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STRONGLY TRANSITIVE FUZZY RELATIONS: A MORE ADEQUATE WAY TO DESCRIBE SIMILARITY

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Abstract. The notion of a transitive closure of a fuzzy relation is very useful for clustering in pattern recognition, for fuzzy databases, etc. It is based on translating the standard definition of transitivity and transitive closure into fuzzy terms. This definition works fine, but to some extent it does not fully capture our understanding of transitivity. The reason is that this definition is based on fuzzifying only the *positive* side of transitivity: if $R(a,b)$ and $R(b,c)$, then $R(a,c)$; but transitivity also includes a negative side: if $R(a,b)$ and not $R(a,c)$, then not $R(b,c)$. In classical logic, this negative statement follows from the standard "positive" definition of transitivity. In fuzzy logic, this negative part of the transitivity has to be formulated as an additional demand.

In the present paper, we define a strongly transitive fuzzy relation as the one that satisfies both the positive and the negative transitivity demands, prove the existence of strongly transitive closure, and find the relationship between strongly transitive similarity and clustering.

1. INTRODUCTION

What this paper is about. The goal of this paper is to present a new way to describe similarity, that we believe to be more adequate than the existing one. Before presenting this definition, we must explain how similarity is defined now, and why we think that the existing definition does not fully capture the intuitive understanding of similarity.

The structure of the paper. Our new definition will be based on the same ideas as the existing one. Therefore, to explain the new definition, we have to explain these ideas and motivations in detail. So, in Section 2, we describe how similarity is defined now, and what are the ideas behind that definition. The following sections contain the drawbacks of this definition (Section 3), the new definition (Section 4), and the mathematical properties of this new definition (Sections 5 and 6). Section 7 talks about the possibility of using a different operator as a fuzzy analog of &. For the readers' convenience, all the proofs are placed at the end, in Section 8.

2. HOW SIMILARITY IS DESCRIBED NOW: ZADEH'S DEFINITION AND ITS MOTIVATIONS

Main idea: similarity is a fuzzy analog of identity. Similarity is a fuzzy notion. We say that two objects a and b are similar if they are to some extent identical. In other words, similarity can be understood as a fuzzy analog of identity. This idea was first proposed and successfully exploited by L. Zadeh in [Z71]. In the present Section, we will describe Zadeh's definition and its motivations in detail.

In view of the above-described idea, a reasonable way to formalize similarity is to formalize identity, and then fuzzify the resulting definition.

Mathematical description of identity. Traditional mathematical description of identity formalizes its main properties:

- every object is identical with itself;
- if a is identical with b, then b is identical with a;
- if a is identical with b, and b is identical with c, then a is identical with c.

If we denote "a is identical with b" by $R(a,b)$, then these properties can be expressed as formulas: $R(a,a)$ (R is *reflexive*), $R(a,b) \leftrightarrow R(b,a)$ (R is *symmetric*), and $(R(a,b) \& R(b,c)) \rightarrow R(a,c)$ (R is *transitive*). A relation that satisfies all these properties is called an *equivalence* relation.

Fuzzification of identity: the notion of similarity. In real life, for some pairs of objects, we are sure that they are identical; for some others, that they have nothing in common. As for the other pairs, we can say that they are to some extent identical. To express this "to some extent", fuzzy logic uses numbers from 0 to 1. So, to define a fuzzy relation means to describe for every pair (a,b) a value $s(a,b) \in [0,1]$ that expresses our estimate of a degree to which a and b are similar. In other words, a fuzzy relation is a function s from $U \times U$ to [0,1], where U is the set of all objects under consideration (in fuzzy logic, it is called a *Universe of discourse*).

Let us describe the above conditions in terms of this function s. The condition $R(a,a)$ means that each object a is absolutely identical with itself; in other words, that our degree of belief $s(a,a)$ that a is identical with a is equal to 1. The second condition (that a is identical with b if and only of b is identical with a) means that our degrees of belief $s(a,b)$ and $s(b,a)$ must coincide. The last condition means that for every a, b, and c, our degree of belief $s(a,c)$ (that a is identical with c) is $\geq$ than the degree of belief that a is identical with b *and* b is identical with c. In standard fuzzy logic, *and* corresponds to the minimum, so we can conclude that $s(a,c) \geq \min(s(a,b), s(b,c))$.

As a result of this analysis, we arrive at the following definition:

Definition 1 [Z71] Assume that a set U is given. It will be called a *Universe of discourse*. A *fuzzy relation* is a function $s : U \times U \rightarrow [0,1]$. We say that a fuzzy relation is *reflexive* if $s(a,a) = 1$ for all a. We say that it is *symmetric* if $s(a,b) = s(b,a)$ for all a, b. We say that s is *transitive* if $s(a,c) \geq \min(s(a,b), s(b,c))$ for arbitrary a, b, and c. A reflexive, symmetric, and transitive relation is called a *similarity relation*.

This is a very useful notion. Zadeh showed [Z71] that a similarity relation allows us to form a reasonable clustering of the objects in pattern recognition problems: namely, the relation $s(a,b) \geq \alpha$ is an equivalence relation for every α; therefore, the equivalence classes that correspond to different α, form a hierarchic clustering (see also [BH78], [LY90], and [BP92]).

Another application of fuzzy transitive closure is in fuzzy databases, where similarity relation is part of our knowledge [BBH86].

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3. WHY IS THIS DEFINITION NOT SUFFICIENT?

Crisp case: negative conclusions are also possible. Let us first consider the crisp case, when the relation is an equivalence relation. The above-given three rules describe the "positive" knowledge that we can deduce from the existing knowledge. In addition to "positive" conclusions, we can also make "negative" ones. For example, if we know that a is identical with b, and a is not identical with c, then we can conclude that b is not identical with c. The ability to make such conclusions is an essential part of our intuitive understanding of identity.

If we use the denotation $R(a,b)$, we can reformulate the above "negative" rules as $(R(a,b)\&\neg R(a,c)) \rightarrow \neg R(b,c)$.

In the crisp case, the negative conclusions automatically follow from the above definition. In the crisp case, the fact that negative conclusions are possible does not mean that anything is wrong with the standard definition of an equivalence relation: Indeed, this property easily follows from transitivity.

In the fuzzy case, negative conclusions do not automatically follow from Zadeh's definition. Let us now turn to a fuzzy version of the above-formulated rule. It means that our degree of belief in the statement "b is not identical with c" must be at least as big as our degree of belief that a is identical with b and a is not identical with c. We have already mentioned that in traditional fuzzy logic, "and" is represented by a minimum. For this negative rule, we must also use the fact that in traditional fuzzy logic, negation is expressed by a function $a \rightarrow 1-a$, i.e., the degree of belief in $\neg A$ is equal to $1-$(degree of belief in A).

So, the negative rule can be formulated as the following inequality: $1 - s(b,c) \geq \min(s(a,b), 1 - s(a,c))$.

Let us show that this inequality is not always true for a similarity relation in the sense of Definition 1. Indeed, let us consider the simplest case, when the Universe of discourse contains exactly three elements: a, b, and c, and assume that a reflexive and symmetric relation is defined as follows: $s(b,c) = 0.8$, $s(a,b) = s(a,c) = 0.6$. It is easy to check that this relation is transitive in the sense of Definition 1. However, in this case, $1 - s(b,c) = 0.2 < \min(s(a,b), 1 - s(a,c)) = \min(0.6, 0.4) = 0.4$, so the above inequality does not hold.

One might think that this is a curious but rare example. However, in the following sections, we will give a complete description of all similarity relations that satisfy this additional inequality, and from this description, one will be able to see that many fuzzy similarities (in the sense of Definition 1) do not satisfy this inequality.

A practical problem with the existing definition. An aditional drawback of Definition 1 is related to its real-life applications. As we have already mentioned, the existing definition of similarity leads to a hierarchy of clusters, that depend on the similarity level α. If we are interested in all possible clusters, then this is a perfect approach. However, if we are solving a real-life problem, and we need to come up with a set of clusters, this freedom becomes unwelcome, because we then need to decide, what exactly value of α to choose.

The problem is not very important for $\alpha < 0.5$. The reason is that if $s(a,b) < 0.5$, then our degree of belief that a and b are similar is less than 0.5, while our degree of belief that a and b are not similar, is equal to $1 - s(a,b)$, and is thus bigger than 0.5 and bigger than $s(a,b)$. In such a situation, we have more reasons to believe that a and b are not similar than to believe that they are. Therefore, it will be strange to assign them to one and the same cluster.

For $\alpha \geq 0.5$, depending on the choice of α, we will get different clusters. Can we somehow modify the above-given definition so that we will be able to choose the unique set of clusters?

Conclusion. Both arguments show that we need a new definition of similarity.

4. A NEW DEFINITION OF SIMILARITY

Definition 2. We say that a fuzzy relation s is *strongly transitive* if it is transitive (in the sense of Definition 1) and for every a, b, and c, $1 - s(b,c) \geq \min(s(a,b), 1 - s(a,c))$. By a *strong similarity* we mean a reflexive, symmetric, and strongly transitive fuzzy relation.

5. PROPERTIES OF STRONG SIMILARITY

Let us first give a complete description of all possible strong similarities. According to our Definition, every strong similarity is a similarity, so in this description, we can use the known results about the similarity relation [Z71]. In particular, we can use the fact that for each similarity s, and for each real number α, the crisp relation $s(a,b) \geq \alpha$ is an equivalence relation, and it divides the Universe of discourse U into equivalence classes.

The following theorem describes the general structure of strong similarity relations.

THEOREM 1. *A similarity relation is a strong similarity if and only if for each equivalence class of the relation $s(a,b) \geq 0.5$, there exists a number $c(E) \geq 0.5$ such that:*
1) if x, y belong to one and the same equivalence class E, then $s(x,y) = c(E)$;
2) if x, y belong to different equivalence classes E and F, then $s(x,y) \leq 1 - c(E)$ and $s(x,y) \leq 1 - c(F)$.

Comment. Since every strong similarity relation is at the same time a similarity relation, we can apply Zadeh's classification theorem to it, and conclude that this relation leads to a hierarchy of clusters. According to Theorem 1, there are two main differences between the general case and this particular case:

- In the general case, one cluster (corresponding to a particular value of α) can dissolve into two or more clusters when we increase the confidence level α. For strong similarity, clusters with $\alpha \geq 0.5$ do not change when we increase α. In particular, *such clusters are not contained in each other*, i.e., if we restrict ourselves to clusters with $\alpha \geq 0.5$ (about which we are more sure that there is a cluster than there is none), *clusters do not intersect* and thus do not form a hierarchy. Since there are no clusters inside

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clusters, we have thus solved a practical problem that we outlined in the previous sections: *a strong similarity relation enables us to choose a unique set of clusters.*

- In the general case, inside a cluster, the degrees of similarity $s(a, b)$ can be different. For strong similarity, if we restrict ourselves to the clusters that correspond to the similarity level 0.5, then inside each of these clusters, the degree of similarity is the same.

6. STRONG TRANSITIVE CLOSURE: DEFINITION AND EXISTENCE THEOREM

Crisp case: transitive closure. It is not necessary to compare all possible pairs to get the similarity relation: it is sufficient to have the results of comparing some pairs (and for big n, it is often simply impossible to ask the user to compare all pairs). If by K we denote the relation that represents our knowledge (i.e., aKb is true if we know that a is similar to b), then we must find the equivalence relation $\sim$ with the property that $aKb \rightarrow a \sim b$. The only natural restriction on K is that aKa for all a (this we know for sure), and $aKb \leftrightarrow bKa$. There may be several equivalence relations $\sim$ with this property; one of them is in which $a \sim b$ for all a and b. We would like to conclude that $a \sim b$ only if the knowledge that we have *forces* us to conclude it. So, we would like to choose as $\sim$ the "smallest" of all possible relations with these properties (a relation is defined in mathematics as a set of pairs; the smallest relation means the relation that is contained in all other relations). Such smallest relation always exists, and is called a *transitive closure* K^* of the initial relation K.

General case: fuzzy transitive closure. In fuzzy case, we may also not have all the information about the similarity of the objects. This partial knowledge can be represented by assigning a number $k(a, b) \in [0, 1]$ that describes to what extent we believe that a is similar to b. Evidently, a is similar to a, and if a is similar to b, then b is similar to a. Hence, $k(a, a) = 1$, and $k(a, b) = k(b, a)$. From this knowledge, we must find the (transitive) similarity relation $s(a, b)$. The natural conditions on s are as follows:

- if we know that a and b are similar, then they are similar (i.e., $s(a, b) \geq k(a, b)$);
- we claim that $s(a, b)$ only when we are forced to do it, i.e., s must be the smallest (= weakest) similarity relation that follows from our knowledge.

These ideas, when formalized, lead to the following definitions (proposed by Zadeh):

Definition 3. [Z71] A fuzzy relation k is called *symmetric* if $k(a, b) = k(b, a)$ for all a and b, and *reflexive* if $k(a, a) = 1$ for all a. We say that a relation k is *weaker* than the relation s if $k(a, b) \leq s(a, b)$ for all a, b. In this case, we also say that s *follows from* k. If we are given a family of fuzzy relations, then the weakest among them (if it exists) will also be called the *smallest*. For a given symmetric reflexive fuzzy relation $k(a, b) = k(b, a)$, by its *transitive closure* we mean the smallest of all similarity relations that follow from k. We will denote the transitive closure of the relation k by k^*.

Zadeh proved that every symmetric reflexive relation has a transitive closure.

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Definition of a strong transitive closure and the main result.

Definition 4. For a given symmetric reflexive fuzzy relation $k(a,b) = k(b,a)$, by its *strongly transitive closure* we mean the smallest of all of all strong similarity relations that follow from k.

THEOREM 2. *For every symmetric reflexive fuzzy relation, there exists a strongly transitive closure.*

7. A REMARK ABOUT OTHER &–OPERATIONS

The above results are true if we interpret & as min. In principle, we can use other t–norms. For example, in [BH78], it is proved that an operation $a\Delta b = \max(a+b-1, 0)$ leads to reasonable results. Let us prove that for this operation, the notions of transitivity and strong transitivity coincide.

Definition 4. [BH78] We say that a fuzzy relation is Δ*–transitive* if $s(a,c) \geq s(a,b)\Delta s(b,c)$ for arbitrary a, b, and c. A reflexive, symmetric, and Δ–transitive relation is called a Δ*–similarity relation.*

Definition 5. We say that a Δ–transitive fuzzy relation is *strongly* Δ*–transitive* if for every a, b, and c, $1 - s(b,c) \geq s(a,b)\Delta(1 - s(a,c))$. By a *strong* Δ*–similarity*, we mean a reflexive, symmetric and strongly Δ–transitive fuzzy relation.

THEOREM 3. *Any* Δ*–similarity relation* s *is a strong* Δ*–similarity.*

8. PROOFS

Proof of Theorem 1. Let us first prove that an arbitrary strong similarity relation $s(a,b)$ satisfies the conditions 1)–2) for an appropriate function c.

Let us define the value $c(E)$ for all equivalence classes E. If a class E contains at least two different elements e_1 and e_2, then we can take $c(E) = s(e_1, e_2)$. For a class E that contains only one element, take $c(E) = 0.5$.

We will first prove that $c(E) \geq 0.5$. If E consists only of one element, then, according to our definition, $c(E) = 0.5$, so this inequality is trivially true. Otherwise, $c(E) = s(e_1, e_2)$, where e_1 and e_2 are two elements of the equivalence class E. By definition of the equivalence relation $a \sim b \leftrightarrow (s(a,b) \geq 0.5)$, this means that $s(e_1, e_2) \geq 0.5$, hence $c(E) = s(e_1, e_2) \geq 0.5$.

Let us now prove that if $x \neq y$ and $x, y \in E$, then $s(x,y) = c(E)$. To prove that, we will first prove that if x, y, z are different elements of E, then $s(x,y) = s(y,z) = s(x,z)$. From Definition 2, we conclude that the inequality $1 - s(y,z) \geq \min(s(x,y), 1 - s(x,z))$ is true, as well as the two similar inequalities that are obtained by permuting x, y and z. Now, since x, y, z belong to E, and E is an equivalence class for the relation $s(a,b) \geq 0.5$, we can conclude that $s(x,y) \geq 0.5$, $s(y,z) \geq 0.5$, and $s(x,z) \geq 0.5$. Therefore, $1 - s(x,z) \leq 0.5 \leq s(x,y)$. Hence, $\min(s(x,y), 1 - s(x,z)) = 1 - s(x,z)$. Thus, the above inequality turns into $1 - s(y,z) \geq 1 - s(x,z)$, which is equivalent to $s(y,z) \leq s(x,z)$.

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If we apply the same argument to the similar inequality that was obtained be permuting x and y, we conclude that $s(x, z) \leq s(y, z)$. Since we have already proved that $s(y, z) \leq s(x, z)$, we can conclude that $s(x, z) = s(y, z)$. Likewise, we can prove that $s(x, y) = s(x, z) = s(y, z)$.

Now, we are ready to prove that $s(x, y) = c(E)$ for an arbitrary pair $x \neq y \in E$. Indeed, if one of the elements x, y coincides with either e_1 or e_2, then, as we have just proved, $s(x, y) = s(e_1, e_2) = c(E)$. So, it is sufficient to consider the remaining case when all 4 elements e_1, e_2, x, y are different. In this case, as we have just proved, $s(x, y) = s(x, e_2)$ and $s(x, e_2) = s(e_1, e_2) = c(E)$. Hence, $s(e_1, e_2) = c(E)$.

Let us now prove that if x and y belong to different equivalence classes: $x \in E$, $y \in F$, $E \neq F$, then $s(x, y) \leq 1 - c(E)$.

Since x and y belong to two different equivalence classes, we have $s(x, y) < 0.5$. If E has only one element x, then $c(E) = 0.5$. Therefore, in this case, we already have the desired inequality $s(a, b) \leq 0.5 = 1 - c(E)$.

Suppose now that E has at least 2 different elements, i.e., in addition to x, there exists a $z \in E$, $z \neq x$. From Definition 2, we conclude that $1 - s(x, z) \geq \min(s(x, y), 1 - s(y, z))$. From the property 1), we conclude that $s(x, z) = c(E)$.

Since $x \not\sim y$ and $z \not\sim y$, we conclude that $s(x, y) < 0.5$ and $s(y, z) < 0.5$. Therefore, $s(x, y) < 0.5 < 1 - s(y, z)$, and $\min(s(x, y), 1 - s(y, z)) = s(x, y)$. So, the above inequality turns into $1 - c(E) = 1 - s(x, z) \geq s(x, y)$, or $s(x, y) \leq 1 - c(E)$.

Likewise, we can prove that $s(x, y) \leq 1 - c(F)$.

We have proved that an arbitrary strong similarity relation satisfies the properties 1)–2) for an appropriate function c. Let us now prove that if a similarity relation satisfies these properties for some function c, then it is a strong similarity. In other words, we need to prove that the inequality from Definition 2 is true for arbitrary three elements a, b, and c. To prove that, let us consider all possible cases:

- when all three elements a, b, c are equivalent to each other;
- when two of them are equivalent, and the third one is not; and
- when all three belong to different equivalence classes.

If all three elements a, b, and c belong to one and the same equivalence class E, then according to 1), $s(a, b) = s(b, c) = s(b, c) = c(E)$, where $c(E) \geq 0.5$. In this case, $s(a, b) \geq 0.5 \geq 1 - s(a, c)$, hence $\min(s(a, b), 1 - s(a, c)) = 1 - s(a, c) = 1 - c(E)$, and the desired inequality turns into $1 - c(E) \geq 1 - c(E)$, which is trivially true.

Let us now consider the cases, when two elements are equivalent, and the third is not equivalent to them. There are three such cases, when this "third" element is c, b, or a. We will consider all three cases separately.

The first case is when $a \sim b$, and c is not equivalent to a or b. Let us denote the equivalence class that contains a and b by E. Then, because of property 1), $s(a, b) = c(E)$,

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and because of property 2), $s(a,c) \leq 1 - c(E)$ and $s(b,c) \leq 1 - c(E)$. Hence, $s(a,c) \leq 1 - s(a,b)$. By adding $s(a,b)$ and subtracting $s(a,c)$ from both sides of this inequality, we conclude that $s(a,b) \leq 1 - s(a,c)$. Therefore, $\min(s(a,b), 1 - s(a,c)) = s(a,b)$. Likewise, from property 2), we can conclude that $s(b,c) \leq 1 - c(E)$, $s(b,c) \leq 1 - s(a,b)$, and $s(a,b) \leq 1 - s(b,c)$. Since we have proved that $s(a,b) = \min(s(a,b), 1 - s(a,c))$, we conclude that $1 - s(b,c) \geq \min(s(a,b), 1 - s(a,c))$. This is exactly the desired inequality.

Another possible case is when $a \sim c$ (i.e., a and c belong to one and the same equivalence class E), and b is not equivalent to a and c. In this case, $s(a,c) = c(E)$, $s(a,b) \leq 1 - c(E)$, and $s(b,c) \leq 1 - c(E)$. From the first inequality, we conclude that $s(a,b) \leq 1 - s(a,c)$, hence $\min(s(a,b), 1 - s(a,c)) = s(a,b)$. Therefore, in this case, the desired inequality turns into $s(a,b) \leq 1 - s(b,c)$, or $s(a,b) + s(b,c) \leq 1$. But $a \not\sim b$ and $b \not\sim c$, so we have $s(a,b) < 0.5$ and $s(b,c) < 0.5$, and hence $s(a,b) + s(b,c) < 1$. So in this case, the desired inequality is also true.

The third case if when b and c are equivalent, and a is not, i.e., when for some equivalence class E, $b \in E$, $c \in E$, and $a \notin E$. In this case, $s(b,c) = c(E)$, and from property 2), we conclude that $s(a,b) \leq 1 - c(E) = 1 - s(b,c)$ and, therefore, $1 - s(b,c) \geq s(a,b) \geq \min(s(a,b), 1 - s(a,c))$. So, the desired inequality is true.

Let us consider the only remaining case when none of the elements a, b, and c are equivalent. In this case, $s(a,b) < 0.5$, $s(b,c) < 0.5$, and $s(a,c) < 0.5$. Hence, $1 - s(b,c) > 0.5 > s(a,b) \geq \min(s(a,b), 1 - s(a,c))$. So, the desired inequality is true in all possible cases. Q.E.D.

Proof of Theorem 2. For every k, there exist strong similarity relations that follow from k: for example, we can take $s(a,b) = 1$ for all a and b. Let us define the relation i as follows: for every a and b, $i(a,b)$ is the infimum (greatest lower bound) of $s(a,b)$, where infimum is taken over all strong similarities s that follow from k.

If this relation i is a strong similarity, then it is evidently the smallest of all strong similarity relations that follow from k, and thus it is the desired transitive closure. So, let us prove that i is a strong similarity.

Since $s(a,a) = 1$ for all s, the infimum of the values $s(a,a)$ is also equal to 1, i.e., $i(a,a) = 1$.

For arbitrary a and b, $s(a,b) = s(b,a)$. Therefore, the numbers $i(a,b)$ and $i(b,a)$ are the greatest lower bounds for one and the same set of values $s(a,b)$. Hence, $i(a,b) = i(b,a)$.

Let us prove that i is transitive, i.e., that $i(a,c) \geq \min(i(a,b), i(b,c))$. Indeed, we are taking the infimum over transitive relations s. Therefore, for each of them, $s(a,c) \geq \min(s(a,b), s(b,c))$. Since $i(a,b)$ is the lower bound for all the values $s(a,b)$, we conclude that $s(a,b) \geq i(a,b)$. Likewise, $s(b,c) \geq i(b,c)$. Therefore, the smallest of the two numbers $s(a,b)$ and $s(b,c)$ is not smaller than the minimum of the two numbers $i(a,b)$ and $i(b,c)$: $\min(s(a,b), s(b,c)) \geq \min(i(a,b), i(b,c))$. Since we know that $s(a,c) \geq \min(s(a,b), s(b,c))$, we can conclude that $s(a,c) \geq \min(i(a,b), i(b,c))$. So, for each strong similarity that follows from k, the value of $s(a,c)$ is not smaller than $\min(i(a,b), i(b,c))$. Therefore, the infimum $i(a,c)$ of these values must be also not smaller than this number, i.e., $i(a,c) \geq \min(i(a,b), i(b,c))$. In other words, i is transitive.

Let us now prove that i is strongly transitive. To do that, we will first show that the definition of a strong transitivity is equivalent to the following one: for every a, b, c, if $s(a,b) + s(b,c) > 1$, then $s(b,c) \leq s(a,c)$.

Indeed, suppose that s is strongly transitive, and $s(a,b) + s(b,c) > 1$. This means that $1 - s(b,c) < s(a,b)$. Since s is strongly transitive, we can conclude that $1 - s(b,c)$ is greater than (or equal to) the smallest of the two numbers: $s(a,b)$ and $1 - s(a,c)$. Due to the fact that $s(a,b)$ has been just proven to be greater than $1 - s(b,c)$, we can conclude that $s(a,b)$ cannot be the smallest of these two numbers; so, $1 - s(a,c)$ is the smallest of the two, and therefore, $1 - s(b,c) \geq 1 - s(a,c)$. From this inequality, we conclude that $s(b,c) \leq s(a,c)$.

Vice versa, suppose that a fuzzy relation s satisfies the above property. Let us prove that s is a strong similarity. To prove that, for each a, b, and c, we will consider two possible cases: $s(a,b) + s(b,c) \leq 1$, and $s(a,b) + s(b,c) > 1$. In the first case, $1 - s(b,c) \geq s(a,b) \geq \min(s(a,b), 1 - s(a,c))$, i.e., the desired inequality is true. In the second case, according to our property, $s(b,c) \leq s(a,c)$, hence $1 - s(b,c) \geq 1 - s(a,c) \geq \min(s(a,b), 1 - s(b,c))$. So the inequality that represents strong transitivity is really equivalent to the above-given condition.

Now, let us prove that i satisfies this condition and is, therefore, strongly transitive. Indeed, assume that $i(a,b) + i(b,c) > 1$. Let us prove that $i(b,c) \leq i(a,c)$. Since $i(a,b) \leq s(a,b)$, we conclude that $s(a,b) + s(b,c) \geq i(a,b) + i(b,c)$ and hence, $s(a,b) + s(b,c) > 1$ for each strong similarity that follows from k. Since s is a strong similarity, we conclude that $s(b,c) \leq s(a,c)$. Therefore, for the infimums the same inequality holds, i.e., $i(a,b) \leq i(a,c)$. So, i is a strong transitivity. Q.E.D.

Proof of Theorem 3. Assume that s is a $\Delta-$similarity. Then, according to Theorem 3.1 from [BH78], the function $r(a,b) = 1 - s(a,b)$ is a metric; in particular, it satisfies the triangle inequality $r(a,c) \leq r(a,b) + r(b,c)$ for all a, b, and c. Let us prove that s is a strong $\Delta-$similarity, i.e., that $1 - s(b,c) \geq s(a,b)\Delta(1 - s(a,c)) = \max(s(a,b) + (1 - s(a,c)) - 1, 0)$. In other words, we need to prove that $1 - s(b,c)$ is bigger than (or equal to) the biggest of the two numbers: 0 and $s(a,b) + (1 - s(a,c)) - 1 = s(a,b) - s(a,c)$. To prove that, it is sufficient to prove that $1 - s(b,c)$ is not smaller than each of them. Since $s(b,c) \geq 1$, we always have $1 - s(b,c) \geq 0$. Therefore, it is sufficient to prove that $1 - s(b,c) \geq s(a,b) - s(a,c)$. If we substitute $s(a,b) = 1 - r(a,b)$, this inequality turns into $r(b,c) \geq r(a,c) - r(a,b)$, which is equivalent to $r(a,b) + r(b,c) \geq r(a,c)$, i.e., to the triangle inequality. So, $1 - s(b,c)$ is bigger than $\max(s(a,b) + (1 - s(a,c)) - 1, 0) = s(a,b)\Delta(1 - s(a,c))$, and hence s is strongly $\Delta-$transitive. Q.E.D.

CONCLUSIONS

The existing definition of a (fuzzy) similarity relation is based on translating the standard definition of an equivalence relation into fuzzy terms. We see two problems with this definition:

- some properties of equivalence relation that automatically follow from its crisp definition, are not preserved under the existing translation; therefore, the existing definition does not completely capture the intuitive idea of a similarity as a fuzzy analog of equivalence;
- if we fix a number α from 0.5 to 1 as a reasonable similarity level, then for an arbitrary fuzzy similarity $s(a,b)$, the elements that are sufficiently similar (i.e., for which $s(a,b) \geq \alpha$) form clusters; however, these clusters essentially depend on what exactly value α we use; it would be nice to have a clustering that is uniquely determined by a a relation s.

In the present paper, we propose a new definition of similarity. To get this definition, we translate both the standard definition of equivalence, and the additional properties of equivalence, into fuzzy logic. The resulting definition leads to a unique choice of clusters.

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