Stability Margins for Multilinear Interval Systems via Phase Conditions: A Unified Approach

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Abstract

This paper gives a simple way of checking the stability with respect to an arbitrary stability region of a family of polynomials containing a vector of parameters varying within prescribed intervals. It is assumed that the parameters appear affine multilinearly in the characteristic polynomial coefficients. The condition proposed here is simply to check the phase difference of the vertex polynomials. This test based on the mapping theorem significantly reduces computational complexity. We omit mathematical proofs in this version. The results can be used to determine various stability margins of control systems containing interconnected interval subsystems. These include the gain, phase, time-delay, H^∞ and nonlinear sector bounded stability margins of multilinear interval systems.

1. INTRODUCTION

The problem of determining whether a control system remains stable or not when some of its parameters undergo perturbations within prescribed intervals is one of the central problems in robust parametric stability. The case where the parameters appear affine linearly in the coefficients of the characteristic polynomial has been solved effectively by the Edge Theorem [1] and the Box Theorem [2,3] The case where the coefficients are affine multilinear functions of the parameters is of current interest (see [4] - [9]). This case arises in control systems where transfer function coefficients perturb or state space parameters perturb or matrix fraction factors perturb. The present paper deals with the multilinear case also and shows that a sufficient condition for the stability of such systems is that the phase difference between various vertex polynomials evaluated along the stability boundary be less than 180°. This is an "image set" approach using the convex hull property of multilinear image set [10,11,12]. We show how this result can be used in conjunction with the multilinear version of the Box Theorem [5] to develop a highly efficient phase test for stability where only the Kharitonov vertices are tested. These conditions which are sufficient conditions can be tightened by increasing the number of vertices and this eventually leads to necessary and sufficient conditions. Using these results we show how various worst case stability margins such as gain, phase, time-delay, H^∞ and sector bounded nonlinear stability margins for multilinear interval systems can be found.

2. NOTATION AND MAIN RESULTS

Let $\mathbf{p} = [p_1, p_2, \cdots, p_l]$ denote a vector of real parameters. Consider the polynomial

$$\delta(s, \mathbf{p}) := \delta_0(\mathbf{p}) + \delta_1(\mathbf{p})s + \delta_2(\mathbf{p})s^2 + \cdots + \delta_n(\mathbf{p})s^n. \quad (1)$$

wherein the coefficients $\delta_i(\mathbf{p})$ are affine multilinear functions of $\mathbf{p}$, $i = 0, 1, \cdots n$. The vector $\mathbf{p}$ lies in an uncertainty set

$$\mathbf{\Pi} := \{\mathbf{p} \mid p_i^- \le p_i \le p_i^+, \quad i = 1, 2, \cdots, l\}. \quad (2)$$

The corresponding set of polynomials is denoted by

$$\mathbf{\Delta} := \{\delta(s, \mathbf{p}) \mid \mathbf{p} \in \mathbf{\Pi}\}. \quad (3)$$

Let $\mathbf{V}$ denote the vertices of $\mathbf{\Pi}$, i.e.,

$$\mathbf{V} := \{\mathbf{p} \mid p_i = p_i^+ \text{ or } p_i = p_i^-, \quad i = 1, 2, \cdots, l\} \quad (4)$$

and

$$\mathbf{\Delta_V} := \{\delta(s, \mathbf{p}) \mid \mathbf{p} \in \mathbf{V}\}. \quad (5)$$

denotes the set of vertex polynomials.

Fixing $s = s^*$, we let $\mathbf{\Delta}(s^*)$ denote the set of points $\delta(s^*, \mathbf{p})$ in the complex plane obtained by letting $\mathbf{p}$ range over $\mathbf{\Pi}$:

$$\mathbf{\Delta}(s^*) := \{\delta(s^*, \mathbf{p}) \mid \mathbf{p} \in \mathbf{\Pi}\}. \quad (6)$$

Likewise

$$\mathbf{\Delta_V}(s^*) := \{\delta(s^*, \mathbf{p}) \mid \mathbf{p} \in \mathbf{V}\}. \quad (7)$$

The convex hull of a set of points $\mathcal{P}$ in the complex plane is *co* $\mathcal{P}$. Denote the set of convex combinations of the vertex polynomials by

$$\mathbf{E} := \{\lambda_{ij}\delta_i(s) + (1 - \lambda_{ij})\delta_j(s) \mid \lambda_{ij} \in [0, 1] \text{ and } \delta_i(s), \delta_j(s) \in \mathbf{\Delta_V}\}. \quad (8)$$

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Let $S \subset \mathbb{C}$ denote an open stability region and ∂S the boundary of S. Let $\Delta(\lambda)$ denote the image set evaluated at $s = \lambda$ of the set of polynomials $\mathbf{\Delta}$. We can now state a first result based on the Mapping Theorem of [13].

Theorem 1. *The set of polynomials $\mathbf{\Delta}$ is stable with respect to S if it has at least one stable polynomial and i) $0 \notin \Delta(\lambda)$ for some $\lambda \in \partial S$ ii) the set of polynomials $\mathbf{E}$ is stable with respect to S.*

The problem of checking the stability of the set $\mathbf{E}$ can be solved computationally relatively easily in view if the fact that $\mathbf{E}(\lambda)$ consists of straight lines whose end points are contained in the set $\mathbf{\Delta_V}(\lambda)$. To formalize this let $\delta(s)$ be a polynomial and $s = \lambda$ a point in the complex plane. Let $\phi_\delta(\lambda)$ denote the argument of the complex number $\delta(\lambda)$:

$$\delta(\lambda) := |\delta(\lambda)| e^{j\phi_\delta(\lambda)}. \tag{9}$$

For a given set of polynomials $\mathbf{T}$ let

$$\Phi_{\mathbf{T}}(\lambda) := \sup_{\delta_1, \delta_2 \in \mathbf{T}} |\phi_{\delta_1}(\lambda) - \phi_{\delta_2}(\lambda)|. \tag{10}$$

We can now state the following useful corollary.

Corollary 1. (Theorem 1) *The set of polynomials $\mathbf{\Delta}$ is stable with respect to S if it contains one stable polynomial:*

a) $0 \notin \Delta(\lambda)$

b) $\sup_{\lambda \in \partial S} \Phi_{\mathbf{\Delta_V}}(\lambda) < \pi$ *where ∂S denotes the boundary of the region S.*

The above result states that $\mathbf{\Delta}$ is stable if the net phase difference of the vertex polynomials evaluated along the stability boundary is less than 180°. We shall refer to this as the *Phase Condition.* The *Phase Condition* can be easily verified by the Segment Lemma [13] without sweeping over frequency.

3. CALCULATION OF STABILITY MARGINS

Exact calculation of stability margins for a multilinear interval control system is computationally difficult. In this section, we give simple techniques to "linearize" the problem so that the *Phase Condition* can be used to estimate various stability margins without excessive calculations.

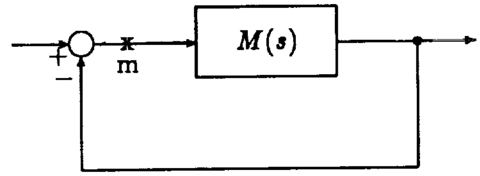

Figure 1. Unity Feedback System

3.1. Gain Margin

Consider the family of feedback systems as above with

$$M(s, \mathbf{p}, \mathbf{q}) := \frac{N(s, \mathbf{p})}{D(s, \mathbf{q})} \tag{11}$$

where $N(s, \mathbf{p})$ and $D(s, \mathbf{q})$ are multilinear functions of $\mathbf{p}$ and $\mathbf{q}$ and $(\mathbf{p}, \mathbf{q})$ varies in a polytope $\mathcal{P}$ with vertex set $\mathcal{P}_V$. The upper gain margin of the family of the family of feedback systems with M is defined as the largest value $L^* \geq 0$ of L so that

$$D(s, \mathbf{q}) + (1 + L) N(s, \mathbf{p}) \tag{12}$$

remains Hurwitz for all $L \in [0, L^*)$ and for all $(\mathbf{p}, \mathbf{q}) \in \mathcal{P}$. The lower gain margin of the family may be similarly defined by replacing $(1 + L)$ by $(1 - L)$. Let us first define the following sets corresponding to each fixed real value of L:

$$\mathcal{I}_L(\omega) \;:=\; \{D(j\omega, \mathbf{q}) + (1 + L) N(j\omega, \mathbf{p}) \mid (\mathbf{p}, \mathbf{q}) \in \mathcal{P}\}. \tag{13}$$

Let

$$\bar{\mathcal{I}}_L(\omega) \;:=\; co\,\{D(j\omega, \mathbf{q}) + (1 + L) N(j\omega, \mathbf{p}) \mid (\mathbf{p}, \mathbf{q}) \in \mathcal{P}_V\}. \tag{14}$$

Now let

$$L(\omega) := \inf\{L \mid 0 \in \mathcal{I}_L(\omega)\} \tag{15}$$

and similarly,

$$\bar{L}(\omega) := \inf\{L \mid 0 \in \bar{\mathcal{I}}_L(\omega)\}. \tag{16}$$

The corresponding gain margins are:

$$L^* \;:=\; \inf_\omega L(\omega) \tag{17}$$

$$\bar{L}^* \;:=\; \inf_\omega \bar{L}(\omega) \tag{18}$$

Theorem 2. *The gain margin L^* of $M(s)$ is bounded from below by $\bar{L}^*$.*

The point here is that lower bound $\bar{L}^*$ can of course be found from the Phase condition. since it involves only the vertices.

3.2. Phase Margin

Similarly, the phase margin of the family with $M(s)$ is defined as the largest value $\theta^* \geq 0$ of θ so that

$$D(s,\mathbf{q}) + e^{j\theta}N(s,\mathbf{p}) \qquad (19)$$

is Hurwitz for all $\theta \in [0,\theta^*)$ and for all $(\mathbf{p},\mathbf{q}) \in \mathcal{P}$. Consequently, we define the sets corresponding to each fixed value of θ:

$$\begin{aligned} \mathcal{I}_\phi(\omega) &:= \{D(j\omega,\mathbf{q}) + e^{j\theta}N(j\omega,\mathbf{p})| \\ &\qquad (\mathbf{p},\mathbf{q}) \in \mathcal{P}\} \qquad (20) \\ \bar{\mathcal{I}}_\phi(\omega) &:= co\,\{D(j\omega,\mathbf{q}) + e^{j\theta}N(j\omega,\mathbf{p})| \\ &\qquad (\mathbf{p},\mathbf{q}) \in \mathcal{P}_V\}. \qquad (21) \end{aligned}$$

And let

$$\begin{aligned} \phi(\omega) &:= \inf\{\phi \mid 0 \in \mathcal{I}_\phi(\omega)\} \qquad (22) \\ \bar{\phi}(\omega) &:= \inf\{\phi \mid 0 \in \bar{\mathcal{I}}_\phi(\omega)\}. \qquad (23) \end{aligned}$$

The corresponding phase margins are:

$$\begin{aligned} \phi^* &:= \inf_\omega \phi(\omega) \qquad (24) \\ \bar{\phi}^* &:= \inf_\omega \bar{\phi}(\omega) \qquad (25) \end{aligned}$$

Theorem 3. *The phase margin ϕ^* of $M(s)$ is bounded from below by $\bar{\phi}^*$.*

Again this lower bound can be found using the *Phase Condition.*

3.3. Time-Delay Margin

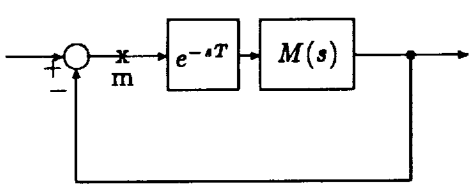

Figure 2. Time-Delay System

Consider the system with the time delay T. The time delay margin of the system is the maximum allowable time delay preserving closed loop stability. It can be calculated as before from the convex hull of the corresponding image sets. Let us define the sets corresponding to each fixed real value of T:

$$\begin{aligned} \mathcal{I}_T(\omega) &:= \{D(j\omega,\mathbf{q}) + e^{-j\omega T}N(j\omega,\mathbf{p}) \mid \\ &\qquad (\mathbf{p},\mathbf{q}) \in \mathcal{P}\} \qquad (26) \\ \bar{\mathcal{I}}_T(\omega) &:= \{D(j\omega,\mathbf{q}) + e^{-j\omega T}N(j\omega,\mathbf{p}) \mid \\ &\qquad (\mathbf{p},\mathbf{q}) \in \mathcal{P}_V\} \qquad (27) \end{aligned}$$

and define

$$\begin{aligned} T(\omega) &:= \inf\{T \mid 0 \in \mathcal{I}_T(\omega)\} \qquad (28) \\ \bar{T}(\omega) &:= \inf\{T \mid 0 \in \bar{\mathcal{I}}_T(\omega)\}. \qquad (29) \end{aligned}$$

The corresponding maximum time delays are:

$$\begin{aligned} T^* &:= \inf_\omega T(\omega) \qquad (30) \\ \bar{T}^* &:= \inf_\omega \bar{T}(\omega) \qquad (31) \end{aligned}$$

Theorem 4. *The time delay margin T^* of $M(s)$ is bounded from below by $\bar{T}^*$.*

As before $\bar{T}^*$ can be found from the *Phase Condition.*

3.4. H^∞ Stability Margin

Consider the following configuration with $Q(s)$ fixed $M(s,\mathbf{p},\mathbf{q})$ as before and ΔM representing an unstructured perturbation. Let

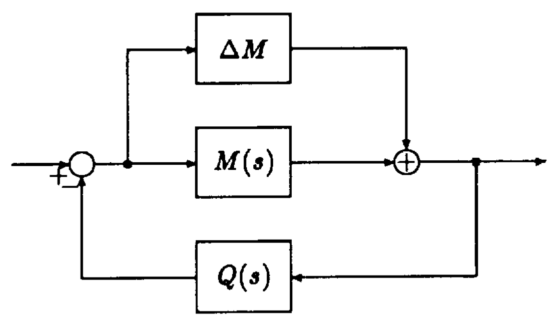

Figure 3. Additive Perturbations

$$Q(s) = \frac{Q_1(s)}{Q_2(s)} \qquad (32)$$

then the H^∞ stability margin of the system is defined as follows:

$$\begin{aligned} \beta^* &:= \frac{1}{\sup_{(\mathbf{p},\mathbf{q})\in\mathcal{P}} \left\| Q(s)[1 + Q(s)\frac{N(s,\mathbf{p})}{D(s,\mathbf{q})}]^{-1} \right\|_\infty} \\ &= \frac{1}{\sup_{(\mathbf{p},\mathbf{q})\in\mathcal{P}} \left\| \frac{Q_1(s)D(s,\mathbf{q})}{Q_2(s)[D(s,\mathbf{q}) + Q_1(s)N(s,\mathbf{p})]} \right\|_\infty}. \end{aligned}$$

From the Lemma in [14], we have

$$\forall\, \beta > 0, \qquad \|Q(s)[1 + M(s,\mathbf{p},\mathbf{q})Q(s)]^{-1}\|_\infty < \frac{1}{\beta}$$

for all $M(s,\mathbf{p},\mathbf{q})$ iff

$$\begin{aligned} &Q_2(s)D(s,\mathbf{q}) + Q_1(s)N(s,\mathbf{p}) + \beta e^{j\theta}Q_1(s)D(s,\mathbf{q}) \\ &= [Q_2(s) + \beta e^{j\theta}Q_1(s)]D(s,\mathbf{q}) + Q_1(s)N(s,\mathbf{p}) \end{aligned}$$

is Hurwitz for all $\theta \in [0, 2\pi)$, and $(\mathbf{p}, \mathbf{q}) \in \mathcal{P}$. Now let us define the sets corresponding to each fixed real value of α and θ:

$$\begin{aligned} \mathcal{I}_{\beta,\theta}(\omega) &:= \{[Q_2(j\omega) + \beta e^{j\theta} Q_1(j\omega)] D(j\omega, \mathbf{q}) + \\ &\quad Q_1(j\omega) N(j\omega, \mathbf{p}) \mid (\mathbf{p}, \mathbf{q}) \in \mathcal{P}\} \qquad (33) \\ \bar{\mathcal{I}}_{\beta,\theta}(\omega) &:= co\ \{[Q_2(j\omega) + \beta e^{j\theta} Q_1(j\omega)] D(j\omega, \mathbf{q}) + \\ &\quad Q_1(j\omega) N(j\omega, \mathbf{p}) \mid (\mathbf{p}, \mathbf{q}) \in \mathcal{P}_V\}. \qquad (34) \end{aligned}$$

If we let

$$\begin{aligned} \beta(\omega) &:= \inf\{\beta \mid 0 \in \mathcal{I}_{\beta,\theta}(\omega) \text{ for some } \theta\} \quad (35) \\ \bar{\beta}(\omega) &:= \inf\{\beta \mid 0 \in \bar{\mathcal{I}}_{\beta,\theta}(\omega) \text{ for some } \theta\} \quad (36) \end{aligned}$$

The corresponding stability margins are:

$$\beta^* = \inf_\omega \beta(\omega) \qquad \text{and} \qquad \bar{\beta}^* = \inf_\omega \bar{\beta}(\omega). \qquad (37)$$

Theorem 5. *The H^∞ stability margin β^* of the configuration in Figure 3 is bounded from below by $\bar{\beta}^*$.*

The bound $\bar{\beta}^*$ can be found from the *Phase Condition.*

3.5. Nonlinear Sector Bounded Margin

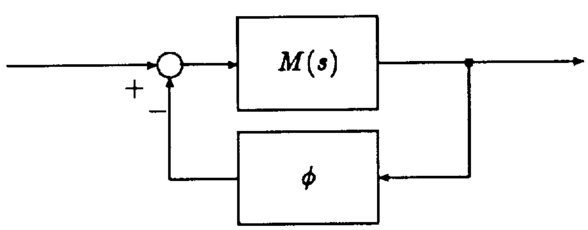

Figure 4. Luré Problem

Now let us consider Luré problem. According to [15], if $g(s)$ is a stable transfer function and ϕ belongs to the sector $[0, k]$, then a sufficient condition for absolute stability is

$$\text{Re}\left\{\frac{1}{k} + g(j\omega)\right\} > 0, \qquad \forall\ \omega \in \mathbb{R}. \qquad (38)$$

In this problem we want to find the maximum value k^* of k so that

$$\text{Re}\left\{\frac{1}{k} + M(j\omega, \mathbf{p}, \mathbf{q})\right\} > 0, \qquad \forall\ \omega \in \mathbb{R} \qquad (39)$$

for all $M(s, \mathbf{p}, \mathbf{q})$.

From the theorem given in [15], we know that $g(s) = \frac{n(s)}{d(s)}$ is SPR iff the following three conditions are satisfied:

1) $\text{Re}\{g(0)\} > 0$

2) $n(s)$ is Hurwitz stable

3) $d(s) + j\alpha n(s)$ is Hurwitz stable for all $\alpha \in R$.

Consider

$$\begin{aligned} \frac{1}{k} + M(j\omega, \mathbf{p}, \mathbf{q}) &= \frac{1}{k} + \frac{N(s, \mathbf{p})}{D(s, \mathbf{q})} \\ &= \frac{D(s, \mathbf{q}) + kN(s, \mathbf{p})}{kD(s, \mathbf{q})}. \qquad (40) \end{aligned}$$

Suppose the transfer function satisfies the above two conditions:

1) $\text{Re}\left\{\frac{D(0,\mathbf{q}) + kN(0,\mathbf{p})}{kD(0,\mathbf{q})}\right\} > 0$ for all $k \in [0, k^*]$ and for $(\mathbf{p}, \mathbf{q}) \in \mathcal{P}$

2) $D(s, \mathbf{q}) + kN(s, \mathbf{p}) \in \mathcal{H}$ for all $k \in [0, k^*]$ and for $(\mathbf{p}, \mathbf{q}) \in \mathcal{P}$.

Let us define the following sets corresponding to each fixed value of k and α:

$$\begin{aligned} \mathcal{I}_{k,\alpha}(\omega) &:= \{kD(j\omega, \mathbf{q}) + j\alpha[D(j\omega, \mathbf{q}) \\ &\quad + kN(j\omega, \mathbf{p})] \mid (\mathbf{p}, \mathbf{q}) \in \mathcal{P}\} \\ &= \{k[D(j\omega, \mathbf{q}) + j\alpha N(j\omega, \mathbf{p})] \\ &\quad + j\alpha D(j\omega, \mathbf{q}) \mid (\mathbf{p}, \mathbf{q}) \in \mathcal{P}\} \quad (41) \\ \bar{\mathcal{I}}_{k,\alpha}(\omega) &:= co\ \{k[D(j\omega, \mathbf{q}) + j\alpha N(j\omega, \mathbf{p})] \\ &\quad + j\alpha D(j\omega, \mathbf{q}) \mid (\mathbf{p}, \mathbf{q}) \in \mathcal{P}_V\} (42) \end{aligned}$$

If we let

$$\begin{aligned} k(\omega) &:= \inf\{k \mid 0 \in \mathcal{I}_{k,\alpha}(\omega), \text{for some } \alpha\} \quad (43) \\ \bar{k}(\omega) &:= \inf\{k \mid 0 \in \bar{\mathcal{I}}_{k,\alpha}(\omega), \text{for some } \alpha\}, (44) \end{aligned}$$

the corresponding nonlinear sector bounded margins are:

$$k^* = \inf_\omega k(\omega) \qquad \text{and} \qquad \bar{k}^* = \inf_\omega \bar{k}(\omega). \qquad (45)$$

Theorem 6. *The nonlinear sector bounded margin k^* of $M(s)$ is bounded from below by $\bar{k}^*$.*

4. CONCLUDING REMARKS

We have given a simple computational method to check the stability and compute various stability margins of polynomial families containing parameters which appear affine multilinearly in the characteristic polynomial coefficients and which vary in prescribed intervals. Furthermore, the proposed method completely eliminates frequency sweeping. Although the test is only a sufficient condition we expect it to be useful especially in view of the practical necessity of maintaining adequate margins of stability. Additionally the bounds obtained on the stability margins can be made as exact as desired by introducing additional vertices. Finally it is worth emphasizing that the computations given here avoid construction of image sets, a task which is enormously unwieldy from a computational standpoint.

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