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WELL-POSEDNESS OF A MODEL FOR STRUCTURAL ACOUSTIC COUPLING IN A CAVITY ENCLOSED BY A THIN CYLINDRICAL SHELL [1]

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ABSTRACT

A fully coupled mathematical model describing the interactions between a vibrating thin cylindrical shell and an enclosed acoustic field is presented. Because the model will ultimately be used in control applications involving piezoceramic actuators, the loads and material contributions resulting from piezoceramic patches bonded to the shell are included in the discussion. Theoretical and computational issues lead to the consideration of a weak form of the modeling set of partial differential equations (PDE's) and through the use a semigroup formulation, well-posedness results for the system model are obtained.

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1 Introduction

The active control of noise generated by structural vibrations has been studied for several years in various applications. One recent motivation leading to an intense study of problems involving the reduction of structure-borne noise has resulted from the development of a new class of turboprop and turbofan engines. These engines, although very fuel efficient, are also very noisy. Specifically, the low frequency, high magnitude exterior noise fields produced by these new engines cause vibrations in the fuselage which, due to structural acoustic coupling between the fuselage and the interior acoustic field, lead to unacceptably high cabin noise levels. The problem is exacerbated by the increased use of lightweight, composite materials in the cabin walls. In order to make the use of the engines feasible in commercial aircraft, a great deal of research has been aimed at developing control techniques to reduce this unwanted interior noise.

Control techniques for structural acoustics problems of this type have been studied from a variety of perspectives [2, 7, 12, 13, 14, 15, 16, 20, 21, 25], one of which is through the use of piezoceramic patches which are bonded to the enclosing structure [2, 7, 14, 16]. As detailed in [8, 9], the patches create bending moments and in-plane strains when a voltage is applied, and through these actions, can be used to alter the structural dynamics in a manner which ultimately reduces interior noise.

In this paper, we develop a fully coupled mathematical model, as well as well-posedness results for this model, which can be used when applying parameter estimation and PDE-based control strategies to problems involving the use of piezoceramic actuators. This model extends the 2-D results in [2, 7] and the 3-D results in [10] to the cylindrical domain of interest (the 3-D domain in [10] consisted of a hard-walled cavity with a thin plate at one end). While ultimately motivated by the cabin noise problem mentioned above, the model is designed to be consistent with an experimental apparatus being used at the Acoustic Division, NASA Langley Research Center. This setup consists of a thin-walled hollow cylindrical shell which is supported by rigid caps at the ends. Piezoceramic patches bonded to the walls of the shell are then used to control interior noise which has been generated by the vibrations of the walls.

This mathematical model differs from previously used 3-D shell/acoustic models in that it fully incorporates the backpressure and momentum conditions which couple the structural dynamics with the interior acoustic response, thus yielding a time-dependent system of partial differential equations which describe the system dynamics. Because the acoustic/structure interactions are fully incorporated, this model is useful for control strategies which utilize the natural "feedback" loop due to the coupling between the acoustic fields and the structural vibrations. Moreover, by incorporating the piezoceramic patch/shell interaction results of [8], the model can be employed in devising control techniques utilizing piezoceramic actuators.

After the development of the mathematical model, well-posedness results are presented. These are important not only for determining the existence and continuous dependence of solutions, but also in providing an initial framework that can be used when determining suitable approximation schemes. By carefully noting the Hilbert spaces containing the state variables and test functions, appropriate choices can be made for the approximating subspaces which contain the basis functions used in finite element or spectral approximations. This is important not only when considering forward simulations but also when considering theoretical issues concerning the parameter estimation and control problems.

Section 2 of this presentation contains a description of the components of this system along with the strong form of the equations of motion for the system. The disadvantages of the strong form are discussed in Section 3 and a weak or variational form of the system equations is presented. An abstract framework amenable to forward simulations, the estimation of physical parameters, and the implementation of PDE-based feedback control strategies is also developed in this section. In Section 4, the abstract model described in the third section is shown to be well-posed. This is accomplished using "extrapolation space" ideas and arguments similar to those presented in [5, 6, 17]. Having obtained the existence and continuous dependence of solutions for the model, approximation and LQR optimal control techniques similar to those discussed in [2, 7, 10] can be applied to the problem in an effort to suitably reduce interior pressure levels.

2 Strong Form of the System Equations

As motivated by the experimental setup described in the Introduction, the structural acoustics problem under consideration is assumed to consist of a cylindrical acoustic cavity $\Omega(t)$ which is enclosed by a thin cylindrical shell (see Figure 1). Hard wall conditions are taken at the ends Γ of the cavity in order to model the rigid end caps used in the experimental apparatus. These end caps also dictate the use of clamped boundary conditions when approximating the shell dynamics.

As outlined in the discussion in the previous section, the dynamics of the coupled system are composed of an acoustic response, shell dynamics, piezoceramic patch/shell interactions, and the coupling acoustic/structure interactions. Each of these components is briefly described and the results are summarized in a coupled system of PDE's which describe the system dynamics.

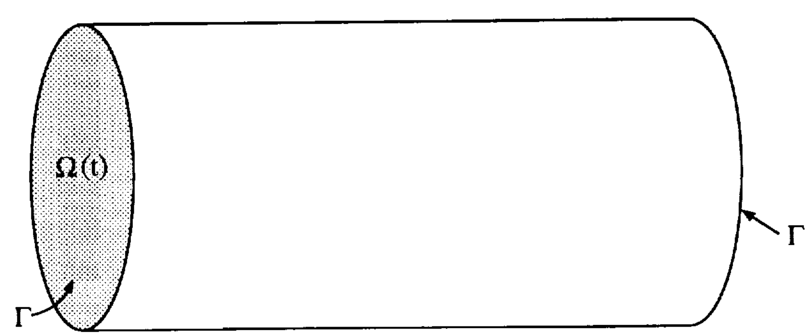

Figure 1. Cylindrical acoustic cavity $\Omega(t)$ with hard end caps Γ.

2.1. Acoustic response

The acoustic wave motion inside the cavity can be described either in terms of a velocity potential ϕ or the acoustic pressure p (the two are related through the relationship $p = \rho_f \phi_t$ where ρ_f is the equilibrium density of the atmosphere); motivated by control theoretic considerations and to simplify the presentation which follows, we choose the former as the second-order state variable. The acoustic dynamics inside the cavity are then modeled by the undamped wave equation

$$\begin{aligned} \phi_{tt} &= c^2 \Delta \phi \qquad , \quad (r, \theta, x) \in \Omega(t) \; , t > 0 \; , \\ \nabla \phi \cdot \hat{n} &= 0 \qquad , \quad (r, \theta, x) \in \Gamma \; , t > 0 \end{aligned}$$

where c is the speed of sound in the cavity, $\hat{n}$ is the outward axial unit normal to the cavity end caps, and the Laplacian in cylindrical coordinates is given by

$$\Delta \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial x^2} \, .$$

Again, Γ denotes the circular caps at the ends of the cavity $\Omega(t)$ (the cavity is variable in time due to the fact that the enclosing shell is vibrating), and x is the axial direction in the cavity. Finally, we note that air damping inside the cavity was omitted due to the relatively small dimensions of the experimental cavity, and could readily be incorporated for significantly larger cavities.

2.2. Shell dynamics

In order to specify the equations of motion of the bounding structure, we consider a thin cylindrical shell having length ℓ, thickness h and radius R with axial direction taken along the x-axis (see Figure 2). The displacements of the middle surface in the axial, tangential and radial directions are taken to be u, v and w, respectively. Furthermore, the shell is assumed to have density ρ, Young's modulus E, Poisson ration ν, and damping coefficient C_D. Finally, we assume that the shell's length is relatively short in relation to it radius and hence the Donnell-Mushtari equations can be used when approximating its motion [18]. This last assumption is made purely for ease of presentation and higher-order theories such as the Byrne-Flügge-Lur'ye model can be substituted for that of Donnell and Mushtari as warranted by the shell dimensions.

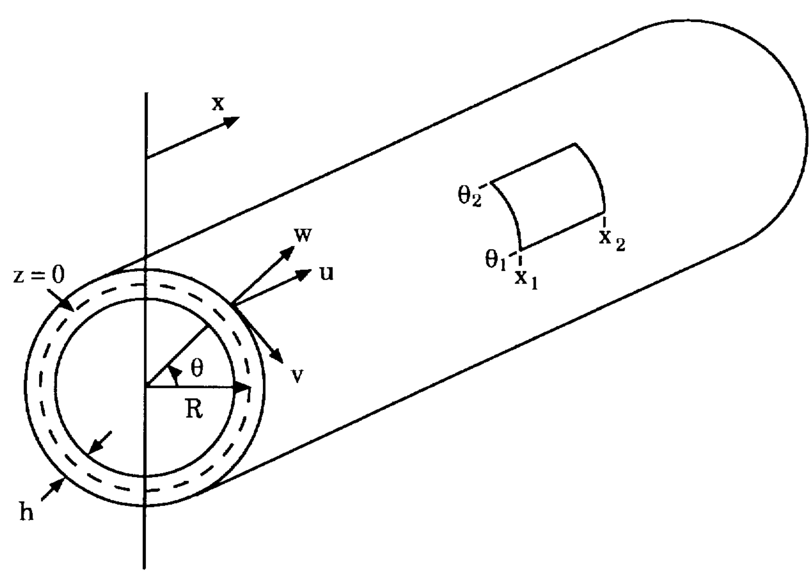

Figure 2. The cylindrical thin shell.

As presented in [8, 19], the Donnell-Mushtari equations for a thin cylindrical shell are given by

$$
\begin{aligned}
R\rho h \frac{\partial^2 u}{\partial t^2} - R\frac{\partial N_x}{\partial x} - \frac{\partial N_{\theta x}}{\partial \theta} &= -R\frac{\partial (N_x)_{pe}}{\partial x} S_{1,2}(x)\hat{S}_{1,2}(\theta) \\
R\rho h \frac{\partial^2 v}{\partial t^2} - \frac{\partial N_\theta}{\partial \theta} - R\frac{N_{x\theta}}{\partial x} &= -\frac{\partial (N_\theta)_{pe}}{\partial \theta} S_{1,2}(x)\hat{S}_{1,2}(\theta) \\
R\rho h \frac{\partial^2 w}{\partial t^2} - R\frac{\partial^2 M_x}{\partial x^2} - \frac{1}{R}\frac{\partial^2 M_\theta}{\partial \theta^2} - 2\frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} + N_\theta &= R\hat{q}_n - R\frac{\partial^2 (M_x)_{pe}}{\partial x^2} - \frac{1}{R}\frac{\partial^2 (M_\theta)_{pe}}{\partial \theta^2} \; .
\end{aligned}
\tag{1}
$$

Here M_x, M_θ, N_x and N_θ are internal moments and force resultants, $(M_x)_{pe}, (M_\theta)_{pe}, (N_x)_{pe}$ and $(N_\theta)_{pe}$ are the resultants for the loads generated by the piezoceramic patches when a voltage is applied, and $\hat{q}_n$ is the external, normal load on the shell. For a patch with uniform thickness and bounding values x_1, x_2, θ_1 and θ_2 as depicted in Figure 2, the presence of the indicator function

$$
S_{1,2}(x) = \begin{cases} 1 & , \quad x < (x_1 + x_2)/2 \\ 0 & , \quad x = (x_1 + x_2)/2 \\ -1 & , \quad x > (x_1 + x_2)/2 \end{cases}
\tag{2}
$$

derives from the fact that the forces generated by the patch in the x-direction are antisymmetric (equal in magnitude but opposite in sign) about the line $\bar{x} = (x_1 + x_2)/2$. The same holds true for the forces in the θ-direction and $\hat{S}_{1,2}(\theta)$ is defined in an analogous manner.

From the discussion in [9], internal damping can be incorporated in the shell equations by assuming a constitutive law which posits that stress is proportional to a linear combination of strain and strain rate. This yields a a Kelvin-Voigt type of damping in the shell with internal resultants for shell regions not covered by patches given by

$$
\begin{aligned}
N_x &= \frac{Eh}{(1-\nu^2)}\left[\frac{\partial u}{\partial x} + \nu\left(\frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{w}{R}\right)\right] + \frac{C_D h}{(1-\nu^2)}\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} + \nu\left(\frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{w}{R}\right)\right] \\
N_\theta &= \frac{Eh}{(1-\nu^2)}\left[\frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{w}{R} + \nu\frac{\partial u}{\partial x}\right] + \frac{C_D h}{(1-\nu^2)}\frac{\partial}{\partial t}\left[\frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{w}{R} + \nu\frac{\partial u}{\partial x}\right] \\
N_{x\theta} = N_{\theta x} &= \frac{Eh}{2(1+\nu)}\left[\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right] + \frac{C_D h}{2(1+\nu)}\frac{\partial}{\partial t}\left[\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right] \\
M_x &= -\frac{Eh^3}{12(1-\nu^2)}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\nu}{R^2}\frac{\partial^2 w}{\partial \theta^2}\right] - \frac{C_D h^3}{12(1-\nu^2)}\frac{\partial}{\partial t}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\nu}{R^2}\frac{\partial^2 w}{\partial \theta^2}\right] \\
M_\theta &= -\frac{Eh^3}{12(1-\nu^2)}\left[\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] - \frac{C_D h^3}{12(1-\nu^2)}\frac{\partial}{\partial t}\left[\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] \\
M_{x\theta} = M_{\theta x} &= -\frac{Eh^3}{12R(1+\nu)}\frac{\partial^2 w}{\partial x \partial \theta} - \frac{C_D h^3}{12R(1+\nu)}\frac{\partial}{\partial t}\left[\frac{\partial^2 w}{\partial x \partial \theta}\right] \; .
\end{aligned}
\tag{3}
$$

Similar expressions (e.g., see [9]) can be derived for those regions of the structure in which piezoceramic patches are bonded to the shell. In those areas, the internal force and moment resultants contain contributions not only from the shell but also from the internal stresses in the piezoceramic patches. The contributions resulting from the external loads $(M_x)_{pe}, (M_\theta)_{pe}, (N_x)_{pe}, (N_\theta)_{pe}$ and $\hat{q}_n$ will be discussed further in the next two subsections.

Finally, we note that the boundary conditions for the shell must be consistent with the rigid end cap conditions on the experimental apparatus. Depending on the exact nature of the end caps, one may enforce either the clamped edge condition

$$u = v = w = \frac{\partial w}{\partial x} = 0$$

or the simply supported edge condition

$$u = v = w = M_x = 0$$

at the ends of the shell. For definiteness, we will employ the first while simply noting that the latter often adequately models the conditions being maintained at the edges of the shell.

2.3. Piezoceramic patch/cylindrical shell interactions

In order to specify the moment and force resultants which result from the activation of the piezoceramic patches, we consider patches which are bonded to the inner and outer surfaces of the shell in a manner such that their edges are parallel to lines of constant x and θ as shown in Figure 2. When it is necessary to differentiate between the two patches, the outer will be denoted with a subscript pe_1 with a subscript pe_2 being used to denote the inner patch in each pair. We point out that in order to keep this presentation compact, we have assumed that both patches are active, and we refer the reader to [8] for the case when only one patch is bonded to the shell.

As discussed in [8], the total line moments and forces for a single patch pair with bounding values x_1, x_2, θ_1 and θ_2 are

$$\begin{aligned}
(M_x)_{pe} &= [(M_\theta)_{pe_1} + (M_\theta)_{pe_2}]\,[H_1(x) - H_2(x)]\,[H_1(\theta) - H_2(\theta)] \\
(M_\theta)_{pe} &= [(M_x)_{pe_1} + (M_x)_{pe_2}]\,[H_1(x) - H_2(x)]\,[H_1(\theta) - H_2(\theta)] \\
(N_x)_{pe} &= [(N_x)_{pe_1} + (N_x)_{pe_2}]\,[H_1(x) - H_2(x)]\,[H_1(\theta) - H_2(\theta)]\,S_{1,2}(x)\hat{S}_{1,2}(\theta) \\
(N_\theta)_{pe} &= [(N_\theta)_{pe_1} + (N_\theta)_{pe_2}]\,[H_1(x) - H_2(x)]\,[H_1(\theta) - H_2(\theta)]\,S_{1,2}(x)\hat{S}_{1,2}(\theta)\ .
\end{aligned} \tag{4}$$

Here H is the Heaviside function with $H_i(x) \equiv H(x - x_i)\,, i = 1,2\,,$ and $S_{1,2}$ is the indicator function defined in (2) with similar definitions being used in θ. The individual components are given by

$$
\begin{aligned}
(M_x)_{pe_1} &= -\frac{E_{pe}}{1-\nu_{pe}^2}\left[\frac{1}{8}\left(4\left(\frac{h}{2}+T\right)^2-h^2\right)+\frac{1}{R}\frac{1}{24}\left(8\left(\frac{h}{2}+T\right)^3-h^3\right)\right](1+\nu_{pe})\,e_{pe_1}\\
(M_x)_{pe_2} &= \frac{E_{pe}}{1-\nu_{pe}^2}\left[\frac{1}{8}\left(4\left(\frac{h}{2}+T\right)^2-h^2\right)-\frac{1}{R}\frac{1}{24}\left(8\left(\frac{h}{2}+T\right)^3-h^3\right)\right](1+\nu_{pe})\,e_{pe_2}\\
(M_\theta)_{pe_1} &= -\frac{E_{pe}}{1-\nu_{pe}^2}\left[\frac{1}{8}\left(4\left(\frac{h}{2}+T\right)^2-h^2\right)\right](1+\nu_{pe})\,e_{pe_1}\\
(M_\theta)_{pe_2} &= \frac{E_{pe}}{1-\nu_{pe}^2}\left[\frac{1}{8}\left(4\left(\frac{h}{2}+T\right)^2-h^2\right)\right](1+\nu_{pe})\,e_{pe_2}\\
(N_x)_{pe_1} &= -\frac{E_{pe}}{1-\nu_{pe}^2}\left[T+\frac{1}{R}\frac{1}{8}\left(4\left(\frac{h}{2}+T\right)^2-h^2\right)\right](1+\nu_{pe})\,e_{pe_1}\\
(N_x)_{pe_2} &= -\frac{E_{pe}}{1-\nu_{pe}^2}\left[T-\frac{1}{R}\frac{1}{8}\left(4\left(\frac{h}{2}+T\right)^2-h^2\right)\right](1+\nu_{pe})\,e_{pe_2}\\
(N_\theta)_{pe_1} &= -\frac{E_{pe}}{1-\nu_{pe}^2}T(1+\nu_{pe})\,e_{pe_1}\\
(N_\theta)_{pe_2} &= -\frac{E_{pe}}{1-\nu_{pe}^2}T(1+\nu_{pe})\,e_{pe_1}\ .
\end{aligned}
\tag{5}
$$

where E_{pe}, ν_{pe} and T are the Young's modulus, Poisson ratio and thickness of the patches, respectively (it is assumed that the inner and outer patches have the same material properties). With d_{31}, V_1 and V_2 denoting the piezoceramic strain constant and voltages into the two patches, the terms $e_{pe_1} = \frac{d_{31}}{T}V_1$ and $e_{pe_2} = \frac{d_{31}}{T}V_2$ provide a relationship between the applied voltages and the resulting in-plane mechanical strains.

From (4) and (5), it can be seen that the resultants due to the activation of the patches depend on the material and geometric properties of the patches, the radius of the shell, and the voltage being applied to the patches. Once determined, these resultants are substituted into (1) as the external loads on the shell. If multiple patch pairs are present, the resultants for each pair are incorporated in the shell equations in a similar manner.

2.4. Normal loads and coupling conditions

It can be seen in (1) that the shell equations also contain the term $\hat{q}_n$ which represents the normal load on the shell. In writing the equations in this form, we have made the assumption that the only loads on the shell are the contributions due to the patches, and the normal load which is due to the exterior noise field f as well as backpressure generated by the interior acoustic field. This assumption was made merely to simplify the discussion, and more complex acoustic or mechanical loads on the shell can be incorporated as additional external forces and moments in the shell equations. Because $\hat{q}_n$, in this case, consists of contributions from the exterior and interior acoustic fields, the first coupling relation is

$$\hat{q}_n(t,\theta,x) = f(t,\theta,x) - \rho_f \phi_t(t, w(t,\theta,x), v(t,\theta,x), u(t,\theta,x))$$

and is in general nonlinear since the backpressure onto the shell occurs at the shell's surface.

The second coupling condition is the velocity constraint

$$\frac{\partial \phi}{\partial r}(t, w(t,\theta,x), v(t,\theta,x), u(t,\theta,x)) = w_t(t,\theta,x)$$

which simply states that the shell is impermeable to air. Note that this second constraint also provides a boundary condition for the acoustic response.

2.5. Fully coupled system: strong form

Under the assumption of small displacements which is inherent in the linear shell theories, the variable domain $\Omega(t)$ is approximated by the fixed domain Ω, and the general nonlinear coupling conditions are approximated by their linear approximations. Let the boundary Γ_0 denote the shell in its unperturbed state. For s pairs of piezoceramic patches, this then yields the linear model

$$\begin{array}{ll}
\phi_{tt} = c^2 \Delta\phi \quad , & (r,\theta,x) \in \Omega \; , t > 0 \; , \\
\nabla\phi \cdot \hat{n} = 0 \quad , & (r,\theta,x) \in \Gamma \; , t > 0
\end{array}$$

$$\frac{\partial \phi}{\partial r}(t,R,\theta,x) = w_t(t,\theta,x) \quad , \quad (\theta,x) \in \Gamma_0 \; , t > 0$$

$$\left.\begin{array}{l}
R\rho h \dfrac{\partial^2 u}{\partial t^2} - R\dfrac{\partial N_x}{\partial x} - \dfrac{\partial N_{\theta x}}{\partial \theta} = -R\displaystyle\sum_{i=1}^{s} \dfrac{\partial\left[(N_x)_{pe}\right]_i}{\partial x}\left[S_{1,2}(x)\right]_i\left[\hat{S}_{1,2}(\theta)\right]_i \\[2ex]
R\rho h \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial N_\theta}{\partial \theta} - R\dfrac{N_{x\theta}}{\partial x} = -\displaystyle\sum_{i=1}^{s} \dfrac{\partial\left[(N_\theta)_{pe}\right]_i}{\partial \theta}\left[S_{1,2}(x)\right]_i\left[\hat{S}_{1,2}(\theta)\right]_i \\[2ex]
R\rho h \dfrac{\partial^2 w}{\partial t^2} - R\dfrac{\partial^2 M_x}{\partial x^2} - \dfrac{1}{R}\dfrac{\partial^2 M_\theta}{\partial \theta^2} - 2\dfrac{\partial^2 M_{x\theta}}{\partial x \partial \theta} + N_\theta \\[1ex]
\quad = R\left[f(t,\theta,x) - \rho_f \phi_t(t,R,\theta,x)\right] \\[1ex]
\quad -R\displaystyle\sum_{i=1}^{s}\left\{\dfrac{\partial^2\left[(M_x)_{pe}\right]_i}{\partial x^2} + \dfrac{1}{R^2}\dfrac{\partial^2\left[(M_\theta)_{pe}\right]_i}{\partial \theta^2}\right\}
\end{array}\right\} \quad , \quad (\theta,x) \in \Gamma_0 \; , t > 0$$

$$u = v = w = \frac{\partial w}{\partial x} = 0 \quad , \quad x = 0, \ell$$

$$\phi(0,r,\theta,x) = \phi_0(r,\theta,x) \quad , \quad \phi_t(0,r,\theta,x) = \phi_1(r,\theta,x) \quad , \quad (r,\theta,x) \in \Omega$$

$$\left.\begin{array}{ll}
u(0,\theta,x) = u_0(\theta,x) \quad , & u_t(0,\theta,x) = u_1(\theta,x) \\
v(0,\theta,x) = v_0(\theta,x) \quad , & v_t(0,\theta,x) = v_1(\theta,x) \\
w(0,\theta,x) = w_0(\theta,x) \quad , & w_t(0,\theta,x) = w_1(\theta,x)
\end{array}\right\} \quad , \quad (\theta,x) \in \Gamma_0 \; . \tag{6}$$

Recall that the internal shell moments and forces are summarized in (3) while the expressions for the external loads generated by the i^{th} pair of piezoceramic patches are given in (4). The notation $[S_{1,2}(x)]_i$ and $[\hat{S}_{1,2}(\theta)]_i$ denotes the indicator functions (see (2)) which are centered over the i^{th} pair of patches. Note that this representation admits different voltages and geometries for the s patches, thus increasing the flexibility of the model for control purposes.

We also reiterate that the only damping in the system is the internal Kelvin-Voigt damping in the shell although medium damping inside the cavity can be added if cavity dimensions become significant. Without medium damping, however, the system is only weakly damped, and this must be considered when discussing theoretical convergence and well-posedness results for the problem.

3 Weak Form of the System Equations

As can be seen in the model (6), the use of the strong form of the system equations leads to first and second derivatives of both the internal and external moment and force resultants. This leads to difficulties both in approximating the behavior of the system and in solving the control problem. The first problem results from the presence and differing material properties of the piezoceramic patches. As noted in [11] where piezoceramic patches are bonded to a beam, the material parameters of the combined structure must be modeled as piecewise constants in order to accurately match structural frequencies. Hence the parameters ρ, E, ν and C_D are expressed in terms of a Heaviside basis with the edges of the patches defining the support of the functions. This leads to difficulties in the strong form since it necessitates the differentiation of discontinuous material parameters. A similar problem arises when including the moments generated by the patches. As seen in (4), the support of the contributions is given in terms of Heaviside functions which implies that the use of the strong form yields an unbounded control operator (it involves the differentiation of the Heaviside functions as well as the Dirac delta). To avoid these difficulties, it is advantageous to formulate the problem in weak or variational form (the use of the variational form also admits the use of basis functions having less smoothness than those used when approximating the solution to the strong form of the equations).

3.1. Weak formulation

The second-order state for the problem is taken to be $y = (\phi, u, v, w)$ in the Hilbert space $H = \bar{L}^2(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_0) \times L^2(\Gamma_0)$. The choice of the space $\bar{L}^2(\Omega)$, defined as the quotient of $L^2(\Omega)$ over the constant functions, results from the fact that the potentials are determined only up to a constant.

To provide a class of functions which are considered when defining a variational form of the problem, we also define the Hilbert space $V = \bar{H}^1(\Omega) \times H_0^1(\Gamma_0) \times H_0^1(\Gamma_0) \times H_0^2(\Gamma_0)$ where $\bar{H}^1(\Omega)$ is the quotient space of $H^1(\Omega)$ over the constant functions. The subscript 0 in the remaining components of the product space denotes the subset of functions in the traditional Sobolev spaces which satisfy the essential boundary conditions $u = v = w = \frac{\partial w}{\partial x} = 0$ at $x = 0, \ell$.

A complete discussion concerning the derivation of the a variational formulation of the shell equations from energy principles can be found in [8]. For our purposes here, we simply note that integration in combination with the use of Green's theorem yields the second-order variational system

$$
\begin{aligned}
&\int_\Omega \frac{\rho_f}{c^2}\frac{\partial^2\phi}{\partial t^2}\overline{\xi}d\omega + \int_\Omega \rho_f \nabla\phi\cdot\overline{\nabla\xi}d\omega + \int_{\Gamma_0}\left\{\rho h\frac{\partial^2 u}{\partial t^2}\overline{\eta_1} + N_x\overline{\frac{\partial\eta_1}{\partial x}} + \frac{1}{R}N_{\theta x}\overline{\frac{\partial\eta_1}{\partial\theta}}\right\}d\gamma \\
&+ \int_{\Gamma_0}\left\{\rho h\frac{\partial^2 v}{\partial t^2}\overline{\eta_2} + \frac{1}{R}N_\theta\overline{\frac{\partial\eta_2}{\partial\theta}} + N_{x\theta}\overline{\frac{\partial\eta_2}{\partial x}}\right\}d\gamma \\
&+ \int_{\Gamma_0}\left\{\rho h\frac{\partial^2 w}{\partial t^2}\overline{\eta_3} + \frac{1}{R}N_\theta\overline{\eta_3} - M_x\overline{\frac{\partial^2\eta_3}{\partial x^2}} - \frac{1}{R^2}M_\theta\overline{\frac{\partial^2\eta_3}{\partial\theta^2}} - \frac{2}{R}M_{x\theta}\overline{\frac{\partial^2\eta_3}{\partial x\partial\theta}}\right\}d\gamma \\
&+ \int_{\Gamma_0}\rho_f\left\{\frac{\partial\phi}{\partial t}\overline{\eta_3} - \frac{\partial w}{\partial t}\overline{\xi}\right\}d\gamma \\
&= \int_{\Gamma_0}\sum_{i=1}^{s}\left\{[(N_x)_{pe}]_i\overline{\frac{\partial\eta_1}{\partial x}} + \frac{1}{R}[(N_\theta)_{pe}]_i\overline{\frac{\partial\eta_2}{\partial\theta}} - [(M_x)_{pe}]_i\overline{\frac{\partial^2\eta_3}{\partial x^2}} - \frac{1}{R^2}[(M_\theta)_{pe}]_i\overline{\frac{\partial^2\eta_3}{\partial\theta^2}}\right\}d\gamma \\
&+ \int_{\Gamma_0} f\overline{\eta_3}d\gamma
\end{aligned}
\tag{7}
$$

for all $(\xi,\eta_1,\eta_2,\eta_3)\in V$ (note that $d\omega = r\,dr\,d\theta\,dx$ and $d\gamma = R\,d\theta\,dx$). The complex L^2 inner products are used in anticipation of the possibility of employing complex Fourier expansions in θ when developing suitable approximation schemes for the problem.

We note that, as is usual, in this variational form, the derivatives have been transferred from the plate and patch moments onto the test functions. This eliminates the problem of having to approximate the derivatives of the Heaviside function and the Dirac delta which arises in the case of the strong form of the equations.

The system (7) can be formally approximated by replacing the state variables by their finite dimensional approximations and constructing the resulting matrix system. In order to discuss convergence results for the approximation, parameter estimation, and control problems, however, it is advantageous to pose the problem in terms of sesquilinear forms and the bounded operators which they define.

3.2. Abstract weak formulation

Before proceeding with the abstract formulation, it is necessary to describe the inner products for the spaces H and V since these will be needed when determining the continuity of various operators. From energy considerations, it is appropriate to use the inner products

$$\langle\Phi,\Psi\rangle_H = \int_\Omega \frac{\rho_f}{c^2}\phi\overline{\xi}d\omega + \int_{\Gamma_0}\rho h u\overline{\eta_1}d\gamma + \int_{\Gamma_0}\rho h v\overline{\eta_2}d\gamma + \int_{\Gamma_0}\rho h w\overline{\eta_3}d\gamma$$

and

$$
\begin{aligned}
\langle \Phi, \Psi \rangle_V &= \int_\Omega \rho_f \nabla\phi \cdot \overline{\nabla\xi} d\omega \\
&+ \int_{\Gamma_0} \frac{Eh}{(1-\nu^2)} \left\{ \left[\frac{\partial u}{\partial x} + \frac{\nu}{R}\left(\frac{\partial v}{\partial \theta} + w\right)\right] \overline{\frac{\partial \eta_1}{\partial x}} + \frac{1}{2R}(1-\nu)\left[\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right] \overline{\frac{\partial \eta_1}{\partial \theta}} \right\} d\gamma \\
&+ \int_{\Gamma_0} \frac{Eh}{(1-\nu^2)} \left\{ \frac{1}{R}\left[\frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{w}{R} + \nu\frac{\partial u}{\partial x}\right] \overline{\frac{\partial \eta_2}{\partial \theta}} + \frac{1}{2}(1-\nu)\left[\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right] \overline{\frac{\partial \eta_2}{\partial x}} \right\} d\gamma \qquad (8) \\
&+ \int_{\Gamma_0} \frac{Eh}{(1-\nu^2)} \left\{ \frac{1}{R}\left[\frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{w}{R} + \nu\frac{\partial u}{\partial x}\right] \overline{\eta_3} + \frac{h^2}{12}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\nu}{R^2}\frac{\partial^2 w}{\partial \theta^2}\right] \overline{\frac{\partial^2 \eta_3}{\partial x^2}} \right. \\
&\qquad \left. + \frac{h^2}{12R^2}\left[\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] \overline{\frac{\partial^2 \eta_3}{\partial \theta^2}} + \frac{h^2}{6R^2}(1-\nu)\frac{\partial^2 w}{\partial x \partial \theta}\overline{\frac{\partial^2 \eta_3}{\partial x \partial \theta}} \right\} d\gamma
\end{aligned}
$$

where $\Phi = (\phi, u, v, w)$ and $\Psi = (\xi, \eta_1, \eta_2, \eta_3)$. We remark that the inner product for the state space H contains terms which arise when considering the kinetic energy for the shell while the shell contributions in the V inner product are motivated by the form of the strain energy for a thin cylindrical shell (see [8]). Also, we note that with this choice of spaces and inner products, we can form the Gelfand triple $V \hookrightarrow H \simeq H^* \hookrightarrow V^*$ with pivot space H (see [27] for a complete discussion of these ideas).

To define appropriate sesquilinear forms $\sigma_i : V \times V \to \mathbb{C}$, $i = 1, 2$, we group the stiffness and wave contributions separately from damping and coupling terms, thus leading to the definitions

$$\sigma_1(\Phi, \Psi) = \langle \Phi, \Psi \rangle_V$$

and

$$
\begin{aligned}
\sigma_2(\Phi, \Psi) &= \int_{\Gamma_0} \frac{C_D h}{(1-\nu^2)} \left\{ \left[\frac{\partial u}{\partial x} + \frac{\nu}{R}\left(\frac{\partial v}{\partial \theta} + w\right)\right] \overline{\frac{\partial \eta_1}{\partial x}} + \frac{1}{2R}(1-\nu)\left[\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right] \overline{\frac{\partial \eta_1}{\partial \theta}} \right\} d\gamma \\
&+ \int_{\Gamma_0} \frac{C_D h}{(1-\nu^2)} \left\{ \frac{1}{R}\left[\frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{w}{R} + \nu\frac{\partial u}{\partial x}\right] \overline{\frac{\partial \eta_2}{\partial \theta}} + \frac{1}{2}(1-\nu)\left[\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right] \overline{\frac{\partial \eta_2}{\partial x}} \right\} d\gamma \\
&+ \int_{\Gamma_0} \frac{C_D h}{(1-\nu^2)} \left\{ \frac{1}{R}\left[\frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{w}{R} + \nu\frac{\partial u}{\partial x}\right] \overline{\eta_3} + \frac{h^2}{12}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\nu}{R^2}\frac{\partial^2 w}{\partial \theta^2}\right] \overline{\frac{\partial^2 \eta_3}{\partial x^2}} \right. \qquad (9) \\
&\qquad \left. + \frac{h^2}{12R^2}\left[\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] \overline{\frac{\partial^2 \eta_3}{\partial \theta^2}} + \frac{h^2}{6R^2}(1-\nu)\frac{\partial^2 w}{\partial x \partial \theta}\overline{\frac{\partial^2 \eta_3}{\partial x \partial \theta}} \right\} d\gamma \\
&+ \int_{\Gamma_0} \rho_f \left(\phi\overline{\eta_3} - w\overline{\xi}\right) d\gamma \, .
\end{aligned}
$$

To account for the control contributions, we let U denote the Hilbert space containing the control inputs, and we define the control operator $B \in \mathcal{L}(U, V^*)$ by

$$\langle Bu, \Psi \rangle_{V^*,V} = \int_{\Gamma_0} \sum_{i=1}^{s} \left\{ [(N_x)_{pe}]_i \overline{\frac{\partial \eta_1}{\partial x}} + \frac{1}{R}[(N_\theta)_{pe}]_i \overline{\frac{\partial \eta_2}{\partial \theta}} - [(M_x)_{pe}]_i \overline{\frac{\partial^2 \eta_3}{\partial x^2}} - \frac{1}{R^2}[(M_\theta)_{pe}]_i \overline{\frac{\partial^2 \eta_3}{\partial \theta^2}} \right\} d\gamma$$

for $\Psi \in V$, where $\langle \cdot, \cdot \rangle_{V^*,V}$ is the usual duality pairing. Finally, by letting $F = \left(0, 0, 0, \frac{f}{\rho h}\right)$, we can write the system in the abstract variational form

$$\langle y_{tt}(t), \Psi \rangle_{V^*,V} + \sigma_2(y_t(t), \Psi) + \sigma_1(y(t), \Psi) = \langle Bu(t) + F, \Psi \rangle_{V^*,V} \ . \tag{10}$$

We reiterate that the state for this second-order system is $y(t) = (\phi(t, \cdot, \cdot, \cdot), u(t, \cdot, \cdot), v(t, \cdot, \cdot), w(t, \cdot, \cdot))$ in $V \hookrightarrow H$.

In order to write the system in terms of associated bounded operators, we first note that σ_1 and σ_2 are bounded (there exist c_1 and c_2 such that $|\sigma_1(\Phi, \Psi)| \leq c_1 |\Phi|_V |\Psi|_V$ and $|\sigma_2(\Phi, \Psi)| \leq c_2 |\Phi|_V |\Psi|_V$), and hence we can define operators $A_1, A_2 \in \mathcal{L}(V, V^*)$ by

$$\langle A_i \Phi, \Psi \rangle_{V^*,V} = \sigma_i(\Phi, \Psi)$$

for $i = 1, 2$. This then yields the abstract second-order system

$$y_{tt}(t) + A_2 y_t(t) + A_1 y(t) = Bu(t) + F$$

in V^*.

To apply infinite dimensional control results for periodic forcing functions to this problem, it is advantageous to write the system in first-order form. This is accomplished by defining the product spaces $\mathcal{H} = V \times H$ and $\mathcal{V} = V \times V$ with the norms

$$|(\Phi, \Psi)|_{\mathcal{H}}^2 = |\Phi|_V^2 + |\Psi|_H^2$$

and

$$|(\Phi, \Psi)|_{\mathcal{V}}^2 = |\Phi|_V^2 + |\Psi|_V^2 \ .$$

We point out that $\mathcal{V} \hookrightarrow \mathcal{H} \simeq \mathcal{H}^* \hookrightarrow \mathcal{V}^*$ again forms a Gelfand triple.

The sesquilinear form $\sigma : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$ is then defined by

$$\sigma(\Theta, \chi) = \sigma((\Upsilon, \Lambda), (\Phi, \Psi)) = -\langle \Lambda, \Phi \rangle_V + \sigma_1(\Upsilon, \Psi) + \sigma_2(\Lambda, \Psi)$$

where $\chi = (\Phi, \Psi)$ and $\Theta = (\Upsilon, \Lambda)$.

For the state $\mathcal{Y}(t) = (y(t), y_t(t))$ in $\mathcal{H}$, we can subsequently write the system in the first-order variational form

$$\langle \mathcal{Y}_t(t), \chi \rangle_{\mathcal{V}^*,\mathcal{V}} + \sigma(\mathcal{Y}(t), \chi) = \langle \mathcal{B}u(t) + \mathcal{F}(t), \chi \rangle_{\mathcal{V}^*,\mathcal{V}} \tag{11}$$

where $\mathcal{F}(t) = (0, F(t))$ and $\mathcal{B}u(t) = (0, Bu(t))$. As usual, the relation (11) must hold for all $\chi \in \mathcal{V}$. Finally, the weak form (11) is *formally* equivalent to the system

$$\mathcal{Y}_t(t) = \mathcal{A}\mathcal{Y}(t) + \mathcal{B}u(t) + \mathcal{F}(t) \tag{12}$$

in $\mathcal{H}$ where

$$\text{dom}\,\mathcal{A} = \{\Theta = (\Upsilon, \Lambda) \in \mathcal{H} : \Lambda \in V, A_1\Upsilon + A_2\Lambda \in H\}$$

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_1 & -A_2 \end{bmatrix} . \tag{13}$$

Hence we see that the system corresponding to the original model can be written in various forms whose utility depend on the applications of interest. For approximating the dynamics of the system, the formulation (7), or equivalently (10) or (11), is useful and ultimately leads to a first-order matrix system when considering the resulting finite dimensional problem. Moreover, as will be shown in the next section, the weak form (10) provides a setting in which one can determine well-posedness results for the problem. Finally, the first-order infinite dimensional formulation (12) is an abstract Cauchy equation which facilitates the application of infinite dimensional LQR results to the problem of controlling interior acoustic pressure levels.

4 Well-Posedness of the System Model

In this section, the system model described in strong form in Section 2 and weak form in Section 3 is shown to be well-posed. The strategy used to do this can be summarized as follows. By employing the properties of the sesquilinear forms σ_1 and σ_2, it is demonstrated that the operator $\mathcal{A}$ of (13) generates a C_0-semigroup $\mathcal{S}(t)$ on $\mathcal{H}$. Due to the fact that the control and forcing terms lie in $\mathcal{V}^*$ rather than $\mathcal{H}$, however, one is prevented from simply applying a variation of parameters approach to define mild solutions as is often done in problems of this type. Instead, a construction due to Haraux [17] is used to extend the semigroup $\mathcal{S}(t)$ to the space $\mathcal{W}^* = [\text{dom}\, \mathcal{A}^*]^*$ where an extension of the variation of parameters technique for describing a mild solution to the problem can be employed. By formulating the problem in this generalized manner, the well-posedness of the system solution can be established in a form that can be used in parameter estimation and control problems.

4.1. Generation of the C_0-semigroup $\mathcal{S}(t)$ on $\mathcal{H}$

Before proving that the operator $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$, we recall several definitions that will be needed (we note that these definitions vary slightly among authors, e.g. see [24, 27], and that the definitions given here will be used in the discussion which follows).

We say that the sesquilinear form $\sigma : V \times V \to \mathbb{C}$ is *V-elliptic* if there exists a constant $c > 0$ such that

$$\text{Re}\, \sigma(\Phi, \Phi) \geq c|\Phi|_V^2 \quad \text{for all } \Phi \in V \ .$$

The sesquilinear form $\sigma : V \times V \to \mathbb{C}$ is called *H-semielliptic* if there exists a constant $b \geq 0$ such that

$$\text{Re}\, \sigma(\Phi, \Phi) \geq b|\Phi|_H^2 \quad \text{for all } \Phi \in V \ .$$

Finally, we say that the sesquilinear form σ is *symmetric* if

$$\sigma(\Phi, \Psi) = \overline{\sigma(\Psi, \Phi)} \quad \text{for all } \Phi, \Psi \in V \ .$$

The following theorem can be used to establish that the operator $\mathcal{A}$ given in (13) generates a C_0-semigroup on $\mathcal{H}$. The proof depends on the Lumer-Philips Theorem and arguments can be found in [1] and pages 82-84 of [3].

Theorem 1. Let $V \hookrightarrow H \simeq H^* \hookrightarrow V^*$ and suppose that σ_1 and σ_2 of (10) satisfy the properties: σ_1 is V-elliptic, continuous, symmetric, and σ_2 is continuous, H-semielliptic. Then $\mathcal{A}$ defined in (13) generates a C_0-semigroup on $\mathcal{H}$.

To see that our system (10) satisfies the hypotheses of this theorem, we first consider the sesquilinear form $\sigma_1(\Phi,\Psi) = \langle\Phi,\Psi\rangle_V$ given in (8). The boundedness of σ_1 (which results from Schwarz's inequality for inner products in conjunction with equivalence results for various Sobolev norms) was noted in the last section. The V-ellipticity and symmetry of σ_1 follow directly from the definition of the sesquilinear form as the V norm. We note that the symmetry of σ_1 depends on the symmetry of the Donnell-Mushtari shell operator, and while some other shell theories such as that of Byrne, Flügge and Lur'ye also provide symmetric operators and resulting sesquilinear forms, others such as the Love-Timoshenko theory do not yield symmetric operators (see [19]) and hence would not fall directly into our framework here. From a physical perspective, the symmetry of the shell operator and resulting sesquilinear form guarantees real vibration frequencies. This symmetry also guarantees that the shell model satisfies the Maxwell-Betti Reciprocity Theorem which essentially states that for a linearly elastic body, the general displacement at a point m resulting from a load at the point n is equal in magnitude to the displacement at n resulting from an equal magnitude load at m (see for example, [23]).

We next turn to the sesquilinear form σ_2 of (9) and recall that the boundedness of σ_2 was also noted in the last section. This follows from the definition of the σ_2 and the equivalence of various norms which arise from the components of the V norm. The H-semiellipticity of σ_2 follows directly with the choice $b = 0$.

The results from Theorem 1 guarantee that the operator $\mathcal{A}$ given in (13) generates a C_0-semigroup $\mathcal{S}(t)$ on the state space $\mathcal{H}$. Moreover, the semigroup satisfies the exponential bound $|\mathcal{S}(t)| \leq e^{\omega t}$ for $t \geq 0$ (where in fact, $\omega = 0$ due to the fact that $\mathcal{A}$ is dissipative as shown in [3]).

In the case of a bounded (in $\mathcal{H}$) control input operator, the usual procedure is to use a variation of parameters approach to define a mild solution to the system. As noted in the last section, however, the control input $B \in \mathcal{L}(U, V^*)$ defines the product space control term $\mathcal{B}u(t) = (0, Bu(t)) \in \{0\} \times V^* \subset V \times V^* = \mathcal{V}^*$. Since $\mathcal{B}u(t)$ lies in $\mathcal{V}^*$ rather than in $\mathcal{H}$, the usual variation of parameters ideas are not feasible. We are therefore motivated to extend the semigroup $\mathcal{S}(t)$ on $\mathcal{H}$ to a semigroup $\tilde{\mathcal{S}}(t)$ on a larger space $\mathcal{W}^* \supset \{0\} \times V^*$ so that the mild solution

$$\mathcal{Y}(t) = \tilde{\mathcal{S}}(t)\mathcal{Y}_0 + \int_0^t \tilde{\mathcal{S}}(t-s)\begin{pmatrix} 0 \\ Bu(s) + F(s) \end{pmatrix} ds \tag{14}$$

is well-defined for $Bu + F \in L^2((0,T), V^*)$ (in the following development, it will be shown that the space $\mathcal{W}^*$ is given by $\mathcal{W}^* = [\mathrm{dom}\,\mathcal{A}^*]^*$). This then provides a setting in which to guarantee the well-posedness (including continuous dependence on initial data and nonhomogeneous input terms) of the solution.

4.2. Description of $\mathcal{A}^*$ and dom $\mathcal{A}^*$

Before discussing the extension of $\mathcal{S}(t)$ from $\mathcal{H}$ to $\mathcal{W}^*$, it is useful to describe $\mathcal{A}^*$ and dom $\mathcal{A}^*$. We first recall from (13) that $\mathcal{A}$ and dom $\mathcal{A}$ are given by

$$\text{dom}\,\mathcal{A} = \{\Theta = (\Upsilon, \Lambda) \in \mathcal{H} : \Lambda \in V, A_1\Upsilon + A_2\Lambda \in H\}$$

$$\mathcal{A}\Theta = \begin{pmatrix} \Lambda \\ -A_1\Upsilon - A_2\Lambda \end{pmatrix} .$$

From the usual definition of the adjoint $\mathcal{A}^*$, we now want to find dom $\mathcal{A}^*$ and $\mathcal{A}^*$ satisfying

$$\text{dom}\,\mathcal{A}^* = \{\chi = (\Phi, \Psi) \in \mathcal{H} | \Theta \mapsto (\mathcal{A}\Theta)(\chi) \text{ is continuous on dom}\,\mathcal{A}\}$$

and

$$\langle \mathcal{A}\Theta, \chi\rangle_{\mathcal{H}} = \langle \Theta, \mathcal{A}^*\chi\rangle_{\mathcal{H}}$$

for all $\Theta = (\Upsilon, \Lambda) \in$ dom $\mathcal{A}$ and $\chi = (\Phi, \Psi) \in$ dom $\mathcal{A}^*$. It follows directly from the definition of the adjoint operator that $\chi \in$ dom $\mathcal{A}^*$ if and only if there exists $\Gamma = (\gamma_1, \gamma_2) \in \mathcal{H} = V \times H$ such that

$$\langle \mathcal{A}\Theta, \chi\rangle_{\mathcal{H}} = \langle \Theta, \Gamma\rangle_{\mathcal{H}}$$

for all $\Theta \in$ dom $\mathcal{A}$. The expansion of this relation yields the condition

$$\langle \Lambda, \Phi\rangle_V + \langle(-A_1\Upsilon - A_2\Lambda), \Psi\rangle_H = \langle \Upsilon, \gamma_1\rangle_V + \langle \Lambda, \gamma_2\rangle_H \tag{15}$$

for all $\Theta = (\Upsilon, \Lambda) \in$ dom $\mathcal{A}$ and $\chi = (\Phi, \Psi) \in$ dom $\mathcal{A}^*$. By noting that (15) must hold for $\Lambda = 0$, it follows that

$$\langle -A_1\Upsilon, \Psi\rangle_H = \langle \Upsilon, \gamma_1\rangle_V = \langle A_1\Upsilon, \gamma_1\rangle_H$$

for all $\Upsilon \in$ dom A_1. The second inequality results from the observation that $\langle \Upsilon, \gamma_1\rangle_V = \sigma_1(\Upsilon, \gamma_1) = \langle A_1\Upsilon, \gamma_1\rangle_{V^*,V} = \langle A_1\Upsilon, \gamma_1\rangle_H$ for $\Theta = (\Upsilon, 0) \in$ dom $\mathcal{A}$. It follows immediately that $\langle A_1\Upsilon, -\Psi - \gamma_1\rangle_H = 0$ for all $\Upsilon \in$ dom A_1, which implies that

$$\gamma_1 = -\Psi$$

due to the V-ellipticity of σ_1. Moreover, $\gamma_1 \in V$ implies that $\Psi \in V$.

When $\gamma_1 = -\Psi$ is substituted into (15), the relationship

$$\langle A_1\Lambda, \Phi\rangle_H + \langle -A_2\Lambda, \Psi\rangle_H = \langle \Lambda, \gamma_2\rangle_H$$

results for all $\Lambda \in V$. From the last equality, it follows that the mapping $\Lambda \mapsto \langle A_1\Lambda, \Phi\rangle_H + \langle -A_2\Lambda, \Psi\rangle_H$ is continous in H. This in turn implies that $A_1^*\Phi - A_2^*\Psi \in H$ and that

$$\langle \Lambda, A_1^*\Phi - A_2^*\Psi\rangle_H = \langle \Lambda, \gamma_2\rangle_H$$

for all $\Lambda \in V$. Hence we have

$$\gamma_2 = A_1^*\Phi - A_2^*\Psi \ .$$

Thus $\mathcal{A}^*$ and dom $\mathcal{A}^*$ are given by

$$\text{dom}\,\mathcal{A}^* = \{\chi = (\Phi, \Psi) \in \mathcal{H} | \Psi \in V, A_1^*\Phi - A_2^*\Psi \in H\}$$

$$\mathcal{A}^*\chi = \begin{pmatrix} -\Psi \\ A_1^*\Phi - A_2^*\Psi \end{pmatrix} . \tag{16}$$

Finally, we note that because $\Psi \in V$ for any $\chi \in \text{dom}\,\mathcal{A}^*$, it follows immediately that

$$\text{dom}\,\mathcal{A}^* \subset V \times V = \mathcal{V} .$$

4.3. Extension of $\mathcal{S}(t)$ to $[\text{dom}\,\mathcal{A}^*]^*$

With $\mathcal{A}^*$ and dom $\mathcal{A}^*$ described as above, we now consider the extension of the semigroup $\mathcal{S}(t)$ generated by $\mathcal{A}$ from $\mathcal{H}$ to a larger space $\mathcal{W}^*$ which is defined below. This will be accomplished by employing the extrapolation space techniques discussed by Haraux in [17]. Other, more detailed examples using these techniques can be found in [5, 6].

The space $\mathcal{W} = [\text{dom}\,\mathcal{A}^*]$ is taken to be dom $\mathcal{A}^*$ with the inner product

$$\langle \Phi, \Psi \rangle_{\mathcal{W}} = \langle (\lambda_0 - \mathcal{A}^*)\Phi, (\lambda_0 - \mathcal{A}^*)\Psi \rangle_{\mathcal{H}}$$

for some arbitrary but fixed λ_0 with $\lambda_0 > \omega$ (recall that the original solution semigroup satisfies the bound $|\mathcal{S}(t)| \le e^{\omega t}$). As proven in [6], the resulting $\mathcal{W}$ norm is equivalent to the graph norm corresponding to $\mathcal{A}^*$; that is, there exist constants c_1 and c_2 such that for any $\Phi \in W$, we have

$$\begin{aligned} |(\lambda_0 - \mathcal{A}^*)\Phi|_{\mathcal{H}} &\le c_1 \left(|\Phi|_{\mathcal{H}} + |\mathcal{A}^*\Phi|_{\mathcal{H}} \right) \\ |\Phi|_{\mathcal{H}} + |\mathcal{A}^*\Phi|_{\mathcal{H}} &\le c_2 \left| (\lambda_0 - \mathcal{A}^*)\Phi \right|_{\mathcal{H}} . \end{aligned} \tag{17}$$

As discussed in [5, 6], the space $\mathcal{W}$ defined in this manner and with this norm is densely and continuously embedded in $\mathcal{H}$. Hence we can formulate the Gelfand triple $\mathcal{W} \hookrightarrow \mathcal{H} \simeq \mathcal{H}^* \hookrightarrow \mathcal{W}^*$ with pivot space $\mathcal{H}$. Moreover, the dual space $\mathcal{W}^*$ is isomorphic to the completion of $\mathcal{H}$ with respect to the norm $|h|_{\mathcal{W}^*} = |(\lambda_0 - \mathcal{A})^{-1}h|_{\mathcal{H}}$.

To facilitate the arguments for extending the operator $\mathcal{A}$ from dom $\mathcal{A} \subset \mathcal{H}$ to all of $\mathcal{H}$, we define the sesquilinear form $\tilde{\sigma} : \mathcal{H} \times \mathcal{W} \to \mathbb{C}$ by

$$\tilde{\sigma}(\Theta, \chi) \equiv \langle \Theta, \mathcal{A}^*\chi \rangle_{\mathcal{H}}$$

for all $\Theta \in \mathcal{H}$, $\chi \in \mathcal{W}$. Due to the equivalence of the $\mathcal{W}$ norm and the graph norm corresponding to $\mathcal{A}^*$, it follows that

$$|\tilde{\sigma}(\Theta, \chi)| \le |\Theta|_{\mathcal{H}} |\mathcal{A}^*\chi|_{\mathcal{H}} \le c|\Theta|_{\mathcal{H}} |\chi|_{W} .$$

This then implies that for each $\Theta \in \mathcal{H}$, the mapping $\chi \mapsto \tilde{\sigma}(\Theta, \chi)$ is in $\mathcal{W}^*$. Hence we can define $\tilde{\mathcal{A}}\Theta \in \mathcal{W}^*$ by $\left(\tilde{\mathcal{A}}\Theta\right)(\chi) = \tilde{\sigma}(\Theta, \chi)$ or equivalently,

$$\left(\tilde{\mathcal{A}}\Theta\right)(\chi) = \langle \Theta, \mathcal{A}^*\chi \rangle_{\mathcal{H}}$$

for all $\Theta \in \mathcal{H}$, $\chi \in \mathcal{W}$. From this definition, it follows that $\tilde{\mathcal{A}} \in \mathcal{L}(\mathcal{H}, \mathcal{W}^*)$ with dom $\tilde{\mathcal{A}} = \mathcal{H}$. Moreover, if $\Theta \in$ dom $\mathcal{A}$, it follows from the Riesz representation theorem that

$$\left(\tilde{\mathcal{A}}\Theta\right)(\chi) = \langle\Theta, \mathcal{A}^*\chi\rangle_{\mathcal{H}} = \langle\mathcal{A}\Theta, \chi\rangle_{\mathcal{H}} = (\mathcal{A}\Theta)(\chi)$$

for all $\chi \in$ dom $\mathcal{A}^*$. Thus we see that $\tilde{\mathcal{A}}$ is an extension of the original operator $\mathcal{A}$ from dom $\mathcal{A} \subset \mathcal{H}$ to all of $\mathcal{H}$.

Having extended the operator to the full state space, our final objective in this subsection is to observe that $\tilde{\mathcal{A}}$ is the infinitesimal generator of a C_0-semigroup on $\mathcal{W}^*$. The proof of this result is given in [6], and we simply state the conclusion in the following theorem while directing the reader to that paper for details.

Theorem 2. The operator $\tilde{\mathcal{A}}$ is the infinitesimal generator of a C_0-semigroup $\tilde{\mathcal{S}}(t)$ on $\mathcal{W}^*$ which is an extension of $\mathcal{S}(t)$ from $\mathcal{H}$ to $\mathcal{W}^*$.

Having described how the original semigroup $\mathcal{S}(t)$ can be extended to a semigroup $\tilde{\mathcal{S}}(t)$ on $\mathcal{W}^* = [\text{dom}\,\mathcal{A}^*]^*$, we now want to show that the space $\{0\} \times V^*$ which contains the control and force input terms is itself contained in $\mathcal{W}^*$. It is tempting at this point to recall that we have the Gelfand triples $\mathcal{V} \hookrightarrow \mathcal{H} \simeq \mathcal{H}^* \hookrightarrow \mathcal{V}^*$ and $\mathcal{W} \hookrightarrow \mathcal{H} \simeq \mathcal{H}^* \hookrightarrow \mathcal{W}^*$ as well as the embedding $\mathcal{W} \subset \mathcal{V}$, and from this assume that the desired result is obvious. We point out, however, that while $\mathcal{W} \subset \mathcal{V}$, the embedding is not in general continuous which implies that $\mathcal{W} \hookrightarrow \mathcal{V}$ can not be assumed to be true (if $\mathcal{A}^*$ is V-elliptic for example, one can obtain the desired continuity of the embedding; however, this is not the case in this problem). Hence more care must be taken in order to argue that $\mathcal{W}^* \supset \{0\} \times V^*$

To this end, let $h = (0, \Lambda^*)$ be an arbitrary functional in $\{0\} \times V^*$. In order to show that $h \in \mathcal{W}^*$, we need to show that $\chi = (\Phi, \Psi) \mapsto h(\chi)$ is bounded on $\mathcal{W}$. Now, it follows that

$$|h(\chi)| = |\Lambda^*(\Psi)| \leq |\Lambda^*|_{V^*}|\Psi|_V \leq k|\Psi|_V \ . \tag{18}$$

Moreover, from the definition (16) of $\mathcal{A}^*$, we see that $\mathcal{A}^*\chi = (-\Psi, A_1^*\Phi - A_2^*\Psi)$ which implies that

$$|\Psi|_V \leq |\mathcal{A}^*\chi|_{\mathcal{H}} \leq |\mathcal{A}^*\chi|_{\mathcal{H}} + |\chi|_{\mathcal{H}} \leq c_2|\chi|_{\mathcal{W}} \ . \tag{19}$$

(The latter inequality results from the equivalence of the $\mathcal{W}$ norm and the graph norm corresponding to $\mathcal{A}^*$ as described in (17).) From (18) and (19) it follows that there exists a constant c such that

$$|h(\chi)| \leq c|\chi|_{\mathcal{W}}$$

therefore showing that $h \in \mathcal{W}^*$. We have thus established the following result which is essential in developing expression (14).

Lemma 1. Under the assumption of Theorem 1, we have $\{0\} \times V^* \subset \mathcal{W}^* = [\text{dom}\,\mathcal{A}^*]^*$.

In summary, we are now able to show the following. The system (11) is well-posed in the sense that (14) is a well-defined entity that can be taken as the mild solution to (11). Moreover, because (14) is well-defined, the usual theorems for continuous dependence with respect to initial data and nonhomogeneous terms (control and external forces) follow immediately. This facilitates the discussion of approximation and control ideas for these problems (see for example, [3, 4]).

In the event that σ_2 of (10) is also V-elliptic, one can argue (see [5] for details) existence of a weak or variational solution to the system (10) which agrees with the mild solution obtained from (14). In the example of Section 3, the form σ_2 is *not* V-elliptic; nevertheless, one can turn to standard results from the theory of C_0-semigroups, [22, 26], to obtain existence and uniqueness of solutions for the system under consideration in this paper. For example, we have the following result.

Theorem 3. Consider the system represented by (10), (12) or (14) and suppose that the mappings $t \mapsto u(t)$ and $t \mapsto F(t)$ from $[0,T]$ to $\mathbb{R}^1$ and V^*, respectively, are Lipschitz continuous. Then for each $\mathcal{Y}_0 \in \mathcal{H} = \text{dom}\,\tilde{\mathcal{A}}$, we have that (12) taken with $\mathcal{Y}(0) = \mathcal{Y}_0$ has a unique strong solution given by (14).

This theorem is readily established by appealing to Corollary 2.11, p. 109 of [22], in the context of our arguments above. Letting $G(t) = (0, Bu(t) + F(t))^T$ and considering $\tilde{\mathcal{S}}(t)$ on the reflexive Banach space $X = \mathcal{W}^*$, we can argue that G is Lipschitz continuous under the hypotheses of the theorem. In particular, the inequalities (18) and (19) imply

$$|G(t)|_{\mathcal{W}^*} \leq c_2 |Bu(t) + F(t)|_{V^*} ,$$

so that

$$|G(t) - G(s)|_{\mathcal{W}^*} \leq c_2 \left\{ |B|_{\mathcal{L}(U,V^*)} \, |u(t) - u(s)| + |F(t) - F(s)|_{V^*} \right\} .$$

It follows that (14) provides the strong solution (i.e., differentiable a.e. in the $\mathcal{W}^*$ sense - see [22]) to (12) interpreted in the $\mathcal{W}^*$ sense.

5 Conclusion

In this paper, a modeling set of partial differential equations describing the dynamics of a structural acoustics problem was presented, and a mathematical framework amenable to the development of approximation schemes for forward simulations, parameter estimation, and the application of PDE-based feedback control strategies was developed. The structure consists of a thin cylindrical shell with hard caps at the ends, and this component of the system was modeled by the Donnell-Mushtari shell equations. A constitutive law postulating that stress is proportional to a linear combination of strain and strain rate was assumed which yields a Kelvin-Voigt type of damping in the shell. The structural dynamics were then coupled to the interior acoustic field through pressure and momentum balance conditions. Finally, this model includes the contributions due to the activation of piezoceramic patches which are bonded to the shell and ultimately will be used to control the interior acoustic pressure levels.

To accommodate the presence of the piezoceramic patches and differing material properties of these patches, the material parameters for the combined structure should be taken to be piecewise constants. This leads, however, to difficulties in the strong form of the system equations since these parameters are contained in moment and force resultants which are differentiated when forming the equations of motion for the shell. Moreover, the discontinuities introduced by the patches lead to an unbounded control operator since it involves derivatives of the Heaviside function and Dirac delta. To avoid these problems, the weak form of the system equations was also presented.

In the weak form, the derivatives appear on the test functions instead of on the moments, thus eliminating the difficulties associated with the discontinuous parameters and patch inputs. The weak form is also advantageous for many approximation schemes since it reduces the smoothness requirements for the basis elements. Finally, when the weak form is posed in the context of sesquilinear forms, convergence and well-posedness issues can be considered.

The first step in proving the well-posedness of the system model was to verify that the first order system operator $\mathcal{A} : \text{dom}\,\mathcal{A} \subset \mathcal{H} \to \mathcal{H}$ generated a C_0-semigroup $\mathcal{S}(t)$ on $\mathcal{H}$. This semigroup was then extended via the extrapolation techniques described by Haraux to a larger space $\mathcal{W}^* = [\text{dom}\,\mathcal{A}^*]^*$ so as to be compatible with the force and control inputs which lie in $\{0\} \times V^*$. Finally, the mild solution in terms of the extended semigroup was formulated and existence and uniqueness results were obtained. Hence the framework and model presented here is mathematically well-posed as well as amenable to the development of computational strategies.

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A fully coupled mathematical model describing the interactions between a vibrating thin cylindrical shell and an enclosed acoustic field is presented. Because the model will ultimately be used in control applications involving piezoceramic actuators, the loads and material contributions resulting from piezoceramic patches bonded to the shell are included in the discussion. Theoretical and computational issues lead to the consideration of a weak form of the modeling set of partial differential equations (PDE's) and through the use a semigroup formulation, well-posedness results for the system model are obtained.

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