feedback control laws that *are* both asymptotic and two-sided as desired. Thus, the idea that dominated the second half of the third reporting period was that, if we were to retain the asymptoticity and the two-sidedness of our control laws when solving optimal control problems involving active state-variable inequality constraints, then, we somehow had to build these two features explicitly into the problem. This meant that we would have to reformulate the *boundary layer problem* so that the optimization is carried over all controllers that lead asymptotically to the state constraint boundary (corresponding to the reduced solution) and do not break down when a perturbation throws the system on the prohibited side of the boundary.

To accomplish the above task one will first have to isolate the asymptotic, two-sided controllers from the rest. It is in this specific area that we have made significant progress during our third reporting period³. Although the details are given in our draft paper prepared for the 1993 AIAA Guidance and Control Conference (see Appendix B), we will hint at our ideas here by considering the dynamical system:

$$\frac{dx}{dt} = f(x, y, u) \quad x(t_0) = x_0 \quad (1)$$

$$\frac{dy}{dt} = g(x, y, u) \quad y(t_0) = y_0 \quad (2)$$

where x, f are vectors of the same dimension, and y, g and u are scalars. We would like to describe the set C of all piecewise continuous (in time) control laws $u(x(t), y(t), t)$ that track a given hypersurface in the state space of the above system, given by the scalar equation:

$$S(x, y) = 0 \quad (3)$$

that is, if $u=u(x, y, t)$ is a specific control law belonging to C , then the system (1), (2), driven by $u(x, y, t)$ for $t > t_0$ (and assuming that $S(x_0, y_0)$ is not zero) will eventually reach the hypersurface given in (3) and stay on it thereafter. A control law can drive the system onto the hypersurface (3) either in finite time or asymptotically. Accordingly, the set C is the union of two disjoint sets \mathcal{F} and \mathcal{A} . The set \mathcal{F} contains all control laws that track hypersurface (3) in a finite time. The set \mathcal{A} contains all control laws that track hypersurface (3) asymptotically. We would like to give a complete description of the sets \mathcal{F} and \mathcal{A} .

Let us denote by Z the set of all piecewise differentiable, scalar functions of the real variable α , defined and invertible for all α in $[0, 1]$, and satisfying the boundary conditions:

$$z(0) = 0 ; \quad z(1) = -S(x_0, y_0) \quad (4)$$

We assume that the range of S is contained in the range of z for all z in Z . We consider α as a time variable and make the transformation from y to α :

$$z(\alpha) + S(x, y) = 0 \quad (5)$$

Differentiating (5) with respect to time and using (1) and (2) results in:

$$\frac{dz}{d\alpha} \frac{d\alpha}{dt} + \frac{\partial S}{\partial x} f(x, y, u) + \frac{\partial S}{\partial y} g(x, y, u) = 0 \quad (6)$$

We now let $d\alpha/dt$ play the role of a new control, β , by defining:

$$\frac{d\alpha}{dt} = \beta \quad (7)$$

then, (6) becomes:

$$\frac{dz}{d\alpha} \beta + \frac{\partial S}{\partial x} f(x, y, u) + \frac{\partial S}{\partial y} g(x, y, u) = 0 \quad (8)$$

We will assume for the sake of simplicity that the hypersurface (3) is first order in u , that is, the total time derivative of $S(x,y)$ is explicitly dependent on u . We also assume that (5) is invertible in y and that (8) is invertible in u , resulting in the two equations:

$$y = h(x, z) \quad (9)$$

$$u = k(x, z, \frac{dz}{d\alpha} \beta) \quad (10)$$

for y and u respectively. System (1) and (2) has now been transformed through (5)-(10) to the equivalent system:

$$\frac{dx}{dt} = f(x, h(x, z(\alpha)), k(x, z(\alpha), \frac{dz}{d\alpha} \beta)) \quad x(t_0) = x_0 \quad (11)$$

$$\frac{d\alpha}{dt} = \beta \quad \alpha(t_0) = 1 \quad (12)$$

If we now want to drive our original system (1), (2) onto the hypersurface (3) in a specified finite time t_f , then all we have to do is use the control law for the transformed system

$$\beta = -\frac{1}{t_f - t_0} \quad \text{for} \quad t_0 < t < t_f \quad (13)$$

$$\beta = 0 \quad \text{for} \quad t_f < t \quad (14)$$

that leads to the time variation for α :

$$\alpha = \frac{t_f - t}{t_f - t_0} \quad \text{for} \quad t_0 < t < t_f \quad (15)$$

$$\alpha = 0 \quad \text{for} \quad t_f < t \quad (16)$$

Thus, α is driven from 1 to 0, in $t_0 < t < t_f$ and stays at zero for $t_f < t$. Accordingly, due to the boundary conditions (4) on the function $z(\alpha)$, our original system (1), (2), is driven from its initial state at $t=t_0$ onto the hypersurface (3) in $t_0 < t < t_f$ and stays on the hypersurface for $t_f < t$. The feedback controller $u(x,y,t)$ that will perform this task for system (1), (2) can be found from (10): β is given by (13), (14), z is equal to $-S(x,y)$ from (5), and since $z(\alpha)$ is invertible on $[0,1]$, $dz/d\alpha$ can be expressed as a function of z and therefore of $-S(x,y)$.

Similarly, if we want to drive our original system (1), (2) onto the hypersurface (3) asymptotically, then we can use the control law for the transformed system

$$\beta = -\alpha \quad (17)$$

that leads to the exponential time variation for α :

$$\alpha = e^{-t} \quad (18)$$

Now, α is driven from 1 to 0 exponentially. Again, due to the boundary conditions (4) on the function $z(\alpha)$, our original system (1), (2), is driven asymptotically from its initial state at $t=t_0$ onto the hypersurface (3). The feedback controller $u(x,y,t)$ that will perform this task for system (1), (2) can be found from (10): β is given by (31), $-S(x,y)$ is equal to z from (5), and since $z(\alpha)$ is invertible on $[0,1]$, $dz/d\alpha$ can be expressed as a function of z and therefore of $-S(x,y)$.

We can thus claim that fixing the control β as in (17), or as in (13), (14) establishes a correspondence between functions $z(\alpha)$ in \mathcal{Z} and controllers $u(x,y,t)$ in \mathcal{A} or \mathcal{F} that bring system (1), (2) onto the hypersurface (3) asymptotically or in finite time respectively. As proven in the draft paper³ for example (see Appendix B), once β is fixed as in (17), for every $z(\alpha)$ in \mathcal{Z} there corresponds a controller $u(x,y,t)$ in \mathcal{A} , and, more importantly, for every controller $u(x,y,t)$ in \mathcal{A} , there corresponds a function $z(\alpha)$ in \mathcal{Z} . The correspondence therefore established between \mathcal{A} and \mathcal{Z} through the selection of β as in (17) is *onto*. Similar facts hold for the sets \mathcal{Z} and \mathcal{F} once β is fixed as in (13), (14).

These results suggest that in an optimization problem in which the asymptotic tracking of a hypersurface is desired (like for example in a boundary layer problem that corresponds to an optimal control problem involving an active state-variable inequality constraint) we can forget about the "real" controllers $u(x,y,t)$, fix the control β as in (17), and optimize over all functions $z(\alpha)$ in \mathcal{Z} . This amounts to carrying out the optimization for the original problem only over all (asymptotic) controllers in \mathcal{A} . The resulting off-line optimization problem is guaranteed to have an infimum if the original problem does have a minimum³. An example that we supply in the draft paper³ (see Appendix B) suggests that it may be possible for this off-line optimization problem not to achieve its infimum, that is, the

infimum may correspond to a finite-time controller (that belongs to \mathcal{F} but not \mathcal{A}). The question that arises naturally in such a case is how to find an asymptotic controller that somehow approximates this infimum. Although there are no general answers yet, in our draft paper³ we were able to show that by imposing an additional isoperimetric constraint on the reformulated problem one can, at least for this particular example, select an asymptotic controller that approximates this infimum.

Conclusions and Future Research

The class of all piecewise continuous (in time) controllers that track a given hypersurface in the state space of a dynamical system can be split into two disjoint classes. The first class contains all controllers that track the hypersurface in finite time. The second class contains all controllers that track the hypersurface asymptotically. A transformation technique can be used to give a complete description of both classes. This splitting of the two classes can be used to reformulate optimal control problems involving active state-variable inequality constraints. The optimization in the reformulated problem is carried out over the class of asymptotic controllers only and not over the class of all controllers. If a minimum over all controllers exist, then the reformulated problem is guaranteed to have an infimum. An example suggests that the reformulated problem does not achieve its infimum, that is, the infimum corresponds to a finite-time controller. By imposing an additional isoperimetric constraint on the reformulated problem one can, at least for this particular example, select an asymptotic controller that approximates this infimum.

Our primary goal for the immediate future is to generalize the results obtained for the particular example of the draft paper³ to a general boundary layer system, corresponding to an optimal control problem involving active state-variable inequality constraints. We plan to examine in this general setting whether and when the off-line optimization problem corresponding to such a system achieves its infimum, that is, whether and when can one find an *asymptotic* controller that results in this infimum. For the cases in which this infimum is not achievable with an asymptotic controller, we would like to find out under what additional conditions can one approximate it (with an asymptotic controller). The isoperimetric constraint that we used in the example in our draft paper should provide us with some insight. Furthermore, we plan to extend our theory to the cases in which the hypersurface described by $S(x,y)$ is n -th order in u , that is, one needs to differentiate $S(x,y)$ n times with respect to time to get explicit dependence in u . Finally, we plan to examine whether the correspondence established in our draft paper between functions $z(\alpha)$ in \mathcal{Z} and controllers $u(x,y,t)$ in \mathcal{A} or \mathcal{F} is one-to-one. Since we showed that it is onto, if it turns out to be also one-to-one, this would mean that it is a bijection. Such a result would be important because it would imply a complete equivalence between the set \mathcal{Z} and the sets \mathcal{A} or \mathcal{F} .

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Appendix A

Progress for the First Half of the Third Reporting Period

In this Appendix we supply a more detailed account of the work conducted during the first half of the third reporting period, namely, between 7/5/92 and 10/1/92. Our main objective during this effort was to somehow combine the classical asymptotic boundary layer theory with optimal control problems involving active state-variable inequality constraints. We supply the work as an Appendix because it proved unsatisfactory. However, it made us look into the problem from a new perspective, which eventually, as we explain in the main part of our report, did lead to some fruitful results and a conference paper. The reference numbers in this Appendix refer to the reference list on page 6.

During the first half of the third reporting period our efforts were focused upon developing a method for applying the asymptotic boundary layer theory to optimal control problems involving active state variable inequality constraints. The main theoretical issue is that when the reduced solution for such problems lies on a state constraint boundary, the corresponding boundary layer transitions are of finite time in the stretched time scale. Thus, it has been impossible so far to apply the traditional asymptotic boundary layer theory to such problems. In our second reporting period we tried to remedy this problem by using two successive transformations². The first (Valentine's) transformation converts the state-variable inequality constraint into an equality constraint but at the expense of introducing a singular arc along the reduced solution. The second (Goh's) transformation then converts to a regular optimization problem. Although this procedure appeared to supply a mechanism for the construction of linearized boundary layer corrections (in Goh's variables), the resulting solutions (in Valentine's variables) consisted of impulse like controls and step-function-like fast states in the (linearized) boundary layer. This meant that the resulting (linearized) boundary layers (in Valentine's variables) supplied no additional information over that supplied by the reduced solution. Closer examination of the whole procedure in a general setting reveals that this behavior is rather not surprising and intimately connected with the singular arc. Specifically, any attempt to obtain a linearized boundary layer correction about a singular arc results in a (trivial) zero-time boundary layer which is nothing else but a step function.

The above difficulties encountered during the second reporting period led us to keep Goh's transformation aside and try to invent an alternative near-optimal procedure for treating such problems. The idea that we came up with and that we will try to expand upon in this report is the following:

Suppose that in an optimal control problem we are imposed with a state-variable inequality constraint of the form:

$$S(x, y) \leq 0 \tag{1}$$

where x and y are the slow and fast state variables respectively. Then, we can eliminate the constraint by using the transformation

$$e^\alpha + S(x, y) = 0 \tag{2}$$

to transform from the (x,y) plane to the (x,α) plane. Any trajectory in the (x,α) plane is a trajectory that does not violate (1). Moreover, in the (x,α) plane α plays the role of the new slow state variable. A reduced solution in the (x,y) plane that rides the constraint (1) corresponds to a singular arc at infinity in the (x,α) plane. Linearized boundary layer

expansions about such a singular arc are just (trivial) step functions. However, if in the (x, α) plane we instead track an arc $\alpha = \alpha_0$, where α_0 is a *finite* negative number, then, due to the properties of the transformation (2), even for moderate magnitudes of α_0 , in the (x, y) plane we are still practically tracking the reduced solution $S(x, y) = 0$ (that is, we are nearly optimal) and more importantly, since the arc $\alpha = \alpha_0$ is non-singular, it affords a non-trivial, asymptotic, linearized boundary layer expansion about it. For example, if in the (x, α) plane we track the arc $\alpha = \alpha_0 = -3$, then in the (x, y) plane we are tracking the arc $S(x, y) = -0.05$, which is pretty close to the optimal arc $S(x, y) = 0$.

In order to demonstrate the applicability of the above idea we will now try to apply it to Example 1 of Ref. 1. In this example we are given the singularly perturbed system:

$$\frac{dx}{dt} = y - u^2 \quad (3)$$

$$\varepsilon \frac{dy}{dt} = u \quad (4)$$

and the state-variable inequality constraint

$$S = y - 1 \leq 0 \quad (5)$$

The initial conditions are $x(0) = y(0) = 0$. The final value of x , $x(t_f) = x_f$ is specified and strictly positive, and the objective is to minimize the time required for x to reach this specified final value. Transformation (2) for the above system reads:

$$e^\alpha + y - 1 = 0 \quad (6)$$

Differentiating (6) with respect to time, and using (4) we obtain:

$$e^\alpha \alpha_1 + u = 0 \quad (7)$$

where the new control variable α_1 is defined as:

$$\varepsilon \frac{d\alpha}{dt} = \alpha_1 \quad (8)$$

The resulting equivalent unconstrained system is:

$$\frac{dx}{dt} = 1 - e^\alpha - e^{2\alpha} \alpha_1^2 \quad (9)$$

$$\varepsilon \frac{d\alpha}{dt} = \alpha_1 \quad (10)$$

with initial conditions $x(0) = 0$, $\alpha(0) = 0$, and final condition $x(t_f) = x_f > 0$. The Hamiltonian for system (9), (10) is:

$$H = \lambda_x (1 - e^\alpha - e^{2\alpha} \alpha_1^2) + \lambda_\alpha \alpha_1 + 1 = 0 \quad (11)$$

resulting in the optimality condition:

$$\frac{\partial H}{\partial \alpha_1} = -2\lambda_x e^{2\alpha} \alpha_1 + \lambda_\alpha = 0 \quad (12)$$

and the adjoint equations:

$$\lambda_x = \text{constant} \quad (13)$$

$$\varepsilon \frac{d\lambda_\alpha}{dt} = \lambda_x e^\alpha (1 + 2e^\alpha \alpha_1^2) \quad (14)$$

Using $\varepsilon=0$ we find that there is only one possible reduced solution for the above system, given by:

$$\alpha_1^0 = 0, \lambda_\alpha^0 = 0, \alpha^0 = -\infty, \lambda_x^0 = -1, x^0(t) = t, y^0 = 1, u^0 = 0 \quad (15)$$

Thus, as expected, in the the (x, α) plane the reduced solution corresponds to an arc at infinity. This arc is singular because the coefficient of α_1 vanishes along it in the optimality condition (12).

At this point we are ready to make our major approximation. Instead of using (15) itself, we will approximate the singular arc at infinity by the nonsingular arc:

$$\alpha^0 = \alpha_0 \quad (16)$$

where α_0 is a strictly negative *finite* number the value of which is left to our selection. Note that as α_0 tends to $-\infty$ our reduced solution in the (x, y) plane rides the state constraint. Note also that even for moderate magnitudes of α_0 our reduced solution in the (x, y) plane will practically ride the state constraint. For example if we select $\alpha_0 = -3$ then from (6) y^0 is approximately 0.95, which is very close to 1. The important point here is of course the fact that the arc given by (16) is a non-singular arc, which implies the possibility of an asymptotic boundary layer expansion about it. In summary, if we approximate the singular arc at infinity by the non-singular arc given by (16), we can easily verify that the exact reduced solution given in (15) will have to be approximated as:

$$\alpha_1^0 = 0, \lambda_\alpha^0 = 0, \alpha^0 = \alpha_0, \lambda_x^0 = \frac{1}{e^{\alpha_0} - 1}, x^0(t) = (1 - e^{\alpha_0})t, y^0 = y_0, u^0 = 0 \quad (17)$$

where we define y_0 as:

$$y_0 = 1 - e^{\alpha_0} \quad (18)$$

We are now ready to examine the possibility of an asymptotic boundary layer expansion about the approximate reduced solution given in (17). Stretching the time scale near $t=0$ by:

$$\tau = \frac{t}{\varepsilon} \quad (19)$$

the boundary layer equations for t near 0 can be written as:

$$\frac{d\alpha}{d\tau} = \alpha_1 \quad (20)$$

$$\frac{d\lambda_\alpha}{d\tau} = \left(\frac{1}{e^{\alpha_0} - 1} \right) e^\alpha (1 + 2e^\alpha \alpha_1^2) \quad (21)$$

The Hamiltonian evaluated in the boundary layer is given as:

$$H^{BL} = \left(\frac{1}{e^{\alpha_0} - 1} \right) (1 - e^\alpha - e^{2\alpha} \alpha_1^2) + \lambda_\alpha \alpha_1 + 1 = 0 \quad (22)$$

The boundary layer system can therefore be stated in the equivalent form of an optimization problem:

$$\text{minimize: } J = \int_0^{\tau_f} \left(\frac{1}{e^{\alpha_0} - 1} \right) (1 - e^\alpha - e^{2\alpha} \alpha_1^2) d\tau \quad \tau_f \text{ free} \quad (23)$$

$$\text{subject to: } \frac{d\alpha}{d\tau} = \alpha_1 \quad \alpha(0)=0 \quad (24)$$

Note that our approximate reduced solution given in (17) serves as an approximate equilibrium solution of the boundary layer system (20), (21). In order to linearize the boundary layer system about the reduced solution given in (17) we define:

$$\delta\alpha = \alpha - \alpha^0 = \alpha - \alpha_0 \quad (25)$$

$$\delta\alpha_1 = \alpha_1 - \alpha_1^0 = \alpha_1 \quad (26)$$

Then, the linearized boundary layer system can be stated in the equivalent form of the optimization problem:

$$\text{minimize: } J = \frac{1}{2} \int_0^{\tau_f} \left(H_{\alpha\alpha}^{BL} \delta\alpha^2 + 2H_{\alpha\alpha_1}^{BL} \delta\alpha \delta\alpha_1 + H_{\alpha_1\alpha_1}^{BL} \delta\alpha_1^2 \right) d\tau \quad \tau_f \text{ free} \quad (27)$$

$$\text{subject to: } \frac{d(\delta\alpha)}{d\tau} = \delta\alpha_1 \quad \delta\alpha(0)=-\alpha_0 \quad (28)$$

where the second partials of H^{BL} with respect to α and α_1 , evaluated along the approximate reduced solution (17) are found to be:

$$H_{\alpha\alpha}^{BL} = \frac{\partial^2 H^{BL}}{\partial \alpha^2} = \frac{e^{\alpha_0}}{1 - e^{\alpha_0}} \quad (29)$$

$$H_{\alpha\alpha_1}^{BL} = \frac{\partial^2 H^{BL}}{\partial \alpha \partial \alpha_1} = 0 \quad (30)$$

$$H_{\alpha_1\alpha_1}^{BL} = \frac{\partial^2 H^{BL}}{\partial \alpha_1^2} = \frac{2e^{2\alpha_0}}{1 - e^{\alpha_0}} \quad (31)$$

The adjoint equation associated with (28) is:

$$\frac{d(\delta\lambda_\alpha)}{d\tau} = -H_{\alpha\alpha}^{BL} \delta\alpha - H_{\alpha\alpha_1}^{BL} \delta\alpha_1 \quad (32)$$

where $\delta\lambda_\alpha$ is defined as:

$$\delta\lambda_\alpha = \lambda_\alpha - \lambda_\alpha^0 = \lambda_\alpha \quad (33)$$

The optimal control $\delta\alpha_1$ is determined from the optimality condition:

$$H_{\alpha\alpha}^{\text{BL}} \delta\alpha + H_{\alpha_1\alpha_1}^{\text{BL}} \delta\alpha_1 + \delta\lambda_\alpha = 0 \quad (34)$$

Solving (34) for $\delta\alpha_1$, substituting into (28) and (32), and taking into account (29) through (31) results in the Hamiltonian system:

$$\frac{d(\delta\alpha)}{d\tau} = \left(\frac{e^{\alpha_0} - 1}{2e^{2\alpha_0}} \right) \delta\lambda_\alpha \quad (35)$$

$$\frac{d(\delta\lambda_\alpha)}{d\tau} = \left(\frac{e^{\alpha_0}}{e^{\alpha_0} - 1} \right) \delta\alpha \quad (36)$$

The eigenvalues associated with this system are:

$$s_1 = \frac{e^{-\frac{\alpha_0}{2}}}{\sqrt{2}} \quad s_2 = -\frac{e^{-\frac{\alpha_0}{2}}}{\sqrt{2}} \quad (37)$$

Note that as α_0 tends to minus infinity the eigenvalues in (37) tend to plus or minus infinity, reasserting the fact that at this limit the boundary layer degenerates to a step function.

The solution of system (35), (36) can be found using standard methods. First, in order to cancel the instability arising from s_1 the initial value of $\delta\lambda_\alpha$ will have to be selected as:

$$\delta\lambda_\alpha(0) = \frac{2\alpha_0 e^{\frac{3\alpha_0}{2}}}{\sqrt{2}(e^{\alpha_0} - 1)} \quad (38)$$

Then, the solution of (35), (36) can be shown to be:

$$\delta\alpha(\tau) = -\alpha_0 \exp \left\{ -\frac{\tau e^{-\frac{\alpha_0}{2}}}{\sqrt{2}} \right\} \quad (39)$$

$$\delta\lambda_\alpha(\tau) = \frac{2\alpha_0 e^{\frac{3\alpha_0}{2}}}{\sqrt{2}(e^{\alpha_0} - 1)} \exp \left\{ -\frac{\tau e^{-\frac{\alpha_0}{2}}}{\sqrt{2}} \right\} \quad (40)$$

The optimal control $\delta\alpha_1$ is determined from the optimality condition (34), using (29) through (31), (39), and (40):

$$\delta\alpha_1(\tau) = \frac{\alpha_0 e^{-\frac{\alpha_0}{2}}}{\sqrt{2}} \exp \left\{ -\frac{\tau e^{-\frac{\alpha_0}{2}}}{\sqrt{2}} \right\} \quad (41)$$

Comparing (41) with (39), $\delta\alpha_1$ can also be written in the form of a linear feedback control law:

$$\delta\alpha_1(\tau) = -\frac{e^{-\frac{\alpha_0}{2}}}{\sqrt{2}} \delta\alpha(\tau) \quad (42)$$

Using now (25), (26), and (33), and replacing τ by t/ε , the composite solution in the (x, α) plane, resulting from the linearized asymptotic boundary layer expansion (39) through (42) can be found to be:

$$\alpha(t) = \alpha_0 \left[1 - \exp \left\{ -\frac{t e^{-\frac{\alpha_0}{2}}}{\varepsilon \sqrt{2}} \right\} \right] \quad (43)$$

$$\alpha_1(t) = \frac{\alpha_0 e^{-\frac{\alpha_0}{2}}}{\sqrt{2}} \exp \left\{ -\frac{t e^{-\frac{\alpha_0}{2}}}{\varepsilon \sqrt{2}} \right\} \quad (44)$$

$$\lambda_\alpha(t) = \frac{2\alpha_0 e^{\frac{3\alpha_0}{2}}}{\sqrt{2}(e^{\alpha_0} - 1)} \exp \left\{ -\frac{t e^{-\frac{\alpha_0}{2}}}{\varepsilon \sqrt{2}} \right\} \quad (45)$$

The linear feedback control law found in (42) now becomes:

$$\alpha_1(t) = -\frac{e^{-\frac{\alpha_0}{2}}}{\sqrt{2}} (\alpha(t) - \alpha_0) \quad (46)$$

Note that as the boundary layer time t/ε tends to infinity, α , α_1 and λ_α approach their (approximate) reduced solution values given in (17). Note also that as α_0 tends to $-\infty$, the solutions given in (43) through (45) degenerate into step functions.

In order to transform the above solution to the (x, y) plane we use (6), (7), and (18). The result for $y(t)$ is:

$$y(t) = 1 - (1 - y_0)^{1 - \exp\left\{-\frac{t}{\epsilon\sqrt{2(1-y_0)}}\right\}} \quad (47)$$

The linear feedback control law in (46) is now transformed to the nonlinear feedback control law:

$$u(t) = \frac{1 - y(t)}{\sqrt{2(1 - y_0)}} \ln\left(\frac{1 - y(t)}{1 - y_0}\right) \quad (48)$$

Note that since y_0 is less than 1, the nonlinear feedback control law in (48) is able to handle two-sided perturbations about y_0 , as long as these perturbations remain small and do not exceed 1. To complete the solution in the (x,y) plane we will also have to have the solution for $x(t)$. But this is just the reduced solution found in (17):

$$x(t) = x^0(t) = (1 - e^{\alpha_0})t \quad (49)$$

Our approximate minimum final time is the time at which x assumes its final value x_f , which from (49) is:

$$t_f = \frac{x_f}{(1 - e^{\alpha_0})} \quad (50)$$

Note that as α_0 tends to $-\infty$, t_f tends to its exact value¹ x_f , which comes as a confirmation of the near-optimality of our procedure.

In order to assess further this near-optimality we will now try to find the solution to the nonlinear boundary layer equations ((23), (24)) in the (x,α) plane, and then compare it with the exact solution given in reference 2 as α_0 tends to $-\infty$.

The optimality condition for the nonlinear case is obtained by differentiating the boundary layer Hamiltonian given in (22) with respect to α_1 :

$$\frac{\partial H^{BL}}{\partial \alpha_1} = -\frac{2e^{2\alpha}\alpha_1}{e^{\alpha_0} - 1} + \lambda_\alpha = 0 \quad (51)$$

Substituting for λ_α from (51) into (22) we obtain:

$$\alpha_1^2 = \frac{e^\alpha - e^{\alpha_0}}{e^{2\alpha}} \quad (52)$$

Substituting for α_1 from (52) into (24) results in a nonlinear differential equation for α :

$$\left(\frac{d\alpha}{d\tau}\right)^2 = \frac{e^\alpha - e^{\alpha_0}}{e^{2\alpha}} \quad \alpha(0)=0 \quad (53)$$

Using (6), and (18) we can show that (53) is equivalent to:

$$\left(\frac{dy}{d\tau}\right)^2 = y_0 - y \quad y(0)=0 \quad (54)$$

Note that a solution does not exist, unless y is less than y_0 . Assuming that this is the case, and taking into account that in order for y to reach 1 $dy/d\tau$ near $t=0$ will have to be positive, we can show that the unique solution of (54) is:

$$y(\tau) = -\frac{\tau^2}{4} + \tau\sqrt{y_0} \quad (55)$$

From reference 1 we can easily verify that the exact nonlinear boundary layer solution is:

$$y(\tau) = -\frac{\tau^2}{4} + \tau \quad (56)$$

Comparing (55) with (56), and taking into account (18), we see that as α_0 tends to $-\infty$, y_0 tends to 1, and our solution (55) tends to the exact solution (56) as expected.

In order to derive a nonlinear feedback control law we now use (4), (19), and (54) which results in:

$$u(\tau) = (y_0 - y(\tau))^{1/2} \quad (57)$$

From reference 1 the nonlinear feedback control law that corresponds to the exact nonlinear boundary layer solution is:

$$u(\tau) = (1 - y(\tau))^{1/2} \quad (58)$$

A comparison of (57) with (58) shows that as y_0 tends to 1 the two control laws become identical as expected.

Comments

From a practical point of view the most important product of a procedure such as the above would be the two feedback control laws found in (48), and (57). Although both laws are nonlinear, there are important differences between the two. Specifically, the law supplied in (48) was found by using the *linearized* boundary layer system in the (x,α) plane. The law supplied by (57) on the other hand was found by using the full *nonlinear* boundary layer system in the (x,α) plane. In order to keep this distinction clear we will from now on use the notations LBL and NBL to refer to the linearized and nonlinear boundary layer systems respectively. Thus, (48) will be referred to as the LBL law and (57) will be referred to as the NBL law. The striking difference between the two laws is that the LBL law is asymptotic, while the NBL law is finite time. This can be concluded by examining (47) and (55), representing the solutions for y that the two laws give rise to respectively. In (47) y reaches y_0 as the time $\tau=t/\epsilon$ goes to infinity. In (55) y reaches y_0 at the time $\tau_f=2(y_0)^{1/2}$ which in turn tends to the exact value¹ $\tau_f=2$ as y_0 tends to 1. The asymptoticity of the LBL law makes it attractive from a practical viewpoint. Moreover, and this is the single most important point of the present work, the LBL law can handle two-sided perturbations about y_0 , while the NBL law can't. However, as y_0 tends to 1 the NBL

law tends to the exact feedback control law given in reference 1, while the LBL law breaks down. The reason for this breakdown is that at this limit the linearized boundary layer expansion in the (x,α) plane is about the singular arc at infinity, supplying no new information over that supplied by the reduced solution. This also suggests that the composite solution obtained using the LBL equations in the (x,α) plane cannot be a uniformly valid approximation to the exact solution as y_0 tends to 1.

The ability of the LBL law to handle two-sided perturbations about y_0 is the major justification for our introduction of the above procedure. Our procedure is nearly-optimal in the same sense that procedures employing feedback linearization are nearly-optimal, that is, because we track a nearly-optimal reduced solution that usually dominates the performance index if the final time is large compared to the characteristic times of the boundary layers. However, our procedure is not as ad hoc as feedback linearization, and, more importantly, the LBL law is *nonlinear* in the state perturbations, suggesting that there might be a hope of avoiding control saturation problems that are sometimes encountered when using feedback linearization. Of course, the major setback in our procedure is the fact that it cannot supply us with a composite solution that uniformly approximates the exact solution. Thus, the question of combining the classical asymptotic boundary layer theory with optimal control problems involving active state-variable inequality constraints still remains unresolved.

Appendix B

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Near-Optimal, Asymptotic Tracking in Control Problems Involving State-Variable Inequality Constraints

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Abstract

A transformation technique is used that splits the class of all piecewise continuous (in time) controllers that track a given hypersurface in the state space of a dynamical system into two disjoint classes. The first class contains all controllers that track the hypersurface in finite time. The second class contains all controllers that track the hypersurface asymptotically. Four theorems are presented that describe the two classes completely. The results are applied to the study of optimal control problems involving active state variable inequality constraints. The controllers obtained from the traditional formulation of such problems suffer from serious practical flaws. They typically tend to track the hypersurface representing a state constraint boundary in a finite time, which makes the traditional asymptotic boundary layer theory non-applicable, and they break down whenever a

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disturbance causes the system to violate the active state constraint. As an alternative, this paper proposes a reformulation of such problems in which the optimization is carried out with respect to asymptotic controllers only and not with respect to all controllers. If a minimum over all controllers exists, then this reformulated problem is guaranteed to have an infimum. An example suggests that the reformulated problem does not achieve its infimum, that is, the infimum corresponds to a finite-time controller. By imposing an additional isoperimetric constraint on the reformulated problem it is shown that one can, at least for this particular example, select an asymptotic controller that approximates this infimum.

I. Introduction

State-variable inequality constraints are commonly encountered in the study of dynamic systems. The study of rigid body aircraft dynamics and control is certainly no exception. For instance, a maximum allowable value of dynamic pressure is usually prescribed for aircraft with supersonic capability. This limit is required to ensure that the vehicle's structural integrity is maintained. Given a typical state-space description of the vehicle dynamics, this limit constitutes an inequality constraint on the vehicle state. Such dynamic pressure bounds are commonly encountered during fuel-optimal climb for supersonic transports¹, for rocket powered launch vehicles such as the U.S. space shuttle², and for single-stage-to-orbit air-breathing launch vehicles³.

State-variable inequality constraints have been studied extensively by researchers in the field of optimal control. First-order necessary conditions for optimality when general functions of the state are constrained have been obtained⁴⁻⁶. However, the direct construction of solutions via this set of conditions proves difficult. Moreover, the controllers derived from such traditional formulations of the problem suffer from serious practical flaws. They typically tend to track the hypersurface representing a state constraint boundary in a finite time, which makes the traditional asymptotic boundary layer theory

non-applicable (see below), and they break down whenever a disturbance causes the system to violate an active state constraint. Accordingly, most practitioners seeking an open loop control solution rely on direct approaches to optimization that employ penalty functions for satisfaction of state-variable inequality constraints.⁷ As a rule however, algorithms employing such methods are computationally intense and slow to converge. Consequently, they are not well suited for real-time implementation.

From a Singular Perturbations point of view^{8,9}, in the absence of a state-variable inequality constraint (i.e. when no constraint is active), the initial boundary layer solution for the class of systems being considered is an infinite time process. A solution is sought which asymptotically approaches the reduced solution. However, when a state constraint is active in the reduced solution, the boundary layer problem can be of finite time in the stretched time variable^{10,11}. Thus, traditional techniques concerning the asymptotic stability of the boundary layer system are not applicable, and cannot be used to construct an approximate boundary layer solution. The presence of an active state-variable inequality constraint also introduces the possibility of discontinuous costate variables at the juncture between constrained and unconstrained arcs. A Valentine transformation can be used to convert the constrained problem to an equivalent unconstrained problem of increased dimension¹². Smoothness is regained in the process, but at the expense of introducing a singular arc along the state constraint boundary^{12,13} and to little or no advantage when seeking a solution for real-time implementation.

As an alternative, this paper proposes a complete reformulation of optimal control problems involving active state variable inequality constraints. Since in practice it is always the asymptotic controllers that have the most desirable properties, maybe the optimization of such problems should be carried out not over the class of all controllers, but only over the class of asymptotic controllers that track a given active state constraint boundary. It is shown in Section II that a transformation technique can be used to isolate and describe completely this class of asymptotic controllers. If a minimum over the class of all

controllers exists, then the reformulated problem is guaranteed to have an infimum. Our results in Section III suggest however that this infimum for the reformulated problem corresponds to a finite-time controller and is not achieved over the class of all asymptotic controllers. The situation is somewhat reminiscent of H-infinity control theory for linear systems in which we seek a proper, stabilizing controller to minimize the H-infinity norm of a closed-loop transfer function. The minimum of this norm over all proper, stabilizing controllers does not exist. Its infimum however does exist and corresponds to an improper controller. Thus, just as in H-infinity theory, the question arises naturally in our case as to how we can find an asymptotic controller that somehow approximates this infimum. Although there are no general answers yet, a procedure is presented in Section III that does supply us with an answer at least for a simple example.

II. Construction of arbitrary nonlinear feedback control laws for a dynamical system that tracks a given hypersurface

Consider the dynamical system:

$$\frac{dx}{dt} = f(x, y, u) \quad x(t_0) = x_0 \quad (1)$$

$$\frac{dy}{dt} = g(x, y, u) \quad y(t_0) = y_0 \quad (2)$$

where x , f are vectors of the same dimension, and y , g and u are scalars. It will be of interest to describe the set \mathcal{C} of all piecewise continuous (in time) control laws $u(x(t), y(t), t)$ that track a given hypersurface in the state space of the above system, given by the scalar equation:

$$S(x, y) = 0 \quad (3)$$

that is, if $u=u(x,y,t)$ is a specific control law belonging to \mathcal{C} , then the system (1), (2), driven by $u(x,y,t)$ for $t > t_0$ (and assuming that $S(x_0, y_0)$ is not zero) will eventually reach

the hypersurface given in (3) and stay on it thereafter. A control law can drive the system onto the hypersurface (3) either in finite time or asymptotically. Accordingly, the set C is the union of two disjoint sets \mathcal{F} and \mathcal{A} . The set \mathcal{F} contains all control laws that track hypersurface (3) in finite time. The set \mathcal{A} contains all control laws that track hypersurface (3) asymptotically. Our purpose in this section is to give a complete description of the sets \mathcal{F} and \mathcal{A} .

We denote by Z the set of all piecewise differentiable, scalar functions of the real variable α , defined and invertible for all α in $[0,1]$, and satisfying the boundary conditions:

$$z(0) = 0 ; \quad z(1) = -S(x_0, y_0) \quad (4)$$

We assume that the range of S is contained in the range of z for all z in Z . We consider α as a time variable and make the transformation from y to α :

$$z(\alpha) + S(x, y) = 0 \quad (5)$$

Differentiating (5) with respect to time and using (1) and (2) results in:

$$\frac{dz}{d\alpha} \frac{d\alpha}{dt} + \frac{\partial S}{\partial x} f(x, y, u) + \frac{\partial S}{\partial y} g(x, y, u) = 0 \quad (6)$$

We now let $d\alpha/dt$ play the role of a new control, β , by defining:

$$\frac{d\alpha}{dt} = \beta \quad (7)$$

then, (6) becomes:

$$\frac{dz}{d\alpha} \beta + \frac{\partial S}{\partial x} f(x, y, u) + \frac{\partial S}{\partial y} g(x, y, u) = 0 \quad (8)$$

We now assume that the hypersurface (3) is first order in u , that is, the total time derivative of $S(x,y)$ is explicitly dependent on u . We also assume that (5) is invertible in y and that (8) is invertible in u , resulting in the two equations:

$$y = h(x, z) \quad (9)$$

$$u = k(x, z, \frac{dz}{d\alpha} \beta) \quad (10)$$

for y and u respectively. System (1) and (2) has now been transformed through (5)-(10) to the equivalent system:

$$\frac{dx}{dt} = f(x, h(x, z(\alpha)), k(x, z(\alpha), \frac{dz}{d\alpha} \beta)) \quad x(t_0) = x_0 \quad (11)$$

$$\frac{d\alpha}{dt} = \beta \quad \alpha(t_0) = 1 \quad (12)$$

Finite time tracking

Theorem 1 : For any given finite time t_f and any function $z(\alpha)$ in Z , there exists a control law $u=u(x,y,t)$ in \mathcal{F} that drives system (1), (2) from its initial state at $t=t_0$ onto the hypersurface (3) at $t=t_f$ and keeps it on the hypersurface thereafter.

Proof : Let t_f be a finite time and $z(\alpha)$ be a function in Z . We can use $z(\alpha)$ to obtain the equivalent transformed system (11), (12). Then, we can use as our control β the function:

$$\beta = -\frac{1}{t_f - t_0} \quad \text{for} \quad t_0 < t < t_f \quad (13)$$

$$\beta = 0 \quad \text{for} \quad t_f < t \quad (14)$$

leading to the time variation for α :

$$\alpha = \frac{t_f - t}{t_f - t_0} \quad \text{for} \quad t_0 < t < t_f \quad (15)$$

$$\alpha = 0 \quad \text{for} \quad t_f < t \quad (16)$$

Thus, α is driven from 1 to 0, in $t_0 < t < t_f$ and stays at zero for $t_f < t$. Accordingly, due to the boundary conditions (4) on the function $z(\alpha)$, our original system (1), (2), is driven from its initial state at $t=t_0$ onto the hypersurface (3) in $t_0 < t < t_f$ and stays on the hypersurface for $t_f < t$. The feedback controller $u(x,y,t)$ that will perform this task for system (1), (2) can be found from (10): β is given by (13), (14), z is equal to $-S(x,y)$ from (5) and since $z(\alpha)$ is invertible on $[0,1]$, $dz/d\alpha$ can be expressed as a function of z and therefore of $-S(x,y)$. The procedure is better illustrated through an example.

Example 1

Consider the system:

$$\frac{dx}{dt} = y - u^2 \quad x(t_0) = x_0 \quad (17)$$

$$\frac{dy}{dt} = u \quad y(t_0) = y_0 \quad (18)$$

and assume that we want to track the line

$$y - 1 = 0 \quad (19)$$

in a specified final time t_f . As our function $z(\alpha)$ we select:

$$z(\alpha) = \left(\frac{1 - y_0}{\ln 2} \right) \ln(1 + \alpha) \quad (20)$$

note that $z(0)=0$, and $z(1)=1-y_0$ as required by (4). From (20) we are led to the transformation:

$$\left(\frac{1 - y_0}{\ln 2} \right) \ln(1 + \alpha) + y - 1 = 0 \quad (21)$$

Differentiating (21) with respect to time we obtain:

$$\left(\frac{1 - y_0}{\ln 2} \right) \left(\frac{\beta}{1 + \alpha} \right) + u = 0 \quad (22)$$

resulting in the transformed system:

$$\frac{dx}{dt} = 1 - \left(\frac{1 - y_0}{\ln 2} \right) \ln(1 + \alpha) + \left[\left(\frac{1 - y_0}{\ln 2} \right) \left(\frac{\beta}{1 + \alpha} \right) \right]^2 \quad x(t_0) = x_0 \quad (23)$$

$$\frac{d\alpha}{dt} = \beta \quad \alpha(t_0) = 1 \quad (24)$$

If we now use the control β given in (13), then, from (22) we obtain:

$$u = \frac{1 - y_0}{\ln 2 (t_f - t_0)} \exp \left\{ -\ln 2 \left(\frac{1 - y}{1 - y_0} \right) \right\} \quad (25)$$

This open loop control can be shown to drive system (17), (18) from its original state at $t=t_0$ onto the line $y=1$ at $t=t_f$. To obtain a closed loop (feedback) controller we simply replace y_0 by y and t_0 by t in (25), to obtain:

$$u = \frac{1 - y}{\ln 4 (t_f - t)} \quad (26)$$

which again drives the system toward $y=1$ at $t=t_f$.

Theorem 2 : Let $u=u(x,y,t)$ be any control law in \mathcal{F} that drives system (1), (2) from its initial state at $t=t_0$ onto the hypersurface (3) at $t=t_f$ and keeps it on the hypersurface thereafter. Then, there exists a function $z(\alpha)$ in \mathcal{Z} , such that, when system (1), (2) is transformed using $z(\alpha)$ to the system given by (11), (12), the control β for the transformed system corresponding to $u(x,y,t)$ is given by (13), (14).

Proof : The function $z(\alpha)$ for $0 < \alpha < 1$ is found by solving the system of differential equations:

$$\frac{dx}{d\alpha} = - (t_f - t_0) f(x, y, u(x, y, t)) \quad (27)$$

$$\frac{dz}{d\alpha} = - (t_f - t_0) \left[\frac{\partial S}{\partial x} f(x, y, u(x, y, t)) + \frac{\partial S}{\partial y} g(x, y, u(x, y, t)) \right] \quad (28)$$

subject to the boundary conditions:

$$x(\alpha = 1) = x_0 \quad ; \quad z(\alpha = 1) = - S(x_0, y_0) \quad (29)$$

where y in (27), (28) is a function of x and z through (9) and t in $u(x,y,t)$ is a function of α through (15), that is,

$$t = t_f - \alpha(t_f - t_0) \quad (30)$$

Note that the function $z(\alpha)$ found from the solution of (27) through (29) satisfies (5) for all $t > t_0$ (α and t are related by (15)). Thus, it also satisfies the boundary condition $z(0)=0$, since at $t = t_f$, α and $S(x,y)$ are both equal to zero.

The preceding two theorems tell us that once the control β for the transformed system (11), (12) is fixed as in (13), (14), there is a complete correspondence between functions in

Z and piecewise continuous control laws $u(x,y,t)$ in \mathcal{F} that track hypersurface (3) in a finite time t_f . That is, for every element of Z there exists an element of \mathcal{F} and, more importantly, for every element of \mathcal{F} there exists an element of Z . The correspondence therefore established between \mathcal{F} and Z through the selection of β as in (13), (14) is *onto*.

Asymptotic tracking

Theorem 3 : For any function $z(\alpha)$ in Z , there exists a control law $u=u(x,y,t)$ in \mathcal{A} that drives system (1), (2) from its initial state at $t=t_0$ onto the hypersurface (3) asymptotically.

Proof : Let $z(\alpha)$ be a function in Z . We use $z(\alpha)$ to obtain the equivalent transformed system (11), (12). We use as our control β the function:

$$\beta = -\alpha \quad (31)$$

leading to the exponential time variation for α :

$$\alpha = e^{-t} \quad (32)$$

Thus, α is driven exponentially from 1 to 0. Accordingly, due to the boundary conditions (4) on the function $z(\alpha)$, our original system (1), (2), is driven asymptotically from its initial state at $t=t_0$ onto the hypersurface (3). The feedback controller $u(x,y,t)$ that will perform this task for system (1), (2) can be found from (10): β is given by (31), z is equal to $-S(x,y)$ from (5) and since $z(\alpha)$ is invertible on $[0,1]$, $dz/d\alpha$ can be expressed as a function of z and therefore of $-S(x,y)$. Again, we illustrate through an example.

Example 2

Consider again the system:

$$\frac{dx}{dt} = y - u^2 \quad x(t_0) = x_0 \quad (33)$$

$$\frac{dy}{dt} = u \quad y(t_0) = y_0 \quad (34)$$

and assume that we want to track asymptotically the line

$$y - 1 = 0 \quad (35)$$

As our function $z(\alpha)$ we again select:

$$z(\alpha) = \left(\frac{1 - y_0}{\ln 2} \right) \ln(1 + \alpha) \quad (36)$$

satisfying $z(0)=0$, and $z(1)=1-y_0$ as required by (4). From (36) we are led to the transformation:

$$\left(\frac{1 - y_0}{\ln 2} \right) \ln(1 + \alpha) + y - 1 = 0 \quad (37)$$

Differentiating (37) with respect to time we obtain:

$$\left(\frac{1 - y_0}{\ln 2} \right) \left(\frac{\beta}{1 + \alpha} \right) + u = 0 \quad (38)$$

resulting in the transformed system:

$$\frac{dx}{dt} = 1 - \left(\frac{1 - y_0}{\ln 2} \right) \ln(1 + \alpha) + \left[\left(\frac{1 - y_0}{\ln 2} \right) \left(\frac{\beta}{1 + \alpha} \right) \right]^2 \quad x(t_0) = x_0 \quad (39)$$

$$\frac{d\alpha}{dt} = \beta \quad \alpha(t_0) = 1 \quad (40)$$

If we now use the control β given in (31), then, from (38) we obtain:

$$u = \frac{e^{k(1-y)} - 1}{k e^{k(1-y)}} \quad (41)$$

where k is defined as:

$$k = \frac{\ln 2}{1 - y_0} \quad (42)$$

This open loop control can be shown to drive system (33), (34) from its original state at $t=t_0$ asymptotically onto the line $y=1$. To obtain a closed loop (feedback) controller we simply replace y_0 by y in (41), to obtain:

$$u = \frac{1-y}{\ln 4} \quad (43)$$

which again drives the system (33), (34) asymptotically toward $y=1$. Note that as expected, in the asymptotic case u does not depend explicitly on t .

Theorem 4 : Let $u=u(x,y,t)$ be any control law that drives system (1), (2) from its initial state at $t=t_0$ onto the hypersurface (3) asymptotically. Then, there exists a function $z(\alpha)$ in Z , such that, when system (1), (2) is transformed using $z(\alpha)$ to the system given by (11), (12), the control β for the transformed system corresponding to $u(x,y,t)$ is given by (31).

Proof : The function $z(\alpha)$ for $0 < \alpha < 1$ is found by solving the system of differential equations:

$$\alpha \frac{dx}{d\alpha} = -f(x, y, u(x, y, t)) \quad (44)$$

$$\alpha \frac{dz}{d\alpha} = \frac{\partial S}{\partial x} f(x, y, u(x, y, t)) + \frac{\partial S}{\partial y} g(x, y, u(x, y, t)) \quad (45)$$

subject to the boundary conditions:

$$x(\alpha = 1) = x_0 \quad ; \quad z(\alpha = 1) = -S(x_0, y_0) \quad (46)$$

where y in (44), (45) is a function of x and z through (9) and t in $u(x,y,t)$ is a function of α through (32), that is,

$$t = \ln \left(\frac{1}{\alpha} \right) \quad (47)$$

Note that the function $z(\alpha)$ found from the solution of (44) through (46) satisfies (5) for all $t > t_0$ (α and t are related by (32)). Thus, it also satisfies the boundary condition $z(0)=0$, since as t approaches infinity α and $S(x,y)$ both tend to zero.

Theorems 3, and 4 tell us that once the control β for the transformed system (11), (12) is fixed as in (31), there is a complete correspondence between functions in Z and piecewise continuous control laws $u(x,y,t)$ in \mathcal{A} that track hypersurface (3) asymptotically.

That is, for every element of Z there exists an element of \mathcal{A} and, more importantly, for every element of \mathcal{A} there exists an element of Z . The correspondence therefore established between \mathcal{A} and Z through the selection of β as in (31) is *onto*.

III. Significance for optimal control problems involving active state-variable inequality constraints

The ideas presented in the last section can be appropriately utilized in the study of optimal control problems involving active state-variable inequality constraints. A common feature of such problems is that when a portion of the optimal trajectory rides the hypersurface representing a state constraint boundary, the optimal transition to this hypersurface from an initial point that does not lie on it occurs in finite time. Consequently, the corresponding optimal feedback controllers for such problems, that can be obtained by well-known analytical or numerical methods, suffer from two flaws that tend to eliminate almost completely their practical usefulness. First, such feedback controllers are finite-time, meaning that, the traditional asymptotic boundary layer theory is not applicable. Second, if a disturbance throws the system instantaneously toward the prohibited side of the hypersurface, the feedback scheme breaks down and there is no "optimal" way of returning to the hypersurface¹⁰.

Asymptotic controllers on the other hand, capable of tracking from both sides a hypersurface representing a state constraint boundary, presumably won't suffer from either of the above two flaws. This observation, and the ideas presented in the last section suggest that once we know that a portion of the optimal trajectory for a problem rides such a hypersurface, we can change our point of view and try to optimize the system over all asymptotic controllers capable of tracking that hypersurface from both sides.

Consider therefore that we are to minimize the performance index:

$$J = \Phi[x(t_f), y(t_f)] + \int_{t_0}^{t_f} L(x, y, u) dt \quad (48)$$

subject to the dynamical equations,

$$\frac{dx}{dt} = f(x, y, u) \quad (49)$$

$$\frac{dy}{dt} = g(x, y, u) \quad (50)$$

the boundary conditions,

$$x(t_0) = x_0 ; \quad y(t_0) = y_0 ; \quad t_0 \text{ fixed} ; \quad t_f \text{ free} \quad (51)$$

and the state variable inequality constraint:

$$S(x, y) \leq 0 \quad (52)$$

As in the last section, x, f are vectors of the same dimension, and L, S, y, g and u are scalars. We will assume that

$$S(x_0, y_0) < 0 \quad (53)$$

and that the optimal trajectory reaches the hypersurface

$$S(x, y) = 0 \quad (54)$$

at a finite time $t=t_1$ and stays on it for $t > t_1$. Thus, in order to avoid the problems with the finite-time controllers mentioned above we would now like to optimize J over the class of all asymptotic controllers, which we have already denoted by \mathcal{A} , capable of tracking hypersurface (54) from both sides. In order to formulate this problem we first use an arbitrary function $z(\alpha)$ from the set of functions Z defined in the last section to transform our problem as follows:

Our transformation reads:

$$z(\alpha) + S(x, y) = 0 \quad (55)$$

with the function $z(\alpha)$ subject to the boundary conditions:

$$z(0) = 0 ; \quad z(1) = -S(x_0, y_0) \quad (56)$$

This leads as in the last section (see equations (5) through (12)) to:

$$\frac{dz}{d\alpha} \beta + \frac{\partial S}{\partial x} f(x, y, u) + \frac{\partial S}{\partial y} g(x, y, u) = 0 \quad (57)$$

and to the transformed system:

$$y = h(x, z) \quad (58)$$

$$u = k(x, z, \frac{dz}{d\alpha} \beta) \quad (59)$$

$$\frac{dx}{dt} = f(x, h(x, z(\alpha)), k(x, z(\alpha), \frac{dz}{d\alpha} \beta)) \quad x(t_0) = x_0 \quad (60)$$

$$\frac{d\alpha}{dt} = \beta \quad \alpha(t_0) = 1 \quad (61)$$

The performance index to be minimized assumes the form:

$$J = \Phi[x(t_f), h(x(t_f), z(\alpha(t_f)))] + \int_{t_0}^{t_f} L(x, h(x, z), k(x, z, \frac{dz}{d\alpha} \beta)) dt \quad (62)$$

As we saw in the last section (Theorems 3 and 4), once the control β for the transformed system (60), (61) is fixed at $\beta = -\alpha$, there is a complete correspondence between functions $z(\alpha)$ in Z and piecewise continuous control laws $u(x, y, t)$ in \mathcal{A} that track hypersurface (3) asymptotically. That is, for every element of Z there exists an element of \mathcal{A} and, more importantly, for every element of \mathcal{A} there exists an element of Z . To find the "best" asymptotic controller therefore it is natural to fix the control β at

$$\beta = -\alpha \quad (63)$$

and then try to determine the "best" function $z(\alpha)$ in Z that minimizes J . This leads directly to the off-line optimization problem: Minimize

$$J_1 = \Phi[x(0), h(x(0), 0)] + \int_0^1 \left(\frac{1}{\alpha} \right) L(x, h(x, z), k(x, z, -\alpha v)) d\alpha \quad (64)$$

subject to:

$$\frac{dx}{d\alpha} = - \left(\frac{1}{\alpha} \right) f(x, h(x, z), k(x, z, -\alpha v)) \quad x(1) = x_0 \quad (65)$$

$$\frac{dz}{d\alpha} = v \quad z(0) = 0 ; \quad z(1) = -S(x_0, y_0) \quad (66)$$

We can see from (64) that for every function $z(\alpha)$ in Z there corresponds a number $J_1(z)$.

Thus, we can define the set of real numbers:

$$J = \{ J_1(z) : z \text{ is in } Z \} \quad (67)$$

and state our off-line optimization problem (64) through (66) by asking for the minimum of J . Although we do not know at the present if and when such a minimum exists, we can state the following theorem that provides us with an important partial answer:

Theorem 5 : If the original problem (48)-(52) involving an active state variable inequality constraint has a minimum, then the off-line optimization problem (64)-(66) has an infimum.

Proof : Assume that problem (48)-(52) has a minimum, which we denote by J_{\min} . This immediately implies that the set J is bounded below by J_{\min} , that is, $J_{\min} \leq J_1(z)$ for all z in Z . Since any set of real numbers that is bounded below has an infimum, J has an infimum.

Example 3

To illustrate the above idea we will apply it to the following problem: Minimize

$$J = \int_0^{\infty} (1 - y + u^2) dt \quad (68)$$

subject to:

$$\frac{dy}{dt} = u ; \quad y(0) = 0 \quad (69)$$

$$y - 1 \leq 0 \quad (70)$$

The solution to this problem for $0 < t < 2$ can be shown to be¹⁰:

$$y = -\frac{t^2}{4} + t ; \quad u = -\frac{t}{2} + 1 \quad (71)$$

leading to the finite-time controller:

$$u = (1 - y)^{1/2} \quad (72)$$

Although this controller is optimal, it is clearly useless from a practical point of view since it breaks down when y exceeds 1. It cannot be used to track the line $y=1$ in the presence of two-sided perturbations about $y=1$. At $t = 2$ y reaches the value $y = 1$. For $t > 2$, y and u stay constant at 1 and 0 respectively and there is no more contribution to the performance index J . In order to optimize J over all asymptotic controllers that track $y = 1$, we now make the transformation:

$$z(\alpha) + y - 1 = 0 \quad (73)$$

$$z(0) = 0 ; \quad z(1) = 1 \quad (74)$$

which leads to:

$$\frac{dz}{d\alpha} \beta + u = 0 \quad (75)$$

and to the transformed optimization problem: Minimize

$$J = \int_0^{\infty} \left(z(\alpha) + \left(\frac{dz}{d\alpha} \right)^2 \beta^2 \right) dt \quad (76)$$

subject to:

$$\frac{d\alpha}{dt} = \beta \quad \alpha(0) = 1 \quad (77)$$

Using now the control:

$$\beta = -\alpha \quad (78)$$

leads to the off-line optimization problem for the function $z(\alpha)$: Minimize

$$J_1 = \int_0^1 \left(\frac{z}{\alpha} + \alpha v^2 \right) d\alpha \quad (79)$$

subject to:

$$\frac{dz}{d\alpha} = v \quad z(0) = 0 \quad ; \quad z(1) = 1 \quad (80)$$

The Hamiltonian associated with this problem is:

$$H = \frac{z}{\alpha} + \alpha v^2 + \lambda v \quad (81)$$

which results in the optimality condition for v:

$$v = -\frac{\lambda}{2\alpha} \quad (82)$$

and in the adjoint equation for λ :

$$\frac{d\lambda}{d\alpha} = -\frac{1}{\alpha} \quad (83)$$

The solution for λ is:

$$\lambda = \ln \left| \frac{A}{\alpha} \right| \quad (84)$$

where A is an integration constant. Combining (82) and (84) with (80) we obtain the differential equation for z:

$$\frac{dz}{d\alpha} = -\left(\frac{1}{2\alpha} \right) \ln \left| \frac{A}{\alpha} \right| \quad (85)$$

which has the general solution:

$$z = \frac{1}{4} \left(\ln \left| \frac{A}{\alpha} \right| \right)^2 + B \quad (86)$$

B being a second integration constant. A quick inspection now reveals that there are no values of A, B, for which the function z(a) in (86) can satisfy both of the boundary conditions $z(0)=0$, $z(1)=1$. The off-line optimization problem therefore posed in (79), (80) has no minimum. As guaranteed however by Theorem 5, it does have an infimum, since it

is bounded below by J_{\min} , where, J_{\min} is the minimum value of the performance index in (68), that is,

$$J_{\min} = 2 \int_0^2 \left(1 + \frac{t^2}{4} - t\right) dt = \frac{4}{3} \quad (87)$$

In order to see what kind of controller u the function $z(\alpha)$ found in (86) implies, we use (75), (78), and (85) to find:

$$u = -\frac{1}{2}(t + \ln|A|) \quad (88)$$

Comparing with (71), we see that (88) implies the optimal, finite-time controller found before. Indeed, (88) can be shown to lead to (71) if the boundary conditions $y(0)=0$ and $y(2)=1$ are imposed and the equation $dy/dt=u$ is integrated. There is no contradiction however with either Theorem 3 or Theorem 4, since there is no function z in Z corresponding to the finite-time controller u given in (88).

The above result implies that the minimum value of J , $J_{\min}=4/3$, found in (87) is not only a lower bound of J_1 but the actual infimum itself. This situation is reminiscent of H-infinity control theory in which we seek a proper, stabilizing controller to minimize the H-infinity norm of a closed-loop transfer function. The minimum of this norm over all proper, stabilizing controllers does not exist. Its infimum however does exist and corresponds to an improper controller. Thus, just as in H-infinity, the question arises naturally in our case as to how we can find an asymptotic controller that somehow approximates this infimum. Although there are no general answers yet, we will present a procedure that does supply us with an answer at least for the above example:

First we can show using (85) that for the off-line optimization problem (79), (80), the integral

$$\int_0^1 v^2 d\alpha = \int_0^1 \left(\frac{1}{2\alpha}\right)^2 \ln^2 \left|\frac{A}{\alpha}\right| d\alpha \quad (89)$$

representing the "total energy" stored in the signal $v(\alpha)$ diverges. This suggests that if we impose the isoperimetric constraint

$$\int_0^1 v^2 d\alpha = k \quad (90)$$

on the off-line optimization problem (79), (80), where k is a given, finite, strictly positive number, we may have a hope of finding a function $z(\alpha)$ in Z , that is, one that does satisfy the boundary conditions $z(0)=0$ and $z(1)=1$. The Hamiltonian associated with this new problem (79), (80), and (90) will read:

$$H = \frac{z}{\alpha} + (\mu + \alpha) v^2 + \lambda v \quad (91)$$

where μ is a constant Lagrange multiplier. For each value of k there corresponds a specific value of μ and vice versa. The corresponding optimality condition for v now becomes:

$$v = - \frac{\lambda}{2(\mu + \alpha)} \quad (92)$$

The adjoint equation for λ remains unchanged as in (83) and leads to the same solution for λ given in (84), while the differential equation for z now reads:

$$\frac{dz}{d\alpha} = - \frac{1}{2(\mu + \alpha)} \ln \left| \frac{A}{\alpha} \right| \quad (93)$$

(93) leads to the solution:

$$z(\alpha) = - \frac{1}{2} \int_0^\alpha \frac{1}{(\mu + \xi)} \ln \left| \frac{A}{\xi} \right| d\xi \quad (94)$$

satisfying the boundary condition $z(0)=0$. The boundary condition $z(1)=1$ is satisfied by the choice of A (as a function of μ) that guarantees that:

$$1 = - \frac{1}{2} \int_0^1 \frac{1}{(\mu + \xi)} \ln \left| \frac{A}{\xi} \right| d\xi \quad (95)$$

finally, the particular value of μ is evaluated from (90) once a value for k has been specified. That the improper integrals on the right-hand-sides of (94), (95) exist will be

argued upon more rigorously in the final paper. Here we will supply a small indication to convince the reader that this is indeed so: Consider the value of k corresponding to the value $\mu=1$. Then, the integral on the right-hand-side of (95) can be evaluated easily from any Table of Definite Integrals¹⁴, and (95) can be solved for A to yield:

$$\ln|A| = -\frac{(24 + \pi^2)}{12 \ln 2} \quad (96)$$

The corresponding asymptotic controller u can be found from (75), (77), (78), (93), (96) and the fact that we set $\mu=1$. The result is:

$$u = \frac{e^{-t}}{2(1 + e^{-t})} \left(\frac{24 + \pi^2}{12 \ln 2} - t \right) \quad (97)$$

It can be shown that this controller drives y asymptotically from $y=0$ at $t=0$ to $y=1$ as t approaches infinity. This fact however is already guaranteed by Theorem 3. To construct an asymptotic feedback controller, one will have to use (97) to integrate the equation $dy/dt=u$. This will lead to an expression for y as a function of t . Elimination of t between this expression and (97) will lead to the desired asymptotic feedback controller. Or, alternatively, one can use the transformation (73) with $z(\alpha)$ supplied by (94) and express α as a function of y . Then, using (93) and (75) with $\beta=-\alpha$ should result in the same asymptotic feedback controller as before. Although the procedure cannot be carried out analytically for either case, it can be carried out numerically and relevant results and details will be given in the final paper.

IV. Conclusions

The class of all piecewise continuous (in time) controllers that track a given hypersurface in the state space of a dynamical system can be split into two disjoint classes. The first class contains all controllers that track the hypersurface in finite time. The second class contains all controllers that track the hypersurface asymptotically. A transformation

technique can be used to give a complete description of both classes. This splitting of the two classes can be used to reformulate optimal control problems involving active state variable inequality constraints. The optimization in the reformulated problem is carried out over the class of asymptotic controllers only and not over the class of all controllers. If a minimum over all controllers exist, then the reformulated problem is guaranteed to have an infimum. An example suggests that the reformulated problem does not achieve its infimum, that is, the infimum corresponds to a finite-time controller. By imposing an additional isoperimetric constraint on the reformulated problem one can, at least for this particular example, select an asymptotic controller that approximates this infimum.

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