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# ROBUST STABILIZER SYNTHESIS FOR INTERVAL PLANTS USING NEVANLINA-PICK THEORY

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## ABSTRACT

This paper deals with the synthesis of robustly stabilizing compensators for interval plants, i.e., plants whose parameters vary within prescribed ranges. Well-known  $H^\infty$  methods are used to establish robust stabilizability conditions for a family of plants and also to synthesize controllers that would stabilize the whole family. Though conservative, these methods give a very simple way to come up with a family of robust stabilizers for an interval plant.

### 1. INTRODUCTION

In parametric robust control theory, a plant with uncertain parameters is modeled by an interval transfer function, i.e., a transfer function whose numerator and denominator polynomial coefficients vary within prescribed ranges. Thus, we have a family of plants and the whole family needs to be stabilized with the help of a fixed controller. Stability analysis, in the area of parametric robust control, is well - developed. The problem of determining whether an interval control system is robustly stable, is solved for single input multiple output and multiinput single output systems by an extremal theorem that appeared in [1]. Given an interval plant and a controller, this theorem reduces the problem of checking the stability of the system to the problem of checking the stability of a prescribed set of *extremal* line segments, the number of which is independent of the dimension of the uncertainty box in parameter space.

However, a synthesis theory for robust stability of interval plants is yet to be developed. Here an attempt is made to convert the synthesis problem for an interval plant to an  $H^\infty$  synthesis problem which is solvable. The extremal theorem is used to convert the parametric uncertainty to an uncertainty band which is used in the  $H^\infty$  synthesis method. The method developed in [2] is used. This method uses Nevanlinna - Pick theory to derive a condition for robust stabilizability and to synthesize a family of robust stabilizers.

The method developed in this paper, though simple and elegant, is conservative. This conservativeness gets introduced while choosing the uncertainty band function needed for the  $H^\infty$  methods. However, this conservativeness could possibly be overcome in the future by using some loop-shaping concepts or some results of approximation theory.

### 2. PROBLEM FORMULATION

We consider a single input single output (SISO) interval control system as shown in Figure 1. The plant  $P(s)$  is an interval plant built around the nominal plant  $P_0(s)$ . Let the entire family of plants be denoted by  $\mathcal{P}$ . Let the perturbations of the coefficients of the plant be parametrized by  $\epsilon$ , i.e., the perturbations are characterized by a box in the parameter space. Each side of this box has a length  $\epsilon$ , weighted by a weighting factor. The values of these weights depend on the scaling inherent in the physical parameters.

The problem of obtaining robust stabilizability conditions and also that of synthesis of robust stabilizers for interval plants is as yet unsolved. In this paper, the

$H^\infty$  robust stabilizability conditions, as dealt with in the following sections, are used to obtain similar conditions for interval plants. Also, the  $H^\infty$  synthesis methods are used to design robust stabilizers for interval plants.

In  $H^\infty$  control theory, a plant with perturbations is characterized by a nominal model and an uncertain band function, i.e., the perturbed plant  $P(s)$  is said to belong to the class  $C(P_0(s), r(s))$  where it is assumed that

- 1)  $P(s)$  has the same number of unstable poles as  $P_0(s)$ ,
- 2)  $|P(j\omega) - P_0(j\omega)| \leq |r(j\omega)|$ ,  $|r(j\omega)| > 0$ ,  $\forall \omega$ .

So, we find that the  $H^\infty$  uncertainty is modeled by the stable, uncertainty band function,  $r(s)$ , while the parametric uncertainty appears in the coefficients of the transfer function model of the plant. If, in any way, we could link up the two, then the  $H^\infty$  synthesis methods could also be extrapolated to the parametric robust control case. So the essential problem here is to obtain a stable  $r(s)$  corresponding to an  $\epsilon$ . Then, we could go ahead and find the maximum  $\epsilon$ ,  $\epsilon_{\max}$ , for which the plant is robustly stabilizable and also find a family of controllers that stabilizes the entire family of plants,  $\mathcal{P}$ , in the  $\epsilon_{\max}$  - box.

### 3. ROBUST STABILIZATION USING NEVANLINNA-PICK THEORY

In  $H^\infty$  synthesis, the problem of robust stabilization of a class of plants  $C(P_0(s), r(s))$  is reduced to the problem of finding an SBR function  $u(s)$  which interpolates to

$$u(\alpha_i) = \beta_i, \quad i = 1, \dots, l; \quad \text{Re}[\alpha_i] > 0, \quad |\beta_i| < 1 \quad (1)$$

This synthesis method is discussed in [4]. The general closed-loop system as shown in Figure 1 is considered. A robust stabilizer  $C(s)$  needs to be synthesized for the

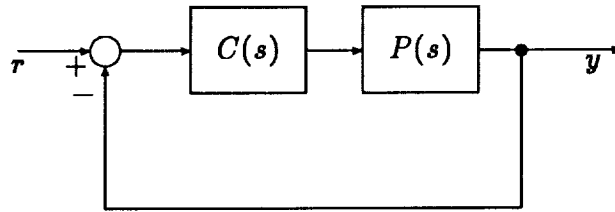


Figure 1: General Closed-loop System Configuration

interval plant  $P(s)$ . Here,

$$u(\alpha_i) = \frac{r_m(\alpha_i)}{\bar{P}_0(\alpha_i)} = \beta_i \quad (2)$$

where the  $\alpha_i$  are the poles of  $P_0(s)$  in  $\text{Re}[s] > 0$ . It is assumed here that  $P_0(s)$  has no poles on the  $j\omega$  axis and  $r_m(s)$  is nothing but a minimum-phase  $H^\infty$  function such that

$$|r_m(j\omega)| = |r(j\omega)|$$

Table 1: Fenyves Array

$\alpha_1$	$\alpha_2$	$\alpha_3$	$\dots$	$\alpha_l$	
$\rho_1$	$\rho_{1,2}$	$\rho_{1,3}$	$\dots$	$\rho_{1,l}$	$u_1(s)$
	$\rho_2$	$\rho_{2,3}$	$\dots$	$\rho_{2,l}$	$u_2(s)$
		$\dots$	$\dots$	$\dots$	$\dots$
				$\rho_l$	$u_l(s)$

and

$$\bar{P}_0(s) = B(s)P_0(s)$$

where

$$B(s) = \prod \left( \frac{\alpha_i - s}{\bar{\alpha}_i + s} \right) \quad (\text{Blaschke product}). \quad (3)$$

Now, the problem of finding the SBR function  $u(s)$  which interpolates to  $u(\alpha_i) = \beta_i$  is solved by using the Nevanlinna - Pick interpolation technique. This technique is discussed in detail in [2]. In this technique, the algorithm for finding such an  $u(s)$  is as follows:

1. First the Fenyves array is formed as shown in Table 1. where  $u_1(s) = u(s)$ ,  $\rho_1 = \beta_1$  and  $\rho_{1,j} = \beta_j$ ; for the other rows

$$\rho_i = \frac{\rho_{i-1,i} - \rho_{i-1}}{1 - \bar{\rho}_{i-1}\rho_{i-1,i}} \quad \frac{\alpha_i + \bar{\alpha}_{i-1}}{\alpha_i - \alpha_{i-1}}; \quad 1 < i \leq l \quad (4)$$

and

$$\rho_{i,j} = \frac{\rho_{i-1,j} - \rho_{i-1}}{1 - \bar{\rho}_{i-1}\rho_{i-1,j}} \quad \frac{\alpha_j + \bar{\alpha}_{i-1}}{\alpha_j - \alpha_{i-1}}; \quad 1 < i < j \leq l \quad (5)$$

2. A Schur function  $u_l(s)$  is parametrized in terms of an arbitrary Schur function  $u_{l+1}(s)$ ,

$$u_l(s) = \frac{\rho_l + u_{l+1}(s) \left( \frac{s - \alpha_l}{s + \bar{\alpha}_l} \right)}{1 + \bar{\rho}_l \left( \frac{s - \alpha_l}{s + \bar{\alpha}_l} \right) u_{l+1}(s)}, \quad (6)$$

such that  $u_l(s)$  interpolates to

$$u_l(\alpha_l) = \rho_l$$

3. The functions  $u_{l-1}(s)$ ,  $u_{l-2}(s)$ ,  $\dots$ ,  $u_2(s)$ ,  $u_1(s)$  are computed by repeating the linear fractional transformations

$$u_{i-1}(s) = \frac{\rho_{i-1} + u_i(s) \left( \frac{s - \alpha_{i-1}}{s + \bar{\alpha}_{i-1}} \right)}{1 + \bar{\rho}_{i-1} \left( \frac{s - \alpha_{i-1}}{s + \bar{\alpha}_{i-1}} \right) u_i(s)} \quad (7)$$

A family of  $u(s)$ , parametrized in terms of an arbitrary Schur function  $u_{l+1}(s)$ , can be obtained this way.

A condition to test whether a solution exists is given by the following theorems.

**Theorem 1.** *The Nevanlinna-Pick problem admits a solution iff the modulus of all the elements of the Fenyves array are less than one, i.e.,*

$$|\rho_i| < 1, \quad |\rho_{i,j}| < 1.$$

*From the Maximum Modulus Theorem, if  $|\rho_k| = 1$  for some  $k$  then only one solution  $u_k(s) = \rho_k$  exists if  $\rho_{k,j} = \rho_k$ . In this case we have*

$$|u_{k-1}(j\omega)| \equiv |u_{k-2}(j\omega)| \equiv \dots \equiv |u_1(j\omega)| \equiv 1 \quad \forall \omega$$

**Theorem 2.** *The Nevanlinna-Pick problem admits a solution iff the Pick matrix  $P$ , whose elements  $p_{ij}$  are given by*

$$p_{ij} = \frac{1 - \beta_i \bar{\beta}_j}{\alpha_i + \bar{\alpha}_j} \quad (8)$$

*is non-negative definite.*

This technique can be modified for plants with poles at the origin and is dealt with in [4]. Also, in the case where some points  $\alpha_i$  are complex, the Nevanlinna-Pick algorithm can be modified accordingly and is reported in [5].

Now, once  $u(s)$  is obtained using Nevanlinna-Pick interpolation we can compute  $q(s)$  from

$$q(s) = \frac{B(s)u(s)}{r_m(s)} \quad (9)$$

and the controller  $C(s)$  from

$$C(s) = \frac{q(s)}{1 - P_0(s)q(s)}. \quad (10)$$

To get a proper controller,  $u(s)$  needs to be strictly proper (i.e.,  $u(\infty) = 0$ ) when the relative degree of  $r_m(s)$  is 1. If  $r_m(s)$  has a relative degree greater than one the interpolation problem at  $\infty$  is more complex, but solvable.

When  $P_0(s)$  is stable, there are no interpolation conditions and it is only necessary to choose  $u(s)$  as any SBR function. Then  $q(s)$  can be obtained from

$$q(s) = \frac{u(s)}{r_m(s)} \quad (11)$$

and the compensator is

$$C(s) = \frac{q(s)}{1 - P_0(s)q(s)}. \quad (12)$$

## 4. MAIN RESULT

First, let's see how we could find a stable uncertainty function  $r(s)$  in the  $H^\infty$  control domain that would correspond to an  $\epsilon$  in the parametric domain. For this, the parametric perturbation of the plant is represented as an additive unstructured perturbation,  $\Delta P(s)$ , i.e.,

$$\Delta P(s) = P(s) - P_0(s) \quad (13)$$

where  $P(s)$  is the interval plant. Obviously,  $\Delta P(s)$  will be a family of transfer functions. Now, we can say from [1] that at each frequency, the maximum magnitude, in the family of transfer functions  $P(s)$ , will correspond to a point on one of the extremal segments, denoted as the E - segments of  $P(s)$ . The same result can be applied here to show that the maximum magnitude of  $\Delta P(s)$ , at each frequency, will also correspond to a point on one of the E-segments of  $P(s)$ . This way, we can search over the E - segments of  $P(s)$  at all frequencies and get  $\max_{P \in \mathcal{P}} |\Delta P(j\omega)|$  for all  $\omega$ .

Now, one way of getting an  $r(s)$  is to choose

$$r(s) = \left\| \max_{P \in \mathcal{P}} |\Delta P_\epsilon(j\omega)| \right\|_\infty, \quad (14)$$

the maximum uncertainty ball over all frequencies. The problem with the choice of constant  $r(s)$  is that it gives a conservative bounding of  $\Delta P(s)$ , thus producing a conservative result, which means a lower value of  $\epsilon_{\max}$ . The conservativeness can be minimized by introducing poles and zeros in  $r(s)$  and shaping it such that  $|r(j\omega)|$  approximates  $\max_{P \in \mathcal{P}} |\Delta P(j\omega)|$  at each  $\omega$  as closely as possible.

So, for each  $\epsilon$ , the maximum magnitude plot of  $\Delta P(s)$  can be obtained by searching through the E - segments of  $P(s)$ , the maximum magnitude of  $\Delta P(s)$  increasing with  $\epsilon$  at all frequencies, and an  $r(s)$  can be loop - shaped such that the  $|r(j\omega)|$  plot approximates the  $\max_{P \in \mathcal{P}} |\Delta P(j\omega)|$  plot from above.

Let us choose  $r(s)$  to be a constant as shown in eq. (14). Then  $r$  can be obtained for each  $\epsilon$ . The plot of  $r$  vs  $\epsilon$  will have the controller design information.

### 4.1. Using Nevanlinna - Pick Method

The Nevanlinna-Pick technique uses this  $r$  to check for the existence of a robustly stabilizing controller. Then, with the help of Theorems 1 and 2, it is possible to find a maximum  $r$ ,  $r_{\max}$  such that the  $H^\infty$  problem has a solution for all  $r < r_{\max}$ . This means that  $r_{\max}$  is the largest value of  $r$  for which there exists a compensator  $C^*(s)$  which stabilizes all plants in the class  $C(P_0(s), r)$ . The elements of the Pick matrix and Fenyves array being functions of  $r$ , this happens iff the Pick matrix is non - negative definite or the Fenyves array has all its elements of modulo less than 1 for all  $r < r_{\max}$ . Therefore,  $r_{\max}$  is that  $r$  for which  $|\rho_k| = 1$  for some  $k$  in the Fenyves array.

So the following algorithm can be used to find  $r_{\max}$ :

- 1) All the elements of the Fenyves array are computed, starting with the first row. Obviously, the elements are going to be functions of  $r$ . An initial arbitrary choice is made for  $r$ .

- 2) If  $|\rho_k| = 1$ , for some row  $k$ , go to step 3. If  $|\rho_i| < 1$  for all  $i$ , increase  $r$  and go to step 1. If  $|\rho_k| > 1$  for some  $k$ , decrease  $r$  and go to back to step 1.
- 3) Let  $u_k(s) = \rho_k$ , and compute the other functions  $u_{k-1}(s), u_{k-2}(s), \dots, u_1(s)$ .
- 4) A solution to the interpolation problem is then given by

$$u(s) = u_1(s)$$

with an  $H^\infty$  norm equal to 1.

Now let  $P(s)$  be an interval plant with an  $\epsilon$  - box perturbation around the nominal plant  $P_0(s)$ . For each  $\epsilon$ , the E - segments of  $P(s)$  can be searched at each frequency to get  $\max_{P \in \mathcal{P}} |\Delta P(j\omega)|$  for all  $\omega$ .  $r$  is nothing but the maximum of these values over all  $\omega$ , and obviously increases with increasing  $\epsilon$ . So a maximum  $\epsilon, \epsilon_{\max}$  can be reached for which  $r = r_{\max}$ , i.e.,

$$r_{\max} = \left\| \max_{P \in \mathcal{P}} |\Delta P_{\epsilon_{\max}}(j\omega)| \right\|_{\infty}.$$

Hence, by the Nevanlinna - Pick conditions,  $\epsilon_{\max}$  is the largest box in the coefficient space for which the interval plant  $P(s)$  is stabilizable. Of course, this value of  $\epsilon_{\max}$  is a conservative estimate as  $r(s)$  is chosen to be a constant. The compensator  $C^*(s)$  will stabilize the family of plants  $\mathcal{P}_\epsilon$  where

$$\mathcal{P}_\epsilon = \{P_0 + \Delta P_\epsilon \mid \epsilon < \epsilon_{\max}\}. \quad (15)$$

The following examples will illustrate the above theory. Let us start with an example for a stable nominal plant.

*Example 1.* Let the nominal plant be

$$P_0(s) = \frac{s + 1}{s^3 + 8s^2 + 22s + 20}$$

with poles at  $-2, -3 \pm j$ , all in the LHP. The perturbed plant is

$$P(s) = \frac{s + a}{s^3 + bs^2 + cs + d}$$

and the perturbation of the coefficients about the nominal is given by

$$\begin{aligned} a &\in [1 - \epsilon, 1 + \epsilon], \\ b &\in [8 - \epsilon, 8 + \epsilon], \\ c &\in [22 - \epsilon, 22 + \epsilon], \\ d &\in [20 - \epsilon, 20 + \epsilon]. \end{aligned}$$

Since, it is assumed that the number of unstable poles of the plant should remain unchanged, in this case it is required that  $\epsilon < \epsilon_{\max}$  such that the entire family of

plants is stable. This initial bound,  $\epsilon_{\max}$  on  $\epsilon$  can be found by checking the Hurwitz stability of the Kharitonov polynomials

$$\begin{aligned} K_{d1} &= s^3 + (8 + \epsilon)s^2 + (22 - \epsilon)s + (20 - \epsilon) \\ K_{d2} &= s^3 + (8 + \epsilon)s^2 + (22 + \epsilon)s + (20 - \epsilon) \\ K_{d3} &= s^3 + (8 - \epsilon)s^2 + (22 - \epsilon)s + (20 + \epsilon) \\ K_{d4} &= s^3 + (8 - \epsilon)s^2 + (22 + \epsilon)s + (20 + \epsilon) \end{aligned}$$

From the Routh - Hurwitz table for  $K_{d1}$ ,  $K_{d2}$ ,  $K_{d3}$ ,  $K_{d4}$ , it can be shown that  $\epsilon_{\max} = 6.321$ , i.e., the entire family of plants will be stable as long as  $\epsilon < 6.321$ .

Now for each  $\epsilon$ , all the E - segments will have to be searched for all  $\omega$  to get  $\max_{P \in \mathcal{P}} |\Delta P(j\omega)|$ , the  $H^\infty$  norm of which will be chosen as  $r$  corresponding to that  $\epsilon$ . Once an  $r$  is obtained, a robustly stabilizing compensator can be synthesized for the interval plant with perturbations in that  $\epsilon$  - box. This can be done for any  $\epsilon < 6.321$ .

The Kharitonov polynomials of the numerator are

$$\begin{aligned} K_{n1} &= K_{n2} = s + (1 - \epsilon) \\ K_{n3} &= K_{n4} = s + (1 + \epsilon) \end{aligned}$$

The plants corresponding to the E-segments are

$$\frac{K_{ni}}{S_{dj}} \quad \text{and} \quad \frac{S_n}{K_{dj}}; \quad i = 1, 3; j = 1, 2, 3, 4$$

where  $S_{dj}$  are the four segments

$$[K_{d1}(s), K_{d2}(s)], [K_{d1}(s), K_{d3}(s)], [K_{d2}(s), K_{d4}(s)], [K_{d3}(s), K_{d4}(s)]$$

and  $S_n$  is the segment  $[K_{n1}(s), K_{n3}(s)]$ .

By segment  $[K_{d1}(s), K_{d2}(s)]$ , is meant all convex combinations of the form

$$(1 - \lambda)K_{d1}(s) + \lambda K_{d2}(s), \quad \lambda \leq 1.$$

Here, it can be seen that there are 12 plants which are to be searched for the maximum magnitude.

Let  $\epsilon = 3$ . MATLAB functions are used to search through the E-segment plants to get the plot of  $\max_{P \in \mathcal{P}} |\Delta P(j\omega)|$  vs  $\omega$  as shown in Figure 2. This quantity is found to be 0.189. Therefore  $r = 0.189$  Choosing  $u = 0.3 < 1$ , we have

$$q = \frac{u}{r} = \frac{0.3}{0.189} = 1.587$$

Therefore,

$$\begin{aligned} C(s) &= \frac{q}{1 - P_0 q} = \frac{u(s)}{r(s) - P_0(s)u(s)} \\ &= \frac{1.587(s^3 + 8s^2 + 22s + 20)}{s^3 + 8s^2 + 20.413s + 18.413} \end{aligned}$$



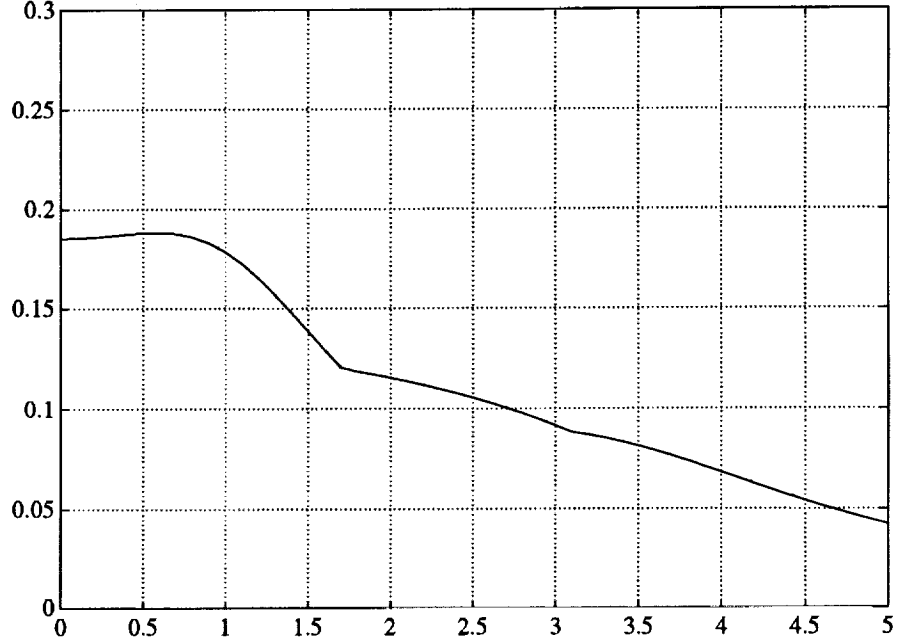


Figure 2: Maximum Unstructured Perturbation vs frequency for  $\epsilon = 3$

So, for the stable plant case, the controller can be parametrized in terms of an arbitrary Schur function  $u(s)$ . This  $C(s)$  stabilizes the interval plant  $P(s)$  with coefficients varying in the intervals

$$\begin{aligned}
 a &\in [-2, 4] \\
 b &\in [5, 11] \\
 c &\in [19, 25] \\
 d &\in [17, 23].
 \end{aligned}$$

This can be verified by checking if  $C(s)$  stabilizes the plants corresponding to the E-segments. The numerator and denominator Kharitonov polynomials are:

$$\begin{aligned}
 K_{n1} &= K_{n2} = s - 2 \\
 K_{n3} &= K_{n4} = s + 4 \\
 K_{d1} &= s^3 + 11s^2 + 19s + 17 \\
 K_{d2} &= s^3 + 11s^2 + 25s + 17 \\
 K_{d3} &= s^3 + 5s^2 + 19s + 23 \\
 K_{d4} &= s^3 + 5s^2 + 25s + 23.
 \end{aligned}$$

The numerator and denominator Kharitonov segments are

$$\begin{aligned}
 S_n &= (1 - \lambda)K_{n1} + \lambda K_{n3} = s + (6\lambda - 2) \\
 S_{d1} &= (1 - \lambda)K_{d1} + \lambda K_{d2} = s^3 + 11s^2 + (6\lambda + 19)s + 17
 \end{aligned}$$

$$\begin{aligned}
S_{d2} &= (1 - \lambda)K_{d1} + \lambda K_{d3} = s^3 + (11 - 6\lambda)s^2 + 19s + (17 + 6\lambda) \\
S_{d3} &= (1 - \lambda)K_{d2} + \lambda K_{d4} = s^3 + (11 - 6\lambda)s^2 + 25s + (17 + 6\lambda) \\
S_{d4} &= (1 - \lambda)K_{d3} + \lambda K_{d4} = s^3 + 5s^2 + (19 + 6\lambda)s + 23
\end{aligned}$$

where  $\lambda \in [0, 1]$ . So, the E-segments are

$$\begin{aligned}
P_1 &= (K_{n1}, S_{d1}) \\
P_2 &= (K_{n1}, S_{d2}) \\
P_3 &= (K_{n1}, S_{d3}) \\
P_4 &= (K_{n1}, S_{d4}) \\
P_5 &= (K_{n3}, S_{d1}) \\
P_6 &= (K_{n3}, S_{d2}) \\
P_7 &= (K_{n3}, S_{d3}) \\
P_8 &= (K_{n3}, S_{d4}) \\
P_9 &= (S_n, S_{d1}) \\
P_{10} &= (S_n, S_{d2}) \\
P_{11} &= (S_n, S_{d3}) \\
P_{12} &= (S_n, S_{d4}).
\end{aligned}$$

A MATLAB function was used to check if the controller  $C(s)$  stabilizes these segments.  $C(s)$  was found to stabilize all the segments and hence, will robustly stabilize the system for  $\epsilon = 3$ .

For  $\epsilon = 6$ , the plot of  $\max_{P \in \mathcal{P}} |\Delta P(j\omega)|$  vs  $\omega$  is shown in Figure 3.

$$r = \left\| \max_{P \in \mathcal{P}} |\Delta P(j\omega)| \right\|_{\infty} = 1.453$$

For  $u = 0.3$ ,

$$q = \frac{0.3}{1.453} = 0.207.$$

Then

$$C(s) = \frac{q}{1 - P_0 q} = \frac{0.207(s^3 + 8s^2 + 22s + 20)}{s^3 + 8s^2 + 21.793s + 19.793}$$

This compensator  $C(s)$  stabilizes  $P(s)$  with coefficient intervals

$$\begin{aligned}
a &\in [-5, 7] \\
b &\in [2, 14] \\
c &\in [16, 28] \\
d &\in [14, 26]
\end{aligned}$$

This is again verified by checking if  $C(s)$  stabilizes another set of E-segments which correspond to the above coefficient intervals.

So, it can be seen from the above example that by choosing a Schur  $u(s)$  it is always possible to come up with a robustly stabilizing compensator for an interval

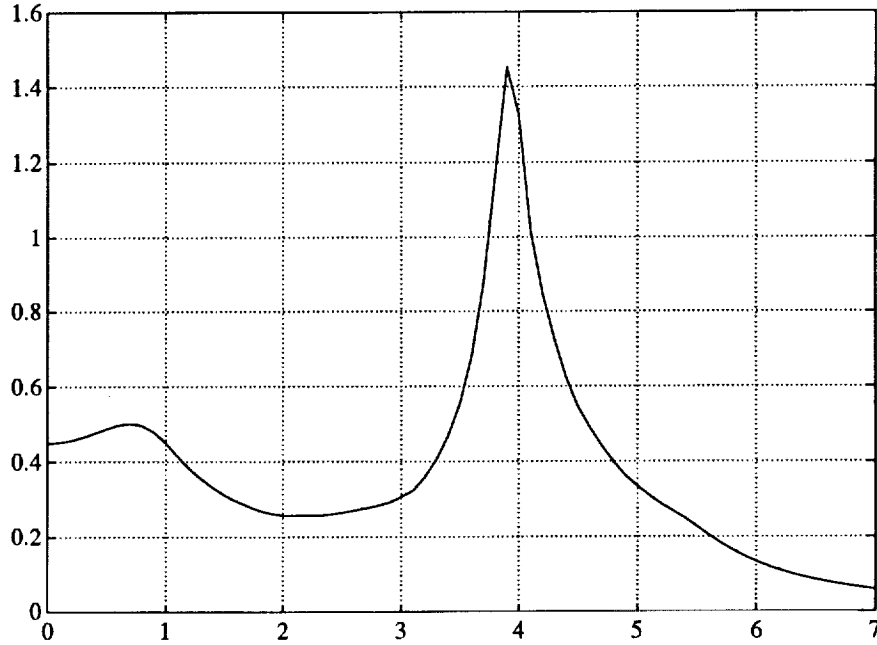


Figure 3: Maximum Unstructured Perturbation vs frequency for  $\epsilon = 6$

plant built around a stable nominal plant as long as the coefficient perturbation,  $\epsilon$  is less than some  $\epsilon_{\max}$ , so that there is no unstable pole over the whole family.

Now let us consider an example for an unstable nominal plant.

*Example 2.*

$$P_0(s) = \frac{5s + 4}{(s - 3)(s + 5)} = \frac{5s + 4}{s^2 + 2s - 15}$$

The interval plant is

$$P(s) = \frac{5s + a}{s^2 + bs + c}$$

with the intervals as

$$\begin{aligned} a &\in [4 - \epsilon, 4 + \epsilon] \\ b &\in [2 - \epsilon, 2 + \epsilon] \\ c &\in [-15 - \epsilon, -15 + \epsilon]. \end{aligned}$$

The Kharitonov polynomials for the numerator and denominator are:

$$\begin{aligned} K_{n1} &= K_{n2} = 5s + (4 - \epsilon) \\ K_{n3} &= K_{n4} = 5s + (4 + \epsilon) \\ K_{d1} &= s^2 + (2 - \epsilon)s - (15 + \epsilon) \\ K_{d2} &= s^2 + (2 + \epsilon)s - (15 + \epsilon) \\ K_{d3} &= s^2 + (2 - \epsilon)s + (-15 + \epsilon) \\ K_{d4} &= s^2 + (2 + \epsilon)s + (-15 + \epsilon) \end{aligned}$$

Using Routh-Hurwitz criterion, it can be shown that the number of unstable zeros of  $K_{di}$ ,  $i = 1, 2, 3, 4$  (and hence the number of unstable poles of the family of plants) does not change for any  $\epsilon < 15$ .

Now, the Blaschke product,

$$B(s) = \frac{3-s}{3+s}$$

$$\tilde{P}_0(s) = B(s)P_0(s) = -\frac{5s+4}{(s+3)(s+5)}$$

$$\tilde{P}_0(3) = -\frac{5 \cdot 3 + 4}{(3+3)(3+5)} = -0.395$$

We have to find a Schur function  $u(s)$  such that

$$u(3) = \frac{r(3)}{\tilde{P}_0(3)} = \frac{-r(3)}{0.395}$$

and

$$|u(3)| < 1 \Rightarrow |r(3)| < 0.395$$

Here, if the design is to be done with a constant  $r$ , it is necessary that  $r < 0.395$  for a robust stabilizer to exist. Therefore  $r_{\max} = 0.395$ . A plot of  $r$  vs  $\epsilon$  is shown in Figure 4. From the plot,  $\epsilon_{\max}$  comes out to be equal to 2.83. So the maximum  $\epsilon$  for which the plant can be robustly stabilized, with a constant  $r$  is 2.83. Let  $\epsilon = 2.83$ . Then, from the plot of  $\max_{P \in \mathcal{P}} |\Delta P(j\omega)|$  vs  $\omega$ , as shown in Figure 5, we have

$$\left\| \max_{P \in \mathcal{P}} |\Delta P(j\omega)| \right\|_{\infty} = 0.395 = r.$$

Hence  $u(3) = 1 = u(s)$ . Now

$$\tilde{q} = \frac{u}{r} = -\frac{1}{0.395} = -2.532$$

$$q(s) = B(s)\tilde{q}(s) = \frac{2.532(s-3)}{(s+3)}.$$

Therefore,

$$C(s) = \frac{q(s)}{1 - P_0(s)q(s)} = \frac{2.532(s+5)}{(s-1.584)}.$$

This  $C(s)$  stabilizes  $P(s)$  with

$$a \in [1.17, 6.83]$$

$$b \in [-0.83, 4.83]$$

$$c \in [-17.83, -12.17].$$

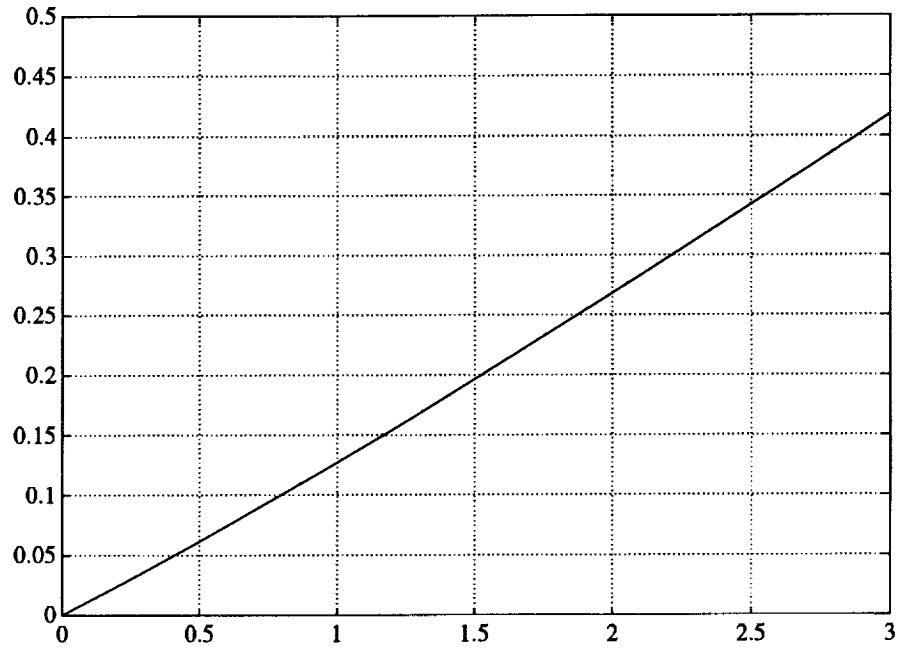


Figure 4:  $\tau$  vs.  $\epsilon$

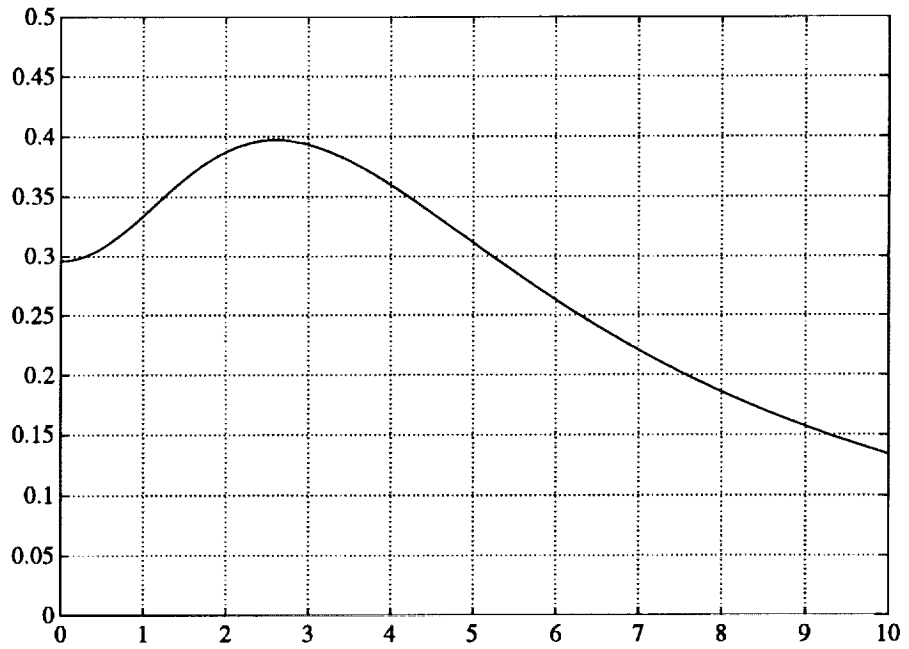


Figure 5: Maximum Unstructured Perturbation vs frequency for  $\epsilon = 2.83$

This is tested by checking if  $C(s)$  stabilizes the E-segments of  $P(s)$ . The Kharitonov polynomials are:

$$\begin{aligned}
K_{n1} &= K_{n2} = 5s + 1.17 \\
K_{n3} &= K_{n4} = 5s + 6.83 \\
K_{d1} &= s^2 - 0.83s - 17.83 \\
K_{d2} &= s^2 + 4.83s - 17.83 \\
K_{d3} &= s^2 - 0.83s - 12.17 \\
K_{d4} &= s^2 + 4.83s - 12.17
\end{aligned}$$

and the Kharitonov segments are:

$$\begin{aligned}
S_n &= 5s + (5.66\lambda + 1.17) \\
S_{d1} &= s^2 + (5.66\lambda - 0.83)s - 17.83 \\
S_{d2} &= s^2 - 0.83s + (5.66\lambda - 17.83) \\
S_{d3} &= s^2 + 4.83s + (5.66\lambda - 17.83) \\
S_{d4} &= s^2 + (5.66\lambda - 0.83)s - 12.17.
\end{aligned}$$

So the E-segments are:

$$\begin{aligned}
P_1 &= (K_{n1}, S_{d1}) \\
P_2 &= (K_{n1}, S_{d2}) \\
P_3 &= (K_{n1}, S_{d3}) \\
P_4 &= (K_{n1}, S_{d4}) \\
P_5 &= (K_{n3}, S_{d1}) \\
P_6 &= (K_{n3}, S_{d2}) \\
P_7 &= (K_{n3}, S_{d3}) \\
P_8 &= (K_{n3}, S_{d4}) \\
P_9 &= (S_n, K_{d1}) \\
P_{10} &= (S_n, K_{d2}) \\
P_{11} &= (S_n, K_{d3}) \\
P_{12} &= (S_n, K_{d4})
\end{aligned}$$

Again MATLAB is used to verify that  $C(s)$  stabilizes all the segments.

For  $\epsilon = 4$ :

$$\left\| \max_{P \in \mathcal{P}} |\Delta P(j\omega)| \right\|_{\infty} = 0.578.$$

Since

$$\left\| \max_{P \in \mathcal{P}} |\Delta P(j\omega)| \right\|_{\infty} > 0.395,$$

hence  $r = 0.578$  is not the right choice. With the plot of  $\max_{P \in \mathcal{P}} |\Delta P(j\omega)|$  as in Figure 6 in mind,  $r(s)$  is loop - shaped to approximate it from above. One such  $r(s)$  is :

$$r(s) = \frac{2.52(s + 0.6)}{(s + 1.3)(s + 2.4)}.$$

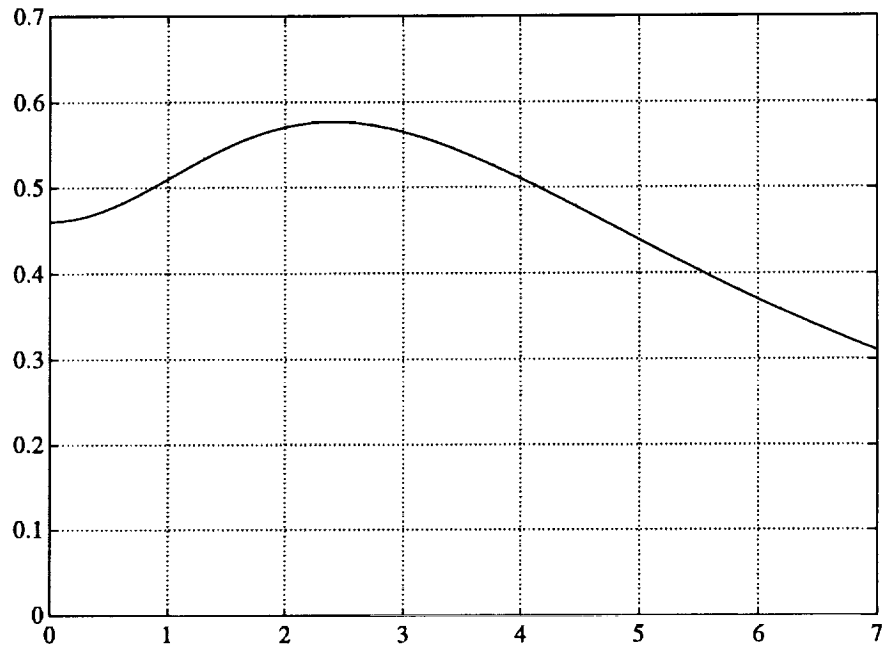


Figure 6: Maximum Unstructured Perturbation vs frequency for  $\epsilon = 4$

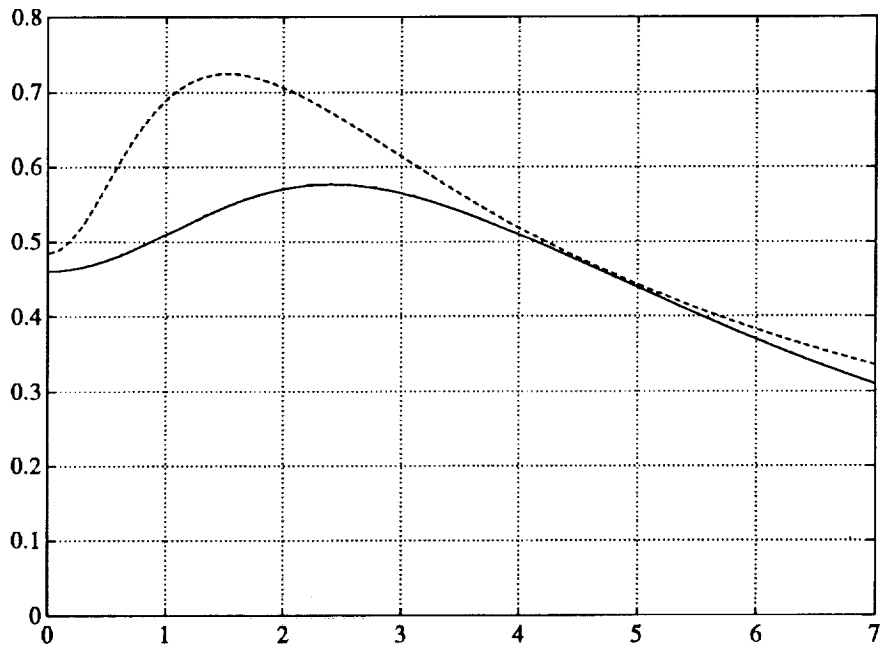


Figure 7: Maximum Unstructured Perturbation and  $r(s)$  vs frequency for  $\epsilon = 4$

Table 2: Fenyves Array

3	$\infty$	
-0.987	0	$u(s)$
	0.987	$u_2(s)$

Figure 7 shows the plot of  $r(s)$ . Then

$$u(3) = \frac{r(3)}{\bar{P}_0(3)} = \frac{-2.52 \times 3.6}{4.3 \times 5.4 \times 0.395} = -0.987.$$

Since  $r(s)$  is of relative degree 1, another interpolation condition on  $u(s)$  is  $u(\infty) = 0$  so that a proper controller is obtained.

Now, the second row of the Fenyves array will be

$$u_2(\infty) = \frac{u(\infty) + 0.987}{1 + 0.987u(\infty)} \left[ \frac{s+3}{s-3} \right]_{s \rightarrow \infty} = 0.987.$$

Hence the Fenyves array is as shown in Table 2.

Since the elements of the Fenyves array are of modulo less than 1, hence an SBR function  $u(s)$  exists which interpolates to

$$u(3) = -0.987, \quad u(\infty) = 0$$

This can also be verified by checking for the non-negative definiteness of the Pick matrix. Here

$$\begin{aligned} \alpha_1 &= 3, & \alpha_2 &= \infty \\ \beta_1 &= -0.987, & \beta_2 &= 0 \end{aligned}$$

so that the Pick matrix is

$$\begin{aligned} P &= \begin{bmatrix} \frac{1-0.987^2}{8} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 4.305 \times 10^{-3} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Since  $P$  is symmetric and has non-negative eigenvalues, hence it is non-negative definite. This also confirms the existence of  $u(s)$ .

Now,  $u_2(s)$  can be parametrized in terms of an arbitrary Schur function  $u_3(s)$ , i.e.,

$$u_2(s) = \frac{0.987 + u_3(s) \left( \frac{s-\infty}{s+\infty} \right)}{1 + 0.987 \left( \frac{s-\infty}{s+\infty} \right) u_3(s)} = \frac{0.987 - u_3(s)}{1 - 0.987u_3(s)}$$

Choosing  $u_3(s) = 0$ , we get  $u_2(s) = 0.987$ .



Then

$$u(s) = \frac{-0.987 + 0.987 \left( \frac{s-3}{s+3} \right)}{1 + 0.987(s - \infty s + \infty) u_3(s)} = \frac{-229.535}{s + 229.574}$$

$$\tilde{q}(s) = \frac{u(s)}{r(s)} = \frac{-91.0853(s + 1.3)(s + 2.4)}{(s + 0.6)(s + 229.574)}$$

$$q(s) = B(s)\tilde{q}(s) = \frac{91.0853(s + 1.3)(s + 2.4)(s - 3)}{(s + 0.6)(s + 3)(s + 229.574)}$$

Therefore, the robust stabilizer is

$$C(s) = \frac{q(s)}{1 - P_0(s)q(s)} = \frac{91.0853(s + 1.3)(s + 2.4)(s + 5)}{(s - 217.47)(s + 2.687)(s + 0.53)}$$

for the interval plant

$$P(s) = \frac{5s + a}{s^2 + bs + c}$$

$$a \in [0, 8]$$

$$b \in [-2, 6]$$

$$c \in [-19, -11].$$

The stability check is again verified with the E-segments. In this way,  $\epsilon$  can be increased in steps and for each  $\epsilon$  an attempt is made to find an  $u(s)$  which is Schur and satisfies the interpolation constraints. Then a robust stabilizer can be obtained from this  $u(s)$ .

In the case when the nominal plant is non-minimum phase, the magnitude of  $\tilde{P}_0(\alpha_i)$  is lower, i.e.  $|u(\alpha_i)|$  is higher for a  $P_0(s)$  with zeroes in the LHP than for a  $P_0(s)$  with zeroes of the same magnitude but in the RHP. So  $r(\alpha_i)$  needs to be lower in the non-minimum phase case to keep  $|u(\alpha_i)|$  less than 1, i.e. less amount of unstructured perturbation can be taken care of.

Let us consider Example 2 again with the zero of the nominal plant in the RHP.

*Example 3.*

$$P_0(s) = \frac{5s - 4}{s^2 + 2s - 15}$$

$$P(s) = \frac{5s + a}{s^2 + bs + c}$$

where

$$a \in [-4 - \epsilon, -4 + \epsilon]$$

$$b \in [2 - \epsilon, 2 + \epsilon]$$

$$c \in [-15 - \epsilon, -15 + \epsilon]$$

The Kharitonov polynomials for the denominator remain the same but those for the numerator are:

$$\begin{aligned} K_{n1} &= K_{n2} = 5s + (-4 - \epsilon) \\ K_{n3} &= K_{n4} = 5s + (-4 + \epsilon) \end{aligned}$$

The initial bound on  $\epsilon$  so that the number of RHP poles of the plant does not change, is also the same, i.e. 15.

$$\begin{aligned} B(s) &= \frac{3 - s}{3 + s} \\ \tilde{P}_0(s) &= \frac{-5s + 4}{(s + 3)(s + 5)} \end{aligned}$$

so that  $\tilde{P}_0(3) = -0.23$ . So we find that  $|\tilde{P}_0(3)|$  has decreased due to the zero of the plant being in the RHP. Therefore

$$u(3) = \frac{-r(3)}{0.23}$$

and it is required that

$$|r(3)| < 0.23$$

for a robust stabilizer to exist, i.e.,  $r_{\max} = 0.23$ . Again a plot of  $r$  vs.  $\epsilon$ , as shown in Figure 8, can be made to obtain  $\epsilon_{\max}$ . It comes out to be 1.8. Here, for  $\epsilon = 2$ ,

$$\left\| \max_{P \in \mathcal{P}} |\Delta P(j\omega)| \right\|_{\infty} = 0.251 > 0.23$$

Hence  $r = 0.251$  is not the right choice. So poles and zeros are chosen so that  $r(s)$  approximates  $\max_{P \in \mathcal{P}} |\Delta P(j\omega)|$  from above,

$$r(s) = \frac{1.591(s + 0.872)}{(s + 1.387)(s + 5)}$$

This is shown in Figure 9. Then

$$u(3) = \frac{-r(3)}{0.23} = -0.763.$$

The other interpolation condition is  $u(\infty) = 0$ . So the Fenyves array is as shown in Table 3. The elements of the Fenyves array are of modulo less than 1 which guarantees the existence of an  $u(s)$ .

Here the Pick matrix comes out to be

$$P = \begin{bmatrix} 0.0696 & 0 \\ 0 & 0 \end{bmatrix}$$

which is obviously non-negative definite as it has non-negative eigenvalues. So  $u(s)$  exists. Choosing the free parameter function  $u_3(s)$  as 0, we have  $u_2(s) = 0.763$ . Then

$$u(s) = \frac{-0.763 + 0.763 \left( \frac{s-3}{s+3} \right)}{1 - 0.763 \left( \frac{s-3}{s+3} \right) 0.763} = \frac{-10.952}{s + 11.354}$$

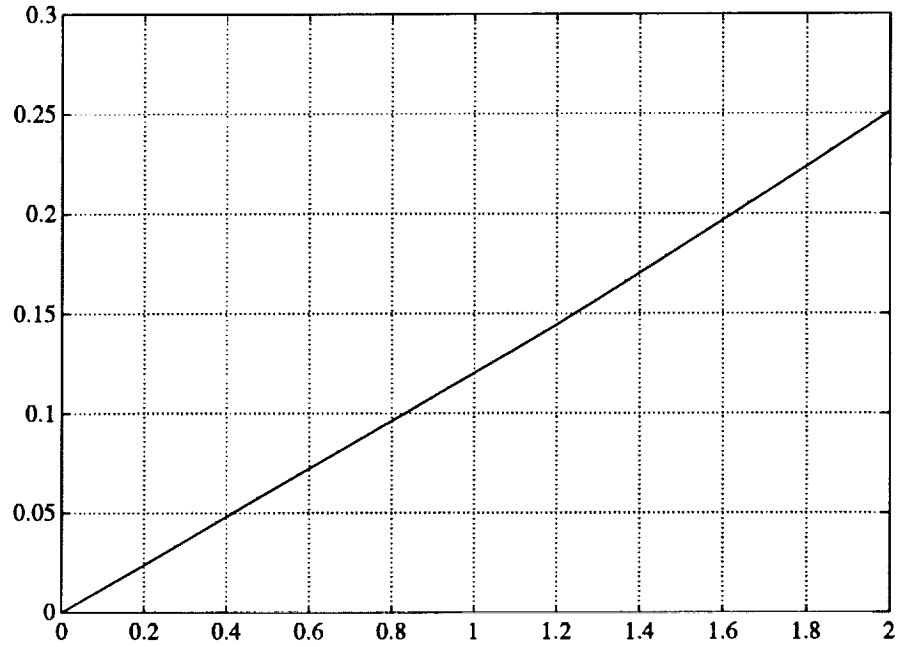


Figure 8:  $\tau$  vs.  $\epsilon$

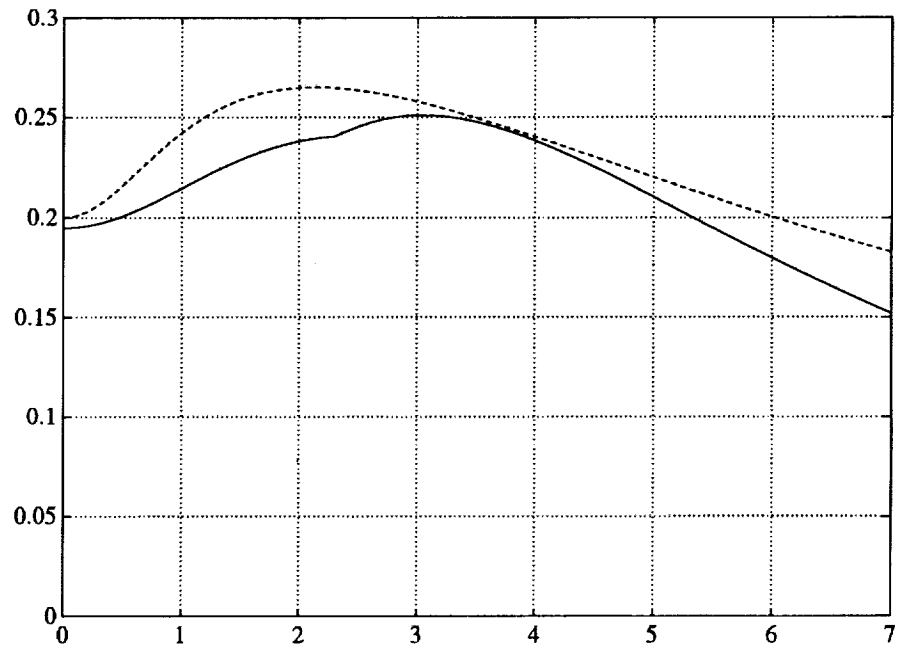


Figure 9: Maximum Unstructured Perturbation and  $\tau(s)$  vs frequency for  $\epsilon = 2$

Table 3: Fenyves Array

3	$\infty$	
-0.763	0	$u(s)$
	0.763	$u_2(s)$

From this the robust stabilizer is found out to be

$$C(s) = \frac{6.884(s + 1.387)(s + 5)}{(s + 1.287)(s - 17.461)}$$

which is found to stabilize all the E-segments of the given interval plant with coefficient intervals

$$\begin{aligned} a &\in [-6, -2] \\ b &\in [0, 4] \\ c &\in [-17, -13] \end{aligned}$$

The next example is given to show the element of conservativeness involved in this method. The choice of constant  $r(s)$  can limit  $\epsilon_{\max}$  to pretty small values in some cases.

*Example 4.*

$$P_0(s) = \frac{30s + 10}{(s + 2)(s - 3)(s - 2)} = \frac{30s + 10}{s^3 - 3s^2 - 4s + 12}$$

The interval plant is

$$P(s) = \frac{30s + a}{s^3 + bs^2 + cs + d}$$

with the intervals as

$$\begin{aligned} a &\in [10 - \epsilon, 10 + \epsilon] \\ b &\in [-3 - \epsilon, -3 + \epsilon] \\ c &\in [-4 - \epsilon, -4 + \epsilon] \\ d &\in [12 - \epsilon, 12 + \epsilon]. \end{aligned}$$

The Kharitonov polynomials for the numerator and denominator are:

$$\begin{aligned} K_{n1} &= K_{n2} = 30s + (10 - \epsilon) \\ K_{n3} &= K_{n4} = 30s + (10 + \epsilon) \\ K_{d1} &= s^2 + (2 - \epsilon)s - (15 + \epsilon) \\ K_{d2} &= s^2 + (2 + \epsilon)s - (15 + \epsilon) \\ K_{d3} &= s^2 + (2 - \epsilon)s + (-15 + \epsilon) \\ K_{d4} &= s^2 + (2 + \epsilon)s + (-15 + \epsilon). \end{aligned}$$

Table 4: Fenyves Array

3	$\infty$	
$\frac{r}{0.875}$	$\frac{r}{0.667}$	$u(s)$
	$\frac{1.0395r}{0.584-r^2}$	$u_2(s)$

Using Routh-Hurwitz criterion, it can be shown that the number of unstable zeros of  $K_{di}$ ,  $i = 1, 2, 3, 4$  (and hence the number of unstable poles of the family of plants) does not change for any  $\epsilon < 6$ .

Now, the Blaschke product,

$$B(s) = \frac{(s-3)(s-2)}{(s+2)(s+3)}$$

$$\tilde{P}_0(s) = B(s)P_0(s) = -\frac{30s+10}{(s+2)^2(s+3)}$$

so that  $\tilde{P}_0(2) = 0.875$  and  $\tilde{P}_0(3) = 0.667$ . We have to find a Schur function  $u(s)$  such that

$$u(2) = \frac{r}{\tilde{P}_0(2)} = \frac{r}{0.875}$$

$$u(3) = \frac{r}{\tilde{P}_0(3)} = \frac{r}{0.667}$$

for a constant  $r$ . It is necessary that

$$|u| < 1 \Rightarrow |r| < 0.667.$$

Now the Nevanlinna-Pick condition needs to be applied to get  $r_{\max}$ . First, the Fenyves array is formed, each element of which is a function of  $r$ . The element of the second row is given by

$$u_2(3) = \frac{\frac{r}{0.667} - \frac{r}{0.875}}{1 - \frac{r^2}{0.584}} \left[ \frac{3+2}{3-2} \right]$$

$$= \frac{1.0395r}{0.584 - r^2}.$$

So the Fenyves array is

Now  $r_{\max}$  is obtained by equating the modulus of the element in the second row of the Fenyves array to 1, and solving the resulting quadratic equation.

$$\left| \frac{1.0395r}{0.584 - r^2} \right| \leq 1$$

The above equation when solved under the constraint of  $|r|$  being already less than 0.667, gives  $r < 0.404$ . So we have  $r_{\max} = 0.404$  and  $u_2(3) = 1$ . Now using Nevanlinna Interpolation mapping functions, we get

$$u(s) = \frac{s - 0.736}{s + 0.736}$$

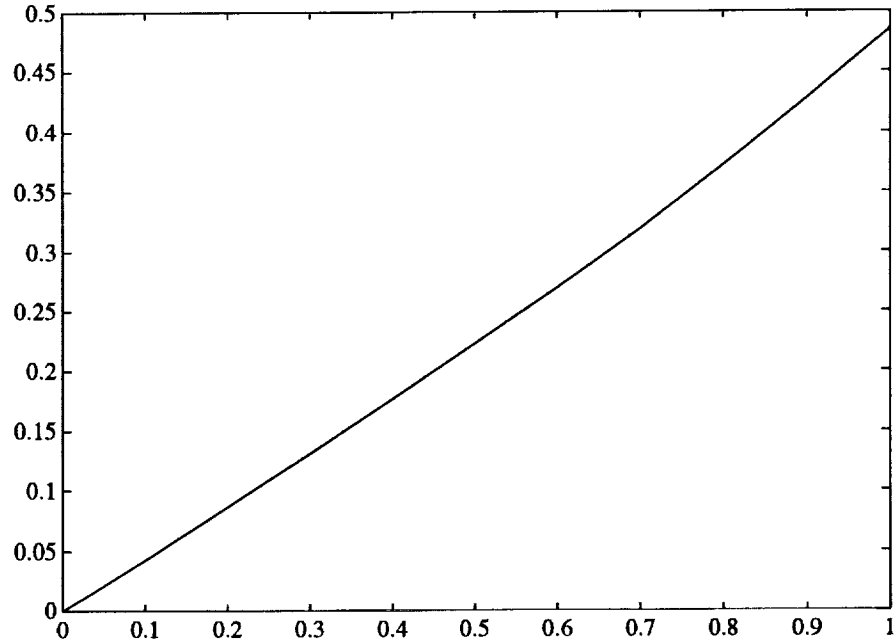


Figure 10:  $r$  vs.  $\epsilon$

from which, we get

$$C(s) = \frac{2.475(s - 0.736)(s + 2)}{(s + 12.366)(s + 0.365)}$$

From the  $\epsilon - r$  plot as shown in Figure 10, we find that  $\epsilon_{\max} = 0.85$  which is pretty small. Coefficient perturbations greater than this can be stabilized only if the choice of  $r(s)$  is less conservative. Figure 11 shows the maximum unstructured perturbation vs frequency plot for  $\epsilon = 0.85$ . This  $C(s)$  stabilizes  $P(s)$  with

$$\begin{aligned} a &\in [9.15, 10.85] \\ b &\in [-3.85, -2.15] \\ c &\in [-4.85, -3.15] \\ d &\in [11.15, 12.85] \end{aligned}$$

This is again tested by checking if  $C(s)$  stabilizes the E-segments of  $P(s)$ . The test was successful, i.e.,  $C(s)$  indeed stabilized all the E-segments.

So, we find that the Nevanlinna-Pick synthesis method developed in [2] can be used to form a robust stabilizability condition in the parametric framework. Though this condition is dependent on the procedure and is not absolute, it can be used as an important synthesis tool for interval plants within reasonable limits.

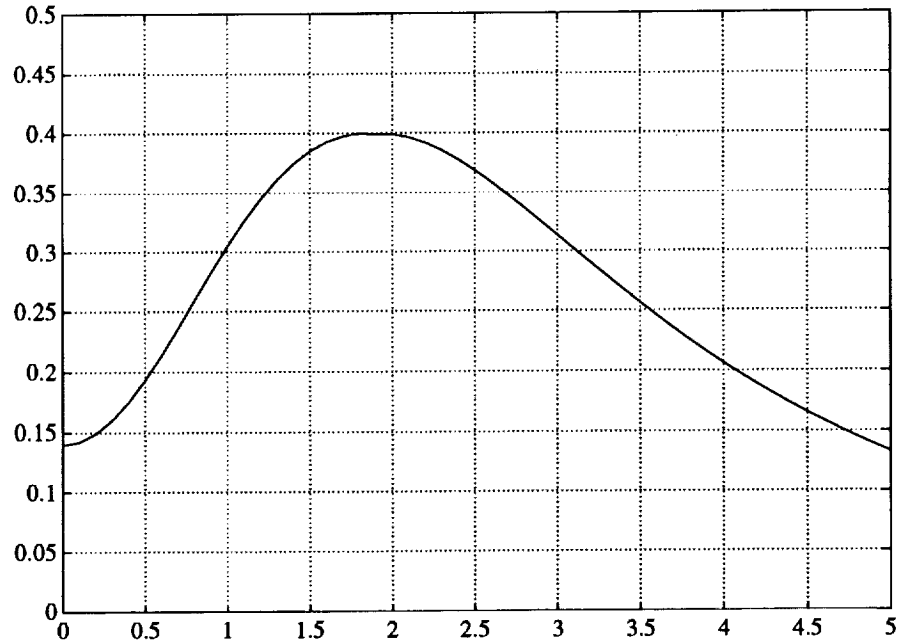


Figure 11: Maximum Unstructured Perturbation vs frequency for  $\epsilon = 0.85$

## 5. CONCLUSION

The methods used in this paper for establishing robust stabilizability conditions for interval plants and the synthesis of robust stabilizers are well-known in the area of H-Infinity Robust Control. The synthesis problem in the Parametric Robust Control area has never been attacked before. So the parametric problems are modified and adapted to a form so that the H-infinity methods could be easily used for them. The results obtained are conservative and procedure - dependent. In spite of all this, the methods developed in this paper are neat and simple and shows one approach to solving the synthesis problem.

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