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# THE FINAL REPORT

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### ROBUST CONTROL WITH STRUCTURED PERTURBATIONS

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## 1. SUMMARY

This final report summarizes the recent results obtained by the principal investigator and his coworkers on the robust stability and control of systems containing parametric uncertainty. The starting point is a generalization of Kharitonov's theorem obtained by Chapellat and Bhattacharyya in 1989 and its generalization to the multilinear case, the singling out of extremal stability subsets and other ramifications now constitutes an extensive and coherent theory of robust parametric stability that is summarized in the results contained here.

## 2. INTRODUCTION

The stability of a linear time invariant continuous time feedback control system is characterized by the root locations of its characteristic polynomial  $\delta(s)$ ; for stability the polynomial  $\delta(s)$  must be Hurwitz i.e. have all its roots in the open left half of the complex plane. Since control systems operate under large uncertainties it is important to determine if stability is robust, that is, preserved under various perturbations. Despite its practical importance the subject of robust parametric stability lay dormant for about 100 years since the Routh-Hurwitz criterion was developed. The field was revived with the advent of Kharitonov's theorem which appeared in 1978. In 1989 Chapellat and Bhattacharyya generalized this result to make it applicable to control systems. This generalization has given rise to a great many useful and insightful results related to stability margin calculations, mixed parametric and unstructured uncertainty, nonlinear perturbations mixed with parametric perturbations, gain and phase margin optimization, development of Bode and Nyquist envelopes and classical design theory for robust systems etc [1,2,3,4,5,6,7,8]. This report gives a summary of these results without proofs. We expect these results to have an important influence on future developments in this field.

## 3. LINEAR INTERVAL SYSTEMS

We begin this section by describing the theorem of Kharitonov which deals with the family of real polynomials

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \dots + \delta_n s^n \quad (1)$$

with coefficient vector  $\underline{\delta} = [\delta_0, \delta_1 \dots \delta_n]$  lying in the box

$$\Delta = [x_0, y_0] \times [x_1, y_1] \times \dots \times [x_n, y_n]. \quad (2)$$

The Kharitonov polynomials associated with the above family of interval polynomials are defined as

$$\begin{aligned} K^1(s) &= x_0 + x_1 s + y_2 s^2 + y_3 s^3 + \dots \\ K^2(s) &= x_0 + y_1 s + y_2 s^2 + x_3 s^3 + \dots \\ K^3(s) &= y_0 + x_1 s + x_2 s^2 + y_3 s^3 + \dots \\ K^4(s) &= y_0 + \beta^1 s + x_2 s^2 + x_3 s^3 + \dots \end{aligned} \quad (3)$$

**Theorem 1. (Kharitonov's Theorem [9])** *The family  $\Delta$  contains only Hurwitz polynomials if and only if  $K^1(s), K^2(s), K^3(s)$  and  $K^4(s)$  are Hurwitz.*

This remarkable theorem unlocked the door leading to the development of a large number of interesting results in the area of real parametric uncertainty. However Kharitonov's theorem itself is of somewhat limited applicability in control problems. To explain this consider the control system shown below in Figure 1.

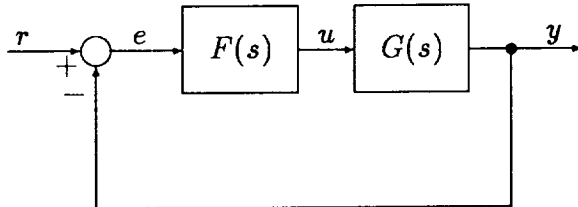


Figure 1. Feedback System

Let

$$F(s) := \frac{F_1(s)}{F_2(s)} \quad G(s) := \frac{N(s)}{D(s)} \quad (4)$$

and  $N(s), D(s)$  and  $F_i(s), i = 1, 2$  are polynomials. The characteristic polynomial of the above system is given by

$$\delta(s) = F_1(s)N(s) + F_2(s)D(s). \quad (5)$$

In (5) the  $F_i(s)$  may denote fixed polynomials corresponding to the controller and  $N(s), D(s)$  may be uncertain polynomials corresponding to the plant. Kharitonov's Theorem cannot deal with this situation because of the assumption implicit in it's statement that the coefficients of  $\delta(s)$  perturb independently. Motivated by these considerations Chapellat and Bhattacharyya [10] formulated and solved the problem of determining the Hurwitz stability of the family

$$\delta(s) = F_1(s)P_1(s) + F_2(s)P_2(s) \cdots \cdots + F_m(s)P_m(s) \quad (6)$$

where the polynomials  $P_i(s)$  are interval. This form of the characteristic polynomial (6) occurs in m-input (m-output) single output (input) systems. We shall refer to such families of polynomials as *linear interval systems*. The uncertain polynomials may of course correspond to the plant (perturbations) or to the controller (design parameters to be chosen from prescribed intervals). In the following sections we describe various recent results, obtained by the authors and their coworkers on the robust stability of the above types of control systems for both parametric as well as unstructured perturbations. In particular we show the importance of certain segments where the extremal values of various types of stability margins occur. These line segments capture the most important structural information for the analysis and design of robust control systems.

#### 4. THE EXTREMAL SEGMENTS

We first state the result of of Chapellat and Bhattacharyya [10], [11] for the special case (5).  $F(s)$  will be assumed to be fixed and  $G(s)$  to lie in an uncertainty set described as follows.

Write

$$\begin{aligned} N(s) &:= n_p s^p + n_{p-1} s^{p-1} + \cdots + n_1 s + n_0 \\ D(s) &:= d_q s^q + d_{q-1} s^{q-1} + \cdots + d_1 s + d_0 \end{aligned}$$

and let the coefficients lie in prescribed intervals

$$\begin{aligned} n_i &\in [n_i^-, n_i^+], \quad i \in \{0, 1, \dots, p\} := \underline{p} \\ d_i &\in [d_i^-, d_i^+], \quad i \in \{0, 1, \dots, q\} := \underline{q} \end{aligned}$$

Introduce the interval polynomial sets

$$\begin{aligned} \mathcal{N}(s) &:= \{N(s) = n_p s^p + n_{p-1} s^{p-1} + \cdots + n_0 : n_i \in [n_i^-, n_i^+], i \in \underline{p}\} \\ \mathcal{D}(s) &:= \{D(s) = d_q s^q + d_{q-1} s^{q-1} + \cdots + d_0 : d_i \in [d_i^-, d_i^+], i \in \underline{q}\} \end{aligned}$$

and the corresponding set of interval transfer functions (or interval systems)

$$\mathbf{G}(s) = \left\{ \frac{N(s)}{D(s)} : (N(s), D(s)) \in \mathcal{N}(s) \times \mathcal{D}(s) \right\}. \quad (7)$$

The four Kharitonov vertex polynomials associated with  $\mathcal{N}(s)$  are

$$\begin{aligned} K_n^1(s) &:= n_0^- + n_1^- s + n_2^+ s^2 + n_3^+ s^3 + n_4^- s^4 + n_5^- s^5 + \cdots \\ K_n^2(s) &:= n_0^- + n_1^+ s + n_2^+ s^2 + n_3^- s^3 + n_4^- s^4 + n_5^+ s^5 + \cdots \\ K_n^3(s) &:= n_0^+ + n_1^- s + n_2^- s^2 + n_3^+ s^3 + n_4^+ s^4 + n_5^- s^5 + \cdots \\ K_n^4(s) &:= n_0^+ + n_1^+ s + n_2^- s^2 + n_3^- s^3 + n_4^+ s^4 + n_5^+ s^5 + \cdots \end{aligned}$$

and we write

$$\mathcal{K}_{\mathcal{N}}(s) := \{K_n^1(s), K_n^2(s), K_n^3(s), K_n^4(s)\}. \quad (8)$$

Similarly the four Kharitonov polynomials associated with  $\mathcal{D}(s)$  are denoted  $K_d^i(s)$ ,  $i = 1, 2, 3, 4$  and

$$\mathcal{K}_{\mathcal{D}}(s) := \{K_d^1(s), K_d^2(s), K_d^3(s), K_d^4(s)\} \quad (9)$$

The four Kharitonov polynomial segments associated with  $\mathcal{N}(s)$  are defined as follows:

$$\mathcal{S}_{\mathcal{N}}(s) :=$$

$$[\lambda K_n^i(s) + (1 - \lambda) K_n^j(s) : \lambda \in [0, 1], (i, j) \in \{(1, 2), (1, 3), (2, 4), (3, 4)\}] \quad (10)$$

and the four Kharitonov polynomial segments associated with  $\mathcal{D}(s)$  are denoted

$$\mathcal{S}_{\mathcal{D}}(s) :=$$

$$[\mu K_d^i(s) + (1 - \mu) K_d^j(s) : \mu \in [0, 1], (i, j) \in \{(1, 2), (1, 3), (2, 4), (3, 4)\}] \quad (11)$$

Following Chapellat and Bhattacharyya [10], [11] we introduce the extremal subset as follows:

$$(\mathcal{N}(s) \times \mathcal{D}(s))_{\mathbf{E}} = \{(N(s), D(s)) : N(s) \in \mathcal{K}_{\mathcal{N}}(s), D(s) \in \mathcal{S}_{\mathcal{D}}(s) \\ \text{or } N(s) \in \mathcal{S}_{\mathcal{N}}(s), D(s) \in \mathcal{K}_{\mathcal{D}}(s)\}. \quad (12)$$

The extremal subset of the family of interval systems  $\mathbf{G}(s)$  is naturally defined as:

$$\mathbf{G}_{\mathbf{E}}(s) := \left\{ \frac{N(s)}{D(s)} : (N(s), D(s)) \in (\mathcal{N}(s) \times \mathcal{D}(s))_{\mathbf{E}} \right\}. \quad (13)$$

**Theorem 2.** *The control system of Figure 1 is stable for all  $G(s) \in \mathbf{G}(s)$  if and only if it is stable for all  $G(s) \in \mathbf{G}_{\mathbf{E}}(s)$ .*

We note that each element of  $\mathbf{G}_{\mathbf{E}}(s)$  is a one parameter family of transfer functions and there are at most 32 such distinct elements. The above Theorem therefore gives a constructive solution to the problem of checking robust stability by reducing it to a set of (at most) 32 root locus problems. An alternative way to check the stability of linear interval systems would be to use the Edge Theorem [12] which requires that the exposed edges of the polytope of uncertain polynomials in the coefficient space be checked for stability. While this result also leads to a set of line segments to be checked the number of segments (exposed edges) increases exponentially with the dimension of the uncertain parameter. In the theorem [10] the number of segments to be checked is independent of the dimension of the uncertainty set  $\mathcal{N}(s) \times \mathcal{D}(s)$ . More importantly perhaps these extremal segments enjoy many extremal properties that are critical in determining stability margins. We now give an example to illustrate the Theorem [10].

*Example 1.* Let us consider the following single input single output plant

$$G(s) = \frac{n(s)}{d(s)} = \frac{s^3 + \alpha s^2 - 2s + \beta}{s^4 + 2s^3 - s^2 + \gamma s + 1},$$

with

$$\alpha \in [-1, -2], \quad \beta \in [0.5, 1], \quad \gamma \in [0, 1].$$

There are two Kharitonov polynomials associated with  $n(s)$ , namely

$$0.5 - 2s - s^2 + s^3 \text{ and } 1 - 2s - 2s^2 + s^3$$

and also two Kharitonov polynomials associated with  $d(s)$

$$1 - s^2 + 2s^3 + s^4 \text{ and } 1 + s - s^2 + 2s^3 + s^4.$$

In order to check that a given controller  $F(s)$  stabilizes the entire family of plants, we only need to check that the controller stabilizes the following four plant segments making up the set  $\mathbf{G}_{\mathbf{E}}(s)$ :

$$\frac{0.5(1 + \lambda) - 2s - (1 + 2\lambda)s^2 + s^3}{1 - s^2 + 2s^3 + s^4} \quad \lambda \in [0, 1]$$

$$\frac{0.5(1 + \lambda) - 2s + (1 + 2\lambda)s^2 + s^3}{1 + s - s^2 + 2s^3 + s^4} \quad \lambda \in [0, 1]$$

$$\frac{0.5 - 2s - s^2 + s^3}{1 + \lambda s - s^2 + 2s^3 + s^4} \quad \lambda \in [0, 1]$$

$$\frac{1 - 2s - 2s^2 + s^3}{1 + \lambda s - s^2 + 2s^3 + s^4} \quad \lambda \in [0, 1].$$

On the other hand if one uses the Edge theorem, it is necessary to check the 12 plant segments corresponding to

$$\begin{aligned} \alpha = -2, \quad \beta = 0.5, \quad \gamma \in [0, 1] \\ \alpha = -2, \quad \beta = 1, \quad \gamma \in [0, 1] \\ \alpha = -1, \quad \beta = 0.5, \quad \gamma \in [0, 1] \\ \alpha = -1, \quad \beta = 1, \quad \gamma \in [0, 1] \end{aligned}$$

$$\begin{aligned} \alpha = -2, \quad \beta \in [0.5, 1], \quad \gamma = 0 \\ \alpha = -2, \quad \beta \in [0.5, 1], \quad \gamma = 1 \\ \alpha = -1, \quad \beta \in [0.5, 1], \quad \gamma = 0 \\ \alpha = -1, \quad \beta \in [0.5, 1], \quad \gamma = 1 \end{aligned}$$

$$\begin{aligned} \alpha \in [-2, -1], \quad \beta = 0.5, \quad \gamma = 0 \\ \alpha \in [-2, -1], \quad \beta = 0.5, \quad \gamma = 1 \\ \alpha \in [-2, -1], \quad \beta = 1, \quad \gamma = 0 \\ \alpha \in [-2, -1], \quad \beta = 1, \quad \gamma = 1. \end{aligned}$$

The problem of checking the stability of a line segment of polynomials was solved by Chapellat and Bhattacharyya in [13]; this result called the Segment Lemma is described next. The lemma basically checks for the occurrence of a  $j\omega$  root along a line segment of polynomials.

#### 4.1. Stability of Segments

Let  $\delta_1(s)$ , and  $\delta_2(s)$  be two polynomials of degree  $n$  and let

$$\delta_\lambda(s) = \lambda\delta_1(s) + (1 - \lambda)\delta_2(s).$$

Denote the segment

$$\{\delta_\lambda(s) : \lambda \in [0, 1]\} := [\delta_1(s), \delta_2(s)]. \quad (14)$$

Also write

$$\begin{aligned} \delta_1(j\omega) &= \delta_1^e(\omega) + j\omega\delta_1^o(\omega) \\ \delta_2(j\omega) &= \delta_2^e(\omega) + j\omega\delta_2^o(\omega). \end{aligned}$$

where  $\delta_i^e(\omega)$  and  $\delta_i^o(\omega)$  are real.



**Lemma 1. (Segment Lemma [13])** *Suppose that  $\delta_1(s)$  and  $\delta_2(s)$  are Hurwitz polynomials of degree  $n$  with positive coefficients. Then there exists an unstable polynomial on the line segment  $[\delta_1(s), \delta_2(s)]$  iff there exists  $\omega_0 > 0$ , such that*

$$\delta_1^e(\omega_0)\delta_2^o(\omega_0) - \delta_2^e(\omega_0)\delta_1^o(\omega_0) = 0$$

and

$$\begin{aligned}\delta_1^e(\omega_0)\delta_2^e(\omega_0) &\leq 0 \\ \delta_1^o(\omega_0)\delta_2^o(\omega_0) &\leq 0.\end{aligned}$$

The Segment Lemma completely solves the problem of checking the Hurwitz stability of a line segment of polynomials. The idea of checking for a root on the stability boundary can of course be extended to other stability regions besides the left half plane.

In general it is known that the stability of the endpoints (vertices) of a line segment does not guarantee that of the entire segment. It is therefore useful to know if there exist some simple additional conditions (simpler than the Segment Lemma) under which the Hurwitz stability of a segment could be guaranteed. It turns out that by restricting the form of the difference polynomial

$$\delta_0(s) = \delta_2(s) - \delta_1(s) \tag{15}$$

it is possible to conclude segment stability from vertex stability. There exist several known results on this problem. Peterson [14] derived the case when the difference polynomial is antiHurwitz (all roots in the closed right half plane), Chapellat and Bhattacharyya [10] dealt with the case when it is even or odd, Hollot and Yang [15] and Mansour and Kraus [16] proved the result for the difference polynomial being of the form  $s^t(as + b)P(s)$  where  $P(s)$  is even or odd. The general result given below encompasses all the above cases; the proof may be found in Bhattacharyya [17].

**Lemma 2. (Vertex Lemma) [17]**

a) *Let  $\delta_1(s)$  and  $\delta_2(s)$  be polynomials with positive coefficients and let*

$$\delta_0(s) = A(s)s^t(as + b)P(s)$$

*where  $A(s)$  is antiHurwitz  $t \geq 0$  is an integer,  $a, b$  are arbitrary real numbers, and  $P(s)$  is even or odd. Then stability of the segment  $[\delta_1(s), \delta_2(s)]$  is implied by that of the endpoints  $\delta_1(s), \delta_2(s)$ .*

b) *When  $\delta_0(s)$  is not of the form specified in a), stability of the endpoints is not sufficient to guarantee that of the segment.*

Using the above Lemma in conjunction with the linear case theorem [10] it is possible to show that if  $F_i(s)$  are of the same form as  $\delta_0(s)$  the stability of the extremal segments (and therefore the robust stability of a linear interval system) can be ascertained from the Kharitonov vertex polynomials of the system. To state this result let

$$\mathbf{G}_K(s) := \left\{ \frac{N(s)}{D(s)} : (N(s), D(s)) \in (\mathcal{K}_N(s) \times \mathcal{K}_D(s)) \right\}. \tag{16}$$

**Corollary 1.** (Theorem 2)[17],[11] *Under the conditions of Theorem 2, if*

$$F_i(s) = A_i(s)s^{t_i}(a_i s + b_i)Q_i(s), i = 1, 2$$

where  $A_i(s)$  is antiHurwitz,  $t_i$  is an arbitrary nonnegative integer,  $a_i, b_i$  are arbitrary real numbers and  $Q_i(s)$  is either an odd or an even polynomial, then the system of Figure 1 is stable for all  $G(s) \in \mathbf{G}(s)$  if and only if it is stable for all  $G(s) \in \mathbf{G}_K(s)$ . Moreover if the  $F_i(s)$  do not satisfy the above conditions stability of the system can not be guaranteed by verifying stability for  $G(s) \in \mathbf{G}_K(s)$ .

This result is obviously useful in control problems where  $F(s)$  the compensator consists of integrators, first order lags and leads and unstable and nonminimum phase elements. For instance in Example 1 if we consider the problem of stabilizing the interval family given, with a controller of the form

$$F(s) = \frac{as + b}{s^t(cs + d)}$$

it is only necessary to check that the controller simultaneously stabilizes the set of plants  $\mathbf{G}_K(s)$  shown below:

$$\begin{array}{cc} \frac{0.5 - 2s - s^2 + s^3}{1 - s^2 + 2s^3 + s^4} & \frac{0.5 - 2s + s^2 + s^3}{1 + s - s^2 + 2s^3 + s^4} \\ \frac{0.5 - 2s - s^2 + s^3}{1 - s^2 + 2s^3 + s^4} & \frac{1 - 2s - 2s^2 + s^3}{1 - s^2 + 2s^3 + s^4} \\ \frac{1 - 2s - 3s^2 + s^3}{1 - s^2 + 2s^3 + s^4} & \frac{1 - 2s + 3s^2 + s^3}{1 + s - s^2 + 2s^3 + s^4} \\ \frac{0.5 - 2s - s^2 + s^3}{1 + s - s^2 + 2s^3 + s^4} & \frac{1 - 2s - 2s^2 + s^3}{1 + s - s^2 + 2s^3 + s^4} \end{array}$$

To state the theorem for the general case (6) we introduce some notation. For any positive integer  $n$  let  $\underline{n}$  denote the set of integers  $\{1, 2, \dots, n\}$ . Referring to (6) let  $d^o(P_i)$  denote the degree of  $P_i(s)$  and let  $p_i^l$  denote the coefficient of  $s^l$  in  $P_i(s)$ :

$$P_i(s) = p_i^0 + p_i^1 s + \dots + p_i^l s^l + \dots + p_i^{d^o(P_i)} s^{d^o(P_i)}. \quad (17)$$

Write  $\underline{p}_i := (p_i^0, p_i^1, \dots, p_i^{d^o(P_i)})$  for each  $i \in \underline{m}$ , and let

$$\mathbf{P} := [\underline{p}_1, \underline{p}_2, \dots, \underline{p}_m] \quad (18)$$

denote the vector of coefficients of the polynomials  $P_i(s)$ ,  $i \in \underline{m}$ . Each such coefficient belongs to a given interval:

$$p_i^l \in [\alpha_i^l, \beta_i^l] \quad l = 0, \dots, d^o(P_i), i \in \underline{m}. \quad (19)$$

We let  $\mathcal{P}_i(s)$  denote the set of interval polynomials to which  $P_i(s)$  belongs and introduce the uncertainty set

$$\mathbf{\Pi} := \{(\mathcal{P}_1(s) \times \mathcal{P}_2(s) \times \dots, \times \mathcal{P}_m(s))\}$$

Alternatively the uncertainty set can be described in the space of the polynomial coefficients. With mild abuse of notation we use  $\Pi$  to also denote the box of uncertain parameters:

$$\Pi := \{\mathbf{p} \mid p_i^l \in [\alpha_i^l, \beta_i^l], l = 0, \dots, d^o(P_i), i \in \underline{m}\}. \quad (20)$$

Each point  $\mathbf{p} \in \Pi$  corresponds to a particular choice of the ordered set of polynomials  $P_i(s), i \in \underline{m}$ . We write  $\delta(s, \mathbf{p})$  for the polynomial family (6) to display the explicit dependence of  $\delta(s)$  on  $\mathbf{p}$ . For a given fixed set of polynomials  $[F_1(s), F_2(s), \dots, F_m(s)] := \underline{F}$  let  $\Delta$  denote the family of polynomials generated by the map  $\underline{F} : \Pi \implies \delta$  as in (6) and obtained by letting the parameter vector  $\mathbf{p}$ , (equivalently, the polynomials  $P_i(s)$ ), range over the box  $\Pi$  described in (20). In other words

$$\Delta := \{\delta(s, \mathbf{p}) \mid \mathbf{p} \in \Pi\}. \quad (21)$$

Following the previous notation the four Kharitonov polynomials and polynomial segments associated with  $\mathcal{P}_i(s)$  are denoted  $\mathcal{K}_i(s)$  and  $\mathcal{S}_i(s)$  respectively and these definitions hold for each  $i \in \underline{m}$ . We now introduce some special subsets of  $\Pi$ . The linear manifolds  $\Pi_l, l \in \underline{m}$  are defined:

$$\begin{aligned} \Pi_l := \\ \{(\mathcal{K}_1(s) \times \dots \times \mathcal{K}_{l-1}(s) \times \mathcal{S}_l(s) \times \mathcal{K}_{l+1}(s) \times \dots, \times \mathcal{K}_m(s))\} \end{aligned} \quad (22)$$

Also

$$\Delta_l := \{\delta(s, \mathbf{p}) \mid \mathbf{p} \in \Pi_l\}. \quad (23)$$

Finally, let

$$\Pi_E := \bigcup_{l=1}^m \Pi_l \quad (24)$$

and

$$\Delta_E := \bigcup_{l=1}^m \Delta_l = \{\delta(s, \mathbf{p}) \mid \mathbf{p} \in \Pi_E\}. \quad (25)$$

Since there is a one to one correspondence between the elements of  $\Pi_E$  and of  $\Delta_E$  we refer to both sets as extremal segments. We also define the *Kharitonov vertices*  $\mathbf{K}(\Pi)$ , of  $\Pi$  to be the subset of all vertices of  $\Pi$  corresponding to the Kharitonov polynomials of the  $P_i(s)$ . It is not difficult to see that the number of distinct segments in  $\Pi_E$  in the most general case when all the Kharitonov polynomials associated with each polynomial  $P_i(s)$  are distinct, is  $m4^m$ . With these preliminaries we are ready to state the main result of this section.

**Theorem 3.** [10,11]  $\underline{F}$  stabilizes  $\Pi$  if and only if  $\underline{F}$  stabilizes  $\Pi_E$ .

In the next section we show that these extremal segments are useful in determining how close one is to instability in the parameter space over a stable set of parameters  $\Pi$ .

## 5. EXTREMAL PARAMETRIC STABILITY PROPERTIES: LINEAR CASE

We now turn to the question of relative stability of the family  $\Pi$ . In other words given a family of polynomials  $\Pi$  which is stable, we wish to know the “distance” to the closest unstable polynomial as the point  $\mathbf{p}$  (equivalently the set of polynomials  $P_i(s)$ ) varies over the box  $\Pi$ . Before discussing the case of a control system we deal with the special case of a single interval polynomial. In this case we first establish an important extremal property of the Kharitonov polynomials, namely that the closest point to instability over a stable box in coefficient space lies at one of the Kharitonov vertices. The proof of this result was first given in [18].

### 5.1. Extremal Property of the Kharitonov Polynomials

Suppose that we have proved the stability of the family of polynomials

$$\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n, \quad (26)$$

with coefficients in the box

$$\Delta = [x_0, y_0] \times [x_1, y_1] \times \cdots \times [x_n, y_n]. \quad (27)$$

Write  $\underline{\delta} = [\delta_0, \delta_1, \dots, \delta_n]$ , and regard  $\underline{\delta}$  as a point in  $R^{n+1}$ . Let  $\|\underline{\delta}\|_p$  denote the  $p$  norm in  $R^{n+1}$  and let this be associated with  $\delta(s)$ . The set of polynomials which are unstable of degree  $n$  or of degree less than  $n$  is denoted by  $\mathcal{U}$ . Then the radius of the stability ball centered at  $\delta$  is

$$\rho(\delta) = \inf_{\mathbf{u} \in \mathcal{U}} \|\underline{\delta} - \mathbf{u}\|_p. \quad (28)$$

If the polynomial family is stable it is possible to associate with each element the largest stability hypersphere around it. We thus define a mapping from  $\Delta$  to the set of all positive real numbers:

$$\begin{array}{rcl} \Delta & \xrightarrow{\rho} & \mathcal{R}^+ \setminus \{0\} \\ \delta(s) & \longrightarrow & \rho(\delta) \end{array}$$

and ask the question: Is there a point in  $\Delta$  which is the nearest to instability? Or stated in terms of functions: Has the function  $\rho$  a minimum and is there a precise point in  $\Delta$  where it is reached? The answer to that question is given in the following theorem proved in [18]. In the discussion to follow we drop the subscript  $p$  from the norm since the result holds for any norm chosen.

**Theorem 4. (Extremality property of the Kharitonov polynomials [18]).**  
*The function*

$$\begin{array}{rcl} \Delta & \xrightarrow{\rho} & \mathcal{R}^+ \setminus \{0\} \\ \delta(s) & \longrightarrow & \rho(\delta) \end{array}$$

*has a minimum which is reached at one of the four Kharitonov polynomials associated with  $\Delta$ .*

The above optimal property of Kharitonov polynomials is extremely important and has many uses. Using it it is possible to prove the extremal property of the stability segments occurring in the linear case result given in the previous section.

## 5.2. Extremal Parametric Property of the extremal Segments

Consider the family  $\Delta$  and the segments  $\Pi_E$  and  $\Delta_E$  which occur in the linear case result of the previous section. As before consider the family of polynomials lying in the uncertainty set  $\Pi$  and let the coefficients of the polynomials  $P_i(s)$ ,  $\mathbf{p} \in R^n$  vary in the prescribed box  $\Pi$ . Let  $\|\cdot\|$  denote any norm in  $R^n$  and let  $\mathcal{P}_u$  denote the set of points  $\mathbf{u}$  in  $R^n$  for which  $\delta(s, \mathbf{u})$  is unstable or loses degree (relative to its generic degree over  $\Pi$ ). Let

$$\rho(\mathbf{p}) = \inf_{\mathbf{u} \in \mathcal{P}_u} \|\mathbf{p} - \mathbf{u}\|_p \quad (29)$$

denote the radius of the stability ball (measured in the norm  $\|\cdot\|$ ) and centered at the point  $\mathbf{p}$ . This number serves as the stability margin associated with the point  $\mathbf{p}$ . If the box  $\Pi$  is stable we can associate a stability margin with each point in  $\Pi$ . A natural counterpart of the question posed in the previous section is: Is there a point in  $\Pi$  which is closest to instability in the norm  $\|\cdot\|$  and where is it? The answer to that question is provided in the following theorem first proved in [19].

As before we define a mapping from  $\Pi$  to the set of all positive real numbers:

$$\begin{aligned} \Pi & \xrightarrow{\rho} \mathcal{R}^+ \setminus \{0\} \\ \mathbf{p} & \longrightarrow \rho(\mathbf{p}). \end{aligned} \quad (30)$$

**Theorem 5. (Extremal property of the Extremal Segments [19])** *The function*

$$\begin{aligned} \Pi & \xrightarrow{\rho} \mathcal{R}^+ \setminus \{0\} \\ \mathbf{p} & \longrightarrow \rho(\mathbf{p}) \end{aligned}$$

*has a minimum which is reached at a point on the extremal manifolds  $\Pi_E$ .*

We remark that this optimality property enjoyed by the extremal segments is very useful to find the worst case parametric robustness margin associated with a given controller.

## 6. PARAMETRIC AND UNSTRUCTURED PERTURBATIONS: LINEAR CASE

We now turn our attention to problems where parametric as well as unstructured uncertainty is simultaneously in operation. The main results in this section will deal with the calculation of the  $H_\infty$  stability margin for systems containing parameter uncertainty as defined above. In the following we will use the standard notation:  $\mathbf{C}_+ := \{s \in C : \text{Re}(s) \geq 0\}$ , and  $H_\infty(\mathbf{C}_+)$  will represent the space of functions  $f(s)$  that are bounded and analytic in  $\mathbf{C}_+$  with the standard  $H_\infty$  norm,

$$\|f\|_\infty = \sup_{\omega \in R} |f(j\omega)|. \quad (31)$$

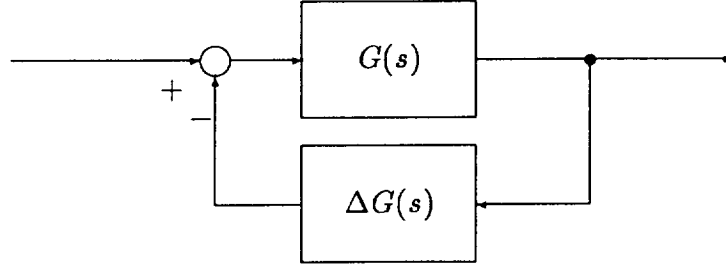


Figure 2.

Consider first the feedback system below where the fixed stable system with transfer function  $G(s)$  is perturbed by  $H_\infty$  norm bounded feedback perturbations  $\Delta G$ . According to the small gain theorem the perturbed system remains stable as long as  $\|\Delta G\|_\infty < \frac{1}{\|G\|_\infty}$ . We note that  $\frac{1}{\|G\|_\infty}$  can be regarded as a “complex gain margin” for the system. The obvious first step is to generalize this by letting  $G(s)$  lie in an uncertainty set. We consider the case when  $G(s) = \frac{N(s)}{D(s)}$  belongs to an interval family  $\mathbf{G}(s)$  as in Section 4. We adopt the notation of Section III and let  $\mathbf{G}_K(s)$  be the set of 16 *Kharitonov systems* associated with  $\mathbf{G}(s)$ . Our robust version of the small gain theorem can be stated as follows.

**Theorem 6. (Robust Small Gain Theorem [20])** *Given the interval family  $\mathbf{G}(s)$  of stable proper systems, the closed-loop system in Figure 2 remains stable for all stable perturbation  $\Delta P$  such that  $\|\Delta P\|_\infty < \alpha$  if and only if,*

$$\alpha \leq \frac{1}{\max_{G(s) \in \mathbf{G}_K(s)} \|G\|_\infty}. \quad (32)$$

The proof of this theorem given in [20] was based on the following fundamental characterization of  $H_\infty$  norms in terms of Hurwitz stability of polynomials. This lemma was also proved in [20].

**Lemma 3.** [20] *Let  $h(s) = n(s)/d(s)$  be a proper (real or complex) rational function in  $H_\infty(\mathbf{C}_+)$ , with  $\deg(d(s)) = q$ . Then  $\|h\|_\infty < 1$  if and only if*

- a1)  $|n_q| < |d_q|$ ,
- b1)  $d(s) + e^{j\theta}n(s)$  is Hurwitz for all  $\theta$  in  $[0, 2\pi)$ .

We now give an example to illustrate the above theorem

**Example 2.** Consider the following stable family  $\mathbf{G}(s)$  of interval systems whose generic element is given by,

$$g(s) = \frac{n_0 + n_1s + n_2s^2 + n_3s^3}{d_0 + d_1s + d_2s^2 + d_3s^3}$$

with

$$n_0 \in [1, 2], \quad n_1 \in [-3, 1], \quad n_2 \in [2, 4], \quad n_3 \in [1, 3]$$

and

$$d_0 \in [1, 3], \quad d_1 \in [2, 4], \quad d_2 \in [6, 7], \quad d_3 \in [1, 2].$$

$\mathbf{G}_K(s)$  consists of the following 16 rational functions,

$$\begin{aligned} g_1(s) &= \frac{1 - 3s + 4s^2 + 3s^3}{1 + 2s + 7s^2 + 2s^3}, & g_2(s) &= \frac{1 - 3s + 4s^2 + 3s^3}{1 + 4s + 7s^2 + s^3}, \\ g_3(s) &= \frac{1 - 3s + 4s^2 + 3s^3}{3 + 2s + 6s^2 + 2s^3}, & g_4(s) &= \frac{1 - 3s + 4s^2 + 3s^3}{3 + 4s + 6s^2 + s^3}, \\ g_5(s) &= \frac{1 + s + 4s^2 + s^3}{1 + 2s + 7s^2 + 2s^3}, & g_6(s) &= \frac{1 + s + 4s^2 + s^3}{1 + 4s + 7s^2 + s^3}, \\ g_7(s) &= \frac{1 + s + 4s^2 + s^3}{3 + 2s + 6s^2 + 2s^3}, & g_8(s) &= \frac{1 + s + 4s^2 + s^3}{3 + 4s + 6s^2 + s^3}, \\ g_9(s) &= \frac{2 - 3s + 2s^2 + 3s^3}{1 + 2s + 7s^2 + 2s^3}, & g_{10}(s) &= \frac{2 - 3s + 2s^2 + 3s^3}{1 + 4s + 7s^2 + s^3}, \\ g_{11}(s) &= \frac{2 - 3s + 2s^2 + 3s^3}{3 + 2s + 6s^2 + 2s^3}, & g_{12}(s) &= \frac{2 - 3s + 2s^2 + 3s^3}{3 + 4s + 6s^2 + s^3}, \\ g_{13}(s) &= \frac{2 + s + 2s^2 + s^3}{1 + 2s + 7s^2 + 2s^3}, & g_{14}(s) &= \frac{2 + s + 2s^2 + s^3}{1 + 4s + 7s^2 + s^3}, \\ g_{15}(s) &= \frac{2 + s + 2s^2 + s^3}{3 + 2s + 6s^2 + 2s^3}, & g_{16}(s) &= \frac{2 + s + 2s^2 + s^3}{3 + 4s + 6s^2 + s^3}. \end{aligned}$$

The  $H_\infty$  norms of the above functions are given by,

$$\begin{aligned} \|g_1\|_\infty &= 2.112, & \|g_2\|_\infty &= 3.0, & \|g_3\|_\infty &= 5.002, & \|g_4\|_\infty &= 3.0, \\ \|g_5\|_\infty &= 1.074, & \|g_6\|_\infty &= 1.0, & \|g_7\|_\infty &= 1.710, & \|g_8\|_\infty &= 1.0, \\ \|g_9\|_\infty &= 3.356, & \|g_{10}\|_\infty &= 3.0, & \|g_{11}\|_\infty &= 4.908, & \|g_{12}\|_\infty &= 3.0, \\ \|g_{13}\|_\infty &= 2.848, & \|g_{14}\|_\infty &= 2.0, & \|g_{15}\|_\infty &= 1.509, & \|g_{16}\|_\infty &= 1.0. \end{aligned}$$

Therefore, by the above theorem the entire family of systems remains stable under any unstructured feedback perturbations of  $H_\infty$  norm less than  $\alpha = \frac{1}{5.002} = 0.19992$  which is the smallest “complex stability margin” over the given box of parameters.

### 6.1. Computation of the Structured Margin

The converse problem is: given a prescribed bound on the level of unstructured perturbations that are to be tolerated determine the amount of parameter perturbations permissible.

In this case one starts with a nominal stable system

$$g^\circ(s) = \frac{n_0^\circ + n_1^\circ s + \dots + n_p^\circ s^p}{d_0^\circ + d_1^\circ s + \dots + d_q^\circ s^q} \quad (33)$$

which satisfies  $\|g^\circ\|_\infty = \alpha$ . A bound  $\frac{1}{\beta} < \frac{1}{\alpha}$  is then set on the desired level of unstructured perturbations. It is then possible to fix the structure of the parametric

perturbations and to maximize a weighted  $l_\infty$  ball around the parameters of  $g^\circ(s)$ . More precisely, one can allow the parameters  $n_i, d_j$  of the plant to vary in intervals of the form

$$n_i \in [n_i^\circ - \epsilon\nu_i, n_i^\circ + \epsilon\nu_i], \quad d_j \in [d_j^\circ - \epsilon\mu_j, d_j^\circ + \epsilon\mu_j],$$

where the weights  $\nu_i, \mu_j$  are fixed and non negative. For each  $\epsilon$  we get a family of interval systems  $\mathbf{G}(\epsilon)$  and its associated Kharitonov systems  $\mathbf{G}_K(\epsilon)$ . The structured stability margin is then given by the largest  $\epsilon$ , say  $\epsilon_{max}$ , for which every system  $g(s)$  in the corresponding interval family  $\mathbf{G}(\epsilon_{max})$  satisfies  $\|g\|_\infty \leq \beta$ .

An upper bound  $\epsilon_1$  for  $\epsilon_{max}$  is easily found by letting  $\epsilon_1$  be the smallest number such that the interval family,

$$\{d(s) = d_0 + \dots + d_q s^q : d_j \in [d_j^\circ - \epsilon_1 \mu_j, d_j^\circ + \epsilon_1 \mu_j]\},$$

contains an unstable polynomial. This upper bound is easily calculated, using for example the method proposed in [20] which is straightforward. This can be used to initiate a bisection algorithm. The reader may consult [20] for the details and an example.

We now consider the case where the fixed transfer function  $F(s)$  in Figure 1 is not necessarily unity. Let  $\mathbf{G}(s)$  be a family of strictly proper interval transfer functions, Assume also that we have found a stabilizing controller  $F(s)$  for the entire family. We therefore have a family of stable closed-loop systems and we consider unstructured additive perturbations as shown in Figure 3. Here also we want to determine the amount of unstructured perturbations that can be tolerated by this family of interval plants. In order to do so we have to find the maximum of the  $H_\infty$  norm of the closed-loop transfer function  $F(s)(1 + G(s)F(s))^{-1}$  over all elements  $G(s)$  in  $\mathbf{G}(s)$ . This result is also reported from [20].

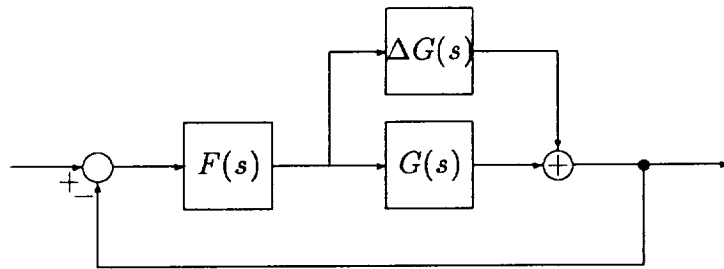


Figure 3.

**Theorem 7.** [20] *Given an interval family  $\mathbf{G}(s)$  of strictly proper plants and a stabilizing controller  $F(s)$  for  $\mathbf{G}(s)$ , the closed loop system in Figure 3 remains stable for all stable perturbations  $\Delta G$  such that  $\|\Delta G\|_\infty < \alpha$  if and only if,*

$$\alpha \leq \frac{1}{\max_{G \in \mathbf{G}_B} \|F(s)(1 + G(s)F(s))^{-1}\|_\infty}.$$



We illustrate this result with an example.

*Example 3.* Consider the following family of interval plants,

$$g_{\beta,\gamma}(s) = \frac{n^p(s)}{d^p(s)} = \frac{\beta s}{1 - s + \gamma s^2 + s^3}, \quad \beta \in [1, 2], \quad \gamma \in [3.4, 5].$$

Using the theorem [10] one can easily check that the controller  $F(s) = \frac{3}{s+1}$  stabilizes the entire family. The transfer function of interest is given by,

$$F(s)(1 + G_{\beta,\gamma}(s)F(s))^{-1} = \frac{3(1 - s + \gamma s^2 + s^3)}{1 + 3\beta s + (\gamma - 1)s^2 + (\gamma + 1)s^3 + s^4}.$$

Following Theorem 7, we have to find the maximum  $H_\infty$  norm of four one-parameter families of rational functions, namely,

$$\begin{aligned} r_\lambda(s) &= \frac{3(1 - s + \lambda s^2 + s^3)}{1 + 3s + (\lambda - 1)s^2 + (\lambda + 1)s^3 + s^4}, \quad \lambda \in [3.4, 5], \\ r_\mu(s) &= \frac{3(1 - s + \mu s^2 + s^3)}{1 + 6s + (\mu - 1)s^2 + (\mu + 1)s^3 + s^4}, \quad \mu \in [3.4, 5], \\ r_\nu(s) &= \frac{3(1 - s + 3.4s^2 + s^3)}{1 + 3\nu s + 2.4s^2 + 4.4s^3 + s^4}, \quad \nu \in [1, 2], \\ r_\xi(s) &= \frac{3(1 - s + 5s^2 + s^3)}{1 + 3\xi s + 4s^2 + 6s^3 + s^4}, \quad \xi \in [1, 2]. \end{aligned}$$

Consider for example the case of  $r_\lambda(s)$ . We have,

$$|r_\lambda(j\omega)|^2 = \frac{9((1 - \lambda\omega^2)^2 + \omega^2(1 + \omega^2)^2)}{(1 - (\lambda - 1)\omega^2 + \omega^4)^2 + \omega^2(3 - (\lambda + 1)\omega^2)^2}.$$

Letting  $t = \omega^2$  we have to find,

$$\sup_{t \geq 0, \lambda \in [3.4, 5]} f(t, \lambda) = \frac{9((1 - \lambda t)^2 + t(1 + t)^2)}{(1 - (\lambda - 1)t + t^2)^2 + t(3 - (1 + \lambda)t)^2}.$$

Differentiating with respect to  $\lambda$  we get a supremum at,

$$\lambda_1(t) = \frac{-2t + 3 + \sqrt{4t^3 + 12t^2 + 1}}{2t}$$

or

$$\lambda_2(t) = \frac{-2t + 3 - \sqrt{4t^3 + 12t^2 + 1}}{2t}.$$

It is then easy to see that  $\lambda_1(t) \in [3.4, 5]$  iff  $t \in [t_1, t_2] \cup [t_3, t_4]$ , where,

$$t_1 \simeq 0.39796, \quad t_2 \simeq 0.64139, \quad t_3 \simeq 15.51766, \quad t_4 \simeq 32.44715,$$

whereas,  $\lambda_2(t) \in [3.4, 5]$  iff  $t \in [t_5, t_6]$  where,

$$t_5 \simeq 0.15488, \quad t_6 \simeq 0.20095.$$

As a result, the maximum  $H_\infty$  norm for  $r_\lambda(s)$  is given by,

$$\max\left(\|r_{3.4}\|_\infty, \|r_5\|_\infty, \sqrt{\sup_{t \in [t_1, t_2] \cup [t_3, t_4]} f(t, \lambda_1(t))}, \sqrt{\sup_{t \in [t_5, t_6]} f(t, \lambda_2(t))}\right),$$

where one can at once verify that,

$$f(t, \lambda_1(t)) = \frac{9(2t - 1 - \sqrt{4t^3 + 12t^2 + 1})}{2t^2 + 7t - 1 - (t + 1)\sqrt{4t^3 + 12t^2 + 1}},$$

and,

$$f(t, \lambda_2(t)) = \frac{9(2t - 1 + \sqrt{4t^3 + 12t^2 + 1})}{2t^2 + 7t - 1 + (t + 1)\sqrt{4t^3 + 12t^2 + 1}}.$$

This maximum is then easily found to be equal to,

$$\max(34.14944, 7.55235, 27.68284, 1.7028) = 34.14944 .$$

Proceeding in the same way we finally get the following result,

$$\max_{\beta \in [1, 2], \gamma \in [3.4, 5]} \|c(s)(1 + g_{\beta, \gamma}(s)c(s))^{-1}\|_\infty = 34.14944 ,$$

where the maximum is in fact achieved for  $\beta = 1$ ,  $\gamma = 3.4$ .

The results given above lead to the following theorem.

**Theorem 8.** *Let  $\mathbf{G}(s)$  be a family of interval plants of fixed degree and let  $\alpha > 0$  be given. There exists a linear time invariant controller  $F(s)$  that stabilizes  $\mathbf{G}(s)$  and that satisfies,*

$$\sup_{G \in \mathbf{G}} \|F(s)(1 + G(s)F(s))^{-1}\|_\infty \leq \alpha,$$

*if and only if such a controller exists for  $\mathbf{G}_E$ .*

We believe that the above theorem sets the stage for further investigation into the synthesis problem by precisely specifying the role of the controller in the robust stability of a family of interval systems.

In the next section we describe the connection between parametric perturbations and nonlinear feedback perturbations.

## 7. PARAMETRIC AND NONLINEAR PERTURBATIONS: LINEAR CASE

An important stability robustness problem that involves unstructured perturbations is the classical Lur'e problem of nonlinear control theory. This problem considers a fixed linear time invariant system subjected to perturbations in the form of nonlinear feedback gains contained in a prescribed sector. In [21] a robust version of the Lur'e problem was treated where structured and unstructured perturbations are simultaneously present. In this formulation the fixed linear system is replaced by the more

realistic model of a parametrized family of plants. The “nonlinear stability margin” of the system can be determined by finding the infimum, over the parametrized family, of such stability sectors. From standard results on the Lur’e problem, the size of such a sector can be determined by finding the infimum of the real part of  $G(j\omega)$  as  $G(s)$  ranges over a parametrized family  $\mathbf{G}(s)$ . In [21] it was shown how the strict positive realness (SPR) property for a stable family of interval systems can be determined from the set  $\mathbf{G}_K(s)$  of the sixteen *Kharitonov systems*. In addition, in the presence of a fixed controller that stabilizes an entire family of interval systems, the SPR property for the family of transfer functions  $F(s)(1 + G(s)F(s))^{-1}$  is determined from a the extremal subset of systems. These results are described in this section and the reader is referred to [21] for proofs.

We begin by giving a stability characterization of the SPR property proved in [21]. Let  $G(s) = \frac{n(s)}{d(s)}$  be a real proper transfer function with no poles in the closed right-half plane.

**Theorem 9.**  *$G(s)$  is SPR if and only if the following three conditions are satisfied:*

- a)  $\operatorname{Re} G(0) > 0$ ,
- b)  $n(s)$  is Hurwitz stable,
- c)  $d(s) + j\alpha n(s)$  is Hurwitz stable for all  $\alpha$  in  $\mathbb{R}$ .

Based on this result it is possible to formulate a robust SPR result as follows. Let  $G(s)$  now belong to an interval family  $\mathbf{G}(s)$  as in Section 3. Given a real number  $\gamma$  we ask: Under what conditions is  $G(s) + \gamma$  SPR for all  $G(s)$  in  $\mathbf{G}(s)$ ? The answer is given in the following result from [21].

**Lemma 4.** [21]  *$G(s) + \gamma$  is SPR for every element in  $\mathbf{G}(s)$  if and only if it is SPR for the 16 Kharitonov systems in  $\mathbf{G}_K(s)$ .*

This result leads to:

**Theorem 10.** [21] *Given a proper stable family  $\mathbf{G}(s)$  of interval plants, the minimum of  $\operatorname{Re}(G(j\omega))$  over all  $\omega$  and over all  $G(s)$  in  $\mathbf{G}(s)$  is achieved at one of the 16 Kharitonov systems in  $\mathbf{G}_K(s)$*

We illustrate this result with an example.

*Example 4.* Consider the following stable family  $\mathbf{G}(s)$  of interval systems whose generic element is given by

$$G(s) = \frac{1 + \alpha s + \beta s^2 + s^3}{\gamma + \delta s + \epsilon s^2 + s^3}$$

where

$$\alpha \in [1, 2], \beta \in [3, 4], \gamma \in [1, 2], \delta \in [5, 6], \epsilon \in [3, 4].$$

$\mathbf{G}_K(s)$  consists of the following 16 rational functions

$$\begin{aligned}
r_1(s) &= \frac{1 + s + 3s^2 + s^3}{1 + 5s + 4s^2 + s^3}, & r_2(s) &= \frac{1 + s + 3s^2 + s^3}{1 + 6s + 4s^2 + s^3}, \\
r_3(s) &= \frac{1 + s + 3s^2 + s^3}{2 + 5s + 3s^2 + s^3}, & r_4(s) &= \frac{1 + s + 3s^2 + s^3}{2 + 6s + 3s^2 + s^3}, \\
r_5(s) &= \frac{1 + s + 4s^2 + s^3}{1 + 5s + 4s^2 + s^3}, & r_6(s) &= \frac{1 + s + 4s^2 + s^3}{1 + 6s + 4s^2 + s^3}, \\
r_7(s) &= \frac{1 + s + 4s^2 + s^3}{2 + 5s + 3s^2 + s^3}, & r_8(s) &= \frac{1 + s + 4s^2 + s^3}{2 + 6s + 3s^2 + s^3}, \\
r_9(s) &= \frac{1 + 2s + 3s^2 + s^3}{1 + 5s + 4s^2 + s^3}, & r_{10}(s) &= \frac{1 + 2s + 3s^2 + s^3}{1 + 6s + 4s^2 + s^3}, \\
r_{11}(s) &= \frac{1 + 2s + 3s^2 + s^3}{2 + 5s + 3s^2 + s^3}, & r_{12}(s) &= \frac{1 + 2s + 3s^2 + s^3}{2 + 6s + 3s^2 + s^3}, \\
r_{13}(s) &= \frac{1 + 2s + 4s^2 + s^3}{1 + 5s + 4s^2 + s^3}, & r_{14}(s) &= \frac{1 + 2s + 4s^2 + s^3}{1 + 6s + 4s^2 + s^3}, \\
r_{15}(s) &= \frac{1 + 2s + 4s^2 + s^3}{2 + 5s + 3s^2 + s^3}, & r_{16}(s) &= \frac{1 + 2s + 4s^2 + s^3}{2 + 6s + 3s^2 + s^3}.
\end{aligned}$$

The corresponding minima of their respective real parts along the imaginary axis are given by,

$$\begin{aligned}
\inf_{\omega \in \mathbb{R}} \operatorname{Re} r_1(j\omega) &= 0.1385416, & \inf_{\omega \in \mathbb{R}} \operatorname{Re} r_2(j\omega) &= 0.1134093, \\
\inf_{\omega \in \mathbb{R}} \operatorname{Re} r_3(j\omega) &= 0.0764526, & \inf_{\omega \in \mathbb{R}} \operatorname{Re} r_4(j\omega) &= 0.0621581, \\
\inf_{\omega \in \mathbb{R}} \operatorname{Re} r_5(j\omega) &= 0.1540306, & \inf_{\omega \in \mathbb{R}} \operatorname{Re} r_6(j\omega) &= 0.1262789, \\
\inf_{\omega \in \mathbb{R}} \operatorname{Re} r_7(j\omega) &= 0.0602399, & \inf_{\omega \in \mathbb{R}} \operatorname{Re} r_8(j\omega) &= 0.0563546, \\
\inf_{\omega \in \mathbb{R}} \operatorname{Re} r_9(j\omega) &= 0.3467740, & \inf_{\omega \in \mathbb{R}} \operatorname{Re} r_{10}(j\omega) &= 0.2862616, \\
\inf_{\omega \in \mathbb{R}} \operatorname{Re} r_{11}(j\omega) &= 0.3011472, & \inf_{\omega \in \mathbb{R}} \operatorname{Re} r_{12}(j\omega) &= 0.2495148, \\
\inf_{\omega \in \mathbb{R}} \operatorname{Re} r_{13}(j\omega) &= 0.3655230, & \inf_{\omega \in \mathbb{R}} \operatorname{Re} r_{14}(j\omega) &= 0.3010231, \\
\inf_{\omega \in \mathbb{R}} \operatorname{Re} r_{15}(j\omega) &= 0.2706398, & \inf_{\omega \in \mathbb{R}} \operatorname{Re} r_{16}(j\omega) &= 0.2345989.
\end{aligned}$$

Therefore, the entire family is SPR and the minimum is achieved at  $r_8(s)$ .

Turning now to the Lur'e problem let us refer to Figure 4. The class of allowable nonlinearities is described by sector bounded functions. Specifically, the nonlinearity  $\phi(t, \sigma)$  satisfies

$$\begin{aligned}
\phi(t, 0) &= 0 \quad \text{for all } t \geq 0 \\
0 &\leq \sigma\phi(t, \sigma) \leq k\sigma^2.
\end{aligned}$$

This implies that  $\phi(t, \sigma)$  is bounded by the lines  $\phi = 0$  and  $\phi = k\sigma$ . Such nonlinearities are said to belong to a sector  $[0, k]$ . Referring to Figure 4, we state the following well-known classical result on *absolute stability*.

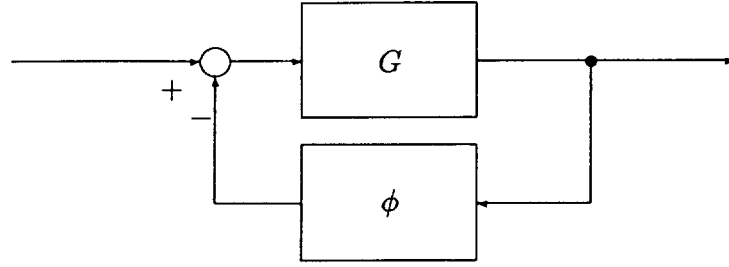


Figure 4.

**Theorem 11.** *If  $G(s)$  is a stable transfer function, and  $\phi$  belongs to the sector  $[0, k]$ , then a sufficient condition for absolute stability is*

$$\operatorname{Re}\left(\frac{1}{k} + g(j\omega)\right) > 0, \text{ for all } \omega \in \mathbb{R}$$

(i.e.  $\frac{1}{k} + g(s)$  is SPR).

Combining this with our previous results we have the robust version of the Lur'e problem shown in Figure 5 below.

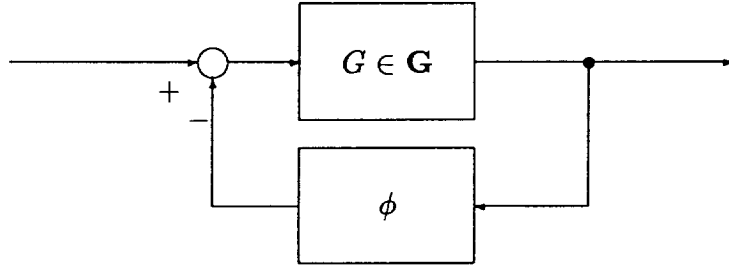


Figure 5.

**Theorem 12.** [21] *Given the interval family  $\mathbf{G}(s)$  of stable proper of stable proper systems and the family of sector bounded nonlinearities  $\phi$  belonging to the sector  $[0, k]$ , a sufficient condition for absolute stability of the closed loop system is that  $k > 0$  is any number such that*

$$k < \infty, \text{ if } \inf_{\mathbf{G}_{K(s)}} \inf_{\omega \in \mathbb{R}} \operatorname{Re}(G(j\omega)) \geq 0$$

otherwise

$$k < -\frac{1}{\inf_{\mathbf{G}_{K(s)}} \inf_{\omega \in \mathbb{R}} \operatorname{Re}(G(j\omega))}$$

where  $\mathbf{G}_{K(s)}$  is the set of sixteen Kharitonov systems corresponding to  $\mathbf{G}(s)$ .

This theorem may be generalized as follows.

**Theorem 13.** *Given the interval family  $\mathbf{G}(s)$  of proper systems stabilized by a fixed system  $F(s)$*

$$\begin{aligned} \inf_{\mathbf{G}(s)} \inf_{\omega \in \mathbb{R}} \operatorname{Re}[F(j\omega)(1 + G(j\omega)F(j\omega))^{-1}] = \\ \inf_{\mathbf{G}_K(s)} \inf_{\omega \in \mathbb{R}} \operatorname{Re}[F(j\omega)(1 + G(j\omega)F(j\omega))^{-1}] \end{aligned}$$

In the last section we describe some frequency domain extremal properties of the extremal segments.

## 8. EXTREMAL FREQUENCY DOMAIN PROPERTIES OF EXTREMAL SEGMENTS

Consider again the feedback system shown in Figure 1. Since the extremal subset characterizes the robust stability of the interval system of Figure 1 it is natural to expect that these subsets also bound the Nyquist and Bode bands of interval systems. This is indeed the case and in this section we present recent results from [8] in this direction. This result was also independently reported in [22]. We expect these results to play a very significant role in synthesis and design issues.

We shall give a quick summary of these results.

### 8.1. Nyquist Envelopes

Referring to the control system in Figure 1 we calculate the following transfer functions of interest in analysis and design problems:

$$\frac{y(s)}{u(s)} = G(s) \quad \frac{u(s)}{e(s)} = F(s) \quad (34)$$

$$\begin{aligned} T^o(s) &:= \frac{y(s)}{e(s)} = G(s)F(s) \\ T^y(s) &:= \frac{y(s)}{r(s)} = \frac{G(s)F(s)}{1 + G(s)F(s)} \\ T^e(s) &:= \frac{e(s)}{r(s)} = \frac{1}{1 + G(s)F(s)} \\ T^u(s) &:= \frac{u(s)}{r(s)} = \frac{F(s)}{1 + G(s)F(s)}. \end{aligned} \quad (35)$$

As  $G(s)$  ranges over the interval uncertainty set  $\mathbf{G}(s)$  (equivalently,  $(N(s), D(s))$  ranges over  $\mathcal{N}(s) \times \mathcal{D}(s)$ ) the transfer functions  $T^o(s)$ ,  $T^y(s)$ ,  $T^u(s)$ ,  $T^e(s)$  range over corresponding uncertainty sets  $\mathbf{T}^o(s)$ ,  $\mathbf{T}^y(s)$ ,  $\mathbf{T}^u(s)$ , and  $\mathbf{T}^e(s)$ , respectively. In other words

$$\mathbf{T}^o(s) := \{G(s)F(s) : G(s) \in \mathbf{G}(s)\}$$

$$\begin{aligned}
\mathbf{T}^y(s) &:= \left\{ \frac{G(s)F(s)}{1 + G(s)F(s)} : G(s) \in \mathbf{G}(s) \right\} \\
\mathbf{T}^u(s) &:= \left\{ \frac{F(s)}{1 + G(s)F(s)} : G(s) \in \mathbf{G}(s) \right\} \\
\mathbf{T}^e(s) &:= \left\{ \frac{1}{1 + G(s)F(s)} : G(s) \in \mathbf{G}(s) \right\}.
\end{aligned} \tag{36}$$

The extremal subsets of the transfer function sets (36) are also introduced:

$$\begin{aligned}
\mathbf{T}_E^o(s) &:= \{G(s)F(s) : G(s) \in \mathbf{G}_E(s)\} \\
\mathbf{T}_E^y(s) &:= \left\{ \frac{G(s)F(s)}{1 + G(s)F(s)} : G(s) \in \mathbf{G}_E(s) \right\} \\
\mathbf{T}_E^u(s) &:= \left\{ \frac{F(s)}{1 + G(s)F(s)} : G(s) \in \mathbf{G}_E(s) \right\} \\
\mathbf{T}_E^e(s) &:= \left\{ \frac{1}{1 + G(s)F(s)} : G(s) \in \mathbf{G}_E(s) \right\}
\end{aligned} \tag{37}$$

In frequency domain analysis and design problems the complex plane image of each of the above sets evaluated at  $s = j\omega$  plays an important role. We denote each of these two dimensional sets in the complex plane by replacing  $s$  in the corresponding argument by  $\omega$ . Thus, for example,

$$\mathbf{T}_E^y(\omega) := \{\mathbf{T}_E^y(s) : s = j\omega\} \tag{38}$$

The Nyquist plot of a set of functions (or polynomials)  $\mathbf{T}(s)$  is denoted by  $\mathbf{T}$ :

$$\mathbf{T} := \cup_{0 \leq \omega < \infty} \mathbf{T}(\omega) \tag{39}$$

The boundary of a set  $S$  is denoted  $\partial S$ .

We shall give the main results here without proof. The proofs are given in Keel, Shaw and Bhattacharyya [23] and also independently by Tesi and Vicino [22].

**Theorem 14.** [8][2] *For every  $\omega \geq 0$ ,*

$$\begin{aligned}
\partial \mathbf{G}(\omega) &\subset \mathbf{G}_E(\omega) \\
\partial \mathbf{T}^o(\omega) &\subset \mathbf{T}_E^o(\omega) \\
\partial \mathbf{T}^y(\omega) &\subset \mathbf{T}_E^y(\omega) \\
\partial \mathbf{T}^u(\omega) &\subset \mathbf{T}_E^u(\omega) \\
\partial \mathbf{T}^e(\omega) &\subset \mathbf{T}_E^e(\omega)
\end{aligned}$$

This result shows that at every  $\omega \geq 0$  the image set of each transfer function in (37) is bounded by the corresponding image set of the extremal segments.

The next result deals with the Nyquist plots of each of the transfer functions in (37).

**Theorem 15.** [8][2] *The Nyquist plots of each of the transfer function sets  $\mathbf{T}^o(s)$ ,  $\mathbf{T}^y(s)$ ,  $\mathbf{T}^u(s)$ , and  $\mathbf{T}^e(s)$  are bounded by their corresponding extremal subsets:*

$$\begin{aligned}\partial\mathbf{T}^o &\subset \mathbf{T}_E^o \\ \partial\mathbf{T}^y &\subset \mathbf{T}_E^y \\ \partial\mathbf{T}^u &\subset \mathbf{T}_E^u \\ \partial\mathbf{T}^e &\subset \mathbf{T}_E^e\end{aligned}$$

## 8.2. Bode Envelopes

For any function say,  $T(s)$  let  $\mu_T(\omega) := |T(j\omega)|$  and  $\phi_T(\omega) := \angle T(j\omega)$  denote the magnitude and phase evaluated at  $s = j\omega$ . If  $\mathbf{T}(s)$  denotes a set of functions we let the extremal values of magnitude and phase at a given frequency be defined as follows:

$$\begin{aligned}\bar{\mu}_{\mathbf{T}}(\omega) &:= \sup_{\mathbf{T}(j\omega)} |T(j\omega)| \\ \underline{\mu}_{\mathbf{T}}(\omega) &:= \inf_{\mathbf{T}(j\omega)} |T(j\omega)|.\end{aligned}\tag{40}$$

Similarly

$$\begin{aligned}\bar{\phi}_{\mathbf{T}}(\omega) &:= \sup_{\mathbf{T}(j\omega)} \angle T(j\omega) \\ \underline{\phi}_{\mathbf{T}}(\omega) &:= \inf_{\mathbf{T}(j\omega)} \angle T(j\omega).\end{aligned}\tag{41}$$

Suppose that  $\mathbf{G}(s)$  is an interval family. To compute

$$\bar{\mu}_{\mathbf{G}}(\omega), \quad \underline{\mu}_{\mathbf{G}}(\omega)\tag{42}$$

and

$$\bar{\phi}_{\mathbf{G}}(\omega), \quad \underline{\phi}_{\mathbf{G}}(\omega),\tag{43}$$

the following two lemmas are necessary.

**Lemma 5.** *Let  $\mathcal{A}$  be a closed polygon in the complex plane, and “ $a$ ” be an arbitrary point in  $\mathcal{A}$ . Let  $V_{\mathcal{A}}$  be the set of vertices and  $E_{\mathcal{A}}$  be the set of edges of  $\mathcal{A}$ . Then the following statements are true.*

- 1)  $\max_{\mathcal{A}} |a| = \max_{V_{\mathcal{A}}} |a|$
- 2)  $\min_{\mathcal{A}} |a| = \min_{E_{\mathcal{A}}} |a|$

**Lemma 6.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be disjoint closed polygons in the complex plane, and “ $a$ ” and “ $b$ ” be arbitrary points on  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Let  $V_{\mathcal{A}}$  and  $V_{\mathcal{B}}$  be the sets of*



vertices and let  $E_A$  and  $E_B$  be the sets of edges of  $A$  and  $B$ , respectively. Then the following statements are true.

- 1)  $\max_{A \times B} \{\angle a - \angle b\} = \max_{V_A \times V_B} \{\angle a - \angle b\}$
- 2)  $\min_{A \times B} \{\angle a - \angle b\} = \min_{V_A \times V_B} \{\angle a - \angle b\}$

Proofs of the above two lemmas are obvious from geometric considerations illustrated in [2].

Let  $\mathcal{N}(\omega)$  denote the complex plane image of the set of polynomials  $N(s) \in \mathcal{N}(s)$  evaluated at  $s = j\omega$ . Similar definitions hold for  $\mathcal{D}(\omega)$ ,  $\mathcal{S}_{\mathcal{N}}(\omega)$  and  $\mathcal{S}_{\mathcal{D}}(\omega)$ .  $\mathcal{N}(\omega)$  is bounded by the set of Kharitonov segments  $\mathcal{S}_{\mathcal{N}}(\omega)$ . Similarly,  $\mathcal{D}(\omega)$  is bounded by the set  $\mathcal{S}_{\mathcal{D}}(\omega)$ . These facts along with Lemmas 5 and 6 lead to the following results. Before we state Theorem 16, let us define the following sets.

$$\mathbf{G}(\omega) := \left\{ G(j\omega) = \frac{N(j\omega)}{D(j\omega)} \mid \right. \\ \left. N(j\omega) \in \mathcal{N}(\omega), D(j\omega) \in \mathcal{D}(\omega) \right\} \quad (44)$$

$$\bar{\mathbf{G}}_{\mathbf{E}}(\omega) := \left\{ G(j\omega) = \frac{N(j\omega)}{D(j\omega)} \mid \right. \\ \left. N(j\omega) \in \mathcal{K}_{\mathcal{N}}(\omega), D(j\omega) \in \mathcal{S}_{\mathcal{D}}(\omega) \right\} \quad (45)$$

$$\underline{\mathbf{G}}_{\mathbf{E}}(\omega) := \left\{ F(j\omega) = \frac{N(j\omega)}{D(j\omega)} \mid \right. \\ \left. N(j\omega) \in \mathcal{S}_{\mathcal{N}}(\omega), D(j\omega) \in \mathcal{K}_{\mathcal{D}}(\omega) \right\}. \quad (46)$$

**Theorem 16.** For every frequency  $\omega \geq 0$ ,

$$\bar{\mu}_{\mathbf{G}}(\omega) = \bar{\mu}_{\bar{\mathbf{G}}_{\mathbf{E}}}(\omega) \\ \underline{\mu}_{\mathbf{G}}(\omega) = \underline{\mu}_{\underline{\mathbf{G}}_{\mathbf{E}}}(\omega)$$

Let us also define the set of systems constructed from Kharitonov vertices as follows:

$$\mathbf{G}_{\mathcal{K}}(\omega) := \left\{ F(j\omega) = \frac{N(j\omega)}{D(j\omega)} \mid N(j\omega) \in \mathcal{K}_{\mathcal{N}}(\omega), D(j\omega) \in \mathcal{K}_{\mathcal{D}}(\omega) \right\}. \quad (47)$$

**Theorem 17.** For every frequency  $\omega \geq 0$ ,

$$\bar{\phi}_{\mathbf{G}}(\omega) = \bar{\phi}_{\mathbf{G}_{\mathcal{K}}}(\omega) \\ \underline{\phi}_{\mathbf{G}}(\omega) = \underline{\phi}_{\mathbf{G}_{\mathcal{K}}}(\omega)$$

Using the above extremal properties it is possible to evaluate the Bode magnitude and phase bands of interval transfer functions. Let us consider the family of transfer functions

$$\mathbf{T}^{\circ}(s) = \left\{ T^{\circ}(s) \mid F(s)G(s), G(s) \in \mathbf{G}(s) \right\}. \quad (48)$$

Since  $F(s)$  is fixed,

$$\begin{aligned}\bar{\mu}_{T^\circ}(\omega) &= |F(j\omega)| \bar{\mu}_G(\omega) \\ \underline{\mu}_{T^\circ}(\omega) &= |F(j\omega)| \underline{\mu}_G(\omega).\end{aligned}\tag{49}$$

Similarly,

$$\begin{aligned}\bar{\phi}_{T^\circ}(\omega) &= \angle F(j\omega) + \bar{\phi}_G(\omega) \\ \underline{\phi}_{T^\circ}(\omega) &= \angle F(j\omega) + \underline{\phi}_G(\omega).\end{aligned}\tag{50}$$

These relations are sufficient to construct the Bode magnitude and phase envelopes.

The Nyquist and Bode envelopes are important tools for solving analysis and design problems in robust parametric stability. In the next section, we show how the previous theory can be used to develop techniques to improve a given controller, by choosing an controller from a given set of stabilizing interval controllers, that provides optimal gain (or phase) margin to the closed loop system.

## 9. DESIGN OF LINEAR INTERVAL CONTROL SYSTEMS

In this section, we consider a nominal plant connected to an interval controller and give some design techniques for improving the closed loop gain and phase margins using the Nyquist envelope described in Theorem 15. From the results of the previous section it is clear that the main computational task is to determine the stability margin over the extremal segments. In the next section, we discuss the problem of determining optimal gain and phase margins over a single segment system.

### 9.1. Segment System

The typical extremal segment is of the form

$$\mathbf{p}(s) := \left\{ \frac{p_0(s)}{p_1(s) + \lambda p_2(s)} \mid \lambda \in [0, 1] \right\}\tag{51}$$

or

$$\mathbf{p}(s) := \left\{ \frac{p_1(s) + \lambda p_2(s)}{p_0(s)} \mid \lambda \in [0, 1] \right\}\tag{52}$$

where  $p_i(s)$  are fixed polynomials. In this section, we develop simple techniques to compute the extremal gain and phase margins over a segment. We also determine the optimal value  $\lambda^*$ , equivalently  $p^*(s)$ , that produces the optimal gain (or phase) margin over the family  $\mathbf{p}(s)$ .

Let us consider the following segment system with

$$p(j\omega, \lambda) = \frac{p_0(j\omega)}{p_1(j\omega) + \lambda p_2(j\omega)}.$$

The problem of computing the extremal gain and phase margins at the loop breaking point “m” over the single segment system is described as follows. Let us denote

$$(\Lambda \times \Omega) := \{(\lambda, \omega) \mid \angle p(j\omega, \lambda) = 180^\circ, \lambda \in [0, 1]\}\tag{53}$$

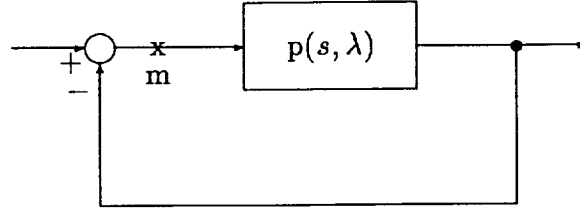


Figure 6. Segment System

and

$$\bar{\mu}_p := \max_{(\Lambda \times \Omega)} |p(j\omega, \lambda)| \quad (54)$$

$$\underline{\mu}_p := \min_{(\Lambda \times \Omega)} |p(j\omega, \lambda)| \quad (55)$$

$$\bar{\phi}_p := \max_{(\Lambda \times \Omega)} \angle p(j\omega, \lambda) \quad (56)$$

$$\underline{\phi}_p := \min_{(\Lambda \times \Omega)} \angle p(j\omega, \lambda) . \quad (57)$$

Then

$$\text{maximum gain margin over } p(s) : \bar{\rho} := \frac{1}{\underline{\mu}_p} \quad (58)$$

$$\text{minimum gain margin over } p(s) : \underline{\rho} := \frac{1}{\bar{\mu}_p} \quad (59)$$

$$\text{maximum phase margin over } p(s) : \bar{\theta} := \bar{\phi}_p - 180^\circ \quad (60)$$

$$\text{minimum phase margin over } p(s) : \underline{\theta} := \underline{\phi}_p - 180^\circ. \quad (61)$$

Similar definitions can be made for the case of gain margins less than 1.

As seen from eqs. (58) - (61), the problem of computing the extremal gain or phase margin over the segment system is two parameter optimization problem. This can be reduced to a simple one parameter problem as follows. Write

$$p_i(j\omega) := p_{iR}(\omega) + jp_{iI}(\omega)$$

Then

$$\begin{aligned} p(j\omega, \lambda) &= \frac{p_o(j\omega)}{p_1(j\omega) + \lambda p_2(j\omega)} \\ &= \frac{p_{oR}(\omega) + jp_{oI}(\omega)}{[p_{1R}(\omega) + \lambda p_{2R}(\omega)] + j[p_{1I}(\omega) + \lambda p_{2I}(\omega)]} \end{aligned}$$

$$\begin{aligned}
&= \frac{p_{0R}(\omega)p_{1R}(\omega) + p_{0I}(\omega)p_{1I}(\omega) + \lambda[p_{0R}(\omega)p_{2R}(\omega) + p_{0I}(\omega)p_{2I}(\omega)]}{\underbrace{[p_{1R}(\omega) + \lambda p_{2R}(\omega)]^2 + [p_{1I}(\omega) + \lambda p_{2I}(\omega)]^2}_{\text{Re}\{p(j\omega, \lambda)\}}} \\
&+ j \frac{p_{0I}(\omega)p_{1R}(\omega) - p_{0R}(\omega)p_{1I}(\omega) + \lambda[p_{0I}(\omega)p_{2R}(\omega) - p_{0R}(\omega)p_{2I}(\omega)]}{\underbrace{[p_{1R}(\omega) + \lambda p_{2R}(\omega)]^2 + [p_{1I}(\omega) + \lambda p_{2I}(\omega)]^2}_{\text{Im}\{p(j\omega, \lambda)\}}}
\end{aligned} \tag{62}$$

In order to determine the gain margin, we set

$$\angle p(j\omega, \lambda) = 180^\circ \tag{63}$$

which implies

$$\text{Im}\{p(j\omega, \lambda)\} = 0. \tag{64}$$

Note that (64) will be satisfied when  $\angle p(j\omega, \lambda) = 0^\circ$  or  $180^\circ$ . We exclude frequencies  $\omega$  for which  $\angle p(j\omega, \lambda) = 0^\circ$ . From eqs.(64) and (62), we have

$$\begin{aligned}
\text{Im}\{p(j\omega, \lambda)\} &= [p_{0I}(\omega)p_{1R}(\omega) - p_{0R}(\omega)p_{1I}(\omega)] \\
&\quad + \lambda[p_{0I}(\omega)p_{2R}(\omega) - p_{0R}(\omega)p_{2I}(\omega)] \\
&= 0
\end{aligned} \tag{65}$$

equivalently

$$\lambda(\omega) = \frac{p_{0R}(\omega)p_{1I}(\omega) - p_{0I}(\omega)p_{1R}(\omega)}{p_{0I}(\omega)p_{2R}(\omega) - p_{0R}(\omega)p_{2I}(\omega)}. \tag{66}$$

From this representation, we can easily conclude that instead of searching both  $\omega \in [0, \infty)$  and  $\lambda \in [0, 1]$ , searching only selected ranges of  $\omega$  that satisfy  $\lambda \in [0, 1]$  is enough. Thus, we let

$$\begin{aligned}
\lambda(\omega) &= \frac{p_{0R}(\omega)p_{1I}(\omega) - p_{0I}(\omega)p_{1R}(\omega)}{p_{0I}(\omega)p_{2R}(\omega) - p_{0R}(\omega)p_{2I}(\omega)} \\
&= 0 \text{ or } 1.
\end{aligned} \tag{67}$$

Without loss of generality, we have

$$\begin{aligned}
\text{for } \lambda = 1 &\quad p_{0R}(\omega)p_{1I}(\omega) - p_{0I}(\omega)p_{1R}(\omega) - p_{0I}(\omega)p_{2R}(\omega) \\
&\quad\quad\quad + p_{0R}(\omega)p_{2I}(\omega) = 0 \\
\text{for } \lambda = 0 &\quad p_{0R}(\omega)p_{1I}(\omega) - p_{0I}(\omega)p_{1R}(\omega) = 0
\end{aligned} \tag{68}$$

The valid ranges of  $\omega$  with respect to the condition  $\lambda \in [0, 1]$  can be easily determined from the roots of the above two equations. Thus, the problem posed in eqs. (58) and (59) is reduced to selection of maximum and minimum magnitudes of  $\lambda$  evaluated over the admissible ranges of  $\omega$  determined from the roots of eq. (68). Furthermore, the optimal value  $\lambda^*$ , equivalently optimal values of parameters over the segment system, can also be easily determined by substituting  $\omega^*$  that corresponds to the maximum gain margin into eq. (66).

If the segment system is of the form in eq. (52), one can follow a similar procedure to determine the extremal margins and the corresponding optimal systems over the segment system. Similar procedures can also be applied for computing extremal phase margins over a single segment. This is easily derived by replacing the condition (63) by

$$|p(j\omega, \lambda)| = 1 \quad (69)$$

## 9.2. Optimal Parameter Selection

Applying the procedure described in the previous section to the entire set of segments systems, the extremal margins over the interval plant are determined. Consequently, the optimal system that produces the maximum gain or phase margin over an interval family is also determined. This procedure may be used to solve the following interesting problem.

Suppose that a fixed system  $F(s)$  and a family of controllers  $\mathbf{G}(s)$  are given, for which the closed loop system is stable. The objective is to select an optimal system  $G_{\text{opt}}(s) \in \mathbb{R}^n(s)$  so that the resulting closed loop system has the maximum possible gain margin or phase margin over the family  $\mathbf{G}(s)$ . Once such an optimal system is found the controller may be reset to the optimal parameter as the new nominal controller. At this point a new family of stabilizing interval controllers can be determined and the previous procedure of selecting the best controller repeated over the new box of parameters. The set of stabilizing interval controllers can be determined by many different methods; for example the locus introduced by Tsytkin and Polyak [24] may be used. This procedure described above can be repeated until 1) improvement of the maximum margin in a given iteration is small or 2) the stability radius in the parameter space is small. Of course there is no guarantee that a globally optimum or even a satisfactory design will be achieved by this method.

In the next section, an illustrative example is given.

## 9.3. Illustrative Example

Suppose

$$F(s) := \frac{N_g(s)}{D_g(s)} = \frac{s^2 + 2s + 1}{s^4 + 2s^3 + 2s^2 + s}$$

$$\mathbf{G}(s) := \left\{ \frac{n_1 s + n_0}{d_2 s^2 + d_1 s + d_0} \right\}$$

where

$$n_0 \in [0.9, 1.1], \quad n_1 \in [0.1, 0.2]$$

$$d_0 \in [1.9, 2.1], \quad d_1 \in [1.8, 2.0], \quad d_2 \in [0.9, 1.0]$$

We first check the stability of the family of closed loop systems with  $F(s)$  and  $\mathbf{G}(s)$ . This can be done by checking the stability of the corresponding extremal segment.

We have

$$\begin{aligned}
K_n^1(s) &= 0.1s + 0.9 & K_n^2(s) &= 0.2s + 0.9 \\
K_n^3(s) &= 0.1s + 1.1 & K_n^4(s) &= 0.2s + 0.9 \\
K_d^1(s) &= s^2 + 1.8s + 1.9 & K_d^2(s) &= s^2 + 2s + 1.9 \\
K_d^3(s) &= 0.9s^2 + 1.8s + 2.1 & K_d^4(s) &= 0.9s^2 + 2s + 2.1
\end{aligned}$$

and the corresponding segments  $\mathbf{G}_E(s)$  are

$$\frac{\lambda K_n^i(s) + (1 - \lambda)K_n^j(s)}{K_d^k(s)} \cup \frac{K_n^k(s)}{\lambda K_d^i(s) + (1 - \lambda)K_d^j(s)}$$

where  $k = 1, 2, 3, 4$

$$(i, j) \in \{(1, 2), (1, 3), (2, 4), (3, 4)\}$$

and  $\lambda \in [0, 1]$  Using the Segment Lemma [13], we verified that all the above extremal segments stabilize the closed loop.

The Bode and Nyquist envelopes associated with  $\mathbf{T}^\circ(s)$ , the forward transfer functions  $F(s)\mathbf{G}(s)$ , are constructed by evaluating the following rational functions over  $\omega \in [0, \infty)$  and  $\lambda \in [0, 1]$ :

$$\frac{N(s)K_n^k(s)}{D(s)[\lambda K_d^i(s) + (1 - \lambda)K_d^j(s)]}$$

and

$$\frac{N(s)[\lambda K_n^i(s) + (1 - \lambda)K_n^j(s)]}{D(s)K_d^k(s)}.$$

This is shown in Figures 7,8 and 9.

From these figures, the minimum and maximum gain margins are found to be:

$$\rho = 1.1240. \quad \bar{\rho} = 1.7582.$$

If we want to improve the gain margin of the system by selecting parameters  $n_i$  and  $d_i$  beyond its previously given intervals, we can repeat the procedure as follows. With the controller designed in the previous part as the new nominal controller we can again construct an interval family of stabilizing controllers centered at the parameters of  $G_{\text{opt}}(s)$ . This can be done by several methods. We used the stability locus introduced by Tsytkin and Polyak [24] which is shown in Figure 10. Note that in this case the locus shows the  $\ell_2$  stability margin.

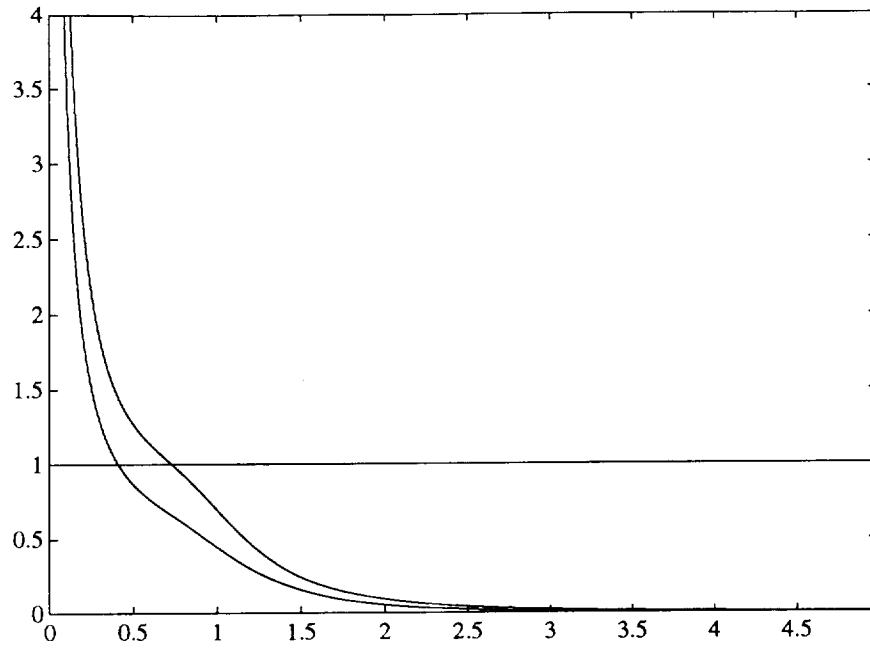


Figure 7. Magintude envelope plot

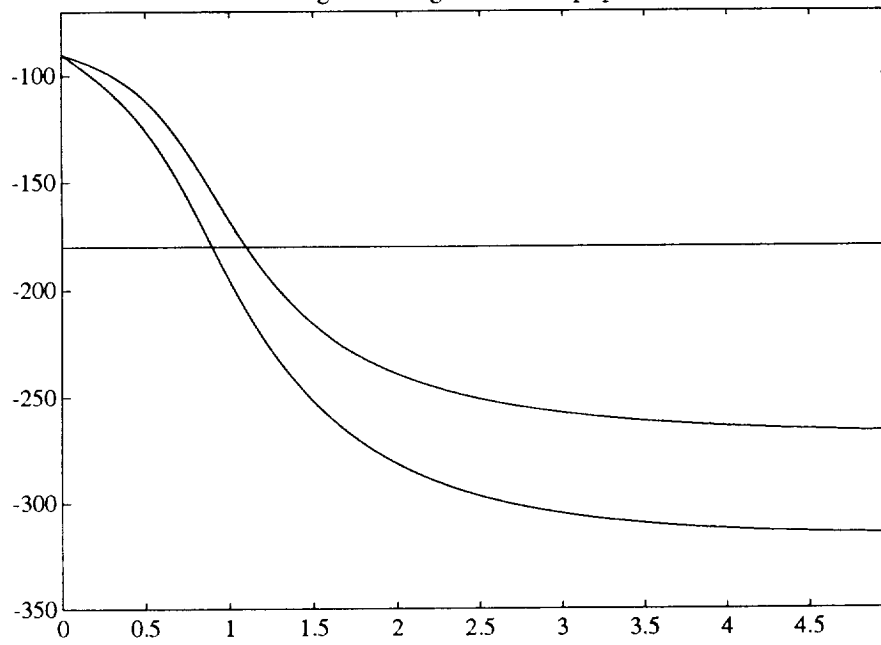


Figure 8. Phase envelope plot

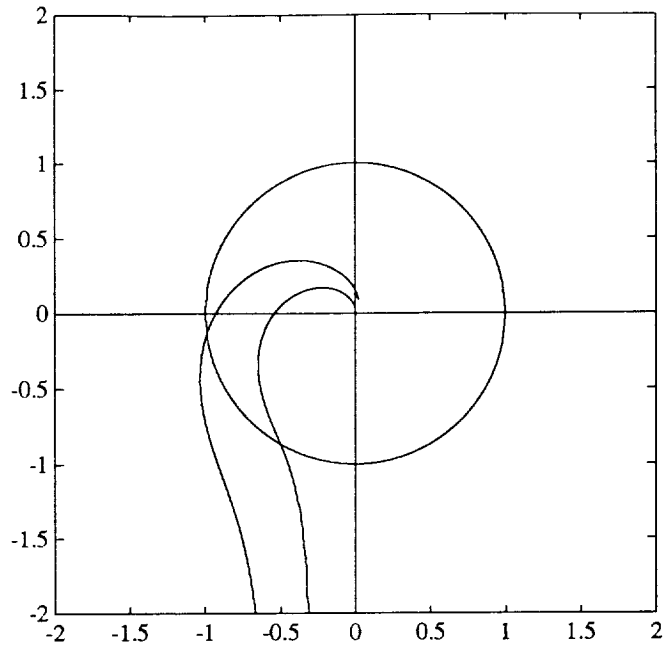


Figure 9. Nyquist envelope plot

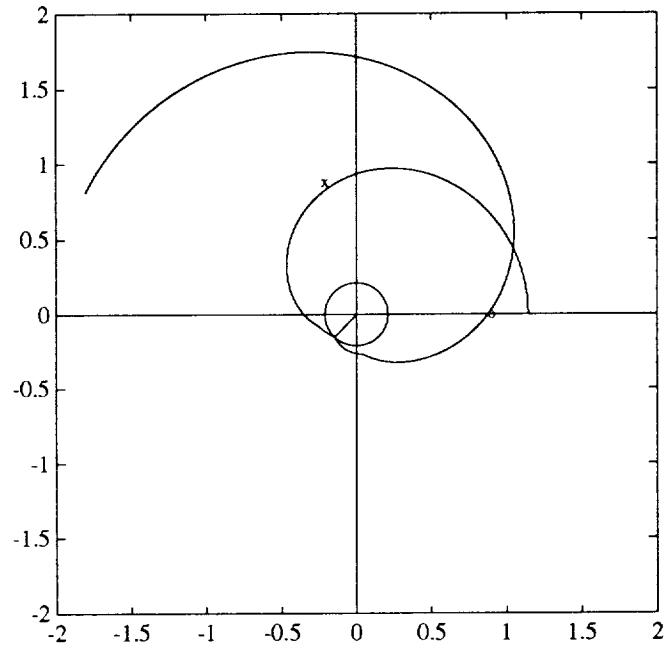


Figure 10. Stability locus (Tsyarkin - Polyak)

From this method, we have the parametric stability margin,  $\gamma = 0.2128$ . Thus we construct the stabilizing intervals around the nominal values  $n_i$  and  $d_i$  as follows:

$$n_i - \frac{\gamma}{2} \leq n_i \leq n_i + \frac{\gamma}{2} \qquad d_i - \frac{\gamma}{2} \leq d_i \leq d_i + \frac{\gamma}{2}$$



Consequently, we obtain

$$\mathbf{G}^{(1)}(s) = \left\{ \frac{n_1^{(1)}s + n_0^{(1)}}{d_2^{(1)}s^2 + d_1^{(1)}s + d_0^{(1)}} \right\}$$

where

$$\begin{aligned} n_0^{(1)} &\in [0.7936, 1.0064], & n_1^{(1)} &\in [0.0839, 0.2967] \\ d_0^{(1)} &\in [1.9936, 2.2064], & d_1^{(1)} &\in [1.8936, 2.1064], & d_2^{(1)} &\in [0.7936, 1.0064]. \end{aligned}$$

Now the previous optimization procedure can be applied to this new family of interval controllers. This yields the result:

$$\begin{aligned} G_{\text{opt}}^{(1)}(s) &= \frac{\lambda K_n^1(s) + (1 - \lambda)K_n^2(s)}{K_d^4(s)} \Big|_{\lambda=0} \\ &= \frac{0.2967s + 0.7936}{0.7936s^2 + 2.1064s + 2.2064} \end{aligned}$$

and the maximum gain margin obtained from this first iteration is

$$\bar{\rho}^{(1)} = 2.4295.$$

The corresponding phase margin is  $71.4506^\circ$  (i.e., clockwise rotation of  $71.4506$  degrees).

By repeating the same procedure until the relative improvement of the gain margin becomes small enough, we can obtain the “optimal” selection of the parameters for this problem.

$$G_{\text{opt}}^*(s) = \frac{.3359s + .0398}{.0398s^2 + 2.8602s + 2.9602}.$$

The maximum gain margin obtained from this second iteration is

$$\bar{\rho}^* = 562.3651$$

and the corresponding phase margin is  $95.77^\circ$  (i.e., clockwise rotation of  $95.77$  degrees). Figure 11 shows the Nyquist plot of the corresponding optimal system for each iteration.

## 10. MULTILINEAR INTERVAL SYSTEMS

In following sections of the report, we consider systems in which the uncertain parameters enter the characteristic polynomial affine multilinearly. As an example consider a feedback control system with a fixed compensator connected to a cascade of two interval plants as in the block diagram below:

We have the following expression for the characteristic polynomial

$$\delta(s) = F_1(s)P_{11}(s)P_{12}(s) + F_2(s)P_{21}(s)P_{22}(s) \quad (70)$$

with  $P_{ij}(s)$  being interval polynomials  $i = 1, 2, j = 1, 2$ . The Theorem derived in [10] cannot deal with the robust stability of the family (70) because it contains products

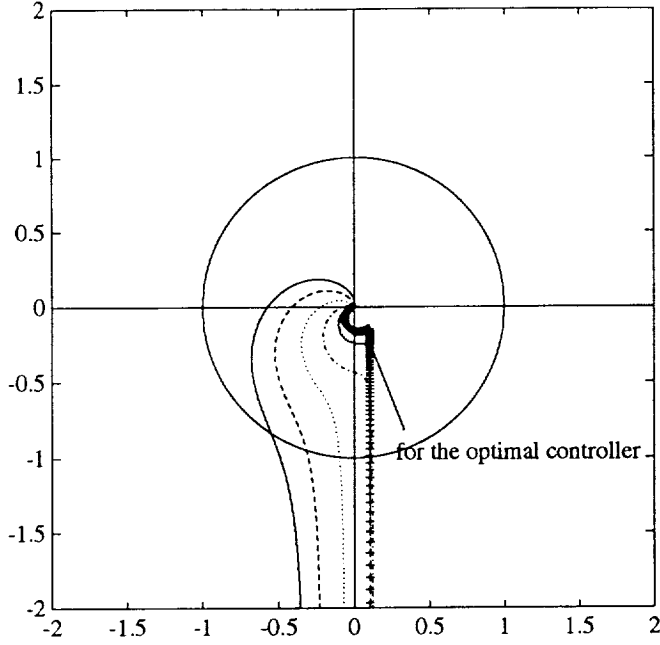


Figure 11. Nyquist plot for various controller parameters.

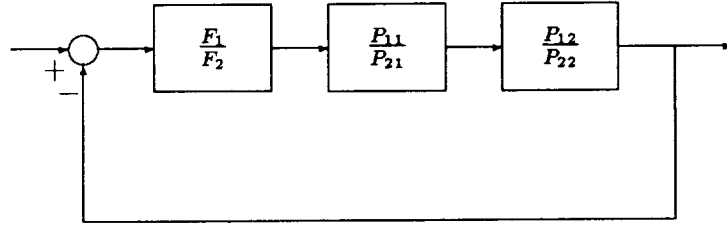


Figure 12. Unity Feedback System.

of interval polynomials. Neither can the stability of the family (70) be checked by using the Edge Theorem of [12] since it is not a polytope.

The above considerations motivate the problem of determining the Hurwitz stability of the family of polynomials

$$\delta(s) = F_1(s)P_{11}(s)P_{12}(s)\cdots P_{1r_1}(s) + \cdots + F_m(s)P_{m1}(s)P_{m2}(s)\cdots P_{mr_m}(s) \quad (71)$$

where  $F_i(s)$  are fixed, and the polynomials  $P_{ij}(s)$  are interval polynomials. The uncertainty set is therefore a box  $\Pi$  in the space of these coefficients. The family (70) is of this type represents the special case of (71) when products of uncertain polynomials do not occur. Characteristic polynomials of the form (71) always occur in control systems containing several interconnected subsystems with uncertain parameters. We remark here that the vector of uncertain parameters, namely the set of coefficients of the polynomials  $P_{ij}(s)$ , enters into the characteristic polynomial coefficients affine

multilinearly. Since we assume that these parameters vary within prescribed intervals we refer to the family (71) as a *multilinearly parametrized interval family*. This form of the characteristic polynomial occurs in state space descriptions with interval matrices and also in matrix fraction description of multivariable systems when the matrix factors contain interval polynomials.

We first introduce some notation. For any positive integer  $n$  let  $\underline{n}$  denote the set of integers  $\{1, 2, \dots, n\}$ . We consider the family of polynomials

$$\begin{aligned} \delta(s) = & F_1(s)P_{11}(s)P_{12}(s)\cdots P_{1r_1}(s) + \cdots \\ & + F_m(s)P_{m1}(s)P_{m2}(s)\cdots P_{mr_m}(s) \end{aligned} \quad (72)$$

where  $F_i(s)$  are fixed, and the polynomials  $P_{ij}(s)$  are interval with  $i \in \underline{m}, j \in \underline{r_i}$ . Let  $d^\circ(P_{ij})$  denote the degree of  $P_{ij}(s)$  and let  $p_{ij}^l$  denote the coefficient of  $s^l$  in  $P_{ij}(s)$ :

$$P_{ij}(s) = p_{ij}^0 + p_{ij}^1 s + \cdots + p_{ij}^l s^l + \cdots + p_{ij}^{d^\circ(P_{ij})} s^{d^\circ(P_{ij})}. \quad (73)$$

Write  $\underline{p}_{ij} := (p_{ij}^0, p_{ij}^1, \dots, p_{ij}^{d^\circ(P_{ij})})$  for each  $i \in \underline{m}, j \in \underline{r_i}$  and let

$$\mathbf{p} := [\underline{p}_{11}, \underline{p}_{12}, \dots, \underline{p}_{mr_m}] \quad (74)$$

denote the vector of coefficients of the polynomials  $P_{ij}(s)$ ,  $i \in \underline{m}, j \in \underline{r_i}$ . Each such coefficient belongs to a given interval:

$$p_{ij}^l \in [\alpha_{ij}^l, \beta_{ij}^l] \quad l = 0, \dots, d^\circ(P_{ij}), i \in \underline{m}, j \in \underline{r_i}. \quad (75)$$

In the space of these coefficients we have the box  $\Pi$  of uncertain parameters:

$$\Pi := \{\mathbf{p} | p_{ij}^l \in [\alpha_{ij}^l, \beta_{ij}^l], l = 0, \dots, d^\circ(P_{ij}), i \in \underline{m}, j \in \underline{r_i}\}. \quad (76)$$

Each point  $\mathbf{p} \in \Pi$  corresponds to a particular choice of the ordered set of polynomials  $P_{ij}(s), i \in \underline{m}, j \in \underline{r_i}$ . We write

$$\begin{aligned} \delta(s, \mathbf{p}) = & F_1(s)P_{11}(s)P_{12}(s)\cdots P_{1r_1}(s) + \cdots \\ & + F_m(s)P_{m1}(s)P_{m2}(s)\cdots P_{mr_m}(s) \end{aligned} \quad (77)$$

to display the explicit dependence of  $\delta(s)$  on  $\mathbf{p}$ . For a given fixed set of polynomials  $[F_1(s), F_2(s), \dots, F_m(s)] := \underline{F}$  let  $\Delta$  denote the family of polynomials generated by the map  $\underline{F} : \Pi \implies \delta$  as in (72) and obtained by letting the parameter vector  $\mathbf{p}$ , (equivalently, the polynomials  $P_{ij}(s)$ ), range over the box  $\Pi$  described in (76). In other words

$$\Delta := \{\delta(s, \mathbf{p}) | \mathbf{p} \in \Pi\}. \quad (78)$$

The four Kharitonov polynomials associated with the family of interval polynomials corresponding to  $P_{ij}(s)$  are

$$\begin{aligned} K_{ij}^1(s) &= \alpha_{ij}^0 + \alpha_{ij}^1 s + \beta_{ij}^2 s^2 + \beta_{ij}^3 s^3 + \cdots \\ K_{ij}^2(s) &= \alpha_{ij}^0 + \beta_{ij}^1 s + \beta_{ij}^2 s^2 + \alpha_{ij}^3 s^3 + \cdots \\ K_{ij}^3(s) &= \beta_{ij}^0 + \alpha_{ij}^1 s + \alpha_{ij}^2 s^2 + \beta_{ij}^3 s^3 + \cdots \\ K_{ij}^4(s) &= \beta_{ij}^0 + \beta_{ij}^1 s + \alpha_{ij}^2 s^2 + \alpha_{ij}^3 s^3 + \cdots, \end{aligned}$$

and these definitions hold for each  $i \in \underline{m}, j \in \underline{r}_i$ . Corresponding to each  $P_{ij}(s)$  we define the four polynomial segments

$$\begin{aligned} S_{ij}^1 &:= [K_{ij}^1(s), K_{ij}^2(s)], & S_{ij}^2 &:= [K_{ij}^1(s), K_{ij}^3(s)] \\ S_{ij}^3 &:= [K_{ij}^2(s), K_{ij}^4(s)], & S_{ij}^4 &:= [K_{ij}^3(s), K_{ij}^4(s)], \end{aligned}$$

which we call the *Kharitonov segments*. These segments were introduced originally in [10]. A typical element of the segment  $S_{ij}^1$ , for example, is a polynomial, denoted by  $S_{ij}^1(\lambda, s)$  which is a convex combination of the form called the *Kharitonov segments*

$$(1 - \lambda)K_{ij}^1(s) + \lambda K_{ij}^2(s) := S_{ij}^1(\lambda, s), \quad \lambda \in [0, 1].$$

We now need to introduce some special subsets of  $\Pi$  called the *extremal manifolds*. Fixing  $i = l$  we let  $\Pi_l \subset \Pi$  denote the union of all the  $r_l$  dimensional linear manifolds obtained by letting  $P_{ij}(s), i \neq l$  range over the corresponding Kharitonov polynomials  $K_{ij}^k(s), k \in \underline{4}, j \in \underline{r}_i, i \in \underline{m}$ , and  $P_{lj}(s)$ , range over the Kharitonov segments  $S_{lj}^k, k \in \underline{4}, j \in \underline{r}_l$ . The extremal manifolds  $\Pi_l, l \in \underline{m}$  map into the corresponding multilinear surfaces  $\Delta_l \subset \Delta, l \in \underline{m}$  in the space of coefficients of the polynomials  $\delta(s)$ , under the previously defined mapping. More concretely,

$$\begin{aligned} \Pi_l &:= \\ &\{[K_{11}^{i(1,1)}(s)K_{12}^{i(1,2)}(s) \cdots K_{1r_1}^{i(1,r_1)}(s), K_{21}^{i(2,1)} \cdots K_{2r_2}^{i(2,r_2)}(s), \dots, \\ &S_{l1}^{i(l,1)}(\lambda_1, s)S_{l2}^{i(l,2)}(\lambda_2, s) \cdots S_{lr_l}^{i(l,r_l)}(\lambda_{r_l}, s), \dots, \\ &K_{m1}^{i(m,1)}(s)K_{m2}^{i(m,2)}(s) \cdots K_{mr_m}^{i(m,r_m)}(s)] \mid \\ &i(k, n) \in \underline{4}, k \in \underline{m}, n \in \underline{r}_k, \lambda_j \in [0, 1], j \in \underline{r}_l.\} \end{aligned}$$

and

$$\begin{aligned} \Delta_l &:= \{ \delta(s) = F_1(s) \cdot K_{11}^{i(1,1)}(s) \cdot K_{12}^{i(1,2)}(s) \cdots K_{1r_1}^{i(1,r_1)}(s) \\ &+ \cdots + F_l(s) \cdot S_{l1}^{i(l,1)}(\lambda_1, s) \cdot S_{l2}^{i(l,2)}(\lambda_2, s) \cdots S_{lr_l}^{i(l,r_l)}(\lambda_{r_l}, s) + \cdots \\ &\cdots + F_m(s) \cdot K_{m1}^{i(m,1)}(s) \cdot K_{m2}^{i(m,2)}(s) \cdots K_{mr_m}^{i(m,r_m)}(s) \\ &\mid i(k, n) \in \underline{4}, k \in \underline{m}, n \in \underline{r}_k, \lambda_j \in [0, 1], j \in \underline{r}_l.\} \end{aligned}$$

Equivalently

$$\Delta_l := \{ \delta(s, \mathbf{p}) \mid \mathbf{p} \in \Pi_l \}.$$

Finally, let

$$\Pi_E := \bigcup_{l=1}^m \Pi_l$$

denote the set of all linear *extremal manifolds* and let the corresponding set of multilinear manifolds

$$\Delta_E := \bigcup_{l=1}^m \Delta_l = \{ \delta(s, \mathbf{p}) \mid \mathbf{p} \in \Pi_E \}.$$

Because there is a one to one correspondence between the elements of  $\Pi_E$  and of  $\Delta_E$  we refer to both sets as extremal manifolds. Define the *Kharitonov vertices*,  $\mathbf{K}(\Pi)$ , of  $\Pi$  to be the subset of all vertices of  $\Pi$  corresponding to the Kharitonov polynomials of the  $P_{ij}(s)$ .

To illustrate the definition of the manifolds consider the special case of the family (70). In this case each manifold in  $\Pi_1$  is the union of polynomial vectors of the form:

$$[S_{11}^i(\lambda_1, s), S_{12}^j(\lambda_2, s), K_{21}^k(s), K_{12}^l(s)], \quad (79)$$

where  $\lambda_t \in [0, 1]$ ,  $t \in \underline{2}$  and  $(i, j, k, l)$  range over  $(\underline{4} \times \underline{4} \times \underline{4} \times \underline{4})$ . Similarly  $\Pi_2$  consists of the union of the polynomial vectors

$$[K_{11}^i(s), K_{12}^j(s), S_{21}^k(\lambda_1, s), S_{22}^l(\lambda_2, s)] \quad (80)$$

where  $(i, j, k, l)$  range over  $(\underline{4} \times \underline{4} \times \underline{4} \times \underline{4})$  and  $\lambda_t \in [0, 1]$ ,  $t \in \underline{2}$ . The polynomial manifold  $\Delta_1$  consists of polynomials of the form:

$$F_1(s)S_{11}^i(\lambda_1, s)S_{12}^j(\lambda_2, s) + F_2(s)K_{21}^k(s)K_{12}^l(s), \lambda_t \in [0, 1], t \in \underline{2}, \quad (81)$$

and  $\Delta_1$  consists of the union of such manifolds obtained by letting  $(i, j, k, l)$  range over  $(\underline{4} \times \underline{4} \times \underline{4} \times \underline{4})$ . Similarly, the manifolds contained in  $\Delta_2$  are of the form:

$$F_1(s)K_{11}^i(s)K_{12}^j(s) + F_2(s)S_{21}^k(\lambda_1, s)S_{22}^l(\lambda_2, s), \lambda_t \in [0, 1], t \in \underline{2}, \quad (82)$$

where  $(i, j, k, l)$  range over  $(\underline{4} \times \underline{4} \times \underline{4} \times \underline{4})$ .

It is not difficult to see that the number of distinct manifolds in  $\Pi_l$  in the most general case when all the Kharitonov polynomials associated with each polynomial  $P_{ij}(s)$  are distinct, is  $4^{(r_1+r_2+\dots+r_m)}$ . Since this holds true for each  $l \in \underline{m}$  the total number of extremal manifolds in  $\Pi_E$  or  $\Delta_E$  is  $m4^R$ ,  $R = r_1 + r_2 + \dots + r_m$ . With these preliminaries we are ready to state the main result of the next section.

## 11. STABILITY OF MULTILINEAR MANIFOLDS

### 11.1. The Multilinear Theorem

In this section we give necessary and sufficient conditions for the Hurwitz stability of the family (71). Using the notation introduced in the last section we shall say that  $\underline{F}$  stabilizes the family  $\Pi$  if and only if each polynomial of the family  $\Delta$  is Hurwitz stable. Similarly we shall say that  $\underline{F}$  stabilizes  $\Pi_E$  if and only if every polynomial in  $\Delta_E$  is Hurwitz stable.

**Theorem 18. (Multilinear Theorem)**  $\underline{F}$  stabilizes  $\Pi$  if and only if  $\underline{F}$  stabilizes  $\Pi_E$ .

The proof of this theorem is based on induction and may be found in [19].

*Remark 1.* The assumption of independence of the perturbations can be easily relaxed. The reader is referred to [19] for the details.

## 11.2. Simple Determination of Stability of Two Dimensional Multilinear Manifolds

In this section we consider the problem of checking the stability of a extremal manifold of dimension 2. This case will arise when  $r_i = 2$  in (71) and is interesting because it can be solved analytically. Consider therefore the following two dimensional manifold

$$\begin{aligned} \delta(\lambda_1, \lambda_2, s) &= p_0(s)\lambda_1\lambda_2 + p_1(s)\lambda_1 + p_2(s)\lambda_2 + p_3(s), \\ \lambda_i &\in [0, 1], i = 1, 2. \end{aligned} \quad (83)$$

Assuming the four vertices are stable, the manifold  $\delta(\lambda_1, \lambda_2, s)$  is unstable if and only if it has a  $j\omega$  root for a set of real values  $(\lambda_1, \lambda_2) \in [0, 1] \times [0, 1]$ . To test for this we separate (83) into real and imaginary parts after substituting  $s = j\omega$ , and set them equal to zero. This gives

$$p_{0r}(\omega)\lambda_1\lambda_2 + p_{1r}(\omega)\lambda_1 + p_{2r}(\omega)\lambda_2 + p_{3r}(\omega) = 0 \quad (84)$$

$$p_{0i}(\omega)\lambda_1\lambda_2 + p_{1i}(\omega)\lambda_1 + p_{2i}(\omega)\lambda_2 + p_{3i}(\omega) = 0 \quad (85)$$

where

$$p_k(s)|_{s=j\omega} := p_{kr}(\omega) + jp_{ki}(\omega), \quad \text{for } k = 0, 1, 2, 3.$$

From (84), we have

$$[p_{0r}(\omega)\lambda_2 + p_{1r}(\omega)]\lambda_1 + p_{2r}(\omega)\lambda_2 + p_{3r}(\omega) = 0 \quad (86)$$

and

$$\lambda_1 = -\frac{p_{2r}(\omega)\lambda_2 + p_{3r}(\omega)}{p_{0r}(\omega)\lambda_2 + p_{1r}(\omega)}. \quad (87)$$

Similarly, from (85),

$$\lambda_1 = -\frac{p_{2i}(\omega)\lambda_2 + p_{3i}(\omega)}{p_{0i}(\omega)\lambda_2 + p_{1i}(\omega)}. \quad (88)$$

Since  $\lambda_1 = \infty \notin [0, 1]$  we can without loss of generality, deal only with the case in which the denominators of (87) and (88) are nonzero. By equating (87) and (88) we have,

$$\begin{aligned} & [p_{2i}(\omega)p_{0r}(\omega) - p_{0i}(\omega)p_{2r}(\omega)]\lambda_2^2 \\ & + [p_{3i}(\omega)p_{0r}(\omega) + p_{2i}(\omega)p_{1r}(\omega) - p_{0i}(\omega)p_{3r}(\omega) - p_{1i}(\omega)p_{2r}(\omega)]\lambda_2 \\ & + [p_{3i}(\omega)p_{1r}(\omega) - p_{1i}(\omega)p_{3r}(\omega)] = 0 \end{aligned} \quad (89)$$

From (89) and (87) we can solve for  $\lambda_1(\omega)$  and  $\lambda_2(\omega)$  and verify if  $(\lambda_1(\omega), \lambda_2(\omega))$  intersects the set  $[0, 1] \times [0, 1]$  for some  $\omega$ . If the intersection is empty the manifold is Hurwitz stable otherwise it is unstable.

## 12. EXTREMAL PARAMETRIC STABILITY PROPERTY: MULTILINEAR CASE

We now consider the family  $\Delta$  and the manifolds  $\Pi_E$  and  $\Delta_E$  which occur in the Multilinear Theorem of the last section. As before let

$$\mathbf{p} := [p_{11}, p_{12} \cdots p_{mr,m}]$$

denote the  $n$  dimensional parameter vector consisting of the ordered set of coefficients of the polynomials  $P_{ij}(s)$  and let  $\mathbf{p} \in R^n$  vary in the prescribed box  $\Pi$  specified by the given upper and lower bounds:

$$p_{ij}^l \in [\alpha_{ij}^l, \beta_{ij}^l] \quad l = 0, \dots, d^\circ(P_{ij}), i \in \underline{m}, j \in \underline{r}$$

Let  $\|\cdot\|$  denote any norm in  $R^n$  and let  $\mathcal{P}_u$  denote the set of points  $\mathbf{u}$  in  $R^n$  for which  $\delta(s, \mathbf{u})$  is unstable or loses degree (relative to its generic degree over  $\Pi$ ). Let

$$\rho(\mathbf{p}) = \inf_{\mathbf{u} \in \mathcal{P}_u} \|\mathbf{p} - \mathbf{u}\|_p$$

denote the radius of the stability ball (measured in the norm  $\|\cdot\|$ ) and centered at the point  $\mathbf{p}$ . This number serves as the stability margin associated with the point  $\mathbf{p}$ . If the box  $\Pi$  is stable we can associate a stability margin with each point in  $\Pi$ . A natural question to ask then is: Is there a point in  $\Pi$  which is closest to instability in the norm  $\|\cdot\|$  and where is it? The answer to that question is provided in the following theorem.

As before we define a mapping from  $\Pi$  to the set of all positive real numbers:

$$\begin{aligned} \Pi &\xrightarrow{\rho} \mathcal{R}^+ \setminus \{0\} \\ \mathbf{p} &\longrightarrow \rho(\mathbf{p}) \end{aligned}$$

Our question stated in terms of functions is: Has the function  $\rho(\mathbf{p})$  a minimum and is there a precise point in  $\Pi$  where it is reached?

**Theorem 19. (Extremal property of the stability manifolds)** *The function*

$$\begin{aligned} \Pi &\xrightarrow{\rho} \mathcal{R}^+ \setminus \{0\} \\ \mathbf{p} &\longrightarrow \rho(\mathbf{p}) \end{aligned}$$

*has a minimum which is reached at a point on the extremal manifolds  $\Pi_E$ .*

The proof of this theorem may be found in [19] and omitted here.

## 13. PARAMETRIC AND UNSTRUCTURED PERTURBATIONS: MULTILINEAR CASE

In this section we will analyze the problem of robust stability in the presence of both parameter variations and unstructured perturbations modelled in the usual way as norm bounded perturbations. The subject of robust stability under mixed types of

perturbations is of current interest (see for example [25], [26], [20], [21] and [27]). We model this situation by considering a multilinear interval plant, namely one whose transfer function is a ratio of polynomials of the type that was introduced in 71. To be specific we will consider single-input, single-output, proper, stable systems with transfer function of the form

$$g(s) = \frac{\gamma(s)}{\delta(s)}$$

Here

$$\begin{aligned} \gamma(s) = & H_1(s)L_{11}(s)L_{12}(s)\cdots L_{1r_1}(s) + \cdots \\ & + H_m(s)L_{m1}(s)L_{m2}(s)\cdots L_{mr_m}(s) \end{aligned}$$

where the polynomials  $H_i(s)$  are fixed and the polynomials  $L_{ij}(s)$  are interval polynomials, that is their coefficients vary in a prescribed box  $\Lambda$ ; the corresponding family of polynomials  $\gamma(s)$  is denoted by  $\Gamma$ . We suppose as before that

$$\begin{aligned} \delta(s) = & F_1(s)P_{11}(s)P_{12}(s)\cdots P_{1r_1}(s) + \cdots \\ & + F_m(s)P_{m1}(s)P_{m2}(s)\cdots P_{mr_m}(s) \end{aligned}$$

where the polynomials  $F_i(s)$  are fixed, the polynomials  $P_{ij}(s)$  are interval polynomials, with coefficients that vary in the prescribed box  $\Pi$  and the resulting family of polynomials  $\delta(s)$  is denoted  $\Delta$ . As in Section 2 we let  $\mathbf{p}$  denote the vector of coefficients of the polynomials  $\{P_{ij}(s)\}$  and we similarly let  $\mathbf{l}$  denote the vector of coefficients  $\{L_{ij}(s)\}$ . We also denote explicitly, the dependence of  $\delta(s)$  on  $\mathbf{p}$  and of  $\nu(s)$  on  $\mathbf{l}$  by writing  $\delta(s, \mathbf{p})$  and  $\nu(s, \mathbf{l})$  whenever necessary. It is assumed that the parameters  $\mathbf{p}$  and  $\mathbf{l}$  perturb independently. From these polynomial families we form the parametrized family of transfer functions

$$\mathcal{G} = \left\{ \frac{\gamma(s, \mathbf{l})}{\delta(s, \mathbf{p})} \mid \mathbf{p} \in \Pi, \text{ and } \mathbf{l} \in \Lambda \right\}. \quad (90)$$

To display the dependence of a typical element  $g(s)$  of  $\mathcal{G}$  on  $\mathbf{l}$  and  $\mathbf{p}$  we write  $g(s, \mathbf{p}, \mathbf{l})$ . Introduce the Kharitonov polynomials and segments associated respectively with the  $P_{ij}(s)$  and  $L_{ij}(s)$  respectively. As in Section 2 these are used to generate the extremal subsets  $\Pi_E$  of  $\Pi$  and  $\Lambda_E$  of  $\Lambda$  respectively. The Kharitonov extreme points of  $\Pi$  and  $\Lambda$  are denoted respectively by  $\mathbf{K}(\Pi)$  and  $\mathbf{K}(\Lambda)$ . Finally, we denote the polynomial manifolds resulting from  $\mathbf{K}(\Pi)$ ,  $\mathbf{K}(\Lambda)$ ,  $\Lambda_E$  and  $\Pi_E$  as follows:

$$\Gamma_E = \{\gamma(s, \mathbf{l}) \mid \mathbf{l} \in \Lambda_E\}, \Gamma_K = \{\gamma(s, \mathbf{l}) \mid \mathbf{l} \in \mathbf{K}(\Lambda)\}$$

$$\Delta_E = \{\delta(s, \mathbf{p}) \mid \mathbf{p} \in \Pi_E\}, \Delta_K = \{\delta(s, \mathbf{p}) \mid \mathbf{p} \in \mathbf{K}(\Pi)\}.$$

The main results in this section will deal with the calculation of the  $H_\infty$  stability margin for systems containing parameter uncertainty as defined above. In the following we will use the standard notation:  $\mathbf{C}_+ := \{s \in \mathbf{C} : \text{Re}(s) \geq 0\}$ , and  $H_\infty(\mathbf{C}_+)$  will represent the space of functions  $f(s)$  that are bounded and analytic in  $\mathbf{C}_+$  with the standard  $H_\infty$  norm,

$$\|f\|_\infty = \sup_{\omega \in \mathbf{R}} |f(j\omega)|.$$



To determine the unstructured stability margin of the family  $\mathcal{G}$  we need to determine the supremum of the  $H_\infty$  norm of certain transfer functions over  $\mathcal{G}$ . Specifically we formulate the following problems: Let  $W(s)$  be a scalar stable weight, with a stable inverse, and write  $W(s) = \frac{n_w(s)}{d_w(s)}$ .

- A) Consider the feedback configuration shown in Figure 13,  $\mathcal{G}$  is a stable family, and  $\Delta P$  is any  $H_\infty$  perturbation that satisfies  $\|\Delta P\| < \alpha$ .

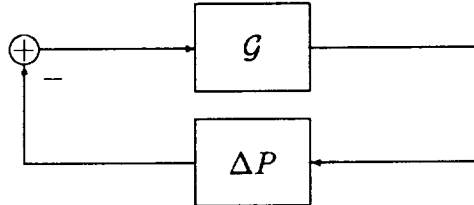


Figure 13. Multiplicative Perturbations

- B) Consider the feedback configuration shown in Figure 14,  $\Delta P$  is any  $H_\infty$  perturbation that satisfies  $\|\Delta P\| < \alpha$ , and  $C$  is a controller that simultaneously stabilizes every element in the set  $\mathcal{G}$ .

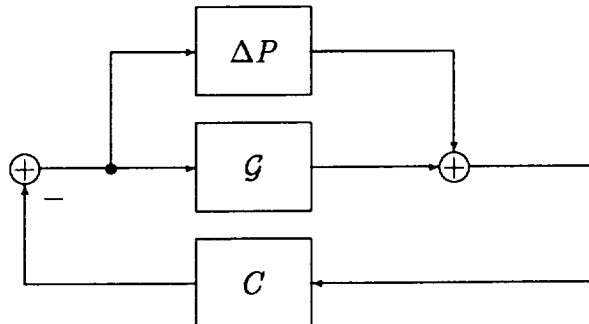


Figure 14. Additive Perturbations

The above problems are generalized versions of standard  $H_\infty$  robust stability problems (see [28]) where a fixed plant is considered. The solution is accomplished once again by showing that the  $H_\infty$  norms in question attain their supremum value over a certain extremal set of transfer functions  $\mathcal{G}_E \subset \mathcal{G}$ . This set is defined as follows:

$$\mathcal{G}_E := \left\{ \frac{\gamma(s, \mathbf{l})}{\delta(s, \mathbf{p})} \mid (\mathbf{l} \in \mathbf{K}(\Lambda), \mathbf{p} \in \Pi_E) \text{ or } (\mathbf{l} \in \Lambda_E, \mathbf{p} \in \mathbf{K}(\Pi)) \right\}.$$

We can now state the main result of this Section.

**Theorem 20. (Extremal properties)**

$$A) \quad \sup_{g \in \mathcal{G}} \|Wg\|_\infty = \sup_{g \in \mathcal{G}_B} \|Wg\|_\infty,$$

$$B) \quad \sup_{g \in \mathcal{G}} \|WC(1 + gC)^{-1}\|_\infty = \sup_{g \in \mathcal{G}_B} \|WC(1 + gC)^{-1}\|_\infty.$$

**Corollary 2. (Unstructured Margins)**

1) *The configuration of Figure 13 will be stable if and only if  $\alpha$  satisfies*

$$\alpha \leq \frac{1}{\sup_{g \in \mathcal{G}_B} \|g\|_\infty} := \alpha_o^*.$$

2) *The configuration of Figure 14 will be stable if and only if  $\alpha$  satisfies*

$$\alpha \leq \frac{1}{\sup_{g \in \mathcal{G}_B} \|C(1 + gC)^{-1}\|_\infty} := \alpha_c^*.$$

The proof of this theorem is similar to that used in [20] and details are omitted here. The idea behind this approach is to replace the question of finding an upper bound of the  $H_\infty$  norm of a transfer function by an equivalent question concerning the stability of a certain parametrized family of polynomials, for which the results of the previous sections apply. For this purpose we need the following lemma [20] which gives a characterization of proper rational functions  $g(s)$  which are in  $H_\infty(\mathbf{C}_+)$  and which satisfy  $\|g\|_\infty < 1$ .

**Lemma 7.** *Let  $h(s) = n(s)/d(s)$  be a proper (real or complex) rational function in  $H_\infty(\mathbf{C}_+)$ , with  $\deg(d(s)) = q$ , then  $\|h\|_\infty < 1$  if and only if*

$$a1) |n_q| < |d_q|,$$

$$b1) d(s) + e^{j\theta} n(s) \text{ is Hurwitz for all } \theta \text{ in } [0, 2\pi).$$

*Remark 2.* The quantities  $\alpha_o^*$  and  $\alpha_c^*$  serve as unstructured  $H_\infty$  stability margins for the respective open and closed loop parametrized systems treated in Problems I and II.

## 14. PARAMETRIC AND NONLINEAR PERTURBATIONS: MULTI-LINEAR CASE

Another stability robustness problem that involves structured and unstructured perturbations is the classical Lur'e problem of nonlinear control theory. This problem considers a fixed linear time invariant system subjected to perturbations in the form of nonlinear feedback gains contained in a prescribed sector. In [21] a robust version of the Lur'e problem was treated. In this formulation the fixed linear system is replaced by the more realistic model of a parametrized family of plants. The "nonlinear stability margin" of the system can be determined by finding the infimum, over the parametrized family, of such stability sectors. From standard results on the Lur'e

problem, the size of such a sector can be determined by finding the infimum of the real part of  $g(j\omega)$  as  $g$  ranges over the parametrized family. In [21] it was shown how the strict positive real (SPR) property for a stable family of interval systems can be determined from a set of sixteen plants called the *Kharitonov systems*. In addition, in the presence of a fixed controller that stabilizes an entire family of interval systems, the SPR property for the family of transfer functions  $C(1 + gC)^{-1}$  is determined from a set of 32 one parameter family of systems. Here we consider the more general situation where the parametrized family considered is the family  $\mathcal{G}$  defined in the previous section. Using the extremal properties established in the last section and the proof developed in [21], it is possible to establish the following theorem. The proof is omitted as it is very similar to that of the last section.

**Theorem 21. (Extremal properties)**

1) Let  $\mathcal{G}$  be the multilinear family defined above, and assume that  $\mathcal{G}$  is stable then

$$\inf_{g \in \mathcal{G}} \inf_{\omega \in \mathbb{R}} \operatorname{Re}(W(j\omega)g(j\omega)) = \inf_{g \in \mathcal{G}_{\mathbb{B}}} \inf_{\omega \in \mathbb{R}} \operatorname{Re}(W(j\omega)g(j\omega)).$$

2) If  $C$  is a controller that stabilizes the entire family  $\mathcal{G}$ , then

$$\begin{aligned} \inf_{g \in \mathcal{G}} \inf_{\omega \in \mathbb{R}} \operatorname{Re}(WC(1 + gC)^{-1}(j\omega)) = \\ \inf_{g \in \mathcal{G}_{\mathbb{B}}} \inf_{\omega \in \mathbb{R}} \operatorname{Re}(WC(1 + gC)^{-1}(j\omega)). \end{aligned}$$

**15. CONCLUDING REMARKS**

The summary of results presented here form the beginnings of a complete theory of interval control systems. We expect such a theory to develop over the next few years. We expect such a theory to impact on the design of control systems filters and communication systems.

**REFERENCES**

- [1] S. P. Bhattacharyya and L. H. Keel (Eds.), *Control of Uncertain Dynamic Systems*. Littleton, MA: CRC Press, 1991.
- [2] L. H. Keel and S. P. Bhattacharyya, "Frequency domain design of interval controllers," in *Control of Uncertain Dynamic Systems*, (S. P. Bhattacharyya and L. H. Keel, eds.), Littleton, MA: CRC Press, September 1991.
- [3] L. H. Keel and S. P. Bhattacharyya, "Extremal gain and phase margins for interval systems," in *Proceedings of 1991 AIAA Guidance, Navigation, and Control Conference*, (New Orleans, LA.), August 1991.
- [4] S. Ahamd and L. H. Keel, "Robust lead-lag compensation for uncertain linear systems," in *Proceedings of the 1992 IEEE Symposium on Circuits and Systems*, (San Diego, CA.), 1992.

- [5] S. P. Bhattacharyya and L. H. Keel, "Robust stability and control of linear and multilinear interval systems," *Control and Dynamic Systems*, vol. 51, pp. 31 – 78, 1992. Academic Press.
- [6] H. Chapellat, S. P. Bhattacharyya, and L. H. Keel, "Stability margin for Hurwitz polynomials," in *Proceedings of the 27th IEEE Conference on Decision and Control*, (Austin, TX.), pp. 1392 – 1398, December 1988.
- [7] H. Chapellat, L. H. Keel, and S. P. Bhattacharyya, "Stability margins for multilinear interval control systems," in *Proceedings of the 30th IEEE Conference on Decision and Control*, (Brighton, U.K.), December 1991.
- [8] L. H. Keel, J. Shaw, and S. P. Bhattacharyya, "Robust control of interval systems," in *Proceedings of 1991 Robust Control Workshop*, Tokyo, Japan: Springer-Verlag, June 1992.
- [9] V. L. Kharitonov, "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations," *Differential Uravnen*, vol. 14, pp. 2086 – 2088, 1978.
- [10] H. Chapellat and S. P. Bhattacharyya, "A generalization of Kharitonov's theorem: robust stability of interval plants," *IEEE Transactions on Automatic Control*, vol. AC - 34, pp. 306 – 311, March 1989.
- [11] S. P. Bhattacharyya, "Robust parametric stability: the role of the CB segments," in *Control of Uncertain Dynamic Systems*, (S. P. Bhattacharyya and L. H. Keel, eds.), Littleton, MA: CRC Press, September 1991.
- [12] A. C. Bartlett, C. V. Hollot, and H. Lin, "Root location of an entire polytope of polynomials: it suffices to check the edges," *Mathematics of Controls, Signals and Systems*, vol. 1, pp. 61 – 71, 1988.
- [13] H. Chapellat and S. P. Bhattacharyya, "An alternative proof of Kharitonov's theorem," *IEEE Transactions on Automatic Control*, vol. AC - 34, pp. 448 – 450, April 1989.
- [14] I. R. Petersen, "A new extension to Kharitonov's theorem," in *Proceedings of IEEE Conference on Decision and Control*, (Los Angeles, CA.), December 1987.
- [15] C. V. Hollot and F. Yang, "Robust stabilization of interval plants using lead or lag compensators," *Systems and Control Letters*, vol. 14, pp. 9 – 12, 1990.
- [16] M. Mansour and F. J. Kraus, "Argument conditions for Hurwitz and Schur stable polynomials and the robust stability problem," Tech. Rep., ETH, Zurich, Tech. Report, 1990.
- [17] S. P. Bhattacharyya, "Vertex results in robust stability," Tech. Rep., ISE Report, Tennessee State University, April 1991.
- [18] H. Chapellat and S. P. Bhattacharyya, "Calculation of maximal stability domains using an optimal property of Kharitonov polynomials," in *Analysis and Optimization of Systems, Lecture Notes in Control and Information Sciences*, pp. 22 – 31, Springer - Verlag, 1988.

- [19] H. Chapellat, M. Dahleh, and S. P. Bhattacharyya, "Extremal manifolds in robust stability," Tech. Rep., ISE Report, Tennessee State University, July 1990.
- [20] H. Chapellat, M. Dahleh, and S. P. Bhattacharyya, "Robust stability under structured and unstructured perturbations," *IEEE Transactions on Automatic Control*, vol. AC - 35, pp. 1100 - 1108, October 1990.
- [21] H. Chapellat, M. Dahleh, and S. P. Bhattacharyya, "On robust nonlinear stability of interval control systems," *IEEE Transactions on Automatic Control*, vol. AC - 36, pp. 59 - 67, January 1991.
- [22] A. Tesi and A. Vicino, "Kharitonov segments suffice for frequency response analysis of plant-controller families," in *Control of Uncertain Dynamic Systems*, (S. P. Bhattacharyya and L. H. Keel, eds.), Littleton, MA: CRC Press, September 1991.
- [23] L. H. Keel, J. Shaw, and S. P. Bhattacharyya, "Robust control under parameter variations," Tech. Rep., ISE Tech. Report, Tennessee State University, July 1991. Also in TCSP Tech. Report, Texas A&M University, July, 1991.
- [24] Ya. Z. Tsypkin and B. T. Polyak, "Frequency domain criteria for  $\ell_p$ -robust stability of continuous linear systems," *IEEE Transactions on Automatic Control*, vol. AC-36, pp. 1464 - 1469, December 1991.
- [25] M. K. H. Fan and A. L. Tits, "Robustness in the presence of joint parametric uncertainty and unmodeled dynamics," in *Proceedings of 1988 American Control Conference*, (Atlanta, GA.), June 1988.
- [26] D. Hinrichsen and A. J. Prichard, "Robustness of stability of linear state space systems with respect to time-varying, nonlinear and dynamic perturbations," in *Proceedings of the 28th IEEE Conference on Decision and Control*, (Tampa, FL.), pp. 52 - 53, December 1989.
- [27] M. K. H. Fan, A. L. Tits, and J. C. Doyle, "Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics," *IEEE Transactions on Automatic Control*, vol. AC - 36, pp. 25 - 38, January 1991.
- [28] B. Francis, *A Course in  $H^\infty$  Control Theory*, *Lecture Notes in Control and Information Sciences*. Springer - Verlag, 1987.

## 16. Appendix: Publications partially supported by NAG-1-863

### Book

1. S.P. Bhattacharyya and L.H. Keel (Eds.) *Control of Uncertain Dynamic Systems*, CRC Press, 1991.

### Publications

1. H. Chapellat, S.P. Bhattacharyya, and L.H. Keel, "Stability margin for Hurwitz polynomials," *Proceedings of the IEEE Conference on Decision and Control*, Austin, TX, December 1988. also in *Recent Advances in Robust Control* (P. Dorato and R.K. Yedavalli, Eds), IEEE Press, 1989.
2. L.H. keel, K.B. Lim, and J.-N. Juang, "Structured robustness with suboptimal responses for linear state space models," *Proceedings of 1989 AIAA Guidance, Navigation, and Control Conference*, Boston, MA, August 1989.
3. L.H. Keel and S.P. Bhattacharyya, "State space design of low order stabilizers," *IEEE Transactions on Automatic Control*, vol. 35, no. 2, pp. 182 - 186, February 1990.
4. L.H. Keel and S.P. Bhattacharyya, "Matrix equation approach for designing low order regulator," *SIAM Journal on Matrix Analysis and Applications*, vol. 11, no. 2, pp. 180 - 199, April 1990.
5. B. Shafai and L.H. Keel, "New pole placement algorithm: polynomial matrix approach," *Proceedings of the 1990 American Control Conference*, San Diego, CA, May 23, 1990.
6. B. Shafai and L.H. Keel, "Theorem of alternative: engenvalue distribution and block companion form in linear system application," *1990 Conference on New Trends in Systems Theory*, Genoa, Italy, July 1990.
7. B.Y. Harris and L.H. keel, "Reliable controller design against arbitrary loop failures," presented at *1990 SIAM Annual Meeting*, Chicago, IL, July 1990.
8. B. Shafai, L.H. Keel, and S. Beale, "Zero assignment and loop transfer recovery in LQG design," *Proceedings of the 29th IEEE Conference on Decision and Control*, Honolulu, Hawaii, December 1990.
9. L.H. keel, K.B. Lim, and J.-N. Juang, "Robust eigenstructure assignment with maximum tolerance to system uncertainties," *AIAA Journal of Guidance, Control and Dynamics*, vol. 14, no. 3, pp. 615 - 620, May - June 1991.
10. L.H. Keel, J. Shaw, and S.P. Bhattacharyya, "Robust control of interval systems," in *Robust Control*, Springer Verlag, 1991.
11. L.H. Keel and S.P. Bhattacharyya, "Frequency domain design of interval controllers," in *Control of Uncertain Dynamic Systems*, CRC Press, September 1991.

12. L.H. Keel and S.P. Bhattacharyya, "Optimum design of robust one - parameter controllers for interval plants," *Proceedings of the 1991 IFAC Symposium on Design Methods for Control Systems*, Zurich, Switzerland, September 1991.
13. L.H. Keel and S.P. Bhattacharyya, "Extremal gain and phase margins for interval systems," *Proceedings of the 1991 AIAA Guidance, Navigation, and Control Conference*, New Orleans, LA, August 1991.
14. H. Chapellat, L.H. Keel, and S.P. Bhattacharyya, "Stability margin for multilinear interval control systems," *Proceedings of the 1991 IEEE Conference on Decision and Control*, Brighton, United Kingdom, December 1991.
15. S.P. Bhattacharyya and L.H. Keel, "Robust stability and control of linear and multilinear interval systems," in *Control and Dynamic Systems*, volume on Advances in Robust Systems techniques and Applications, Academic Press, 1992.
16. S. Ahmad and L.H. Keel, "Robust lead-lag compensation for the family of uncertain linear systems," *Proceedings of the 1992 IEEE Symposium on Circuits and Systems*, San Diego, CA, May 1992.
17. S. Ahmad, L.H. Keel, and S.P. Bhattacharyya, "Computer aided robust control design for interval control systems," *Proceedings of the 1992 IEEE Symposium on Computer Aided Control System Design*, Napa, CA, March 1992.
18. L.H. Keel and S.P. Bhattacharyya, "Control system design for parametric uncertainties," *Proceedings of the 1st Quantitative Feedback Theory Symposium*, Wright Patterson Air Force Base, OH, August 3-4, 1992.
19. L.H. Keel, J. Shaw, and S.P. Bhattacharyya, "Robust control of interval systems," *Robust Control*, Springer - Verlag, 1992.
20. S. Ahmad, L.H. Keel, and S.P. Bhattacharyya, "Robust PID control and lead-lag compensation for linear interval systems," *Robustness of Dynamic Systems with Parametric Uncertainties*, pp. 251 - 260, Birhauser Press, 1992.
21. H. Chapellat, L.H. Keel, and S.P. Bhattacharyya, "Robustness properties of multilinear interval systems," *Robustness of Dynamic Systems with Parametric Uncertainties*, pp. 73 - 80, Birhauser Press, 1992.