POLAR DECOMPOSITION FOR ATTITUDE DETERMINATION FROM VECTOR OBSERVATIONS

by

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Abstract

This work treats the problem of weighted least squares fitting of a 3D Euclidean-coordinate transformation matrix to a set of unit vectors measured in the reference and transformed coordinates. A closed-form analytic solution to the problem is re-derived. The fact that the solution is the closest orthogonal matrix to some matrix defined on the measured vectors and their weights is clearly demonstrated. Several known algorithms for computing the analytic closed form solution are considered. An algorithm is discussed which is based on the polar decomposition of matrices into the closest unitary matrix to the decomposed matrix and a Hermitian matrix. A somewhat longer improved algorithm is suggested too. A comparison of several algorithms is carried out using simulated data as well as real data from the Upper Atmosphere Research Satellite. The comparison is based on accuracy and time consumption. It is concluded that the algorithms based on polar decomposition yield a simple although somewhat less accurate solution. The precision of the latter algorithms increase with the number of the measured vectors and with the accuracy of their measurement.

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1. INTRODUCTION

The problem of attitude determination from vector observations is as follows. A sequence, $\mathbf{b}_i$, i=0,1,2,...K of unit vectors is given. These unit vectors are the result of measurements performed in vehicle body axes of the directions to known objects. The sequence, $\mathbf{r}_i$, i=0,1,2,...K of unit vectors, is the sequence of the corresponding representation of these directions with respect to some reference coordinate system. We wish to find the attitude matrix, A, such that the cost functional p(A) defined as follows

$$p(A) = \frac{1}{2} \sum_{i=1}^{K} a_i ||\mathbf{b}_i - A\mathbf{r}_i||^2 \tag{1}$$

is minimized. This problem, which is basically a least-squares fit problem for the attitude matrix, A, was posed in [1] and is generally known as Wahba's problem. This problem has been treated extensively [see, e.g. 2-11].

In the next section we derive an analytic solution to Wahba's problem, then in Section III we show, in a rather simple way, that this solution is actually the closest orthogonal matrix to a matrix defined on the reference and measured unit vectors $\mathbf{r}_i$ and $\mathbf{b}_i$ respectively, and on their relative weight. Several algorithms for computing the attitude matrix are considered in that section. The connection between polar decomposition of matrices and the solution to Wahba's problem is then discussed in Section IV. Two algorithms for computing the solution, which are based on the polar decomposition, are considered. A numerical comparison between these algorithms and other suggested ones, using simulated as well as real satellite data, is presented in Section V. The conclusions of this work are finally presented in Section VI.

II. DIRECT SOLUTION OF WAHBA'S PROBLEM

Since only the relative value of the weights, a_i, matter, we may, with no loss of generality, normalize the weights to give

$$\sum_{i=1}^{K} a_i = 1$$

It can be shown [2] that

$$p(A) = 1 - \mathrm{tr}(AB^T) \tag{2}$$

where tr denotes the trace of a matrix and

$$B = \sum_{i=1}^{K} a_i \mathbf{b}_i \mathbf{r}_i^T \tag{3}$$

We seek the orthogonal matrix, A, which minimizes p(A). Obviously, that matrix maximizes $\mathrm{tr}(AB^T)$. Using the method of Lagrange multipliers, we can incorporate the orthogonality constraint on A into the maximization problem of $\mathrm{tr}(AB^T)$ by defining the new functional p*(A)

$$p^*(A) = tr(BA^T) + tr[\tfrac{1}{2}L(AA^T - I)] \tag{4}$$

where I is the 3x3 identity matrix. The matrix L is a matrix of Lagrange multipliers scaled to enable the inclusion of the one half factor which is added for simplicity of the ensuing derivation. Also note that with no loss of generality, we may choose L to be symmetric. The new cost function, p*(A), can be written as follows

$$p^*(A) = tr[(BA^T) + \frac{1}{2} L(AA^T - I)] \tag{5}$$

Use now the directional derivative to maximize p*(A). To accomplish this, express A as follows

$$A = A_o + eH \tag{6}$$

where A_o is the A matrix which maximizes p*(A), e is a scalar variable, and H is any 3x3 real matrix. Note that A in (6) is expressed as a sum of the maximizing matrix, A_o, and a "step", e, in the "direction" of H. Also note that any real 3x3 matrix can be expressed in this way. Substitution of (6) into (5) gives

$$p'(e) = tr\{B(A_o + eH)^T + \frac{1}{2} L[(A_o + eH)(A_o + eH)^T - I]\} \tag{7}$$

Next differentiate p'(e) with respect to e to obtain

$$\frac{dp'(e)}{de} = tr[BH^T + \frac{1}{2} LH(A_o^T + eH^T) + \frac{1}{2} L(A_o + eH)H^T] \tag{8}$$

A necessary condition for p'(e) to have a maximum at A_o is

$$\left.\frac{dp'(e)}{de}\right|_{e=0} = 0 \qquad \text{for all H} \tag{9}$$

Applying (9) to (8) yields

$$tr[(B + LA_o)H^T] = 0 \qquad \text{for all H} \tag{10}$$

The latter can exist only if

$$B + LA_o = 0$$

or, assuming L is non-singular,

$$A_o = L^{-1}B \tag{11}$$

Using (11) in the orthogonality constraint on A_o and making use of the fact that L is symmetric we obtain

$$A_oA_o^T = L^{-1}BB^TL^{-1} = I$$

which yields

$$BB^T = L^2$$

The matrix BB^T is a positive definite matrix thus it can be decomposed as follows

$$BB^T = V \operatorname{diag}\{\beta_1^2, \beta_2^2, \beta_3^2\}\ V^T$$

where $\operatorname{diag}\{\beta_1^2, \beta_2^2, \beta_3^2\}$ is a diagonal matrix whose elements are β_1^2, β_2^2 and β_3^2. Consequently

$$L = \pm(BB^T)^{1/2} = V \operatorname{diag}\{\pm\beta_1,\ \pm\beta_2,\ \pm\beta_3\}\ V^T \tag{12}$$

Substitution of (12) into (11) yields

$$A_o = \mp(BB^T)^{-1/2}B = V \operatorname{diag}\{\pm\beta_1,\ \pm\beta_2,\ \pm\beta_3\}\ V^T\ B \tag{13}$$

It can be verified that to obtain maximum of p*(A) we need to choose the plus signs in (13). We designate it by choosing the plus sign in front of $(BB^1)^{1/2}$; that is

$$\boxed{A_o = (BB^T)^{-1/2}B} \tag{14}$$

which is the sought solution of Wahba's problem.

The expression given in (14) is also the solution of another problem as discussed next.

III. THE CLOSEST ORTHOGONAL MATRIX

Consider the following problem. Given a real matrix, B, what is the closest (in the Euclidean-norm sense) orthogonal matrix to it? To solve this problem denote the square of the Euclidean norm of the difference between B and any same order real matrix, A, by s(A); that is

$$s(A) = ||B - A||^2$$

(where $||.||$ denotes the Euclidean-norm) and find the 3x3 orthogonal matrix, A, which minimizes s(A). It can be easily shown that

$$s(A) = tr[(B - A)(B - A)^T]$$

thus

$$s(A) = tr(BB^T - BA^T - AB^T + AA^T)$$

Using the fact that A has to be orthogonal and the properties of the trace operation it can be easily shown that

$$s(A) = tr(BB^T) + 3 - 2tr(AB^T) \tag{15}$$

Obviously, that A which minimizes s(A) is the A which maximizes the term $tr(AB^T)$. An inspection of (2) reveals that this particular A is also the solution to Wahba's problem. This result can be stated as follows. *The closest orthogonal matrix to B, where B is as defined in (3), is the solution to Wahba's problem.* Indeed if we proceed with finding that orthogonal A which minimizes (15), we will obtain the result given in (14); namely,

$$A_o = (BB^T)^{-1/2}B \tag{16}$$

Consequently, any solution to the closest orthogonal matrix problem is also a solution to Wahba's problem. This conclusion will be exploited in the ensuing.

The solution expressed in (16) to the closest orthogonal matrix problem was obtained and investigated quite extensively in the past [12 - 19]. The solution of A_o using (16) is cumbersome. Various iterative solutions have been investigated [15 - 19].

Another solution to the closest orthogonal matrix problem, and hence to Wahba's problem, makes use of the singular value decomposition (SVD) of A_o. This solution is presented next. It is well known [20] that any matrix, and therefore also B, can be decomposed as follows

$$B = USV^T$$

where U and V are 3x3 orthogonal matrices and S is a diagonal matrix whose elements are the nonnegative square roots of the eigenvalues of B^TB. It can be shown that

$$A_o = UV^T$$

The latter was used in [21] to solve Wahba's problem.

IV. POLAR DECOMPOSITION

It is well known [22] that B can be decomposed as follows

$$B = PH \tag{17}$$

where P is orthogonal and H is symmetric. This decomposition is known as polar decomposition (PD). It was shown [23] that P is precisely the orthogonal matrix closest to B; that is, P of the polar decomposition is the solution to Wahba's problem when B is as defined in (3). We can write therefore

$$B = A_oH$$

(where A_o is, of course, the closest orthogonal matrix to B). This yields

$$A_o = BH^{-1} \tag{18}$$

We wish now to utilize the PD concept for solving Wahba's problem. We consider two cases as follows.

IV. 1: The Error Free Case

Assume now that both sequences of vectors $\mathbf{b}_i$ and $\mathbf{r}_i$ i=1,2,3,...K are error free. We can then write $\mathbf{b}_i=A\mathbf{r}_i$. Substitution of this equation into (3) yields

$$B = \sum_{i=1}^{K} a_i \mathbf{b}_i \mathbf{r}_i^T = \sum_{i=1}^{K} a_i A\mathbf{r}_i \mathbf{r}_i^T = A \sum_{i=1}^{K} a_i \mathbf{r}_i \mathbf{r}_i^T \tag{19}$$

Define now the matrix R as follows

$$R = \sum_{i=1}^{K} a_i \mathbf{r}_i \mathbf{r}_i^T \tag{20}$$

then from (19) we obtain

$$B = AR \tag{21}$$

where R is a symmetric matrix. Comparing (21) with (17) it is easy to see that in this case (21) is the PD of B where A=P and R=H. It is clear then that A_o=A. In this case A_o can be found as follows

$$A_o = BR^{-1} \tag{22}$$

provided that in constructing R, according to (20), we use at least 3 non-collinear vectors. (This assures that R is invertible.)

IV. 2: The Actual Case

In practice the vectors $\mathbf{b}_i$ are contaminated by measurement noise. However, since the position of the body and the time of measurement are known within a high degree of precision, the error in the determination of the $\mathbf{r}_i$ vectors is negligible. Denote the error in $\mathbf{b}_i$ by $\mathbf{n}_i$ then we can write that

$$\mathbf{b}_i = \mathbf{n}_i + A\mathbf{r}_i$$

Using the last equation in (3) we obtain

$$B = \sum_{i=1}^{K} a_i(\mathbf{n}_i + A\mathbf{r}_i)\mathbf{r}_i^T$$

This can be written as

$$B = \sum_{i=1}^{K} a_i \mathbf{n}_i \mathbf{r}_i^T + \sum_{i=1}^{K} a_i A\mathbf{r}_i \mathbf{r}_i^T$$

which yields

$$B - \sum_{i=1}^{K} a_i \mathbf{n}_i \mathbf{r}_i^T = A \sum_{i=1}^{K} a_i \mathbf{r}_i \mathbf{r}_i^T$$

Using (20) we obtain from the last equation

$$A = \left[B - \sum_{i=1}^{K} a_i \mathbf{n}_i \mathbf{r}_i^T \right] R^{-1} \tag{23}$$

We can now use the last equation to obtain the "best" estimate of A. We note that B contains all measured information, therefore we compute $\hat{A}$, the "best" estimate of A, as the conditional expectation of A given B [24]. Performing the conditional expectation on both sides of (23) yields

$$E\{A/B\} = \left[B - \sum_{i=1}^{K} a_i E\{\mathbf{n}_i/B\} \mathbf{r}_i^T \right] R^{-1} \tag{24}$$

It is assumed that the measurement errors are unbiased, therefore

$$E\{\mathbf{n}_i/B\} = 0 \tag{25}$$

(The latter assumption is based on the premise that the measurement biases have been removed or else are very small. If this is not the case, there is no way to obtain the correct attitude from the biased measured vector no matter what algorithm is used.) Substitution of (25) into (24) yields

$$E\{A/B\} = BR^{-1}$$

thus

$$\boxed{\hat{A} = BR^{-1}} \tag{26}$$

where B and R are computed according to (3) and (20) respectively.

Note that this result was first obtained by Brock [13, eq. (5)] in a way unrelated to the notion of polar decomposition and with no consideration of the randomness of **n**.

If $\mathbf{n}_i$ are very small or the number of measurements is large such that the particular realization of $\mathbf{n}_i$ has a negligible mean, which complies with the assumption in (25), then the computation of $\hat{A}$ according to (26) yields an accurate estimate of A. When this is not the case, the estimate can be quite erroneous. It is interesting to note that when K<4, $\hat{A}$ zeros the cost function of Wahba's problem which is given in (1) as follows

$$p(A) = \frac{1}{2} \sum_{i=1}^{K} a_i || \mathbf{b}_i - A\mathbf{r}_i ||^2$$

even if A is not equal to $\hat{A}$. This is a result of the approximation $\mathbf{b}_i = A\mathbf{r}_i$ which was made in the derivation of $\hat{A}$. However, while $\hat{A}$ drives p(A) to its minimal value, $\hat{A}$ is not necessarily orthogonal. (Recall that we seek the *orthogonal* matrix which minimizes p(A)). We can correct the non-orthogonality of $\hat{A}$ by the application of one orthogonalization iteration as follows [17, 18]

$$\hat{A}' = 0.5(\hat{A}^{-T} + \hat{A}) \tag{27}$$

This operation yields a close to orthogonal matrix, $\hat{A}'$, which is usually also closer to A. (The superscript -T denotes the inverse of the transpose.) We can, of course, bypass the computation of $\hat{A}$ by using (26) in (27) to obtain

$$\boxed{\hat{A}' = 0.5(B^{-T}R + BR^{-1})} \tag{28}$$

V. NUMERICAL COMPARISON

Five possible solutions to Wahba's problem are considered as follows.

(1) **QUEST**

Use the algorithm QUEST [6] to obtain, **q**, the quaternion which corresponds to the solution matrix of Wahba's problem, and then use **q** to compute the solution matrix itself which we denote by A_{qst}.

(2) **ITERATIVE ALGORITHM (IA)**

Apply the iterative orthogonalization algorithm [17, 18] starting with the computation of B according to (3) and then continue with

$$\hat{A}_o = B \tag{29.a}$$

$$\hat{A}_{j+1} = 0.5(\hat{A}_j^{-T} + \hat{A}_j) \tag{29.b}$$

which converges to the solution of Wahba's problem given in (13). We denote the final matrix by A_{itr}.

(3) **SINGULAR VALUE DECOMPOSITION (SVD)**

Apply the SVD algorithm to decompose B into

$$B = USV^T \tag{30.a}$$

and compute

$$A_{svd} = UV^T \tag{30.b}$$

As explained in Section III, A_{svd} too is the solution of Wahba's problem.

(4) **FAST OPTIMAL MATRIX ALGORITHM (FOAM)**

Use the FOAM algorithm [25] to obtain the solution matrix to Wahba's problem. We denote the computed solution by A_{fom}.

(5) **POLAR DECOMPOSITION (PD)**

Compute the matrices B and R, the latter according to (20), and then calculate the *estimate* of the solution to Wahba's problem according to (26)

$$\hat{A} = BR^{-1} \quad (26)$$

(6) IMPROVED POLAR DECOMPOSITION (IPD)

Compute an *improved estimate* of the solution to Wahba's problem by performing one orthogonalization iteration on the preceding estimate. The overall algorithm is as in (28)

$$\hat{A}' = 0.5(B^{-T}R + BR^{-1}) \quad (28)$$

V.1 Results with Simulated Data

The five algorithms were tested with simulated data. The importance of tests with simulated data stems from the fact that using real data we do not know the correct attitude. This constitutes a major difficulty since the difference between algorithms may be smaller than the difference between the correct attitude and the computed ones. Only when we use simulated data can we observe the difference between the computed attitude and the correct one. The simulated measurements of vectors in body axes were obtained by transforming the reference, $\mathbf{r}_i$, vectors to body axes using A, the correct attitude matrix, addition of a noise component to each component of the transformed vector and normalization of the resultant vectors. The added noise components had a zero mean and a standard deviation value of 0.144. Typical simulation results are shown next for four and three measured vectors. Three cost values were computed in order to evaluate the accuracy of the results. The cost p is Wahba's cost function computed according to (3) for the particular solution matrix. The cost f is the Euclidean norm of the difference between the particular solution matrix and the correct attitude matrix. Finally, the cost J is a measure of the non-orthogonality of the solution matrix. It is the Euclidean norm of the matrix XX^T-I where X is the particular solution matrix.

V.1.1 Four reference vectors

$$\mathbf{r}_1 = \begin{bmatrix} .267261 \\ .534522 \\ .801784 \end{bmatrix} \quad \mathbf{r}_2 = \begin{bmatrix} -.666667 \\ -.666667 \\ -.333333 \end{bmatrix} \quad \mathbf{r}_3 = \begin{bmatrix} .267261 \\ -.801784 \\ .534522 \end{bmatrix} \quad \mathbf{r}_4 = \begin{bmatrix} -.447214 \\ .894427 \\ .000000 \end{bmatrix}$$

Four "measured" body vectors

$$\mathbf{b}_1 = \begin{bmatrix} .815399 \\ .577901 \\ -.033975 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} -.872214 \\ -.075280 \\ .483296 \end{bmatrix} \quad \mathbf{b}_3 = \begin{bmatrix} .290203 \\ -.206009 \\ .934528 \end{bmatrix} \quad \mathbf{b}_4 = \begin{bmatrix} -.118959 \\ .679197 \\ -.706940 \end{bmatrix}$$

Four weights

$a_1 = .100000 \qquad a_2 = .300000 \qquad a_3 = .400000 \qquad a_4 = .200000$

The correct attitude matrix

$$A = \begin{bmatrix} .764744 & .293558 & .573576 \\ -.636031 & .486370 & .599090 \\ -.103103 & -.822963 & .558660 \end{bmatrix}$$

Solutions

$$A_{qst} = \begin{bmatrix} .770135 & .274589 & .575754 \\ -.629265 & .474889 & .615228 \\ -.104485 & -.836111 & .538518 \end{bmatrix} \qquad \begin{aligned} p_{qst} &= .44078E\text{-}03 \\ f_{qst} &= .37578E\text{-}01 \\ J_{qst} &= .20588E\text{-}06 \end{aligned}$$

$$A_{itr} = \begin{bmatrix} .761290 & .266299 & .591204 \\ -.639697 & .457436 & .617689 \\ -.105948 & -.848432 & .518593 \end{bmatrix} \qquad \begin{aligned} p_{itr} &= .22933E\text{-}03 \\ f_{itr} &= .67264E\text{-}01 \\ J_{itr} &= .19037E\text{-}06 \end{aligned}$$

$$A_{svd} = \begin{bmatrix} .761290 & .266299 & .591204 \\ -.639697 & .457436 & .617689 \\ -.105948 & -.848432 & .518593 \end{bmatrix} \qquad \begin{aligned} p_{svd} &= .22933E\text{-}03 \\ f_{svd} &= .67264E\text{-}01 \\ J_{svd} &= .18014E\text{-}15 \end{aligned}$$

$$A_{fom} = \begin{bmatrix} .770135 & .274589 & .575754 \\ -.629265 & .474889 & .615228 \\ -.104485 & -.836111 & .538518 \end{bmatrix} \qquad \begin{aligned} p_{fom} &= .44078E\text{-}03 \\ f_{fom} &= .37578E\text{-}01 \\ J_{fom} &= .86667E\text{-}16 \end{aligned}$$

$$\hat{A} = \begin{bmatrix} .770556 & .263174 & .561689 \\ -.654729 & .455528 & .628148 \\ -.143061 & -.851596 & .551271 \end{bmatrix} \qquad \begin{aligned} p &= .67846E\text{-}04 \\ f &= .75595E\text{-}01 \\ J &= .11190E\text{+}00 \end{aligned}$$

$$\hat{A}' = \begin{bmatrix} .768038 & .263582 & .584080 \\ -.630931 & .473739 & .615274 \\ -.115625 & -.840565 & .530461 \end{bmatrix} \qquad \begin{aligned} p' &= .33350E\text{-}03 \\ f' &= .52240E\text{-}01 \\ J' &= .25319E\text{-}02 \end{aligned}$$

V.1.2 Three reference vectors

$$\mathbf{r}_1 = \begin{bmatrix} .267261 \\ .534522 \\ .801784 \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} -.666667 \\ -.666667 \\ -.333333 \end{bmatrix} \qquad \mathbf{r}_3 = \begin{bmatrix} .267261 \\ -.801784 \\ .534522 \end{bmatrix}$$

Three "measured" body vectors

$$\mathbf{b}_1 = \begin{bmatrix} .815399 \\ .577901 \\ -.033975 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} -.872214 \\ -.075280 \\ .483296 \end{bmatrix} \qquad \mathbf{b}_3 = \begin{bmatrix} .290203 \\ -.206009 \\ .934528 \end{bmatrix}$$

Three weights

$$a_1 = .125000 \qquad a_2 = .375000 \qquad a_3 = .500000$$

The correct attitude matrix

$$A = \begin{bmatrix} .764744 & .293558 & .573576 \\ -.636031 & .486370 & .599090 \\ -.103103 & -.822963 & .558660 \end{bmatrix}$$

Solutions

$$A_{qst} = \begin{bmatrix} .767731 & .276451 & .578069 \\ -.631488 & .479442 & .609392 \\ -.108683 & -.832893 & .542658 \end{bmatrix} \qquad \begin{aligned} p_{qst} &= .55884E\text{-}03 \\ f_{qst} &= .29705E\text{-}01 \\ J_{qst} &= .67617E\text{-}07 \end{aligned}$$

$$A_{itr} = \begin{bmatrix} .758264 & .271018 & .592946 \\ -.643834 & .454336 & .615676 \\ -.102537 & -.848604 & .518997 \end{bmatrix} \qquad \begin{aligned} p_{itr} &= .23600E\text{-}03 \\ f_{itr} &= .67219E\text{-}01 \\ J_{itr} &= .32845E\text{-}06 \end{aligned}$$

$$A_{svd} = \begin{bmatrix} .758264 & .271018 & .592946 \\ -.643834 & .454336 & .615676 \\ -.102537 & -.848604 & .518997 \end{bmatrix} \qquad \begin{aligned} p_{svd} &= .23600E\text{-}03 \\ f_{svd} &= .67219E\text{-}01 \\ J_{svd} &= .11102E\text{-}15 \end{aligned}$$

$$A_{fom} = \begin{bmatrix} .767731 & .276451 & .578069 \\ -.631488 & .479442 & .609392 \\ -.108683 & -.832893 & .542658 \end{bmatrix} \qquad \begin{aligned} p_{fom} &= .55884E\text{-}03 \\ f_{fom} &= .29705E\text{-}01 \\ J_{fom} &= .30626E\text{-}15 \end{aligned}$$

$$\hat{A} = \begin{bmatrix} .739265 & .275664 & .586784 \\ -.664499 & .459428 & .635984 \\ -.172692 & -.839769 & .575035 \end{bmatrix} \qquad \begin{aligned} p &= .11783E\text{-}14 \\ f &= .97131E\text{-}01 \\ J &= .16640E{+}00 \end{aligned}$$

$$\hat{A}' = \begin{bmatrix} .753716 & .268839 & .600058 \\ -.645610 & .483007 & .593789 \\ -.131702 & -.833708 & .539069 \end{bmatrix} \qquad \begin{aligned} p' &= .60457E\text{-}03 \\ f' &= .53687E\text{-}01 \\ J' &= .53638E\text{-}02 \end{aligned}$$

We observe that, as expected, A_{itr} and A_{svd} are practically identical. We also observe that as expected, for three measured vectors (K=3) Wahba's cost, p, for $\hat{A}$ is practically zero. The single normalization cycle which generates $\hat{A}'$ improves the orthogonality (reduces J) considerably. This comes at the expense of an increase in p. For four measured vectors (K=4), p for $\hat{A}$ is similar in value to that of the other algorithms, and again, the single normalization cycle improves orthogonality considerably at the expense of Wahba's cost.

V.2 Results with UARS Data

The following are results of the application of the five algorithms to data measured on-board the Upper Atmosphere Research Satellite (UARS). UARS was deployed on September 15, 1991 at 04:23 GMT by the shuttle spacecraft Discovery which was launched on September 12, 1991, at 23:12 GMT. The data were measured on September 30, 1991 at 18:32:31.206749916 GMT. The first vector corresponds to the Sun Sensor, the second to the triad of Magnetometers, and the third to the Infra-Red Horizon Sensor.

The reference vectors

$$\mathbf{r}_1 = \begin{bmatrix} -.992324 \\ -.113458 \\ -.049192 \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} -.814177 \\ .550862 \\ -.183487 \end{bmatrix} \qquad \mathbf{r}_3 = \begin{bmatrix} .543295 \\ -.542620 \\ .640619 \end{bmatrix}$$

The measured body vectors

$$\mathbf{b}_1 = \begin{bmatrix} -.810765 \\ -.294952 \\ -.411403 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} -.455867 \\ .186491 \\ -.870291 \end{bmatrix} \qquad \mathbf{b}_3 = \begin{bmatrix} .002528 \\ .003031 \\ .999992 \end{bmatrix}$$

Three weights

$$a_1 = .243291 \qquad a_2 = .002506 \qquad a_3 = .754203$$

Solutions

$$A_{qst} = \begin{bmatrix} .826549 & .178850 & -.533694 \\ .178119 & .816336 & .549426 \\ .533939 & -.549189 & .642885 \end{bmatrix} \qquad p_{qst} = .96423\text{E-}03 \qquad J_{qst} = .20183\text{E-}06$$

$$A_{itr} = \begin{bmatrix} .832537 & .172669 & -.526372 \\ .180280 & .814010 & .552166 \\ .523814 & -.554593 & .646564 \end{bmatrix} \qquad p_{itr} = .89246\text{E-}03 \qquad J_{itr} = .15599\text{E-}07$$

$$A_{svd} = \begin{bmatrix} .832537 & .172669 & -.526372 \\ .180280 & .814010 & .552166 \\ .523814 & -.554593 & .646564 \end{bmatrix} \qquad p_{svd} = .89246\text{E-}03 \qquad J_{svd} = .19700\text{E-}15$$

$$A_{fom} = \begin{bmatrix} .826549 & .178850 & -.533694 \\ .178119 & .816336 & .549426 \\ .533939 & -.549189 & .642885 \end{bmatrix} \qquad p_{fom} = .96423\text{E-}03 \qquad J_{fom} = .46516\text{E-}15$$

$$\hat{A} = \begin{bmatrix} .818163 & .211577 & -.510709 \\ .182985 & .778550 & .508996 \\ .466235 & -.700019 & .572641 \end{bmatrix} \qquad p = .13611E-12 \qquad J = .28575E+00$$

$$\hat{A}' = \begin{bmatrix} .837978 & .176650 & .517931 \\ .217866 & .767101 & .613477 \\ .503300 & -.622646 & .606208 \end{bmatrix} \qquad p' = .39206E-02 \qquad J' = .16305E-01$$

Here too we observe the identity between A_{itr} and A_{svd}. As before, we also observe the reduction in J at the expense of an increase in p when a single orthogonalization cycle is applied to $\hat{A}$ to generate $\hat{A}'$.

V.3 Time Consumption Analysis

A computation-time measurement was performed on all five algorithms using the simulated three and four measured vectors. The runs were made on a VAX 9210 computer employing the VMS Version 5.4-2 operating system. The time measurement routine used the internal machine clock at a resolution of 10 msec. In order to increase the resolution, the runs were performed over 50000 successive solutions and the total time was then divided by 50000. The results are presented in Table I.

Table I: Algorithm Computation Time (msec).

	A_{qst}	A_{itr}	A_{svd}	A_{fom}	$\hat{A}$	$\hat{A}'$
Three measured vectors	0.0890	0.790	0.548	0.060	0.058	0.084
Four measured vectors	0.1060	0.694	0.526	0.070	0.068	0.094

Note the decrease in computation time of A_{itr} when the number of measured vectors increased from 3 to 4. This is due to the fact that in the four vector case the convergence criterion was met after only 7 iterations whereas in the 3 vector case 8 iterations were performed until the same convergence criterion was met. In all our tests it was found that when a fourth measured vector was added, less iterations were required. This stemmed from the fact that when a fourth measured vector is added the orthogonality of B increases provided the fourth vector is not a linear combination of the other three. The decrease of the computation time of A_{svd} with the increase of the number of measured vectors is not consistent. It depends on the number of iterations needed for the completion of the SVD calculations.

VI. CONCLUSIONS

It was shown that the solution to Wahba's problem is the closest orthogonal matrix to B where B is defined on the measured vectors and on weights associated with their measurements. The weights signify the confidence assigned to the measurements. The matrix B includes all the information contained in the measurement.

Once it was established that the sought solution is the orthogonal matrix closest to B, algorithms for computing that orthogonal matrix were considered, and an algorithm was discussed which is based on the polar decomposition of matrices into the closest unitary (in our case: orthogonal) matrix and a Hermitian (in our case symmetric) matrix. The accuracy of the algorithm increases with the accuracy of the measurements and with their unbiasedness. If the measurements are error free the algorithm yields the exact solution.

When only three measured vectors are used the new algorithm yields a solution which zeros Wahba's cost; however, the solution is not necessarily orthogonal. An application of one orthogonalization iteration to the solution matrix constitutes a modified algorithm which yields a better solution. Although the latter algorithm generates a matrix which increases Wahba's cost. The new matrix is closer to orthogonality. We note that the same iteration cycle if applied repeatedly to B itself, yields eventually the optimal solution as shown in the examples; however, since B is usually quite far from orthogonality, it takes several iterations to obtain the solution.

The advantage of the algorithm is in its simplicity which enables its use for obtaining first cut solutions using "back-of the envelop" like programs such as MathCAD. Another advantage of the first new algorithm is its ability to indicate the precision of the measurements. This stems from the fact that generally the closeness of the solution matrix to orthogonality is indicative of the precision of the measurements. It is interesting to note that the fact that the two PD algorithms yield the exact solution in the noise-free case is analogous to the fact that the largest eigenvalue of the 4x4 K matrix used in the QUEST algorithm is precisely 1 in the noise-free case.

The two PD algorithms were tested vis-a-vis other popular algorithms using simulated and real UARS data.

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