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Workshop on Harmonic Oscillators

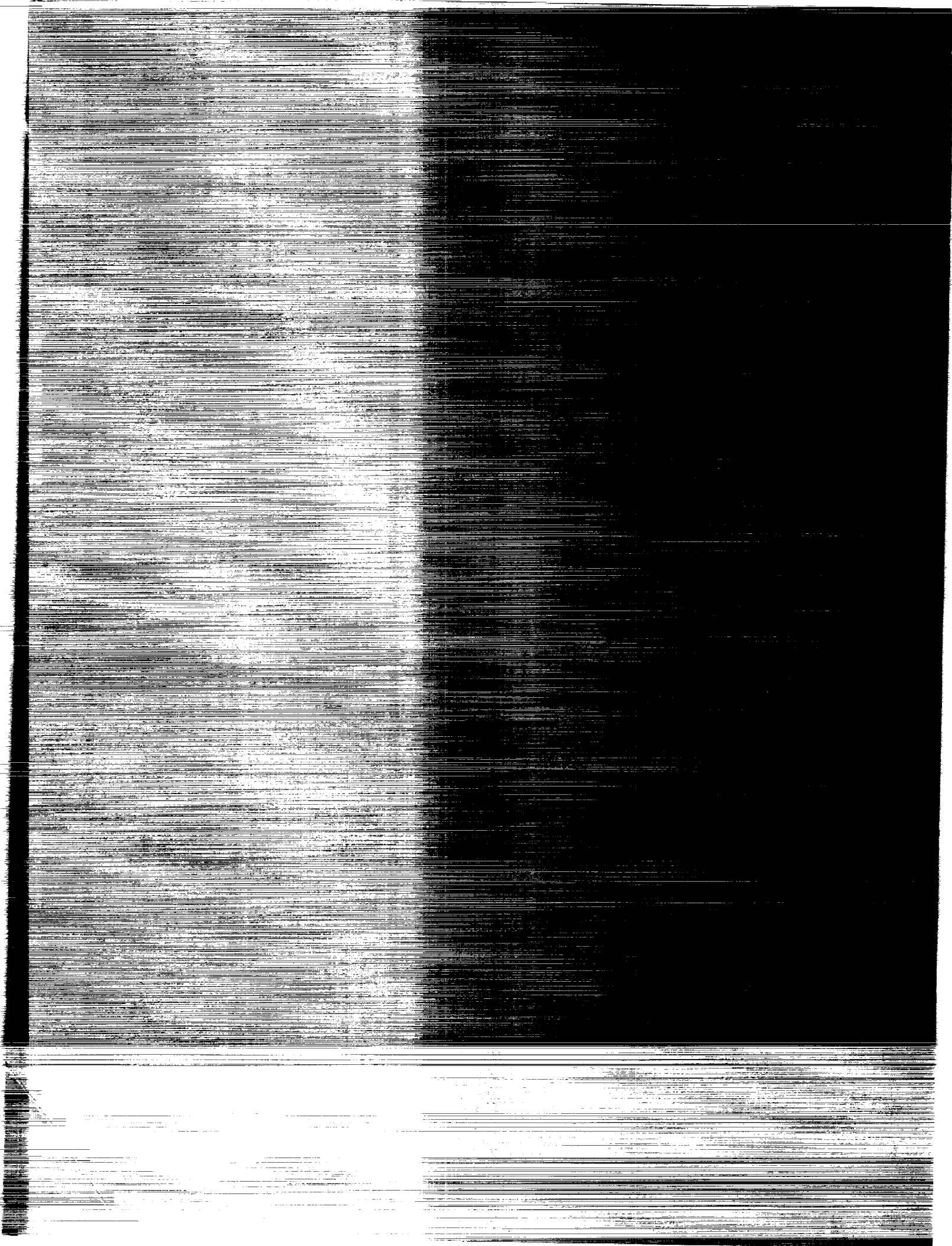Proceedings of a conference held at the
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NASA

Workshop on Harmonic Oscillators

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PREFACE

The Workshop on Harmonic Oscillators was held at the College Park Campus of the University of Maryland on March 25 - 28, 1992. This Workshop was mostly supported by the Goddard Space Flight Center of the National Aeronautics and Space Administration.

The harmonic oscillator formalism has been and still is playing an important role in many branches of physics. This is the simplest mathematical device which can connect the basic principle of physics with what we observe in the real world. The oscillator formalism is, therefore, a very useful language in establishing communications among

(1) The physicists interested in fundamental principles and those interested in describing what we observe in laboratories.

(2) Researchers in different branches of physics, such as atomic, nuclear and particle physics, quantum optics, statistical and thermal physics, foundations of quantum mechanics and quantum field theory, and group representations for possible future theories.

The Workshop brought together active researchers in harmonic oscillators in many different fields, and provided the opportunity for them to learn new ideas and techniques from their colleagues in the fields of specialization different from their own.

The Second International Workshop on Harmonic Oscillators will be held in Mexico in 1993. The Principal Organizer for this important meeting will be Kurt Bernardo Wolf of the Universidad Nacional Autonoma de Mexico.

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INTRODUCTION

The harmonic oscillator is the basic scientific language for physics. It was Einstein who predicted the existence of quantized energy levels using the harmonic oscillator model for the specific head of solids. The role of harmonic oscillators in the development of quantum mechanics and quantum field theory is well known. Indeed, because of its mathematical simplicity, the harmonic oscillator model often precedes new physical theories. It also acts as an approximation in many of the existing theories.

Among the most respected physicists of our century, Paul A. M. Dirac and Richard P. Feynman were quite fond of harmonic oscillators. It was Dirac who started using harmonic oscillators for representing the Lorentz group. It was Feynman who said that we should try an oscillator formalism, instead of Feynman diagrams, to understand relativistic bound-state problems. Feynman's path integral formulation of quantum mechanics is also based on harmonic oscillators. These two physicists have left a profound influence on what we are doing now.

In spite of its past role, it is important to realize that we do not study harmonic oscillators to learn the history of physics. Our major concern is the future of physics. Let us look at one of the cases of what we are doing today. Since the development of lasers in the late 1950's and early 1960's, the theory of coherent radiation has become a major branch of modern physics. It is generally agreed that this new theory is more or less the physics of harmonic oscillators or the study of the Lorentz group using harmonic oscillators which Dirac pioneered (J. Math. Phys. Vol. 4, page 901, 1963). Let us look at another example. The question of thermal excitations and the lack of coherence is of current interest. Here also the basic theoretical tool is the harmonic oscillator as Feynman noted in his book on statistical mechanics (Benjamin/Cummings, 1972).

In view of the past and present roles of harmonic oscillators in physics, it is fully justified to develop new oscillator formalisms for possible new physical theories in the future, even though their immediate physical applications are not clear. This typically takes the form of constructing representations of groups using harmonic oscillators. Developing a mathematical formalism before the birth of a new physical theory is the most sacred role of mathematical physics. The theory of squeezed states of light is a case in point. It was possible to construct this theory very quickly because all relevant mathematical techniques were available when its physical idea was conceived. The harmonic oscillator indeed occupies a very important place in mathematical physics.

The Workshop on Harmonic Oscillators was the first scientific meeting of its kind. The Workshop was attended by many of the researchers in harmonic oscillators, including those in atomic, nuclear, and particle physics, quantum optics, statistical physics, as well as mathematical physics. It was also attended by many students who are potential developers of new theories. Many interesting papers were presented.

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There were many lively informal discussions. However, the Workshop was by no means a perfect meeting. Many potential participants did not attend the Workshop because the purpose of the meeting was not clear enough to them. Yet, those who came to the Workshop have set the tone for future meetings in the same series. It is the participants, not the organizers, who decide the success or failure of any given scientific meeting. Indeed, the participants of the Workshop on Harmonic Oscillators did very well, and well enough to generate the second meeting in the same series.

I. QUANTUM MECHANICS

QUONS,
AN INTERPOLATION BETWEEN
BOSE AND FERMI OSCILLATORS

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Abstract

After a brief mention of Bose and Fermi oscillators and of particles which obey other types of statistics, including intermediate statistics, parastatistics, paronic statistics, anyon statistics and infinite statistics, I discuss the statistics of "quons" (pronounced to rhyme with muons), particles whose annihilation and creation operators obey the q-deformed commutation relation (the quon algebra or q-mutator) which interpolates between fermions and bosons. I emphasize that the operator for interaction with an external source must be an effective Bose operator in all cases. To accomplish this for parabose, parafermi and quon operators, I introduce parabose, parafermi and quon Grassmann numbers, respectively. I also discuss interactions of non-relativistic quons, quantization of quon fields with antiparticles, calculation of vacuum matrix elements of relativistic quon fields, demonstration of the TCP theorem, cluster decomposition, and Wick's theorem for relativistic quon fields, and the failure of local commutativity of observables for relativistic quon fields. I conclude with the bound on the parameter q for electrons due to the Ramberg-Snow experiment.

1 Introduction

I start by reviewing the (Bose) harmonic oscillator. I want to emphasize that the commutation relation,

$$[a_i, a_j^\dagger]_- \equiv a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij},$$

(1)

and the vacuum condition which characterizes the Fock representation

$$a_i|0\rangle = 0$$

(2)

suffice to calculate all vacuum matrix elements of polynomials in the annihilation and creation operators. The strategy is to move annihilation operators to the right, picking up terms with a contraction of an annihilation and a creation operator. When the annihilation operator gets to the vacuum on the right, it annihilates it. For example,

$$\langle 0|a_{i_1}a_{i_2}\cdots a_{i_n}a_{j_m}^\dagger\cdots a_{j_2}^\dagger a_{j_1}^\dagger|0\rangle = \delta_{i_n j_m}\langle 0|a_{i_1}a_{i_2}\cdots a_{i_{n-1}}a_{j_{m-1}}^\dagger\cdots a_{j_2}^\dagger a_{j_1}^\dagger|0\rangle$$
$$+\langle 0|a_{i_1}a_{i_2}\cdots a_{i_{n-1}}a_{j_m}^\dagger a_{i_n}a_{j_{m-1}}^\dagger\cdots a_{j_2}^\dagger a_{j_1}^\dagger|0\rangle.$$

(3)

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Continuing this reduction, it is clear that this vacuum matrix element vanishes, unless the set $\{i_1, i_2, \cdots, i_n\}$ is a permutation of the set $\{j_1, j_2, \cdots, j_m\}$ (this includes $n = m$). In particular, *no relation is needed between two a's or between two $a^\dagger$'s*. As we know, it turns out that

$$[a_i, a_j]_- = 0 = [a_i^\dagger, a_j^\dagger]_-, \tag{4}$$

but these relations are redundant in the Fock representation. Also, only the totally symmetric (one-dimensional) representations of the symmetric (i.e., permutation) group $\mathcal{S}_n$ occur.

To construct observables in the free theory we can use the number operator, n_k, or the transition operator, n_{kl},

$$n_k = n_{kk} = a_k^\dagger a_k, \quad n_{kl} = a_k^\dagger a_l. \tag{5}$$

The commutation relation,

$$[n_{kl}, a_m^\dagger]_- = \delta_{lm} a_k^\dagger, \tag{6}$$

follows from Eq.(1). The number operator has integer eigenvalues,

$$n_k (a_k^\dagger)^{\mathcal{N}} |0\rangle = \mathcal{N} (a_k^\dagger)^{\mathcal{N}} |0\rangle. \tag{7}$$

Using n_k and n_{kl} we can construct the Hamiltonian,

$$H = \sum_k \epsilon_k n_k, \tag{8}$$

and other observables for the free theory. The Hamiltonian obeys

$$[H, a_l^\dagger]_- = \epsilon_l a_l^\dagger. \tag{9}$$

Analogous formulas of higher degree in the a's and $a^\dagger$'s give interaction terms.

I want to pay special attention to couplings to external sources in the quon theory; in preparation for that I write the external Hamiltonian in the Bose case,

$$H_{ext} = \sum_k (j_k^\star a_k + a_k^\dagger j_k), \tag{10}$$

where j_k is a c-number; i.e.,

$$[j_k, a_l^\dagger]_- = [j_k, j_l^\star]_- = 0, \text{ etc.} \tag{11}$$

This satisfies the commutation relation

$$[H_{ext}, a_l^\dagger]_- = j_l^\star. \tag{12}$$

Equations (9) and (12) state that H and H_{ext} are "effective Bose operators" in the context of a free theory with an external source. In particular, Eq. (9) and (12) imply

$$[H, a_{l_1}^\dagger a_{l_2}^\dagger \cdots a_{l_n}^\dagger]_- = \sum_i \epsilon_i a_{l_1}^\dagger a_{l_2}^\dagger \cdots a_{l_n}^\dagger \tag{13}$$

and

$$[H_{ext}, a_{l_1}^\dagger a_{l_2}^\dagger \cdots a_{l_n}^\dagger]_- = \sum_i a_{l_1}^\dagger a_{l_2}^\dagger a_{l_{i-1}}^\dagger j_{l_i}^\star a_{l_{i+1}}^\dagger \cdots a_{l_n}^\dagger, \tag{14}$$

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so the energy is additive for a system of free particles. The general definition of an effective Bose operator is that the Hamiltonian density commutes with the field when the points are separated by a large spacelike distance,

$$[\mathcal{H}(\mathbf{x}), \phi(\mathbf{y})]_- \to 0, |\mathbf{x} - \mathbf{y}| \to \infty. \tag{15}$$

This definition holds for all cases, including quons.

Everything I stated for the Bose harmonic oscillator can be repeated for the Fermi oscillator, with obvious modifications. The commutation relation Eq. (1) is replaced by the anticommutation relation;

$$[a_i, a_j^\dagger]_+ \equiv a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij} \tag{16}$$

that, together with the vacuum condition which characterizes the Fock representation,

$$a_i|0\rangle = 0, \tag{17}$$

again suffices to calculate all vacuum matrix elements of multinomials in the annihilation and creation operators. For example,

$$\langle 0|a_{i_1} a_{i_2} \cdots a_{i_n} a_{j_m}^\dagger \cdots a_{j_2}^\dagger a_{j_1}^\dagger |0\rangle = \delta_{i_n j_m} \langle 0|\langle 0|a_{i_1} a_{i_2} \cdots a_{i_{n-1}} a_{j_{m-1}}^\dagger \cdots a_{j_2}^\dagger a_{j_1}^\dagger |0\rangle$$
$$- \langle 0|a_{i_1} a_{i_2} \cdots a_{i_{n-1}} a_{j_m}^\dagger a_{i_n} a_{j_{m-1}}^\dagger \cdots a_{j_2}^\dagger a_{j_1}^\dagger |0\rangle. \tag{18}$$

Continuing this reduction, it is clear that this vacuum matrix element vanishes, unless the set $\{i_1, i_2, \cdots, i_n\}$ is a permutation of the set $\{j_1, j_2, \cdots, j_m\}$. In particular, again *no relation is needed between two a's or between two a†'s*. As we know, it turns out that

$$[a_i, a_j]_+ = 0 = [a_i^\dagger, a_j^\dagger]_+, \tag{19}$$

but these relations again are redundant in the Fock representation. Also, as we know, only the totally antisymmetric (one-dimensional) representations of the symmetric group occur.

To construct observables in the free theory we again use the number operator, n_k, or the transition operator, n_{kl},

$$n_k = n_{kk} = a_k^\dagger a_k, \quad n_{kl} = a_k^\dagger a_l. \tag{20}$$

The commutation relation

$$[n_{kl}, a_m^\dagger]_- = \delta_{lm} a_k^\dagger \tag{21}$$

follows from the commutation relation Eq.(16). The number operator again has integer eigenvalues; now, however, the number of particles in a single quantum state can only be zero or one, since Eq.(19), which holds in the Fock representation, implies $a_i^{\dagger 2} = 0$,

$$n_k(a_l^\dagger)^{\mathcal{N}}|0\rangle = \delta_{kl} \mathcal{N}(a_l^\dagger)^{\mathcal{N}}|0\rangle, \mathcal{N} = 0, 1. \tag{22}$$

Using n_k and n_{kl} we again can construct the Hamiltonian,

$$H = \sum_k \epsilon_k n_k \tag{23}$$

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and other observables for the free theory. The Hamiltonian obeys

$$[H, a_l^\dagger]_- = \epsilon_l a_l^\dagger. \tag{24}$$

Analogous formulas of higher degree in the a's and $a^\dagger$'s give interaction terms.

I again pay special attention to couplings to external sources; the external Hamiltonian in the Fermi case is

$$H_{ext} = \sum_k (f_k^\star a_k + a_k^\dagger f_k), \tag{25}$$

where f_k is an anticommuting (Grassmann) number,

$$[f_k, f_l]_+ = [f_k, f_l^\star]_+ = [f_k, a_l]_+ = [f_k, a_l^\dagger]_+ = 0. \tag{26}$$

The external Hamiltonian satisfies the commutation relation,

$$[H_{ext}, a_l^\dagger]_- = f_l^\star. \tag{27}$$

The commutation relations Eq.(24) and Eq.(27) state that H and H_{ext} are "effective Bose operators" in the context of a free theory with an external source. In particular, Eq.(24) and Eq.(27) imply

$$[H, a_{l_1}^\dagger a_{l_2}^\dagger \cdots a_{l_n}^\dagger]_- = \sum_i \epsilon_i a_{l_1}^\dagger a_{l_2}^\dagger \cdots a_{l_n}^\dagger \tag{28}$$

and

$$[H_{ext}, a_{l_1}^\dagger a_{l_2}^\dagger \cdots a_{l_n}^\dagger]_- = \sum_i a_{l_1}^\dagger a_{l_2}^\dagger a_{l_{i-1}}^\dagger f_{l_i}^\star a_{l_{i+1}}^\dagger \cdots a_{l_n}^\dagger, \tag{29}$$

so that the energy is additive for a system of free particles.

Notice that Eq.(2,5,6,8,9,13) for the Bose case are *identical* to Eq.(17,20,21,23,24,28) for the Fermi case. Eq.(7,10,12,14) for the Bose case are analogous to Eq.(22,25,27,29) for the Fermi case. Finally, Eq.(1,3,4,11) for the Bose case and Eq.(16,18,19,26) for the Fermi case differ only by minus signs.

2 Generalizations of Bose and Fermi Statistics

As far as I know, the first attempt to go beyond Bose and Fermi statistics was made by G. Gentile [1]. He suggested "intermediate statistics," in which up to n particles can occupy a given quantum state. Clearly Fermi statistics is recovered for $n = 1$ and Bose statistics is recovered in the limit $n \to \infty$. As formulated by Gentile, intermediate statistics is not a proper quantum statistics, because the condition of having at most n particles in a quantum state is not invariant under change of basis.

H.S. Green [2] invented a generalization which is invariant under change of basis. I later dubbed his invention "parastatistics" [3]. Green noticed that the number operator and transition operator, Eq.(5, 20), have the same form for both bosons and fermions, as do the commutation

relations between the transition operator and the creation and annihilation operators, Eq.(6, 21). Green generalized the transition operator by writing

$$n_{kl} = (1/2)([a_k^\dagger, a_l]_\pm \mp p\delta_{kl}),$$ (30)

where the upper signs are for the generalization of bosons ("parabosons") and the lower signs are for the generalization of fermions ("parafermions"). Since Eq.(30) is trilinear, two conditions the states are necessary to fix the Focklike representation: the usual vacuum condition is

$$a_k|0\rangle = 0;$$ (31)

the new condition

$$a_k a_l^\dagger|0\rangle = p\,\delta_{kl}, p\ \text{integer},$$ (32)

contains the parameter p which is the order of the parastatistics. The Hamiltonian for free particles obeying parastatistics has the same form, in terms of the number operators, as for Bose and Fermi statistics,

$$H = \sum_k \epsilon_k n_k, \text{ where, as usual } [H, a_l^\dagger]_- = \epsilon_l a_l^\dagger.$$ (33)

For interactions with an external source, we must introduce para-Grassmann numbers which make the interaction Hamiltonian an effective Bose operator. This is in analogy with the cases of external Bose and Fermi sources discussed above. We want

$$[H_{ext}, a_l^\dagger] = c_l^\star.$$ (34)

We accomplish this by choosing $H_{ext} = \sum_{kl} H_{kl}^{ext}$, with

$$H_{kl}^{ext} = (1/2)([c_k^\star, a_l]_\pm + [a_k^\dagger, c_l]_\pm),$$ (35)

where the para-Grassmann numbers c_k and $c_k^\dagger$ obey

$$[[c_k^\star, c_l]_\pm, c_m^\star]_- = 0, \ [[c_k^\star, a_l]_\pm, a_m^\dagger]_- = 2\delta_{lm} c_k^\star, \text{ etc.,}$$ (36)

and the upper (lower) sign is for parabose-Grassmann (parafermi-Grassmann) numbers. The "etc." in Eq.(36) means that when some of the c's or $c^\dagger$'s is replaced by an a or an $a^\dagger$, the relation retains its form, except when the a and $a^\dagger$ can contract, in which case the contraction appears on the right-hand-side.

It is worthwhile to make explicit the fact that in theories with parastatistics states belong to many-dimensional representations of the symmetric group. This contrasts with the cases of Bose and Fermi statistics in which only the one dimensional representations occur.

I emphasize that parastatistics is a perfectly consistent local quantum field theory. The observables, such as the current, are local provided the proper symmetrization or antisymmetrization is used; for example,

$$j^\mu(x) = (1/2)[\bar\psi(x), \gamma^\mu\psi(x)]_-$$ (37)

for the current of a spin-1/2 field. Further, all norms in a parastatistics theory are positive; there are no negative probabilities. On the other hand, parastatistics of order $p > 1$ gives a gross

violation of statistics; for example, for a parafermi theory of order $p > 1$ each quantum state can be occupied p times. A precise experiment is not needed to rule out such a gross violation.

Within the last three years two new approaches to particle statistics (in three or more space dimensions) have been studied in order to provide theories in which the Pauli exclusion principle (i.e., Fermi statistics) and/or Bose statistics can be violated by a small amount. One of these approaches uses deformations [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] of the trilinear commutation relations of H.S. Green [2] and Volkov. [14]. (Deformations of algebras and groups, in particular, quantum groups, are a subject of great interest and activity at present. The extensive literature on this subject can be traced from [15].) The particles, called "parons," which obey this type of statistics have a quantum field theory which is local, but some states of such theories must have negative squared norms (i.e., there are negative probabilities in the theory). The negative squared norms first appear in many-particle states: in the model considered in [5] the first negative norm occurs in the state with Young tableau (3,1). It does not seem that the negative norm states decouple from those with positive squared norms (as, in contrast, the corresponding states do decouple in manifestly covariant quantum electrodynamics). Thus theories with parons seem to have a fatal flaw.

The other approach uses deformations of the bilinear Bose and Fermi commutation relations [16, 17, 18, 19, 20, 21, 23]. The particles which obey this type of statistics, called "quons," have positive-definite squared norms for a range of the deformation parameter, but the observables of such theories fail to have the desired locality properties. This failure raises questions about the validity of relativistic quon theories, but, in contrast to the paron theories, does not cause a problem with non-relativistic quon theories. (As I prove below, the TCP theorem and clustering hold for free relativistic quon theories, so even relativistic quon theories may be interesting.)

Still other approaches to violations of statistics were given in [24, 25, 26]. An interpolation between Bose and Fermi statistics using parastatistics of increasing order was studied in [27]; this also does not give a small violation.

Yet another type of statistics, anyon statistics, has been extensively discussed recently, and applied to the fractional Hall effect and to high-temperature superconductivity. For anyons, the transposition of two particles can give *any* phase,

$$\psi(1,2) = e^{i\phi}\psi(2,1). \tag{38}$$

In the form usually considered, anyons only exist in two space dimensions, and are outside the framework I am considering. I will not discuss them further here; rather I give two relevant references [29, 30].

3   The Quon Algebra

3.1   *The $q = 0$ case*

In their general classification of possible particle statistics, Doplicher, Haag and Roberts [31] included bosons, fermions, parabosons, parafermions and one other case, infinite statistics, in which all representations of the symmetric group could occur, but did not give an algebra which

led to this last case. Roger Hegstrom, a chemist at Wake Forest University, suggested averaging the Bose and Fermi commutation relations to get

$$a_k a_l^\dagger = \delta_{kl}, \quad a_k|0\rangle = 0. \tag{39}$$

(Unknown to Hegstrom and me, this algebra had been considered earlier by Cuntz [28].) With Hegstrom's permission, I followed up his idea and showed that this algebra gives an example of infinite statistics. Consider a general scalar product,

$$(a_{k_1}^\dagger \cdots a_{k_n}^\dagger|0\rangle, a_{P^{-1}k_1}^\dagger \cdots a_{P^{-1}k_m}^\dagger|0\rangle). \tag{40}$$

This vanishes unless $n = m$ and P is the identity, and then it equals one. From this it follows that one can choose coefficients $c(P)$ to project into states in each irreducible of S_n and that the norm will be positive,

$$\| \sum_P c(P) a_{P^{-1}k_1}^\dagger \cdots a_{P^{-1}k_n}^\dagger|0\rangle \|^2 > 0; \tag{41}$$

thus every representation of S_n occurs. Note that there is no relation between two a's or two $a^\dagger$'s; as before, the Fock vacuum condition makes such relations unnecessary.

To construct observables, we want a number operator and a transition operator which obey

$$[n_k, a_l^\dagger]_- = \delta_{kl} a_l^\dagger, \quad [n_{kl}, a_m^\dagger]_- = \delta_{lm} a_k^\dagger. \tag{42}$$

Once Eq.(42) holds, the Hamiltonian and other observables can be constructed in the usual way; for example,

$$H = \sum_k \epsilon_k n_k, \quad \text{etc.} \tag{43}$$

The obvious thing is to try

$$n_k = a_k^\dagger a_k. \tag{44}$$

Then

$$[n_k, a_l^\dagger]_- = a_k^\dagger a_k a_l^\dagger - a_l^\dagger a_k^\dagger a_k. \tag{45}$$

The first term in Eq.(45) is $\delta_{kl} a_k^\dagger$ as desired; however the second term is extra and must be canceled. This can be done by adding the term $\sum_t a_t^\dagger a_k^\dagger a_k a_t$ to the term in Eq.(44). This cancels the extra term, but adds a new extra term, which must be canceled by another term. This procedure yields an infinite series for the number operator and for the transition operator,

$$n_{kl} = a_k^\dagger a_l + \sum_t a_t^\dagger a_k^\dagger a_l a_t + \sum_{t_1,t_2} a_{t_2}^\dagger a_{t_1}^\dagger a_k^\dagger a_l a_{t_1} a_{t_2} + \cdots \tag{46}$$

As in the Bose case, this infinite series for the transition or number operator defines an unbounded operator whose domain includes states made by polynomials in the creation operators acting on the vacuum. (As far as I know, this is the first case in which the number operator, Hamiltonian, etc. for a free field are of infinite degree.)

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For nonrelativistic theories, the x-space form of the transition operator is

$$\rho_1(\mathbf{x};\mathbf{y}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{y}) + \int d^3z\,\psi^\dagger(\mathbf{z})\psi^\dagger(\mathbf{x})\psi(\mathbf{y})\psi(\mathbf{z})$$

$$+ \int d^3z_1 d^3z_2\,\psi(\mathbf{z_2})\psi^\dagger(\mathbf{z_1})\psi^\dagger(\mathbf{x})\psi(\mathbf{y})\psi(\mathbf{z_1})\psi(\mathbf{z_2}) + \cdots, \tag{47}$$

which obeys the nonrelativistic locality requirement

$$[\rho_1(\mathbf{x};\mathbf{y}),\psi^\dagger(\mathbf{w})]_- = \delta(\mathbf{y}-\mathbf{w})\psi^\dagger(\mathbf{x}), \quad \text{and} \quad \rho(\mathbf{x};\mathbf{y})|0\rangle = 0. \tag{48}$$

The apparent nonlocality of this formula associated with the space integrals has no physical significance. To support this last statement, consider

$$[Qj_\mu(x),Qj_\nu(y)]_- = 0, \quad x \sim y, \tag{49}$$

where $Q = \int d^e x j^0(x)$. Equation (49) seems to have nonlocality because of the space integral in the Q factors; however, if

$$[j_\mu(x),j_\nu(y)]_- = 0, \quad x \sim y, \tag{50}$$

then Eq.(49) holds, despite the apparent nonlocality. What is relevant is the commutation relation, not the representation in terms of a space integral. (The apparent nonlocality of quantum electrodynamics in the Coulomb gauge is another such example.)

In a similar way,

$$[\rho_2(\mathbf{x},\mathbf{y};\mathbf{y}',\mathbf{x}'),\psi^\dagger(\mathbf{z})]_- = \delta(\mathbf{x}'-\mathbf{z})\psi^\dagger(\mathbf{x})\rho_1(\mathbf{y},\mathbf{y}') + \delta(\mathbf{y}'-\mathbf{z})\psi^\dagger(\mathbf{y})\rho_1(\mathbf{x},\mathbf{x}'). \tag{51}$$

Then the Hamiltonian of a nonrelativistic theory with two-body interactions has the form

$$H = (2m)^{-1}\int d^3x\,\nabla_x \cdot \nabla_{x'}\rho_1(\mathbf{x},\mathbf{x}')|_{\mathbf{x}=\mathbf{x}'} + \frac{1}{2}\int d^3x\,d^3y\,V(|\mathbf{x}-\mathbf{y}|)\rho_2(\mathbf{x},\mathbf{y};bfy,\mathbf{x}). \tag{52}$$

$$[H,\psi^\dagger(\mathbf{z_1})\ldots\psi^\dagger(\mathbf{z_n})]_- = -(2m)^{-1}\sum_{j=1}^n \nabla_{z_i}^2 + \sum_{i<j}V(|\mathbf{z_i}-\mathbf{z_j}|)]\psi^\dagger(\mathbf{z_1})\ldots\psi^\dagger(\mathbf{z_n})$$

$$+ \sum_{j=1}^n \int d^3x\,V(|\mathbf{x}-\mathbf{z_j}|)\psi^\dagger(\mathbf{z_1})\cdots\psi^\dagger(\mathbf{z_n})\rho_1(\mathbf{x},\mathbf{x}'). \tag{53}$$

Since the second term on the right-hand-side of Eq.(53) vanishes when the equation is applied to the vacuum, this equation shows that the usual Schrödinger equation holds for the n-particle system. Thus the usual quantum mechanics is valid, with the sole exception that any permutation symmetry is allowed for the many-particle system. This construction justifies calculating the energy levels of (anomalous) atoms with electrons in states which violate the exclusion principle using the normal Hamiltonian, but allowing anomalous permutation symmetry for the electrons [32].

In general, an arbitrary many-particle state is in a mixture of inequivalent irreducible representations of S_n. If $\mathcal{O}$ is any observable and Ψ is any state, the cross terms between irreducibles in the matrix element $\langle\Psi|\mathcal{O}|\Psi\rangle$ automatically vanish, since observables keep states inside their irreducible representation of S_n.

3.2 *The general quon algebra for $-1 \le q \le 1$.*

The quon algebra,

$$a_k a_l^\dagger - q a_l^\dagger a_k = \delta_{kl},$$
(54)

which is a deformation of the Bose and Fermi algebras and interpolates between these algebras as q goes from 1 to -1 on the real axis, shares many qualitative features with the special case of $q = 0$ just discussed. In particular, the quon algebra also allows all representations of S_n. This algebra, supplemented by the vacuum condition

$$a_k |0\rangle = 0,$$
(55)

determines a (Fock-like) representation in a linear vector space. For $-1 < q < 1$, the squared norms of all vectors made by limits of polynomials of the creation operators, $a_k^\dagger$, are strictly positive[19, 20, 21]. Among other things, this means that there are $n!$ linearly independent states of n particles with distinct quantum numbers, and all representations of the symmetric group occur. Also, as in the case of $q = 0$, Eqs.(54,55) allow the calculation of the vacuum to vacuum matrix element of any polynomial in the a's and $a^\dagger$'s. As before, no commutation relation between two a's or between two $a^\dagger$'s is needed. Further, in this case, no such rule can be imposed on aa or $a^\dagger a^\dagger$. The relation,

$$a_k a_l - q a_l a_k = 0,$$
(56)

between two a's which one might guess in analogy with the Bose and Fermi commutation rules holds only when $q^2 = 1$; and requires that $q = \pm 1$ in Eq.(54); i.e., Eq.(56) can hold only in the Bose and Fermi cases. To see this, interchange k and l in Eq.(56) and put the result back in the initial relation. (Commutation relations between two a's or between two $a^\dagger$'s are also not needed for normal ordering, i.e., to expand a product of a's and $a^\dagger$'s as a sum of terms in which creation operators always stand to the left of annihilation operators. Wick's theorem for quon operators is similar to the usual Wick's theorem; the only difference is that the terms acquire powers of q. I gave the precise algorithm in [33].) As q approaches -1 from above, the more antisymmetric representations become more heavily weighted and at -1 only the antisymmetric representation survives. As q approaches 1 from below, the more symmetric representations become more heavily weighted and at 1 only the symmetric representation survives. Outside the interval $[-1, 1]$, the squares of some norms become negative.

Now I discuss the construction of observables both without and with an external source. Without an external source, one again needs a set of number operators n_k such that

$$[n_k, a_l^\dagger]_- = \delta_{kl} a_l^\dagger.$$
(57)

Like the $q = 0$ case, the expression for n_k or n_{kl} is an infinite series in creation and annihilation operators; unlike the $q = 0$ case, the coefficients are complicated. The first two terms are

$$n_{kl} = a_k^\dagger a_l + (1 - q^2)^{-1} \sum (a_l^\dagger a_k^\dagger - q a_k^\dagger a_l^\dagger)(a_l a_l - q a_l a_l) + \cdots.$$
(58)

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Here I gave the transition number operator n_{kl} for $k \to l$ since this takes no extra effort. The general formula for the number operator is given in [22] following a conjecture of Zagier [19]. As before, the Hamiltonian is

$$H = \sum_k \epsilon_k n_k, \quad \text{with} \quad [H, a_l^\dagger]_- = \epsilon_l a_l^\dagger. \tag{59}$$

For an external source, it is crucial to insure that H_{ext} is an effective Bose operator. In order to do this, one must choose the external source to be a quon analog of a Grassmann number, i.e., to obey

$$c_k c_l^* - q c_l^* c_k = 0; \quad c_k a_l^\dagger - q a_l^\dagger c_k = 0; \quad a_k c_l^* - q c_l^* a_k = 0. \tag{60}$$

Then H_{ext} must be chosen to obey

$$[H_{ext}, a_l^\dagger]_- = c_l^* \tag{61}$$

For example, for $q = 0$, the first two terms are

$$H_{ext} = \sum_k (c_k^* a_k + a_k^\dagger c_k) + \sum_k \sum_t a_t^\dagger (c_k^* a_k + a_k^\dagger c_k) a_t + \cdots \tag{62}$$

For the general case, I give the first two terms of H_{kl}^{ext}, subject to

$$[H_{kl}^{ext}, a_m^\dagger]_- = \delta_{lm} c_k^* \tag{63}$$

and hermiticity, $(H_{kl}^{ext})^\dagger = H_{lk}^{ext}$,

$$H_{kl}^{ext} = c_k^* a_l + a_k^\dagger c_l + (1 - q^2)^{-1} \sum_t (a_t^\dagger c_k^* - q c_k^* a_t^\dagger)(a_l a_t - q a_t a_l)$$

$$+ \sum_k (1 - q^2)^{-1} (a_t^\dagger a_k^\dagger - q a_k^\dagger a_t^\dagger)(c_l a_t - q a_t c_l) + \cdots \tag{64}$$

If, instead, we incorrectly choose $H_{ext} = \sum_k (j_k^* a_k + a_k^\dagger j_k)$, where j is a c-number, then the energy of widely separated states is not additive,

$$H_{ext} a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_n}^\dagger |0\rangle = [j_{k_1}^* a_{k_2}^\dagger \cdots a_{k_n}^\dagger + q a_{k_1}^\dagger j_{k_1}^* \cdots a_{k_n}^\dagger + \cdots q^{n-1} a_{k_1}^\dagger a_{k_2}^\dagger \cdots j_{k_n}^*]|0\rangle \tag{65}$$

Although this point is transparent for the case of fermions where powers of negative one replace powers of q in Eq.(65), it seems to be less clear in the quon case. Because this point was not recognized, the bound on validity of Bose statistics for photons given in [34] is incorrect.

Again one- and two-body observables can be constructed from $\rho_1(\mathbf{x}, \mathbf{x})$ and $\rho_2(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_2, \mathbf{y}_1)$. The formula for n can be translated into a formula for ρ_1, and at least the first non-trivial term is known for ρ_2. With these, a valid nonrelativistic theory of identical particles with (small) violations of Fermi of Bose statistics can be formulated [35].

The condition that observables must be effective Bose operators leads to conservation of statistics which states that all interactions must involve an even number of fermions or para-fermions and an even number of para particles (except for cases in which p para fields can occur when the order of the parastatistics is p)[36]. I expect that conservation of statistics must also hold for quons and, in particular, that a single quon cannot couple to normal fields [37]. I plan to discuss the conservation of statistics for quons in detail elsewhere. I have discussed the simple case of a single oscillator elsewhere[33], so I will not repeat this discussion here.

To summarize, all irreducible representations of S_n have positive (norm)^2 in this interval. As $q \to \pm 1$ the more symmetric (antisymmetric) irreducibles occur with higher weight. At the endpoints, $q = \pm 1$, only the symmetric (antisymmetric) representation survives.

4 The quon algebra in the presence of antiparticles

The pattern is established by discussing the spin-zero case. Since

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} (b_k e^{-ik\cdot x} + d_k^\dagger e^{ik\cdot x}), \tag{66}$$

$$\phi^\dagger(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} (d_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}), \tag{67}$$

$\omega_k = k^0 = \sqrt{\mathbf{k}^2 + m^2}$, to preserve charge conjugation symmetry one should supplement the commutation relation for the b's and $b^\dagger$'s by

$$d_k d_l^\dagger - q d_l^\dagger d_k = \delta_{kl}, \tag{68}$$

$$d_k b_l^\dagger - q b_l^\dagger d_k = 0, \tag{69}$$

With this choice, the positivity of the norms is preserved in the presence of antiparticles. If, instead, one chooses the x-space relation,

$$\phi(x)\phi^\dagger(y) - q\phi^\dagger(y)\phi(x) = F(x-y) \equiv \text{vev}(lhs) \tag{70}$$

then one finds the usual quon commutation relation for the b's, but

$$d_k d_l^\dagger - q^{-1} d_l^\dagger d_k = \delta_{kl} \tag{71}$$

for the d's. Since Eq.(71) gives positive norms only *outside* $|q| < 1$, this choice is inconsistent. In [39, 40] this last choice has been argued to imply breaking of charge conjugation invariance.

It is amusing to note that the TCP theorem and clustering hold, at least for free quon fields, despite the failure of locality [33].

5 Experiments

In a conference devoted to issues related to harmonic oscillators, it is worthwhile to make some comments about the experimental relevance of the quon oscillator. The quon oscillator provides a parametrization of possible small departures from Fermi or Bose statistics. The simplest way to detect small violations of statistics is to find a state which either Fermi or Bose statistics would not allow. For Fermi (Bose) statistics, this would be a state in which identical particles are not totally antisymmetric (symmetric). The path-breaking high-precision experiment of Ramberg and Snow[41] searches for transitions to a state in which the electrons of the copper atom are not totally antisymmetric. The failure to detect such transitions (above background) leads to the following upper bound on violation of the exclusion principle,

$$\rho_2 = \frac{1}{2}(1 - \beta^2)\rho_a + \frac{1}{2}\beta^2 \rho_s, \quad \frac{1}{2}\beta^2 \leq 1.7 \times 10^{-26}, \tag{72}$$

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ρ_2 is the two-electron density matrix, $\rho_{a(s)}$ is the antisymmetric (symmetric) two-electron density matrix. For two electrons in different states ρ_2 can be expressed in terms of q of the q-mutator as

$$\rho_2 = \frac{1}{2}(1 - q)\rho_a + \frac{1}{2}(1 + q)\rho_s, \tag{73}$$

so the Ramberg Snow bound is

$$0 \leq (1 + q)/2 \leq 1.7 \times 10^{-26}. \tag{74}$$

A high-precision experiment to detect or bound violations of the exclusion principle for electrons in helium is being conducted by D. Kelleher, et al.[42]

I conclude this brief discussion of experimental bounds on small violations of statistics by remarking that there are three types of such experiments: (1) to detect an accumulation of particles in anomalous states, (2) to detect transitions to anomalous states and (3) to detect deviations from the usual statistical properties of many-particle systems. Here and in [8] type (2) experiments are discussed, because they allow detection of single transitions to anomalous states. Type (1) experiments require detection of a small concentration of anomalous states in a macroscopic system; for that reason they are generally less sensitive than type (2) experiments. I have not analyzed type (3) experiments; however it seems likely that they will fail to provide high-precision tests for the same reason that type (1) experiments fail: it will be difficult to detect the modification of the statistical properties of a macroscopic sample due to a small concentration of anomalous states.

6   Summary

The quon oscillator serves as an interpolation between Fermi and Bose statistics. This interpolation preserves positivity of norms and the non-relativistic form of locality, but fails to allow local observables in a relativistic theory. Nonetheless, the TCP theorem and clustering hold in relativistic quon theories. Terms in the Hamiltonian for both self-interacting systems and systems interacting with an external source must be effective Bose operators in order for the additivity of the energy for widely separated subsystems to hold. The quon theory provides a parametrization of possible deviations from Bose or Fermi statistics.

7   Acknowledgement

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THE UNIVERSAL PROPAGATOR

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Abstract

For a general Hamiltonian appropriate to a single canonical degree of freedom, we characterize and define a universal propagator with the property that it correctly evolves the coherent-state Hilbert space representatives for an arbitrary fiducial vector. The universal propagator is explicitly constructed for the harmonic oscillator, with a result that differs from the conventional propagators for this system.

1   Introduction

Canonical coherent states, and the coherent state propagator they engender, have been around for over three decades.[1-3] In essence, their construction is simplicity itself. Let P and Q denote an irreducible pair of self-adjoint Heisenberg operators satisfying $[Q, P] = i(\hbar = 1)$, and let

$$|p, q; \eta\rangle = e^{-iqP} e^{ipQ} |\eta\rangle$$

denote a family of normalized states defined for a fixed fiducial vector $|\eta\rangle$, $\langle \eta|\eta\rangle = 1$, and for all $(p, q) \in \mathbf{R}^2$. These states are the canonical coherent states and they admit a resolution of unity in the form

$$\int |p, q; \eta\rangle\langle p, q; \eta| dpdq/2\pi = I,$$

for any $|\eta\rangle$, when integrated over all phase space.[2] These states lead to a representation of Hilbert space $\mathbf{H}$ by bounded, continuous functions,

$$\psi_\eta(p, q) \equiv \langle p, q; \eta|\psi\rangle,$$

defined for all $|\psi\rangle \in \mathbf{H}$, that evidently depend on the choice of $|\eta\rangle$, although that dependence is often left implicit. An inner product in this representation is afforded by

$$\langle \phi|\psi\rangle = \int \phi_\eta^*(p, q)\psi_\eta(p, q) dpdq/2\pi,$$

an integral which removes all trace of the fiducial vector $|\eta\rangle$.

1.1 Propagators

The abstract Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}|\psi\rangle,$$

involving the self-adjoint Hamiltonian $\mathcal{H}$, is formally solved with the aid of the evolution operator $U(t) = \exp(-it\mathcal{H})$, namely

$$|\psi(t'')\rangle = e^{-i(t''-t')\mathcal{H}}|\psi(t')\rangle.$$

In a coherent-state representation the evolution is effected by an integral kernel

$$K_\eta(p'',q'',t'';p',q',t') \equiv \langle p'',q'';\eta|e^{-i(t''-t')\mathcal{H}}|p',q';\eta\rangle$$

in the form

$$\psi_\eta(p'',q'',t'') = \int K_\eta(p'',q'',t'';p',q',t')\psi_\eta(p',q',t')dp'dq'/2\pi.$$

Clearly, K_η depends strongly on the fiducial vector as does ψ_η.

Our goal in this paper is to formulate a *universal propagator* $K(p'',q'',t'';p',q',t')$, a single function independent of any particular fiducial vector, which, nevertheless, has the property that

$$\psi_\eta(p'',q'',t'') = \int K(p'',q'',t'';p',q',t')\psi_\eta(p',q',t')dp'dq'/2\pi \qquad (1)$$

holds just as before for any choice of fiducial vector.

The functions K_η and K are qualitatively different as is clear from their behavior as $t'' \to t'$. In particular

$$\lim_{t''\to t'} K_\eta(p'',q'',t'';p',q',t') = \langle p'',q'';\eta|p',q';\eta\rangle, \qquad (2)$$

which clearly retains a strong dependence on the fiducial vector. On the other hand, if (1) is to hold for any η, we must require that

$$\lim_{t''\to t'} K(p'',q'',t'';p',q',t') = 2\pi\delta(p''-p')\delta(q''-q'). \qquad (3)$$

Next let us turn our attention to a suitable differential equation satisfied by K_η and K. It is straightforward to see that

$$\left(-i\frac{\partial}{\partial q}\right)\langle p,q;\eta|\psi\rangle = \langle p,q;\eta|P|\psi\rangle,$$

$$\left(q+i\frac{\partial}{\partial p}\right)\langle p,q;\eta|\psi\rangle = \langle p,q;\eta|Q|\psi\rangle$$

hold quite independently of $|\eta\rangle$. Thus if $\mathcal{H} = \mathcal{H}(P,Q)$ denotes the Hamiltonian it follows that Schrödinger's equation takes the form

$$i\frac{\partial}{\partial t}\psi_\eta(p,q,t) = \langle p,q;\eta|\mathcal{H}(P,Q)|\psi(t)\rangle$$

$$= \mathcal{H}(-i\frac{\partial}{\partial q},q+i\frac{\partial}{\partial p})\psi_\eta(p,q,t)$$

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valid for any $|\eta\rangle$.[3] The propagators are also solutions of Schrödinger's equation so it follows that

$$i\frac{\partial}{\partial t}K_\#(p,q,t;p',q',t') = \mathcal{H}(-i\frac{\partial}{\partial q},q+i\frac{\partial}{\partial p})K_\#(p,q,t;p',q',t'),\qquad(4)$$

where $K_\#$ denotes either K_η or K. What distinguishes which function is under consideration is the *initial condition* (at $t=t'$) of the solution, namely, either (2) or (3).

When K_η is under consideration, the operators $-i\frac{\partial}{\partial q}$ and $q+i\frac{\partial}{\partial p}$ refer to a *single* degree of freedom made irreducible by confining attention to the subspace of $L^2(\mathbf{R}^2,dpdq/2\pi)$ spanned by $\psi_\eta(p,q)$ for a fixed $|\eta\rangle$ and for all $|\psi\rangle \in \mathbf{H}$. This restriction is implicit in K_η because as $t''\to t'$ the resultant integral kernel $\langle p'',q'';\eta|p',q';\eta\rangle$ is a *projection operator* onto the subspace of an irreducible representation.

1.2   The Universal Propagator

In contrast to the former case, when the universal propagator K is under consideration the resultant Schrödinger equation (4) is interpreted as one appropriate to *two* degrees of freedom. In this view $y_1 = q$ and $y_2 = p$ denote two "coordinates", and one is looking at the irreducible Schrödinger representation of a special class of two-variable Hamiltonians, ones where the classical Hamiltonian is restricted to have the form $H_c(p_1,y_1-p_2)$, rather than the most general form $H_c(p_1,p_2,y_1,y_2)$.

In the case of K, and based on the interpretation described above, a standard phase-space path integral solution may be given for the universal propagator. In particular, and for a sufficiently wide class of Hamiltonians, it follows that

$$K(p'',q'',t'';p',q',t') = \mathcal{M}\int e^{i\int[x\dot{p}+k\dot{q}-\mathcal{H}(k,q-x)]dt}\mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x.$$

Note that "x" and "k" are "momenta" conjugate to the "coordinates" "p" and "q", and also that the special form of the Hamiltonian has been used. In the standard phase-space path integral there is always one more (k,x) pair of integrals compared to the (p,q) family, and the (k,x) integrals are unrestricted. This situation is made explicit in the regularized prescription for the path integral given, in standard notation, by

$K(p'',q'',t'';p',q',t')$

$$= \lim_{\substack{\epsilon\to0\\L\to\infty}} 2\pi\int\cdots\int e^{i\sum_{\ell=0}^{L}\left[x_{\ell+\frac{1}{2}}(p_{\ell+1}-p_\ell)+k_{\ell+\frac{1}{2}}(q_{\ell+1}-q_\ell)-\epsilon\mathcal{H}(k_{\ell+\frac{1}{2}},(q_{\ell+1}+q_\ell)/2-x_{\ell+\frac{1}{2}})\right]}$$

$$\times \prod_{\ell=1}^{L}dp_\ell dq_\ell\prod_{\ell=0}^{L}dk_{\ell+\frac{1}{2}}dx_{\ell+\frac{1}{2}}\ /\ (2\pi)^2,$$

where $p_{L+1},q_{L+1} = p'',q''$, $p_0,q_0 = p',q'$, and where $(L+1)\epsilon = (t''-t')$ is held fixed. Let us first change the variables $x_{\ell+\frac{1}{2}} \to x_{\ell+\frac{1}{2}} + (q_{\ell+1}+q_\ell)/2$, followed by a second change $x_{\ell+\frac{1}{2}} \to -x_{\ell+\frac{1}{2}}$. The resultant regularized path integral reads

$K(p'',q'',t'';p',q',t')$

$$= \lim_{\substack{\epsilon \to 0 \\ L \to \infty}} 2\pi \int \cdots \int e^{i\sum_{\ell=0}^{L}[\frac{1}{2}(q_{\ell+1}+q_\ell)(p_{\ell+1}-p_\ell)+k_{\ell+\frac{1}{2}}(q_{\ell+1}-q_\ell)-x_{\ell+\frac{1}{2}}(p_{\ell+1}-p_\ell)-\epsilon\mathcal{H}(k_{\ell+\frac{1}{2}},x_{\ell+\frac{1}{2}})]}$$

$$\times \prod_{\ell=1}^{L} dp_\ell dq_\ell \prod_{\ell=0}^{L} dk_{\ell+\frac{1}{2}} dx_{\ell+\frac{1}{2}} \Big/ (2\pi)^2 \,,$$

or the formal path integral is given by

$$K(p'',q'',t'';p',q',t') = \mathcal{M} \int e^{i\int [q\dot{p}+k\dot{q}-x\dot{p}-\mathcal{H}(k,x)]dt} \mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x \,, \tag{5}$$

which is our final expression for the universal propagator in the present case. From this formula it is clear that the dependence on p'' and p' is always of the form $p'' - p'$, and the dependence on q'' and q' is always of the form $q'' - q'$ save for the universal phase factor $\frac{1}{2}(q'' + q')(p'' - p')$. In other words,

$$K(p'',q'',t'';p',q',t') = F(p'' - p', q'' - q', t'' - t')e^{i\frac{1}{2}(q''+q')(p''-p')}$$

for some function F. Of course, if $\mathcal{H}$ depends explicitly on time then F is not simply a function of the time difference $t'' - t'$.

2 Examples of the Universal Propagator

2.1 Vanishing Hamiltonian

Let us evaluate the universal propagator in three soluble examples. The simplest case is that of a vanishing Hamiltonian which leads to

$$\begin{aligned}
K(p'',q'',t'';p',q',t') &= \mathcal{M} \int e^{i\int [q\dot{p}+k\dot{q}-x\dot{p}]dt} \mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x \\
&= \mathcal{N} \int e^{i\int q\dot{p}dt} \delta\{\dot{q}\}\delta\{\dot{p}\} \mathcal{D}p\mathcal{D}q \\
&= 2\pi\delta(p'' - p')\delta(q'' - q') \,,
\end{aligned}$$

where the normalization follows from the initial condition. Evidently this is the correct result.

2.2 Free Particle

The next case is the free particle where $\mathcal{H}(k,x) = k^2/2m$. In this event

$$\begin{aligned}
K(p'',q'',t'';p',q',t') &= \mathcal{M} \int e^{i\int (q\dot{p}+k\dot{q}-x\dot{p}-k^2/2m)dt} \mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x \\
&= \mathcal{N} \int e^{i\int (q\dot{p}+m\dot{q}^2/2)dt} \delta\{\dot{p}\} \mathcal{D}p\mathcal{D}q \\
&= \sqrt{\frac{2\pi m}{i(t'' - t')}} \delta(p'' - p')e^{im(q''-q')^2/2(t''-t')} \,.
\end{aligned}$$

2.3 Harmonic Oscillator

The last case we consider is the harmonic oscillator where $\mathcal{H}(k,x) = (k^2 + \omega^2 x^2)/2$. Now

$$
K(p'', q'', t''; p', q', t')
$$

$$
= \mathcal{M} \int e^{i \int [q\dot{p} + k\dot{q} - x\dot{p} - (k^2 + \omega^2 x^2)/2] dt} \mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x
$$

$$
= \mathcal{N} \int e^{i \int [q\dot{p} + (\dot{q}^2 + \dot{p}^2/\omega^2)/2] dt} \mathcal{D}p\mathcal{D}q
$$

$$
= (2i)^{-1} \csc(\omega T/2) \exp\left(i \left\{ \frac{1}{2}(q'' + q')(p'' - p') + \frac{1}{4}\cot(\omega T/2) \left[\frac{1}{\omega}(p'' - p')^2 + \omega(q'' - q')^2 \right] \right\} \right)
$$

$$
\tag{6}
$$

where $T \equiv t'' - t'$. Observe that this result is rather different from conventional propagators for harmonic oscillator Hamiltonians. Indeed, (6) is more like the propagator for a two-dimensional free particle in a uniform magnetic field.[4] This result also applies even when $\omega \to i\omega$, or with a suitable limit, even when $\omega \to 0$ leading to the free particle solution.

3 Propagation with the Universal Propagator

In order to check our results for the universal propagator let us put them to the test. For ease of computation we choose as the initial state for our propagation the coherent-state overlap function

$$
\langle p', q'; \eta | p, q; \eta \rangle,
$$

and additionally we choose the fiducial vector $|\eta\rangle$ to be the ground state of an oscillator with frequency Ω for which $\langle \eta | Q | \eta \rangle = 0 = \langle \eta | P | \eta \rangle$. In that case the initial state reads

$$
\psi_\eta(p', q', t') \equiv \langle p', q'; \eta | p, q; \eta \rangle
$$

$$
= \exp\left(i \left\{ \frac{1}{2}(p' + p)(q' - q) - \frac{1}{4}[\Omega^{-1}(p' - p)^2 + \Omega(q' - q)^2] \right\} \right).
$$

3.1 Free Particle

For the free particle case we need to compute $(T = t'' - t')$

$$
\frac{\sqrt{m}}{\sqrt{2\pi i T}} \int \exp\left\{ \frac{i}{2}mT^{-1}(q'' - q')^2 + \frac{i}{2}(p'' + p)(q' - q) - \frac{1}{4}[\Omega^{-1}(p'' - p)^2 + \Omega(q' - q)^2] \right\} dq',
$$

which is readily found to be $[\bar{p} \equiv (p'' + p)/2, \bar{q} \equiv (q'' + q)/2, p^* \equiv p'' - p, q^* \equiv q'' - q]$

$$
\psi_\eta(p'', q'', t'')
$$

$$
= \frac{\sqrt{m}}{\sqrt{m + i\Omega T/2}} \exp\left\{ \frac{im}{2} \frac{(\Omega^2 T q^{*2}/4 - T\bar{p}^2 + 2m\bar{p}q^*)}{(m^2 + \Omega^2 T^2/4)} - \frac{p^{*2}}{4\Omega} - \frac{\Omega(mq^* - T\bar{p})^2}{4(m^2 + \Omega^2 T^2/4)} \right\}.
$$

$$
\tag{7}
$$

This result agrees with one obtained elsewhere[5] thereby establishing its validity.

In addition, as $\Omega \to \infty$ or $\Omega \to 0$, Eq. (7), apart from a suitable scale factor, yields the result in the sharp q or sharp p representation, respectively.[3,5] In particular, consider

$$
\begin{aligned}
\psi(q'', t'') &= \lim_{\Omega \to \infty} \frac{1}{2}\sqrt{\frac{\Omega}{\pi}} \psi_\eta(p'', q'', t'') \\
&= \sqrt{\frac{m}{2\pi i T}} e^{i\frac{m}{2T}(q''-q)^2} ,
\end{aligned}
$$

which is the appropriate Schrödinger representation solution for the free particle. Likewise, consider

$$
\begin{aligned}
\tilde{\psi}(p'', t'') &= \lim_{\Omega \to 0} \frac{1}{2\sqrt{\pi \Omega}} \psi_\eta(p'', q'', t'') e^{-i(p''q''-pq)} \\
&= \lim_{\Omega \to 0} \frac{1}{2\sqrt{\pi \Omega}} e^{-i\frac{Tp^2}{2m} - \frac{1}{4\Omega}p^{*2} - ip^*\overline{q}} \\
&= \delta(p'' - p) e^{-i\frac{Tp^2}{2m}} ,
\end{aligned}
$$

which is the proper answer in momentum space for the free particle.

3.2   Harmonic Oscillator

Finally, let us consider the time evolution for the harmonic oscillator as given by

$$
\begin{aligned}
(4\pi i)^{-1}\csc(\omega T/2) \int & \quad \exp\left(i\left\{\frac{1}{2}(q''+q')(p''-p') + \frac{1}{4}\cot(\omega T/2)\left[\frac{1}{\omega}(p''-p')^2 + \omega(q''-q')^2\right]\right\}\right) \\
\times & \quad \exp\left\{i\frac{1}{2}(p'+p)(q'-q) - \frac{1}{4}[\Omega^{-1}(p'-p)^2 + \Omega(q'-q)^2]\right\} dp'dq' ,
\end{aligned}
$$

which is evaluated as $[s \equiv \sin(\omega T/2), \ c \equiv \cos(\omega T/2)]$.

$$
\begin{aligned}
\psi_\eta(p'', q'', t'') &= C(T)^{-\frac{1}{2}} \exp\left(i\overline{q}p^* \frac{\omega^2 s^2}{\Omega^2 c^2 + \omega^2 s^2} + i\overline{p}q^* \frac{\omega^2 c^2}{\omega^2 c^2 + \Omega^2 s^2}\right) \\
&\quad \times \exp\left[-\frac{i\omega sc}{4(\omega^2 c^2 + \Omega^2 s^2)}(4\overline{p}^2 - \Omega^2 q^{*2}) - \frac{i\omega \Omega^2 sc}{4(\Omega^2 c^2 + \omega^2 s^2)}(4\overline{q}^2 - \Omega^{-2}p^{*2})\right] \\
&\quad \times \exp\left[-\frac{\Omega(cp^* + 2\omega s\overline{q})^2}{4(\Omega^2 c^2 + \omega^2 s^2)} - \frac{\Omega\omega^2(cq^* - 2\omega^{-1}s\overline{p})^2}{4(\omega^2 c^2 + \Omega^2 s^2)}\right] ,
\end{aligned}
\tag{8}
$$

where

$$
C(T) = \cos\omega T + i\frac{1}{2}\left(\frac{\omega}{\Omega} + \frac{\Omega}{\omega}\right)\sin\omega T .
$$

As in the free particle case, we can obtain the sharp q propagator by the same kind of limit, namely

$$
\begin{aligned}
\psi(q'', t'') &= \lim_{\Omega \to \infty} \frac{1}{2}\sqrt{\frac{\Omega}{\pi}} \psi_\eta(p'', q'', t'') \\
&= \sqrt{\frac{\omega}{2\pi i \sin\omega T}} \exp\left\{\frac{i\omega}{2\sin\omega T}\left[\cos\omega T(q''^2 + q^2) - 2q''q\right]\right\} ,
\end{aligned}
$$

which, of course, is the standard result. Likewise, the sharp p propagator is given by

$$\tilde{\psi}(p'', t'') = \lim_{\Omega \to 0} \frac{1}{2} \frac{1}{\sqrt{\pi\Omega}} \psi_\eta(p'', q'', t'') e^{-i(p''q'' - pq)}$$

$$= \frac{1}{\sqrt{2\pi i\omega \sin \omega T}} \exp\left\{ \frac{i}{2\omega \sin \omega T} \left[\cos \omega T (p''^2 + p^2) - 2p''p \right] \right\},$$

which, again, is the standard result.

We may also observe that the harmonic oscillator evolution simplifies considerably when $\Omega = \omega$. In that case[1]

$$\psi_\eta(p'', q'', t'') = \exp\left[\frac{i}{2}(q''p'' - qp) + \frac{i}{2}(q''p_T - p''q_T) - \frac{1}{4\omega}(p'' - p_T)^2 - \frac{\omega}{4}(q'' - q_T)^2 \right],$$

where

$$q_T = q\cos(\omega T) + \omega^{-1} p \sin(\omega T),$$
$$p_T = p\cos(\omega T) - \omega q \sin(\omega T),$$

evolution equations that are seen to follow the classical solution.

3.3 Generalization

Although we have only shown that a limited set of fiducial vectors are correctly propagated by the universal propagator, it should be fairly clear that the stated properties of the universal propagator hold true. Indeed, the general case may be discussed by considering as initial condition

$$\langle \tilde{p}, \tilde{q}; \eta | e^{-ip'Q} e^{iq'P} | p, q; \eta \rangle = e^{-i\tilde{q}p'} \langle p' + \tilde{p}, q' + \tilde{q}; \eta | p, q; \eta \rangle$$
$$= e^{i(\tilde{p}q' - \tilde{q}p') + i\tilde{p}(\tilde{q} - q)} \langle p', q'; \eta | p - \tilde{p}, q - \tilde{q}; \eta \rangle$$

for just a *single* $|\eta\rangle$, say a Gaussian with $\Omega = 1$. Then a suitable superposition over $\tilde{p}, \tilde{q}$ leads to any fiducial vector of interest, while a second and independent suitable superposition over p, q leads to any initial state $|\psi\rangle$ of interest.

4 Classical Limit

Although the universal propagator has been derived by identifying the relevant Schrödinger equation as one for two degrees of freedom, it should nevertheless be true that the classical limit refers to a single degree of freedom. This is possible, in the present case, because of the limited form of the quantum or classical Hamiltonian.

Recall, under standard assumptions, that the classical action for a conventional coherent state path integral reads, in the limit $\hbar \to 0$, as

$$I = \int [p\dot{q} - \mathcal{H}(p, q)]dt.$$

Extremal variation of this expression holding the end points fixed leads to the usual Hamiltonian equations of motion,

$$\dot{q} = \partial\mathcal{H}(p,q)/\partial p,$$
$$\dot{p} = -\partial\mathcal{H}(p,q)/\partial q,$$

appropriate to a single degree of freedom. Let us denote a generic solution of these equations by $q_c(t)$ and $p_c(t)$.

Before proceeding it is important to reexamine the "standard assumptions" that lead to this result. For finite $\hbar$ the expression that represents the classical Hamiltonian in a coherent state path integral is traditionally given by

$$H(p,q) = \langle p,q;\eta|\mathcal{H}(P,Q)|p,q;\eta\rangle$$
$$= \langle\eta|\mathcal{H}(P+p,Q+q)|\eta\rangle.$$

Normally, one restricts $|\eta\rangle$ so that $\langle\eta|Q|\eta\rangle = 0 = \langle\eta|P|\eta\rangle$, and $\langle\eta|Q^2|\eta\rangle \to 0$ and $\langle\eta|P^2|\eta\rangle \to 0$ as $\hbar \to 0$. In this case

$$\lim_{\hbar\to 0} H(p,q) = \mathcal{H}(p,q).$$

However, in the present paper we want to deal with more general fiducial vectors $|\eta\rangle$ such that

$$\langle\eta|Q|\eta\rangle = q_\eta,$$
$$\langle\eta|P|\eta\rangle = p_\eta,$$

are arbitrary real variables. We still insist on vanishing dispersion as $\hbar \to 0$, namely, that

$$\langle\eta|(Q-q_\eta)^2|\eta\rangle \to 0,$$
$$\langle\eta|(P-p_\eta)^2|\eta\rangle \to 0,$$

as $\hbar \to 0$. This more general situation leads to the result

$$\lim_{\hbar\to 0} H(p,q) = \mathcal{H}(p+p_\eta, q+q_\eta)$$

as the representative of the classical Hamiltonian.

In this more general case the classical action appropriate to the coherent state path integral becomes

$$I = \int [(p+p_\eta)\dot{q} - q_\eta\dot{p} - \mathcal{H}(p+p_\eta, q+q_\eta)]dt.$$

In this expression $p = p(t)$ and $q = q(t)$, while p_η and q_η are time-independent constants. The term $\int(p_\eta\dot{q} - q_\eta\dot{p})dt = p_\eta(q''-q') - q_\eta(p''-p')$ is a pure surface term and will not affect the equations of motion; it could be eliminated simply by a phase change of the coherent states. Extremal variation leads to the equations of motion

$$\dot{q} = \partial\mathcal{H}(p+p_\eta, q+q_\eta)/\partial p$$
$$\dot{p} = -\partial\mathcal{H}(p+p_\eta, q+q_\eta)/\partial q,$$

which have as their solutions

$$q(t) = q_c(t) - q_\eta \,,$$
$$p(t) = p_c(t) - p_\eta \,,$$

where $q_c(t)$ and $p_c(t)$ denote a generic solution of Hamilton's equations when $q_\eta = p_\eta = 0$, as discussed above.

Finally, we note that although the *dispersion* of $|\eta\rangle$ vanishes as $\hbar \to 0$, the generally nonvanishing values of q_η and p_η are *vestiges of the coherent-state representation induced by $|\eta\rangle$ that remain even after $\hbar \to 0$.*

4.1   Classical Limit of the Universal Propagator

In the case of the universal propagator the expression that serves as the classical action is identified as [cf. (5)]

$$I = \int [q\dot{p} + k\dot{q} - x\dot{p} - \mathcal{H}(k, x)]dt \,. \tag{9}$$

Extremal variation of this expression holding the end points fixed leads to the set of equations

$$\dot{q} = \dot{x} \,,$$
$$\dot{p} = \dot{k} \,,$$
$$\dot{q} = \partial \mathcal{H}(k, x)/\partial k \,,$$
$$\dot{p} = -\partial \mathcal{H}(k, x)/\partial x \,.$$

Consequently

$$\dot{x} = \partial \mathcal{H}(k, x)/\partial k \,,$$
$$\dot{k} = -\partial \mathcal{H}(k, x)/\partial x \,,$$

which show that (k, x) satisfy exactly the same equations of motion as do (p_c, q_c) in the usual classical theory. Thus we may identify the solution $k(t), x(t)$ with $p_c(t), q_c(t)$. In addition, we have

$$q(t) = q_c(t) - c_q \,,$$
$$p(t) = p_c(t) - c_p \,,$$

where c_q and c_p denote two arbitrary integration constants. Among all possible values of c_q and c_p are those that coincide with q_η and p_η for a general $|\eta\rangle$.

Thus we find that the set of solutions of the universal classical equations of motion appropriate to the universal propagator includes *every possible solution* of the classical equations of motion appropriate to the most general coherent-state propagator (with $|\eta\rangle$ having vanishing dispersion as $\hbar \to 0$). Not only does the quantum dynamics (universal propagator) correctly evolve the state vectors in a canonical coherent-state representation for a general $|\eta\rangle$, but the classical dynamics (universal classical equations of motion) correctly evolves the classical phase space points according to the coherent-state induced classical equations imprinted with arbitrary values of the only remnant of the fiducial vector after $\hbar \to 0$, namely its average coordinate and momentum values.

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5 Extension to Other Coherent States

We observe that the procedure to introduce a Schrödinger equation and a path integral solution for the universal propagator applies for other sets of coherent states, such as the spin coherent states, the affine [or SU(1,1)] coherent states, etc.[6] In each of these cases it becomes possible to introduce an appropriate universal propagator just by following the procedure we have given for the canonical coherent state case.

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PHASE-SPACE QUANTUM MECHANICS STUDY
OF TWO IDENTICAL PARTICLES
IN AN EXTERNAL OSCILLATORY POTENTIAL

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Abstract

We use this simple example to show how the formalism of Moyal works when it is applied to systems of identical particles. The symmetric and antisymmetric Moyal propagators are evaluated for this case; from them, the correct energy levels of energy are obtained, as well as the Wigner functions for the symmetric and antisymmetric states of the two identical particle system. Finally, the solution of the Bloch equation is straightforwardly obtained from the expressions of the Moyal propagators.

1 Phase-space Q M formalism

The original ideas of this approach to Q M are due to Weyl [1], Wigner [2] and Moyal [3]. States and observables are no longer operators on a Hilbert space but functions on an adequate phase space. The Weyl mapping relates both formalisms: given a function f defined over the phase space R^{2n}, the corresponding operator $\hat{F}$ is given by

$$\hat{F} = W(f) = \frac{1}{(2\pi\hbar)^n} \int_{R^{2n}} f(\mathbf{u})\,\Pi(\mathbf{u})\,d\mathbf{u}; \quad \mathbf{u} = (\mathbf{q}, \mathbf{p}). \tag{1}$$

Reciprocally, given an operator $\hat{A}$ the associated function in the phase space is

$$f_{\hat{A}}(\mathbf{u}) = tr\left\{\hat{A}\Pi(\mathbf{u})\right\} = W^{-1}\left(\hat{A}\right). \tag{2}$$

As we can see, a central role is played by the "Grossman-Royer" operators [4, 5]:

$$[\Pi(\mathbf{q},\mathbf{p})\Psi](\boldsymbol{\eta}) = 2^n \exp\left[\frac{2i}{\hbar}\mathbf{p}(\boldsymbol{\eta}-\mathbf{q})\right]\Psi(2\mathbf{q}-\boldsymbol{\eta}). \tag{3}$$

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The twisted product of two functions is defined as the non-commutative operation that corresponds to the product of operators:

$$(f \times g)(\mathbf{u}) = W^{-1}\big(W(f)\,W(g)\big)$$

$$= \frac{1}{(\pi\hbar)^{2n}} \int_{R^{4n}} f(\mathbf{v})g(\mathbf{w}) \exp\left[\frac{2i}{\hbar}(\mathbf{u}J\mathbf{v} + \mathbf{v}J\mathbf{w} + \mathbf{w}J\mathbf{u})\right] d\mathbf{v}\,d\mathbf{w}, \qquad (4)$$

where the matrix J is simply

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad (5)$$

being I the n-dimensional identity matrix.

In the Schrödinger representation of quantum mechanics, the information about dynamics is contained in the evolution operator $\hat{U}(t)$. Its counterpart in this formalism is the "Moyal propagator", defined as

$$\Xi(\mathbf{u}, t) = W^{-1}\big(\hat{U}(t)\big). \qquad (6)$$

It verifies Schrödinger equation:

$$i\hbar \frac{\partial \Xi}{\partial t} = H \times \Xi. \qquad (7)$$

The Fourier transform of this function with respect to t gives the spectral projections parametrized by E:

$$\Gamma(\mathbf{u}, E) = \frac{1}{2\pi\hbar} \int_R \Xi(\mathbf{u}, t)\, e^{iEt/\hbar}\, dt. \qquad (8)$$

If the Hamiltonian is time independent, the support on E of Γ coincides with the spectrum of $\hat{H}$ [6]. If E_0 belongs to the discrete spectrum of $\hat{H}$, $\Gamma(\mathbf{u}, E_0)$ is, but for a constant factor, the Wigner function of the orthogonal projector into the proper subspace E_0 [6]:

$$\mathcal{W}_\psi(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi\hbar)^n} W^{-1}\big(|\psi\rangle\langle\psi|\big) = \frac{1}{(2\pi\hbar)^n} \int_{R^n} e^{i\mathbf{p}\mathbf{y}/\hbar} \psi^*(\mathbf{q} + \mathbf{y}/2)\psi(\mathbf{q} - \mathbf{y}/2)\, d\mathbf{y}. \qquad (9)$$

2   Phase-space Q M formalism for identical particles

In the standard formalism of quantum mechanics, to deal with a system of N identical particles, we introduce a superselection rule: the space of physical states is a closed subspace of the initial Hilbert space. The Hilbert space is splitted [7]

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_w, \qquad (10)$$

where $\mathcal{H}_+$ is the Hilbert space of the wave functions symmetric under the exchange of any two particles and $\mathcal{H}_-$ the Hilbert space of the antisymmetric functions. The functions in $\mathcal{H}_w$ have no symmetry of this kind. The orthogonal projectors are given by

$$P_+ = \frac{1}{N!} \sum_{\sigma \in P_N} P_\sigma , \qquad P_- = \frac{1}{N!} \sum_{\sigma \in P_N} (-1)^{\pi(\sigma)} P_\sigma , \qquad (11)$$

where

$$(P_\sigma \psi)(x_1, \ldots, x_n) = \psi(x_{\sigma(1)}, \ldots, x_{\sigma(N)}). \tag{12}$$

for any σ in the group P_N of permutations of N elements; $\pi(\sigma)$ is the parity of σ.

If B is either an observable or an state in $\mathcal{H}$, the corresponding operators for a system of N fermions or bosons are:

$$P_- B P_- \quad \text{and} \quad P_+ B P_+. \tag{13}$$

If B is invariant under the exchange of particles, we have

$$P_\pm B P_\pm = B P_\pm = P_\pm B. \tag{14}$$

We use the Weyl transformation to translate these ideas into the language of phase space. Therefore, the function for an state or observable when we consider N bosons or fermions is

$$W^{-1}(P_\pm B P_\pm) = W^{-1}(P_\pm) \times W^{-1}(B) \times W^{-1}(P_\pm). \tag{15}$$

Due to the fact that the Weyl map is linear, all we need is the function for any permutation σ. As σ can be written as the product of cyclic permutations with no common elements [8], it is enough to compute the function corresponding to such a cycle. If we consider a general cycle $\sigma = (1, 2, 3, \ldots, M)$ we get:

$$\tilde{\sigma} = 2^{(M-1)n} \exp\left\{ -\frac{2i}{\hbar} \sum_{k=1; l > k}^{M} (-1)^{k+l} \, \mathbf{u}_k J \mathbf{u}_l \right\}, \quad M \text{ odd}; \tag{16}$$

$$\tilde{\sigma} = (2^{M-1} \pi \hbar)^n \delta(\mathbf{u}_1 - \mathbf{u}_2 + \ldots - \mathbf{u}_M) \exp\left\{ -\frac{2i}{\hbar} \sum_{k=1; l > k}^{M} (-1)^{k+l} \, \mathbf{u}_k J \mathbf{u}_l \right\}, \quad M \text{ even.} \tag{17}$$

As an example, for a two cycle that exchanges the particles i and j we have:

$$\tilde{\sigma}_{ij}(\mathbf{u}_1, \ldots, \mathbf{u}_N) = \tilde{\sigma}_{ij}(\mathbf{u}_i, \mathbf{u}_j) = (2\pi\hbar)^n \, \delta(\mathbf{u}_i - \mathbf{u}_j), \tag{18}$$

and it can be checked that

$$(\tilde{\sigma}_{ij} \times \rho \times \tilde{\sigma}_{ij}) (\ldots \mathbf{u}_i, \ldots \mathbf{u}_j, \ldots) = \rho(\ldots \mathbf{u}_j, \ldots \mathbf{u}_i, \ldots). \tag{19}$$

The functions corresponding to the orthogonal projectors for a system of two onedimensional particles are

$$p_+(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{2}(1 + 2\pi\hbar \, \delta(\mathbf{u}_1 - \mathbf{u}_2)), \tag{20}$$

$$p_-(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{2}(1 - 2\pi\hbar \, \delta(\mathbf{u}_1 - \mathbf{u}_2)). \tag{21}$$

3 Two onedimensional identical particles under an external oscillatory potential

Along the present section, we intend to presenting an example of particular interest in order to illustrate the preceeding discussion. We shall study the behavior of a two onedimensional particle system subjected to oscillatory forces of the same frequency. If we do not take into account the identity of the particles, the Hamiltonian will be simply:

$$H(\mathbf{u}_1, \mathbf{u}_2) = H(\mathbf{u}_1) + H(\mathbf{u}_2) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(q_1^2 + q_2^2). \tag{22}$$

The corresponding Moyal propagator has been already evaluated [6], and is:

$$\Xi(\mathbf{u}_1, \mathbf{u}_2) = \Xi(\mathbf{u}_1)\Xi(\mathbf{u}_2) = \frac{1}{\cos^2 \frac{\omega t}{2}} \exp\left\{ -\frac{2i}{\hbar\omega}(H(\mathbf{u}_1) + H(\mathbf{u}_2)) \tan \frac{\omega t}{2} \right\}. \tag{23}$$

Let us now introduce the statistics. As $H(\mathbf{u}_1, \mathbf{u}_2)$ is invariant under permutations of the two particles, the Hamiltonian for our system of two identical particles is:

$$H_\pm(\mathbf{u}_1, \mathbf{u}_2) = (H \times p_\pm)(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{2}\left\{ H(\mathbf{u}_1) + H(\mathbf{u}_2) \mp 2\pi\hbar^3 \left(\frac{\delta(\mathbf{u}_1 - \mathbf{u}_2)}{2m(q_1 - q_2)^2} + \frac{m\omega^2}{2}\frac{\delta(\mathbf{u}_1 - \mathbf{u}_2)}{(p_1 - p_2)^2} \right) \right\}; \tag{24}$$

We see that, after symmetrization or antisymmetrization, the Hamiltonian on phase space of our system is not longer (22) but (24). Equation (24) includes (22) plus an extra term. From this term results an extra potential, due to the introduction of the statistics, which has a quite different action depending whether the particles are fermions or bosons. In the first case, this potential is preceeded by a plus sign and, therefore, it is equivalent to a delta barrier preventing that $q_1 = q_2$ and $p_1 = p_2$. This already suggests that both particles cannot remain in the same state and, hence, that they fulfill the Pauli principle. This idea will be confirmed by our calculations for the lowest energy levels. On the contrary, if the particles are bosons the extra term has a minus sign and, consequently, it represents the apparition of a delta well. This delta well would rather favor the presence of particles in the same quantum state. In a clear opposition to the case of fermions, no exclusion principle can exist here.

The symmetrized Moyal propagator is obtained in a similar way:

$$\begin{aligned} \Xi_\pm(\mathbf{u}_1, \mathbf{u}_2) &= \frac{1}{2}\left\{ \frac{1}{\cos^2 \frac{\omega t}{2}} \exp\left(\frac{-2i(H(\mathbf{u}_1) + H(\mathbf{u}_2))}{\hbar\omega} \tan \frac{\omega t}{2} \right) \right. \\ &\left. \pm(-i) \frac{\exp\left\{ \frac{-i}{\hbar\omega}H(\mathbf{u}_1 + \mathbf{u}_2) \tan \frac{\omega t}{2} \right\}}{\cos \frac{\omega t}{2}} \frac{\exp\left\{ \frac{i}{\hbar\omega}H(\mathbf{u}_1 - \mathbf{u}_2) \cot \frac{\omega t}{2} \right\}}{\sin \frac{\omega t}{2}} \right\}. \end{aligned} \tag{25}$$

Comparing with (23) we see that there is also an extra term due to the statistics.

The spectral projections are obtained from (8) and (25); in this case we obtain

$$\Gamma_\pm(\mathbf{u}_1, \mathbf{u}_2, E) = 2e^{-2(H(\mathbf{u}_1) + H(\mathbf{u}_2))/\hbar\omega} \sum_{k=0}^{\infty} (-1)^k \delta\left(E - \hbar\omega(k+1)\right)$$

$$\times \sum_{n=0}^{k} \left[L_n\left(\frac{4H(\mathbf{u}_1)}{\hbar\omega}\right) L_{k-n}\left(\frac{4H(\mathbf{u}_2)}{\hbar\omega}\right) \pm (-1)^{k-n} L_n\left(\frac{2H(\mathbf{u}_1 + \mathbf{u}_2)}{\hbar\omega}\right) L_{k-n}\left(\frac{2H(\mathbf{u}_1 - \mathbf{u}_2)}{\hbar\omega}\right) \right]. \tag{26}$$

From here, the well know energy levels are obtained for the fermionic and bosonic cases. Let us notice the coefficient of $\delta(E - \hbar\omega)$, that vanishes for fermions but not for bosons.

We can evaluate the Wigner functions corresponding to states of two particles, in states i and j. Let us write those functions as W_{ij}^a, the corresponding to the antisymmetric state, and W_{ij}^s the associated to the symmetric state. We then have:

$$\Gamma_+(\mathbf{u}_1, \mathbf{u}_2, E) = (2\pi\hbar)^2[\; W_{00}^s(\mathbf{q}, \mathbf{p})\,\delta(E - \hbar\omega) + W_{01}^s(\mathbf{q}, \mathbf{p})\,\delta(E - 2\hbar\omega)$$
$$+ (W_{11}^s(\mathbf{q}, \mathbf{p}) + W_{02}^s(\mathbf{q}, \mathbf{p}))\,\delta(E - 3\hbar\omega) + \cdots], \tag{27}$$

$$\Gamma_-(\mathbf{u}_1, \mathbf{u}_2, E) = (2\pi\hbar)^2[\; W_{01}^a(\mathbf{q}, \mathbf{p})\,\delta(E - 2\hbar\omega) + W_{02}^a(\mathbf{q}, \mathbf{p})\,\delta(E - 3\hbar\omega) + \cdots]. \tag{28}$$

The coefficients of the δ are the Wigner functions of the orthogonal projector on the corresponding eigenspaces.

To finish, let us solve the Bloch equation, that is, let us find the Wigner function corresponding to the density matrix of the canonical ensemble for the system we are considering. Bloch equation reads simply:

$$\frac{\partial \Omega}{\partial \beta} = -H \times \Omega = -\Omega \times H, \quad \beta = 1/kT, \tag{29}$$

that is, it is Schrödinger's equation with the change $t \to -i\hbar\beta$.

But, as we already know the form of the Moyal propagator, we can write inmediately the solution for $\Omega(\mathbf{u}_1, \mathbf{u}_2, \beta)$ by making the change $t \to -i\hbar\beta$ in $\Xi_\pm(\mathbf{u}_1, \mathbf{u}_2, t)$. We get:

$$\Omega_\pm(\mathbf{u}_1, \mathbf{u}_2, \beta) = \frac{1}{2}\left\{ \frac{1}{\cosh^2\frac{\hbar\omega\beta}{2}} \exp\left(\frac{-2(H(\mathbf{u}_1) + H(\mathbf{u}_2))}{\hbar\omega} \tanh\frac{\hbar\omega\beta}{2} \right) \right. \tag{30}$$

$$\left. \pm \frac{\exp\left\{ \frac{-1}{\hbar\omega} H(\mathbf{u}_1 + \mathbf{u}_2) \tanh\frac{\hbar\omega\beta}{2} \right\}}{\cosh\frac{\hbar\omega\beta}{2}} \frac{\exp\left\{ \frac{-1}{\hbar\omega} H(\mathbf{u}_1 - \mathbf{u}_2) \coth\frac{\hbar\omega\beta}{2} \right\}}{\sinh\frac{\hbar\omega\beta}{2}} \right\}. \tag{31}$$

After integration of $\Omega(\mathbf{u}_1, \mathbf{u}_2, \beta)$ over the phase space, we get

$$Z_\pm(\beta) = \frac{\exp(\pm\hbar\omega\beta/2)}{8\cosh(\hbar\omega\beta/2)\sinh^2(\hbar\omega\beta/2)}. \tag{32}$$

From this partition function, we can obtain the thermodynamical quantities, for example the internal energy, the free energy and the entropy

$$E_\pm(\beta) = \frac{\hbar\omega}{2}\left\{ \tanh\frac{\hbar\omega\beta}{2} + 2\coth\frac{\hbar\omega\beta}{2} \right\} \mp \frac{\hbar\omega}{2}, \tag{33}$$

$$F_\pm(\beta) = \frac{1}{\beta}\left\{ \log\left[\cosh\frac{\hbar\omega\beta}{2} \right] + 2\log\left[\sinh\frac{\hbar\omega\beta}{2} \right] + \log 8 \right\} \mp \frac{\hbar\omega}{2}, \tag{34}$$

$$S_\pm(\beta) = k\left\{ \frac{\hbar\omega\beta}{2}\left[\tanh\frac{\hbar\omega\beta}{2} + 2\coth\frac{\hbar\omega\beta}{2} \right] - \ln\left[\cosh\frac{\hbar\omega\beta}{2} \right] - 2\ln\left[\sinh\frac{\hbar\omega\beta}{2} \right] - \ln 8 \right\}. \tag{35}$$

Notice that the entropy is the same in both cases (bosonic and fermionic).

4 Acknowledgments

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WAVE PACKET MOTION IN HARMONIC POTENTIAL
AND
COMPUTER VISUALIZATION

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Abstract

Wave packet motions of a single electron in harmonic potentials or a magnetic field are obtained analytically. The phase of the wave function which depends on both time and space is also presented explicitly. The probability density of the electron changes its width and central position periodically. These results are visualized using computer animation techniques.

1 Introduction

We investigate a time evolution of the electron wave packet through analytical methods. The time evolutions of restricted initial wave packets were obtained [1]-[3]. Here, we consider a general initial wave packet and obtain a classical harmonic oscillation of the center of mass of the probability density and an oscillation of its variance. We have also obtained the analytic form of the phase of the wave packet.

2 One-dimensional harmonic potential

We consider the Schrödinger equation for the one-dimensional harmonic potential

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi + \frac{k}{2}x^2\psi. \tag{1}$$

The stationary solution is

$$\varphi_n(x,t) = u_n(x)\exp(-i\omega(n+\frac{1}{2})t), \tag{2}$$

where $\omega = \sqrt{\frac{k}{m}}$ and $u_n(x)$ is expressed using the Hermite polynominal $H_n(x)$

$$u_n(x) = N_n H_n(\alpha x) \exp(-\frac{1}{2}\alpha^2 x^2) \tag{3}$$

with $\alpha = \sqrt[4]{\frac{mk}{\hbar^2}}$ and the normalization factor $N_n(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}$.

Next, we shall expand an initial wave function by these functions and trace its time evolution. Hereafter, the unit length $\alpha = 1$ is used. Without loss of generality, we choose the initial wave packet as

$$\psi(x,0) = \sqrt{\frac{1}{\sqrt{2\pi}\sigma}} \exp(-\frac{(x-z)^2}{4\sigma^2}) \exp(-\frac{z_1^2}{4\sigma^2}), \tag{4}$$

where z is a complex number $z = z_0 + iz_1$. We shall expand this wave packet in terms of the stationary solutions

$$\psi(x,0) = \sum_{n=0}^{\infty} C_n u_n(x). \tag{5}$$

We calculate a expansion coefficient C_n, with the help of the generating function of the Hermite polynominal and obtain the following expression.

$$C_n = \sqrt{\frac{4\sigma n!}{\sqrt{2\pi}2^n(1+2\sigma^2)}} \exp(-\frac{z^2}{2(1+2\sigma^2)} - \frac{z_1^2}{4\sigma^2}) \sum_{m=0}^{[\frac{n}{2}]} \frac{(-\frac{1-2\sigma^2}{1+2\sigma^2})^m (\frac{2z}{1+2\sigma^2})^{n-2m}}{m!(n-2m)!}. \tag{6}$$

Thus we obtain the time evolution of the wave packet by the following infinite series

$$\psi(x,t) = \sqrt{\frac{4\sigma}{\sqrt{2\pi}(1+2\sigma^2)}} \exp(-\frac{z^2}{2(1+2\sigma^2)} - \frac{z_1^2}{4\sigma^2}) \exp(-\frac{x^2}{2}) \exp(-\frac{i\omega t}{2})$$

$$\times \sum_{n=0}^{\infty} H_n(x)(\frac{\exp(-i\omega t)}{2})^n \sum_{m=0}^{[\frac{n}{2}]} \frac{(-\frac{1-2\sigma^2}{1+2\sigma^2})^m (\frac{2z}{1+2\sigma^2})^{n-2m}}{m!(n-2m)!}. \tag{7}$$

When σ^2 equals $\frac{1}{2}$, this summation is evaluated easily.

$$\psi(x,t) = \sqrt[4]{\frac{1}{\pi}} \exp(-\frac{z^2}{4} - \frac{z_1^2}{2} - \frac{x^2}{2}) \exp(-\frac{i\omega t}{2}) \exp(-\frac{z^2}{4}\exp(-2i\omega t) + zx\exp(-i\omega t)). \tag{8}$$

The center of mass of the probability density of the wave packet oscillates sinusoidally. On the other hand, the variance of the probability density is constant during the motion.

When the variance σ^2 is not $\frac{1}{2}$, we shall eliminate the time dependent phase factor in eq.(7) by the following transformations.

$$\frac{1-2\sigma^2}{1+2\sigma^2}\exp(-2i\omega t) = \frac{1-2\delta}{1+2\delta}, \tag{9}$$

$$\frac{z^2}{1-4\sigma^4} = \frac{w^2}{1-4\delta^2}. \tag{10}$$

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From eqs.(9) and (10) we obtain

$$\delta = \delta_0 + i\delta_1 = \frac{4\sigma^2 + i(1 - 4\sigma^4 \sin(2\omega t))}{2((1 + 4\sigma^4) + (1 - 4\sigma^4)\cos(2\omega t))}, \tag{11}$$

$$w = w_0 + iw_1 = \frac{2(z_0 \cos(\omega t) + 2z_1\sigma^2 \sin(\omega t)) + 2i(z_1 \cos(\omega t) - 2z_0\sigma^2 \sin(\omega t))}{(1 + 4\sigma^4) + (1 - 4\sigma^4)\cos(2\omega t)}. \tag{12}$$

Inserting these values into the expression (7), we have

$$\psi(x,t) = \sqrt{\frac{4\sigma}{\sqrt{2\pi}(1 + 2\sigma^2)}} \exp\left(-\frac{z^2}{2(1 + 2\sigma^2)} - \frac{z_1^2}{4\sigma^2}\right) \exp\left(-\frac{x^2}{2}\right) \exp\left(-\frac{i\omega t}{2}\right)$$

$$\times \sum_{n=0}^{\infty} H_n(x) \frac{1}{2^n} \sum_{m=0}^{[\frac{n}{2}]} \frac{(-\frac{1-2\delta}{1+2\delta})^m (\frac{2w}{1+2\delta})^{n-2m}}{m!(n - 2m)!}. \tag{13}$$

Comparing this and the expression (7) at $t = 0$, we see that this is also the expansion formula of a Gaussian wave packet. After a straight forward but lengthy calculation we obtain

$$\psi(x,t) = \exp(i\gamma)\sqrt{\frac{\sqrt{\delta_0}}{\sqrt{2\pi}\,|\,\delta\,|}} \exp\left(-\frac{(x - w)^2}{4\delta}\right) \exp\left(-\frac{w_1^2}{4\delta_0}\right), \tag{14}$$

where $\exp(i\gamma)$ is a phase factor which depends only on time

$$\gamma = \frac{-(z_0^2 - z_1^2)\sin(2\omega t) + 4z_0z_1\sigma^2(\cos(2\omega t) - 1)}{2(\cos(2\omega t)(1 - 4\sigma^4) + 1 + 4\sigma^4)} - \frac{1}{2}\arctan\left(\frac{\tan(\omega t)}{2\sigma^2}\right). \tag{15}$$

The center of mass of the wave packet oscillates sinusoidally between $-\frac{\sqrt{4z_0^2\sigma^4 + z_1^2}}{2\sigma^2}$ and $\frac{\sqrt{4z_0^2\sigma^4 + z_1^2}}{2\sigma^2}$. The variance of the probability density changes periodically in the range between $\frac{1}{4\sigma^2}$ and σ^2 ($\sigma^2 > \frac{1}{2}$) or between σ^2 and $\frac{1}{4\sigma^4}$ ($\sigma^2 < \frac{1}{2}$). The period of its change is half of that of the oscillatory motion of the center of mass[5]. The motion of the probability density function is presented in FIG. 1.
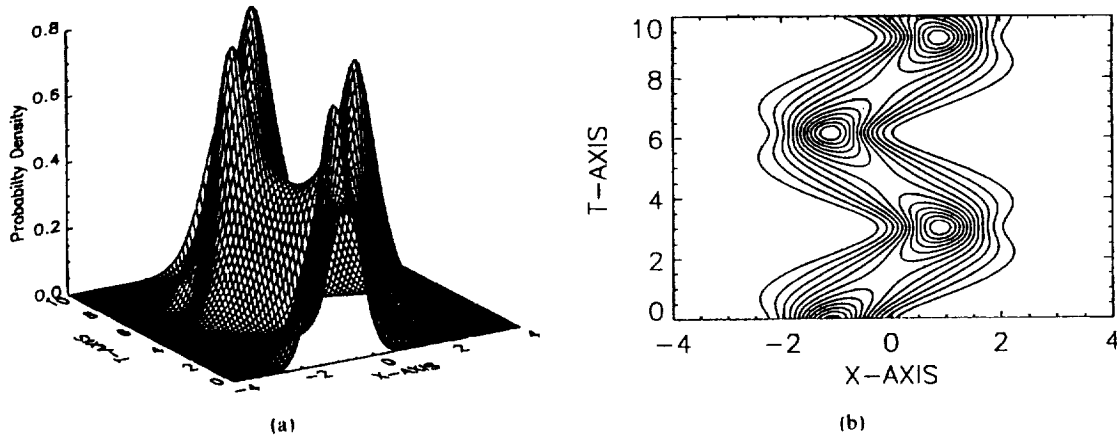
FIG. 1. The motion of the probability density function. Here, we choose the variance of the probability density of the initial wave packet as $\frac{1}{4}$. (a) bird's-eye view. (b) contour line.

3 Two-dimensional Harmonic Potential

Next, we consider the two dimensional Schrödinger equation for an isotropic harmonic potential

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2 \nabla^2}{2m}\psi + \frac{k^2(x^2 + y^2)}{2}\psi. \tag{16}$$

We choose the initial wave packet as

$$\psi(x,y,0) = N\exp(-\xi(x-x_0)^2 + ik_{x0}(x-x_0) - \eta(y-y_0)^2 + ik_{y0}(y-y_0) + \lambda(x-x_0)(y-y_0)), \tag{17}$$

where ξ, η and λ are complex constants,

$$\xi = \xi_0 + i\xi_1, \quad \eta = \eta_0 + i\eta_1, \quad \lambda = \lambda_0 + i\lambda_1, \tag{18}$$

which satisfy the following inequalities

$$\xi_0 > 0, \quad \eta_0 > 0, \quad 4\xi_0\eta_0 - \lambda_0^2 > 0, \tag{19}$$

and N is a normalization constant

$$N = \sqrt{\frac{\sqrt{4\xi_0\eta_0 - \lambda_0^2}}{\pi}}, \tag{20}$$

Using the same techniques and procedure in the one-dimensional case, we obtain the time evolution of the wave packet in terms of an infinite series.

$$\psi(x,y,t) = \sum_{n=0}^{\infty}[u_n(y)\sum_{m=0}^{\infty} C_{m,n}u_m(x)\exp(-i\omega(m+n+1)t)], \tag{21}$$

The expansion coefficients $C_{m,n}$ are also calculated explicitly.

For an uncorrelated initial condition

$$\psi(x,y,0) = \sqrt{\frac{\sqrt{4\xi_0\eta_0}}{\pi}}\exp(-\xi(x-x_0)^2 + ik_{x0}(x-x_0) - \eta(y-y_0)^2 + ik_{y0}(y-y_0)), \tag{22}$$

we can evaluate the infinite series

$$\psi(x,y,t) = \sqrt{\frac{1}{2\pi\sigma_{xt}\sigma_{yt}}}\exp(-\frac{(x-x_t)^2}{4\sigma_{xt}^2} - \frac{(y-y_t)^2}{4\sigma_{yt}^2})\exp(ik_{xt}x + ik_{yt}y)$$

$$\times \exp(i\frac{(1-4\sigma_x^4)\sin(2\omega t + 2\gamma_x)(x-x_t)^2}{16\sigma_x^2\sigma_{xt}^2} + i\frac{(1-4\sigma_y^4)\sin(2\omega t + 2\gamma_y)(y-y_t)^2}{16\sigma_y^2\sigma_{yt}^2})$$

$$\times \exp(-i\frac{k_{xt}x_t}{2} - i\frac{k_{x0}x_0}{2} - i\frac{1}{2}\arctan(\frac{\tan(\omega t)}{2\sigma_x^2}) + i\theta_x)$$

$$\times \exp(-i\frac{k_{yt}y_t}{2} - i\frac{k_{y0}y_0}{2} - i\frac{1}{2}\arctan(\frac{\tan(\omega t)}{2\sigma_y^2}) + i\theta_y), \tag{23}$$

where

$$\sigma_x^2 = \frac{(1 - 4(\xi_0^2 + \xi_1^2))\xi_0}{(-16(\xi_0^2 + \xi_1^2)^2 + 4(\xi_0^2 - \xi_1^2))\cos^2(\gamma_x) + (1 - 4(\xi_0^2 - \xi_1^2))\sin^2(\gamma_x)} , \qquad (24)$$

$$\sigma_y^2 = \frac{(1 - 4(\eta_0^2 + \eta_1^2))\eta_0}{(-16(\eta_0^2 + \eta_1^2)^2 + 4(\eta_0^2 - \eta_1^2))\cos^2(\gamma_y) + (1 - 4(\eta_0^2 - \eta_1^2))\sin^2(\gamma_y)} , \qquad (25)$$

$$\gamma_x = \frac{1}{2}\arctan(\frac{4\xi_1}{1 - 4(\xi_0^2 + \xi_1^2)}) \; , \quad \gamma_y = \frac{1}{2}\arctan(\frac{4\eta_1}{1 - 4(\eta_0^2 + \eta_1^2)}), \qquad (26)$$

$$\sigma_{xt}^2 = \frac{\sin^2(\omega t + \gamma_x) + 4\sigma_x^4\cos^2(\omega t + \gamma_x)}{4\sigma_x^2} \; , \quad \sigma_{yt}^2 = \frac{\sin^2(\omega t + \gamma_y) + 4\sigma_y^4\cos^2(\omega t + \gamma_y)}{4\sigma_y^2}, \qquad (27)$$

Here θ_x and θ_y are time independent phase factors

$$\theta_x = -(\frac{(1 - 4\sigma_x^4)\sin(2\gamma_x)}{4(\sin^2(\gamma_x) + 4\sigma_x^4\cos^2(\gamma_x))} + \xi_1)x_0^2 + \frac{1}{2}\arctan(\frac{\tan(\gamma_x)}{2\sigma_x^2}), \qquad (28)$$

$$\theta_y = -(\frac{(1 - 4\sigma_y^4)\sin(2\gamma_y)}{4(\sin^2(\gamma_y) + 4\sigma_y^4\cos^2(\gamma_y))} + \eta_1)y_0^2 + \frac{1}{2}\arctan(\frac{\tan(\gamma_y)}{2\sigma_y^2}). \qquad (29)$$

We obtain the explicit time dependence of the following parameters

$$x_t = x_0\cos(\omega t) + k_{x0}\sin(\omega t) \; , \quad y_t = y_0\cos(\omega t) + k_{y0}\sin(\omega t), \qquad (30)$$

$$k_{xt} = k_{x0}\cos(\omega t) - x_0\sin(\omega t) \; , \quad k_{yt} = k_{y0}\cos(\omega t) - y_0\sin(\omega t) . \qquad (31)$$

The trajectory of the center of mass of the probability density function is an elliptic motion around the origin with an angular frequency ω.

4　Uniform magnetic field

The Schrödinger equation for a single electron in a uniform magnetic field perpendicular to the two dimensional flat plane is

$$i\hbar\frac{\partial}{\partial t}\psi = \frac{1}{2m}(-i\hbar\nabla + \frac{e\boldsymbol{A}}{c})^2\psi, \qquad (32)$$

where the vector potential $\boldsymbol{A}$ in Landau gauge is

$$\boldsymbol{A} = (-By, 0) \qquad (33)$$

We separate a special solution of the wave equation as

$$\psi(x, y, t) = \exp(ikx)f(k, y, t) \qquad (34)$$

The wave equation for $f(k, y, t)$ becomes

$$i\hbar\frac{\partial}{\partial t}f(k, y, t) = \frac{\hbar^2}{2m}(-\frac{\partial^2}{\partial y^2} + \alpha^2(y - \frac{k}{\alpha})^2)f(k, y, t), \qquad (35)$$

where

$$\alpha = \frac{eB}{c\hbar} \tag{36}$$

This is the one dimensional Schrödinger equation for the harmonic potential centered at $y = k/\alpha$. Thus above techniques and procedures can be applied in order to obtain the time evolution of the wave packet [6]. We choose the initial wave packet eq.(17). The comlete descriptions are presented in the literatutre [6]. The major difference between two dimensional isotropic harmonic potential and magnetic field is the period of the change of the variance. The former is the half of the latter. This fact is also interpreted by the pass integration technique [6], [8].

For the following initial condition

$$\xi = \eta \;,\;\; \lambda = \frac{i}{2}. \tag{37}$$

the shape of the contour lines of the probability density function remains circular during the motion.

For the following initial condition

$$\xi = \eta = \frac{1}{4} \;,\;\; \lambda = \frac{i}{2}, \tag{38}$$

the shape of the probability density remains unchanged.

5 Conclusion

Using a frame buffer NVS2000 and video recorder BVW-75, we have made CG animations which can give us an intuitive understanding of the wave packet motions.

The potentials are simple but due to the quantum mecanical property the analytic form of the wave packet motions are very complicated.

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$SU(2)$ ACTION-ANGLE VARIABLES

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Abstract

Operator angle-action variables are studied in the frame of the $SU(2)$ algebra, and their eigenstates and coherent states are discussed. The quantum mechanical addition of action-angle variables is shown to lead to a novel non commutative Hopf algebra. The group contraction is used to make the connection with the harmonic oscillator.

1 Introduction

Action-angle variables in quantum mechanics one known to lack, in the operator level, some of properties of their classical analogues [1,2]. Especially the exponential phase operators for the harmonic oscillator, occuring in the polar decomposition of the bosonic creation and annihilation operators (an operator analogon of the polar decomposition of a complex number), lack the unitary and satify the weaker condition of one side-unitary or isometry operator. Based on the mathematical fact that, unlike in finite dimensional Hilbert spaces as the Fock space of harmonic oscillator, in finite spaces an isometry is equivalent to a unitary operator, we have in recent works, suggested a group theoretical construction of a unitary phase operator by introducing action-angle variables for the $SU(2)$ algebra and going over to their oscillator counterparts via the Inönü-Wigner method of group contraction [3-6]. In this report we will briefly review and then expand this work with respect to two aspects: first, a set of coherent states will be introduced along the lines of the displacement operator creating the usual coherent states from the vacuum state and second, we will show that addition of spins in terms of their action-angles (polar) operators, unlike the usual addition in terms of the step (cartesian) operators, involves a genuine no commutative, no co-commutative Hopf algebra structure and relates interestingly the phase operators subject to the subject of quantum groups.

2 Action-angle Variables and States

Let us start with the $SU(2)$ action-angle operators

$$J_- = e^{i\Phi}\sqrt{J_+J_-} = \sqrt{J_-J_+}e^{i\Phi} \tag{1}$$

$$J_+ = e^{-i\Phi}\sqrt{J_-J_+} = \sqrt{J_+J_-}e^{-i\Phi} \tag{2}$$

where

$$J_+ = \sum_{m=0}^{2j} \sqrt{m(2j - m + 1)}|J; m + 1 >< J; m| \qquad , \qquad J_- = J_+^+ \tag{3}$$

$$J_3 = \sum_{m=0}^{2j} (m - j)|J; m >< J; m| \tag{4}$$

and

$$e^{i\Phi} = \sum_{\ell=0}^{2j} |J; \ell >< J; \ell + 1| , \tag{5}$$

mod$(2j + 1)$, and $hh^+ = h^+h = 1$ with $h \equiv e^{i\Phi}$, $h^+ \equiv e^{-i\Phi}$ the unitary angle operator. Then from the fact that h, generates the cyclic group Z_{2j+1} acting as a cyclic permutation in the weight space of the algebra we can construct phase states

$$|\Phi; k >= F|J; k >= \frac{1}{\sqrt{2j + 1}} \sum_{m=0}^{2j} \omega^{kn}|J; n > \tag{6}$$

through the finite Fourier transform $FF^+ = F^+F = 1$, which maps action eigenstates to angle eigenstates and conjugates the respective variables, where $\omega = \exp i(2\pi/2j + 1)$. Indeed, if $g := \omega^{J_3+j1}$ then $FgF^+ = h$, $FhF^+ = g^{-1}$ and $g(h)$ acts as step operator in the angle (action) state basis, i.e,

$$h|J; n >= |J; n + 1 > \quad , \quad h|\Phi; m >= \omega^m|\Phi; m > \tag{7}$$

while

$$g^{-1}|\Phi; n >= |\Phi; n + 1 > \quad , \quad g|J; m >= \omega^m|J; m > \tag{8}$$

mod$(2j + 1)$ and $h^{2j+1} = g^{2j+1} = 1$, (notice that the state $|J; n >$ and $|\Phi; m >$ where denoted as $|n >$ and $|\varphi_m >$ respectively, in Refs. 3-6). The noncommutativity between the action and the angle variables is best expressed by the formula

$$\omega g h = h g \tag{9}$$

which resembles the exponential form of the Heisenberg canonical commutation relations (CR) as were originally written by Weyl with the association that here the action operator J_3 is a finite version of the position operator and the angle operator stands for the momentum operator. By virtue of this analogy we may interpret eqs. (7-8) as the translations along the two different directions of the phase space of our problem, which due to the module condition is a lattice torus, parametrized by the discrete action and angle values. Also eq. (9), exhibits the unusual noncommutative character of two succesive translations along different directions. Moreover, the effect of group contraction which is discussed below, is to increase the density of the lattice points until the continous limit $j \to \infty$. Furthermore this association to position and momentum suggests

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that we should look for the "number states" $|N; m>$, $m = 0, 1, ..., 2j$ in our finite system. Indeed by diagonalizing the finite Fourier transform $F|N; m> = i^m|N; m>$, we find the number states $|N; m>$, related e.g. with the orthonormal action states as:

$$|N; k> = \sum_{m=0}^{2j} |J; m> < J; m|N; k>,$$

(10)

with expansion coefficients given in terms of the Hermite polynomial, H_k with discrete argument,

$$< J; m|N; k> = \sum_{p=-\infty}^{\infty} e^{-\frac{\pi}{2j+1}(p(2j+1)+m)^2} H_k\left(\sqrt{\frac{2\pi}{2j+1}}(p(2j+1)+m)\right)$$

(11)

This situation is akin to that of the harmonic oscillator number states which are similarly eigenstates of the usual Fourier transform operator which conjugates position and momentum operators, a fact that stems from the property of the oscillator eigenstates $\exp(-\frac{1}{2}x^2)H_k(x)$, to be their own Fourier transforms. Especially the vacuum or lowest number state is,

$$|N; 0> = \sum_{m=0}^{2j} \omega^{\frac{1}{2}m^2} \theta_3(im|i(2j+1))|J; m>$$

(12)

where θ_3 is the theta-Jacobi function [7]:

$$\theta_3(z|\tau) = \sum_{s=-\infty}^{\infty} e^{\pm i2\pi s + \tau i\pi s^2}$$

(13)

Having the action $|J; m>$, the angle $|\Phi; n>$ and the number states $|N; k>$ as were given above, we can further built, as have been outlined in Ref. 4, the quantum theory of action-angle variables by introducing the corresponding coherent states acting on the vacuum $|N; 0>$, with a displacement operator. Such an operator is furnished by the unitary traceless elements $J_{m_1, m_2} := \omega^{m_1 m_2/2} g^{m_1} h^{m_2}$, where $J^+_{m_1, m_2} = J_{-m_1, -m_2} = J_{2j+1-m_1, 2j+1-m_2}$, with (m_1, m_2) pairs belonging to the square index-lattice $0 \le m_1, m_2 \le 2j$ with boundary conditions and the $(0, 0)$ pair excluded.

The following interesting properties of these operators suggest them as the Glauber displacement operator of our case; first they constitute an orthonormal set of $(2j+1)^2 - 1$ elements obeying the relation

$$< J_{\vec{m}}, J_{\vec{n}} > := Tr\ J_{\vec{m}} J_{\vec{n}} = (2j+1)\delta_{\vec{m}+\vec{n}, \vec{0}},$$

(14)

where e.g. $J_{\vec{m}} = J_{m_1 m_2}$, and further,

$$J_{\vec{m}} J_{\vec{n}} = \omega^{-\frac{1}{2}\vec{m}\times\vec{n}} J_{\vec{m}+\vec{n}}$$

(15)

and

$$J_{\vec{n}} J_{\vec{m}} = \omega^{\vec{m}\times\vec{n}} J_{\vec{m}} J_{\vec{n}}$$

(16)

and finally

$$[J_{\vec{m}}, J_{\vec{n}}] = -2i \sin\left[\frac{\pi}{2j+1}\vec{m}\times\vec{n}\right] J_{\vec{m}+\vec{n}}$$

(17)

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$\text{mod}(2j+1)$, while $\vec{m} \times \vec{n} = m_1 n_2 - m_2 n_1$. With the aid of these operators we now introduce coherent states $|\vec{\ell}>$, for the action-angle system by acting on the vacum:

$$|\vec{\ell}> := J_{\vec{\ell}}|N;0> = \omega^{\frac{1}{2}\ell_1\ell_2} \sum_{m=0}^{2j} \omega^{\ell_1 m + \frac{i}{2} m^2} \theta_3(im|i(2j+1))|J;m+\ell_2> \tag{18}$$

These are now coherent states defined on the lattice phase space which is the appropriate phase space of the quantum action-angle variables. They involve the Jacobi theta functions which are also appearing in the case of the ordinary coherent states when, looking for a complete subset out of the over complete set of coherent states we lattice the phase space. Elsewhere, the normalization and minimum uncertainty properties of the states will be studied in detail.

3   Quantum Angles Addition

Let us now turn to the case where there are several action-angle degrees of freedom and search for the way we combine them quantum mechanically. The similar problem for the "cartesian" generators J_i, with $[J_i, J_j] = 2i\epsilon_{jk}J_k$ is the fundamental theme of addition of spins and customanily is solved by tensoring the generators,

$$\Delta J_i := J_i \otimes 1 + 1 \otimes J_i \tag{19}$$

which again satisfy the commutation relations, $[\Delta J_i, \Delta J_j] = 2i\epsilon_{ijk}\Delta J_k$. In our case, for the "polar" generators $g = \omega^{(J_3+j1)}$ and $h = \omega^{F(J_3+j1)F^+}$ with $\omega gh = hg$ we must find an appropriate tensoring (coproduct in the jargon of Hopf algebras), which provides such Δg and Δh that $\omega\Delta g = \Delta h$. Two such coproducts we have found,

$$\Delta g = g \otimes g \quad , \quad \Delta h = h \otimes 1 + g \otimes h \tag{20}$$

and

$$\Delta g = g \otimes g \quad , \quad \Delta h = h \otimes g + g^{-1} \otimes h \tag{21}$$

which both have the remarkable property of not been the same under permutation of their components involved in the tensor products. This is distingly different to the usual addition of spins, where there is no sence of order in the tensoring the spins. Technically speaking we have here a natural case of no co-commutativity unlike in eq. (19), where the product is co-commutative [8-11]. We end here this discussion, as we intent to expand it elsewhere, by saying that it is also possible to show the Hopf and quasi triangular Hopf algebra structure of the above tensoring and then to find the R-matrix and to verify the Yang-Baxter equation.

4   Contraction to the Oscillator

Before we came to conclusions let us mention that as was shown in Ref. 3 via the group contraction that the $SU(2)$ action-angle variables can be contructed to those of the oscillator and the dynamical aspects of this procces could be exemplified by studing the Jaynes-Cummings model. We illustrate

now this idea be contracting the $SU(2)$ generators to the oscillator generators in the Bargmann analytic realization. In the space of analytic polynomials of degree $2j$ the $SU(2)$ algebra is realized as,

$$J_+ = -z^2 \frac{d}{dz} + z2j \qquad J_- = \frac{d}{dz} \qquad J_3 = z\frac{d}{dz} - j \qquad (22)$$

where z is the complex label of the spin coherent states, and geometrically stands for the projective coordinate of the coset sphere $SU(2)/U(1) \sim S^1$. Transforming now the generators like $J_\pm \rightarrow J_\pm/\sqrt{2j}$ and $J_3 \rightarrow J_3 + j1$ we find in the large j limit, the oscillator generators in their Bargmann form as follows:

$$\frac{J_+}{\sqrt{2j}} = -\frac{(\sqrt{2j}z)^2}{2j}\frac{d}{d(\sqrt{2j}z)} + \sqrt{2j}z \approx \alpha \equiv a^+ \qquad (23)$$

$$\frac{J_-}{\sqrt{2j}} = \frac{d}{d(\sqrt{2j}z)} \approx \frac{d}{d\alpha} = a \qquad (24)$$

and

$$J_3 + j = \sqrt{2j}z\frac{d}{d(\sqrt{2j}z)} \approx \alpha\frac{d}{d\alpha} = N \qquad (25)$$

where $\sqrt{2j}z \approx \alpha$ is the complex variable of the Glauber coherent states which is now becoming the coordinate of the tangent phase plane of the harmonic oscillator. One can further show that the overlap, the completeness relation and all other notions of the spin coherent states can be contracted to their respective oscillator counterparts. Moreover in Ref. 5 has been shown how a q-deformed oscillator with q deformation parameter to be root of unity can be employed to define action-angles variables in a finite Fock Hilbert space and a number of their properties have been worked out. In such an approach we have shown [5], that the contraction method is substituted by the limit procedure of undeforming the q-oscillator to the usual ocillators.

5 Conclusion

In conclusion, we have shown that the quantization of action-angle classical variables can be developed in the framework of the $SU(2)$ algebra in a manner which allows for the classical properties of these variables to find well defined operator analogues. Interesting relations to the quantum groups and Hopf algebras are naturally emerge from the present method of angle quantization which will be pursued further, together with the introduction of the Wigner function for the action-angles variables and the star and Moyal product defined between functions of the phase space of our problem.

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ALTERNATIVE DESCRIPTIONS OF WAVE AND PARTICLE ASPECTS OF THE HARMONIC OSCILLATOR

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Abstract

The dynamical properties of the wave and particle aspects of the harmonic oscillator can be studied with the help of the time-dependent Schrödinger equation (SE). Especially the time-dependence of maximum and width of Gaussian wave packet solutions allow to show the evolution and connections of those two complementary aspects. The investigation of the relations between the equations describing wave and particle aspects leads to an alternative description of the considered systems. This can be achieved by means of a Newtonian equation for a complex variable in connection with a conservation law for a nonclassical angular momentum-type quantity. With the help of this complex variable it is also possible to develop a Hamiltonian formalism for the wave aspect contained in the SE, which allows to describe the dynamics of the position and momentum uncertainties. In this case the Hamiltonian function is equivalent to the difference between the mean value of the Hamiltonian operator and the classical Hamiltonian function.

1 Introduction

In wave mechanics a complex equation, the Schrödinger equation (SE), is used to describe the dynamics and energetics of the particle and wave aspects of a material system under the influence of conservative forces, e.g., the harmonic force of an undamped oscillator. In classical mechanics Newton's equation of motion is a *real* equation which is only capable of describing the particle aspect. It will be shown that it is possible to also take into account the wave aspect by changing to a *complex* Newtonian equation. However, real and imaginary parts of the new complex variable are not independent of each other, but are coupled by a well-defined relation which is connected with a conservation law for a nonclassical angular momentum-type quantity. With the help of this new complex variable it is also possible to express the groundstate energy $\tilde{E}$ in a way that it can serve as a Hamiltonian function for the position and momentum uncertainties.

2 Dynamics of Particle and Wave Aspects

The wave mechanical equation (SE) for the harmonic oscillator (HO)

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2\} \Psi(x,t),$$ (1)

possesses exact analytic solutions of the form of Gaussian wave packets (WP). The dynamics of the particle aspect is reflected by the fact that the maximum of the WP follows the classical trajectory of the corresponding particle. The wave aspect is expressed by the finite width of the WP. This width can also be time-dependent. This time-dependence is closely connected with a contribution to the convective current density in the continuity equation for the (real) density function corresponding to the (complex) WP.

Inserting the Gaussian WP given in the form

$$\Psi_L(x,t) = N_L(t)exp \left\{ i \left[y(t)\tilde{x}^2 + \frac{1}{\hbar}\langle p\rangle\tilde{x} + K(t) \right] \right\} , \tag{2}$$

(where $\tilde{x} = x - \langle x\rangle = x - \eta(t)$ and $\langle p\rangle = m\frac{d}{dt}\langle x\rangle$ denotes the mean value of momentum p, the explicit form of N(t) and K(t) is not relevant for the following discussion), into the SE(1) shows that the maximum at position $\langle x\rangle = \eta(t)$ fulfills the classical Newtonian equation of motion

$$\ddot{\eta} + \omega^2\eta = 0 . \tag{3}$$

The WP width, $\sqrt{\langle \tilde{x}^2\rangle}$ (where $\langle \tilde{x}^2\rangle = \langle x^2\rangle - \langle x\rangle^2$), is connected with the imaginary part of the complex coefficient of $\tilde{x}^2$ in the exponent, y(t), via

$$\frac{2\hbar}{m} y_I = \frac{\hbar}{2m\langle \tilde{x}^2\rangle} = \frac{1}{\alpha^2(t)} . \tag{4}$$

To determine the time-dependence of the WP width, the complex (quadratically) nonlinear equation of Ricatti-type

$$\frac{2\hbar}{m} \dot{y} + (\frac{2\hbar}{m} y)^2 + \omega^2 = 0 \tag{5}$$

has to be solved.

With the aid of the variable $\alpha(t)$ as defined in Eq. (4) (which is apart from a constant factor identical with the WP width), the corresponding real part turns into

$$\frac{2\hbar}{m}\, y_R = \frac{\dot{\alpha}}{\alpha} \tag{6}$$

and Eq. (5) yields the (real) nonlinear Newtonian equation

$$\ddot{\alpha} + \omega^2\, \alpha = \frac{1}{\alpha^3} \ . \tag{7}$$

The only difference between this equation, determining the dynamics of the WP width, and Eq. (3) for the dynamics of the WP maximum is the inverse cubic term on the rhs of Eq. (7). In order to elucidate the meaning of this additional term, the Ricatti Eq. (5) has to be reconsidered. Using the substitution

$$\frac{2\hbar}{m}\, y = \frac{\dot{\lambda}}{\lambda} \tag{8}$$

with the new complex variable $\lambda = \hat{u} + i\hat{z}$, Eq. (5) can be linearized to yield the *complex* linear Newtonian equation

$$\ddot{\lambda} + \omega^2\, \lambda = 0 \ . \tag{9}$$

This equation is formally indentical with the Newtonian Eq. (3) for the WP maximum. It can be shown (e.g. by expressing the WP(2) in terms of λ or with the help of a Green-function, see [1-3]) that the imaginary part of λ is directly proportional to the classical trajectory, i.e.

$$\frac{\hat{z}\alpha_0 p_0}{m} = \langle x \rangle = \eta(t) \ , \tag{10}$$

(where α_0 and p_0 are the initial values of $\alpha(t)$ and $\langle p \rangle(t)$, respectively). Furthermore, in the same way it can be shown (see e.g. [1-3]) that real and imaginary parts of λ are uniquely connected via the relation

$$\dot{\hat{z}}\hat{u} \ - \ \dot{\hat{u}}\hat{z} \ = 1 \ . \tag{11}$$

Equation (8) for the time evolution of λ was obtained from the Ricatti Equation (5), which describes the evolution of the WP width, as shown in Eq. (7) for $\alpha(t)$. In order to show how the wave aspect is contained in λ, it shall be written in polar coordinates,

$$\lambda = \alpha \ e^{i\varphi} \ = \alpha \cos\varphi \ + i \ \alpha \sin\varphi \ . \tag{12}$$

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Inserting this form into Eq. (8), comparison with the definitions given in Eqs. (4) and (6) shows that the quantity α in Eq. (12) denoting the absolute value of λ is identical with the quantity α denoting the WP width in Eq. (7), if the relation

$$\dot{\varphi} = \frac{1}{\alpha^2} \tag{13}$$

is fulfilled. However, the validity of Eq. (13) can easily be proven by inserting (12) into Eq. (11).

The physical meaning of Eq. (13) becomes more transparent, when the motion of $\lambda(t)$ in the *complex* plane is compared with the motion of a two-dimensional harmonic oscillator in the *real* physical space, written in polar coordinates (see e.g. [1-3]). This comparison shows that relation (13) (and thus the equivalent relation (11) in cartesian coordinates) corresponds to the *conservation of angular momentum* in real space.

Furthermore, it shows that the inverse cubic term on the rhs of Eq. (7) corresponds to a *centrifugal force* in real space.

So, it can be stated that the complex quantity $\lambda(t)$ fulfilling the Newtonian Eq. (9) contains the information about the dynamics of both particle and wave aspects of the system. Written in cartesian coordinates, the imaginary part of λ directly provides the information about the dynamics of the particle aspect, the WP maximum, written in polar coordinates, the absolute value of λ directly provides the information about the dynamics of the wave aspect, the WP width.

3 Energetics of Particle and Wave Aspects

It shall be mentioned only briefly here (for further details see e.g. [2,3]) that this new complex variable λ can also provide new information contained in the groundstate energy of the harmonic oscillator, usually only given in the form $\bar{E} = \frac{1}{2}\hbar\omega$. The notation $\bar{E}$ is used to already indicate that this energy contribution is just the difference between the mean value of the Hamiltonian operator (calculated with the WP-solution (2)) and the classical energy E_{class}, respectively

$$
\begin{aligned}
\langle E \rangle &= \langle H \rangle = \frac{1}{2m}\langle p^2 \rangle + \frac{m}{2}\omega^2 \langle x^2 \rangle \\
&= \left(\frac{1}{2m}\langle p \rangle^2 + \frac{m}{2}\omega^2 \langle x \rangle^2 \right) + \left(\frac{1}{2m}\langle \tilde{p}^2 \rangle + \frac{m}{2}\omega^2 \langle \tilde{x}^2 \rangle \right) \\
&= E_{class} + \bar{E} \ .
\end{aligned}
\tag{14}
$$

In classical mechanics the energy E_{class} of the HO is identical with the classical Hamiltonian function, H_{class}, which also provides the equations of motion for the particle aspect.

Writing the difference of kinetic and potential energy uncertainties in terms of the polar coordinates of λ, i.e.

$$\bar{\mathcal{L}}(\alpha, \varphi, \dot{\alpha}, \dot{\varphi}) = \frac{\hbar}{4}(\dot{\alpha}^2 + \alpha^2\dot{\varphi}^2 - \omega^2\alpha^2) \ , \tag{15}$$

this quantity can be used as Lagrangian function for the position and momentum uncertainties. From $\bar{\mathcal{L}}$ the canonically conjugate momenta to the coordinates α and φ can be obtained in the usual way and the groundstate energy can be written in the form of a Hamiltonian function that provides the equations of motion for the position and momentum uncertainties.

In this context it is interesting that the canonical angular momentum p_φ, obtained from the Lagrangian (15), is not only constant, as already mentioned in connection with Eqs. (11) and (13), but has the value

$$p_\varphi = \frac{\hbar}{2} \cdot$$

(16)

Thus, the complex variable $\lambda(t)$, obeying the simple Newtonian Eq. (9), follows a path in the *complex* plane similar to the path of a particle in a two-dimensional HO-field. However, the quantity corresponding to the conserved classical angular momentum in *real* space is the quantity $p_\varphi = \hbar/2$ in *complex* space. This is rather surprising, because even in wave mechanics orbital angular momenta are integer multiples of $\hbar$. Half-integer multiples of $\hbar$ are usually connected with the purely quantum mechanical property spin. Whether there are any relations between p_φ and spin shall not be further discussed in this work.

4   Conclusions

The information on the dynamics of the considered system contained in the time-dependent SE can also be obtained from a corresponding Newtonian equation for this system, if a complex variable is used, where the imaginary part of this variable is proportional to the classical trajectory and the real part is uniquely connected with the imaginary part. The connecting relation expresses a kind of conservation of angular momentum for the two-dimensional motion in the *complex* plane. In addition, the value of this conserved nonclassical angular momentum property is $\hbar/2$, usually only known from the quantum mechanical property spin.

With the help of this complex quantity $\lambda = \hat{u} + i\hat{z} = \alpha \exp(i\varphi)$, it is possible to obtain equations of motion for the particle aspect, $\langle x \rangle = \eta = (\alpha_0 p_0/m)\hat{z}$, as well as for the wave aspect, $\langle x^2 \rangle = \hbar/2m)\alpha^2$.

Furthermore, it is possible to express the difference between the mean value of the Hamiltonian operator and the classical energy E_{class} in terms of the coordinates α and φ and the corresponding canonically conjugate momenta. Thus, it is possible to write $\tilde{E}$ in the form of a Hamiltonian function, wherefrom the correct equations of motion for the "wave properties" (uncertainties) can be obtained in exactly the same way as the equations of motion for the particle properties can be obtained from the classical energy, respectively Hamiltonian function.

5   Acknowledgments

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SYMMETRIES OF COUPLED HARMONIC OSCILLATORS

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Abstract

It is shown that the system of two coupled harmonic oscillators possesses many interesting symmetries. It is noted that the symmetry of a single oscillator is that of the three-parameter group $Sp(2)$. Thus two uncoupled oscillator exhibits a direct product of two $Sp(2)$ groups, with six parameters. The coupling can be achieved through a rotation in the two–dimensional space of two oscillator coordinates. The closure of the commutation relations for the generators leads to the ten–parameter group $Sp(4)$ which is locally isomorphic to the deSitter group $O(3,2)$.

1 Introduction

Since the classical mechanics of two coupled harmonic oscillators is known to every physicist, there is a tendency to believe that this oscillator problem is completely understood and that nothing new can be learned from it. We are writing this note because there are so many new lessons to learn from the system of coupled oscillators. The system shares symmetries with a number of physical models of current interest, such as the Lee model in quantum field theory [1], the Bogoliubov transformation in superconductivity [2, 3], two-mode squeezed states of light [4, 5, 6], the covariant harmonic oscillator model for the parton picture [7], and models in molecular physics [8]. There are also models of current interest in which one of the variables is not observed, including thermo-field dynamics [9], two-mode squeezed states [10, 11], the hadronic temperature [12].

From the mathematical point of view, the standard approach is to construct a suitable representation of the symmetry group after writing down its generators. The symmetry group in the present case is $Sp(4)$ with ten generators [4, 6]. However, it is extremely difficult to study physics in terms of ten parameters. We should somehow start with a smaller number of parameters.

For example, let us consider the three–dimensional rotation group with three generators. We need only two generators to describe rotations [13]. The third generator is produced during the process of constructing a closed set of commutation relations. For the coupled oscillators, a reasonable approach is to start with simpler groups describing two uncoupled oscillators. We can then introduce an additional generator to couple the two oscillators. The number of generators of the resulting group may be larger than the sum of those for the two starting groups plus the additional generator to couple them.

The process of constructing a larger group from two smaller groups is quite common in physics. We are quite familiar with the "direct product" and "semi-direct product," where the number of generators is the sum of those for the two smaller groups. We shall use the word "construction of the group by soldering two subgroups," when the resulting group has more generators than all those for the starting groups and the additional generator(s) to couple them. We need this new word "soldering" in order to reduce the number of input parameters in physics.

The soldering process takes different forms. We can construct the three dimensional rotation group by soldering the one-parameter rotation group around the x axis and another rotation group around the y axis. The resulting group has three generators. As we shall see in this paper, we can construct the $O(3,2)$ deSitter group by soldering two $O(3,1)$ Lorentz groups. In this case, we solder them by observing that the two $O(3,1)$ groups share the same rotation group. We start with nine generators, but the resulting $O(3,2)$ deSitter group has ten generators.

Since the symmetry group of each uncoupled oscillator is the three-parameter $Sp(2)$ group, and since it is likely that one more group operation is needed to couple the system, we start here with seven generators. We shall see in this paper that the resulting group is $Sp(4)$ with ten generators. It is easier to study physics with seven generators than with ten.

It is also shown in this paper that the $Sp(4)$ symmetry does not exhaust all possible symmetries of the coupled oscillator system. It is noted that the group $Sp(4)$ is a subgroup of a lager group $SL(4,r)$. Possible physical implications of this larger symmetry group are discussed.

In Sec. 2, we shall study linear canonical transformations in the four-dimensional phase space consisting of two pairs of canonical variables. It is noted that the symmetry group is $Sp(4)$ which is locally isomorphic to the $O(3,2)$ deSitter group. In Sec. 3, we shall see how the $O(3,2)$ group can be constructed from two $(3+1)$-dimensional Lorentz groups. In Sec. 3, we shall construct the symmetry group of two coupled oscillators from the symmetry group of each oscillator.

Section 5 contains a new parametrization of the coupled oscillator system which is consistent with that of the symmetry group. In Sec. 6, we discuss the quantum mechanics of the oscillator system and the unitary transformations which correspond to canonical transformations in classical mechanics. In Sec. 7, we discuss physical applications of the formalism developed in this note. Finally, in Sec. 8, we discuss scale transformations in phase space and their implications in measurement theory.

2 Linear Canonical Transformations in Classical Mechanics

For a dynamical system consisting of two pairs of canonical variables x_1, p_1 and x_2, p_2, we can introduce the four-dimensional coordinate system:

$$(\eta_1, \eta_2, \eta_3, \eta_4) = (x_1, x_2, p_1, p_2).$$ (1)

Then the transformation of the variables from η_i to ξ_i is canonical if

$$MJ\tilde{M} = J,$$ (2)

where

$$M_{ij} = \frac{\partial}{\partial \eta_j} \xi_i,$$

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and

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

For linear canonical transformations, we can work with the group of four-by-four real matrices satisfying the condition of Eq.(2). This group is called the four-dimensional symplectic group or $Sp(4)$. While there are many physical applications of this group, we are interested here in constructing the representations relevant to the study of two coupled harmonic oscillators.

It is more convenient to discuss this group in terms of its generators G, defined as

$$M = \exp(-i\alpha G), \tag{3}$$

where G represents a set of purely imaginary four-by-four matrices. The symplectic condition of Eq.(2) dictates that G be symmetric and anticommute with J or be antisymmetric and commute with J.

In terms of the Pauli spin matrices and the two-by-two identity matrix, we can construct the following four antisymmetric matrices which commute with J of Eq.(2).

$$J_1 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \qquad J_2 = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

$$J_3 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \qquad J_0 = \frac{i}{2}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{4}$$

The following six symmetric generators anticommute with J.

$$K_1 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad K_2 = \frac{i}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad K_3 = -\frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

and

$$Q_1 = \frac{i}{2}\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad Q_2 = \frac{i}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad Q_3 = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \tag{5}$$

These generators satisfy the commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = [Q_i, Q_j] = -i\epsilon_{ijk}J_k,$$

$$[J_i, J_0] = 0, \qquad [K_i, Q_j] = i\delta_{ij}J_o,$$

$$[J_i, Q_j] = i\epsilon_{ijk}Q_k, \qquad [K_i, J_o] = iQ_i, \qquad [Q_i, J_o] = -iK_i. \tag{6}$$

The group of homogeneous linear transformations with this closed set of generators is called the symplectic group $Sp(4)$. The J matrices are known to generate rotations while K and Q matrices generate squeezes [6]

It is often more convenient to study the physics of four–dimensional phase space using the coordinate system

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (x_1, p_1, x_2, p_2). \tag{7}$$

The transformation from $(\eta_1, \eta_2, \eta_3, \eta_4)$ is

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3, \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix}, \tag{8}$$

and the J matrix becomes

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & .0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{9}$$

In this new coordinate system, the rotation generators take the form

$$J_1 = \frac{-1}{2}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad J_2 = \frac{i}{2}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

$$J_3 = \frac{-1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \qquad J_0 = \frac{-1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{10}$$

The squeeze generators become

$$K_1 = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \qquad K_2 = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad K_3 = -\frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

$$Q_1 = \frac{i}{2}\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad Q_2 = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad Q_3 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{11}$$

When we deal with canonical transformations of functions of the coordinate variables, we have to use the differential operators. The rotation generators are [6]

$$J_1 = +\frac{i}{2}\left\{ \left(x_1\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial x_1} \right) + \left(x_2\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial x_2} \right) \right\},$$

$$J_2 = -\frac{i}{2}\left\{ \left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} \right) + \left(\frac{p_1\partial}{\partial p_2} - p_2\frac{\partial}{\partial p_1} \right) \right\},$$

$$J_3 = +\frac{i}{2}\left\{ \left(x_1\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial x_1} \right) - \left(x_2\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial x_2} \right) \right\},$$

$$J_0 = +\frac{i}{2}\left\{ \left(x_1\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial x_1} \right) + \left(x_2\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial x_2} \right) \right\}, \tag{12}$$

and the six squeeze generators are

$$K_1 = -\frac{i}{2}\left\{ \left(x_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial x_1} \right) - \left(x_2\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial x_2} \right) \right\},$$

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$$K_2 = -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial x_1} - p_1\frac{\partial}{\partial p_1}\right) + \left(x_2\frac{\partial}{\partial x_2} - p_2\frac{\partial}{\partial p_2}\right)\right\},$$

$$K_3 = +\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial x_2}\right)\right\},$$

$$Q_1 = +\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial x_1} - p_1\frac{\partial}{\partial p_1}\right) - \left(x_2\frac{\partial}{\partial x_2} - p_2\frac{\partial}{\partial p_2}\right)\right\},$$

$$Q_2 = -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial x_2}\right)\right\},$$

$$Q_3 = -\frac{i}{2}\left\{\left(x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right) - \left(p_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial p_2}\right)\right\}. \tag{13}$$

It was noted that there are two convenient coordinate systems in phase space, namely those of Eq.(1) and Eq.(7). The above differential forms are applicable to both coordinate systems. They of course satisfy the commutation relations given in Eq.(6).

It is remarkable that these operators are also applicable to the Wigner phase–space distribution function which is constructed from the Schrödinger wave function [6]. It is also remarkable that there are unitary transformations on the wave function which lead to canonical transformations of the Wigner function in phase space [6].

3   Construction of the O(3,2) deSitter Group by Soldering Two Lorentz Groups

In Sec. 2, we constructed the ten generators of canonical transformations acting on two pairs of canonical variables. The mathematics is straight-forward, but it is not too comfortable to study physics with ten independent parameters. We can have a better physical picture if we can study the problem in terms of concrete physical examples with smaller symmetries.

The deSitter group $O(3,2)$ is known to be locally isomorphic to the group $Sp(4)$. Indeed, as we shall see in this section, the notations for the generators of $Sp(4)$ given in Sec. 2 are the natural notations for the deSitter group. Thus, one way to study $Sp(4)$ is to study $O(3,2)$. In this section, we shall study $O(3,2)$ by constructing it by soldering two $O(3,1)$ Lorentz groups.

In the space-time of (x, y, z, t, s), where x, y, z are three space-like variables and t and s are two time-like variables, we can consider two $O(3,1)$-like Lorentz groups in the spaces of (x, y, z, t) and (x, y, z, s) respectively. The generators of rotations applicable to the three-dimensional space of x, y and z are

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

The Lorentz boosts in the subspace of (x, y, z, t) are generated by

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (15)$$

These three boost generators, together with the rotation generators of Eq.(14), form a closed Lie algebra for the Lorentz group applicable to the four-dimensional Minkowski space of (x, y, z, t). The same is true for the space of (x, y, z, s) with the boost generators:

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}. \quad (16)$$

The above two Lorentz groups have nine generators. If we attempt to form a closed set of commutation relations, we end up with an additional generator

$$J_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}, \quad (17)$$

which will generate rotations in the two-dimensional space of s and t. These ten generators form a closed set of commutations relations.

We started with two $O(3, 1)$ Lorentz groups. Each Lorentz group has its own rotation subgroup. In the present case, both Lorentz groups share the same rotation subgroup. This is how these two groups are soldered.

It is remarkable that this set of commutation relations is identical to that of Eq.(6). The group $O(3, 2)$ is locally isomorphic to $Sp(4)$. The group $O(3, 2)$ occupies a very important place in relativity and elementary particle physics simply because it contains two Lorentz groups as its subgroups. The local isomorphism between $O(3, 2)$ and $Sp(4)$ enables us to study this group in terms of linear canonical transformations in classical mechanics or in the Wigner phase–space picture of quantum mechanics.

4 Construction of the Sp(4) Symmetry Group for Coupled Oscillators by Soldering two Sp(2) Groups

For two uncoupled oscillators, we can start with the coordinate system:

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (x_1, p_1, x_2, p_2). \quad (18)$$

Since the two oscillators are independent, it is possible to perform linear canonical transformations on each coordinate separately. The canonical transformation in the first coordinate system is generated by

$$A_1 = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_1 = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \qquad C_1 = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{19}$$

These generators satisfy the well-known commutation relations:

$$[A_1, B_1] = iC_1, \qquad [B_1, C_1] = -iA_1, \qquad [C_1, A_1] = iB_1. \tag{20}$$

It is also well known that this set of commutation relations is identical to that for the (2 + 1)–dimensional Lorentz group. Linear canonical transformations on the second coordinate are generated by

$$A_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad B_2 = \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad C_2 = \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{21}$$

These generators also satisfy the commutation relations of Eq.(20). We are interested here in constructing the symmetry group for the coupled oscillators by soldering two $Sp(2)$ groups generated by A_1, B_1, C_1 and A_2, B_2, C_2 respectively.

It will be more convenient to use the linear combinations:

$$A_+ = A_1 + A_2, \qquad B_+ = B_1 + B_2, \qquad C_+ = C_1 + C_2,$$

$$A_- = A_1 - A_2, \qquad B_- = B_1 - B_2, \qquad C_- = C_1 - C_2, \tag{22}$$

These matrices take the form

$$A_+ = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad B_+ = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad C_+ = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

$$A_- = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \qquad B_- = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \qquad C_- = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \tag{23}$$

The sets (A_+, B_+, C_+) and (A_+, B_-, C_-) satisfy the commutation relations of Eq.(20). The same is true for (A_-, B_+, C_-) and (A_-, B_-, C_+).

Next, let us couple the oscillators through a rotation generated by

$$A_0 = \frac{i}{2}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{24}$$

Then, A_0 commutes with A_+, B_+, C_+, and the following commutation relations generate new operators A_3, B_3 and C_3:

$$[A_0, A_-] = iA_3, \qquad [A_0, B_-] = iB_3, \qquad [A_0, C_-] = iB_3, \tag{25}$$

where

$$A_3 = \frac{1}{2}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad B_3 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad C_3 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \tag{26}$$

There are now ten generators. They form the closed set of commutation relations of Eq.(6), if we identify these matrices as

$$A_+ = -J_0, \qquad A_- = -J_3, \qquad A_3 = -J_1, \qquad A_0 = J_2,$$

$$B_+ = K_2, \qquad B_- = -Q_1, \qquad B_3 = Q_3,$$

$$C_+ = Q_2, \qquad C_- = K_1, \qquad C_3 = -K_3, \tag{27}$$

where the J, K and Q matrices are given in Eq.(10) and Eq.(11).

In this section, we started with the generators of the symmetry groups for two independent oscillators. They are A_1, B_1, C_1 and A_1, B_1, C_1. We then introduced another generator A_0 to solder them up. This processes produced three additional generators A_3, B_3, C_3 which are $-J_1, Q_3, and -K_3$ respectively. It is remarkable that K_3, Q_3 and J_0 form the set of generators for another $Sp(2)$ group. They satisfy the commutation relations

$$[Q_3, K_3] = -iJ_0, \qquad [K_3, J_0] = -iQ_3, \qquad [Q_3, J_0] = iK_3 \tag{28}$$

This symmetry group will play the major role in decoupling the coupled oscillator problem.

5 Reparametrization of Coupled Oscillators

Let us consider a system of two coupled harmonic oscillators. The Hamiltonian for this system is

$$H = \frac{1}{2} \left\{ \frac{1}{m_1} p_1^2 + \frac{1}{m_2} p_2^2 + A' x_1^2 + B' x_2^2 + C' x_1 x_2 \right\}. \tag{29}$$

where

$$A' > 0, \qquad B' > 0, \qquad 4A'B' - C'^2 > 0. \tag{30}$$

By making scale changes of x_1 and x_2 to $(m_1/m_2)^{1/4} x_1$ and $(m_2/m_1)^{1/4} x_2$ respectively, it is possible to make a canonical transformation of the above Hamiltonian to the form [14, 15]

$$H = \frac{1}{2m} \left\{ p_1^2 + p_2^2 \right\} + \frac{1}{2} \left\{ A x_1^2 + B x_2^2 + C x_1 x_2 \right\}, \tag{31}$$

with $m = (m_1 m_2)^{1/2}$. This transformation is generated We can decouple this Hamiltonian by making the coordinate transformation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{32}$$

Under this rotation, the kinetic energy portion of the Hamiltonian in Eq.(31) remains invariant. Thus we can achieve the decoupling by diagonalizing the potential energy. Indeed, the system becomes diagonal if the angle α becomes

$$\tan \alpha = \frac{C}{B - A}. \tag{33}$$

This diagonalization procedure is well known.

We now introduce the new parameters K and η defined as

$$K = \sqrt{AB - C^2/4}, \qquad \exp(-2\eta) = \frac{A + B + \sqrt{(A-B)^2 + C^2}}{\sqrt{4AB - C^2}}, \tag{34}$$

in addition to the rotation angle α. In terms of this new set of variables, A, B and C take the form

$$A = K \left(e^{2\eta} \cos^2 \frac{\alpha}{2} + e^{-2\eta} \sin^2 \frac{\alpha}{2} \right),$$

$$B = K \left(e^{2\eta} \sin^2 \frac{\alpha}{2} + e^{-2\eta} \cos^2 \frac{\alpha}{2} \right),$$

$$A = K \left(e^{-2\eta} - e^{2\eta} \right) \sin \alpha. \tag{35}$$

the Hamiltonian can be written as

$$H = \frac{1}{2m} \left\{ q_1^2 + q_2^2 \right\} + \frac{K}{2} \left\{ e^{2\eta} y_1^2 + e^{-2\eta} y_2^2 \right\}, \tag{36}$$

where y_1 and y_2 are defined in Eq.(32), and

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{37}$$

This form will be our starting point. The above rotation together with that of Eq.(32) is generated by J_0.

If we measure the coordinate variable in units of $(mK)^{1/4}$, and use $(mK)^{-1/4}$ for the momentum variables, the Hamiltonian takes the form

$$H = \frac{\omega}{2} e^{\eta} \left(e^{-\eta} q_1^2 + e^{\eta} y_1^2 \right) + \frac{\omega}{2} e^{-\eta} \left(e^{\eta} q_2^2 + e^{-\eta} y_2^2 \right), \tag{38}$$

where $\omega = \sqrt{K/m}$. If $\eta = 0$, the system becomes decoupled, and the Hamiltonian becomes

$$H = \frac{\omega}{2} \left(p_1^2 + x_1^2 \right) + \frac{\omega}{2} \left(p_2^2 + x_2^2 \right). \tag{39}$$

In Sec. 8, we will be dealing with the problem of what happens when no observations are made on the second coordinate. If the system is decoupled, as the above Hamiltonian indicates, the physics in the first coordinate is solely dictated by the Hamiltonian

$$H_1 = \frac{\omega}{2} \left(p_1^2 + x_1^2 \right). \tag{40}$$

It is important to note that the Hamiltonian of Eq.(39) cannot be obtained from Eq.(38) by canonical transformation. For this reason, the Hamiltonian of the form

$$H' = \frac{\omega}{2} \left(e^{-\eta} q_1^2 + e^{\eta} y_1^2 \right) + \frac{\omega}{2} \left(e^{\eta} q_2^2 + e^{-\eta} y_2^2 \right) \tag{41}$$

may play a useful role in our discussion. This Hamiltonian can be transformed into the decoupled form of Eq.(39) through a canonical transformation.

6 Quantum Mechanics of Coupled Oscillators

It is remarkable that both the Hamiltonian H of Eq.(38) and H' of Eq.(41) lead to the same Schrödinger wave function. If y_1 and y_2 are measured in units of $(mK)^{1/4}$, the ground-state wave function for this oscillator system is

$$\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}(e^\eta y_1^2 + e^{-\eta} y_2^2)\right\}. \tag{42}$$

The wave function is separable in the y_1 and y_2 variables. However, for the variables x_1 and x_2, the story is quite different. If we write this wave function in terms of x_1 and x_2, then

$$\psi(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}\left[e^\eta(x_1 \cos\frac{\alpha}{2} - x_2 \sin\frac{\alpha}{2})^2 \right.\right.$$
$$\left.\left. + e^{-\eta}(x_1 \sin\frac{\alpha}{2} + x_2 \cos\frac{\alpha}{2})^2\right]\right\}. \tag{43}$$

If $\eta = 0$, this wave function becomes

$$\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}(x_1^2 + x_2^2)\right\}. \tag{44}$$

For other values of η, the wave function of Eq.(43) can be obtained from the above expression by a unitary transformation.

$$\sum_{m_1 m_2} A_{m_1 m_2}(\alpha, \eta)\psi_{m_1}(x_1)\psi_{m_2}(x_2), \tag{45}$$

where $\psi_m(x)$ is the mth excited state wave function. The coefficients $A_{m_1 m_2}(\eta)$ satisfy the unitarity condition

$$\sum_{m_1 m_2} |A_{m_1 m_2}(\alpha, \eta)|^2 = 1. \tag{46}$$

It is possible to carry out a similar expansion in the case of excited states [16].

As for unitary transformations applicable to wave functions, let us go back the generators of canonical transformations in classical mechanics in Eq.(12) and Eq.(13). As was stated before, they are also applicable to the Wigner phase–space distribution function. The canonical transformation of the Wigner function is translated into a unitary transformation of the Schrödinger wave function. There are therefore ten generators of unitary transformations applicable to Schrödinger wave functions [6, 4]. They are

$$\hat{J}_1 = \frac{1}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \qquad \hat{J}_2 = \frac{1}{2i}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right),$$

$$\hat{J}_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \qquad \hat{J}_0 = \frac{1}{2}\left(a_1^\dagger a_1 + a_2 a_2^\dagger\right),$$

$$\hat{K}_1 = -\frac{1}{4}\left(a_1^\dagger a_1^\dagger + a_1 a_1 - a_2^\dagger a_2^\dagger - a_2 a_2\right),$$

$$\hat{K}_2 = \frac{i}{4}\left(a_1^\dagger a_1^\dagger - a_1 a_1 + a_2^\dagger a_2^\dagger - a_2 a_2\right),$$

$$\hat{K}_3 = \frac{1}{2}\left(a_1^\dagger a_2^\dagger + a_1 a_2\right),$$

$$\hat{Q}_1 = -\frac{i}{4}\left(a_1^\dagger a_1^\dagger - a_1 a_1 - a_2^\dagger a_2^\dagger + a_2 a_2\right),$$

$$\hat{Q}_2 = -\frac{1}{4}\left(a_1^\dagger a_1^\dagger + a_1 a_1 + a_2^\dagger a_2^\dagger + a_2 a_2\right),$$

$$\hat{Q}_3 = \frac{i}{2}\left(a_1^\dagger a_2^\dagger - a_1 a_2\right). \tag{47}$$

where $a^\dagger$ and a are the step-up and step-down operators applicable to harmonic oscillator wave functions. The above operators also satisfy the commutation relations given in Eq.(6).

7 Wigner Functions and Uncertainty Relations

The Wigner phase–space picture of quantum mechanics is often more convenient for studying the uncertainty relations. Unitary transformations in the Schrödinger picture can be achieved through canonical transformations in phase space. It has been known that canonical transformations are uncertainty–preserving transformations. They are also entropy–preserving transformations [17]. Are there then non-canonical transformations in quantum mechanics?

In his book on statistical mechanics [18], Feynman raises the issue of the rest of the universe in connection with the density matrix. Feynman divides the universe into two parts. We make measurements in the first part, but are not able to measure anything in the second part. The second part is Feynman's rest of the universe. Indeed, the density matrix plays the essential role when we are not able to measure all the variables in quantum mechanics [19, 20].

In the present case of coupled harmonic oscillators, we assume that we are not able to measure the x_2 coordinate. It is often more convenient to use the Wigner phase-space distribution function to study the density matrix, especially when we want to study the uncertainty products in detail [15, 18].

For two coordinate variables, the Wigner function is defined as [15]

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \int \exp\left\{-2i(p_1 y_1 + p_2 y_2)\right\}$$

$$\times \psi^*(x_1 + y_1, x_2 + y_2)\psi(x_1 - y_1, x_2 - y_2)dy_1 dy_2. \tag{48}$$

The Wigner function corresponding to the oscillator wave function of Eq.(43) is

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \exp\left\{-e^\eta(x_1 \cos\frac{\alpha}{2} - x_2 \sin\frac{\alpha}{2})^2\right.$$

$$-e^{-\eta}(x_1 \sin \frac{\alpha}{2} + x_2 \cos \frac{\alpha}{2})^2 - e^{-\eta}(p_1 \cos \frac{\alpha}{2} - p_2 \sin \frac{\alpha}{2})^2$$

$$-e^{\eta}(p_1 \sin \frac{\alpha}{2} + p_2 \cos \frac{\alpha}{2})^2 \Big\} . \qquad (49)$$

If we do not make observations in the $x_2 p_2$ coordinates, the Wigner function becomes

$$W(x_1, p_1) = \int W(x_1, x_2; p_1, p_2) dx_2 dp_2. \qquad (50)$$

The evaluation of the integral leads to

$$W(x_1, x_2; p_1, p_2) = \left\{ \frac{1}{\pi^2(1 + \sinh^2 \eta \sin^2 \alpha)} \right\}^{1/2}$$

$$\times \exp\left\{ -\left(\frac{x_1^2}{\cosh \eta - \sin \eta \cos \alpha} + \frac{p_1^2}{\cosh \eta + \sin \eta \cos \alpha} \right) \right\}. \qquad (51)$$

This Wigner function gives an elliptic distribution in the phase space of x_1 and p_1. This distribution gives the uncertainty product of

$$(\Delta x)^2 (\Delta p)^2 = \frac{1}{4}(1 + \sinh^2 \eta \sin^2 \alpha). \qquad (52)$$

This expression becomes $1/4$ if the oscillator system becomes uncoupled with $\alpha = 0$. Because x_1 is coupled with x_2, our ignorance about the x_2 coordinate, which in this case acts as Feynman's rest of the universe, increases the uncertainty in the x_1 world which, in Feynman's words, is the system in which we are interested.

In the Wigner phase–space picture, the uncertainty is measured in terms of the area in phase space where the Wigner function is sufficiently different from zero. According to the Wigner function for a thermally excited oscillator state, the temperature and entropy are also determined by the degree of the spread of the Wigner function phase space.

8 Scale Transformations in Phase Space

In addition to the ten generators given in Eq.(10) and also in Eq.(11), we can consider the scale transformation in which both the position and momentum of the first coordinate are expanded and those of the second coordinate contracted. The Hamiltonian given in Eq.(38) suggests such a transformation, and the transformation can be generated by

$$S_0 = \frac{i}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \qquad (53)$$

This matrix generates scale transformations in phase space. The transformation leads to a radial expansion of the phase space of the first coordinate [21] and contracts the phase space of the second coordinate. What is the physical significance of this operation? As we discussed in Sec. 7, the expansion of phase space leads to an increase in uncertainty and entropy. Mathematically

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speaking, the contraction of the second coordinate should cause a decrease in uncertainty and entropy. Can this happen? The answer is clearly No, because it will violate the uncertainty principle. This question will be addressed in future publications.

In the meantime, let us study what happens when the matrix S_0 is introduced into the set of matrices given in Eq.(10) and Eq.(11). It commutes with J_0, J_3, K_1, K_2, Q_1, and Q_2. However, its commutators with the rest of the matrices produce four more generators:

$$[S_0, J_1] = \frac{i}{2} \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad [S_0, J_2] = \frac{1}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$[S_0, K_3] = \frac{1}{2} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \qquad [S_0, Q_3] = \frac{1}{2} \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{54}$$

If we take into account the above five generators in addition to the sixteen generators of $Sp(4)$, there are fifteen generators. They form the closed set of commutation relations for the the group $SL(4, r)$. This $SL(4, r)$ symmetry of the coupled oscillator system may have interesting physical implications.

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II. QUANTUM GROUPS

q-HARMONIC OSCILLATORS, q-COHERENT STATES
AND THE q-SYMPLECTON[†][‡]

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Abstract

The recently introduced notion of a quantum group is discussed conceptually and then related to deformed harmonic oscillators ("q-harmonic oscillators"). Two developments in applying q-harmonic oscillators are reviewed: q-coherent states and the q-symplecton.

1 Introduction

It is not unfamiliar in physics that a new theory appears in the form of a 'deformation' of a previous 'classical' theory; thus, for example, quantum mechanics can be considered to be a deformation of classical mechanics (which is recovered in the limit that the 'deformation parameter' $\hbar \rightarrow 0$), and Einsteinian relativity to be a deformation of Newtonian relativity (which is recovered when the 'deformation parameter' $c \rightarrow \infty$). Recently this notion of deformation has been applied [1,2] to symmetry itself, leading to the concept of a 'quantum group' as a deformation of a classical (Lie) group with a deformation parameter denoted by q. This new development has had numerous important applications in both physics and mathematics [3,4]. Since harmonic oscillators have played a fundamental—and pervasive!—rôle in the applications of symmetry in quantum physics, it is not surprising that the concepts of quantum groups, and

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deformations, are important here also, and hence relevant to the present conference. Accordingly, it is our purpose to discuss here deformed harmonic oscillators ("q-harmonic oscillators"), deformed coherent states ("q-coherent states") and the deformed algebraic structure (based on harmonic oscillators) called the "q-symplecton".

We will begin by discussing, in conceptual and motivational terms, the simplest of quantum groups—$SU_q(2)$, the q-deformed quantal rotation group—to set the stage for introducing deformed harmonic oscillators, and then the remaining topics mentioned above.

2 The Quantum Group SU$_q$(2)

The commutation relations for the three generators $\{J_+^q, J_-^q, J_z^q\}$ defining the quantum group $SU_q(2)$ are given by:

$$[J_z^q, J_\pm^q] = \pm J_\pm^q, \tag{2.1}$$

$$[J_+^q, J_-^q] = \frac{q^{J_z^q} - q^{-J_z^q}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad q \in I\!R^+. \tag{2.2}$$

These defining relations for $SU_q(2)$ differ from those of ordinary angular momentum ($SU(2)$) in two ways:

(a) the commutator in (2.2) is not $2J_z$ as usual, but an infinite series (for generic q) involving *all odd powers*: $(J_z^q)^1, (J_z^q)^3, \ldots$. Each such power is a linearly independent operator in the enveloping algebra; accordingly, the Lie algebra of $SU_q(2)$ is *not* of finite dimension.

(b) For $q \to 1$, the right hand side of (2.2) becomes $2J_z$. Thus we recover in the limit the usual commutation relations for angular momentum.

The differences noted in (a) and (b) are expressed by saying that the quantum group $SU_q(2)$ is a *deformation* of the enveloping algebra of $SU(2)$.

The deformation parameter q occurs in $SU_q(2)$ in a characteristic way, as *q-integers* denoted by $[n]_q$ such that:

$$\begin{aligned}
[n]_q &= \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \\
&= q^{\frac{(n-1)}{2}} + q^{\frac{(n-3)}{2}} + \ldots q^{-\frac{(n-1)}{2}}, \quad n \in \mathbb{Z}.
\end{aligned} \tag{2.3}$$

These q-integers, $[n]_q$ obey the rule: $[-n]_q = (-1)[n]_q$, with $[0]_q = 0$ and $[1]_q = 1$. Note that $[n]_q = [n]_{q^{-1}}$, so that the defining relations (2.1) and (2.2) are invariant to $q \leftrightarrow q^{-1}$.

The quantum group concept involves much more than just deforming the commutation relations of the classical group generators. Actually an interesting new algebraic structure is also imposed, that of a *Hopf algebra* [5]. Let us first define this new structure and then discuss its meaning.

Consider an associative algebra A, with a unit element, $\mathbf{1}$, over a field say, $\mathcal{C}$. Then the algebra involves the operations:

$$\text{multiplication:} \quad m : A \otimes A \to A, \quad \text{and,} \tag{2.4}$$

$$\text{unit:} \quad \mathbf{1} : \mathcal{C} \to A, \tag{2.5}$$

subject to the familiar axioms of associativity and the compatibility of addition and multiplication.

We can extend this algebra to become a Hopf algebra if we can "reverse the arrows" in (2.4) and (2.5) above, that is, if we can define two new operations:

$$co\text{-}multiplication: \quad \Delta : A \to A \otimes A, \quad \text{and}, \tag{2.6}$$

$$co\text{-}unit: \quad \epsilon : A \to \mathbb{C}. \tag{2.7}$$

Since for a quantum group the algebra A is a group algebra, it is reasonable to require that one have a third new operation:

$$\gamma : \quad A \to A, \tag{2.8}$$

called "antipode", (the analog to the inverse in the group).

These three new operations must satisfy the requirement that Δ and ϵ are homomorphisms of the algebra A and that γ is an anti-homomorphism. In addition, the operations must satisfy the compatibility axioms:

Associativity of co-multiplication: $(id \otimes \Delta)\Delta(a) = (\Delta \otimes id)\Delta(a), \quad a \in A$ (2.9)

Antipode axiom: $m(id \otimes \gamma)\Delta(a) = m(\gamma \otimes id)\Delta(a) = \epsilon(a)1,$ (2.10)

Co-unit axiom: $(\epsilon \otimes id)\Delta(a) = (id \otimes \epsilon)\Delta(a) = a.$ (2.11)

• For a physicist, the introduction of such complicated and heavy algebraic machinery "out of the blue" is very disconcerting. Certainly it requires motivation. The obvious question is: "why a Hopf algebra"? Let us try to answer this.

Physicists are already very familiar with the algebraic approach to symmetry in quantum mechanics; what is needed is a physical reason for "reversing the arrows". What this really means, in effect, is that all one needs is a simple motivating physical example.

Here is that example. Consider angular momentum: there is a natural, classical, concept for adding angular momenta, which is taken over in quantum mechanics. Consider J_{total} as the total angular momentum operator which is to be the sum of two independent constituent angular momenta J_1 and J_2. Writing the total angular momentum operator J_{total} as an action on the two constituent state vectors we have:

$$\mathbf{J}_{total}|\psi\rangle_{total} = \mathbf{J}_1|\varphi\rangle_1 \otimes 1|\chi\rangle_2 + 1|\varphi\rangle_1 \otimes \mathbf{J}_2|\chi\rangle_2, \tag{2.12}$$

where we have been careful to use a precise notation for the tensor product $\otimes$ of the two independent systems.

Writing this same result in an abstract formal manner, we discover that what we have really done by "adding angular momentum" is to define a co-multiplication:

$$\Delta(J) = \mathbf{J} \otimes 1 + 1 \otimes \mathbf{J}, \tag{2.13}$$

where $\mathbf{J}$ denotes a generic angular momentum (defined as obeying the commutation relations).

In other words: *The vector addition of angular momenta defines a commutative co-product in a Hopf algebra.* One sees accordingly that a (commutative) Hopf algebra structure is not only

very natural in quantum physics, but actually implicit, and in fact essential—unfamiliar only because unrecognized. The remaining Hopf algebra axioms are required to make the structures compatible and well-defined, and in a sense analogous to group concepts.

What we wish to emphasize is that the deformation of the algebraic structure in a quantum group is only part of the basic concept—requiring the additional Hopf algebra structure, which is natural to quantum mechanics, provides an important constraint on the freedom to deform the commutation relations.

One can now understand intuitively from our example the fundamental significance of quantum groups for physics: one now has the new possibility of defining a *non-commutative* co-multiplication, as actually occurs for the quantum group $SU_q(2)$. This means that:

(i) the fundamental commutation relations are changed ("deformed"); that is, one has *kinematic symmetry breaking*. (Recall that Hamiltonian perturbation theory is dynamical and leaves commutation relations (which are kinematical) *invariant*);

(ii) the "addition of q-angular momentum" depends on the *order* of addition.

There is one other feature of the commutation relations for $SU_q(2)$ that deserves comment: the relations (2.1) and (2.2) single out J_z^q and thus appear to break the rotational symmetry. For generic values of q this seeming result is incorrect: the degeneracy structure of q-group irreps is in fact *preserved*, a consequence of the Rosso-Lusztig theorem. (We take this opportunity to note that ref. [6] is misleading on this particular point.)

For completeness, since we have emphasized the importance of the complete Hopf algebra structure, let us give explicitly the remaining Hopf algebra operations for the quantum group $SU_q(2)$:

$$\Delta(J_z^q) = J_z^q \otimes 1 + 1 \otimes J_z^q, \tag{2.14}$$

$$\Delta(J_\pm^q) = J_\pm^q \otimes q^{\frac{J_z^q}{4}} + q^{-\frac{J_z^q}{4}} \otimes J_\pm^q, \tag{2.15}$$

$$\epsilon(1) = 1, \quad \epsilon(J_\pm^q) = \epsilon(J_z^q) = 0, \tag{2.16}$$

$$\gamma(J_\pm^q) = -q^{\mp\frac{1}{2}} J_\pm^q, \quad \gamma(J_z^q) = -J_z^q. \tag{2.17}$$

3 q-Boson operators

In order to understand the meaning of the deformed commutation relations (2.1) and (2.2) it is natural to look for representations of the operators $J_\pm^q, J_z^q$ as finite-dimensional matrices. For the usual angular momentum group, there is a standard way to do this: one uses the *Jordan-Schwinger map* [7], which maps the 2×2 matrices $\{J_\pm, J_z\}$ of the fundamental irrep into *boson operators*.

Let us recall how this works. One begins with a realization of the operators $J_\pm, J_z$ in terms of a pair of commuting boson creation operators (a_1, a_2) and annihilation operators, $(\bar{a}_1, \bar{a}_2)$, and defines the Jordan-Schwinger map:

$$J_+ \rightarrow a_1 \bar{a}_2, \quad J_- \rightarrow a_2 \bar{a}_1, \quad J_z \rightarrow \tfrac{1}{2}(a_1 \bar{a}_1 - a_2 \bar{a}_2). \tag{3.1a,b,c}$$

This map preserves the angular momentum commutation relations (that is, the Jordan-Schwinger map is a homomorphism) and from this map one can explicitly construct *all* unitary irreps of $SU(2)$.

Is there a q-analog to the Jordan-Schwinger map? There is indeed! (Refs. [8,9,10]). *The basic idea is to construct q-analogs to the boson operators.* To do so introduce the *q-creation operator a^q*, its Hermitian conjugate the *q-destruction operator $\bar{a}^q$*, and the *q-boson vacuum ket vector* $|0\rangle$ defined by the equation

$$\bar{a}^q|0\rangle = 0. \tag{3.2}$$

Instead of the Heisenberg relation, $[\bar{a}, a] = 1$, let us postulate the algebraic relation:

$$\bar{a}^q a^q - q^{\frac{1}{2}} a^q \bar{a}^q = q^{-\frac{N^q}{2}}, \tag{3.3}$$

where N^q is the Hermitian *number operator* satisfying

$$[N^q, a^q] = a^q, \quad [N^q, \bar{a}^q] = -\bar{a}^q, \quad \text{with } N^q|0\rangle \equiv 0. \tag{3.4a,b,c}$$

This algebra is a deformation of the Heisenberg-Weyl algebra, which is recovered in the limit $q \to 1$. (Note that the q-number operator N^q is now no longer the operator $a\bar{a}$ as in the Heisenberg case.)

Orthonormal ket vectors corresponding to states of n q-quanta are given by:

$$|n\rangle_q \equiv ([n]_q!)^{-\frac{1}{2}} (a^q)^n |0\rangle, \tag{3.5}$$

$$\text{with}: \quad N^q|n\rangle_q = n|n\rangle_q. \tag{3.6}$$

It is now easy to define a q-analog for the algebra of the generators of the quantum group $SU_q(2)$. In the language of q-boson operators, one defines a pair of *mutually commuting q-bosons* a_i^q for $i = 1, 2$. That is, for each, i, a_i^q and $\bar{a}_i^q$ obey equations (3.3), (3.4) and, in addition, the relations:

$$\text{for } i \neq j : \quad [a_i^q, a_j^q] = [\bar{a}_i^q, \bar{a}_j^q] = [a_i^q, \bar{a}_j^q] = 0. \tag{3.7}$$

The generators $\{J_+^q, J_-^q, J_z^q\}$ of $SU_q(2)$ are then realized by

$$J_+^q \to a_1^q \bar{a}_2^q, \quad J_-^q \to a_2^q \bar{a}_1^q, \quad J_z^q \to \frac{1}{2}(N_1^q - N_2^q). \tag{3.8a,b,c}$$

The construction of all unitary irreps of the quantum group $SU_q(2)$—for generic q—is now straightforward [6] but will be omitted.

Remarks: (1) We have emphasized in Section 2 that the Hopf algebra structure—more particularly co-multiplication—is an important constraint on possible deformations. Let us note that the deformation of q-bosons given by eq. (3.3) does allow a (non-commutative) co-product to be defined. However, as shown by Prof. T. Palev (private communication), a complete Hopf algebra structure is not possible.

(2) The deformation given in eq. (3.3) can be put into many differently appearing, but equivalent, forms. For example, if we define $A^q = a^q q^{\frac{1}{4}N^q}$ and $\bar{A}^q = q^{\frac{1}{4}N^q}\bar{a}^q$, then eq. (3.3) becomes:

$$\bar{A}^q A^q = q A^q \bar{A}^q + 1, \tag{3.9}$$

a form often found in the literature.

4 The q-Harmonic Oscillator

We have motivated the introduction of q-deformed bosons as a way to implement the concept of a quantum group.

Let us now examine the q-harmonic oscillator on its own merits. From the q-boson operators $a, \bar{a}$ we can define q-momentum (P) and q-position (Q) operators in the same way as for boson operators. That is, we define:

$$P \equiv i\sqrt{\frac{m\hbar\omega}{2}}(a^q - \bar{a}^q), \tag{4.1}$$

$$Q \equiv \sqrt{\frac{\hbar}{2m\omega}}(a^q + \bar{a}^q). \tag{4.2}$$

The commutator $[P, Q]$ is then (using (3.3)):

$$i[P, Q] = \hbar[\bar{a}^q, a^q] = \hbar([N+1]_q - [N]_q). \tag{4.3}$$

The eigenvalues $(N \to n)$ of the right hand side are therefore

$$\hbar([n+1]_q - [n]_q) = \hbar \frac{\cosh(\frac{1}{4}(2n+1)\log q)}{\cosh(\frac{1}{4}\log q)}. \tag{4.4}$$

One sees that the Heisenberg uncertainty in the q-harmonic oscillator is minimal (and independent of q) *only* in the limit $q \to 1$; the uncertainty *increases* with n for $q \neq 1$.

The q-harmonic oscillator Hamiltonian is defined from P, Q according to

$$\begin{aligned}
\mathcal{H} &\equiv \frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2, \\
&= \frac{\hbar\omega}{2}(\bar{a}^q a^q + a^q \bar{a}^q).
\end{aligned} \tag{4.5}$$

From (3.3) we find

$$\mathcal{H} = \frac{\hbar\omega}{2}([N+1]_q + [N]_q), \tag{4.6}$$

showing that the eigenvalues of $\mathcal{H}$ are

$$E(n) = \frac{\hbar\omega}{2}([n+1]_q + [n]_q). \tag{4.7}$$

The normalized eigenstates $|n\rangle$ are:

$$|n\rangle = ([n]!)^{-\frac{1}{2}}(a^q)^n|0\rangle. \tag{4.8}$$

The energy spectrum for the q-harmonic oscillator is uniformly spaced only for $q = 1$, the undeformed case. For q large, one sees that the spectrum becomes exponential: $E(n) \sim \hbar\omega q^{\frac{n}{2}}(1 + \mathcal{O}(\frac{1}{q}))$.

5 Coherent States

It is natural to ask, once one has defined q-deformed bosons, whether or not coherent states exist for this new harmonic oscillator structure. The answer is yes [11], as one might expect. Let us review this structure briefly here.

There are two key characteristics of the (usual) coherent states, as identified by Klauder and Skagerstam [12]:

(a) *continuity* of the coherent state $|z\rangle$ as a function of z.

and (b) *the resolution of unity:*

$$1 = \int |z\rangle\langle z| \ d\mu(z), \tag{5.1}$$

where the integration takes place with respect to a positive measure $d\mu(z)$.

The best known examples of coherent states, which certainly satisfy these two characteristics, are the canonical coherent states generated by the (usual) creation and annihilation operators a and $\bar{a}$. These canonical coherent states are defined by [8]

$$
\begin{aligned}
|z\rangle &\equiv e^{-|z|^2/2} e^{za} |0\rangle \\
&= e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle,
\end{aligned}
\tag{5.2}
$$

where $|n\rangle$ denotes the orthonormal vectors generated by the creation operator a.

We can immediately write down q-coherent states $|z\rangle_q$ by replacing the boson operator of (5.2) by its q-boson analog, and replacing the exponential in (5.2) by the *q-exponential function* $\exp_q$:

$$
\begin{aligned}
|z\rangle_q &= \left(\exp_q(|z|^2)\right)^{-\frac{1}{2}} \exp_q(za^q)|0\rangle_q \\
&= \left(\exp_q\left(|z|^2\right)\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_q!}} |n\rangle_q.
\end{aligned}
\tag{5.3}
$$

These states satisfy:

$$\bar{a}_q |z\rangle_q = z|z\rangle_q, \tag{5.4}$$

showing that the q-coherent state $|z\rangle_q$ is an eigenstate of the annihilation operator $\bar{a}^q$ with eigenvalue z and, since $z = {}_q\langle z|\bar{a}^q|z\rangle_q$ (assuming the states $|z\rangle_q$ are normalized), the label z is the *mean* of $\bar{a}^q$ in the state $|z\rangle_q$. The definition (5.3) is not a unique q-extension of (5.2), for we could have chosen any one of the family e_q^λ of exponential functions in [13]; this would introduce explicit q-factors in equations such as (5.4). We outline below how the particular q-harmonic oscillator model of Section 4 (above) leads naturally to these q-coherent states. (The states (5.3) were first considered in Ref. [8] and subsequently also in Refs. [14-17]. In fact, as with many q-analogs of classical and quantum concepts, some q-generalizations were obtained before the appearance of quantum groups [18]).

Let us now consider the two characteristic properties of coherent states, continuity and completeness. (a) The continuity properties of $|z\rangle_q$, as a function of z, follow immediately from the continuity of the deformed exponential function, $\exp_q$ in (5.3).

(b) The resolution of unity within the Hilbert space, in terms of the states $|z\rangle_q$, has been considered by Gray and Nelson [15] and also Bracken et al [17]. The q-analog of Euler's formula for $\Gamma(x)$ is required, and is expressed in terms of the q-integration defined in [13]:

$$\int_0^\zeta \exp_q(-x)x^n d_q x = [n]_q!$$ (5.5)

where ζ is the largest zero of $\exp_q(x)$ (note that, unlike e^x, $\exp_q(x)$ alternates in sign as $x \to -\infty$). A natural restriction is $|z|^2 < \zeta$ and then, with the help of (3.5), the resolution of unity can be derived [17],

$$1 = \int |z\rangle_q \,_q\langle z| \, d\mu(z)$$ (5.6)

where the measure $d\mu(z)$ is given by

$$d\mu(z) = \frac{1}{2\pi} \exp_q(|z|^2) \exp_q(-|z|^2) d_q |z|^2 d\theta,$$ (5.7)

where $\theta = \arg(z)$. It follows from (5.6) that an arbitrary state can be expanded in terms of the states $|z\rangle_q$. (In fact, q-coherent states are overcomplete, for an arbitrary q-coherent state is non-orthogonal to $|z\rangle_q$, for any z.)

Coherent states arise naturally within the framework of the harmonic oscillator of Section 4, by defining boson operators from position and momentum operators, Q, P, putting dimensional factors to unity:

$$a^q = \tfrac{1}{\sqrt{2}}(Q - iP), \quad \overline{a}^q = \tfrac{1}{\sqrt{2}}(Q + iP).$$ (5.8)

Conversely, we can use these formulas to define momentum and position operators and so, given q-boson operators, these formulas also provide convenient q-analog definitions of q-momentum and position operators [8].

Alternatively, one can define a q-harmonic oscillator by starting with Schrödinger's equation and replacing the derivative by a finite difference operator which provides an alternative form for the deformation. We use the following q-derivative,

$$\nabla_q f(x) = \frac{f(xq) - f(x)}{x(q-1)},$$ (5.9)

and the q-harmonic oscillator states are now determined by the equation

$$\tfrac{1}{2}(-\nabla_q^2 + qx^2)\psi(x) = E\psi(x).$$ (5.10)

Effectively, we have chosen q-momentum and q-position operators Q_q, P_q satisfying

$$qQ_q P_q - P_q Q_q = i,$$ (5.11)

with the realization $Q_q = x$, $P_q = i\nabla_q$. (This is yet another realization different from (3.9) for the deformation.)

C-2

Solutions of the difference equation (5.10) have been given by several authors [19,20], and involve q-extensions of the Hermite polynomials. The ground state ψ_0 is given by

$$\psi_0(x) = \sum_{n=0}^{\infty} \frac{(-)^n q^{-\frac{n^2}{2}} x^{2n}}{[2n]_q!!}, \tag{5.12}$$

where $[2n]_q!! = [2n]_q[2n-2]_q \ldots [2]_q$. Upon using the identity $[2n]_q = [2]_q[n]_{q^2}$ we can identify the function (5.12) as one of the family of q-exponential functions given by Exton [13].

The eigenstates ψ_n of the deformed Schrödinger equation (5.10) are labelled by an integer n, and the energy levels are $E_n = \frac{1}{2}[2n+1]_q$. (For comparison, note that in the model defined in Section 4, the energy levels are different: $E_n = \frac{1}{2}([n+1]_q + [n]_q) = \frac{1}{2}[2n+1]_{q^{1/2}}$). The eigenstates of (5.10), ψ_n, take the form

$$\psi_n(x) = H_n^q(x)\psi_0\left(xq^{-\frac{n}{2}}\right), \tag{5.13}$$

where ψ_0 is given by (5.12) and $H_n^q(x)$ denotes a q-extension of the classical Hermite polynomial, with the explicit formula:

$$H_n^q(x) = \sum_{r=0}^{n} \frac{C_r x^r q^{-\frac{(2n+1)r}{4}}}{[r]_q!}, \tag{5.14}$$

where the coefficients C_r are given (for even or odd r) by

$$C_{2m} = (-)^m q^{(2n+1)m/2}[2n]_q[2n-4]_q \ldots [2n-4m+4]_q \tag{5.15a}$$

$$C_{2m+1} = (-)^m q^{(2n+1)m/2}[2n-2]_q[2n-6]_q \ldots [2n-4m+2]_q. \tag{5.15b}$$

From the explicit eigenstates one can identify q-boson operators which step between the eigenstates $\psi_n(x)$, from which one can form the q-coherent states of this model of the q-harmonic oscillator [20].

6 The q-Symplecton

The idea behind the symplecton construction has a close relationship to harmonic oscillators. In the Jordan-Schwinger realization of angular momentum one obtains uniformly all unitary irreps in terms of *two* independent harmonic oscillators. This naturally suggests the question: can one do better and realize all irreps *uniformly* in terms of *one* harmonic oscillator? The answer is (of course) yes—this is the symplecton realization [7,21], which uses the creation operator (a) as the spin-$\frac{1}{2}$ "up" state and the destruction operator $(\bar{a})$ as the "down" state. This implies that there is no longer a vacuum ket $|0\rangle$ annihilated by $\bar{a}$. Instead we define a *formal ket* $|\rangle$ *and seek to interpret both* $a|\rangle$ *and* $\bar{a}|\rangle$ *as non-vanishing vectors.*

Operators in this symplecton calculus will be defined as polynomials over $(a, \bar{a})$ with complex numbers as scalars. State vectors will be defined as operators multiplied on the right by the basic formal ket, i.e.,

$$|\nu\rangle \equiv \mathcal{O}_\nu|\rangle, \tag{6.1}$$

where $|\nu\rangle$ is a vector and $\mathcal{O}_\nu$ the operator creating this vector. The action of the generators on state vectors will be defined as commutation on the relevant operator $\mathcal{O}_\nu$, that is,

$$J_i(|\nu\rangle) \equiv [J_i, \mathcal{O}_\nu]|\rangle. \tag{6.2}$$

To be completely explicit we are considering (for the undeformed symplecton) a single boson operator a and its conjugate $\bar{a}$ obeying:

$$[\bar{a}, a] = 1, \tag{6.3}$$

all other commutators zero. The generators of $SU(2)$ are realized by:

$$J_+ \to -\tfrac{1}{2}a^2, \quad \text{(note the sign!)} \quad J_- \to \tfrac{1}{2}\bar{a}^2, \quad J_0 \to \tfrac{1}{4}(a\bar{a} + \bar{a}a). \tag{6.4}$$

It is easily verified that this realization obeys the desired commutation relations:

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0. \tag{6.5}$$

Note that the action of these generators on symplecton state vectors, verifying the commutation relations, succeeds precisely because of the Jacobi identity. Using commutation under the generators, the labels J and M can be assigned to define characteristic polynomials $\mathcal{P}_J^M$. The angular momentum irrep eigenvectors are then given by the set of vectors $\mathcal{P}_J^M|\rangle$.

The *adjoint polynomial* $(\mathcal{P}_J^M)^{\mathrm{adj}}$ is defined by:

$$(\mathcal{P})^{\mathrm{adj}} = (-1)^{J-M}\mathcal{P}_J^{-M}, \tag{6.6}$$

with $\bar{a}$ taken to be adjoint to a. The adjoint (dual space) vector to $\mathcal{P}_J^M|\rangle$ is defined as $\langle|(\mathcal{P}_J^M)^{\mathrm{adj}}$.

The crucial problem in this (undeformed) symplecton construction is the proper definition of an inner product for the Hilbert space of the irreps. Omitting details [7], the answer is obtained from the *multiplication law for symplecton eigen-polynomials.*

THEOREM [21]: *Let $\mathcal{P}_a^\alpha$ and $\mathcal{P}_b^\beta$ be normalized eigen-polynomials of the generators J_i. Then these polynomials obey the product law:*

$$\mathcal{P}_a^\alpha \mathcal{P}_b^\beta = \sum_{c=|a-b|}^{a+b} \langle c|a|b\rangle \ C_{\beta\alpha}^{bac} \ P_c^{\alpha+\beta}, \tag{6.7a}$$

where

$$\langle c|a|b\rangle = (2c+1)^{-\frac{1}{2}} \cdot \Delta(abc), \tag{6.7b}$$

$$\Delta(abc) \equiv \left[\frac{(a+b+c+1)!}{(a+b-c)!(a-b+c)!(-a+b+c)!} \right]^{\frac{1}{2}}, \tag{6.7c}$$

and $C_{\beta\alpha}^{bac}$ is the usual Wigner-Clebsch-Gordan coefficient for $SU(2)$.

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Using this theorem it is now easy to understand the inner product $\langle\mu|\nu\rangle$: one applies the product law to the polynomials $\mathcal{O}_\mu^{\text{adj}}$ and $\mathcal{O}_\nu$ and then *projects onto* the $J = 0$ part. The Wigner-Clebsch-Gordan coefficient (for $J = 0$) quite literally defines here a metric!

Remark: It is clear also that one can extend this structure by adjoining additional symplectons. That is, one considers a symplecton having n "internal" states: $a_1, a_2, \ldots, a_n$ and their conjugates $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$. Just as the adjunction of a boson with n "internal" states suffices to realize $SU(n)$, so does an n state symplecton suffice to realize the structure $Sp(2n)$.

An important consequence of the symplecton construction is the definition of a new invariant angular momentum function: *the triangle coefficient* $\Delta(abc)$, eq. (6.7c). This triangle function, $\Delta(abc)$, has gratifyingly simple properties. It is a function defined symmetrically on three "lengths" or "sides" a, b, c, which (from the properties of the factorial function) vanishes unless the triangle conditions (that the sum of any two sides equals or exceeds the third side) are fulfilled. The symplecton realization of angular momentum yields the triangle rule of vector addition in a particularly graphic way.

The triangle function is clearly a rotationally *invariant* function defined on three angular momenta; as such, it fits very nicely into the series of invariant functions defined on $3n$ angular momenta: $(6j)$ [Racah coefficient] and $(9j)$ [Fano coefficient]. The Wigner coefficients are often called "$(3j)$ symbols", but in view of the fact—emphasized by Wigner—that these coefficients are coordinate frame dependent (i.e., involve magnetic quantum numbers) one might consider the triangle function as the more appropriate to designate as the $(3j)$ symbol.

The triangle function obeys the following transformation law, Ref. [21]:

$$\Delta(acf)\Delta(bdf) = (2f + 1)\sum \Delta(abe)\Delta(cde)W(abcd; ef). \tag{6.8}$$

It is quite remarkable that the Racah function appears here as a *tetrahedral* function coupling four *triangles* by pairs.

Having reviewed now the symplecton construction it is time to return to our main theme: can one define a deformed symplecton ("q-symplecton") using a single deformed harmonic oscillator? The answer (of course) is *yes*, but there are some surprises [22]. We will develop the deformed structure using finite q-transformations, which provides further insights into the deformation process [23]. (The infinitesimal approach—which obtains the q-generators $\{J_i^q\}$ using a single q-boson, the q-boson analogs to eqs. (6.4)—was developed earlier in ref. [24].)

Let a_q and $\bar{a}_q$ be q-boson creation and annihilation operators obeying:

$$\bar{a}_q a_q - q^{\frac{1}{2}} a_q \bar{a}_q = 1. \tag{6.9}$$

This q-commutation relation is invariant under the transformation of q-spaces [23]:

$$(a, \bar{a})' = (a, \bar{a})\begin{pmatrix} x & u \\ v & y \end{pmatrix}, \tag{6.10}$$

where:

$$ux = q^{\frac{1}{2}}xu, \quad vx = q^{\frac{1}{2}}xv, \quad yv = q^{\frac{1}{2}}vy, \tag{6.11a,b,c}$$

$$yu = q^{\frac{1}{2}}uy, \quad uv = vu, \tag{6.11d,e}$$

$$xy - q^{-\frac{1}{2}}vu = yx - q^{\frac{1}{2}}vu = 1. \tag{6.11f}$$

The *adjoint* to $(a, \overline{a})$ is: $\left(q^{\frac{1}{4}}\overline{a}, -q^{-\frac{1}{4}}a\right)$ and obeys:

$$\left(q^{\frac{1}{4}}\overline{a}, -q^{-\frac{1}{4}}a\right)' = \left(q^{\frac{1}{4}}\overline{a}, -q^{-\frac{1}{4}}a\right)\begin{pmatrix} x^* & u^* \\ v^* & y^* \end{pmatrix}, \tag{6.12}$$

with:
$$x^* = y, \quad u^* = -q^{-\frac{1}{2}}v, \quad v^* = -q^{\frac{1}{2}}u, \quad y^* = x. \tag{6.13a,b,c,d}$$

Let us denote the q-symplecton eigenpolynomials by: Q_j^m. Then Q_j^m is a polynomial of order $j + m$ in a and $j - m$ in $\overline{a}$ and defined to transform as:

$$Q_j^m(a', \overline{a}') = \sum_n d_{nm}^j(x, u, v, y) Q_j^n(a, \overline{a}). \tag{6.14}$$

Here $d_{n,m}^j(x, u, v, y)$ is the *q-rotation matrix* which obeys:

$$d_{m'k'}^{j'}(x, u, v, y) \ d_{m''k''}^{j''}(x, u, v, y) = \sum_j {}_qC_{m'm''m}^{j'j''j}$$

$$\times \ {}_qC_{k'k''k}^{j'j''j} \ d_{mk}^j(x, u, v, y), \tag{6.15}$$

where ${}_qC_{\cdots}^{\cdots}$ are q-WCG coefficients. It follows that the set $\{Q_j^m, \ m = -j, -j+1, \ldots, j\}$ is an irreducible tensor of rank j. Moreover Q_j^m is a q-symmetric function:

$$\left(\begin{bmatrix} 2j \\ j+m \end{bmatrix}\right)^{\frac{1}{2}} Q_j^m = q^{-\frac{(j+m)(j-m)}{4}} a^{j+m}\overline{a}^{j-m}$$

$$+ q^{-\frac{(j+m)(j-m)}{4}+\frac{1}{2}} a^{j+m-1}\overline{a}a\overline{a}^{j-m-1} + \ldots$$

$$+ q^{-\frac{(j+m)(j-m)}{4}+\frac{\ell}{2}} f(a, \overline{a}) + \ldots + q^{\frac{(j-m)(j+m)}{4}} \overline{a}^{j-m} a^{j+m}. \tag{6.16}$$

Here ℓ is the least number of transpositions needed to put $f(a, \overline{a})$ in normal-ordered form.

Example: $\quad [4]^{\frac{1}{2}} Q_2^1 = q^{-\frac{3}{4}} a^3\overline{a} + q^{-\frac{1}{4}} a^2\overline{a}a + q^{\frac{1}{4}} a\overline{a}a^2 + q^{\frac{3}{4}} \overline{a}a^3. \tag{6.17}$

As is clear from our review (of the usual symplectons), the major task is to prove a product law for the deformed q-eigenpolynomials, Q_j^m.

THEOREM [23]: *Let $Q_{j'}^{m'}$ and $Q_{j''}^{m''}$ be normalized q-eigenpolynomials. Then:*

$$Q_{j'}^{m'}(a, \overline{a}) Q_{j''}^{m''}(a, \overline{a}) = \sum_j N(j'j''j) \cdot {}_qC_{m'm''m}^{j'j''j} \cdot Q_j^m(a, \overline{a}), \tag{6.18}$$

where: ${}_qC_{m'm''m}^{j'j''j}$ is the *q-Wigner-Clebsch-Gordan* coefficient, and $N(j'j''j)$ is a *scalar function of q dependent only on j', j'', j*.

$N(j'j''j)$ obeys the recursion relation:

$$([2j''][2j+1])^{\frac{1}{2}}N(j'j''j) = ([j'-j''+j+1]_q[j'+j''-j]_q)^{\frac{1}{2}}$$
$$\times N\left(j',j''-\tfrac{1}{2},j+\tfrac{1}{2}\right)N\left(j+\tfrac{1}{2},\tfrac{1}{2},j\right)$$
$$+([j'+j''+j+1]_q[-j'+j''+j]_q)^{\frac{1}{2}}$$
$$\times N\left(j',j''-\tfrac{1}{2},j-\tfrac{1}{2}\right). \tag{6.19}$$

The determination of the coefficient $N(j'j''j)$ is very difficult. It helps to see a few special cases. We find:

$$N(j\ 0\ j) = 1, \tag{6.20}$$

$$N(j_1,j_2,j_1+j_2) = 1 \tag{6.21}$$

$$N\left(j,\tfrac{1}{2},j-\tfrac{1}{2}\right) = \frac{-q^{-\frac{1}{4}}F(2j)}{([2j][2j+1])^{\frac{1}{2}}}, \tag{6.22}$$

$$\text{with:}\quad F(n) \equiv [1]+[2]+\ldots+[n], \quad F(0) \equiv 0. \tag{6.23}$$

We remark that the appearance of the function $F(n)$ is characteristic of relations involving the q-symplecton [23].

One can prove the further property, at this stage, that the function $N(j',j'',j)$ is symmetric in the first two indices. One of the surprising properties [23] is that the (q-rotationally invariant) function $N(j',j'',j)$ is *not* symmetric under $q \to q^{-1}$.

These results show that $N(j',j'',j)$ is not the proper q-analog to the triangle function $\Delta(a,b,c)$, despite the fact that the q-symplecton product law seemingly appears to define $N(j',j'',j)$ in the proper form. It has been shown in Ref. [22], that the proper way to proceed is via the definition:

$$\Delta_q(abc) \equiv (-1)^{a+b+c}N(abc)q^{\frac{(a+b-c)}{4}}\sqrt{\frac{F(2c)![2a+1]![2b+1]!}{F(2a)!F(2b)![2c]!}}. \tag{6.24}$$

This q-triangle coefficient has the desired symmetry. As shown in Ref. [22], $\Delta_q(j_1j_2j_3)$ *is totally symmetric in its arguments* j_1,j_2,j_3—precisely the same property possessed by the (undeformed) triangle coefficient $\Delta(j_1j_2j_3)$ in (6.7c).

Moreover, it is now possible [22] to obtain the proper q-analog of (6.8):

$$\Delta_q(acf)\Delta(bdf) = [2f+1]\sum_e \Delta_q(abe)\Delta_q(cde)W_q(abcd;ef). \tag{6.25}$$

Let us conclude by citing the product law for q-eigenpolynomials in the proper form now to show the desired q-analog structure [22]:

$$Q_a^\alpha Q_b^\beta = \sum_{c=|a-b|}^{a+b} [2c+1]^{-\frac{1}{2}}\Delta_q(abc)(ba\beta\alpha|c\alpha+\beta)_q Q_c^{\alpha+\beta}. \tag{6.26}$$

Note the surprising appearance of the q-WCG coefficient involving q^{-1} as the proper form to show the analogy.

Space is lacking for more than this brief survey of the q-symplecton and the associated subtleties of q-analysis. More detail can be found in [22], and related discussions—from the aspect of Weyl-ordered boson polynomials—is given in [25] and [26].

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WHICH Q-ANALOGUE OF THE SQUEEZED OSCILLATOR?

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1 Introduction and Content

The noise (variance squared) of a component of the electromagnetic field - considered as a quantum oscillator -in the vacuum is equal to one half, in appropriate units (taking Planck's constant and the mass and frequency of the oscillator all equal to 1). A practical definition of a squeezed state is one for which the noise is less than the vacuum value - and the amount of squeezing is determined by the appropriate ratio. Thus the usual coherent (Glauber) states are not squeezed, as they produce the same variance as the vacuum. However, it is not difficult to define states analogous to coherent states which do have this noise-reducing effect. In fact, they are coherent states in the more general group sense but with respect to groups other than the Heisenberg-Weyl Group which defines the Glauber states. The original, conventional squeezed state in quantum optics is that associated with the group $SU(1,1)$. Just as the annihilation operator a of a single photon mode (and its hermitian conjugate $a^\dagger$, the creation operator) generates the Heisenberg-Weyl algebra, so the pair-photon operator a^2 and its conjugate generates the algebra of the group $SU(1,1)$. Another viewpoint, more productive from the calculational stance, is to note that the automorphism group of the Heisenberg-Weyl algebra is $SU(1,1)$. Needless to say, each of these viewpoints generalizes differently to the quantum group context. In this talk we shall discuss both. The structure of the talk is as follows:

- Conventional Coherent and Squeezed States

- Eigenstate Definitions

- Exponential Definitions

- Algebra (Group) Definitions

- Automorphism Group Definition

- Example: Signal-to-Noise Ratio

- q-Coherent and q-Squeezed States

- M and P q-bosons

- Eigenstate Definitions

- Exponential Definitions

- Algebra (q-Group) Definitions

- Example: Signal-to-Noise Ratio

- Automorphism q-Group Definition

2   Conventional Coherent and Squeezed States

The elementary treatment of (a single frequency) of the quantized electromagnetic field leads to the identification of its components

$$E \sim x, \quad B \sim p, \quad [x, p] = i$$

in suitably chosen units. We may introduce boson operators $b, b^\dagger$ by

$$x = (b + b^\dagger)/\sqrt{2} \quad p = (b - b^\dagger)/\sqrt{2}i$$

which then satisfy the Heisenberg-Weyl Algebra

$$
\begin{aligned}
[b, b^\dagger] &= 1 \\
\left[N, b^\dagger\right] &= b^\dagger
\end{aligned}
\tag{1}
$$

where $N = b^\dagger b$. The interpretation of these operators is that they annihilate (*resp.* create) photons; the *vacuum state* $|0>$ satisfies

$$b|0> = 0.$$

The quantum noise of the x-component (E-field) in the vacuum state is given by

$$(\Delta x)^2 \equiv <x^2> - <x>^2 \ = 1/2$$

with a similar result for the B-component. The vacuum signal ($<x>^2$) vanishes.

   The conventional *coherent* states (Glauber [1] states) are defined as eigenstates of the operator b,

$$b|\lambda> = \lambda|\lambda> . \tag{2}$$

For these states one readily evaluates

$$(\Delta x)^2 = 1/2 \quad <x>^2 = (\lambda + \bar{\lambda})^2/2.$$

An alternative, suggestive definition of the coherent states which readily lends itself to generalization, is that they are obtained by the action of the realizations of the group corresponding to the Heisenberg-Weyl Algebra generated by $\{b, b^\dagger, 1\}$ on the vacuum, thus;

$$|\lambda> = \exp(\lambda b^\dagger)|0> . \tag{3}$$

It is an important practical problem to maximize the signal-to-noise ratio ρ for radiation states; here we are of course only considering the quantum noise. What we see from the preceding is that ρ vanishes for the vacuum; it attains the value $4N_s$ for a coherent state, taking (real) $\lambda^2 = N_s$ where N_s is the number of photons in the signal.

In a classic paper, Yuen [2] showed that for any radiation field the maximum signal-to-quantum noise ratio ρ for fixed energy has the value $\rho_{max} = 4N_s(N_s + 1)$, where N_s gives the upper limit on the number of photons in the signal (effectively a maximum power per unit frequency constraint). He further showed that this value is attained by the *squeezed* states [3], two-photon coherent light states generated as eigenstates of the operator $\mu b + \nu b^\dagger$ where $|\mu|^2 - |\nu|^2 = 1$. The only mathematical input to this result consists of the canonical commutation relations Equation (1). The term "squeezed" derives from the fact that in these states the quantum dispersion may attain values below the vacuum (or coherent) state value of $1/2$. Such states have been produced experimentally. These squeezed states may also be defined by the action on the vacuum (more generally, on Glauber coherent states) of the group corresponding to the algebra generated by

$$\{b^2, (b^\dagger)^2, (bb^\dagger + b^\dagger b)\}. \tag{4}$$

Thus a typical squeezed state (up to normalization) may be written

$$|\xi, z> = \exp(\frac{1}{2}\xi(b^\dagger)^2)\exp(zb^\dagger)|0>. \tag{5}$$

The state $|\xi, z>$ is an eigenstate of $(b - \xi b^\dagger)$ with eigenvalue z, in agreement with the definition of squeezed state above ($\mu = 1, \nu = -\xi$ and for convergence we require that $|\xi| \leq 1$.) The operators in Equation (4) satisfy the commutation relations of $SU(1,1)$

$$
\begin{aligned}
[K_+, K_-] &= -2K_0 \\
[K_0, K_\pm] &= \pm K_\pm.
\end{aligned}
\tag{6}
$$

An alternative definition which results in states exhibiting squeezing is to define them as (normalized) eigenstates of the of the lowering operator $K_- \equiv b^2$. These states have the form

$$
\begin{aligned}
|\xi_1> &= \sum_{i=0}^{\infty} \frac{\xi_1^i}{\sqrt{2i!}}|2i> \\
|\xi_2> &= \sum_{i=0}^{\infty} \frac{\xi_2^i}{\sqrt{(2i+1)!}}|2i+1>.
\end{aligned}
\tag{7}
$$

An appropriate sum of these squeezed states recovers a Glauber coherent state.

A more basic definition of squeezed states arises from the observation that the automorphism group of the H-W algebra is $SU(1,1)$; thus a unitary transformation U on b gives

$$b \mapsto U b U^\dagger = \mu b + \nu b^\dagger \tag{8}$$

where $|\mu|^2 - |\nu|^2 = 1$. The conventional squeezed state is then defined, exactly as above, as an eigenstate of the transformed bose destruction operator

$$(\mu b + \nu b^\dagger)|\xi> = \xi|\xi>. \tag{9}$$

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More generally, a conventional squeezed state is defined as the action of the unitary operator $U(\mu, \nu)$ on a coherent state $|z> = D(z)|0>$, thus:

$$|\xi> = U(\mu, \nu)D(z)|0> = U(\mu, \nu)|z> \qquad (10)$$

This definition is not only elegant but, by applying the inverse transformation, enables calculations in squeezed states to be made as readily as in the coherent states.

For example; using

$$U(\mu, \nu)^{-1} \, b \, U(\mu, \nu) = \bar{\mu} b - \nu b^\dagger$$

one may readily evaluate the dispersion of x in the squeezed state $|\xi>$ to be

$$(\Delta x)^2 = \frac{1}{2}|\mu - \nu|^2$$

and the signal to be

$$<x>^2 = \{(\mu - \nu)\bar{z} + \overline{(\mu - \nu)z)}\}^2.$$

For real values of the parameters, the maximum of the signal-to-quantum noise ratio

$$\rho \equiv <x>^2 /(\Delta x)^2 = 4z^2$$

may readily be seen to be attained at $\rho_{max} = 4N_s(N_s + 1)$ as cited above [2].

3 q-Coherent and q-Squeezed States

A deformation a_M of the standard boson operator b was introduced some years ago by Arik and Coon [4]. Their deformed bosons satisfy

$$a_M a_M{}^\dagger - q a_M{}^\dagger a_M = 1. \qquad (11)$$

More recently, the deformed q-boson operator a_P satisfying the Quantum Heisenberg-Weyl Algebra ($H - W_q$ Algebra)

$$\begin{aligned} a_P a_P{}^\dagger - q a_P{}^\dagger a_P &= q^{-N} \\ \left[N, a_P{}^\dagger\right] &= a_P{}^\dagger \end{aligned} \qquad (12)$$

has been introduced [5, 6]. (I have used the subscript M to denote the relation to the mathematician's classical q-analysis, a study which goes back at least as far as Gauss, in contrast to the more recently introduced physicist's form, subscript P. The second equation of (12) is satisfied by both forms. There is no need to subscript the operator N for the reason given below.)

In principle, either Equation (2) or Equation (3) can be used as a starting point for an eigenstate definition of q-coherent states for both types of deformed bosons. It is easily shown that an attempt to use Equation (3) does not lead to a normalizable state (for $q \neq 1$) in either case. Starting from Equation (2), q-coherent states for the deformed boson operator of Arik and Coon were constructed

by these authors [4]; the same equation was used [6] for the q-bosons defined in Equation (12). Both forms of q-boson lead to the q-coherent state

$$|\beta>_q = \mathcal{N}^{-1}\exp_q(\beta a^\dagger)|0>$$ (13)

where $a = a_M$ or a_P and

$$\mathcal{N}^2 = \exp_q(|\beta|^2).$$ (14)

The q-exponential is defined in both cases by

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^r}{[r]_q!}.$$ (15)

The symbol $[r]_q!$ is defined by $[r]_q! = [r]_q[r-1]_q[r-2]_q \cdots [1]_q$ where, in the case of Equation (11), we define

$$[x]_q = (q^x - 1)/(q - 1)$$ (16)

and in the case of Equation (12), we define

$$[x]_q = (q^x - q^{-x})/(q - q^{-1}).$$ (17)

Equation (11) gives rise to the classical form of the q-exponential usually written as $E_q(x)$, which converges for $|q| > 1$, or for $|x| < |\frac{1}{1-q}|$ when $|q| < 1$. The form of q-exponential corresponding to Equation (12) is convergent for all x and q. In both cases, $\lim_{q \to 1} \exp_q(x) = \exp(x)$, and when $q = 1$ the q-boson operators reduce to standard boson operators.

The q-bosons are related to the conventional bosons b as follows:

$$a = b\sqrt{\frac{[N]_q}{N}}$$ (18)

where $N = b^\dagger b$, using the appropriate definition of $[N]_q$ for "mathematical" bosons Equation (16) [7] or "physical" bosons Equation (17) [8].

The q-coherent states defined above do not give rise to (time-independent) squeezing, just as in the case of the conventional coherent states. In fact, it may be shown [7] that the term which gives rise to squeezing is, in general,

$$<a^2> - <a>^2$$

which is zero for eigenstates of a. However, Buzek [9] has shown that there is time-dependent squeezing, by choice of a suitable analogue of the usual Hamiltonian; and this has also been found by Celeghini, Rasetti and Vitiello [10].

It is not immediately clear how conventional squeezed states can be generalized to the quantum group context. The most direct approach is to use a q-boson realization of the analogous $su_q(1,1)$ algebra; one may then attempt to define the analogous q-squeezed states by the exponential action

of the resulting operators on the vacuum (or on the q-coherent states). A realization of $su_q(1,1)$ given by Kulish and Damaskinsky [8] is

$$K_+ = p(a^\dagger)^2 \quad K_- = p(a)^2 \quad K_0 = \frac{1}{2}(n + \frac{1}{2}) \tag{19}$$

with $p = (q + q^{-1})^{-1}$ and $[K_+, K_-] = -[2K_0]_{q^2}$. However, the exponential action of the operators of this algebra fail to give a normalizable state not only for the conventional exponential (which was to be expected) but also for the q-exponential $\exp_q(x)$ defined above (and also for $\exp_{q^2}(x)$) which one would have thought to be the appropriate function here.

The eigenstates of K_- corresponding to Equations (7), obtained by substituting the "box" factorials for the conventional ones, give normalizable states [11].

We may alternatively carry over the definition

$$(a - \xi a^\dagger)|\xi, z> = z|\xi, z> \tag{20}$$

to the q-boson case. For the choice $z = 0$ we obtain

$$|\xi, z> = \mathcal{N}^{-1} \sum_{i=0}^\infty \xi^i \sqrt{\frac{[2i-1]_q!!}{[2i]_q!!}} |2i> \tag{21}$$

with normalization

$$\mathcal{N}^2 = \sum_{i=0}^\infty |\xi|^{2i} \frac{[2i-1]_q!!}{[2i]_q!!}. \tag{22}$$

The symbol $[r]_q!!$ has the expected meaning $[r]_q!! = [r]_q[r-2]_q[r-4]_q \cdots$ and the first term in (22) is 1. The squeezing properties of states defined in this way, for various values of the parameters ξ and q were calculated in [11].

A more basic definition of squeezed states in the quantum group case arises from generalizing the automorphism group property given in the previous section for the conventional case, Equation (8) and Equation (9). One may seek by analogy to define q-squeezed states in terms of the automorphism quantum group of the quantum Heisenberg-Weyl algebra of q-bosons. Consider the quantum plane *á la Manin* generated by two elements α and γ as defined by Woronowicz [12], satisfying the following commutation relations:

$$\begin{aligned}
\alpha\gamma &= \mu\gamma\alpha \\
\alpha\gamma^* &= \mu\gamma^*\alpha \\
\gamma\gamma^* &= \gamma^*\gamma \\
\alpha^*\alpha - \gamma^*\gamma &= 1 \\
\alpha\alpha^* - \mu^2\gamma^*\gamma &= 1.
\end{aligned} \tag{23}$$

We now introduce a conjugation $A \mapsto \tilde{A}$ defined by its effect on

1. c-numbers $\quad c \mapsto c^*$, (complex conjugation)

2. q-numbers (quantum plane) $\tilde{\alpha} = \alpha^* \quad \tilde{\alpha^*} = \alpha \quad \tilde{\gamma} = \mu\gamma^* \quad \tilde{\gamma^*} = \frac{1}{\mu}\gamma$

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3. operators $\tilde{A} = q^{\frac{1}{2}(N^2-N)} A^\dagger q^{-\frac{1}{2}(N^2-N)}$ (q real).

Under this transformation, $\tilde{\tilde{A}} = A$, $\hat{A}B = \check{B}\hat{A}$;   and the boson a satisfying

$$aa^\dagger - qa^\dagger a = q^{-N}$$

maps to $\tilde{a}$, with the pair $a, \tilde{a}$ satisfying

$$a\tilde{a} - \mu\tilde{a}a = 1. \tag{24}$$

with $\mu = q^2$. The two-dimensional fundamental representation of $SU_\mu(1,1)$ is given by

$$u = \begin{bmatrix} \alpha & \mu\gamma^\star \\ \gamma & \alpha^\star \end{bmatrix} \tag{25}$$

and u satisfies $uJ\hat{u} = J$ where

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -\mu \end{bmatrix}$$

The transformation

$$[a, \tilde{a}] \mapsto [a, \tilde{a}]u \tag{26}$$

is an automorphism which preserves Equation (24). Squeezed states in the quantum group context may now be defined as the eigenstates of the transformed a, thus generalizing the results of [11].

Finally, we note that one may derive an analogue of Yuen's result [2] cited above on the optimal signal-to-Quantum Noise ratio q-photons [13]; the corresponding bound for q-photons may be shown to be

$$\rho_q = 4[N_s]_q[N_s + 1]_q/([N_s + 1]_q - [N_s]_q)^2. \tag{27}$$

that is, for a radiation field in terms of photons satisfying the modified commutation relations of the quantum group version of the Heisenberg-Weyl Algebra. This ratio is always less than the value in the conventional case, attained for the $SU(1,1)$ squeezed states.

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DEFORMATION OF SUPERSYMMETRIC AND CONFORMAL QUANTUM MECHANICS THROUGH AFFINE TRANSFORMATIONS

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Abstract

Affine transformations (dilatations and translations) are used to define a deformation of one-dimensional $N = 2$ supersymmetric quantum mechanics. Resulting physical systems do not have conserved charges and degeneracies in the spectra. Instead, superpartner Hamiltonians are q-isospectral, i.e. the spectrum of one can be obtained from another (with possible exception of the lowest level) by q^2-factor scaling. This construction allows easily to rederive a special self-similar potential found by Shabat and to show that for the latter a q-deformed harmonic oscillator algebra of Biedenharn and Macfarlane serves as the spectrum generating algebra. A general class of potentials related to the quantum conformal algebra $su_q(1,1)$ is described. Further possibilities for q-deformation of known solvable potentials are outlined.

1. Introduction

Standard Lie theory is known to provide very useful tools for description of physical systems. Elegant applications were found in quantum mechanics within the concept of spectrum generating, or, dynamical (super)symmetry algebras [1]. The most famous example is given by the harmonic oscillator problem (so the name of this workshop) where spectrum is generated by the Heisenberg-Weyl algebra. Some time ago a wide attention was drawn to the deformations of Lie algebras which nowdays are loosely called "quantum algebras", or, "quantum groups" [2] (below we do not use the second term because Hopf algebra structure is not relevant in the present context). Spin-chain models were found [3] where Hamiltonian commutes with generators of the quantum algebra $su_q(2)$, deformation parameter q being related to a coupling constant. Thus, an equivalence of a particular perturbation of the interaction between "particles" to the deformation of symmetry algebra governing the dynamics was demonstrated.

Biedenharn and Macfarlane introduced q-deformed harmonic oscillator as a building block of the quantum algebras [4, 5]. Various applications of q-oscillators appeared since that time [6-13] (an overview of the algebraic aspects of q-analysis is given in Ref.[7]). Physical models refering to q-oscillators can be conditionally divided into three classes. The first one is related to systems on lattices [8]. In the second class dynamical quantities are defined on "quantum planes" – the spaces

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with non-commutative coordinates [9]. Although Schrödinger equation in this approach looks similar to the standard one, all suggested explicit realizations of it in terms of the normal calculus result in purely finite-difference equations. Parameter q responsible for the non-commutativity of quantum space coordinates serves as some non-local scale on the continuous manifolds and, therefore, the basic physical principles are drastically changed in this type of deformation. We shall not pursue here the routes of these two groups of models.

The third – dynamical symmetry realization class – is purely phenomenological: one deforms already known spectra by postulating the form of a Hamiltonian as some combination of formal quantum algebra generators [10], or, as an anticommutator of q-oscillator creation and annihilation operators [4, 8]. This application, in fact, does not have straightforward physical meaning because of the non-uniqueness of deformation procedure. Even exact knowledge of a spectrum is not enough for precise reconstruction of an interaction. For a given potential with some number of bound states one can associate another potential containing new parameters and exhibiting the same spectrum [14]. Therefore the physics behind such deformations is not completely fixed. Moreover, for a rich class of spectral problems there are powerful restrictions on the asymptotic growth of discrete eigenvalues [15] so that not any ordered set of numbers can represent a spectrum. All this means that one should more rigorously define physical interaction responsible for a prescribed deformation of a given simple spectrum. q-Analogs of the harmonic oscillators were also used for the description of small violation of statistics of identical particles [13] (general idea on the treatment of this problem on the basis of a parametric deformation of commutation relations was suggested in Ref.[16]). The papers listed above represent only a small fraction of works devoted to quantum algebras and q-analysis. For an account of unmentioned here applications we refer to reviews [17, 18].

Recently Shabat have found one-dimensional reflectionless potential showing peculiar self-similar behavior and describing an infinite number soliton system [19]. Following this development the author proposed [20] to take known exactly solvable Schrödinger potentials and try to deform their shape in such a way that the problem remains to be exactly solvable but the spectrum acquires complicated functional character. So, the Shabat's potential was identified in Ref.[20] as a q-deformation of conformally invariant harmonic and particular forms of Rosen-Morse and Pöschl-Teller potentials. The hidden q-deformed Heisenberg-Weyl algebra was found to be responsible for purely exponential character of the spectrum. In comparison with the discussed above third group of models present approach to "quantum" symmetries is the direct one – physical interaction is fixed first and the question on quantum algebra behind prescribed rule of q-deformation is secondary.

In accordance with this guiding principle a deformation of supersymmetric (SUSY) quantum mechanics [21, 22] was proposed in Ref.[23]. This talk is devoted to description of the results of Refs.[19, 20, 23] and subsequent developments. We start by giving in Sect.2 a brief account of the properties of simplest $(0 + 1)$-dimensional SUSY models. In Sect.3 we describe a deformation of these models on the basis of pure scaling transformation of a superpartner potential, namely, we find q-SUSY algebra following from this rule and analyze its properties. Sect.4 outlines possible extensions of the simplest potential deformation. In Sect.5 we show that mentioned above self-similar potential naturally appears within q-SUSY as that characterized by the simplest structure of Hamiltonian. In this case factorization operators entering the supercharges are well defined on the Hilbert space of square integrable functions and generate q-oscillator algebra. As a result,

a representation of q-deformed conformal algebra $su_q(1,1)$ is obtained. In Sect.6 we give short description of further generalizations of the Shabat's potential which correspond to general q-deformed conformal quantum mechanics and q-deformation of (hyper)elliptic potentials. Sect.7 contains some conclusions. We would like to stress once more that suggested realizations of q-algebras are continuous (i.e. they are not purely finite-difference ones) and they are used within the standard physical concepts.

2. SUSY quantum mechanics

The simplest $N = 2$ SUSY quantum mechanics is fixed by the following algebraic relations between the Hamiltonian of a system H and supercharges $Q^\dagger$, Q [21]

$$\{Q^\dagger, Q\} = H, \quad Q^2 = (Q^\dagger)^2 = 0, \quad [H, Q] = [H, Q^\dagger] = 0. \tag{1}$$

All operators are supposed to be well defined on the relevant Hilbert space. Then, independently on explicit realizations the spectrum is two-fold degenerate and the ground state energy is semipositive, $E_{vac} \geq 0$.

Let us consider a particle moving in one-dimensional space. Below, the coordinate x is tacitly assumed to cover the whole line, $x \in R$, if it is not explicitly stated that it belongs to some cut. Standard representation of the algebra (1) contains one free superpotential $W(x)$ [22]:

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad A = (p - iW(x))/\sqrt{2}, \quad [x, p] = i, \tag{2}$$

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & A A^\dagger \end{pmatrix} = \tfrac{1}{2}(p^2 + W^2(x) - W'(x)\sigma_3), \tag{3}$$

$$W'(x) \equiv \frac{d}{dx} W(x), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It describes a particle with two-dimensional internal space the basis vectors of which can be identified with the spin "up" and "down" states.

The subhamiltonians $H_\pm$ are isospectral as a result of the intertwining relations

$$H_- A^\dagger = A^\dagger H_+, \quad A H_- = H_+ A. \tag{4}$$

The only possible difference concerns the lowest level. Note that the choice $W(x) = x$ corresponds to the harmonic oscillator problem and then $A^\dagger$, A coincide with the bosonic creation and annihilation operators $a^\dagger$, a which satisfy the algebra

$$[a, a^\dagger] = 1, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \tag{5}$$

where N is the number operator, $N = a^\dagger a$. This, and another particular choice, $W(x) = \lambda/x$, correspond to the conformally invariant dynamics [24].

3. q-Deformed SUSY quantum mechanics

Now we shall introduce the tools needed for the quantum algebraic deformation of the above construction. Let T_q be smooth q-scaling operator defined on the continuous functions

$$T_q f(x) = f(qx), \tag{6}$$

where q is a real non-negative parameter. Evident properties of this operator are listed below

$$T_q f(x) g(x) = [T_q f(x)][T_q g(x)], \qquad T_q \frac{d}{dx} = q^{-1} \frac{d}{dx} T_q,$$

$$T_q T_p = T_{qp}, \qquad T_q^{-1} = T_{q^{-1}}, \qquad T_1 = 1. \tag{7}$$

On the Hilbert space of square integrable functions $\mathcal{L}_2$ one has

$$\int_{-\infty}^{\infty} \phi^*(x) \psi(qx) dx = q^{-1} \int_{-\infty}^{\infty} \phi^*(q^{-1}x) \psi(x) dx, \tag{8}$$

where from the hermitian conjugate of T_q can be found

$$T_q^\dagger = q^{-1} T_q^{-1}, \qquad (T_q^\dagger)^\dagger = T_q. \tag{9}$$

As a result, $\sqrt{q}\, T_q$ is a unitary operator. Because we take wave functions to be infinitely differentiable, an explicit realization of T_q is provided by the operator

$$T_q = e^{\ln q\, x\, d/dx} = q^{x\, d/dx}. \tag{10}$$

Expanding (10) into the formal series and using integration by parts one can prove relations (9) on the infinite line and semiline $[0, \infty]$. A special care should be taken for finite cut considerations since T_q moves boundary point(s).

Let us define the q-deformed factorization operators

$$A^\dagger = \frac{1}{\sqrt{2}} (p + iW(x)) T_q, \qquad A = \frac{q^{-1}}{\sqrt{2}} T_q^{-1} (p - iW(x)), \tag{11}$$

where $W(x)$ is arbitrary function and for convinience we use the same notations as in the undeformed case (3). A and $A^\dagger$ are hermitian conjugates of each other on $\mathcal{L}_2$. Now one has

$$A^\dagger A = \tfrac{1}{2} q^{-1} (p^2 + W^2(x) - W'(x)) \equiv q^{-1} H_-, \tag{12}$$

$$\begin{aligned} A A^\dagger &= \tfrac{1}{2} q^{-1} T_q^{-1} (p^2 + W^2(x) + W'(x)) T_q \\ &= \tfrac{1}{2} q (p^2 + q^{-2} W^2(q^{-1}x) + q^{-1} W'(q^{-1}x)) \equiv q H_+. \end{aligned} \tag{13}$$

We define q-deformed SUSY Hamiltonian and supercharges to be

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} q A^\dagger A & 0 \\ 0 & q^{-1} A A^\dagger \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \tag{14}$$

These operators satisfy the following q-deformed version of the $N = 2$ SUSY algebra

$$\{Q^\dagger, Q\}_q = H, \quad \{Q, Q\}_q = \{Q^\dagger, Q^\dagger\}_q = 0, \quad [H, Q]_q = [Q^\dagger, H]_q = 0, \tag{15}$$

where we introduced q-brackets

$$[X, Y]_q \equiv qXY - q^{-1}YX, \qquad [Y, X]_q = -[X, Y]_{q^{-1}}, \tag{16}$$

$$\{X, Y\}_q \equiv qXY + q^{-1}YX, \qquad \{Y, X\}_q = \{X, Y\}_{q^{-1}}. \tag{17}$$

Note that the supercharges are not conserved because they do not commute with the Hamiltonian (in this respect our algebra principally differs from the construction of Ref.[11]). An interesting property of the algebra (15) is that it shares with (1) the semipositiveness of the ground state energy which follows from the observation that $Q^\dagger$, Q and the operator $q^{-\sigma_3}H$ satisfy ordinary SUSY algebra (1). Evidently, in the limit $q \to 1$ one recovers conventional SUSY quantum mechanics.

For the subhamiltonians $H_\pm$ the intertwining relations look as follows

$$H_- A^\dagger = q^2 A^\dagger H_+, \qquad AH_- = q^2 H_+ A. \tag{18}$$

Hence, $H_\pm$ are not isospectral but rather q-isospectral, i.e. the spectrum of H_- can be obtained from the spectrum of H_+ just by the q^2-factor scaling:

$$H_+ \psi^{(+)} = E^{(+)}\psi^{(+)}, \qquad H_- \psi^{(-)} = E^{(-)}\psi^{(-)},$$

$$E^{(-)} = q^2 E^{(+)}, \qquad \psi^{(-)} \propto A^\dagger \psi^{(+)}, \quad \psi^{(+)} \propto A\psi^{(-)}. \tag{19}$$

Possible exception concerns only the lowest level in the same spirit as it was in the undeformed SUSY quantum mechanics. If $A^\dagger, A$ do not have zero modes then there is one-to-one correspondence between the spectra. We name this situation as a spontaneously broken q-SUSY because for it $E_{vac} > 0$. If A (or, $A^\dagger$) has zero mode then q-SUSY is exact, $E_{vac} = 0$, and H_+ (or, H_-) has one level less than its superpartner H_- (or, H_+).

As a simplest physical example let us consider the case $W(x) = qx$. The Hamiltonian takes the form

$$\begin{aligned} H &= \tfrac{1}{2}p^2 + \tfrac{1}{4}(q^2 + q^{-2})x^2 + \tfrac{1}{4}(q^{-1} - q) + \tfrac{1}{4}((q^2 - q^{-2})x^2 - q - q^{-1})\sigma_3 \\ &= \tfrac{1}{2}p^2 + \tfrac{1}{2}q^{2\sigma_3}x^2 - \tfrac{1}{2}q^{\sigma_3}\sigma_3, \end{aligned} \tag{20}$$

and describes a spin-1/2 particle in the harmonic potential and related magnetic field along the third axis. The physical meaning of the deformation parameter q is analogous to that in the XXZ-model [3] – it is a specific interaction constant in the standard physical sense. This model has exact q-SUSY and if q^2 is a rational number then the spectrum exhibits accidental degeneracies.

4. General deformation of superpartner Hamiltonians

Described above q-deformation of the SUSY quantum mechanics is by no means unique. If one chooses in the formulas (11) T_q to be not q-scaling operator but, instead, the shift operator

$$T_q f(x) = f(x + q), \qquad T_q = e^{q\, d/dx}, \tag{21}$$

then SUSY algebra will not be deformed at all. The superpartner Hamiltonians will be isospectral and the presence of T_q-operator results in the very simple deformation of old superpartner potential $U_+(x) \to U_+(x - q)$ (kinetic term is invariant). Evidently such deformation does not change the spectrum of $U_+(x)$ and that is why SUSY algebra remains intact. Nevertheless it creates new physically relevant SUSY quantum mechanical models. The crucial point in generating of them was the implication of essentially infinite order differential operators as the intertwining operators.

A more general T_q is given by the shift operator in arbitrary coordinate system

$$T_q f(z(x)) = f(z(x) + q), \qquad T_q = e^{q\, d/dz(x)}, \quad \frac{d}{dz} = \frac{1}{z'(x)} \frac{d}{dx}. \tag{22}$$

The effects of choices $z = \ln x$ and $z = x$ were already discussed above. In general, operator T_q will not preserve the form of kinetic term in H_+-Hamiltonian. Physically, such change would correspond to the transition from motion of a particle on flat space to the curved space dynamics. Below we shall assume the definition (6) but full affine transformation on the line

$$T_q f(x) = f(qx + a)$$

may be used in all formulas without changes.

An interesting question is whether inversion transformation can be joined to the affine part so that a complete $SL(2)$ group element $z \to (az + b)/(cz + d)$ will enter the formalism in a meaningful way? Application of the described construction to the higher dimensional problems is not so straightforward. If variables separate (spherically symmetric or other special potentials) then it may work in a parallel with the non-deformed models. In the many-body case one can perform independent affine transformations for each of the superselected by fermionic number subhamiltonians and thus to "deform" these SUSY models as well.

5. q-Deformed conformal quantum mechanics

Particular form of the $su(1,1)$ algebra generators can be realized via the harmonic oscillator creation and annihilation operators (5)

$$K_+ = \tfrac{1}{2}(a^\dagger)^2, \qquad K_- = \tfrac{1}{2}a^2, \qquad K_0 = \tfrac{1}{2}(N + \tfrac{1}{2}), \tag{23}$$

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_0. \tag{24}$$

This means that harmonic potential has $su(1,1)$ as the dynamical symmetry algebra, physical states being split into two irreducible representaions according to their parity. Let us show that the potential introduced in Ref.[19] obeys the quantum conformal symmetry algebra $su_q(1,1)$ in complete parallel with (23),(24).

First, we shall rederive this potential within q-SUSY physical situation. Let us consider the Hamiltonian of a spin-1/2 particle in an external potential $\frac{1}{2}U(x)$ and a magnetic field $\frac{1}{2}B(x)$ along the third axis

$$H = \frac{1}{2}(p^2 + U(x) + B(x)\sigma_3) \tag{25}$$

and impose two conditions: we take magnetic field to be homogeneous

$$B = -\beta^2 q^{-2} = constant \tag{26}$$

and require the presence of q-SUSY (15). Equating (25) and (14) we arrive at the potential

$$U(x) = W^2(x) - W'(x) + \beta^2 q^{-2}, \tag{27}$$

where $W(x)$ satisfies the following mixed finite-difference and differential equation

$$W'(x) + qW'(qx) + W^2(x) - q^2 W^2(qx) = 2\beta^2. \tag{28}$$

This is the condition of a self-similarity [19] which bootstraps the potential in different points (in Ref.[20] $\beta^2 = \gamma^2(1 + q^2)/2$ parametrization was used). Smooth solution of (28) for symmetric potentials $U(-x) = U(x)$ is given by the following power series

$$W(x) = \sum_{i=1}^{\infty} c_i x^{2i-1}, \qquad c_i = \frac{q^{2i} - 1}{q^{2i} + 1} \frac{1}{2i - 1} \sum_{m=1}^{i-1} c_{i-m} c_m, \qquad c_1 = \frac{2\beta^2}{1 + q^2}. \tag{29}$$

In different limits of the parameters several well known exactly solvable problems arise: 1) Rosen-Morse – at $q \to 0$; 2) Pöschl-Teller – at $\beta \propto q \to \infty$; 3) harmonic potential – at $q \to 1$; 4) $1/x^2$-potential – at $q \to 0$ and $\beta \to 0$. However, strictly speaking for all these limits to be valid one has to prove their smoothness, e.g., for 4) there may be solutions for which two limiting procedures do not commute, etc. Note also that for the case 2) the coordinate range should be restricted to finite cut because of the presence of singularities. Infinite soliton solution of Shabat corresponds to the range $0 < q < 1$ at fixed β. If $q \neq 0, 1, \infty$, there is no analytical expression for $W(x)$ but some general properties of this function may be found along the analysis of Ref.[19].

The spectrum can be derived by pure algebraic means. We already know that the spectra of $H_{\pm}$ subhamiltonians are related via the q^2-scaling

$$E_{n+1}^{(-)} = q^2 E_n^{(+)}, \tag{30}$$

where the number n numerates levels from below for both spectra. Because q-SUSY is exact in this model the lowest level of H_- corresponds to the first excited state of H_+. But due to the restriction (26) the spectra differ only by a constant,

$$E_n^{(-)} = E_n^{(+)} - \beta^2 q^{-2}, \tag{31}$$

Conditions (30) and (31) give us the spectrum of H

$$E_{n,m} = \beta^2 \frac{q^{-2m} - q^{2n}}{1 - q^2}, \qquad m = 0, 1; \ n = 0, 1, \ldots, \infty. \tag{32}$$

At $q < 1$ there are two finite accumulation points, i.e. (32) looks similar to two-band spectrum. At $q > 1$ energy eigenvalues seem to grow exponentially to the infinity but there is a catch which does not allow to identify (32) in this case with real physical spectrum. In Ref.[19] it was proven that for $0 < q < 1$ the superpotential is smooth and positive at $x = +\infty$. But then $\psi_0^{(-)}(x) = \exp(-\int^x W(y)dy)$ is a normalizable wave function defining the ground state of H_--subhamiltonian and all other states are generated from it without violation of the normalizability condition. Therefore relation (32) at $0 < q < 1$ defines real physical spectrum.

At $q > 1$ the series defining $W(x)$ converges only on a finite interval $|x| < r < \infty$. From inequalities

$$\rho^2 \equiv \frac{q^2 - 1}{q^2 + 1} < \frac{q^{2i} - 1}{q^{2i} + 1} < 1, \qquad i > 1$$

we have $0 < c_i^{(1)} < c_i < c_i^{(2)}$, where $c_i^{(1,2)}$ are defined by the rule (29) when q-factor on the right hand side is replaced by ρ^2 and 1 respectively ($c_1^{(1,2)} = c_1$). As a result, $1 < 2\sqrt{c_1}r/\pi < \rho^{-1}$, which means that $W(x)$ is smooth only on a cut at the ends of which it has some singularities. From the basic relation (28) it follows that these are simple poles with negative unit residues. In fact there should be an infinite number of simple "primary" and "secondary" poles. The former ones are characterized by negative unit residues and location points x_m tending to $\pi(m + 1/2)/\sqrt{c_1}$, $m \in Z$, at $q \to \infty$ (c_1 is fixed). "Secondary" poles are situated at $x = q^n x_m$, $n \in Z^+$, with corresponding residues defined by some algebraic equations. Unfortunately, general analytical structure of the function $W(x)$ is not known yet, presented above hypothesis needs rigorous proof with exact identification of all singularities and this is quite challenging problem.

On the other hand, existence of singularities in superpotential does not allow to take formal consequences of SUSY as granted. Namely, isospectrality (or, q-isospectrality) of H_+ and H_- for the whole line problem is broken at this point. Hence one is forced to consider Shrödinger operator (25) on a cut $[-r, r]$ with boundary conditions $\psi_n(\pm r) = 0$. Pole character of $W(x)$ singularities leads to $\psi_0^{(-)}(\pm r) = 0$, i.e. $\psi_0^{(-)}$ is true ground state of H_-. It also garantees that $U_-(x)$ is finite on the physical boundaries, $U_-(\pm r) < \infty$. Note, however, that the spectrum E_n for such type of problems can not grow faster than n^2 at $n \to \infty$ [15] in apparent contradiction with (32). This discrepancy is resolved by observation that action of T_q-operator creates singularities inside the interval $[-r, r]$ so that $U_+(x)$ and $q^2 U_+(qx)$ are not isospectral potentials (in ordinary sense) as it was at $q < 1$. Hence, the $q > 1$ case of (32) does not correspond to real physical spectrum of the model.

The number of deformations of a given function is not limited. The crucial property preserved by the presented above q-curling is the property of exact solvability of "undeformed" Rosen-Morse, harmonic oscillator, and Pöschl-Teller potentials. It is well known that potentials at infinitely small and exact zero values of a parameter may obey completely different spectra. In our case, deformation with $q < 1$ converts one-level Rosen-Morse problem into the infinite-level one with exponentially small energy eigenvalues. Whether one gets exactly solvable potential at $q > 1$ is an open question but this is quite plausible because at $q = \infty$ a problem with known spectrum arises.

Derivation of the dynamical symmetry algebra is not difficult. To find that we rewrite relations

(12), (13) for the superpotential (28)

$$A^\dagger A = q^{-1}H + \frac{\beta^2 q^{-1}}{1 - q^2}, \qquad A A^\dagger = qH + \frac{\beta^2 q^{-1}}{1 - q^2}, \tag{33}$$

where H is the Hamiltonian with purely exponential spectrum

$$H = \tfrac{1}{2}(p^2 + W^2(x) - W'(x)) - \frac{\beta^2}{1 - q^2}, \qquad E_n = -\frac{\beta^2}{1 - q^2} q^{2n}. \tag{34}$$

Evidently,

$$AA^\dagger - q^2 A^\dagger A = \beta^2 q^{-1}. \tag{35}$$

Normalization of the right hand side of (35) to unity results in the first relation entering the definition of q-deformed Heisenberg-Weyl algebra.

The shifted Hamiltonian (34) and $A^\dagger$, A operators satisfy braid-type commutation relations

$$[A^\dagger, H]_q = [H, A]_q = 0,$$

or,

$$H A^\dagger = q^2 A^\dagger H, \qquad A H = q^2 H A. \tag{36}$$

Energy eigenfunctions $|n\rangle$ can be uniquely determined from the ladder operators action

$$A^\dagger |n\rangle = \beta q^{-1/2} \sqrt{\frac{1 - q^{2(n+1)}}{1 - q^2}} |n + 1\rangle, \qquad A |n\rangle = \beta q^{-1/2} \sqrt{\frac{1 - q^{2n}}{1 - q^2}} |n - 1\rangle. \tag{37}$$

It is convinient to introduce the formal number operator

$$N = \frac{\ln[(q^2 - 1)H/\beta^2]}{\ln q^2}, \qquad N |n\rangle = n|n\rangle, \tag{38}$$

which is defined only on the eigenstates of the Hamiltonian. Now one can check that operators

$$a_q = \frac{q}{\beta} A q^{-N/2}, \qquad a_q^\dagger = \frac{q}{\beta} q^{-N/2} A^\dagger \tag{39}$$

satisfy original q-deformed harmonic oscillator algebra of Biedenharn and Macfarlane [4, 5]

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{-N}, \quad [N, a_q^\dagger] = a_q^\dagger, \quad [N, a_q] = -a_q. \tag{40}$$

The quantum conformal algebra $su_q(1,1)$ is realized as follows,

$$K^+ = \frac{1}{q + q^{-1}}(a_q^\dagger)^2, \qquad K^- = (K^+)^\dagger, \qquad K_0 = \tfrac{1}{2}(N + \tfrac{1}{2}),$$

$$[K_0, K^\pm] = \pm K^\pm, \qquad [K^+, K^-] = -\frac{q^{4K_0} - q^{-4K_0}}{q^2 - q^{-2}}. \tag{41}$$

Since $H \propto q^{4K_0}$, the dynamical symmetry algebra of the model is $su_q(1,1)$. Generators $K^{\pm}$ are parity invariant and therefore even and odd wave functions belong to different irreducible representations of this algebra. We conclude that quantum algebras have useful applications even within the continuous dynamics described by ordinary differential equations. A different approach to q-deformation of conformal quantum mechanics on the basis of pure finite difference realizations was suggested in Ref.[25].

Let us compare presented model with the construction of Ref.[26]. Kalnins, Levine, and Miller called as the conformal symmetry generator any differential operator $L(t)$ which maps solutions of the time-dependent Schrödinger equation to the solutions, i.e. which satisfies the relation

$$i\frac{\partial}{\partial t}L - [H, L] = R(i\frac{\partial}{\partial t} - H), \tag{42}$$

where R is some operator. On the shell of Schrödinger equation solutions $L(t)$ is conserved and all higher powers of space derivative, entering the definition of $L(t)$, can be replaced by the powers of $\partial/\partial t$ and linear in $\partial/\partial x$ term. But any analytical function of $\partial/\partial t$ is replaced by the function of energy when applied to stationary states. This trick allows to simulate any infinite order differential operator by the one linear in space derivative and to prove that a solution with energy E can always be mapped to the not-necessarily normalizable solution with the energy $E + f(E)$ where $f(E)$ is arbitrary analytical function. "On-shell" raising and lowering operators always can be found if one knows the basis solutions of the Schrödinger equation but sometimes it is easier to find symmetry generators and use them in search of the spectrum. In our construction we have "off-shell" symmetry generators, which map physical states onto each other and satisfy quantum algebraic relations in the rigorous operator sense. In this respect our results are complimentary to those of the Ref.[26].

It is clear that affine transformations provide a particular example of possible potential deformations leading just to scaling of spectra. In general one can try to find a map of a given potential with spectrum E_n to a particular related potential with the spectrum $f(E_n)$ for any analytical function $f(E)$. A problem of arbitrary non-linear deformation of Lie algebras was treated in Ref.[12] using the symbols of operators which were not well defined on proper Hilbert space. Certainly, the method of Ref.[26] should be helpful in the analysis of this interesting problem in a more rigorous fashion and the model presented above shows that sometimes an "off-shell" realization of symmetry generators can be found.

6.   Factorization method and new potentials

SUSY quantum mechanics is related to the factorization method of solving of Schrödinger equation [27-29]. Within the latter approach one has to find solutions of the following nonlinear chain of coupled differential equations for superpotentials $W_j(x)$

$$W_j' + W_{j+1}' + W_j^2 - W_{j+1}^2 = k_{j+1} \equiv \lambda_{j+1} - \lambda_j, \quad j = 0, 1, 2 \ldots \tag{43}$$

where k_j, λ_j are some constants. The Hamiltonians associated to (43) are

$$2H_j = p^2 + U_j(x) = p^2 + W_j^2(x) - W_j'(x) + \lambda_j, \tag{44}$$

$$U_0(x) = W_0^2 - W_0' + \lambda_0, \qquad U_{j+1}(x) = U_j(x) + 2W_j'(x),$$

where λ_0 is an arbitrary energy shift parameter.

SUSY Hamiltonians are obtained by unification of any two successive pairs H_j, H_{j+1} in a diagonal 2×2 matrix. Analogous construction for a piece of the chain (44) with more entries was called an order N parasupersymmetric quantum mechanics [30, 31]. In the latter case relations (43) naturally arise as the diagonality conditions of a general $(N + 1) \times (N + 1)$-dimensional parasupersymmetric Hamiltonian.

If $W_j(x)$'s do not have severe singularities then the spectra of two operators from (44) may differ only by a finite number of lowest levels. Under the additional condition that the functions

$$\psi_0^{(j)}(x) = e^{-\int^x W_j(v)dv} \tag{45}$$

are square normalizable one finds the spectrum

$$H_j \psi_n^{(j)}(x) = E_n^{(j)} \psi_n^{(0)}(x), \qquad E_n^{(j)} = \tfrac{1}{2}\lambda_{j+n}, \tag{46}$$

where subscript n numerates levels from below. In this case (45) represents ground state wave function of H_j from which one can determine lowest excited states of $H_{j'}$, $j' < j$,

$$\psi_n^{(j)}(x) \propto (p + iW_j)(p + iW_{j+1})\dots(p + iW_{j+n-1})\psi_0^{(j+n)}. \tag{47}$$

Any exactly solvable discrete spectrum problem can be represented in the form (43)-(47). Sometimes it is easier to solve Schrödinger equation by direct construction of the chain of associated Hamiltonians (44). If $U_0(x)$ has only N bound states then there does not exist $W_N(x)$ making $\psi_0^{(N)}$ normalizable. If $W_N(x) = 0$, then H_j ($j < N$) has exactly $N - j$ levels, the potential $U_j(x)$ is reflectionless and corresponds to $(N - j)$-soliton solution of the KdV-equation.

In order to solve evidently underdetermined system (43) one has to impose some closure conditions. At this stage it is an art of a researcher to find such an Ansatz which allows to generate infinite number of W_j and k_j from fewer entries. Most of old known examples are generated by the choice $W_j(x) = a(x)j + b(x) + c(x)/j$ where a, b, c are some functions determined from the recurrence relations [27, 28] (see also [19]). New look on the equations (43) was expressed in Ref.[32]. It was suggested to consider that chain as some infinite dimensional dynamical system and to analyze general constraints reducing it to the finite-dimensional integrable cases. In particular, it was shown that very simple periodic closure conditions

$$W_{j+N}(x) = W_j(x), \qquad \lambda_{j+N} = \lambda_j, \tag{48}$$

for N odd lead to all known hyperelliptic potentials describing finite-gap spectra (i.e. those with finite number of permitted bands). In this case parameters λ_j do not, of cause, coincide with the spectrum. The first non-trivial example appears at $N = 3$ and corresponds to Lame equation with one finite gap in the spectrum. Equivalently one can consider arising Schrödinger equation in the Weierstrass form (then periodic potential has singular points where wave functions are required to be equal to zero) and again parameters λ_j do not coincide with (purely discrete) spectrum. Note that in the analysis of parasupersymmetric models [30, 31] constants k_j were naturally treated as arbitrary parameters only occasionally giving the energy levels.

The self-similar potential of Sect.5 was found in Ref.[19] by the following Ansatz in the chain (43)

$$W_j(x) = q^j W(q^j x),$$ (49)

which gives a solution provided $W(x)$ satisfies the equation (28) and constants k_j are related to each other as follows

$$k_j \propto q^{2j}, \quad j \geq 0.$$ (50)

As it was already discussed, the parameters $\lambda_j \propto q^{2j}$ give the spectrum of problem at $0 < q < 1$ and therefore closure (49) seems to be completely different from (48). However, described above q-SUSY quantum mechanics and subsequent derivation of (49),(50) shows that in fact (49) is a q-deformation of the following closure condition

$$W_{j+1}(x) = W_j(x), \qquad k_{j+1} = k_j,$$ (51)

which leads to harmonic oscillator potential. Indeed, one may write

$$W_{j+1}(x) = q W_j(qx), \qquad k_{j+1} = q^2 k_j$$ (52)

and check that (49), (50) follow from these conditions.

As it was announced in Ref.[23] one can easily generalize deformation of SUSY quantum mechanical models to the parasupersymmetric ones. In the particular case defined by $(N + 1)$-member piece of the chain (44) one simply has to act on the successive Hamiltonians by different affine transformation group elements. This would lead to multiparameter deformation of the parasupersymmetric algebraic relations. Following the consideration of Ref.[30] one may impose analogous physical restrictions on the Hamiltonians and look for the explicit form of potentials accepting these constraints. Analyzing such possibilities the author have found the following general q-periodic closure of the chain (43)

$$W_{j+N}(x) = q W_j(qx), \qquad k_{j+N} = q^2 k_j.$$ (53)

These conditions describe q-deformation of the finite-gap and related potentials appearing at $q = 1$. Let us find a symmetry algebra behind (53) at $N = 2$.

First we write out explicitly the system of arising equations

$$W_1'(x) + W_2'(x) + W_1^2(x) - W_2^2(x) = 2\alpha,$$
$$W_2'(x) + q W_1'(qx) + W_2^2(x) - q^2 W_1^2(qx) = 2\beta.$$ (54)

One can check that the operators

$$K^+ = \tfrac{1}{2}(p + iW_1)(p + iW_2)\sqrt{q}T_q, \qquad K^- = (K^+)^\dagger$$ (55)

satisfy the relations

$$K^+ K^- = H(H - \alpha), \qquad K^- K^+ = (q^2 H + \beta)(q^2 H + \alpha + \beta),$$ (56)

$$H = \tfrac{1}{2}(p^2 + W_1^2(x) - W_1'(x)).$$

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The operator H obeys the following commutation relations with $K^{\pm}$

$$HK^+ - q^2K^+H = (\alpha + \beta)K^+, \qquad K^-H - q^2HK^- = (\alpha + \beta)K^-. \tag{57}$$

Note that by adding to H of some constant equations (57) may be rewritten in the form (36).

On the basis of (56) one may define various q-commutation relations between K^+ and K^-. The simplest one would be the following

$$K^-K^+ - q^4K^+K^- = q^2(\alpha(1 + q^2) + 2\beta)H + \beta(\alpha + \beta). \tag{58}$$

The formal map onto the relations (41) is also available. Therefore relations (57),(58) give a particular form of the "quantization" of the algebra $su(1,1)$ which is explicitly recovered at $q = 1$.

Described q-deformation of the conformal quantum mechanics is more general than that presented in Sect.5. Indeed, various limits of q give the following solvable cases: 1) a two-level potential corresponding to two-soliton system appears at $q = 0$; 2) a finite cut analog of two-soliton potential arises at $q \to \infty$; 3) the general conformal potential comprising both oscillator and $1/x^2$ parts is recovered in the limit $q \to 1$ when $W(x) \propto a/x + bx$. In order to find the spectrum of H at arbitrary q it is neccessary to know general properties of the superpotential W_1. Let us suppose that there exists a solution for positive α and β such that $\exp(-\int^x W_{1,2})$ are normalizable wave functions. Then the spectrum consists of two geometric series and by shifting can be represented in the form

$$E_n = \begin{cases} E_0q^{2m}, & \text{for n=2m} \\ E_1q^{2m}, & \text{for n=2m+1} \end{cases} \tag{59}$$

with the $E_n < E_{n+1}$ ordering fulfilled. Even and odd wave functions fall into independent irreducible representations of $su_q(1,1)$. A more detailed consideration of potentials and algebraic structures arising from the q-periodic closure of the chain (43) will be given elsewhere.

7. Conclusions

To conclude, we described a deformation of the SUSY quantum mechanics on the basis of affine transformations. The main feature of the construction is that superpartner Hamiltonians satisfy non-trivial braid-type intertwining relations which remove degeneracies of the original SUSY spectra. Obtained formalism naturally leads to the Shabat's self-similar potential describing slowly decreasing solutions of the KdV equation. The latter is shown to have straightforward meaning as a q-deformation of the harmonic oscillator potential. Equivalently, one may consider it as a deformation of a one-soliton system. Corresponding raising and lowering operators satisfy q-deformed Heisenberg-Weyl algebra atop of which a quantum conformal algebra $su_q(1,1)$ can be built. We also outlined a generalization of the Shabat's potential on the basis of q-deformation of periodic closure condition and presented q-deformation of general conformal quantum mechanics potentials.

In this paper the parameter q was taken to be real but nothing prevents from consideration of complex values as well (this changes only hermicity properties). The most interesting cases appear when q is a root of unity [33]. For example, at $q^3 = 1$ eq. (28) generates a potential proportional to

the so-called equianharmonic Weierstrass functions. More complicated hyperelliptic potentials are generated at higher roots of unity. The nontrivial Hopf algebra structure of the quantum groups was not considered because it is not relevant in the context of quantum mechanics of one particle in one dimension. Perhaps higher dimensional and many body problems shall elucidate this point. In fact, there seems to be no principle obstacles for higher dimensional generalizations although resulting systems may not have direct physical meaning. Another possibility is that described self-similar systems may arise from higher dimensional ones after the similarity reductions.

In order to illustrate various possibilities we rewrite the simplest self-similarity equation without scaling (i.e. at $q = 1$) but with non-trivial translationary part

$$W'(x) + W'(x + a) + W^2(x) - W^2(x + a) = constant. \tag{60}$$

Solutions of this equation provide a realization of the ordinary undeformed Heisenberg-Weyl algebra. The full effect of the presence of the parameter a in (60) is not known to the author but solutions whose absolute values monotonically increase at $x \to \pm\infty$ seem to be forbidden. Note also that in all formulas of SUSY and q-SUSY quantum mechanics superpotential $W(x)$ may be replaced by a hermitian $n \times n$ matrix function. The equations (28), (35), (60) may be equally thought as being the matrix ones with the right hand sides proportional to unit matrices. We end by a speculative conjecture that described machinery may be useful in seeking for q-deformations of the non-linear integrable evolution equations, like KdV, sin-Gordon, etc.

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PHASE OF THE QUANTUM HARMONIC OSCILLATOR WITH APPLICATIONS TO OPTICAL POLARIZATION

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Abstract

The phase of the quantum harmonic oscillator, the temporal distribution of a particle in a square-well potential, and a quantum theory of angles are derived from a general theory of complementarity. Schwinger's harmonic oscillator model of angular momenta [1] is modified for the case of photons. Angular distributions for systems of identical and distinguishable particles are discussed. Unitary and antiunitary time reversal operators are then presented and applied to optical polarization states in birefringent media.

1 General Theory of Complementarity

The fact that linear momentum is the generator of translations in space, leads to the Fourier transform relations between the momentum and spatial representations of Schrodinger's wave-mechanics [2]. Similarly, since energy generates translations in time, there are Fourier transform relations between the energy and temporal representations [3]. For the case of the harmonic oscillator, the energy eigenspectrum is proportional to the integers $n = 0, 1, 2...$ (recall $\hat{H} = \hbar\omega(\hat{n} + 1/2)$, where $\hat{n} = \hat{a}^{\dagger}\hat{a}$ is the photon number operator) and this spectrum is aperiodic (i.e. not periodic). Therefore the temporal distribution of the oscillator will be continuous and periodic. Indeed, the simplest way (that I have found) to describe the phase ($\Phi = \omega t$) of the quantum harmonic oscillator is to form the wavefunction

$$\psi(\Phi) = \sum_{n=0}^{\infty} \psi_n e^{-in\Phi} \tag{1}$$

which is the Fourier series of the n-space wavefunction (or number-ket expansion coefficients) $\psi_n \equiv \langle n|\psi\rangle$, where $\hat{n}|n\rangle = n|n\rangle$). The probability density for finding Φ on any 2π interval (the period of $\psi(\Phi)$) is then simply $|\psi(\Phi)|^2/2\pi$. The wavefunction approach circumvents complications associated with the equally correct perspective [4] that this phase distribution corresponds to the realizable measurement of the Susskind-Glogower (SG) [5] phase operator.

Suppose we wish to study the temporal behavior of a particle in a one dimensional box (the "phase of the infinite square well"). We do not have to start all over, we can simply take the Fourier (series) transform of the discrete energy wavefunction, which underlies the discrete energy eigenspecta:

$$E_i = \frac{\hbar^2 \pi^2}{2mL^2}(i)^2 \quad (i = 1, 2, 3...) \tag{2}$$

where L is the length of the box and m the mass of the particle. In other words, labeling the energy eigenstates, $\{|E_n\rangle\}$, according to the value of $n \equiv (i)^2$, we'd use the $\psi_n \equiv \langle E_n|\psi\rangle$ as the Fourier series coefficients in $\psi(\Phi) = \sum_n \psi_n e^{-in\Phi}$, where $\Phi = t(\hbar\pi^2/2mL^2)$. The temporal distribution is therefore like that of a harmonic oscillator for which $\psi_2 \equiv 0 \equiv \psi_3$, $\psi_5 \equiv 0 \equiv \psi_6 \equiv \psi_7 \equiv \psi_8$, etc. For a well of finite depth, the bound state eigenenergies will be perturbed from being proportional to the squares of integers, but they will still be discrete and we would still sum over the $\langle E_i|\psi\rangle$ with each one weighted by $e^{-iE_i t/\hbar}$, to form $\psi(t)$ which is quasi-periodic (it can't be *exactly* periodic since the E_i are no longer integer multiples of each other, however the difference between $\psi(t)$ and $\psi(t+T)$ can be made as small as we wish by making T large enough — hence the term "quasi-periodic"). The unbound states for this problem, however, have a continuous distribution in energy and for these we would form the aperiodic

$$\psi(t) = \int dE\, \psi(E)\, e^{-iEt/\hbar} \tag{3}$$

where $\psi(E) \equiv \langle E|\psi\rangle$. Notice that the unbound states exhibit an aperiodic temporal distribution, i.e. they can be "here today and gone tomorrow" as they zip past the potential well, whereas the bound states are trapped into quasi-periodic oscillations.

From the general theory of complementarity we can also obtain a quantum theory of angles. The z component of angular momentum, $\hat{J}_z$, is (by definition) the generator of translations in the angle about the z axis, which shall be denoted as ϕ. It is well known that $\hat{J}_z$ has discrete eigenvalues given by $m\hbar$ where $m \in \{-j, -j+1, ...j-1, j\}$ and j is the label of the discrete eigenvalues of the simultaneously measurable $\hat{J}^2 \equiv \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ which are $j(j+1)\hbar^2$. For states in which each value of m is uniquely represented (the degenerate case will be discussed in the next section), such as a particle of spin s (i.e. $j = s = $ a fixed number), we can form the angle representation

$$\psi(\phi) = \sum_m \psi_m e^{-im\phi} \tag{4}$$

where $\psi_m \equiv \langle j, m|\psi\rangle$ and the angular distribution is $p(\phi) = |\psi(\phi)|^2/2\pi$. Since $\psi(\Phi)$ is periodic its transform ψ_m must be discrete, i.e. the quantization of angular momentum (projected onto an axis) is a simple and immediate consequence of the periodicity of the angle (about that axis).

2   Harmonic Oscillator Models of Angular Momenta

In 1952, Schwinger [1] demonstrated a connection between the algebra of two uncoupled harmonic oscillators and the algebra of angular momenta. The key points of Schwinger's model are as follows:

$$\hat{J}_- \equiv \hbar \hat{a}_d^\dagger \hat{a}_u \quad and \quad \hat{J}_z \equiv \frac{\hbar}{2}(\hat{n}_u - \hat{n}_d), \tag{5}$$

where $\hat{a}_u$ and $\hat{a}_d$ denote the annihilation operators for the "up type" and "down type" oscillators. From this we obtain the fundamental commutation relations of angular momentum:

$$[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z \quad and \quad [\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm, \tag{6}$$

where $\hat{J}_+ \equiv (\hat{J}_-)^\dagger$ and $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$, so $[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$ etc. Since the quanta of these oscillators behave like spin 1/2 objects (as seen from eq.(5)), yet only totaly symmetrical states are con-

structed by this method, these quanta are not believed to correspond to actual particles and the connection is merely within the mathematics [2].

We put some physics into this connection by considering a rotation of a single frequency electromagnetic wave about the z axis (along which the $\vec{k}$ vector lies) which leads to the well known result that a right handed circularly polarized photon is an eigenstate of $\hat{J}_z/\hbar$ with eigenvalue $m = +1$. Similarly, a left handed photon is associated with $m = -1$ and since we need only consider transverse components of the vector potential, the photon is said to be a particle of "spin 1 with m=0 missing" [6]. Since the photon is a boson which resembles a spin 1/2 object in the sense that its spin space is two dimensional, it seems reasonable to attempt to reconstruct the algebra of angular momenta from these physically significant photonic primitives. Indeed, taking

$$\hat{J}_- \equiv 2\hbar \hat{a}_l^\dagger \hat{a}_r \quad and \quad \hat{J}_z \equiv \hbar(\hat{n}_r - \hat{n}_l), \tag{7}$$

where $\hat{a}_r$ and $\hat{a}_l$ are the annihilation operators for the right and left circularly polarized modes of a single frequency, z propagating, electromagnetic wave, we obtain

$$[\hat{J}_+, \hat{J}_-] = 4\hbar\hat{J}_z \quad and \quad [\hat{J}_z, \hat{J}_\pm] = \pm 2\hbar\hat{J}_\pm. \tag{8}$$

where as before $\hat{J}_+ = (\hat{J}_+)^\dagger$ and $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$, so $[\hat{J}_x, \hat{J}_y] = 2i\hbar\hat{J}_z$ etc. This is the same group, however $\hat{J}_-$ now lowers m by 2 (rather than by 1) which is exactly what we want for photons. Notice that a differential phase shift between these circularly polarized modes (or between the up and down oscillators for the case of fermions and ordinary, i.e. "non-photonic", bosons) is equivalent to a rotation about the z axis:

$$\hat{R}_z(\phi)|\psi\rangle = \sum_{n_r,n_l} \psi_{n_r,n_l} e^{-i(n_r - n_l)\phi} |n_r, n_l\rangle. \tag{9}$$

We can relable our two-mode number states according to the values of $j \equiv n_r + n_l$ (or $j = (n_u + n_d)/2$) and $m \equiv n_r - n_l$ (or $m = (n_u - n_d)/2$). To obtain the angle representation for the case of identical particles (e.g. all these states are photons, or they are all electrons, etc.) we should allow for quantum interference of all these states (i.e. we should add amplitudes rather than probabilities) and therefore simply use

$$\psi_m \equiv \sum_j \langle j, m | \psi \rangle \tag{10}$$

in eq.(4) for these cases. Since the ψ_m defined in eq.(10) are no longer normalized (for m states degenerate in j) we must renormalize:

$$\psi(\phi) \rightarrow \psi(\phi)/\sqrt{c} \quad where \quad c \equiv \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} |\psi(\phi)|^2. \tag{11}$$

For bosons or photons the minimal non-zero value of $|m|$ is one therefore the period of $\psi(\phi)$ is at most 2π, and since $\Delta m_{min} = 1$ the period of $p(\phi)$ is at most 2π. For fermions m can be 1/2 so the period of $\psi(\phi)$ can be 4π. This indicates the rotational Berry's phase "for fermions"[7], which we now see to be more correctly stated as being "for fermions which have non-zero overlap with $m = \pm 1/2$ states." Since Δm_{min} for fermions is still one, $p(\phi)$ is still mod 2π, indicating that

observation of the "mod 4π Berry's phase" requires interference of this state with another one, as is well known.

Notice that if we allowed for **particles** comprised of mixtures of integer and half-integer spin we could have $\Delta m_{min} = 1/2$ so that $p(\phi)$ (not just $\psi(\phi)$) would be periodic mod 4π! Since no interference with another state is required, the existence of proposed particles of this type would radically alter our conceptualization of space (each point like a Mobius strip?). Alternatively, we might argue that it is physicaly reasonable to require that $p(\phi)$ have at most a period of 2π and therefore we would have a theoretical explanation of the "... empirical fact that a mixed symmetry does not occur" [2]. If however, we had a **system** comprised of a fermion and a boson (e.g. an electron and a photon) then since these distinguishable particles do not interfere, the angular distribution should be (at most) mod 2π.

For the case of distinguishable particles we should add probabilities (rather than amplitudes), i.e. rather than the procedure defined by eq.s (4), (10), and (11), we should do the following. For each distinguishable particle we should form an angular wavefunction, *then* square its magnitude and divide by 2π to form each different particle's individual angular distribution, *then* add these individual distributions to form the angular distribution of the entire system. When these distinguishable particles have distinct values of spin (such as a system comprised of a spin 1/2 electron and a spin 1 photon for example) this proceedure is as follows. For each fixed value of $j = s$ we form

$$\psi^{(j)}(\phi) \equiv \sum_m \psi_{j,m} e^{-im\phi} \quad and \quad p^{(j)}(\phi) \equiv |\psi^{(j)}(\phi)|^2/2\pi \tag{12}$$

from which we obtain the system's angular distribution: $p(\phi) = \sum_j p^{(j)}(\phi)$. This procedure corresponds to the measurement of $\hat{Z} \equiv (\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z)^{-1/2}\hat{J}_-$ where the leading term obviates the $\sqrt{(j+m)(j-m+1)}$ factor from $\hat{J}_-$ so that the lowering of $\hat{Z}$ is "pure": $\hat{Z}|j,m\rangle = |j,m-1\rangle$.

3 Unitary and Antiunitary Time Reversal

Although in the literature to date [2] it has been argued that a time reversal operator must be antiunitary (so that kinetic energy, for example, remains non-negative) it is more appropriate for our purposes to define a *unitary* time reversal operator since we are mainly interested in *relative* (rather than absolute) time and relative time (e.g differential phase) is complementary to the *relative energy* (e.g. photon number difference) which can be negative.

For a quantum mechanical operation to conserve probability the corresponding operator must either be unitary or antiunitary [8] (or some combination thereof). In either case it is reasonable to require that a time reversal operator, $\hat{T}$, should satisfy [2]

$$\hat{U}(t)\hat{T} = \hat{T}\hat{U}(-t), \tag{13}$$

where $\hat{U}(t)$ denotes (unitary) time evolution of an amount t. Equivalently, we could require $\hat{U}(t)\hat{T}\hat{U}(t)\hat{T} = \hat{I}$ and we are neglecting (as we did in eq.(13)) any overall phase which might be aquired in getting back to the "same" state.

Any antiunitary operator can be expressed as a product of a "complex conjugator" (of c-numbers) and a unitary operator [8]. Thus the unitary time reversal operator, $\hat{T}_u$, is simply the

unitary part of the antiunitary time reversal operator

$$\hat{T}_a = \hat{C}\hat{T}_u, \tag{14}$$

where $\hat{C}$ denotes complex conjugation. It has previously been demonstrated that the auxiliary (a) mode (associated with the realizable measurement of the SG operator) must be "time reversed" with respect to the original system (s) mode [9]. Therefore, $\hat{T}_u$, should permute these modes so that $\hat{T}_a$ acting on a two-mode state, $|\psi\rangle \equiv \sum_{n_s} \sum_{n_a} \psi_{n_s,n_a} |n_s\rangle_s |n_a\rangle_a$, yields

$$\hat{T}_a |\psi\rangle = \sum_{n_s} \sum_{n_a} \psi^*_{n_s,n_a} |n_a\rangle_s |n_s\rangle_a = \sum_{n_s} \sum_{n_a} \psi^*_{n_a,n_s} |n_s\rangle_s |n_a\rangle_a. \tag{15}$$

Subsequent time evolution (i.e. *absolute* phase shift) of this state results in

$$\hat{U}(t)\hat{T}_a |\psi\rangle = \sum_{n_s} \sum_{n_a} \psi^*_{n_a,n_s} e^{-i(n_s+n_a+1)\omega t} |n_s\rangle_s |n_a\rangle_a, \tag{16}$$

where $\hat{U}(t) = e^{-i(\hat{n}_s+\hat{n}_a+1)\omega t}$. If instead, we first propagated the initial state $|\psi\rangle$ "backwards" in time, and then time reversed we'd obtain $\hat{T}_a\hat{U}(-t)|\psi\rangle =$

$$\hat{T}_a \sum_{n_s} \sum_{n_a} \psi_{n_s,n_a} e^{+i(n_s+n_a+1)\omega t} |n_s\rangle_s |n_a\rangle_a = \sum_{n_s} \sum_{n_a} \psi^*_{n_a,n_s} e^{-i(n_s+n_a+1)\omega t} |n_s\rangle_s |n_a\rangle_a, \tag{17}$$

which is the same state as in eq.(16) and therefore the requirement of eq.(13) is satisfied.

For unitary time reversal, we simply omit the complex conjugation of the expansion coefficients and we find that in order to satisfy eq.(13) we must consider a *differential* (rather than absolute) phase shift $\hat{U}_d(t) \equiv e^{-i(\hat{n}_s-\hat{n}_a)\omega t}$. Explicitly, we have

$$\hat{U}_d(t)\hat{T}_u |\psi\rangle = \hat{U}_d(t) \sum_{n_s} \sum_{n_a} \psi_{n_a,n_s} |n_s\rangle_s |n_a\rangle_a = \sum_{n_s} \sum_{n_a} \psi_{n_a,n_s} e^{-i(n_s-n_a)\omega t} |n_s\rangle_s |n_a\rangle_a, \tag{18}$$

which is equivalent to $\hat{T}_u\hat{U}_d(-t)|\psi\rangle =$

$$\hat{T}_u \sum_{n_s} \sum_{n_a} \psi_{n_s,n_a} e^{+i(n_s-n_a)\omega t} |n_s\rangle_s |n_a\rangle_a = \sum_{n_s} \sum_{n_a} \psi_{n_a,n_s} e^{-i(n_s-n_a)\omega t} |n_s\rangle_s |n_a\rangle_a. \tag{19}$$

Thus the "time" to be associated with unitary time reversal is the difference time, translations in which are generated by the energy difference $\hbar\omega(\hat{n}_s - \hat{n}_a)$.

We have already demonstrated that the differential phase between the two oscillators of our angular momenta model is equivalent to the angle ϕ. Therefore $\hat{T}_u$ corresponds to angle inversion ($\phi \to -\phi$) when we take the s and a modes to be the right and left circularly polarized electromagnetic modes (or the up and down oscillators), i.e. under $\hat{T}_u$ we have:

$$\psi_{n_r,n_l} \to \psi_{n_l,n_r} \quad or \quad \psi_{j,m} \to \psi_{j,-m} \quad so \quad \psi(\phi) \to \psi(-\phi) \tag{20}$$

(in the antiunitary case, we'd have $\psi(\phi) \to \psi^*(-\phi)$ under $\hat{T}_a$). A $\hat{T}_u$ eigenstate ($\psi(\phi) = \psi(-\phi)$) will therefore have an angular distribution symmetrically centered about $\phi = 0$, so that any vector associated with this state can only be along the x axis. Indeed, from $\psi_{n_r,n_l} = \psi_{n_l,n_r}$ we can show $\langle(\hat{a}_r)^p\rangle = \langle(\hat{a}_l)^p\rangle$ $\forall p \in \{0,1,2,...\}$ and from the $p = 1$ result we have $\langle\hat{E}_y\rangle = 0$. The $\hat{T}_u$ eigenstates

(in the cicularly polarized basis) therefore correspond to polarization which is *linear* in terms of the polarization "signal" (i.e. the $\langle \hat{\bar{E}} \rangle$) so that they resemble (and include) the case of putting one linear polarized mode in the vacuum state, but they can achieve this with a reduction in polarization "noise" (e.g. $\Delta^2 E_x$ or $\Delta^2 E_y$). These states therefore provide a foundation for the study of quantum limits on the performance of devices which utilize circularly birefringent media (e.g. Faraday rotators, optical isolators, etc.).

As a simple example, compare these two $\hat{T}_u$ eigenstates: one an x polarized coherent state (with the y polarization unexcited), $|\alpha\rangle_r |\alpha\rangle_l = |\sqrt{2}\alpha\rangle_x |0\rangle_y$; and the other $(|\alpha\rangle_r |0\rangle_l + |0\rangle_r |\alpha\rangle_l)/\sqrt{2} + (1 - \sqrt{2})e^{-|\alpha|^2/2}|0\rangle_r |0\rangle_l$, which I'll refer to as the pseudo-coherent state. Both states yield similar polarization "signals" $\langle \hat{E}_x \rangle \simeq -2\alpha \sin(\omega t)$ *and* $\langle \hat{E}_y \rangle = 0$, yet, the polarization "noise" of the pseudo-coherent state ($\Delta^2 E_x = 1/2$) is 3 dB below the shot noise limit of the coherent state ($\Delta^2 E_x = 1$), where we assume $|\alpha|^2 >> 1$ (else the pseudo-coherent and coherent states both tend towards the vacuum).

We can also use the phase representation to describe the measurement of the differential phase shift of two linearly polarized modes which is germane to optical polarization states propagating through linearly birefringent media. The sense in which $\psi(\phi)$ would describe the polarization state for the linear mode set would be different however since we lose the connection with the angular measurement as the energy eigenstates in the linear basis are not eigenstates of angular momenta. Nonetheless, the mode exchange eigenstates in this basis correspond to an expected value of the electric field operator that resembles circular polarization and these states provide a foundation for the study of quantum limits on the performance of quarter-wave plates, etc.

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CONDITION FOR EQUIVALENCE OF q-DEFORMED AND ANHARMONIC OSCILLATORS

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Abstract

We discuss the equivalence between the q-deformed harmonic oscillator and a specific anharmonic oscillator model, by which some new insight into the problem of the physical meaning of the parameter q can be attained.

1 Introduction

Recently there has been a great deal of interest in the study of quantum groups. Of particular interest here is the development by Macfarlane [1] and independently by Biedenharn [2] of the realization of the $q$uantum group $SU(2)_q$ in terms of the q-analogue of the quantum harnomic oscillator. Although many aspects of the q-deformation of the bose harmonic oscillator algebra have been investigated, still one of the most appealing issues is perhaps the physics behind the parameter q. Here an attempt is made in this direction.

We show that the q-deformed harmonic oscillator model can be used to describe a specific anharmonic oscillator. Thus a q-deformation can be understood as an effective anharmonic deformation, where q is proportional to the strength of the harmonicity. The anharmonic and the q-deformed oscillator models are presented respectively in section 2 and 3 and their equivalence is therein discussed. The latter can in turn be used to examine interesting non-classical features induced by a q-deformation during the time-evolution of a $SU(2)$ coherent state. This is put forward in section 4, and discussed in [3]

2 Anharmonic oscillator

The *anharmonic* oscillator we wish to discuss has the hamiltonian

$$H_\lambda = H_0 + \frac{\mu}{\omega_0}N^3 \equiv N + \frac{1}{2} + \frac{\mu}{\omega_0}N^3 \tag{1}$$

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where H_0 is the free hamiltonian of the simple *harmonic* oscillator whose fundermental frenquency is ω_0. $N = b^\dagger b$ is the number operator, whereas $b^\dagger$ and b are respectively the lowering and raising bose operators. H_λ is in units of ω_0 when H_0 is in units of ω_0. The anharmonic term is taken proportional to N^3, and the anharmonicity parameter is positive: specifically we take here $\mu \equiv \omega_0 \gamma^2/6$. In the limit of small anharmonic deformations the hamiltonian in Eq.(1) can be discussed in terms of

$$a_\gamma = \sqrt{\Omega_\gamma^{-1}}\,[1 + \gamma^2 \frac{(b^\dagger b + 1)^2}{2 \cdot 3!}] \qquad \Omega_\gamma = \gamma^{-1}\sinh\gamma \tag{2}$$

It is readily seen that in this representation

$$H_\gamma = \Omega_\gamma(a_\gamma^\dagger a_\gamma + 1/2) \tag{3}$$

is indeed equivalent [4] to H_λ in Eq.(1).

States of our anharmonic oscillator can be constructed as quantum states for H_γ. First note that the vacuum $|0\rangle_\gamma$, defined as $a_\gamma|0\rangle_\gamma = 0$, is the same as the vacuum $|0\rangle_\gamma$ for the harmonic oscillator. However, eigenstates of the number operator $N_\gamma = a_\gamma^\dagger a_\gamma$ substantially differ from those for the harmonic oscillator. The former can be defined as

$$|n\rangle_\gamma = \frac{(a_\gamma^\dagger)^n}{\sqrt{c_{n,\gamma}}}|0\rangle_\gamma \qquad N_\gamma|n_\gamma\rangle_\gamma = \frac{c_{n,\gamma}}{c_{n-1,\gamma}}|n\rangle_\gamma \tag{4}$$

while the normalization condition $_\gamma\langle m|n\rangle_\gamma = \delta_{m,n}$ determines the $c_{n,\gamma}$'s:

$$c_{n,\gamma} = n!\Omega_\gamma^{-n}\prod_{k=1}^{n}(1 + \frac{\gamma^2 k^2}{2 \cdot 3!})^2 = n!\Omega_\gamma^{-n}[(1 + \frac{\gamma^2 n^2}{2 \cdot 3!})^2]!, \qquad c_{0,\gamma} = 1 \tag{5}$$

Here we will be concerned, in particular, with coherent states. In the basis $\{|n\rangle_\gamma\}$ $(n = 0, 1, 2, \ldots)$ these can be expressed as [5]

$$|\alpha\rangle_\gamma = C_\gamma\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{c_{n,\gamma}}}|n\rangle_\gamma, \qquad C_\gamma^{-2} = \sum_{n=0}^{\infty}\frac{\alpha^{2n}}{c_{n,\gamma}} \tag{6}$$

Where C_γ derives from the normalization condition $_\gamma\langle\alpha|\alpha\rangle_\gamma = 1$. The resemblance of the $|\alpha\rangle_\gamma$'s with coherent states of the harmonic oscillator is resdily seen: however, we should stress that only in the limit $\gamma \to \infty$ the anharmonic and harmonic oscillator models are exactly the same.

3 q-deformed harmonic oscillator

Let us recall the $(b, b^\dagger)$ bose operators for the harmonic oscillator introduced earlier. They satisfy the Weyl-Heisenberg algebra

$$[b, b^\dagger] = 1 \qquad [N, b^\dagger] = b^\dagger \qquad N = b^\dagger b \tag{7}$$

Macfarlane [1] and Biedenharm [2] have discussed a deformation of this algebra so that

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{-N} \qquad [N, a_q^\dagger] = a_q^\dagger \tag{8}$$

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and, in particular, its realization in terms of a q-deformed harmonic oscillator. The parameter q [6] characterizes the strength of the deformation.

We explore in this section the connection between q-deformations and anharmonic deformations of the harmonic oscillator. We will first study the effect of a q-deformation on the states of the harmonic oscillator, similarly to what was done in the previous section for the anharmonic oscillator model. By recalling that the q-operators can be realized in terms of the bose operators of the form [1, 2]

$$a_q = \sqrt{\frac{[N+1]_q}{N+1}}\, b; \qquad a_q^\dagger = b^\dagger \sqrt{\frac{[N+1]_q}{N+1}}\,, \tag{9}$$

where $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$, we first construct the quantum states for the q-harmonic oscillator. The q-deformed vacuum is defined as $a_q|0\rangle_q = 0$, and since a_q is a function of b and power of $b^\dagger b$, $|0\rangle_q$ and the vacuum $|0\rangle$ of the harmonic oscillator turn out to be the same. Eigenstates of the number operator $N_q = a_q^\dagger a_q$ can be defined as

$$|n\rangle_q = \frac{(a_q^\dagger)^n}{\sqrt{c_{n,q}}}|0\rangle_q \qquad N_q|n_q\rangle_q = \frac{c_{n,q}}{c_{n-1,q}}\,|n\rangle_q \tag{10}$$

With the choice of $c_{n,q} \equiv \sqrt{[n]_q!}$, where $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, the set of eigenvectors $\{|n\rangle_q\}$ $(n = 0, 1, 2, \ldots)$ is orthonormal $({}_q\langle m|n\rangle_q = \delta_{m,n})$ and generates the Fock space for the q-deformed oscillator. On the basis $\{|n\rangle_q\}$ $(n = 0, 1, 2, \ldots)$ one can express the coherent states of the q-deformed harmonic oscillator as

$$|\alpha\rangle_q = C_q \sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{c_{n,q}}}|n\rangle_q \qquad C_q = [exp_q \alpha^2]^{-1/2} \tag{11}$$

where the factor C_q is again set by the normalization condition ${}_q\langle \alpha|\alpha\rangle_q = 1$. Here exp_q stands for the q-exponential, i.e. $exp_q = \sum_{n=0}^\infty x/[n]_q!$. Again note that as $q \to 1$ this q-deformed model exactly reduces to that of a simple harmonic oscillator.

A connection can be established between coherent states of q-deformed harmonic oscillator and coherent states of the anharmonic oscillator in the sense that there exists a condition under which the $|\alpha\rangle_q$'s and the $|\alpha\rangle_\gamma$'s are equivalent. Namely, for oscillator displacements α and γ (or q) such that [3]

$$\alpha(\alpha + 8) < \ln^{-1} q^{1/4} \tag{12}$$

we have $|\alpha\rangle_q \to |\alpha\rangle_\gamma$, provided $\gamma = \ln q$. An analytic proof of this equivalence is beyond the aim of this paper and will be reported elsewhere [3]. However, we can compare here the probability number distribution for the $|\alpha\rangle_\gamma$'s to that for the $|\alpha\rangle_q$'s, that is, $P_n^\gamma(\alpha) = |\langle n|\alpha\rangle_\gamma|^2$ and $P_n^q(\alpha) = |\langle n|\alpha\rangle_q|^2$. Owing to the definition of probability as overlap over the same state $|n\rangle$, equal distributions would infer the equivalence of the states $|\alpha\rangle_\gamma$ and $|\alpha\rangle_q$. A numerical evaluation is reported in Fig.1 for values of q and α respectitively conforming and not conforming with the

condition (12). In this latter case $P_n^{\gamma_2}(\alpha_2)$ is strongly shifted with respect to $P_n^{q_2}(\alpha_2)$, whereas in the former case the two distribution are nearly the same.

In conclusion, for appropriate displacements (α) and anharmonic couplings (μ) coherent states of an oscillator with anharmonicity $\sim N^3$ (N is the number of particles) are correctly described in terms of coherent states of the q-deformed Lie algebra of $SU(2)$, where $q \simeq exp(\mu/\omega_0)^{1/2}$. This result is particularly important because the parameter q can be given a direct physical meaning: it is proportional to the square root of the anharmonic coupling strength.
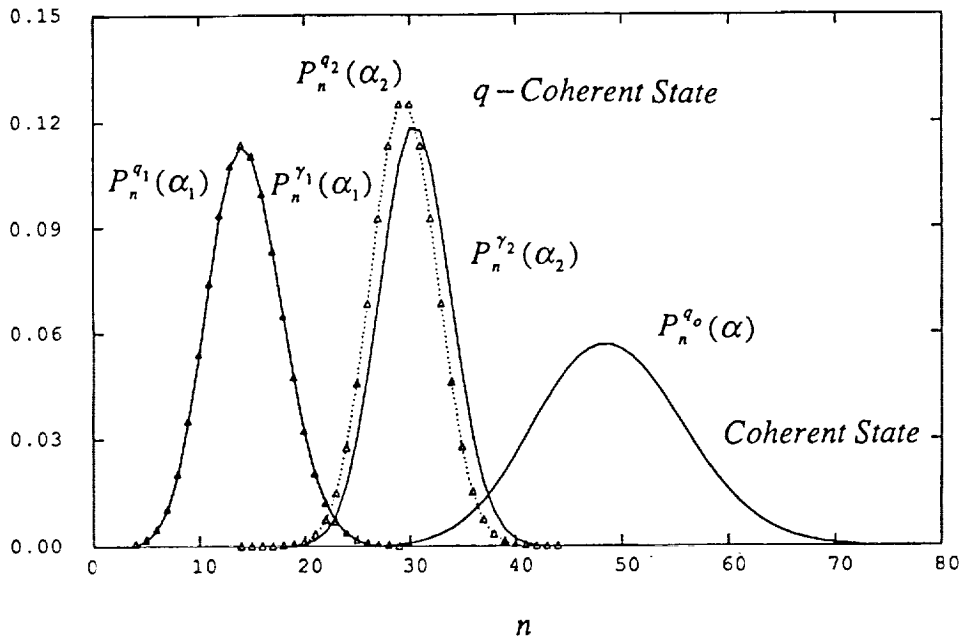
FIG.1. Probability number distributions for coherent states $(|\alpha\rangle_{q_1}, |\alpha\rangle_{q_2})$ of a q-deformed quantum oscillator and for coherent states $(|\alpha\rangle_{\gamma_1}, |\alpha\rangle_{\gamma_2})$ of a quantum oscillator with a third order anharmonicity in the particle number. From their equivalence one can infer the equivalence between the corresponding states, which holds depending on whether the oscillator parameters satisfy ($\alpha_1 = 4$, $\gamma_1 = 0.05$) or do not satisfy ($\alpha_1 = 10$, $\gamma_1 = 0.1$) the condition (12), respectively. Here $q = e^\gamma$. $P_n^{q_0}(\alpha)$ is a reference Poission ($q_0 = 1$) distribution with $\alpha = 7$.

4 q-deformation and non-classical harmonic oscillator

The equivalence we have established between anharmonicity and q-deformation of a harmonic oscillator is a very helpful one: not only does it provide the q parameter with a definite physical meaning, but also dose it turn out to be useful for investigating and attaining a sound physical interpretation of interesting non-calssical effects induced by a q-deformation during time-evolution of a $SU(2)$ coherent state. The most important of these effects is a q-dependent *self-squeezing*: i.e. a reduction of the uncertainty expectations of the two orthogonal components (quadratures) of the oscillator field below their vacuum values that varies with q. A q-deformation does also alter the *minimality* properties of an initial mimimum uncertainty coherent state, but not its *possionian counting statistics*. The connection between q-deformations of the harmonic oscillator and these rather interesting phenonena is however beyond the purpose of this paper and will be discussed elsewhere [3].

5 Acknowledgments

References

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[4] We here retain terms only of the order γ^3 or lower, as typically done for small anharmonic deformations at ordinary energies;

[5] For simplicity, we take α real;

[6] q is in general complex: however, here $q > 1$ and real;

NOVEL PROPERTIES OF THE q-ANALOGUE QUANTIZED RADIATION FIELD

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Abstract

The "classical limit" of the q-analogue quantized radiation field is studied paralleling conventional quantum optics analyses. The q-generalizations of the phase operator of Susskind and Glogower (circa 1964) and that of Pegg and Barnett (circa 1988) are constructed. Both generalizations and their associated number-phase uncertainty relations are manifestly q-independent in the $|n>_q$ number basis. However, in the q-coherent state $|z>_q$ basis, the variance of the generic electric field, $(\Delta E)^2$, is found to be increased by a factor $\lambda(z)$ where $\lambda(z) > 1$ if $q \neq 1$. At large amplitudes, the amplitude itself would be quantized if the available resolution of unity for the q-analogue coherent states is accepted in the formulation. These consequences are remarkable versus the conventional $q = 1$ limit.

1 Introduction

On several occasions during the last fifty years, new mathematical symmetries have been constructed in theoretical physics but only found to be relevant to nature five or more years later. If this is occurring now in the case of quantum algebras, we need to know the physical implications of these new and distinctly novel symmetry structures. If there are q-oscillators in nature which realize these new algebras, surely there must be a quantum field which has such q-oscillators as its normal modes. Until we know the physical properties of such a field , say in its "classical limit", we may not be able to glean its distinct relevance to problems and phenomena in quantum optics, many body physics, particle physics ....

2 A Completeness Relation for the q-Analogue Coherent States by q-Integration

The q-analogue coherent states $|z>_q$ satisfy $a|z>_q= z|z>_q$ where the q-oscillator algebra is [1] ($q \rightarrow 1$, usual bosons)

$$aa^\dagger - q^{1/2}a^\dagger a = q^{-N/2} \tag{1}$$

$$[N, a^\dagger] = a^\dagger \qquad [N, a] = -a \tag{2}$$

It is physically very important that there remains the mathematically trivial bosonic $[a, a] = 0$. In the $|n>_q$ basis, $< m|n >= \delta_{mn}$ and[1]

$$a^\dagger |n >= \sqrt{[n+1]}|n + 1 > \qquad a|n >= \sqrt{[n]}|n - 1 > \qquad a|0 >= 0 \tag{3}$$

where $[x]_q = [x] \equiv (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$ is the "q-deformation" of x. More simply $[x] = \sinh(sx/2)/\sinh(s/2)$ where $q = \exp s$, $0 \leq q \leq 1$.

The q-analogue coherent states $|z>_q$ are good candidates for studying the classical limit of the q-analogue quantized radiation field because (i) there exists a resolution of unity [2]

$$I = \int |z >< z| \, d\mu(z) \tag{4}$$

(ii) they indeed are "minimum uncertainty states" for they do minimize the fundamental commutation relation

$$U_{Q,P} \equiv \frac{2\Delta Q \Delta P - | < [Q, P] > |}{| < [Q, P] > |} \geq 0 \tag{5}$$

with $U|_{|z>} = 0$ but $U|_{|n>\neq|0>} = \frac{(3[n]+[n+1])}{([n+1]-[n])}$, and (iii) the nth order correlation function factorizes, i.e.

$$Tr(\rho E^-(x)E^+(y)) = \mathcal{E}^-(x)\mathcal{E}^+(y), \ldots \tag{6}$$

But, simultaneously, there are intriguing differences in the $|z>_q$ basis for other coherence and uncertainty properties of the q-analogue quantized field. Some of these will be discussed as we go along.

In the $|z>_q$ basis, from $a|z >= z|z >$ it follows that for $< z|z >= 1$

$$|z >_q= N(z) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}}|n >, \qquad N(z) = e_q(|z|^2)^{-1/2} \tag{7}$$

in terms of the "q-exponential function"

$$e_q(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{[n]!}, \qquad [n]! \equiv [n][n-1]\cdots[1], \qquad [0]! = 1 \tag{8}$$

which is an entire function $|e_q(z)| \leq e_q(|z|) \leq exp(|z|)$. For $x > 0$, it's positive, but for $x < 0$ it wildly oscillates within these bounds!

To derive the resolution of unity, we need a lemma which is a q-analoque of Euler's formula: We define the q-derivative

$$\frac{d}{d_q x}f(x) \equiv \frac{f(q^{1/2}x) - f(q^{-1/2}x)}{q^{1/2}x - q^{-1/2}x} \tag{9}$$

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[1]From now on the sub-q's are usually implicit!

and for $f(x)$ on the interval $[0, a]$, the inverse operation

$$\int_0^a f(x)\, d_q x \equiv a(q^{-1/2} - q^{1/2}) \sum_{n=0}^{\infty} q^{\frac{2n+1}{2}} f(q^{(2n+1)/2}a). \tag{10}$$

So, for instance $\frac{d}{d_q x} a x^n = a[n] x^{n-1}$, $\frac{d}{d_q x} e_q(ax) = a\, e_q(ax)$ and inversely $\int a x^{n-1}\, d_q x = a x^n/[n]$, $\int e_q(ax)\, d_q x = e_q(ax)/a$ up to the constants. It follows that there are two integration by parts formulas

$$\int_0^a f(q^{1/2}x)\,(\frac{d}{d_q x} g(x))\, d_q x = f(x)g(x)|_{x=0}^{x=a} - \int_0^a \frac{d}{d_q x} f(x) g(q^{-1/2}x)\, d_q x \tag{11}$$

and the $q \rightarrow 1/q$ expression.

We define $-\zeta$ =largest zero of $e_q(x)$ and restrict $e_q(z) \equiv [\ \sum_{n=0}^{\infty} \frac{z^n}{[n]!}$ for$-\zeta < x$; 0,otherwise] . Then by the first integration by parts formula

$$\int_0^{\zeta} e_q(-x)x^n\, d_q x = [n]! \tag{12}$$

From this the resolution of unity simply follows for the measure

$$d\mu(z) = \frac{1}{2\pi} e_q(|z|^2) e_q(-|z|^2) d_q |z|^2\, d\theta \tag{13}$$

since

$$\int |z><z|\, d\mu(z) = \frac{1}{2\pi} \int \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|z|^n |z^*|^m}{\sqrt{[n]![m]!}} e_q(-|z|^2)\, d_q |z|^2$$

$$\int \exp(i(n-m)\theta) d\theta |n><m| \tag{14}$$

$$= \sum_{n=0}^{\infty} \frac{1}{[n]!} \int_0^{\zeta_i} x^n e_q(-x)\, d_q x\, |n><n|, \qquad x = |z|^2 \tag{15}$$

$$= \sum_{n=0}^{\infty} |n><n| = I \tag{16}$$

Several remarks are appropriate:(i) states with $|z|^2 \geq \zeta_i$ do not contribute,(ii) arbitrary $|z>_q$ coherent states are not orthogonal since $<\alpha|\beta> = N(\alpha)N(\beta) e_q(\alpha^*\beta) \neq 0$,(iii) the $|z>_q$are actually overcomplete, since

$$|\alpha>_q = \int |z><z|\alpha>\, d\mu(z), \qquad <z|\alpha> \neq 0, \tag{17}$$

(iv) with $f(z) \equiv <z|f>$, the $a^\dagger, a$ act $<z|a^\dagger|f> = z^* f(z)$, and $<z|a|f> = N(z)\frac{d}{d_q z^*} N(z)^{-1} f(z)$, (v) any zero of $e_q(-\zeta_i) = 0$ can be the upper endpoint of integration provided something restricts $e_q(x)$ beyond $-\zeta_i$. If not, on the rhs of (12) there is also $r_n = -[n]! \sum_{k=0}^{n} \frac{1}{[n-k]!} (q^{1/2}x_k)^{n-k} e_q(-x_k)$ where $x_k = q^{k/2}|\zeta_i|$. This restriction occurs if there are q-discrete auxiliary states ($|\tilde{z}_k|^2 = x_k$)

$$|\tilde{z}_k>_q = M_k \sum_{j=0}^{\infty} \frac{(q^{1/4}\tilde{z}_k)^j}{\sqrt{[j]!}} |j+k>, \qquad \tilde{a}_k|\tilde{z}_k>_q = (q^{1/4}\tilde{z}_k)|\tilde{z}_k>_q \tag{18}$$

with $k = 0, 1, \ldots$; $M_k = e_q(q^{1/2}|\tilde{z}_k|^2)^{-1/2}$; with a discrete measure $d\tilde{\mu}_k = \frac{1}{2\pi M_k^2} e_q(-|\tilde{z}_k|^2)\, d\theta$.

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3 The q-Analogue Quantized Radiation Field and Its Uncertainty Relations

In analyzing the field in the $|z>_q$ classical limit, we suppress the $\overrightarrow{k}$ mode and $\hat{\epsilon}$ polarization indices for the generic electric and magnetic fields, etc. . There are diagonal representations of operators, e.g. the single-mode density operator

$$\hat{\rho} = \int d\mu(z)\,\phi_N(z,z^*)|z><z| \tag{19}$$

where $\int d\mu(z)\,\phi_N(z,z^*) = 1$ as $Tr(\rho) = 1$; so $< (a^\dagger)^r a^s > = Tr[\rho(a^\dagger)^r a^s] = \int d\mu(z)\,(z^*)^r z^s \phi_N(z,z^*)$. Similarly, $< a^r(a^\dagger)^s > = \int d\mu(z)\, z^r(z^*)^s \psi_N(z,z^*)$ for $\psi_N(z,z^*) \equiv < z|\hat{\rho}|z >$, $\int d\mu(z)\,\psi_N(z,z^*) = 1$, and so

$$\psi_N(z,z^*) = \int d\mu(y)\,\phi_N(y,y^*)N(y)^2 N(z)^2 e_q(yz^*)e_q(zy^*) \tag{20}$$

Note that due to the use of q-integration to obtain (16), a new " q-quantization" in the z complex plane has occurred, e.g. ϕ_N contributes to (19) only when

$$|z|^2 = q^{(2n+1)/2}\zeta_i, \qquad n = 0,1,2,\ldots. \tag{21}$$

Consequently, for the generic electric and magnetic fields

$$\hat{E} = i(\hbar\omega/2\epsilon_0 V)^{1/2}[ae^{i(\overrightarrow{k}\cdot\overrightarrow{r}-\omega t)} - a^\dagger e^{-i(\overrightarrow{k}\cdot\overrightarrow{r}-\omega t)}] \tag{22}$$

with $z = |z|\exp(i\theta)$,

$$< z|\hat{E}|z > = -2(\hbar\omega/2\epsilon_0 V)^{1/2}\,|z|\,\sin(\overrightarrow{k}\cdot\overrightarrow{r} - \omega t + \theta) \tag{23}$$

which indeed *"looks "* like a classical field *but* the possible amplitudes are q-quantized; the modulus squared assumes a geometric series of discrete values.

With the usual definitions $\hat{P} = -i(\hbar\omega/2)^{1/2}(a - a^\dagger)$, $\hat{Q} = (\hbar/2\omega)^{1/2}(a + a^\dagger)$, the fractional uncertainties $\frac{\Delta\hat{Q}}{|<\hat{Q}>|}$ and $\frac{\Delta\hat{P}}{|<\hat{P}>|}$ are of $O(1)$ for $|z| \to \infty$ and

$$< z|[Q,P]|z > = < z|[a,a^\dagger]|z > = i\hbar\lambda(z) \geq i\hbar \tag{24}$$

where the important function ($q = \exp s$)

$$\lambda(z) \equiv N(z)^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}\cosh(s(2n+1)/4)}{[n]!\cosh(s/4)} \tag{25}$$

goes as $(q^{-1/2} - 1)|z|^2 + 1$ as $|z| \to \infty$. However, $\Delta Q\Delta P = 1/2|< [Q,P] >|$ for $|z>_q$ expectation values, per (5).

For the generic electric field, in the $|n>_q$ basis

$$(\Delta\hat{E})^2_{|n>} = (\hbar\omega/2\epsilon_0 V)([n+1] + [n]) \tag{26}$$

Instead, in the $|z>_q$ basis

$$(\Delta \hat{E})^2_{|z>} = (\hbar\omega/2\epsilon_0 V) < z|[a,a^\dagger]|z> = (\hbar\omega/2\epsilon_0 V)\,\lambda(z) \qquad (27)$$

and so the fractional uncertainty in amp $\hat{E}$ (or $\hat{B}$) is also of $O(1)$. Note that from (25) $\lambda(z) = N(z)^2 e_q(|z|^2/q^{1/2}) - |z|^2(1 - q^{1/2})$. There is a curious operator identity for $q \neq 1$

$$(-(i/\hbar)[Q,P]\cosh(s/4))^2 - ((2/\hbar\omega)H\sinh(s/4))^2 = 1 \qquad (28)$$

which fundamentally relates the basic commutation relation and the single-mode hamiltonian[2] , (quadratic in $\hat{P},\hat{Q}$)

$$H = (1/2)\hbar\omega(a^\dagger a + aa^\dagger) = (1/2)(\hat{P}^2 + \omega^2\hat{Q}^2). \qquad (29)$$

We get for $(1-q)$ small, that $\lambda(z) \simeq \sqrt{1 + ((2\bar{E}/\hbar\omega)^2 - 4\bar{E}/\hbar\omega)\tanh^2(s/4)}$ where $\bar{E} = E_{\bar{n}} - \hbar\omega/2$ for $\bar{n} = < z|N|z >$ or $< z|[N]|z >$, so λ depends on the deviation from the vacuum energy.

4 q-Generalizations of the Phase Operators

Since z's magnitude may be q-quantized as in basic analysis, we next consider possible phase operators. Recall $z = |z|\exp(i\theta)$ and that mathematically a hermitian phase operator conjugate to N, to $[N] \equiv a^\dagger a$, or to H does not exist [3].

An $\widehat{exp}(i\phi)_q$ generalization of the phase operator of Susskind-Glogower [3] is defined by [4]

$$a \equiv ([N+1])^{1/2}\widehat{exp}(i\phi) \qquad\qquad a^\dagger \equiv \widehat{exp}(-i\phi)([N+1])^{1/2} \qquad (30)$$

and there are hermitian operators

$$\widehat{cos}(\phi) \equiv (1/2)[\widehat{exp}(i\phi) + \widehat{exp}(-i\phi)] \qquad\qquad \widehat{sin}(\phi) \equiv (1/2i)[\widehat{exp}(i\phi) - \widehat{exp}(-i\phi)]. \qquad (31)$$

These generalizations give many q-independent operator commutation relations , see [4]. So, from $[N,\widehat{cos}(\phi)] = -i\widehat{sin}(\phi),\dots$ the usual number-phase uncertainty relations follow for arbitrary q:

$$\Delta N\,\Delta\widehat{cos}(\phi) \geq (1/2)|<\widehat{sin}(\phi)>| \qquad\qquad \Delta N\,\Delta\widehat{sin}(\phi) \geq (1/2)|<\widehat{cos}(\phi)>| \qquad (32)$$

In the $|n>_q$ basis, these definitions (30-31) correspond to

$$\widehat{exp}(i\phi)_q \equiv \sum_{n=0}^{\infty} |n><n+1| \qquad (33)$$

which is manifestly q-independent in $|n>_q$, non-unitary, and a q-analogue of the SG operator.

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[2]For H, the energy is not additive for two widely separated systems, violating the usual cluster decomposition "axiom" in quantum field theory. But, for q-quanta this is not unreasonable since the fractional uncertainty in the energy based on H is also $O(1)$ in the $|z>$ basis and the quanta by (1) are compelled to be always interacting,i.e. by exclusion-principle-like q-forces! An alternative hamiltonian is $H_N = \hbar\omega(N + 1/2)$ where N is the number operator and it has the usual free-quanta additivity, etc. .

Analogously, a q-generalization of the Pegg and Barnett operator [5] is obtained [4] by introducing a complete, orthonormal basis of $(s+1)$ phase states $|\theta_m>_q = (s+1)^{-1/2}\sum_{m=0}^{s}\exp(in\theta_m)|n>_q$, $\theta_m = \theta_0 + 2m\pi/(s+1)$, with $m = 0, 1, \ldots, s,$. These are eigenstates of the respectively hermitian and unitary

$$\hat{\phi}_q \equiv \sum_{m=0}^{s} \theta_m |\theta_m><\theta_m| \tag{34}$$

$$\exp(i\hat{\phi})_q \equiv |0><1| + \cdots + |s-1><s| + \exp(i(s+1)\theta_0)|s><0| \tag{35}$$

which is manifestly q-independent, unitary, and only differs from (33) by the last term. Chaichian and Ellinas' polar operator is the same as $\exp(i\hat{\phi})_q$ when the reference phase in [6] is chosen to be $\phi_R = (s+1)\theta_0$.

Finally, although the $|z>_q$ coherent states do not minimize the $N, \widehat{\cos}(\phi), \widehat{\sin}(\phi)$ uncertainty relations (32), they do in the PB-case [7] both *give* and *minimize* Dirac's commutation relation, i.e. in $|z>_q$ basis for $|z|$ large

$$[N, \hat{\phi}_q] = i \tag{36}$$

Also $\widehat{\cos}(\phi)_q$ and $\widehat{\sin}(\phi)_q$ show some "correspondence principle" type behavior:

$$\frac{<z|\widehat{\sin}(\phi)|z>}{<z|\widehat{\cos}(\phi)|z>} = \frac{\sin(\theta)}{\cos(\theta)}, \qquad <z|\widehat{\cos}(\phi)^2 + \widehat{\sin}(\phi)^2|z> = 1 - (1/2)e_q(|z|^2)^{-1} \tag{37}$$

and proportionality for $<z|\widehat{\cos}(\phi)^2 - \widehat{\sin}(\phi)^2|z>$.

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III. QUANTUM OPTICS

DISTRIBUTION OF PHOTONS IN "SQUEEZED" POLYMODE LIGHT

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Abstract

The distribution functions of photons in squeezed and correlated light for one-mode and multimode cases are obtained based on the method of integrals of motion. Correlation coefficient and squeezing parameter are calculated. The possibility to generate squeezed light using nonstationary Casimir effect is discussed. Quantum parametric Josephson junction is proposed as quantum vacuum generator of electrical vibrations.

1 Introduction

The aim of this work is to discuss integrals of the motion and uncertainty relations and to obtain the distribution function of photons in squeezed and correlated light for one–mode and multimode cases. The distribution function of photons in squeezed light for one–mode fields was discussed by Schleich and Wheeler [1], by Agarwal and Adam [2], and by Chaturvedi and Srinivasan [3]. The photon distribution function for squeezed and correlated light [4] and [5] was discussed by Dodonov, Klimov and Man'ko [6]. This distribution function depends not only on the squeezing parameter, but also on the correlation parameter connected with Schrödinger uncertainty relation [7] as well,

$$\delta q \delta p \geq \frac{\hbar}{2\sqrt{1 - r^2}},$$ (1)

where the parameter r is the correlation coefficient of the position and momentum

$$r = (\delta q \delta p)^{-1} \left\{ \frac{\langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle}{2} - \langle \hat{q} \rangle \langle \hat{p} \rangle \right\}.$$ (2)

The states with nonzero parameter r we call the correlated states. In the section below we'll consider the problem how to find the states which minimize the Schrödinger uncertainty relation. For such states instead of the Schrödinger inequality we have the equality

$$\delta q \delta p = \frac{\hbar}{2\sqrt{1 - r^2}}.$$ (3)

These states describe squeezed and correlated light. We will demonstrate in the next section how these states are naturally created for quantum parametric oscillator. The case of the photon

distribution function for the two-mode squeezed light was considered by Caves, Zhu, Milburn and Schleich [8]. Multidimensional generalization of the expression for the distribution of photons in squeezed light in terms of Hermite polynomials of several variables may be reformulated. We derive this expression on the bases of the result obtained in [17], [9] and [10] for a nonstationary parametric multidimensional oscillator.

The squeezing phenomenon in quantum optics is closely related to the oscillator models described by relativistic wave equations for elementary particles with mass spectrum. These relativistic models have been studied by Yukawa [11], by Markov [12], by Ginzburg and Man'ko [13], and by Kim and Noz [14]. As shown in [14], the Lorentz boost applied to relativistic oscillator gives the squeezing whose mathematics is identical to that of the squeezing in quantum optics. The statistical properties of such squeezed relativistic oscillators have been studied by Kim and Wigner [15].

To obtain the photon distribution function we will consider the nonstationary multidimensional oscillator. We shall discuss first the one-mode case in Sec. 2.

2 One-mode Light

The Hamiltonian for one-mode light is given by the formula

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}).$$ (4)

This mode of the electromagnetic field in a resonator may be described by the model of the mechanical oscillator with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{q}^2}{2}.$$ (5)

In this case the annihilation and creation operators with boson commutation relations

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{q}}{l} + i\frac{\hat{p}}{p_0}\right),$$ (6)

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\hat{q}}{l} - i\frac{\hat{p}}{p_0}\right),$$ (7)

where

$$l = (\frac{\hbar}{m\omega})^{\frac{1}{2}}, \quad p_0 = (\hbar m\omega)^{\frac{1}{2}},$$ (8)

connect both Hamiltonians and forms, together with the identity operator, the basis of the Heisenberg-Weyl algebra. In coordinate representation the complete set of coherent states $\mid \alpha\rangle$ satisfying the equation

$$\hat{a} \mid \alpha\rangle = \alpha \mid \alpha\rangle,$$ (9)

where α is any complex number, is given by the formula

$$\langle q \mid \alpha\rangle = \pi^{-\frac{1}{4}}l^{-\frac{1}{2}}\exp\left[-\frac{q^2}{2l^2} - \frac{\mid\alpha\mid^2}{2} + \frac{\sqrt{2}\alpha q}{l} - \frac{\alpha^2}{2}\right].$$ (10)

The dispersions of the positions δq and the momentum δp do not depend on the parameter α, and are given by the relation

$$\delta q = \frac{l}{\sqrt{2}}, \tag{11}$$

$$\delta p = \frac{p_0}{\sqrt{2}}. \tag{12}$$

For the coherent states the product of these dispersions minimizes the Heisenberg inequality

$$\delta q \delta p = \frac{\hbar}{2}. \tag{13}$$

The time evolution of the coherent state $\mid \alpha, t \rangle$ may be obtained by simple replacement of the parameter α in the formula (10) by the term $\alpha \exp(-i\omega t)$ and the phase of the wave function. We have

$$\langle q \mid \alpha, t \rangle = \langle q \mid \alpha \exp\left(-i\omega t\right) \rangle \exp\left(\frac{-i\omega t}{2}\right). \tag{14}$$

The correlation coefficient of the position and momentum is equal to zero for arbitrary coherent state. It is also equal to zero for stationary Fock state $\mid n, t \rangle$ satisfying the eigenvalue equation

$$\hat{a}^\dagger \hat{a} \mid n, t \rangle = n \mid n, t \rangle, \qquad n = 0, 1, 2, \ldots \tag{15}$$

This state has the following wave function in the coordinate representation

$$\langle q \mid n, t \rangle = \pi^{-\frac{1}{4}} l^{-\frac{1}{2}} 2^{-\frac{n}{2}} (n!)^{-\frac{1}{2}} H_n\left(\frac{q}{l}\right) \exp\left[-\frac{q^2}{2l^2} - i\omega t\left(n + \frac{1}{2}\right)\right] \tag{16}$$

The photon distribution function $W_n(\alpha)$ for the coherent state $\mid \alpha, t \rangle$ is determined by the overlap integral

$$\mid \langle n, t \mid \alpha, t \rangle \mid^2 = W_n(\alpha) \tag{17}$$

and coincides with the Poisson distribution function

$$W_n(\alpha) = \frac{\mid \alpha \mid^{2n}}{n!} \exp\left(-\mid \alpha \mid^2\right). \tag{18}$$

The mode has the following time-dependent integral of the motion

$$\hat{A}(t) = \exp\left(i\omega t\right)\hat{a}. \tag{19}$$

We now discuss how the influence of the dependence of the oscillator frequency $\Omega(t)$ on time will change the photon distribution function and the dispersions of the conjugate variables $\hat{q}$ and $\hat{p}$. The Hamiltonian of the mechanical parametric oscillator depends on time and has the form

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{m\Omega^2(t)\hat{q}^2}{2}. \tag{20}$$

This system has the linear integral of the motion [16]

$$\hat{A}(t) = \frac{i}{\sqrt{2}} \left(\frac{\epsilon(t)\hat{p}}{p_0} - \frac{\dot{\epsilon}(t)\hat{q}}{l\omega}\right). \tag{21}$$

Here $\omega = \Omega(0)$, and the complex function $\epsilon(t)$ satisfies the equation of classical oscillator motion

$$\ddot{\epsilon} + \Omega^2(t)\epsilon = 0. \tag{22}$$

The integrals of motion $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ satisfy the boson commutation relation

$$\left[\hat{A}(t), \hat{A}^\dagger(t)\right] = 1, \tag{23}$$

if the Wronskian for the equation (22) is given by the equality

$$\epsilon\dot{\epsilon}^* - \epsilon^*\dot{\epsilon} = 2i\omega. \tag{24}$$

The initial condition for the function $\epsilon(t)$ may be taken as follows

$$\epsilon(0) = 1, \qquad \dot{\epsilon}(0) = i\omega. \tag{25}$$

If the frequency of the oscillator is constant, the function $\epsilon = \exp(i\omega t)$ and the formula (21) gives the integral of the motion (19). The normalized state $\mid 0, t\rangle$ satisfying the Schrödinger equation and the relation

$$\hat{A}(t) \mid 0, t\rangle = 0 \tag{26}$$

has the following wave function in the coordinate representation

$$\langle q \mid 0, t\rangle = \pi^{-\frac{1}{4}}(l\epsilon)^{-\frac{1}{2}} \exp\left(\frac{i\dot{\epsilon}q^2}{2\omega\epsilon l^2}\right). \tag{27}$$

The state $\mid \alpha, t\rangle$ which is the eigenstate of the integral of the motion $\hat{A}(t)$ given by formula (21)

$$\hat{A}(t) \mid \alpha, t\rangle = \alpha \mid \alpha, t\rangle \tag{28}$$

has the following wave function in the coordinate representation

$$\langle q \mid \alpha, t\rangle = \langle q \mid 0, t\rangle \exp\left(-\frac{1}{2}\mid \alpha \mid^2 + \frac{\sqrt{2}\alpha q}{l\epsilon} - \frac{\alpha^2\epsilon^*}{2\epsilon}\right). \tag{29}$$

Here α is an arbitrary complex number and

$$\langle \beta, t \mid \alpha, t\rangle = \exp\left(-\frac{\mid \alpha \mid^2}{2} - \frac{\mid \beta \mid^2}{2} + \beta^*\alpha\right). \tag{30}$$

The wave function (29) satisfies the Schrödinger equation. The Fock states of the parametric mode $\mid n, t\rangle$ satisfy the eigenvalue equation

$$\hat{A}^\dagger(t)\hat{A}(t) \mid n, t\rangle = n \mid n, t\rangle, \qquad n = 0, 1, 2, \ldots \tag{31}$$

The solutions to this equation have the following form in the coordinate representation

$$\langle q \mid n, t\rangle = \frac{1}{\sqrt{n!}}\langle q \mid 0, t\rangle\left(\frac{\epsilon^*}{2\epsilon}\right)^{n/2} H_n\left(\frac{q}{l \mid \epsilon \mid}\right). \tag{32}$$

Since the state $\mid \alpha, t\rangle$ is the generating state for the Fock states $\mid m, t\rangle$

$$\mid \alpha, t\rangle = \exp\left(-\frac{\mid \alpha \mid^2}{2}\right) \sum_{m=0}^{\infty} \frac{\alpha^m \mid m, t\rangle}{\sqrt{m!}}, \tag{33}$$

the transition probability from the initial state $\mid n\rangle$ may be calculated

$$W_n^m = W_0^0 \frac{n!}{m!} \left(P_{\frac{(m+n)}{2}}^{\frac{(m-n)}{2}}(W_0^0)\right)^2, \qquad m \geq n. \tag{34}$$

Here the transition probability W_0^0 is the probability to be in the ground state

$$W_0^0 = 2\left(\mid \epsilon \mid^2 + \omega^{-2}\mid \dot{\epsilon} \mid^2 + 2\right)^{-1/2}. \tag{35}$$

For $n > m$ the formula (34) must be changed to

$$W_n^m = W_0^0 \frac{m!}{n!} \left(P_{\frac{(n+m)}{2}}^{\frac{(n-m)}{2}}(W_0^0)\right)^2. \tag{36}$$

The numbers n and m in the formulae (34) and (35) are either both even or both odd. If one of these numbers is even and another number is odd the transition probability between such states is equal to zero

$$W_{2k}^{2p+1} = W_{2k+1}^{2p} = 0, \qquad k, p = 0, 1, 2, \ldots \tag{37}$$

The formulae (34) and (36) describe the photon distribution function for the one-mode electromagnetic field in a resonator either with moving walls or with media with time-dependent refraction index. Thus, we conclude that the squeezing parameters of the parametric oscillator

$$S_q = \left(\frac{2m\omega}{\hbar}\right)^{1/2} \delta q = \mid \epsilon \mid, \tag{38}$$

$$S_p = \left(\frac{2}{\hbar m\omega}\right)^{1/2} \delta p = \mid \dot{\epsilon} \mid \tag{39}$$

are connected with the photon distribution function by the ratio (35) which may be rewritten in the form

$$W_0^0 = 2(S_q^2 + S_p^2 + 2)^{-1/2}. \tag{40}$$

In the case of vacuum light $S_p = S_q = 1$ and the vacuum-vacuum transition probability W_0^0 is equal to unity.

Another photon distribution function corresponds to the excitation of light state which may be described by the model of the forced mechanical oscillator with Hamiltonian

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 - f(t)\hat{q}. \tag{41}$$

This oscillator has the integral of the motion [16]

$$\hat{A}(t) = \exp(i\omega t)\hat{a} + \delta(t), \tag{42}$$

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$$\delta(t) = -\frac{il}{\sqrt{2}\hbar} \int_0^t f(\tau) \exp{(i\omega\tau)} d\tau. \tag{43}$$

If the initial state of a forced oscillator is the coherent state, the squeezing parameters S_q and S_p are time-independent. They are equal to unity. The photon distribution function is described by a Poisson distribution. Thus, if the initial state is the vacuum state the Poisson distribution has the form

$$W_m = \frac{|\delta|^{2m}}{m! \exp(|\delta|^2)}. \tag{44}$$

The physical meaning of the parameter $|\delta|^2$ (43) which determines the integral of the motion (42) is just the mean photon number after the excitation of the vacuum state by the external linear force. The photon distribution function W_n^m in the case when the initial state was the state $|n\rangle$ with n photons is described by the function

$$W_n^m = \frac{n! |\delta|^{2(m-n)}}{m! \exp(|\delta|^2)} \left[L_n^{m-n}(|\delta|^2) \right]^2. \tag{45}$$

Here the function L_N^M is the Laguerre polynomial.

Now consider a general situation when the frequency of an oscillator depends on time and an external force is present. The Hamiltonian of the mechanical oscillator model looks like

$$\hat{H}(t) = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2(t)\hat{q}^2 - f(t)\hat{q}. \tag{46}$$

We have taken $m = \omega = \hbar = 1$. The linear integral of motion $\hat{A}(t)$ is equal in this case to

$$\hat{A}(t) = u(t)\hat{a} + v(t)\hat{a}^\dagger + \delta(t), \tag{47}$$

$$u(t) = \frac{1}{2}\left(\epsilon(t) - i\dot{\epsilon}(t)\right), \tag{48}$$

$$v(t) = \frac{1}{2}\left(\epsilon(t) + i\dot{\epsilon}(t)\right), \tag{49}$$

$$\delta(t) = -\frac{i}{\sqrt{2}} \int f(\tau)\epsilon(\tau)d\tau. \tag{50}$$

The normalized eigenstate $\psi_\alpha(q,t)$ of the integral of motion (47) has the form

$$\psi_\alpha(q,t) = \psi_0(q,t) \exp\left[-\frac{|\alpha|^2}{2} + \frac{\sqrt{2}\alpha q}{\epsilon} + \frac{\alpha(\delta\epsilon^* + \delta^*\epsilon)}{\epsilon} - \frac{\alpha^2\epsilon^*}{2\epsilon} \right], \tag{51}$$

where

$$\psi_0(q,t) = \frac{1}{\sqrt{\sqrt{\pi}\epsilon}} \exp\left[\frac{iq^2\dot{\epsilon}}{2\epsilon} - \frac{\sqrt{2}q\delta}{\epsilon} - \frac{\delta^2\epsilon^*}{2\epsilon} - \frac{|\delta|^2}{2} + \frac{1}{2}\int_0^t (\delta\dot{\delta}^* - \delta\dot{\delta}^*)d\tau \right]. \tag{52}$$

The squeezing parameters S_q and S_p for the states (51) are described by the formulae for the unforced parametric oscillator (38), (39). The correlation coefficient r is given by the expression

$$r = |\epsilon\dot{\epsilon}|^{-1} \left[(\epsilon\dot{\epsilon})^2 - 1 \right]^{1/2}. \tag{53}$$

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The Fock states which are the eigenstates of the integral of motion $\hat{A}^\dagger(t)\hat{A}(t)$ are of the form

$$\psi_n(q,t) = \psi_0(q,t)\frac{1}{\sqrt{n!}}\left(\frac{\epsilon^*}{2\epsilon}\right)^{n/2} H_n\left(\frac{q+(\delta\epsilon^*+\delta^*\epsilon)/\sqrt{2}}{\mid\epsilon\mid}\right). \tag{54}$$

The photon distribution function for the electromagnetic field created due to the nonstationary Casimir effect is expressed in terms of Hermite polynomials of two variables

$$\frac{W_n^m}{W_0^0} = (n!m!)^{-1}\mid H_{nm}^{\{R\}}(x_1,x_2)\mid^2, \tag{55}$$

where

$$x_1 = -\frac{\delta^*}{\zeta^*}, \qquad x_2 = \delta - \frac{\eta\delta^*}{\zeta^*}, \tag{56}$$

and the matrix R has the elements

$$R = \zeta^{-1}\begin{pmatrix} \eta & -1 \\ -1 & -\eta^* \end{pmatrix}.$$

The parameters ζ and η are given by the relation

$$\epsilon(t) = \zeta e^{it} - \eta e^{-it}. \tag{57}$$

The photon distribution function (55) has oscillatory behavior due to the oscillatory behavior of the Hermite polynomial of two variables.

The last photon distribution function describes the influence of the nonstationary Casimir effect on the initially thermal equilibrium state

$$\hat{\rho}(0) = Z^{-1}\exp\left[-\beta(\hat{a}^\dagger\hat{a} + \frac{1}{2})\right], \tag{58}$$

$$Z^{-1} = 2\sinh(\beta/2). \tag{59}$$

The distribution of photons in the light mode is expressed by the density matrix diagonal elements

$$\frac{\rho_{nn}}{\rho_{00}} = \frac{1}{n!}H_{nn}^{\{R\}}(\bar{x}_1,\bar{x}_2), \tag{60}$$

where

$$R = D^{-1}\begin{pmatrix} u^*v(1-z^2) & -z \\ -z & uv^*(1-z^2) \end{pmatrix}$$

and

$$\bar{x}_1 = (1-z)\frac{\delta^*uz+\delta v^*}{\mid u\mid^2 z^2 - \mid v\mid^2}, \qquad \bar{x}_2 = \bar{x}_1^*.$$

The photon distribution function has the following deformed Planck distribution form

$$\langle n \rangle = \frac{1}{e^\beta - 1} + \mid v \mid^2\frac{e^\beta + 1}{e^\beta - 1} + \mid \delta u^* - \delta^*v \mid^2. \tag{61}$$

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If there is no external force the parameter $\delta = 0$. The deformed Planck distribution has the form

$$\langle n \rangle = \frac{1}{e^{\beta} - 1} + \mid v \mid^2 \coth(\beta/2). \tag{62}$$

The squeezing parameters S_q and S_p depend on temperature

$$S_q = \mid \epsilon \mid (\coth{(\beta/2)})^{1/2}, \tag{63}$$

$$S_p = \mid \dot{\epsilon} \mid (\coth{(\beta/2)})^{1/2}. \tag{64}$$

Thus the parametric excitation of the oscillator may produce the squeezing phenomenon when

$$S_q < 1, \tag{65}$$

or

$$S_p < 1. \tag{66}$$

But the higher the temperature the more difficult to obtain the squeezing.

3   Polymode Squeezed Light

We will consider the photon distribution function for polymode squeezed and correlated light using the model of nonstationary parametric multidimensional quantum oscillator with N degrees of freedom. Its Hamiltonian may be written in the form

$$\hat{H} = \frac{1}{2}\hat{\mathbf{q}}B(t)\hat{\mathbf{q}} + \vec{C}(t)\hat{\mathbf{q}}, \tag{67}$$

where the vector

$$\hat{\mathbf{q}} = (\hat{p}_1, \hat{p}_2, ..., \hat{p}_N, \hat{x}_1, \hat{x}_2, ..., \hat{x}_N) = (\hat{\mathbf{p}}, \hat{\mathbf{x}}) \tag{68}$$

contains N momentum projection operators and N position projection operators. The 2Nx2N - matrix $B(t)$ and 2N - vector $\vec{C}(t)$ are time-dependent parameters of the system. The model corresponds to N light modes in the resonator. The interaction of these modes depends on time either due to the motion of the resonator walls or due to the time-dependence of the media refraction index. The system must demonstrate the properties of nonstationary Casimir effect for $N \mapsto \infty$. The oscillator has 2N - vector

$$\hat{\mathbf{Q}}(t) = \Lambda(t)\hat{\mathbf{q}} + \vec{\Delta}(t), \tag{69}$$

which is the linear integral of motion if the 2Nx2N–symplectic matrix Λ satisfies the classical equation of motion

$$\dot{\Lambda} = K\Sigma B(t), \tag{70}$$

where the 2Nx2N–matrix Σ has the form

$$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{71}$$

The 2N – vector $\vec{\Delta}(t)$ obeys the equation

$$\dot{\vec{\Delta}} = \Lambda(t)\Sigma\vec{C}(t). \tag{72}$$

The solution $(\Lambda, \vec{\Delta})$ of equations (70) and (72) describe the classical trajectories of multidimensional oscillators and may be considered as an element of the inhomogeneous real symplectic group ISp(2N,R). The initial conditions for these equations are

$$\Lambda(0) = 1, \qquad \vec{\Delta}(0) = 0. \tag{73}$$

The propagator of the system has the form

$$G(x_2, x_1, t) = \frac{1}{\sqrt{\det(-2i\pi\lambda_3)}} \exp\left\{ -\frac{i}{2}(\mathbf{x}_2\lambda_3^{-1}\lambda_4\mathbf{x}_2 \right.$$

$$-2\mathbf{x}_2\lambda_3^{-1}\mathbf{x}_1 + \mathbf{x}_1\lambda_1\lambda_3^{-1}\mathbf{x}_1 + 2\mathbf{x}_2\lambda_3^{-1}\vec{\delta}_2$$

$$\left. +2\mathbf{x}_1(\vec{\delta}_1 - \lambda_1\lambda_3^{-1}\vec{\delta}_2) + \vec{\delta}_2\lambda_1\lambda_3^{-1}\vec{\delta}_2 - 2\int_0^t \dot{\vec{\delta}}_1\vec{\delta}_2 d\tau)\right\}, \tag{74}$$

with ($\hbar = 1$). Here the matrices λ_i, $i = 1, 2, 3, 4$ are NxN blocks of the matrix Λ

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \tag{75}$$

and N–vectors $\vec{\delta}_1, \vec{\delta}_2$, are the components of the vector $\vec{\Delta}$

$$\vec{\Delta} = (\vec{\delta}_1, \vec{\delta}_2). \tag{76}$$

The Hamiltonian of the system may be rewritten in terms of the boson annihilation and creation operators $\hat{\mathbf{a}} = (\hat{a}_1, ..., \hat{a}_N)$, $\hat{\mathbf{a}}^\dagger = (\hat{a}_1^\dagger, ..., \hat{a}_N^\dagger)$, in the form

$$\hat{H} = \frac{1}{2}(\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger)D(t)\begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{a}}^\dagger \end{pmatrix} + \vec{f}\hat{\mathbf{a}} + \vec{f}^*\hat{\mathbf{a}}^\dagger. \tag{77}$$

If we introduce the 2N – vector

$$\hat{\mathbf{A}}(t) = \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{a}}^\dagger \end{pmatrix} \tag{78}$$

this vector is connected with the 2N – vector $\hat{\mathbf{q}} = (\hat{\mathbf{p}}, \hat{\mathbf{x}})$ by the relation

$$\hat{\mathbf{A}} = V\hat{\mathbf{q}}, \tag{79}$$

where the 2Nx2N – matrix V has the form

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}, \qquad V^\dagger = V^{-1}. \tag{80}$$

Then the matrix $B(t)$ in (67) is connected with the matrix $D(t)$ in (77) by the relation

$$D(t) = V^\dagger B V^\dagger. \tag{81}$$

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The integrals of motion of the multimode nonstationary system are given by the formula

$$\hat{A}(t) = \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \Lambda_1(t) \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} + \vec{\Gamma}(t). \tag{82}$$

The 2N – vector $\vec{\Gamma}(t) = (\vec{\gamma}, \vec{\gamma}^*)$ is connected with the vector $\vec{\Delta}$

$$\vec{\Gamma}(t) = V\vec{\Delta}(t). \tag{83}$$

The 2Nx2N – matrix $\Lambda_1(t)$ with NxN – blocks ξ and η

$$\Lambda_1(t) = \begin{pmatrix} \xi(t) & \eta(t) \\ \xi^*(t) & \xi^*(t) \end{pmatrix} \tag{84}$$

is connected with the matrix $\Lambda(t)$ by the relation

$$\Lambda_1(t) = V\Lambda(t)V^\dagger. \tag{85}$$

The propagator of the system in coherent state representation has the Gaussian form

$$G(\vec{\alpha}^*, \vec{\beta}, t) = \frac{1}{\sqrt{\det \xi(t)}} \exp \left(-\frac{1}{2}\vec{\alpha}^*\xi^{-1}\eta\vec{\alpha}^* + \vec{\alpha}^*\xi^{-1}\vec{\beta} - \vec{\alpha}^*\xi^{-1}\vec{\gamma} + \frac{1}{2}\vec{\beta}\eta^*\xi^{-1}\vec{\beta} \right.$$
$$\left. + \vec{\beta}(\vec{\gamma}^* - \eta^*\xi^{-1}\vec{\gamma}) + \frac{1}{2}\vec{\gamma}\eta^*\xi^{-1}\vec{\gamma} - \int_0^t \vec{\gamma}(\tau)\dot{\vec{\gamma}}^*(\tau)d\tau \right). \tag{86}$$

The photon distribution function for multimode case may be obtained by expansion of the propagator (86) into a series with respect to the parameters α^*. We have the distribution of photons in squeezed and correlated light

$$| \langle \vec{m} \mid \vec{\beta}, t \rangle |^2 = W_{\vec{m}}(\vec{\beta}) = | G(0, \vec{\beta}) |^2 \frac{1}{m_1! \dots m_N!} | H_{\vec{m}}^{\{\xi^{-1}\eta\}} \left[\eta^{-1} \left[\vec{\beta} - \vec{\gamma} \right] \right] |^2. \tag{87}$$

Thus, the distribution function for N – mode system is described by the Hermite polynomial of N variables. For squeezed and correlated light the behavior of the function $W_{\vec{m}}(\vec{\beta})$ is very oscillatory as well as for one– and two– mode cases. The partial cases for two-mode light may be obtained if one uses the formulae for Hermite polynomials of two variables found in [5].

4 Nonstationary Casimir Effect

Now let us discuss some possible applications. One of the possible methods to generate squeezed light is to use the nonstationary Casimir effect when moving resonator walls produce continuous time-dependent reconstruction of the electromagnetic vacuum state. The work against the Casimir forces produces two effects. The first effect is the generation of photons from the vacuum. Thus, the resonator with mechanically trembling walls is a quantum vacuum generator of electromagnetic radiation. The second effect is the squeezing of quantized modes in the resonator due to parametric change of vacuum energy. Both effects exist simultaneously. Thus, the plates in the Casimir effect may be moved by external mechanical forces. The refraction index of the media may vary with

time, the geometrical dimensions of the resonator may be influenced by external mechanical forces. In all these cases the vacuum state energy must be changed. This means that the vacuum state is continuously reconstructed. For each of the time moments the state is no longer the vacuum state due to the change of the parameters. If we have a system with photons or other quanta in a box with changing volume, it corresponds to the process with the creation of quanta, e.g. the photons. So, due to the work against the Casimir forces one form of energy may be converted into the other form. Thus for waving neutral plates (due to mechanical external forces) between the plates the photons must be created and this means that the mechanical energy from external sources is converted into electromagnetic energy of photons. It is interesting that this effect must create the quanta of all other fields existing in nature. Thus, due to Casimir forces we can have the generation of photons in a parametric resonator which may be called a quantum vacuum generator.

It is possible to discuss another reduction of nonstationary Casimir effect using the Josephson junction. If there is no external voltage in the Josephson junction but its parameters are time-dependent, the vibrations of current and voltage will be excited in it. This suggestion [17] is based on the analogy of the Josephson junction and a conventional resonant circuit (quantized resonant circuit). In classical resonant circuits it is impossible to excite electrical vibrations without external sources of voltage. But for a quantum resonator circuit due to Casimir nonstationary forces, it is possible to transform mechanical energy which may be the reason for the change of the circuit parameters into electrical energy of current vibrations. If this idea is realized it will be a quantum vacuum generator of electrical vibrations. The current and voltage in this case play the role of conjugate quantum observables and in parametric Josephson junctions they may be squeezed. Thus, the quantum noise in Josephson junctions may be reduced for current. In this case the voltage will have larger noise. The squeezed and correlated states of Josephson junctions may be also excited by changing its parameters with time.

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TWO PHOTON ANNIHILATION OPERATORS AND SQUEEZED VACUUM

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Abstract

We introduce inverses of the harmonic oscillator creation and annihilaton operators by their actions on the number states. Three of the two photon annihilation operators, viz., $\hat{a}^{+-1}\hat{a}$, $\hat{a}\hat{a}^{+-1}$ and $\hat{a}^2$, have normalizable right eigenstates with non vanishing eigenvalues. We discuss the eigenvalue equation of these operators and obtain their normalized eigenstates. We find that the Fock state representation, in each case separates into two sets of states, one involving only the even number states while the other involving only the odd number states. We show that the even set of eigenstates of the operator $\hat{a}^{+-1}\hat{a}$ is the customary squeezed vacuum $\hat{S}(\sigma)|0\rangle$.

1 Introduction

In quantum optics several different representations of the harmonic oscillator states have been discussed such as number states, coherent states [1], squeezed states [2-4], squeezed number states [5], near number states [6], and photon added coherent states [7]. The basic operators are the boson annihilation and creation operators $\hat{a}$ and $\hat{a}^+$, satisfying the usual commutation relation $[\hat{a}, \hat{a}^+] = 1$. These operators are defined in terms of their actions on number states as

$$\hat{a}|n\rangle = n^{1/2}|n - 1\rangle, \qquad (1.1)$$

$$\hat{a}^+|n\rangle = (n + 1)^{1/2}|n + 1\rangle. \tag{1.2}$$

One may introduce the generalized inverses [8] of $\hat{a}$ and $\hat{a}^+$:

$$\hat{a}^{-1}|n\rangle = (n + 1)^{-1/2}|n + 1\rangle, \tag{1.3}$$

$$\hat{a}^{+-1}|n\rangle = (1 - \delta_{n,0})\,(n)^{-1/2}|n - 1\rangle, \tag{1.4}$$

The operator $\hat{a}^{-1}$ behaves as a creation operator whereas $\hat{a}^{+-1}$ behaves as an annihilation operator. Further $\hat{a}^{-1}$ is the right inverse of $\hat{a}$ and $\hat{a}^{+-1}$ is the left inverse of $\hat{a}^+$, i.e.,

$$\hat{a}\hat{a}^{-1} = \hat{a}^{+-1}\hat{a}^+ = I. \tag{1.5}$$

On the other hand $\hat{a}^{-1}\hat{a}$ and $\hat{a}^+\hat{a}^{+-1}$ give

$$\hat{a}^{-1}\hat{a} = \hat{a}^+\hat{a}^{+-1} = I - |0\rangle\langle 0|, \tag{1.6}$$

where $|0\rangle\langle 0|$ is the projection operator on the vacuum.

Five of the operators exhibiting two photon processes, viz., $\hat{a}^{+2}$, $\hat{a}^{-2}$, $\hat{a}^{+-2}$, $\hat{a}^+\hat{a}^{-1}$ and $\hat{a}^{-1}\hat{a}^+$ do not have any normalizable right eigenstate with non-zero eigenvalue. We can solve the eigenvalue problem for the remaining three, viz., $\hat{a}^{+-1}\hat{a}$, $\hat{a}\hat{a}^{+-1}$ and $\hat{a}^2$. These three are the two photon annihilation operators (TAO). The matrix representation of these TAOs may readily be obtained by noting their actions on the number states $|n\rangle$. Using Eqs.(1.1)-(1.4) we obtain, for $n \geq 2$

$$\hat{a}^{+-1}\hat{a}|n\rangle = [n/(n - 1)]^{1/2}|n-2\rangle, \tag{1.7}$$

$$\hat{a}\hat{a}^{+-1}|n\rangle = [(n - 1)/n]^{1/2}|n-2\rangle, \tag{1.8}$$

$$\hat{a}^2|n\rangle = [n(n - 1)]^{-1/2}|n-2\rangle, \tag{1.9}$$

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whereas their action on $|n\rangle$ with $n = 0$ or 1 gives zero. We, therefore, find the following matrix elements for these operators:

$$\langle m|\hat{a}^{+-1}\hat{a}|n\rangle = [n/(n-1)]^{1/2}\, \delta_{m,n-2}, \tag{1.10}$$

$$\langle m|\hat{a}\hat{a}^{+-1}|n\rangle = [(n-1)/n]^{1/2}\, \delta_{m,n-2}, \tag{1.11}$$

$$\langle m|\hat{a}^2|n\rangle = [n(n-1)]^{1/2}\, \delta_{m,n-2}. \tag{1.12}$$

We now consider the eigenvalue problem for these TAOs in detail.

2 Eigenstates of $\hat{a}^{+-1}\hat{a}$

We write an eigenvalue equation for the operator $\hat{a}^{+-1}\hat{a}$ as:

$$\hat{a}^{+-1}\hat{a}|\lambda, 1\rangle = \lambda|\lambda, 1\rangle, \tag{2.1}$$

where $|\lambda, 1\rangle$ is a right eigenstate of the first TAO $\hat{a}^{+-1}\hat{a}$ with eigenvalue λ and obtain a solution for suitable complex number λ. Expressing $|\lambda, 1\rangle$ in the form

$$|\lambda, 1\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{2.2}$$

we obtain the following recurrence relation for C_n :

$$C_n = \lambda\, [(n-1)/n]^{1/2}\, C_{n-2}. \tag{2.3}$$

From this recurrence relation it is observed that the eigenstates $|\lambda, 1\rangle$ separate into two sets of states involving either even number states or odd number states as follows:

$$|\lambda, +1\rangle = N_+ \sum_{n=0}^{\infty} \frac{[(2n)!]^{1/2}}{2^n n!}\, \lambda^n |2n\rangle \tag{2.4}$$

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and

$$|\lambda,-1\rangle = N_- \sum_{n=0}^{\infty} \frac{2^n\,n!}{[(2n+1)!]^{1/2}} \lambda^n |2n+1\rangle. \tag{2.5}$$

Here N_+ and N_- are the normalization constants given by

$$N_+ = (1 - |\lambda|^2)^{1/4}, \qquad\qquad |\lambda| < 1, \tag{2.6}$$

$$N_- = \left[\frac{|\lambda|(1-|\lambda|^2)^{1/2}}{\sin^{-1}|\lambda|}\right]^{1/2}, \qquad |\lambda| < 1. \tag{2.7}$$

Both of the states $|\lambda,+1\rangle$ and $|\lambda,-1\rangle$ correspond to the same eigenvalue λ and hence any linear combination of these states is the general eigenstate of the TAO $\hat{a}^{+-1}\hat{a}$.

3 Eigenstates of $\hat{a}\hat{a}^{+-1}$

We write the eigenvalue equation for this operator as

$$\hat{a}\hat{a}^{+-1}|\lambda,2\rangle = \lambda|\lambda,2\rangle, \tag{3.1}$$

where $|\lambda,2\rangle$ is the right eigenstate of the second operator $\hat{a}\hat{a}^{+-1}$ with an eigenvalue λ. Proceeding in a manner strictly analogous to that followed in Sec.2, we find that these eigenstates also separate into two sets, one involving even number states and the other involving odd number states

$$|\lambda,+2\rangle = M_+ \sum_{n=0}^{\infty} \frac{2^n n!}{[(2n)!]^{1/2}} \lambda^n |2n\rangle \tag{3.2}$$

and

$$|\lambda,-2\rangle = M_- \sum_{n=0}^{\infty} \frac{[(2n+1)!]^{1/2}}{2^n n!} \lambda^n |2n+1\rangle, \tag{3.3}$$

where M_+ and M_- are the normalization constants given by

$$M_+ = \left[\frac{(1-|\lambda|^2)^{3/2}}{(1-|\lambda|^2)^{1/2} + |\lambda| \sin^{-1} |\lambda|}\right]^{1/2}, \qquad |\lambda| < 1, \qquad (3.4)$$

$$M_- = [1 - |\lambda|^2]^{3/4}, \qquad\qquad |\lambda| < 1. \qquad (3.5)$$

A general eigenstate of the TAO $(\hat{a}\hat{a}^{+-1})$ is a linear combination of the states $|\lambda, +2\rangle$ and $|\lambda, -2\rangle$.

4 Eigenstates of $\hat{a}^2$

Coherent states are the right eigenstates of the annihilation operator $\hat{a}$, and so that of $\hat{a}^2$ also. These states [9] separate neatly into the even and odd parts both being the eigenstates of $\hat{a}^2$, as in the case of the other TAOs. Hence the normalized eigenstates of $\hat{a}^2$ with eigenvalue λ can be expressed in the form

$$|\lambda, +3\rangle = (\cosh |\lambda|)^{-1/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{[(2n)!]^{1/2}} |2n\rangle \qquad (4.1)$$

and

$$|\lambda, -3\rangle = \left[\frac{\sinh |\lambda|}{|\lambda|}\right]^{-1/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{[(2n+1)!]^{1/2}} |2n+1\rangle. \qquad (4.2)$$

Any linear superposition of $|\lambda, +3\rangle$ and $|\lambda, -3\rangle$ states is an eigenstate of $\hat{a}^2$. Of course, a particular linear combination happens to be the coherent state $|(\alpha)^{1/2}\rangle$. Further there is no restriction on the value of $|\lambda|$, whereas in the earlier cases $|\lambda|$ was restricted to be less than 1.

5 Squeezed Vacuum as an Eigenstate of $\hat{a}^{+-1}\hat{a}$

It is interesting to note that the state $|\lambda, +1\rangle$ [Eq. (2.4)] is essentially the squeezed vacuum discussed in literature [3, 10-12].

The squeezed vacuum is generated by the action of the squeeze operator $\hat{S}(\sigma)$ on vacuum

$$\hat{S}(\sigma)|0\rangle = \exp[1/2 \ (\sigma\hat{a}^{+2} - \sigma^{*}\hat{a}^{2})]|0\rangle. \tag{5.1}$$

Using the normal ordered form of the operator $S(\sigma)$ we find the number state representation of the squeezed vacuum as

$$\hat{S}(\sigma)|0\rangle = (\cosh r)^{-1/2} \times \sum_{n=0}^{\infty} \frac{[(2n)!]^{1/2}}{2^{n}n!} (e^{i\theta}\tanh r)^{n}|2n\rangle, \tag{5.2}$$

where the squeeze parameter $\sigma = r\,e^{i\theta}$. Comparing Eqs.(2.4) and (5.2) we find that

$$|\lambda,+1\rangle = \hat{S}(\sigma)|0\rangle, \tag{5.3}$$

where the eigenvalue λ is related to the squeeze parameter σ by

$$\lambda = e^{i\theta}\tanh r. \tag{5.4}$$

Hence we conclude that the squeezed vacuum is an eigenstate of our TAO $\hat{a}^{+-1}\hat{a}$. In a similar manner we can show that the squeezed first number state $\hat{S}(\sigma)|n=1\rangle$ is an eigenstate $|\lambda,-2\rangle$ of the operator $\hat{a}\hat{a}^{+-1}$.

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BOUNDARY CONDITIONS IN TUNNELING
VIA QUANTUM HYDRODYNAMICS

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Abstract

Via the hydrodynamical formulation of quantum mechanics, a novel approach to the problem of tunneling through sharp-edged potential barriers is developed. Above all, it is shown how more general boundary conditions follow from the continuity of mass, momentum, and energy.

1 Introduction

A commonly used assumption in quantum mechanics [1,2,3,4] is that the boundary conditions on a surface σ where the potential undergoes a finite jump reduce to the requirement that both the wave function (ψ) and its derivative ($\partial\psi/\partial x$) be continuous on σ. We show below through the hydrodynamical formulation of quantum mechanics how more general boundary conditions follow from the continuity of mass, momentum, and energy densities. With these new boundary conditions, a novel approach to tunneling through sharp-edged potential barriers is presented.

2 Formulation

Let us consider the dynamics of a quantum particle described by the coupled hydrodynamical equations

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{m}\frac{\partial(V + V_{qu})}{\partial x} = 0, \tag{2}$$

where Equation (1) represents the mass conservation law with mass density $\rho = \phi^2$ and Equation (2) describes trajectories of a particle with velocity $v = (\hbar/m)(\partial S/\partial x)$, subject to an external potential V and the quantum potential $V_{qu} = -(\hbar^2/2m\phi)(\partial^2\phi/\partial x^2)$, which accounts

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for quantum-wave features, such as interference and diffraction [5,6]. The wave function has been expressed in the polar form $\psi = \phi \exp(iS)$. Equations (1) and (2) yield

$$\hbar \frac{\partial S}{\partial t} + \left(\frac{mv^2}{2} + V_{qu} + V \right) = 0,$$

(3)

and the corresponding Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi.$$

(4)

From Equations (1) and (2), we obtain the conservation laws for the momentum and energy densities as follows:

$$\frac{\partial J}{\partial t} + \frac{\partial P}{\partial x} + \frac{\rho}{m} \frac{\partial V}{\partial x} = 0,$$

(5)

$$\frac{\partial U}{\partial t} + \frac{\partial Q}{\partial x} = 0,$$

(6)

where

$$J = \rho v,$$

(7)

$$P = \rho v^2 - \frac{\hbar^2}{4m^2} \left[\frac{\partial^2 \rho}{\partial x^2} - \frac{1}{\rho} \left(\frac{\partial \rho}{\partial x} \right)^2 \right],$$

(8)

$$U = \rho \left(\frac{mv^2}{2} + V_{qu} + V \right),$$

(9)

$$Q = vU + \frac{\hbar^2}{2m^2} \left(\phi \frac{\partial^2 \phi}{\partial x \partial t} - \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} \right)$$

(10)

are the momentum, momentum flux, energy, and energy flux densities, respectively. The momentum density ρv appearing in the hydrodynamical equations can be shown to be the real part of a more general quantum mechanical local momentum field $\mathcal{P}$ defined from the momentum-density operator

$$\mathcal{P} = \frac{\hbar}{i} \psi^* \frac{\partial \psi}{\partial x} = m\rho(v + iu),$$

(11)

where $v = (\hbar/m)(\partial S/\partial x)$ and $u = -(\hbar/2m\rho)(\partial \rho/\partial x)$.

It follows now that the boundary conditions for the continuity of mass, momentum, and energy are:

$$\rho, \ \rho v, \ \rho u, \ \text{and} \ \rho \left(\frac{mv^2}{2} + V_{qu} + V \right).$$

(12)

In terms of the wave function and from Equation (3) the above conditions are equivalent to: $\psi^* \psi$, $\psi^* (\partial \psi / \partial x)$, and $(\partial S / \partial t)$.

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3 Tunneling

Next consider the stationary flow of particles with incident energy E striking a potential barrier of height V and width L: $V(x) = V$ for $0 < x < L$ and zero elsewhere. The wave functions for $x < 0$ (incidence region 1), $0 < x < L$ (tunneling region 2), and $x > L$ (transmission region 3) are given respectively by

$$
\begin{aligned}
\psi_1(x,t) &= \sqrt{\rho_1}\,\exp(iS_1) \\
&= \sqrt{1 + a^2 + 2a\cos(2kx - \alpha)} \\
&\quad \times\ \exp i\left(-\omega t + \frac{\alpha}{2} + \tan^{-1}\left[\frac{1-a}{1+a}\tan(kx - \frac{\alpha}{2})\right]\right),
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\psi_2(x,t) &= \sqrt{\rho_2}\,\exp(iS_2) \\
&= \sqrt{\left[c^2 e^{2\bar{q}x} + d^2 e^{-2\bar{q}x} + 2dc\cos(\gamma - \delta)\right]/\bar{q}} \\
&\quad \times\ \exp i\left(-\omega t + \frac{\gamma + \delta}{2} + \tan^{-1}\left[\frac{ce^{\bar{q}x} - de^{-\bar{q}x}}{ce^{\bar{q}x} + de^{-\bar{q}x}}\tan(\frac{\gamma - \delta}{2})\right]\right),
\end{aligned}
\tag{14}
$$

$$
\psi_3(x,t) = \sqrt{\rho_3}\,\exp(iS_3) = b\,\exp i(-\omega t + kx + \beta),
\tag{15}
$$

where $k^2 = 2mE/\hbar^2$ and $\bar{q}^2 = 2m(V - E)/\hbar^2$.

The boundary conditions from (12) where the potential undergoes a finite jump read:

$$
\rho_1(0) = \rho_2(0),
\tag{16}
$$

$$
\rho_2(L) = \rho_3(L),
\tag{17}
$$

$$
\rho_1'(0) = \rho_2'(0),
\tag{18}
$$

$$
\rho_2'(L) = \rho_3'(L),
\tag{19}
$$

$$
\rho_1(0)v_1(0) = \rho_2(0)v_2(0),
\tag{20}
$$

$$
\rho_2(L)v_2(L) = \rho_3(L)v_3(L),
\tag{21}
$$

$$
\left(\frac{\partial S_1}{\partial t}\right)_0 = \left(\frac{\partial S_2}{\partial t}\right)_0,
\tag{22}
$$

$$
\left(\frac{\partial S_2}{\partial t}\right)_L = \left(\frac{\partial S_3}{\partial t}\right)_L.
\tag{23}
$$

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By applying the above boundary conditions on Equations (13), (14), and (15), we obtain:

$$1 + a^2 + 2a\cos\alpha = \frac{c^2 + d^2 + 2cd\cos(\gamma - \delta)}{\overline{q}}, \tag{24}$$

$$\frac{c^2 e^{2\overline{q}L} + d^2 e^{-2\overline{q}L} + 2cd\cos(\gamma - \delta)}{\overline{q}} = b^2, \tag{25}$$

$$2ak\sin\alpha = (c^2 - d^2), \tag{26}$$

$$c = de^{-2\overline{q}L}, \tag{27}$$

$$1 - a^2 = \frac{2d^2 e^{-2\overline{q}L}\sin(\gamma - \delta)}{k}, \tag{28}$$

$$\frac{2d^2 e^{-2\overline{q}L}\sin(\gamma - \delta)}{k} = b^2. \tag{29}$$

From Equations (25) and (27), we have

$$b^2 = \frac{2d^2 e^{-2\overline{q}L}[1 + \cos(\gamma - \delta)]}{\overline{q}}, \tag{30}$$

which combined with Equation (29) gives

$$\tan\left(\frac{\gamma - \delta}{2}\right) = \frac{k}{\overline{q}}, \tag{31}$$

$$\sin(\gamma - \delta) = \frac{2k\overline{q}}{\overline{q}^2 + k^2}, \tag{32}$$

$$\cos(\gamma - \delta) = \frac{\overline{q}^2 - k^2}{\overline{q}^2 + k^2}. \tag{33}$$

Equations (29) and (33) allow us to write Equation (30) as

$$b^2 = \left(\frac{4\overline{q}}{\overline{q}^2 + k^2}\right) d^2 e^{-2\overline{q}L}, \tag{34}$$

which, in turn, combined with Equations (27) and (33), reduces Equation (24)

$$1 + a^2 + 2a\cos\alpha = b^2 \left(\frac{\overline{q}^2 - k^2}{2\overline{q}^2}\right)\left(1 + \frac{\overline{q}^2 + k^2}{\overline{q}^2 - k^2}\cosh 2\overline{q}L\right). \tag{35}$$

Equations (28) and (29) imply that

$$a^2 = 1 - b^2, \tag{36}$$

which inserted into Equation (35) gives

$$a \cos \alpha = b^2 \left(1 + \left[\frac{\overline{q}^2 + k^2}{2\overline{q}^2} \right] \sinh^2 \overline{q} L \right) - 1. \tag{37}$$

By the same procedure above, Equation (26) can be rewritten as

$$a \sin \alpha = - \left(\frac{\overline{q}^2 + k^2}{4k\overline{q}} \right) b^2 \sinh 2\overline{q} L. \tag{38}$$

Combination of Equations (36), (37), and (38) leads to

$$b^{-2} = \frac{\left[1 + \left(\frac{\overline{q}^2 + k^2}{2\overline{q}^2} \right) \sinh^2 \overline{q} L \right]^2 + \left(\frac{\overline{q}^2 + k^2}{4k\overline{q}} \right)^2 \sinh^2 2\overline{q} L}{1 + \left(\frac{\overline{q}^2 + k^2}{2\overline{q}^2} \right) \sinh^2 \overline{q} L}. \tag{39}$$

Using the identity $\sinh^2 2\overline{q} L = 4(\sinh^2 \overline{q} L + \sinh^4 \overline{q} L)$, and after dividing the numerator by the denominator in Equation (39), we arrive at the known result

$$b^{-2} = 1 + \left(\frac{\overline{q}^2 + k^2}{2k\overline{q}} \right)^2 \sinh^2 \overline{q} L. \tag{40}$$

4   Boundary Conditions for Dissipative Systems

Next we show below that the boundary conditions (12) are not only more general but the assumption that "ψ and $(\partial \psi / \partial x)$ are continuous at σ" is physically incorrect for dissipative systems. To this end, let us consider the dynamics of a quantum particle in the tunneling region described by Equation (1) and

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{m} \frac{\partial (V + V_{qu})}{\partial x} = -\nu v, \tag{41}$$

where ν is the friction coefficient, and the term on the right-hand side of Equation (41) accounts for the dissipation. By expressing the wave function as before [see Equation (3)] we have

$$\hbar \left(\frac{\partial S}{\partial t} + \nu S \right) + \left(\frac{mv^2}{2} + V_{qu} + V \right) = 0. \tag{42}$$

The new boundary conditions now are given by Equations (16) through (21) plus

$$\left(\frac{\partial S_1}{\partial t} \right)_0 = \left(\frac{\partial S_2}{\partial t} + \nu S_2 \right)_0, \tag{43}$$

$$\left(\frac{\partial S_2}{\partial t} + \nu S_2 \right)_L = \left(\frac{\partial S_3}{\partial t} \right)_L, \tag{44}$$

which shows the discontinuity in the phase of the wave function at σ. In an upcoming publication, we will detail the application of the above boundary conditions and show that friction on the tunneling of a particle through a single, sharp-edged rectangular barrier diminishes the transmission coefficient.

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Using Harmonic Oscillators to Determine the Spot Size of Hermite-Gaussian Laser Beams

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Abstract

This paper illustrates the similarity of the functional forms of quantum mechanical harmonic oscillators and the modes of Hermite-Gaussian laser beams. This functional similarity provides a direct correlation to investigate the spot size of large-order mode Hermite-Gaussian laser beams. The classical limits of a corresponding two-dimensional harmonic oscillator provide a definition of the spot size of Hermite-Gaussian laser beams. The classical limits of the harmonic oscillator provide integration limits for the photon probability densities of the laser-beam modes to determine the fraction of photons detected therein. Mathematica is used to integrate the probability densities for large-order beam modes and to illustrate the functional similarities. The probabilities of detecting photons within the classical limits of Hermite-Gaussian laser beams asymptotically approach unity in the limit of large-order modes, in agreement with the Correspondence Principle. The classical limits for large-order modes include all of the nodes for Hermite-Gaussian laser beams; Sturm's theorem provides a direct proof.

1. Introduction

There are many instances in science where different physical models have similar or identical functional forms. Scientists often exploit and glean ideas from other disciplines to better understand new areas of research, especially if the physical models exhibit similar functional forms. The harmonic oscillator is a powerful tool for explaining and understanding many similar disciplines of physics. Since exact solutions exist for the classical and quantum harmonic oscillator, it is a tool and simple model to understand basic principles of vibrational motion and normal modes. In addition, the harmonic oscillator is an excellent pedagogical system to help model and understand the basic properties of quantum mechanics, quantized radiation fields, quantum optics, and other disciplines of physics. Yes— the harmonic oscillator rightfully deserves its place "on a pedestal" [1].

In this paper we will exploit and use the similarity of the functional forms of quantum harmonic oscillators and Hermite-Gaussian laser beams to investigate the

spot size of laser-beam modes and the fractional energy and photons incident therein. As a result of two slightly different definitions for Hermite polynomials [2,3], some references indicate that the spot size, as delimited by the peaks of large-order Hermite-Gaussian beams, does not include most of the energy [4,5]. In view of the Correspondence Principle, the probability of finding the quantum oscillator within the classical limits asymptotically increases to unity for higher-order modes. Since the functional forms of the quantum oscillator and laser-beam mode are similar, we should expect the probability of detecting photons within the corresponding classical limits of Hermite-Gaussian laser-beam modes to similarly approach unity for higher-order modes. Mathematica [6] is used to integrate the laser-beam mode probability densities for small- and large-order modes to illustrate these principles. Sturm's theorem provides a direct proof that the classical limits also contain all of the probability density peaks. The harmonic oscillator's classical limits, therefore, serve to provide a good measure of large-order mode spot size for Hermite-Gaussian laser beams.

The classical oscillator, it's classical limits, and the classical probability density are reviewed in Section 2. Section 3 provides a discussion of the quantum oscillator, the corresponding probability densities, and the Correspondence Principle. The Hermite-Gaussian laser beam modes are reviewed in Section 4 and compared to the quantum oscillator. Section 5 provides a discussion of the Mathematica results from integrating the laser-beam mode probability densities. Sturm's theorem and its application to the peaks and zeros of the probability densities are discussed in Section 6.

2. Classical Limits and Probability Densities

Many systems oscillate by small amounts near a point of stable equilibrium. The motion of a simple system having one degree of freedom and small oscillations can be described by a simple linear harmonic oscillator. Some systems having more than one degree of freedom can also be described by a set of coupled or decoupled harmonic oscillators. Although the Lagrangian formulation is well suited for developing the theory of small oscillations [7], the Hamiltonian formulation provides a direct solution for the simple harmonic oscillator of mass m coupled to a massless spring of force constant k. The force on the mass is given by Hooke's law $F = -kx$ with the corresponding potential $V = kx^2/2$. The Hamiltonian for a harmonic oscillator can be written as the sum of a kinetic and a potential energy quadratic in the momentum p and the position x

$$\mathcal{H} = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \qquad (1)$$

where $\omega^2 = k/m$ and $\omega = 2\pi\nu$ is the angular frequency of oscillation.

The equations of motion for the harmonic oscillator are obtained from Hamilton's canonical equations [7]

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$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}. \tag{2}$$

Using Hamilton's equations (2) with the Hamiltonian given in (1), the time derivatives for the canonical variables x and p are obtained

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -m\omega^2 x. \tag{3}$$

Differentiating $\dot{x}$ with respect to time and substituting for $\dot{p}$ in (3), we obtain the standard harmonic oscillator equation

$$\ddot{x} + \omega^2 x = 0. \tag{4}$$

The solution of this harmonic oscillator equation can be written as

$$x(t) = x_0 \cos(\omega t + \phi). \tag{5}$$

The total energy E_c of the classical harmonic oscillator is a constant of the motion. Using the oscillator Hamiltonian (1) and the relationship between the momentum and velocity, $p = m\dot{x}$, the energy can be written as

$$E_c = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m \omega^2 x^2 = \tfrac{1}{2} m \omega^2 x_0^2. \tag{6}$$

For the classical harmonic oscillator, the amplitude $x_0 = (2E_c / k)^{1/2}$ is a continuous variable. The energy is, therefore, also a non-negative continuous variable; the energy can be zero or a positive value. Solving (6) for the speed of the particle

$$|\dot{x}| = \left(2E_c/m - \omega^2 x^2\right)^{1/2} = \omega\left(x_0^2 - x^2\right)^{1/2}, \tag{7}$$

we see that the particle oscillates between the classical limits. The particle obtains maximum velocity at $x = 0$ and zero velocity at the outer limits of its motion. From (5) we also see that the particle does not classically exceed $\pm x_0$.

If we measure the oscillator's position x at random, any value within the classical limits could be observed in principle. The probability for finding the particle between x and $x + dx$ is equal to the ratio of the time spent between x and $x + dx$ to the total time for one oscillation period $T = 2\pi/\omega$. Noting that the particle passes the same position twice per oscillation, we obtain the classical probability density

$$\wp_c(x)dx = \frac{2dt}{T} = \frac{2dx}{|\dot{x}|}\frac{1}{T} = \frac{2dx}{|\dot{x}|}\frac{\omega}{2\pi} = \frac{dx}{\pi\left(x_o^2 - x^2\right)^{1/2}} \qquad (8)$$

$$\wp_c(x) = \begin{cases} \dfrac{1}{\pi\left(x_o^2 - x^2\right)^{1/2}} & , \quad |x| \le x_o \\[2em] 0 & , \quad |x| > x_o . \end{cases} \qquad (9)$$

We are certain to find the classical harmonic oscillator within the classical limits $\pm x_o$; classically, the oscillator will not be observed outside the classical limits (see Fig. 1). The probability for finding the particle within the classical limits is unity and the probability for finding the particle outside the classical limits is zero, as noted by integration of (9)

$$\int_{-x_o}^{x_o} \wp_c(x)dx = \int_{-x_o}^{x_o} \frac{dx}{\pi\left(x_o^2 - x^2\right)^{1/2}} = 1 \qquad (10a)$$

$$\int_{-\infty}^{-x_o} \wp_c(x)dx = 0 \qquad (10b)$$

$$\int_{x_o}^{\infty} \wp_c(x)dx = 0. \qquad (10c)$$

Oscillator position (x/x_o)
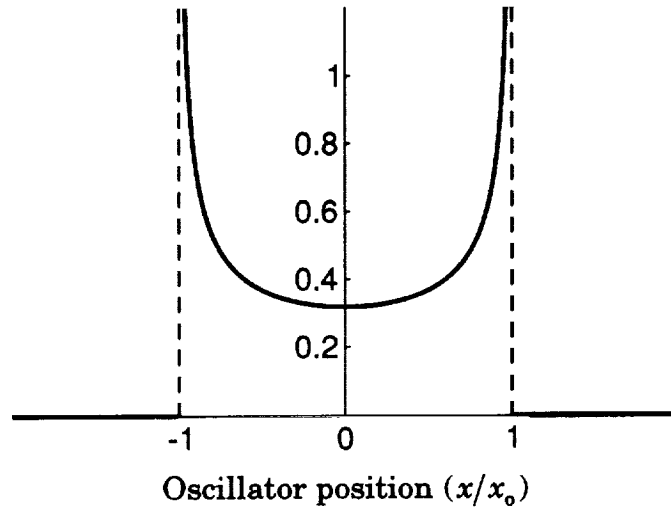
Figure 1. Classical harmonic oscillator probability density $\wp_c(x)$.

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3. Quantum Mechanical Probability Densities

The quantum mechanical harmonic oscillator energy levels and eigenstates are derived from the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle \tag{11}$$

using the same Hamiltonian (1) where the canonical variables (x, p) are replaced with operators (X, P)

$$H = \mathcal{H}(x \to X, p \to P) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2. \tag{12}$$

The eigenstates and discrete energies of the quantum harmonic oscillator are derived and discussed in many older and newer references [1,8-15]. Only the salient features are presented here in order to compare the classical and quantum oscillator probability densities with the Hermite-Gaussian laser-beam mode photon probability densities presented in Section 4.

The time-independent Schrödinger equation, as written in the X-basis representation,

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \right)\psi = E\psi \tag{13}$$

is solved for normalized solutions after tedious operations [1]

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2} \right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar} \right) H_n\left[\left(\frac{m\omega}{\hbar} \right)^{1/2} x \right]. \tag{14}$$

If we use $\alpha = m\omega/\hbar$ and introduce a new dimensionless variable $\xi = \sqrt{\alpha}x$, then the probability amplitude $\psi_n(\xi)$ for finding the quantum oscillator between ξ and $\xi + d\xi$ can be written in a simplified form [9,16]

$$\psi_n(\xi) = \left(\frac{1}{\pi^{1/2}2^n n!} \right)^{1/2} \exp\left(-\frac{\xi^2}{2} \right) H_n(\xi). \tag{15}$$

The Hermite polynomials $H_n(\xi)$ are nth-degree orthogonal polynomials relative to the standard weighting function $w(\xi) = e^{-\xi^2}$

$$\int_{-\infty}^{\infty} H_m(\xi) H_n(\xi) e^{-\xi^2} d\xi = \delta_{mn} \pi^{1/2} 2^n n!. \tag{16}$$

The Hermite polynomials (first four listed here)

$$H_0(\xi) = 1 \qquad\qquad H_2(\xi) = 4\xi^2 - 2$$
$$H_1(\xi) = 2\xi \qquad\qquad H_3(\xi) = 8\xi^2 - 12\xi$$

also satisfy the differential equation [3]

$$y'' + \left(2n + 1 - x^2\right) y = 0$$
$$y(x) = e^{-x^2/2} H_n(x). \tag{17}$$

In contrast to the continuous energy levels (6) of the classical harmonic oscillator, the energies of the quantum harmonic oscillator are discrete. The quantized energy values E_n correspond to the eigenstates (14) of the Schrödinger equation (13)

$$E_n = (n + 1/2)\hbar\omega. \tag{18}$$

Using (6), we see that the corresponding classical limits can be written as

$$x_o = (\hbar/m\omega)^{1/2}(2n + 1)^{1/2}. \tag{19}$$

The smallest energy value $\hbar\omega/2$ of the quantum oscillator corresponds to the zero-state $\psi_0(x)$; the energy increases incrementally by $\Delta E_n = \hbar\omega$. The probability density $|\psi_n|^2 = \psi_n \psi_n^*$ for observing the quantum harmonic oscillator between ξ and $\xi + d\xi$ is obtained from (15)

$$|\psi_n(\xi)|^2 = \left(\frac{1}{\pi^{1/2} 2^n n!}\right) e^{-\xi^2} H_n^2(\xi). \tag{20}$$

The classical (9) and quantum (20) probability densities are plotted together in Fig. 2 for a few of the oscillator modes. As the order of the oscillator mode increases, we observe that the fraction of the area or probability to be outside of the classical limits decreases; the quantum oscillator's probability to be within the classical limits increases. We also see that the classical probability density is near the average of the quantum probability densities; this is more apparent for the large-order modes. The classical limits appear to increase with a corresponding increase in the mode order such that the outer peaks of the probability densities are always contained within the classical limits.

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The classical and quantum probability densities are quite different yet similar in a number of ways. In particular, a position measurement of the quantum oscillator of energy E_n can result in any value between $-\infty$ and $+\infty$. However, when measuring the classical oscillator's position, only values between $-x_o = -\sqrt{2E_n/k}$ and $x_o = \sqrt{2E_n/k}$ will be obtained. If we consider an oscillator having a small mass of 1 gram and oscillating at 1 rad/sec with an amplitude of 1 cm, then the energy would be $m\omega^2 x_o^2/2 = 0.5$ erg. We can compare this to the energy difference between
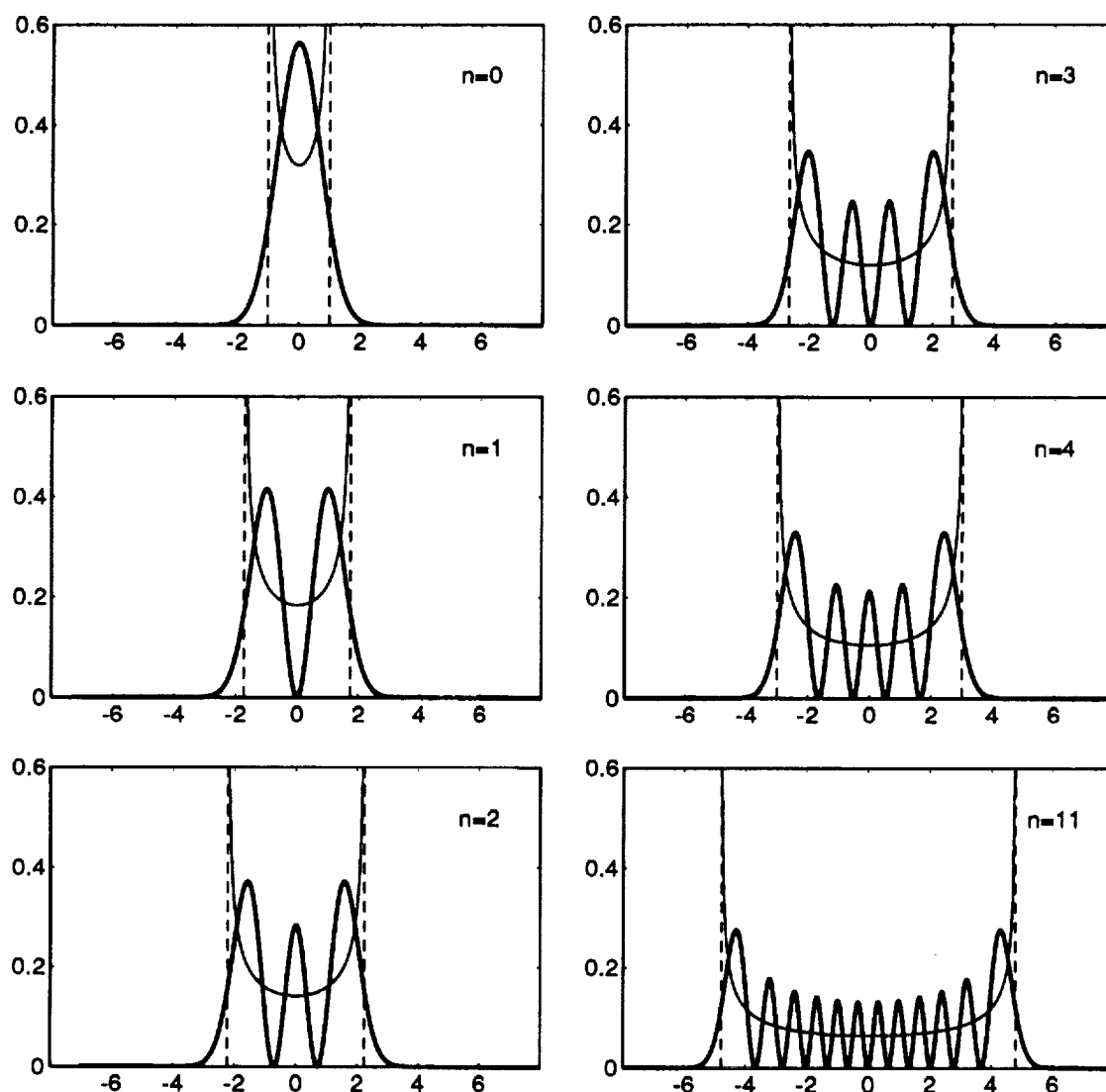

Figure 2. Quantum harmonic oscillator probability densities. (The dashed vertical lines represent the classical limits. The thin curves correspond to the classical probability densities.

the quantum oscillator levels $\Delta E = \hbar\omega \approx 10^{-27}$ erg. Experimentally it would be practically impossible to detect energy differences separated by 10^{-27} erg. Similarly, if we invert (18) to determine the mode level for this small oscillator, we see that $n = (E/\hbar\omega) - 1/2 \approx 10^{27}$. Because the mode order n is equal to the number of nodes in the quantum oscillator's probability density, it would be virtually impossible to observe 10^{27} oscillation nodes within the 2 cm interval. We would, instead, only detect or measure the average of the quantum probability density, which is just the classical result shown previously in Fig. 1. For large n, the classical and quantum results become indistinguishable as required by the Correspondence Principle [1].

In the limit of large-order modes, this special case of the Correspondence Principle illustrates how the classical picture is indeed regained. From the Correspondence Principle and the limit of large-order modes $n \to \infty$, we should expect the quantum mechanical probability densities to be functionally similar to the classical harmonic oscillator probability density. This can be derived in a number of ways [8,15]. If we examine the quantum oscillator's asymptotic functional form when the mode order increases to infinity, we find a rapid oscillatory behavior that averages out to the classical results (9) [8]

$$|\psi_n(x)|^2 \xrightarrow[\substack{\text{small } x \\ \text{and} \\ \text{large } n}]{} \begin{cases} \dfrac{2}{\pi} \dfrac{1}{\left(x_0^2 - x^2\right)^{1/2}} \cos^2\left(\dfrac{x_0 x}{\alpha}\right), & \text{for even } n \\[4mm] \dfrac{2}{\pi} \dfrac{1}{\left(x_0^2 - x^2\right)^{1/2}} \sin^2\left(\dfrac{x_0 x}{\alpha}\right), & \text{for odd } n. \end{cases} \tag{21}$$

4. Hermite-Gaussian Laser Beam Modes

We now consider an Hermite-Gaussian laser beam propagating along the z direction. The laser beam considered can have different beam waists along the x and y directions. The Hermite-Gaussian laser-beam intensity or irradiance at some $+z$ direction is obtained from a scalar wave equation [17,18]. The irradiance distribution of an Hermite-Gaussian laser beam that is focused at $z = 0$ can be written as [4,19]

$$E(x,y,z) = E_0 \frac{w_x(0)w_y(0)}{w_x(z)w_y(z)} \exp\left(-\frac{x^2}{w_x^2} - \frac{y^2}{w_y^2}\right) H_m^2\left(\frac{x}{w_x(z)}\right) H_n^2\left(\frac{y}{w_y(z)}\right). \tag{22}$$

The beam waists w_x and w_y are the distances at which the lowest-order mode intensity drops to e^{-1} times the value on the optical axis (some references use an e^{-2} factor to define a beam waist). The x-axis beam waist

$$w_x(z) = w_x(0)\left(1 + \frac{z^2}{z_{0_x}^2}\right)^{1/2} \tag{23}$$

depends on the beam parameter z_{0_x} that is a function of the wavelength λ

$$z_{0_x} = \frac{2\pi}{\lambda}w_x^2(0) \tag{24}$$

with similar results for the y-axis waist and beam parameter.

We see that (22) is similar to the quantum harmonic oscillator density (20). The functional form of the Hermite-Gaussian laser-beam mode is similar to a two-dimensional quantum harmonic oscillator probability density. Equation (22) provides the irradiance at some position in the laser beam; with proper normalization, (22) could also be interpreted as the probability density to detect photons at some position in the laser beam. If we divide (22) by the total power in the laser beam, then the result is interpreted as a probability density to detect a photon at the corresponding position.

To determine the spot size for large-order beam modes, we consider the mean-squared value (second moment) of x and y. As an example, we look at the mean-squared value of x

$$u_x^2(z)_m = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 E(x,y,z)dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y,z)dxdy}. \tag{25}$$

Substituting (22) into (25) we see that the integral is separable

$$u_x^2(z)_m = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 H_m^2\left(\frac{x}{w_x(z)}\right)H_n^2\left(\frac{y}{w_y(z)}\right)\exp\left(-\frac{x^2}{w_x^2}-\frac{y^2}{w_y^2}\right)dx\,dy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_m^2\left(\frac{x}{w_x(z)}\right)H_n^2\left(\frac{y}{w_y(z)}\right)\exp\left(-\frac{x^2}{w_x^2}-\frac{y^2}{w_y^2}\right)dx\,dy} \tag{26}$$

and reduces to a simpler form by canceling the y-dependent factors

$$u_x^2(z)_m = \frac{\int_{-\infty}^{\infty} x^2 H_m^2\left(\frac{x}{w_x(z)}\right)\exp\left(-\frac{x^2}{w_x^2}\right)dx}{\int_{-\infty}^{\infty} H_m^2\left(\frac{x}{w_x(z)}\right)\exp\left(-\frac{x^2}{w_x^2}\right)dx} \tag{27}$$

with a similar result obtained for the y direction. The integrals in the numerator and denominator of (27) occur frequently in quantum mechanics in relation to the harmonic oscillator problem and are readily solved using the generating function for the Hermite polynomials [13,16]

$$\int_{-\infty}^{\infty} x^2 H_m^2\left(\frac{x}{a}\right)\exp\left(-\frac{x^2}{a^2}\right)dx = 2^m \pi^{1/2} m! a^3\left(m+\frac{1}{2}\right) \tag{28}$$

$$\int_{-\infty}^{\infty} H_m^2\left(\frac{x}{a}\right)\exp\left(-\frac{x^2}{a^2}\right)dx = 2^m \pi^{1/2} m! a. \tag{29}$$

Using (28) and (29) in (27) we obtain

$$u_x^2(z)_m = w_x^2(z)(m + 1/2). \tag{30}$$

Taking the square root of twice the mean-squared value, we then obtain

$$2u_x^2(z)_m \equiv w_x^2(z)_m = w_x^2(z)(2m + 1) \tag{31}$$

$$w_x(z)_m = w_x(z)(2m + 1)^{1/2} \tag{32a}$$

$$w_y(z)_n = w_y(z)(2n + 1)^{1/2}. \tag{32b}$$

Equations (32) define the beam waists for large-order modes and depend on the order m,n of the mode. We see that the beam waists (32) have a mode order dependence that is identical to that of the classical limits of the harmonic oscillator (19). To illustrate the beam waist, two laser-beam modes are plotted in Fig. 3 along with the corresponding limits (32) that define the rectangular region and size of the laser-beam spot.
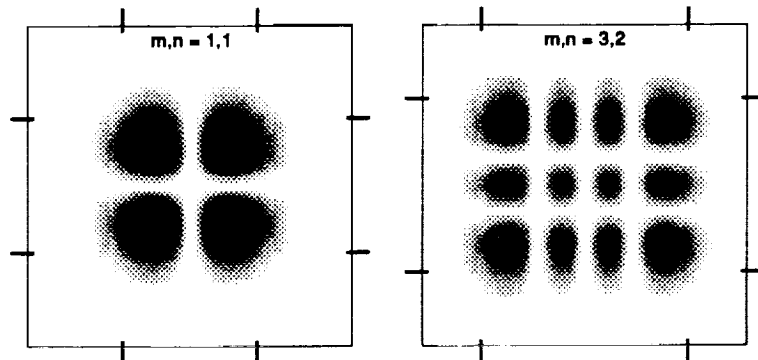

Figure 3. Photon probability density plots and classical limits for TEM_{11} and TEM_{32} modes. (The vertical and horizontal ticks represent the classical limits $w_x(z)_m$ and $w_y(z)_n$.)

We again see that the corresponding beam waists or "classical limits" seem to increase such that the intensity peaks are always contained therein. Since the photon probability density for an Hermite-Gaussian laser beam is identical to a two-dimensional quantum oscillator, it is expected that the probability of detecting photons within the corresponding classical limits of Hermite-Gaussian laser-beam modes will also asymptotically approach unity as the laser-beam mode order increases to infinity, that is as $m,n \to \infty$.

5. Fractional Power and Photon Probabilities

In Section 3 and 4, we saw that the classical limits seem to contain most of the probability to detect the quantum oscillator and the photons for the Hermite-Gaussian laser beams. For the large-order mode spot size to be meaningful and useful, it should contain a large portion of the power or photons of the laser beam. The probability to detect photons within the corresponding classical limits that define the spot size of the laser beam should similarly increase for large-order beams, just as in the quantum oscillator case and in agreement with the Correspondence Principle. To investigate the fraction of the power or the photon probability within the classical limits, as illustrated in Fig. 3, an integration over the classical limits is performed

$$f_{m,n} = \frac{\underset{\text{Classical Limits}}{\iint} H_m^2(\xi)\exp(-\xi^2)H_n^2(\zeta)\exp(-\zeta^2)d\xi d\zeta}{\underset{\infty}{\iint} H_m^2(\xi)\exp(-\xi^2)H_n^2(\zeta)\exp(-\zeta^2)d\xi d\zeta} \tag{33}$$

where $\xi = x/w_x(z)$ and $\zeta = y/w_y(z)$. Using (16) in (33) we obtain

$$f_{m,n} = \frac{\int_{-\sqrt{2m+1}}^{\sqrt{2m+1}} H_m^2(\xi)\exp(-\xi^2)d\xi \int_{-\sqrt{2n+1}}^{\sqrt{2n+1}} H_n^2(\zeta)\exp(-\zeta^2)d\zeta}{2^{m+n}\pi m! n!}. \tag{34}$$

The photon probabilities (34) were computed using Mathematica and are presented in Table 1 for the low-order modes. Mathematica was also used to compute the photon probabilities for higher-order modes. Figure 4 illustrates the asymptotic behavior anticipated for the higher-order Hermite-Gaussian laser beam modes. As the order of the laser-beam mode increases to infinity, we see that the probability to detect photons within the corresponding classical limits asymptotically approaches unity, as expected from the quantum oscillator problem and the Correspondence Principle. In particular, we see from Table 1 and Fig. 4 that $f_{m,n} \to 1$ as $m,n \to \infty$.

Table 1. Probability $f_{m,n}$ of detecting photons within the classical limits.

	$m=0$	1	2	3	4	5	6	7	8	9	10
$n=0$	**0.710**	0.748	0.762	0.770	0.776	0.780	0.783	0.786	0.788	0.790	0.792
1	0.748	**0.789**	0.803	0.812	0.818	0.822	0.826	0.828	0.831	0.833	0.835
2	0.762	0.803	**0.818**	0.827	0.833	0.837	0.841	0.844	0.846	0.848	0.850
3	0.770	0.812	0.827	**0.836**	0.842	0.846	0.850	0.853	0.855	0.857	0.859
4	0.776	0.818	0.833	0.842	**0.848**	0.852	0.856	0.859	0.861	0.863	0.865
5	0.780	0.822	0.837	0.846	0.852	**0.857**	0.861	0.863	0.866	0.868	0.870
6	0.783	0.826	0.841	0.850	0.856	0.861	**0.864**	0.867	0.870	0.872	0.873
7	0.786	0.828	0.844	0.853	0.859	0.863	0.867	**0.870**	0.872	0.875	0.876
8	0.788	0.831	0.846	0.855	0.861	0.866	0.870	0.872	**0.875**	0.877	0.879
9	0.790	0.833	0.848	0.857	0.863	0.868	0.872	0.875	0.877	**0.879**	0.881
10	0.792	0.835	0.850	0.859	0.865	0.870	0.873	0.876	0.879	0.881	**0.883**

6. Probability Density Nodes, Peaks, and Sturm's Theorem

It is not entirely obvious that all probability density peaks of the quantum oscillator or of the large-order Hermite-Gaussian laser beams are contained within the corresponding classical limits. The Hermite functions $y(x) = e^{-x^2/2} H_n(x)$ determine the nodes (zeros) of both the quantum oscillator densities and the Hermite-Gaussian laser-beam intensities for all modes. The nodes of orthogonal
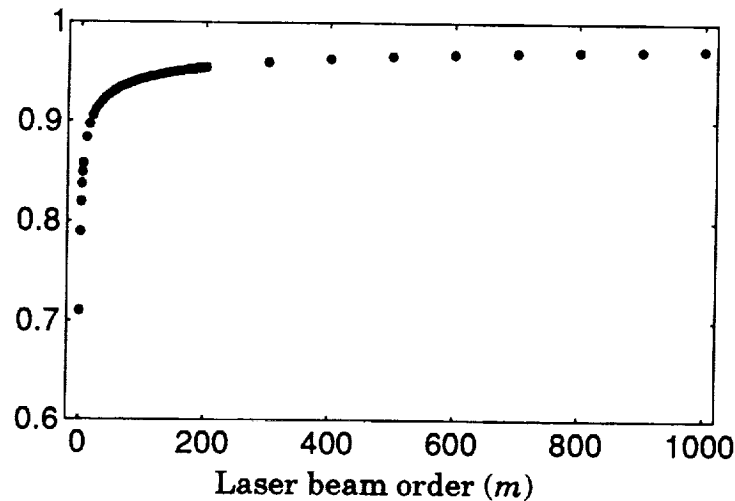
Figure 4. Plot of the probability $f_{m,m}$ of detecting photons within the classical limits of an Hermite-Gaussian laser beam.

polynomials are all real, distinct, and lie within the interior of the orthogonality region [20]. Figure 5 illustrates that the nodes of the Hermite functions also determine the nodes of the quantum oscillator and the Hermite-Gaussian laser beam modes.

The orthogonality region of Hermite polynomials extends from minus infinity to plus infinity as seen from the integral (16). Some method is, therefore, desired that will provide a limit to the extent of the nodes of the Hermite polynomials, the Hermite functions, and consequently the nodes of the probability densities of the quantum harmonic oscillator and of the Hermite-Gaussian laser-beam modes. Sturm's classic work on differential forms and the zeros of functions is one such method for analysis of the nodes of the Hermite functions. Sturm's theorem provides a useful method to determine the limits of the nodes in many functions, especially the classical orthogonal polynomials such as the Hermite polynomials. Direct application of Sturm's theorem [20] and (17), shows that all nodes lie within the classical limits for the quantum oscillator (19) and the Hermite-Gaussian laser beam (32).

In addition to Sturm's method, the concavity and convexity of a function is also useful. Equation (17) can be rewritten as

$$y''/y = x^2 - (2n + 1) \tag{35}$$

where

$$y''/y \quad \begin{cases} < 0, & \text{is concave for} \quad |x| < (2n + 1)^{1/2} \\ > 0, & \text{is convex for} \quad |x| > (2n + 1)^{1/2} \end{cases} \tag{36}$$

determines the concavity or convexity of the Hermite functions as illustrated in Fig. 6 for orders $n = 3$ and $n = 4$.

Figure 5. Quantum oscillator probability density and Hermite function of order $n = 10$. (The thick curve corresponds to the probability density. The light curve is the corresponding normalized Hermite function. Dashed lines correspond to the classical limits.)
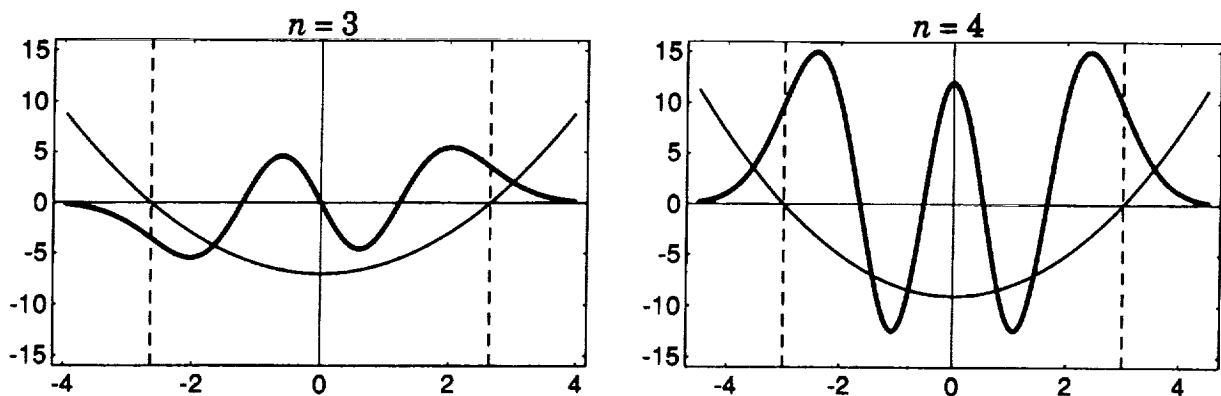
Figure 6. Concavity and convexity of the Hermite functions. (Thick curves denote the Hermite functions; thin curves illustrate equations (35) and (36).)
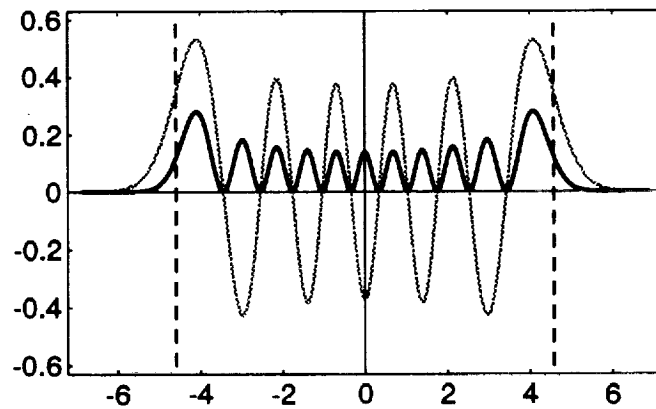
Noting the regions of concavity and convexity for the Hermite functions in (36) and Fig. 6, we see that the classical limits separate the concave and convex regions of the Hermite functions. The concave region lies between the classical limits while the convex regions lie outside of the classical limits. The classical limits always reside at inflection points of the Hermite functions and the peaks always reside within the concave region, that is, within the classical limits. We can therefore assert that the beam waists for large-order Hermite-Gaussian laser beams contain most of the laser beam power and all intensity peaks, as expected from comparison with the quantum harmonic oscillator and the Correspondence Principle.

7. Conclusions

The harmonic oscillator is indeed a useful tool to help model physical systems and, as shown in this paper, to help clarify and better understand some aspects of the probability densities of Hermite-Gaussian laser beams. In particular, the probability densities for two-dimensional quantum harmonic oscillator modes are functionally similar to the probability densities of Hermite-Gaussian laser beam modes. This functional similarity and the Correspondence Principle provide guidance to determine that the corresponding classical limits for Hermite-Gaussian laser beams define a spot size that contains a large portion of the laser beam's power. As computed with Mathematica, the portion of the Hermite-Gaussian laser-beam power or photons contained within the classical limits or beam waists asymptotically increases to unity as the laser-beam order increases to infinity. The classical limits and the corresponding laser-beam spot, as delimited by the beam waists, contain all nodes and probability density peaks of the quantum oscillator and the Hermite-Gaussian laser beams.

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8. Acknowledgments

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SQUEEZED STATES OF DAMPED OSCILLATOR CHAIN

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Abstract

The Caldirola-Kanai model of one-dimensional damped oscillator is extended to the chain of coupled parametric oscillators with damping. The correlated and squeezed states for the chain of coupled parametric oscillators with damping are constructed. Based on the concept of the integrals of motion it is demonstrated how squeezing phenomenon arises due to parametric excitation.

1 Introduction

A number of papers devoted to finding exact solutions of the Schrödinger equation with explicitly time-dependent quadratic Hamiltonians were published over the past quarter of a century. Several different but equivalent approaches are usually exploited for this purpose. In this paper we would like to present the model of quantum chain of coupled parametric oscillators with damping extending the model of known one-dimensional damped oscillator, and to demonstrate how squeezing phenomenon arrives in the chain based on the concept of time-dependent quantum integrals of motion. This method was elaborated in [1, 2, 3], while a detailed review was given in [4]. The exact formulae for propagators, wave functions, coherent states, density matrices, Wigner function, transition amplitudes and probabilities were given in [2, 3, 4] in the most convenient and explicit forms for quite general quadratic systems and numerous special cases.

Here we will apply developed approach to the model describing oscillator chain of coupled parametric oscillators with damping. This model is the partial case of general problem of multi-dimensional parametric oscillator, but the dynamics of these systems may be investigated in the explicit form due to the possibility of using usual normal mode transformation in spite of the coefficients being time-dependent. It is necessary to note that the problem of different kinds of quantum closed chains was discussed in recent papers [5–12] and the problem of of unclosed chain in [13, 14].

2 Integrals of Motion

Let us consider a quantum chain consisting of N coupled harmonic parametric oscillators with damping. All oscillators vibrate with frequency $\omega_0(t)$ which depends on time and interacts linearly with neighbors. The interaction constant $\omega(t)$ depends on time too. When the distance between neighbors approaches zero, and number N tends to infinity, the chain turns into the parametric string.

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The Hamiltonian of this system depends on time and has the form

$$\hat{H} = \frac{1}{2}\sum_{n=1}^{N} \left(\frac{p_n^2 e^{-2\Gamma(t)}}{m} + m\Omega^2(t)e^{2\Gamma(t)}(q_n - q_{n+1})^2 + m\Omega_0^2(t)e^{2\Gamma(t)}q_n^2 \right), \tag{1}$$

where q_n is an operator of a shift from the equilibrium point of a n-th oscillator, p_n is a momentum operator of the oscillator, m is the mass of each oscillator, and Γ is a damping coefficient depending on time.

In this model, damping is described in the framework of a phenomenological Hamiltonian first suggested for one-dimensional quantum oscillators with damping by Caldirola [15] and Kanai [16]. In this model, the mass of the oscillator increases exponentially. That dependence models the interaction of the oscillator with external degrees of freedom. Hamiltonian (1) is an extension of the Caldirola-Kanai Hamiltonian to the case of quantum chain of coupled parametric oscillators with damping.

The equations of motion corresponding to Hamiltonian (1) are

$$p_n = \dot{q}_n e^{2\Gamma(t)}, \qquad \ddot{q}_n = \Omega^2(t)\left(q_{n+1} + q_{n-1} - 2q_n\right) - \Omega_0^2(t)q_n - 2\dot{\Gamma}(t)\dot{q}_n. \tag{2}$$

We take into consideration the closed chain, so we have the condition

$$q_{1+N} = q_1.$$

A property of this model is that the time-dependence of the coefficients does not prevent from the application of the usual normal-mode reduction formulae. So, let us introduce new variables

$$x_s = \sqrt{\frac{2}{N}}\sum_{m=1}^{N} q_m \cos\left(2\pi sm/N\right), \tag{3}$$

$$y_s = \sqrt{\frac{2}{N}}\sum_{m=1}^{N} q_m \sin\left(2\pi sm/N\right), \tag{4}$$

$$x_N = \frac{1}{\sqrt{N}}\sum_{m=1}^{N} q_m. \tag{5}$$

For simplicity, we consider the chain consisting of an odd number of oscillators, so the number s changes from 1 to $p = (N - 1)/2$. The normal mode transformation (3)-(5) reduces the system of N coupled oscillators (2) to a set of N free oscillators vibrating independently according to the equations

$$\ddot{x}_s - 2\dot{\Gamma}(t)\dot{x}_s + \Omega_s^2(t)x_s = 0, \tag{6}$$

$$\ddot{y}_s - 2\dot{\Gamma}(t)\dot{y}_s + \Omega_s^2(t)y_s = 0, \tag{7}$$

$$\ddot{x}_N - 2\dot{\Gamma}(t)\dot{x}_N + \Omega_0^2(t)x_N = 0, \tag{8}$$

where the frequencies are given by the relation

$$\Omega_s^2(t) = 4\Omega^2(t)\sin^2\left(\pi s/N\right) + \Omega_0^2(t). \tag{9}$$

One can see that equations (6)-(8) are the trajectory equations of classical damped oscillators with frequencies (9).

Following the usual procedure (see,for example, [4]) "annihilation" operators for variable-frequency chain with damping can be constructed

$$\hat{A}_s(t) = \frac{i}{\sqrt{N}} \sum_{m=1}^{N} \left(\frac{l_s \epsilon_s p_m}{\hbar} - \frac{\dot{\epsilon}_s e^{2\Gamma(t)} q_m}{l_s \Omega_s(0)} \right) \cos(2\pi sm/N), \tag{10}$$

$$\hat{B}_s(t) = \frac{i}{\sqrt{N}} \sum_{m=1}^{N} \left(\frac{l_s \epsilon_s p_m}{\hbar} - \frac{\dot{\epsilon}_s e^{2\Gamma(t)} q_m}{l_s \Omega_s(0)} \right) \sin(2\pi sm/N), \tag{11}$$

$$\hat{A}_N(t) = \frac{i}{\sqrt{2N}} \sum_{m=1}^{N} \left(\frac{l_0 \epsilon_0 p_m}{\hbar} - \frac{\dot{\epsilon}_0 e^{2\Gamma(t)} q_m}{l_0 \Omega_0(0)} \right), \tag{12}$$

where

$$l_s = \left(\frac{\hbar}{m\Omega_s(0)} \right)^{1/2}, \qquad l_0 = \left(\frac{\hbar}{m\Omega_0(0)} \right)^{1/2}.$$

The complex functions $\epsilon_s(t)$ and $\epsilon_0(t)$ are the solutions of the equations of motion of classical parametric oscillators with damping

$$\ddot{\epsilon}_s(t) + 2\dot{\Gamma}(t)\dot{\epsilon}_s(t) + \Omega_s^2(t)\epsilon_s(t) = 0, \qquad \ddot{\epsilon}_0(t) + 2\dot{\Gamma}(t)\dot{\epsilon}_0(t) + \Omega_0^2(t)\epsilon_0(t) = 0. \tag{13}$$

"Annihilation" operators and their Hermitian conjugate operators satisfy the boson commutation relations

$$\left[\hat{A}_s(t), \hat{A}_i^\dagger(t) \right] = \left[\hat{B}_s(t), \hat{B}_i^\dagger(t) \right] = \delta_{si}, \qquad \left[\hat{A}_N(t), \hat{A}_N^\dagger(t) \right] = 1,$$

and

$$\left[\hat{A}_s(t), \hat{B}_i^\dagger(t) \right] = \left[\hat{A}_s(t), \hat{A}_N^\dagger(t) \right] = \left[\hat{A}_s(t), \hat{B}_i(t) \right] = \left[\hat{A}_s(t), \hat{A}_N(t) \right] = \left[\hat{B}_s(t), \hat{A}_N(t) \right] = 0,$$

if the functions $\epsilon_s(t)$, $\epsilon_0(t)$ satisfy the additional conditions

$$e^{2\Gamma(t)} \left(\dot{\epsilon}_s(t) \epsilon_s^*(t) - \dot{\epsilon}_s^*(t) \epsilon_s(t) \right) = 2i\Omega_s(0), \tag{14}$$

$$e^{2\Gamma(t)} \left(\dot{\epsilon}_o(t) \epsilon_o^*(t) - \dot{\epsilon}_o^*(t) \epsilon_o(t) \right) = 2i\Omega_0(0). \tag{15}$$

One can check that the full derivatives of operators (10)-(12) and their Hermitian conjugates are equal to zero, so they are the linear integrals of the motion of the quantum parametric chain with damping.

3 Squeezed Correlated and Fock States

The ground state of the parametric chain with damping can be constructed with the help of the integrals of motion (10)-(12) using the relations

$$\hat{A}_s(t)\psi_{\vec{0}}(\vec{q}, t) = \hat{B}_s(t)\psi_{\vec{0}}(\vec{q}, t) = \hat{A}_N(t)\psi_{\vec{0}}(\vec{q}, t) = 0, \tag{16}$$

where $\vec{q} = (q_1, ..., q_N)$. So the normalized ground state satisfying the Schrödinger equations with Hamiltonian (1) has the following wave function in coordinate representation

$$\psi_{\vec{0}}(\vec{q}, t) = \pi^{-N/4}(l_0\epsilon_0)^{-1/2}\prod_{s=1}^{p}(l_s\epsilon_s)^{-1}$$

$$\exp\left\{\sum_{m,m'=1}^{N}q_m q_{m'}\left(\frac{i\epsilon_0 e^{2\Gamma(t)}}{2\epsilon_0 N\Omega_0(0)l_0^2} + \sum_{s=1}^{p}\frac{i\epsilon_s e^{2\Gamma(t)}}{\epsilon_s N\Omega_s(0)l_s^2}\cos\left(2\pi s(m - m')/N\right)\right)\right\}. \quad (17)$$

Constructing with the help of integrals of motion (10)-(12) the displacement operator

$$D(\vec{\alpha}) = \prod_{s=1}^{p}\exp\left(\alpha_s \hat{A}_s^\dagger - \alpha_s^* \hat{A}_s + \beta_s \hat{B}_s^\dagger - \beta_s^* \hat{B}_s \xi \hat{A}_N^\dagger - \xi^* \hat{A}_N\right),$$

where components of the vector

$$\vec{\alpha} = (\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_p, \xi)$$

are complex numbers, and acting by displacement operator on ground state $\psi_{\vec{0}}(t)(\vec{q}, t)$ the entire family of correlated squeezed states can be obtained. So the wave functions in coordinate representation have the form

$$\psi_{\vec{\alpha}}(\vec{q}, t) = \psi_{\vec{0}}(\vec{q}, t)\exp\left\{-\frac{|\xi|^2}{2} - \frac{\xi^2\epsilon_0^*}{2\epsilon_0}\right.$$

$$+\sqrt{\frac{2}{N}}\frac{\xi}{l_0\epsilon_0}\sum_{m=1}^{N}q_m + \sum_{s=1}^{p}[-\frac{|\alpha_s|^2}{2} - \frac{|\beta_s|^2}{2} - \frac{\epsilon_s^*}{2\epsilon_s}(\alpha_s^2 + \beta_s^2)]$$

$$\left.+\sum_{s=1}^{p}\frac{2}{\epsilon_s l_s\sqrt{N}}\sum_{m=1}^{N}q_m[\alpha_s\cos\left(2\pi sm/N\right) + \beta_s\sin\left(2\pi sm/N\right)]\right\}. \quad (18)$$

The correlated squeezed states satisfy the Schrödinger equation with Hamiltonian (1) and are eigenstates of the integrals of motion (10)-(12), and components of the vector $\vec{\alpha}$ are eigenvalues of operators (10)-(12)

$$\hat{A}_s(t)\psi_{\vec{\alpha}}(\vec{q}, t) = \alpha_s\psi_{\vec{\alpha}}(\vec{q}, t),$$

$$\hat{B}_s(t)\psi_{\vec{\alpha}}(\vec{q}, t) = \beta_s\psi_{\vec{\alpha}}(\vec{q}, t),$$

$$\hat{A}_N(t)\psi_{\vec{\alpha}}(\vec{q}, t) = \xi\psi_{\vec{\alpha}}(\vec{q}, t).$$

One can see that the wave function of the ground state (17) and squeezed correlated states (18) are gaussian states with time-dependent coefficients in quadratic form of exponent function.

Using the property of squeezed correlated states (18) to be a generating function for Fock states

$$\psi_{\vec{\alpha}}(\vec{q}, t) = \exp\left(-\frac{|\xi|^2}{2} - \frac{1}{2}\sum_{s=1}^{p}(|\alpha_s|^2 + |\beta_s|^2)\right)\sum_{\vec{n}=0}^{\infty}\frac{\xi^{n_0}}{\sqrt{n_0!}}\prod_{s=1}^{p}\frac{\alpha_s^{n_s}\beta_s^{m_s}}{\sqrt{n_s!m_s!}}\psi_{\vec{n}}(\vec{q}, t),$$

where vector $\vec{n}$ has the components $\vec{n} = (n_0, n_1, ..., n_p, m_1, ..., m_p)$, the Fock states of quantum parametric chain with damping can be constructed, and are of the form

$$\psi_{\vec{n}}(\vec{q}, t) = \frac{1}{\sqrt{n_0!}} \left(\frac{\epsilon_0^*}{2\epsilon_0}\right)^{n_0/2} H_{n_0} \left(\frac{1}{\mid \epsilon_0 \mid \sqrt{N}} \sum_{m=1}^{N} q_m\right)$$

$$\prod_{s=1}^{p} \frac{1}{\sqrt{n_s! m_s!}} \left(\frac{\epsilon_s^*}{2\epsilon_s}\right)^{(n_s+m_s)/2} H_{n_s} \left(\frac{2}{\mid \epsilon_s \mid \sqrt{N}} \sum_{m=1}^{N} q_m cos(2\pi sm/N)\right)$$

$$H_{m_s} \left(\frac{2}{\mid \epsilon_s \mid \sqrt{N}} \sum_{m=1}^{N} q_m sin(2\pi sm/N)\right) \psi_{\vec{0}}(\vec{q}, t), \tag{19}$$

where $H_i(x_j)$ are Hermite polynomials.

The Fock states (19) are the eigenstates of the integrals of motion $\hat{A}_s^\dagger(t)\hat{A}_s(t)$, $\hat{B}_s^\dagger(t)\hat{B}_s(t)$ and $\hat{A}_N^\dagger(t)\hat{A}_N(t)$ and components of the vector $\vec{n}$ are eigenvalues of these operators.

$$\hat{A}_s^\dagger(t)\hat{A}_s(t)\psi_{\vec{n}}(\vec{q}, t) = n_s\psi_{\vec{n}}(\vec{q}, t),$$

$$\hat{B}_s^\dagger(t)\hat{B}_s(t)\psi_{\vec{n}}(\vec{q}, t) = m_s\psi_{\vec{n}}(\vec{q}, t),$$

$$\hat{A}_N^\dagger(t)\hat{A}_N(t)\psi_{\vec{n}}(\vec{q}, t) = n_0\psi_{\vec{n}}(\vec{q}, t).$$

4   Squeezing and Correlated Coefficients

Let us calculate the dispersions of coordinates and momenta in squeezed correlated states (18). We define the dispersions and correlations by the formulae

$$\sigma_{q_i q_k} = \langle \psi_{\vec{\alpha}}(\vec{q}, t) \mid \hat{q}_i \hat{q}_k \mid \psi_{\vec{\alpha}}(\vec{q}, t)\rangle - \langle \psi_{\vec{\alpha}}(\vec{q}, t) \mid \hat{q}_i \mid \psi_{\vec{\alpha}}(\vec{q}, t)\rangle\langle \psi_{\vec{\alpha}}(\vec{q}, t) \mid \hat{q}_k \mid \psi_{\vec{\alpha}}(\vec{q}, t)\rangle,$$

$$\sigma_{p_i p_k} = \langle \psi_{\vec{\alpha}}(\vec{q}, t) \mid \hat{p}_i \hat{p}_k \mid \psi_{\vec{\alpha}}(\vec{q}, t)\rangle - \langle \psi_{\vec{\alpha}}(\vec{q}, t) \mid \hat{p}_i \mid \psi_{\vec{\alpha}}(\vec{q}, t)\rangle\langle \psi_{\vec{\alpha}}(\vec{q}, t) \mid \hat{p}_k \mid \psi_{\vec{\alpha}}(\vec{q}, t)\rangle,$$

$$\sigma_{q_i p_k} = \frac{1}{2}\langle \psi_{\vec{\alpha}}(\vec{q}, t) \mid \hat{q}_i\hat{p}_k + \hat{p}_k\hat{q}_i \mid \psi_{\vec{\alpha}}(\vec{q}, t)\rangle - \langle \psi_{\vec{\alpha}}(\vec{q}, t) \mid \hat{q}_i \mid \psi_{\vec{\alpha}}(\vec{q}, t)\rangle\langle \psi_{\vec{\alpha}}(\vec{q}, t) \mid \hat{p}_k \mid \psi_{\vec{\alpha}}(\vec{q}, t)\rangle.$$

So one can calculate that the correlation of coordinates and momenta of different oscillators are not equal to zero and have the form

$$\sigma_{q_i q_k} = \frac{l_0^2 \mid \epsilon_0 \mid^2}{2N} + \sum_{s=1}^{p} \frac{l_s^2 \mid \epsilon_s \mid^2}{N} \cos\left(2\pi sm(i - k)/N\right),$$

$$\sigma_{p_i p_k} = \frac{\hbar^2 e^{4\Gamma(t)} \mid \dot{\epsilon}_0 \mid^2}{2N l_0^2 \Omega_0^2(0)} + \sum_{s=1}^{p} \frac{\hbar^2 e^{4\Gamma(t)} \mid \dot{\epsilon}_s \mid^2}{N l_s^2 \Omega_s^2(0)} \cos\left(2\pi sm(i - k)/N\right).$$

One has for the dispersions of coordinate and momenta of the same oscillator

$$\sigma_{q_i^2} = \frac{l_0^2 \mid \epsilon_0 \mid^2}{2N} + \sum_{s=1}^{p} \frac{l_s^2 \mid \epsilon_s \mid^2}{N}, \tag{20}$$

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$$\sigma_{p_i^2} = \frac{\hbar^2 e^{4\Gamma(t)} \mid \dot{\epsilon}_0 \mid^2}{2N l_0^2 \Omega_0^2(0)} + \sum_{s=1}^{p} \frac{\hbar^2 e^{4\Gamma(t)} \mid \dot{\epsilon}_s \mid^2}{N l_s^2 \Omega_s^2(0)}. \tag{21}$$

The correlated squeezed states (18) and ground state (17) minimize the Schrödinger-Robertson uncertainty relation [17, 18]

$$\sigma_{q_k^2} \sigma_{p_k^2} \geq \frac{\hbar^2}{4(1 - r^2)}$$

with the correlation coefficient

$$r = \sigma_{q_k p_k} \left(\sigma_{q_k^2} \sigma_{p_k^2}\right)^{-1/2}$$

equal to

$$r = \left(1 - N^2 e^{-4\Gamma(t)} \left[\mid \epsilon_0 \mid^2 l_0^2 + 2\sum_{s=1}^{p} \mid \epsilon_s \mid^2 l_s^2\right]^{-1} \left[\frac{\mid \dot{\epsilon}_0 \mid^2}{\Omega_0^2(0) l_0^2} + 2\sum_{s=1}^{p} \frac{\mid \dot{\epsilon}_s \mid^2}{\Omega_s^2(0) l_s^2}\right]^{-1}\right)^{1/2}. \tag{22}$$

One can see from (20)-(21) that changing of the frequencies influence the dispersions and the squeezing coefficient $k = \frac{\sigma_{q_k^2}(t)}{2\sigma_{q_k^2}(0)}$. Namely, by changing the frequencies one can decrease the dispersions of the coordinates due to increasing of the dispersions of momenta, and vice versa, and make squeezing coefficients less then unity. So the squeezing phenomenon arises due to parametric excitation of quantum chain of coupled parametric oscillators with damping. It is necessary to note that due to parametric excitation each oscillator has the additional time-dependent parameter (22), so-called correlation coefficient which is equal to zero in the stationary regime.

5 Acknowledgements

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THE UNCERTAINTY PRINCIPLE IN RESONANT GRAVITATIONAL WAVE ANTENNAE AND QUANTUM NON-DEMOLITION MEASUREMENT SCHEMES

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Abstract

A review on the current efforts to approach and to surpass the fundamental limit in the sensitivity of the Weber type gravitational wave antennae is reported. Applications of quantum non-demolition techniques to the concrete example of an antenna resonant with the transducer are discussed in detail. Analogies and differences from the framework of the squeezed states in quantum optics are finally discussed.

1 Introduction

The importance of detecting gravitational waves, as frequently pointed out, consists not only in verifying one of the most direct and astonishing predictions of the simplest metric theory of gravitation, i.e. General Relativity, but also in the possibility to open new windows on phenomena in the Universe in which only violent releases of gravitational energy occur [1]. Gravitational waves have not yet been directly observed because of the extreme smallness of the energy released in actual detectors even if they are emitted by astronomical systems. The hypothetical sources which are strong candidates for emitting gravitational waves, according to our understanding of them due to information collected via the electromagnetic astronomy, are divided into two classes based upon the time evolution. Impulsive sources can be catastrophic events such as supernovae explosions and collapsing binary systems. The frequency spectrum of gravitational waves of this kind is flat up to 10^3 Hz, these impulsive phenomena having a characteristic duration of the order of milliseconds. One expects a perturbation of the metric tensor $h \approx 10^{-21} - 10^{-18}$ for events in our Galaxy and $h \approx 10^{-23} - 10^{-21}$ for events in the Virgo Cluster. Periodic sources can be pulsars if they deviate substantially from axial symmetry. The expected frequencies range is in this case

between 10^{-2} and 10^2 Hz, while $h \approx 10^{-27} - 10^{-25}$. The efforts to detect gravitational waves have been concentrated from the very beginning on the impulsive events because of the larger expected perturbation to the metric tensor. It turns out that the modulation of the space-time induced by a gravitational wave on an extended body can also be seen as a production of a force field in it. Detecting the gravitational wave is therefore translated into the problem of detecting this small force of geometrical nature and the displacements produced by it in a test mass. The displacement induced in a body of reasonable sizes, $\approx 1\,m$, has therefore an amplitude of the order of 10^{-21} if the event is due to a supernovae in the Virgo Cluster. The accuracy required to measure such a small displacement is so high that the quantum nature of the detector has to be taken into account because the De Broglie wavelenght of a macroscopic test mass is of the same order of magnitude of the expected signal due to the gravitational waves. Here we report on the status of the art of the measurement techniques developed to allow monitoring of a class of gravitational wave detectors in a quantum regime. After a brief introduction for schematizing the detectors of gravitational waves and the sensitivity limit due to the fundamental noise in part 2, we introduce, in part 3, the quantum non-demolition measurement schemes for overcoming these limitations. The applications of stroboscopic and continuous quantum non-demolition schemes for a gravitational bar antenna resonant with the transducer are described respectively in part 4 and part 5. Conclusions deal also with the analogies and the differences from the quantum optics framework and the importance of this topic for understanding quantum mechanics applied to single macroscopic degrees of freedom repeatedly monitored.

2 Weber gravitational antennae:fundamental sensitivity limits

The gravitational wave detectors devised so far are based upon monitoring of the distance between two masses localized at different points. The equivalence principle requires a non-local, extended, structure of a gravitational wave detector because it is possible to nullify locally the effects of a gravitational field by means of a suitable choice of the reference frame.

Let us consider two masses in free fall: what is then measured is their variable distance which is supposed to be much smaller than the gravitational wavelength. The effect of a gravitational wave coming along z axis with proper polarization is to increase of $h/2$ the distance along y axis and to decrease by $h/2$ the distance along x axis. A classification of the gravitational wave detectors divides these into non resonant and resonant detectors if the two masses are respectively free or elastically coupled.

In non resonant detectors the distance between the two masses is measured by means of interferometric devices. The arms of the interferometer proposed so far are of the order of Km and use of multiple reflections allows an increase in the physical path by several orders of magnitude. In this contribution we will not be concerned with this kind of detector but we shall instead consider the resonant detectors (Weber type gravitational wave antennae), the quantum limit in a interferometric antenna being enforced by the shot noise and the momentum fluctuations imparted by the photon flux to the central mirror of the interferometer [2].

Resonant antennae are typically cylindrical bars of materials having low internal dissipation. The materials used are silicon, sapphire, niobium or a particular aluminum alloy (Al 5056) and

the mass of the antennae is a few tons.

One can show that the motion of the ends of a cylindrical bar of mass M and length L oscillating in its fundamental longitudinal mode is equivalent to that of a harmonic oscillator of mass $M/2$ and equivalent length $4L/\pi^2$. If x is the displacement from equilibrium position the equation of motion of the Weber oscillator is

$$\ddot{x} + \frac{\dot{x}}{\tau_0} + \omega_0^2 x = \frac{2}{\pi^2} L \ddot{h}(t) \tag{1}$$

where τ_0 is the damping time, ω_0 is the proper frequency and $h(t)$ is the amplitude of the incoming gravitational wave. The forcing term due to the gravitational field is proportional to the distance between the two masses. From this formula one can calculate the cross section for the transfer of energy from the wave to the antenna and one finds that this is proportional to the mass of the antenna and to L^2. The proper frequency ω_0 is chosen to be tuned with the frequency of the expected wave ($10^3 Hz$) and the corresponding wavelength is very large compared to the size of the antenna. To amplify the extremely small oscillations coupling of the bar with another oscillator of very small mass is used [3],[4]. In this case a system of two coupled harmonic oscillators is obtained in which the energy is continuously transferred back and forth from M to m via beating. If the dissipations in the two oscillators are made negligible the amplitude of the oscillations in the second resonator is increased by a factor $1/\sqrt{\mu}$ with respect to the first resonator, where $\mu = m/M$, provided that the frequencies of the two uncoupled oscillators are made coincident. The motion of the transducer is transformed into an electric signal by means of a variable capacitor and an amplifier schematized as an ideal amplifier of gain A and two noise sources generators with current and voltage spectral densities respectively SI_n and SV_n. The sources of noise are the thermal noise, i.e. Brownian motion of antenna, which gives a contribution $K_B T$ to the energy of the oscillator, being K_B the Boltzmann's constant and T the thermodynamical temperature of the antenna and the amplifier noise, which is expressed by means of the parameter $T_n = (SV_n SI_n)^{1/2}/K_B$, called noise temperature of the amplifier. This last noise has two effects: it contributes directly as an additive noise source at the output and it acts on the transducer leading to an increase of the temperature. In other words every transducer is at the same time an actuator and the amplifier noise gives rise to a back-action force acting on the mechanical oscillator.

If we define a noise temperature T_{eff} as the temperature which corresponds to the minimum detectable energy $E_{eff} = K_B T_{eff}$ transferred to the bar by an impulsive signal with an output signal/noise ratio equal to 1, we find, using a Wiener algorithm in the data analysis [5]

$$T_{eff} = 2 T_n \sqrt{\left(1 + \frac{1}{\lambda_0^2}\right)\left(1 + \frac{2T\lambda_0}{\beta Q T_n}\right)} \tag{2}$$

where $Q = \omega_0 \tau_0$ is the quality factor of the mechanical system, β is substantially the fraction of energy transferred to the electromagnetic circuit by the bar through the capacitive coupling and λ_0 the impedance matching factor defined as

$$\lambda_0 = \frac{SV_n}{SI_n} \frac{1}{Z_0}. \tag{3}$$

For the antenna of the Rome group continously operating since one year at CERN one has a thermodynamical temperature of $\approx 4.2K$; the other parameters are $Q \simeq 5 \cdot 10^6$ and an amplifier

noise temperature $T_n \simeq 10^{-7}$K [6]. It has been possible to achieve this last result making use of a SQUID amplifier. So one gets for T_{eff} a value of $\approx 10\mu K$, which is not far from the quantum limit temperature

$$T_{QL} = \frac{\hbar\omega}{K_B} \simeq 10^{-8}\text{K}. \tag{4}$$

One expects that the force with which a gravitational wave acts on the antenna is by many orders of magnitude below the thermal noise even at thermodynamical temperatures as low as 10mK which is the temperature at which the third generation antennae will operate. However, due to the particular features of the data analysis based on the variation of energy in the oscillator in the time, the quantum regime is reached earlier than as expected by (4). By writing the amount of energy which is exchanged during the measurement time Δt between the harmonic oscillator and the thermal reservoir and the quantized energy introduced by the measuring apparatus is easy to show that the quantum regime is obtained when the following condition is satisfied

$$\frac{K_B T \Delta t}{Q} \ll \hbar \tag{5}$$

This can be also shown by reasoning in terms of displacements instead of energy. The variation of the length of the bar due to a gravitational wave with amplitude h is, according to (1)

$$\frac{\Delta l}{l} \simeq \frac{h}{2}. \tag{6}$$

Because typical values for h are $h = 10^{-21}$ (which corresponds to a supernova explosion in the center of the Galaxy) taking $L = 1$ m, one gets from (5) a variation of the length of the bar $\Delta L \simeq 10^{-19}$ cm which coincides with the standard quantum limit (i.e. the root square mean of the position of a harmonic oscillator in his fundamental mode)

$$\sqrt{\langle \Delta l^2 \rangle} = \sqrt{\frac{\hbar}{2M\omega}}. \tag{7}$$

It follows therefore that if we do not overcome this limit no information can be obtained on the evolution of the harmonic oscillator.

In these conditions one can find a method to measure the position of the quantum oscillator and to see if an external force has acted on it. However in doing this one must take into account that the position operator $\hat{x}(t)$ does not commute with itself at different times. Indeed with a measurement of $\hat{x}(t)$ at time t one puts the oscillator into an eigenstate of $\hat{x}(t)$; if one repeats this measurement at the instant $t + \tau$ one puts the oscillator into another eigenstate. It turns out that it is not possible to know if the change in $\hat{x}(t)$ is caused by a very weak classical external force or by the demolition of the state due to the previous measurement. What is needed is therefore a measurement which does not prevent the execution of the next measurements of the same observable avoiding the demolition of the projection of the state on that observable. This is possible in non-relativistic quantum mechanics as we will discuss in the following considerations, because this theory makes limitations only on a simultaneous, perfect knowledge of two canonical observables.

3 Quantum non demolition measurements

The introduction of the quantum non-demolition measurements (QND) dates back to an article by Landau and Peierls [7] in 1931. However only recently, after understanding the role of quantum mechanics in the fundamental limits to the amplifier sensitivity [8],[9] and under the request to surpass the quantum limit in detectors of small displacements [10],[11], the problem has been studied in detail [12],[13]. The idea of a QND strategy is to perform a series of measurements of one observable of a single object in such a way that the act of the measurement itself does not affect the predictability of the result of the next measurements of the same observable. In order to do this the observable, the instants of time in which it is observed and the interaction Hamiltonian should be all carefully chosen for a given dynamical system. For instance, a first high precision measurement of the position of a free particle implies a large dispersion in the possible values of measurements of momentum. If a second measurement of position is made, due to the Heisenberg evolution, the result will have a large dispersion too. Instead, if a measurement of momentum in a free particle is made at a given instant of time, a second possible measurement will give the same result due to the constant value of the momentum between the two consecutive measurements, provided that the interaction due to the first measurement has not demolished the state. This simple example shows the route to define quantum non-demolition measurements. Only particular observables which satisfy a commutation relation at different times t_i and t_j are allowed to be monitored in a QND way, i.e. if

$$[\hat{x}(t_i), \hat{x}(t_j)] = 0. \tag{8}$$

Moreover, we must also take into account the perturbation on $\hat{x}(t)$ induced by the measuring apparatus which is coupled to the observed system by means of the Hamiltonian operator $\widehat{H}_i$. To avoid changes in the expected value of the observable during the measurement the following condition must be satisfied:

$$[\hat{x}(t), \widehat{H}_i] = 0. \tag{9}$$

This condition assures that the interaction Hamiltonian is simultaneously diagonalizable with the measured observable, no changes are induced in the measured observable during the measurement time in which only the interaction Hamiltonian will be responsible for the time evolution. A sequence of measurements performed under conditions (8) and (9) will give always the same result. This is a definition of a QND measurement. If the instants of time in which it is satisfied (8) are discrete the QND scheme is named stroboscopic or, in a realistic configuration with a duration of the measurement small with respect to the characteristic timescale of the motion of the observed system, quasi-stroboscopic [14],[15],[16]. Otherwise, having a continuous set of instants of time, the QND scheme is named continuous.

In the case of a single oscillator one introduces the two components of the complex amplitude

$$\begin{cases} \widehat{X}_1 = Re[(\hat{x} + i\frac{\hat{p}}{m\omega})e^{i\omega t}] \\ \\ \widehat{X}_2 = Im[(\hat{x} + i\frac{\hat{p}}{m\omega})e^{i\omega t}] \end{cases} \tag{10}$$

such that $\hat{x}(t) = \widehat{X}_1 \cos \omega t + \widehat{X}_2 \sin \omega t$. Their properties are

$$(a) \qquad \frac{d\widehat{X_1}}{dt} = \frac{d\widehat{X_2}}{dt} = 0 \Rightarrow [\widehat{X_1}(t), \widehat{X_1}(t + \tau)] = [\widehat{X_2}(t), \widehat{X_2}(t + \tau)] = 0 \qquad (11)$$

$$(b) \qquad [\widehat{X_1}(t), \widehat{X_2}(t)] = \frac{i\hbar}{m\omega}. \qquad (12)$$

By using (a) and (b) we get

$$[\hat{x}(t), \hat{x}(t + \tau)] = -[\widehat{X_1}, \widehat{X_2}]\{\cos \omega t \, \sin \omega(t + \tau) - \sin \omega t \, \cos \omega(t + \tau)\} = \frac{i\hbar}{m\omega} \sin \omega\tau. \qquad (13)$$

This means that to do a QND measurement of the operator $\hat{x}(t)$ in a single harmonic oscillator one needs the Hamiltonian (here $\hat{q}$ is the variable of the measuring apparatus which couples with the oscillator)

$$\widehat{H_i} = E_0 \, \delta(t - \frac{n\pi}{\omega})\hat{x}\hat{q} \qquad (14)$$

such that the interaction between the system and the measuring apparatus is turned on only when $\hat{x}(t)$ commutes with itself, that is why this kinds of measurements are called stroboscopic Q.N.D..

For a component of the complex amplitude, $\widehat{X_1}$, a QND interaction Hamiltonian should be [12]

$$\widehat{H_i} = E_0\widehat{X_1}\hat{q} \qquad (15)$$

that is approximately obtained by using the interaction Hamiltonian

$$\widehat{H_i} = 2E_0 \cos \omega_m t \, \hat{x} \, \hat{q} \qquad (16)$$

provided a low-pass filter at $\omega_c \ll \omega_m$ is used. For practical reasons a different pumping scheme is used, namely an up-conversion around an electrical frequency ω_e such that the interaction Hamiltonian is now

$$\widehat{H_i} = E_0 \cos \omega_e t \cos \omega_m t \, \hat{x} \, \hat{q} = \frac{E_0}{2}[\cos(\omega_e + \omega_m)t + \cos(\omega_e - \omega_m)t] \, \hat{x} \, \hat{q} \qquad (17)$$

which allows an approximate measurement of $\widehat{X_1}$ if a filtering around ω_e is performed with a selectivity such that the terms oscillating at $\omega_e \pm 2\omega_m$ are made negligible. It has been pointed out that the continuous approximate QND measurement scheme of one component of the complex amplitude is obtained as a first order approximation of the corresponding stroboscopic scheme [17]. If we start from the interaction Hamiltonian of a stroboscopic measurement of $\widehat{X_1}$ expressed in terms of the physical observable $\hat{x}$

$$\widehat{H_i} = E_0 \cos \omega_e t \sum_n \delta(t - \frac{n\pi}{\omega_1})\widehat{X_1}\hat{q} = E_0 \cos \omega_e t \sum_n (-1)^n\delta(t - \frac{n\pi}{\omega_1}) \, \hat{x} \, \hat{q} \qquad (18)$$

we will see that, by Fourier expanding the Dirac-distribution, it is obtained

$$\widehat{H_i} = E_0 \cos \omega_e t \sum_n \cos(2n + 1)\omega_1 t\hat{x} \, \hat{q} \qquad (19)$$

that, at the first order, is

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \omega_1 t \hat{x} \, \hat{q} \qquad (20)$$

i.e. the usual approximate scheme for monitoring of $\widehat{X}_1$. Thus knowing a QND stroboscopic strategy it is simple to write the corresponding QND approximate continuous strategy. This property will be particularly useful in the following considerations, where the more complicated but realistic case of two coupled harmonic oscillators will be treated.

It has been pointed out that also in the classical regime, i.e. when the amplifier is not quantum limited, the QND measurement schemes provide a better sensitivity because one phase of the signal is shielded by the back-action force of the amplifier. A quantitative model in the classical limit has been developed in [18]: it turns out that by writing the noise temperature as

$$T_b = \frac{\omega_m}{\omega_e} T_n \frac{1}{r} \qquad (21)$$

for a standard 'amplitude and phase' monitoring is $r < 1$, and for a QND/BAE scheme r may be greater than unity. This is due to the squeezing of the electrical noise into one mechanical phase. A generalized uncertainty relation for the two classical conjugate observables due to the back-action of the amplifier noise is introduced as

$$\Delta X_1 \Delta X_2 \simeq \frac{K_B T_n}{2m\omega_m\omega_e} \qquad (22)$$

which may be obtained through a replacement on the right hand side in the standard quantum uncertainty relationship

$$\Delta X_1 \Delta X_2 \simeq \frac{\hbar}{2m\omega_m} \qquad (23)$$

of $\hbar$ with $K_B T_n/\omega_2$. If a squeezing factor ρ such that $\Delta X_1 = \rho \Delta X_2$ is introduced ($\rho \to 0$ means a noise-free measurement of $\widehat{X}_1$) the minimum burst noise temperature can be written as

$$T_b = \frac{m\omega_m^2 \Delta X_1^2}{2} \simeq \frac{1}{4} T_n \frac{\omega_1}{\omega_2} \rho \qquad (24)$$

showing that the r figure of merit has a dynamical interpretation in terms of a squeezing factor. Recently, an interpretation of the back-action evasion strategies in which they are seen as an alternative to the usual impedance matching for maximizing the signal to noise ratio has been discussed [19].

The description of the QND measurement suggests how to measure small forces below the standard quantum limit. By means of a simple integration of the Heisenberg equation in presence of an external force $F(t)$, one gets for the QND operator $\widehat{X}_1$

$$\widehat{X}_1(t) = \widehat{X}_1(t_0) - \hat{I} \int_{t_0}^{t} \frac{F(t)}{m\omega} \sin \omega t' \, dt'. \qquad (25)$$

A sequence of measurement of $\widehat{X}_1$ will then give as a result a sequence of eigenstates linked to the value of the external force

$$\xi(t,\tau) = \xi(t_0) - \int_{t_0}^{t} \frac{F(t)}{m\omega} \sin \omega t', \, dt'. \tag{26}$$

By means of successive measurements it is possible to study the form of $F(t)$ simply inverting (26)

$$F(\tau) = -\frac{m\omega}{\sin \omega t} \frac{d}{dt} \xi(t_0,t)\Big|_{t=\tau} \tag{27}$$

The singularities for $t = n\pi/\omega$ correspond to a null information on the force acting on the harmonic oscillator on some instants of time. This can be compensated by using a second oscillator (i.e. a second antenna) with complex amplitude $\hat{Y}_1 + i\hat{Y}_2$ which has eigenvalues

$$\zeta(t,\tau) = \zeta(t_0) - \int_{t_0}^{t} \frac{F(t)}{m\omega} \sin \omega t', \, dt' \tag{28}$$

here obviously the singularities are in $t_n = (2n+1)\pi/2\omega$.

4 QND quasi-stroboscopic scheme for coupled harmonic oscillators

The current generation of gravitational wave antenna of the Weber type operates by means of an antenna coupled to a small mechanical resonator. In such a way the energy deposited in the antenna by a gravitational wave burst is transferred to the transducer. In the case of an ideal transfer of energy, i.e. with both a perfect tuning of the two uncoupled frequencies and negligible dissipations during the beating period, the amplitude of the oscillations in the transducer is larger than that in the antenna by a factor equal to the square root of the ratio of the equivalent masses of the two resonators. All the detectors operating in coincidence as described in [6] were equipped with a resonant transducer and the same is also planned for the third generation of gravitational wave antennas cooled at 50 mK now under development. It is therefore important to generalize the previous considerations on the QND schemes to this situation, as already outlined in [20]. As we have seen, it is possible to schematize the gravitational cryogenic antenna and the resonant transducer with two coupled harmonic oscillator having masses respectively m_x and m_y (with $\mu = \frac{m_y}{m_x} \ll 1$). The two coupled mechanical oscillators are described by the Lagrangian

$$L = \frac{1}{2}m_x\dot{x}^2 + \frac{1}{2}m_y\dot{y}^2 - \frac{1}{2}m_x\omega_x^2 x^2 - \frac{1}{2}m_y\omega_y^2(y-x)^2 = \frac{1}{2}(\dot{\xi}\ \dot{\eta})\, T \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} - \frac{1}{2}(\xi,\eta)\, V \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{29}$$

where the normalized coordinates $\xi = \sqrt{m_x}x$ and $\eta = \sqrt{m_y}y$ have been introduced, together with the matrices T and V

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{30}$$

$$V = \begin{pmatrix} \omega_x^2 + \mu\omega_y^2 & -\sqrt{\mu}\omega_y^2 \\ -\sqrt{\mu}\omega_y^2 & \omega_y^2 \end{pmatrix}. \tag{31}$$

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As we have already cited to obtain the maximum coupling the two oscillators should have the same frequency $\omega_x = \omega_y = \omega$, i.e. they should be tuned. In this case one finds the solutions

$$\omega_{\pm}{}^2 = \omega_0{}^2 \left(1 + \frac{\mu}{2} \pm \sqrt{\mu \left(1 + \frac{\mu}{4} \right)} \right) \tag{32}$$

which we can write more easily introducing $\alpha_{\pm} = \frac{\mu}{2} \pm \sqrt{\mu \left(1 + \frac{\mu}{4} \right)}$ obtaining $\omega_{\pm}^2 = \omega_0^2 (1 + \alpha_{\pm})$.

The normal coordinates $\Xi_{\pm}$ corresponding to the eigenfrequencies $\omega_{\pm}$ are linked to the physical coordinates by means of an orthogonal matrix

$$\begin{pmatrix} \widehat{\Xi}_+ \\ \widehat{\Xi}_- \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{\alpha_- + 2}} & \dfrac{\alpha_-}{\sqrt{\mu}\sqrt{\alpha_- + 2}} \\[3mm] \dfrac{1}{\sqrt{\alpha_+ + 2}} & \dfrac{\alpha}{\sqrt{\mu}\sqrt{\alpha_+ + 2}} \end{pmatrix} \begin{pmatrix} \sqrt{m_x}\,\hat{x} \\ \sqrt{m_y}\,\hat{y} \end{pmatrix}. \tag{33}$$

Let us introduce the complex amplitudes of the normal modes

$$\begin{cases} \widehat{X}_1^{\pm} = \widehat{\Xi}_{\pm} \cos \omega_{\pm} t - \dfrac{\widehat{P}_{\Xi_{\pm}}}{\omega_{\pm}} \sin \omega_{\pm} t \\[4mm] \widehat{X}_2^{\pm} = \widehat{\Xi}_{\pm} \sin \omega_{\pm} t + \dfrac{\widehat{P}_{\Xi_{\pm}}}{\omega_{\pm}} \cos \omega_{\pm} t \end{cases} \tag{34}$$

which satisfy the relations

$$[\widehat{X}_1^+, \widehat{X}_2^+] = \frac{i\hbar}{\omega_+} \qquad [\widehat{X}_1^-, \widehat{X}_2^-] = \frac{i\hbar}{\omega_-} \tag{35}$$

as well as

$$[\widehat{X}_{1,2}^+(t), \widehat{X}_{1,2}^+(t+\tau)] = [\widehat{X}_{1,2}^-(t), \widehat{X}_{1,2}^-(t+\tau)] = 0. \tag{36}$$

We can also rewrite the Hamiltonian $\widehat{H}$ of the system as

$$\widehat{H} = \frac{\omega_+^2}{2}[(\widehat{X}_1^+)^2 + (\widehat{X}_2^+)^2] + \frac{\omega_-^2}{2}[(\widehat{X}_1^-)^2 + (\widehat{X}_2^-)^2]. \tag{37}$$

The commutator $[\hat{y}(t) - \hat{x}(t), \hat{y}(t+\tau) - \hat{x}(t+\tau)]$ is calculated by writing $\hat{y}$ and $\hat{x}$ in terms of the complex amplitudes $\widehat{X}_{1,2}^{\pm}, \widehat{X}_{1,2}^{\pm}$ of the normal modes which are integral of the motion and by using the same computation procedure which led us to formula (13). Using (35),(36) we obtain, finally, the expression

$$[\hat{y}(t) - \hat{x}(t), \hat{y}(t+\tau) - \hat{x}(t+\tau)] = \frac{i\hbar}{M\omega\mu\sqrt{\mu+4}} \left[\frac{\omega_+^2}{\omega^2} \sin \omega_+\tau + \frac{\omega_-^2}{\omega^2} \sin \omega_-\tau \right]. \tag{38}$$

This quantity becomes, in the limit $\mu \to 0$

$$[\hat{y}(t) - \hat{x}(t), \hat{y}(t+\tau) - \hat{x}(t+\tau)] = \frac{i\hbar}{2m_y\omega} \frac{\omega_+^2 + \omega_-^2}{\omega^2} \sin \bar{\omega}\tau \, \cos \omega_B\tau \tag{39}$$

where $\bar{\omega} = \frac{\omega_+ + \omega_-}{2} = \frac{\omega}{2}\sqrt{\mu + 4} \rightarrow \omega$ and $\omega_B = \frac{\omega_+ - \omega_-}{2} = \frac{\omega}{2}\sqrt{\mu}$. Eqns. (38) and (39) show that the commutator of the operator $\hat{y} - \hat{x}$ with itself at different times is time dependent and it has a characteristic beating behaviour. We have seen that in a quasi-stroboscopic scheme for a single harmonic oscillator the commutator is zero each half a period of the motion and the stroboscopicity is defined whenever measurements with a duration small compared to the period of the motion are performed. This implies a measurement time, a duty cycle, very small and a consequent small value of the effective electromechanical quality factor. In the case of a double harmonic oscillator this drawback is less pronounced because the commutativity is assured every half of a beating period for a time of the order of a period of oscillation. Thus quasi-stroboscopic QND schemes already proposed as a generalization of the conventional BAE scheme based upon a continuous monitoring [17] and already tested on a single oscillator system [21] can be adapted to this situation. In the case of a single harmonic oscillator the duration of the measurement must be small compared to the period of the harmonic oscillator T, in the case of two coupled harmonic oscillators this duration is of the order of some periods of the uncoupled oscillator, although the interaction must be turned on every quarter of a beating period. The interaction Hamiltonian for a two coupled harmonic oscillator system is therefore

$$\widehat{H}_i = \frac{E_0}{2} \sum_n [\theta(t - \frac{nT_B}{2} + \frac{\Delta T}{2}) + \theta(-t + \frac{nT_B}{2} + \frac{\Delta T}{2})](\hat{y} - \hat{x})\hat{q} \qquad (40)$$

where T_B is the beat period and ΔT is of the order of the period of a single harmonic oscillator. Practical values are $T_B \simeq 40\text{ms}$ and $\Delta T \simeq 2\text{ms}$. To calculate the error in a quasi stroboscopic measurement of the operator $\hat{y} - \hat{x}$ performed for instance in the interval $\frac{\pi}{2\omega_B} - \frac{2\pi}{\omega}, \frac{\pi}{2\omega_B} + \frac{2\pi}{\omega}$ we identify the conjugate observable of $\hat{y} - \hat{x}$ as the quantity $(\hat{p}_y - \hat{p}_x)/2$. This last can be expressed in terms of the components of the amplitudes of the normal modes and the commutator at different times of the two conjugate observables is obtained as

$$[\hat{y}(t + \tau) - \hat{x}(t + \tau), \frac{1}{2}(\hat{p}_y(t) - \hat{p}_x(t))] = \frac{i\hbar}{2}(\frac{\alpha_- - 1}{\alpha_-(\alpha_- + 2)} \cos \omega_+ \tau + \frac{\alpha_+ - 1}{\alpha_+(\alpha_+ + 2)} \cos \omega_- \tau). \quad (41)$$

When $\tau = 0$ the commutator relationship (41) is written as

$$[\hat{y}(t) - \hat{x}(t), \frac{1}{2}(\hat{p}_y(t) - \hat{p}_x(t))] = i\hbar \qquad (42)$$

which is exactly the quantity $[\hat{x}(t), \frac{1}{2}\hat{p}_x(t)] + [\hat{y}(t), \frac{1}{2}\hat{p}_y(t)]$.

By expressing ω_+ and ω_- in terms of the frequencies $\bar{\omega}$ and ω_B and substituting in $\alpha_\pm$ their expressions in terms of μ we get finally

$$[\hat{y}(t + \tau) - \hat{x}(t + \tau), \frac{1}{2}(\hat{p}_y(t) - \hat{p}_x(t))] = i\hbar(\cos \bar{\omega}\tau \cos \omega_B\tau - \frac{1 + \mu}{\sqrt{\mu(\mu + 4)}} \sin \bar{\omega}\tau \sin \omega_B\tau) \quad (43)$$

If the measurement is performed in the interval $[\frac{\pi}{2\omega_B} - \frac{2\pi}{\omega}, \frac{\pi}{2\omega_B} + \frac{2\pi}{\omega}]$, we can approximate $\cos \omega_B\tau \simeq 1$ and $\sin \omega_B\tau \simeq \omega_B\tau - \frac{\pi}{2}$ and a measurement of infinitesimal duration t' performed in such interval and with a precision $\Delta[\hat{y}(t) - \hat{x}(t)]$ allows to evaluate the error introduced in the measurement process on the uncertainty product as

$$\Delta[\hat{y}(t+t') - \hat{x}(t+t')] \cdot \frac{1}{2}\Delta[\hat{p}_y(t) - \hat{p}_x(t)] \approx \frac{\hbar}{2}|\cos \bar{\omega}t' - \frac{1+\mu}{\sqrt{\mu(\mu+4)}} \sin \bar{\omega}t'(\omega_B t' - \frac{\pi}{2})| \quad (44)$$

from which, under the approximation for the trigonometric functions, we obtain

$$\frac{1}{2}\Delta[\hat{p}_y(t) - \hat{p}_x(t)] \approx \frac{\hbar}{2\Delta[\hat{y}(t) - \hat{x}(t)]}. \quad (45)$$

The error due to a measurement of duration t' on the operator $\hat{y} - \hat{x}$ is calculated starting from $\Delta[\hat{y}(t) - \hat{x}(t)]$ because

$$\Delta[\hat{y}(t+t') - \hat{x}(t+t')] \approx \Delta[\hat{y}(t) - \hat{x}(t)]|\cos \bar{\omega}t' - \frac{1+\mu}{\sqrt{\mu(\mu+4)}} \sin \bar{\omega}t'(\omega_B t' - \frac{\pi}{2})| \quad (46)$$

If the notation now is changed defining $\Delta_t = \Delta[\hat{y}(t) - \hat{x}(t)]$ we have

$$\frac{\Delta_t}{t'}[\cos \bar{\omega}t' - \frac{1+\mu}{\sqrt{\mu(\mu+4)}} \sin \bar{\omega}t'|\omega_B t' - \frac{\pi}{2}|] \quad (47)$$

and in the limit of $t' \to 0$ we get

$$\frac{d\Delta_t}{dt} = \frac{1+\mu}{\sqrt{\mu(\mu+4)}} \frac{\pi\bar{\omega}}{2} \cdot \Delta_t \quad (48)$$

from which, by integrating, we obtain the error on a measurement performed around $t = \frac{\pi}{2\omega_B}$ as

$$\Delta[\hat{y}(t+\tau) - \hat{x}(t+\tau)] \approx \Delta[\hat{y}(t) - \hat{x}(t)] \exp[\frac{\pi(\mu+1)}{2\sqrt{\mu(\mu+4)}}\bar{\omega}\tau]. \quad (49)$$

For instance, for a choice $t = [\frac{\pi}{2\omega_B} - \frac{2\pi}{\bar{\omega}}]$ and $\tau = \frac{4\pi}{\bar{\omega}}$ we obtain

$$\Delta[\hat{y}(\frac{\pi}{2\omega_B} + \frac{2\pi}{\bar{\omega}}) - \hat{x}(\frac{\pi}{2\omega_B} + \frac{2\pi}{\omega})] \approx \Delta[\hat{y}(\frac{\pi}{2\omega_B} - \frac{2\pi}{\bar{\omega}}) - \hat{x}(\frac{\pi}{2\omega_B} - \frac{2\pi}{\omega})] \exp[\frac{2\pi^2(\mu+1)}{\sqrt{\mu(\mu+4)}}] \quad (50)$$

A drawback of these measurement scheme appears when μ is very small and the frequency of the measurement is consequently very small too. To overcome this problem a multimode configuration can be used. In this case the commutator at different times approaches zero more frequently when compared to a two-mode configuration of the same final mass ratio. A more detailed description of this point can be found in [22].

5 QND continuous schemes for coupled harmonic oscillators

Also QND continuous schemes can be used for coupled harmonic oscillator. A first example is given by a monitoring of the complex amplitude of the physical modes $\hat{x}$ and $\hat{y}$ [23]. Introducing the complex amplitudes such that

$$\begin{cases} \hat{y} = Re[(\widehat{Y_1} + i\widehat{Y_2})e^{-i\omega_y t}] \\[2mm] \hat{p}_y/m_y\omega_y = Img[(\widehat{Y_1} + i\widehat{Y_2})e^{-i\omega_y t}] \end{cases} \tag{51}$$

we can rewrite the Hamiltonian in terms of $\widehat{Y_1}$ and $\widehat{Y_2}$ and, by writing the Heisenberg equations for the time evolution of $\widehat{Y_1}$, we obtain

$$\frac{d\widehat{Y_1}}{dt} = -\omega_y \hat{x} \sin \omega_y t. \tag{52}$$

The complex amplitude is not a constant of the motion. However it is easily proved that it is a QND observable. A relationship valid for an infinitesimal time τ is derived for the time evolution

$$\widehat{Y_1}(t + \tau) = \widehat{Y_1}(t) - \omega_y \hat{x} \sin \omega_y t\, \tau \tag{53}$$

and this implies the commutation rule for $\widehat{Y_1}$ at different times

$$[\widehat{Y_1}(t + \tau), \widehat{Y_1}(t)] = [\widehat{Y_1}(t) - \omega_y \hat{x} \sin \omega_y t, \widehat{Y_1}] = 0 \tag{54}$$

because of the commutativity between $\widehat{Y_1}$ and $\hat{x}$. Thus $\widehat{Y_1}$ (or $\widehat{Y_2}$, for which similar relationships hold) is a QND observable, although it is not conserved during the motion. From (52) the coordinate $\hat{x}$ is inferred as

$$\hat{x}(t) = -\frac{1}{\omega_y \sin \omega_y t} \frac{d\widehat{Y_1}}{dt} \tag{55}$$

apart from the singularities already discussed appearing when $\sin \omega_y t = 0$. When a classical force $F(t)$ acts on the system the Hamiltonian operator is modified and the added term is

$$\widehat{H}_f = -(\hat{x} + \hat{y})\, F(t) \tag{56}$$

obtaining, in this case, the following expression for the time evolution of $\widehat{Y_1}$

$$\frac{d\widehat{Y_1}}{dt} = -\omega_y \hat{x} \sin \omega_y t - \frac{\sin \omega_y t}{m_y\omega_y} F(t). \tag{57}$$

However the effect of the external force to be detected, in our case of geometrical nature, on the transducer is negligible compared to the effect on the antenna, due to the smaller size of the transducer. Thus $\widehat{H}_f \simeq -\hat{x}F(t)$ and the second term in (57) can be omitted. In this reasonable approximation, i.e. $F(t)$ acting only on the antenna, $\widehat{Y_1}$ is also QNDF, i.e. QND also in presence of an external force. To obtain a continuous monitoring of $\widehat{Y_1}$ we need a QND interaction Hamiltonian of the type

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \omega_y t\, (\hat{y} - \hat{x})\hat{q} \tag{58}$$

that is a coherent superposition of pumpings at frequencies $\omega_e \pm \omega_y$. Analogous considerations can be made for the monitoring of the real or the imaginary part of the complex amplitude of one normal mode expressed in terms of the physical modes through (33). The advantage in this case is that the quantity $\widehat{X_1}^+$ is a constant of the motion and its monitoring is the standard one

already discussed for a single harmonic oscillator. This is obtained by means of the interaction Hamiltonian

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \omega_+ t\,(\hat{y} - \hat{x})\hat{q} \tag{59}$$

and the analogous for monitoring a component of the complex amplitude $\widehat{X_1^-}$ by substituting ω_+ with ω_-. One drawback of monitoring one component of the complex amplitude of the normal modes is that the information on the other mode is lost, and it is crucial to have information on both the modes to take full advantage of the resonant schemes.

An alternative scheme suggested by the time dependence of the commutator consists in a monitoring corresponding to the following Hamiltonian:

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \bar{\omega} t \cos \omega_B t\,(\hat{y} - \hat{x})\hat{q}. \tag{60}$$

This coupling allows one to infer information on both the modes because, upon filtering around ω_e in such a way to neglect terms oscillating at $\omega_e \pm 2\omega_B$, $\omega_e \pm 2\bar{\omega}$, $\omega_e \pm 2(\bar{\omega} \pm \omega_B)$, it can be rewritten as

$$\widehat{H}_i = \frac{E_0}{4} \cos \omega_e t (\beta_+ \hat{X}_1^+ + \beta_- \hat{X}_1^-)\hat{q} \tag{61}$$

where $\beta_\pm$ are coefficients related to the coefficients of the matrix (33) and are expressed as

$$\beta_\pm = [m_x(2 + \frac{\mu}{2} \mp \sqrt{\mu(1 + \frac{\mu}{4})})]^{-1/2}(\frac{1}{2\sqrt{\mu}} \mp \sqrt{\frac{(1 + \frac{\mu}{4})}{\mu}} - \frac{1}{\mu^{3/2}}) \tag{62}$$

which, in the limit of $\mu \to 0$, goes to $\beta_\pm = \mp 1/\sqrt{2m_x\mu}$. In this limit the interaction Hamiltonian assumes a simple form

$$\widehat{H}_i = \frac{E_0}{\sqrt{32 m_x \mu}} \cos \omega_e t(\hat{X}_1^+ - \hat{X}_1^-)\hat{q} \tag{63}$$

which contains information on both the normal modes and in such a way that QND measurements can be performed on both the modes. In all the three cases here discussed the selectivity requirements on the electrical circuit are more stringent than in the case of a single harmonic oscillator, because now the electrical oscillator must have a quality factor $Q_e >> \omega_e/\omega_B$ in order to avoid detection of sidebands contributions. The interaction Hamiltonian (60) can also be written as

$$\widehat{H}_i = \frac{E_0}{2} \cos \omega_e t(\cos \omega_+ t + \cos \omega_- t)(\hat{y} - \hat{x})\hat{q}. \tag{64}$$

With the analogy to the multipump scheme discussed for a single oscillator we can imagine a interaction Hamiltonian of which (64) is only the first order approximation

$$\widehat{H}_i \simeq \frac{E_0}{2} \cos \omega_e t[\sum_{n=0}^{+\infty} \cos (2n + 1)\omega_+ t + \sum_{m=0}^{+\infty} \cos (2m + 1)\omega_- t](\hat{y} - \hat{x})\hat{q} \tag{65}$$

which corresponds, in the limit of a stroboscopic pumping of the kind

$$\widehat{H}_i = E_0 [\sum_{n=0}^{+\infty}(-1)^n\delta(t - \frac{n\pi}{\omega}) + \sum_{m=0}^{+\infty}(-1)^m\delta(t - \frac{n\pi}{\omega})](\hat{y} - \hat{x})\hat{q} \tag{66}$$

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It is interesting to observe that after a time equal to $T_B/2$ both the trains of Dirac distributions will coincide, i.e. $T_B/2 = n\pi/\omega_+ = m\pi/\omega_-$ where $n = m + 2$ (the fact that n and m have the same parity assures the same sign of the corresponding Dirac pulses at those times). So each half a period the two trains are summed and the quasi-stroboscopic scheme discussed in the previous section can be considered as the first order approximation of the stroboscopic scheme resulting from (66). This completes the connection between the multipump continuous schemes and the quasi-stroboscopic scheme introduced in the previous section.

6 Conclusions

We have shown the scenario under which quantum non-demolition measurement schemes should be demanded for detecting gravitational waves in the generation of resonant gravitational wave antennæ currently under development, particularly ultra-low temperature resonant bar antennæ such as the Rome, Legnaro and Stanford ones which will work at a thermodynamical temperature of $\simeq 50$ mK. Both QND stroboscopic and continuous schemes have been discussed as well as their link and practical schemes to implement them. However the interest of quantum non-demolition measurement schemes goes beyond the detectability of the gravitational radiation, involving also the quantum measurement theory and the predictions of it for repeated measurements on a single macroscopic oscillator. Feasibility of the generation of macroscopically distinguishable states using a QND scheme has been recently discussed in quantum optics [24], [25]. It has been pointed out that the generation of Schroedinger cats using micromechanical oscillators with quantum limited sensitivity is also feasible [26]. Unlike the optical case, in which the QND measurement is obtained with a frequency mixing due to non-linear susceptivity, the QND measurement for the mechanical case is obtained using an electric field which can be large as one wants. Dissipations in a mechanical oscillator also are quite low compared to electrical or optical oscillators. Moreover, analogies to the production and the detection of squeezed states in optics [27] have been shown. We want to point out a fundamental difference between the two topics: in the case of the optical squeezed states we deal with a quantized field in which its quantum nature is responsible for the limitation to the sensitivity, in the case of quantum non-demolition measurements on a harmonic oscillator the eventual force field which has to be monitored is considered classical and the fundamental limitations comes from the process of the measurement and the interaction of the meter with the external environment. What is squeezed in a QND measure is the back-action noise generated by the amplifier and the squeezing is made in a phase orthogonal to the one which is detected [21]. Despite this conceptual difference the formalisms to deal with QND strategies are similar to the one used to deal with squeezed states. This analogy is so narrow that also multipump [28], [29] and quasi-stroboscopic [30], [31] schemes have been independently and successfully implemented for squeezing the light. Further thoughts on the analogies and the differences between quantum non-demolition measurements on a harmonic oscillator and the squeezing of the quantum noise can give rise to a better understanding on the same interpretation of Quantum Electrodynamics and the operative origin of the vacuum fluctuations of the field in terms of a measurement process [32], an aspect of this fascinating and successful theory which has been very little investigated until now.

7 Acknowledgments

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HARMONIC OSCILLATOR STATES
IN ABERRATION OPTICS

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Abstract
The states of the three-dimensional quantum harmonic oscillator classify optical aberrations of axis-symmetric systems due to the isomorphism between the two mathematical structures. Cartesian quanta and angular momentum classifications have their corresponding aberration classifications. The operation of concatenation of optical elements introduces a new operation between harmonic oscillator states.

1 Introduction: Optical Phase Space

Geometric optics uses the following 'screen' coordinates for light rays [1]:

$\mathbf{q} = (q_x, q_y)$: Coordinates of position on the 2-dimensional screen. The intersection of the ray with the screen ranges over $\Re^2$.

$\mathbf{p} = (p_x, p_y)$: Coordinates of momentum (with respect to the same screen). The projecton of the ray 3-vector along the ray [of length n, the refractive index of the medium at the point] on the plane of the screen is the momentm 2-vector; the component normal to the screen is the evolution Hamiltonian (below). See Figure 1.

Geometric optics has Hamiltonian evolution equations between the canonically conjugate variables $\mathbf{p}$ and $\mathbf{q}$. The optical Hamiltonian is

$$h = -p_z = -\sqrt{n(\mathbf{q}, z)^2 - |\mathbf{p}|^2}. \tag{1}$$

Plain geometry [2] provides the first Hamilton equation

$$\frac{d\mathbf{q}}{dz} = \frac{\partial h}{\partial \mathbf{p}} = \frac{\mathbf{p}}{p_z} = \{p_z, \circ\}\, \mathbf{q}, \tag{2}$$

while Snell's law yields the dynamics of the second Hamilton equation

$$\frac{d\mathbf{p}}{dz} = -\frac{\partial h}{\partial \mathbf{q}} = -\frac{n}{p_z}\frac{\partial n}{\partial \mathbf{q}} = \{p_z, \circ\}\, \mathbf{p}. \tag{3}$$

We use the Poisson bracket [3]

$$\{f, g\} = \sum_{i=x, y} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right), \tag{4}$$
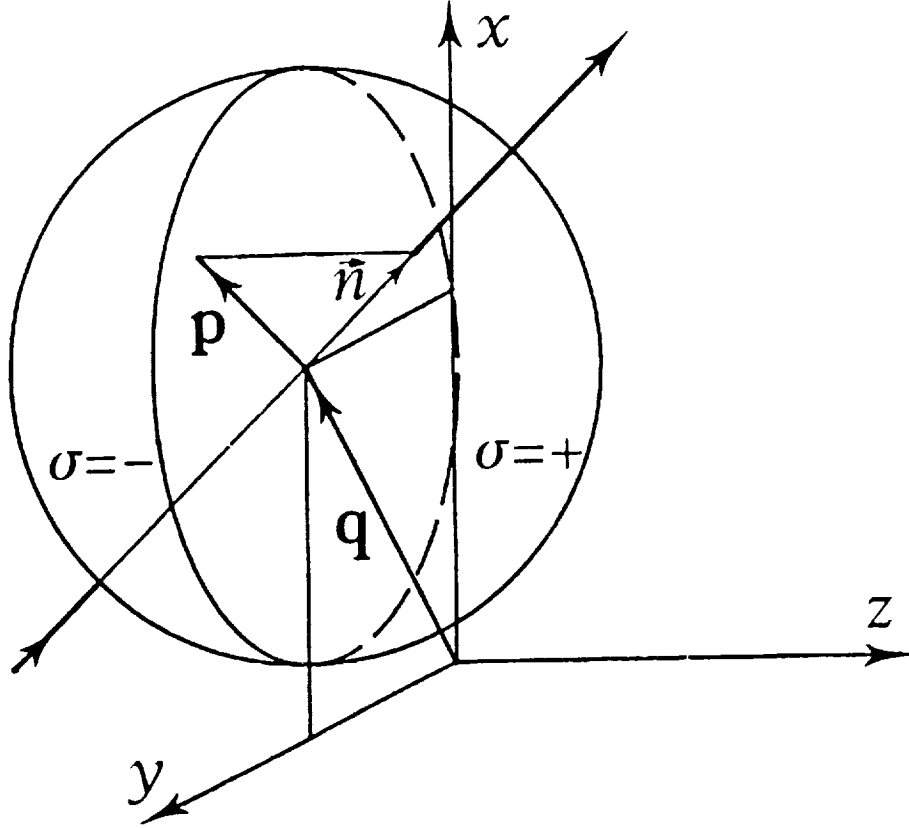
FIG. 1.   The coordinates $(\mathbf{p}, \mathbf{q})$ of optical phase space in geometric optics. The z-axis is traditionally the optical axis.

for functions f, g of $\mathbf{p}$ and $\mathbf{q}$, and the corresponding Lie-Poisson operators $\{f, \circ\}$ as generators of Lie algebras [4]. This space is subject to Hamiltonian evolution equations and constitutes *optical phase space*, subject to the same local symplectic structure as the well known mechanical phase space. Optical phase space *differs* from the mechanical one *globally*, however, in that its momentum ranges over a compact domain: (two) *disks* (forward and backward rays) of radius $n(\mathbf{q})$, the refractive index.

2   Examples

Free flight in homogeneous medium (n constant) yields the solutions

$$\mathbf{q}(z) = \exp z\{p_z, \circ\}\, \mathbf{q} = \mathbf{q} + z\frac{\mathbf{p}}{\sqrt{n^2 - |\mathbf{p}|^2}}, \qquad \mathbf{p}(z) = \exp z\{p_z, \circ\}\, \mathbf{p} = \mathbf{p}. \tag{5}$$

This is shown in Figure 2.

One example we may use to distinguish mechanical from optical phase space transformations pertains the mechanical oscillator *versus* —or *vis-à-vis* [5]— light in a fiber whose refractive index is a function of the radius $|\mathbf{q}|$ to the optical axis: $n(\mathbf{q}) = \sqrt{n_o^2 - \nu|\mathbf{q}|^2}$. We call

this the **elliptic index-profile fiber**. The evolution Hamiltonian is $h = -\sqrt{n_o^2 - (|\mathbf{p}|^2 + \nu|\mathbf{q}|^2)}$. For every z 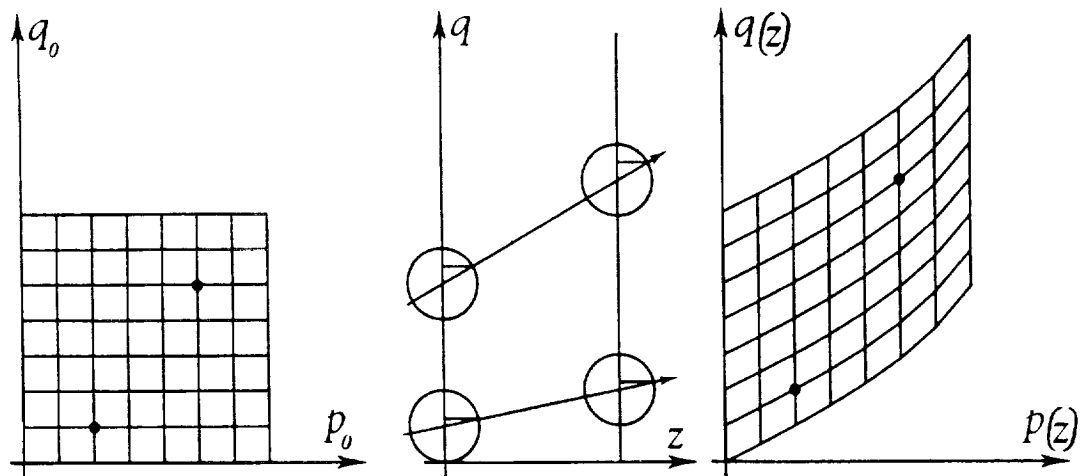phase space is the patch $(|\mathbf{p}|^2 + \nu|\mathbf{q}|^2) \leq n_o^2$. Figure 3 shows the resulting evolution of phase space, compared with that of the mechanical harmonic oscillator.

In a homogeneous optical medium, phase space is a strip $|\mathbf{p}| \leq n$. The Fourier transform is a fundamental ingredient for much of coherent-state (paraxial) optics; yet, the Fourier transformation $(\mathbf{p} \mapsto -\mathbf{q}, \ \mathbf{q} \mapsto \mathbf{p})$ is not an invariance transformation of this space. Nevertheless, one may define a truly optical **Fourier** transform, that in one dimension has the form

$$p^F = -\frac{(1-p^2)^{3/2}}{\sqrt{1+(1-p^2)^3 q^2}}\,q, \qquad q^F = \frac{[1+(1-p^2)^3 q^2]^{3/2}}{\sqrt{1-p^2}}\,p, \qquad (6)$$

that respects the strip of phase space, paraxially rotates by $\frac{1}{2}\pi$ (as the usual Fourier transform), and is nonlinear symplectic. In Figure 4 we show the optical Fourier transform for a quadrant in the phase space strip.

3 Linear Transformations of Phase Space

Although mechanics and optics phase spaces differ in their global properties, their paraxial correspondence motivates that we Taylor-expand all expressions into series where we may truncate the series to some *aberration order* in the powers of the phase space variables. We will thus work with polynomials and so we may ignore the range restriction. In fact, we thus substitute the structure of *Euclidean*-based 4π optics by the *metaxial* Heisenberg–Weyl Lie-algebraic structure suitable for aberration expansions in powers of the phase space coordinates.
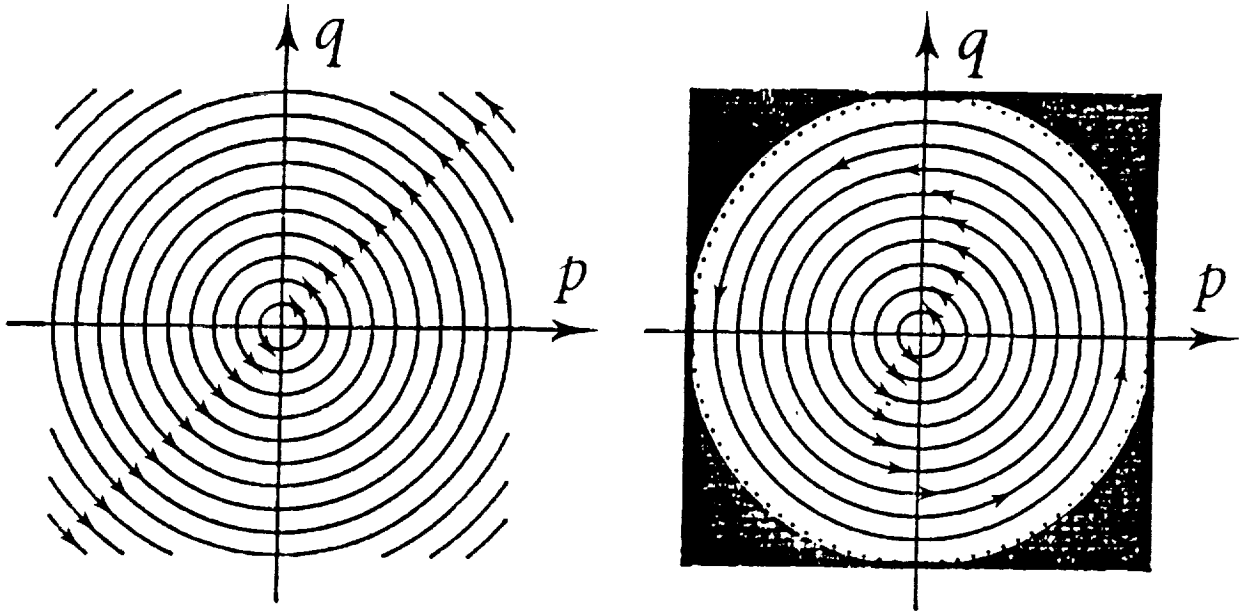
FIG. 3. *Left*: evolution of mechanical phase space under harmonic oscillator time evolution, for one space and one momentum dimension. Phase space rotates rigidly. *Right*: evolution of optical phase space under z-translation. The phase space disk rotates differentially and resembles the harmonic oscillator only for small p and q, *i.e.*, in the *paraxial* optical regime. The circular (in general, elliptic) region is optical phase space: coordinates are meaningless beyond the edge. [Figure by G. KRÖTZSCH, IIMAS–UNAM.]

In effect, we may classify the symplectomorphisms of phase space by the Lie-Poisson generators of the finite transformations.

Translations of phase space are generated by *linear* functions of $(\mathbf{p}, \mathbf{q})$ through exponentiated Lie–Poisson operators:

$$\exp(\mathbf{a} \cdot \{\mathbf{p}, \circ\}) \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} - \mathbf{a} \end{pmatrix}, \qquad \exp(\mathbf{b} \cdot \{\mathbf{q}, \circ\}) \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{p} + \mathbf{b} \\ \mathbf{q} \end{pmatrix}, \qquad (7)$$

as shown in Figure 5.

Linear canonical transformations of phase space are generated by the *quadratic* functions $p_i p_j$, $p_i q_j$, $q_i q_j$, $i = x, y$. These functions close into an $\mathrm{sp}(4, \Re)$ algebra under Poisson brackets. Further, optical systems that have a common axis of rotational (and inversion) symmetry are generated by linear combinations of $p^2 = |\mathbf{p}|^2$, $\mathbf{p} \cdot \mathbf{q}$, and $q^2 = |\mathbf{q}|^2$, that close into an $\mathrm{sp}(2, \Re) = \mathrm{so}(2, 1) = \mathrm{sl}(2, \Re)$ Lie algebra. The corresponding group transformations of phase space are well known and shown in Figure 6. The quadratic polynomial $p^2 + q^2$ generates rigid rotations, as is well known.

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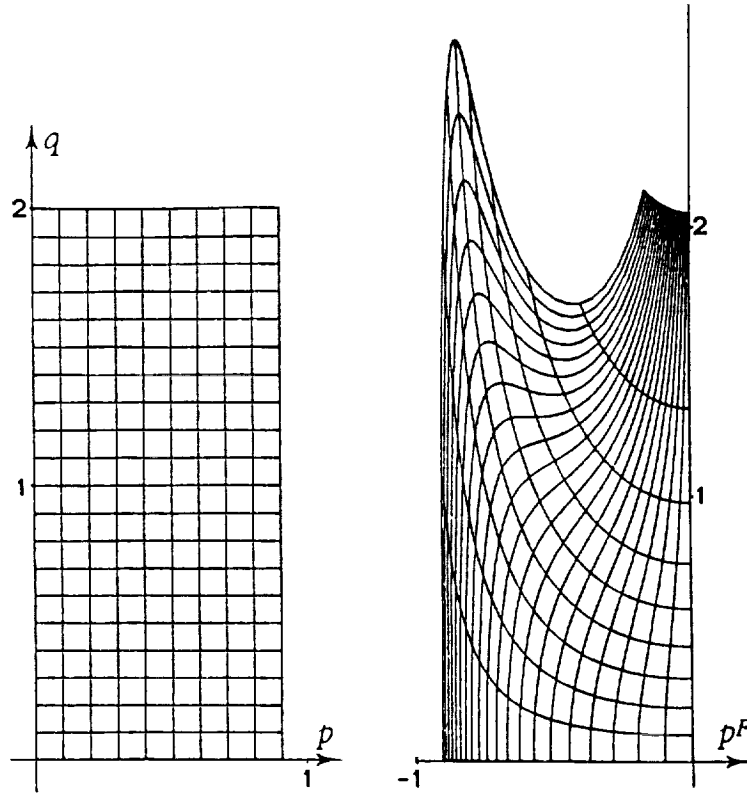
FIG. 4. *Left*: a quadrant in the phase space strip $p < 1$. The usual (mechanical) Fourier transform rotates this patch by $\frac{1}{2}\pi$ in the counterclockwise direction. *Right*: the optical Fourier transform maps points in the strip on points in the same; paraxially it rotates the plane by $\frac{1}{2}\pi$, and is globally symplectic: it conserves area elements. [Figure by G. KRÖTZSCH, IIMAS–UNAM.]

4   General Nonlinear Transformations

Nonlinear symplectic transformations of phase space are generated by polynomials in the components of $(\mathbf{p}, \mathbf{q})$ of degree higher than second. Again, the transformations of axis-symmetric optical systems leads us to concentrate on polynomials of three variables

$$\xi_+ = p^2/\sqrt{2}, \qquad \xi_0 = \mathbf{p} \cdot \mathbf{q}, \qquad \xi_- = q^2/\sqrt{2}. \tag{8a}$$

These variables may be placed into a vector $\vec{\xi}$ of cartesian components

$$\xi_1 = \tfrac{1}{2}(p^2 - q^2), \qquad \xi_2 = \tfrac{i}{2}(p^2 + q^2), \qquad \xi_3 = \mathbf{p} \cdot \mathbf{q}. \tag{8b}$$

They close into an $\mathrm{sp}(2, \Re)$ algebra, as we noted before. The Casimir of this algebra is minus the squared radius of a sphere,

$$(\mathbf{p} \times \mathbf{q})^2 = p^2 q^2 - (\mathbf{p} \cdot \mathbf{q})^2 = -\xi_1^2 - \xi_2^2 - \xi_3^2 = -|\vec{\xi}|^2, \tag{9}$$

and is the well-known Petzval invariant of geometric optics.

In representing the action of optical elements by means of Lie-Poisson transformations generated by polynomials, we are aided by the following theorem:

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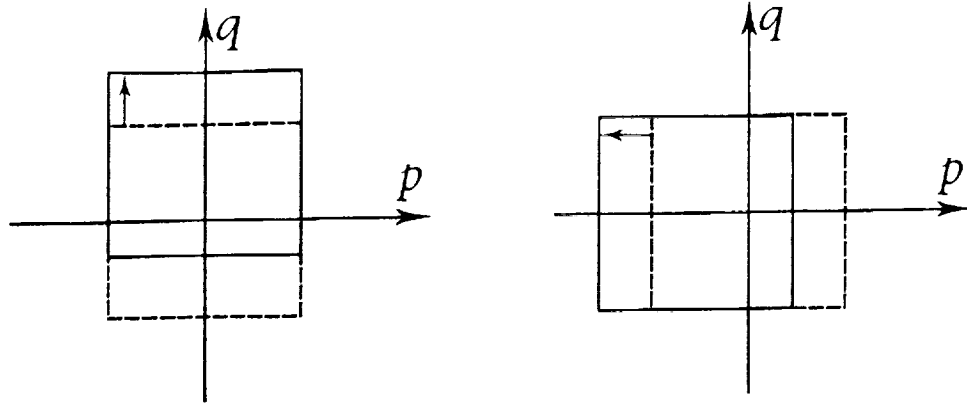
FIG. 5. Translations of phase space are generated by linear functions of the phase space observables. *Left*: $\exp\{p, \circ\}$ translates in q. *Right*: $\exp\{p, \circ\}$ translates in p. The latter does not leave the optical phase space invariant, but serves as a first-order approximation to translation and rotation of the screen. [Figure by G. KRÖTZSCH, IIMAS–UNAM.]
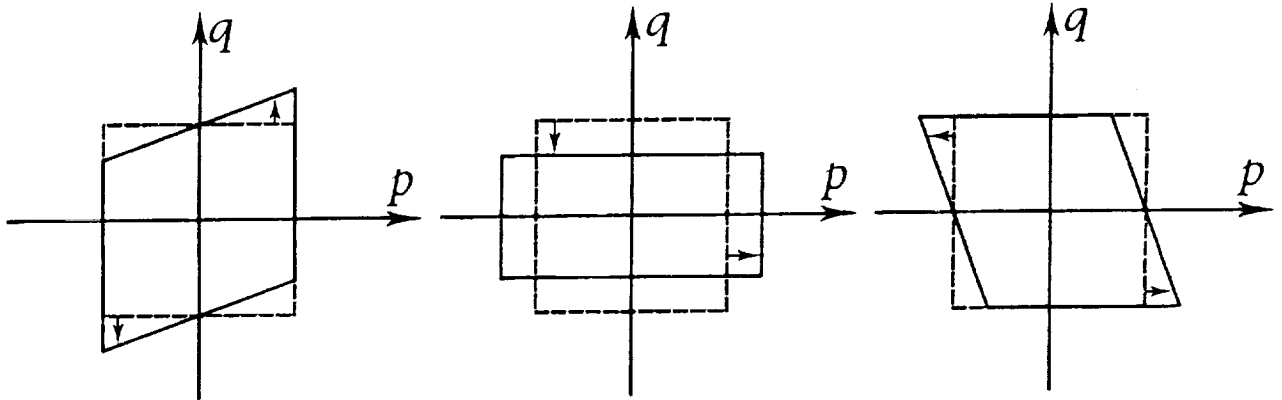
FIG. 6. Linear transformations of phase space are generated by quadratic polynomials in the phase space variables. In one position and one momentum coordinate, the transformations shown above correspond to $\exp\{p^2, \circ\}$, $\exp\{pq, \circ\}$, $\exp\{q^2, \circ\}$, and respectively. [Figure by G. KRÖTZSCH, IIMAS–UNAM.]

Theorem (Dragt & Finn) [6]: Canonical transformations $\mathcal{M}$ leaving the origin invariant, (*i.e.*, *excluding* translations) can be approximated by a truncated product series of Lie transformations

$$\mathcal{M} = \cdots \exp\{f_5, \circ\} \ \exp\{f_4, \circ\} \ \exp\{f_3, \circ\} \ \exp\{f_2, \circ\}, \tag{10}$$

where $f_k(\mathbf{p}, \mathbf{q})$ are homogeneous polynomials of degree k in the components of $\mathbf{p}$ and $\mathbf{q}$.

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Correspondingly, for axis-symmetric systems, the approximation is written as

$$\mathcal{A} = \cdots \exp\{a_4, \circ\} \exp\{a_3, \circ\} \exp\{a_2, \circ\} \exp\{a_1, \circ\}, \tag{11}$$

with $a_k(\vec{\xi})$ polynomials of degree k in the components of ξ as defined in Eqs. (8); k is called the *rank* of the phase space generating monomials.

The rightmost factor in (10) and (11) is the subgroup of linear transformations, and factors to the left of it are nonlinear, called generically *aberrations* in optics. Since the degrees of polynomials in Poisson brackets satisfy $\{f_j, f_k\} = f_{j+k-2}$, the *ranks* of the sp$(2, \Re)$–based polynomials satisfy

$$\{a_j, a_k\} = a_{j+k-1}. \tag{12}$$

it follows that for each rank k, the polynomial set a_k is a (reducible) *ideal* under linear (paraxial) transformations. This is the aberration ideal of *order* $A = 2k - 1$. Axis-symmetric optical systems are described by a paraxial approximation modified by aberrations of orders 3, 5, 7, ..., generated by a_2, a_3, a_4, ..., respectively.

There are legitimate questions about the convergence of the product series and global properties of the *group* of nonlinear symplectic transformations. In any case, by taking functions of phase space *modulo* polynomials of degree higher than the aberration order, we may construct a well-defined finite-parameter group of transformations of phase space truncated to that order. Its best parametrization and, especially, the **product law** must be discovered —once and for all. Thus we construct the *A-th order aberration groups*. The theory developed in optical aberrations serves as well in higher approximation theory.

5 The Monomial Classification of Aberrations

It should be quite evident by now that we may classify aberrations of axis-symmetric systems through proposing complete bases of polynomial functions $a_k(\vec{\xi})$ of degree $2k$ in phase space. (This may also be used to classify non-axis-symmetric aberrations as broken symmetry [7]. We have a sp$(2, \Re)$ Lie algebraic graded covering structure with Poisson (Berezin) brackets between polynomial functions of three variables.

The **monomial basis** is

$$M_{k_+, k_0, k_-} = \text{const} \times \xi_+^{k_+} \xi_0^{k_0} \xi_-^{k_-}, \qquad k_i = 0, 1, \ldots . \tag{13}$$

These monomials have rank $k = k_+ + k_0 + k_-$, and are classified as the harmonic oscillator Cartesian basis states. We may examine the action generated by exponentials of these monomials up to the first Taylor term,

$$\exp\{M_{k_+, k_0, k_-}, \circ\} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{p} + k_0 M_{k_+, k_0-1, k_-} \mathbf{p} + 2k_- M_{k_+, k_0, k_- -1} \mathbf{q} + \cdots \\ \mathbf{q} - 2k_+ M_{k_+ -1, k_0, k_-} \mathbf{p} - k_0 M_{k_+, k_0-1, k_-} \mathbf{q} + \cdots \end{pmatrix}. \tag{14}$$

In this Dragt [8] recognized the traditional third-order Seidel aberrations, generated by and called

$$(p^2)^2 \quad \text{SPHERICAL ABERRATION,}$$
$$p^2 \, \mathbf{p} \cdot \mathbf{q} \quad \text{CIRCULAR COMA,}$$

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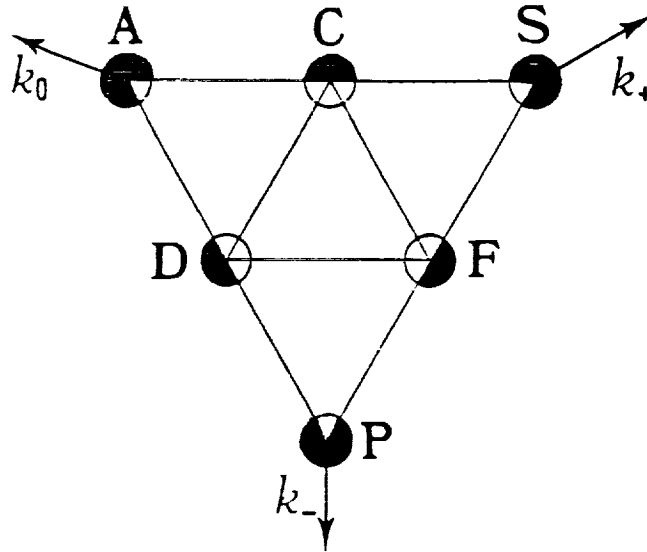
FIG. 7. The third order aberration sextuplet classified in ξ_+, ξ_0, and ξ_- coordinates, and the traditional names. S: spherical aberration, C: coma, A: astigmatism, F: curvature of field, D: distorsion, and P: pocus.

$$
\begin{array}{ll}
(\mathbf{p} \cdot \mathbf{q})^2 & \text{ASTIGMATISM,} \\
p^2 q^2 & \text{CURVATURE OF FIELD,} \\
\mathbf{p} \cdot \mathbf{q} \, q^2 & \text{DISTORSION,} \\
(q^2)^2 & \text{POCUS.}
\end{array}
$$

Actually, *pocus* was excluded from Dragt's list because it does not produce any change in the screen images; rather it only changes the directions of ray arrival; it has a p-unfocusing action that we have playfully called pocus [9]. Yet it is an aberration of phase space on par with the other five, and the Fourier transform of spherical aberration. In Figure 7 we show the familiar harmonic oscillator states with two quanta and its corresponding aberrations of rank two in the monomial basis.

6 The Symplectic Classification of Aberrations

The harmonic oscillator eigenfunction structure naturally suggests another basis, following the number of quanta, total angular momentum, and one projection.

The **symplectic harmonic basis** is [1], [9], [10], [11]

$$
{}^k \chi_m^j(\vec{\xi}) = \text{const} \times [(\vec{\xi})^2]^{(k-j)/2} \, \mathcal{Y}_{j,m}(\vec{\xi}), \tag{15}
$$

where $(\vec{\xi})^2$ is the Petzval and $\mathcal{Y}_{j,m}(\vec{\xi})$ the solid spherical harmonic of angular momentum j and projection m. The total degree of ${}^k \chi_m^j(\vec{\xi})$ in ξ_i is k —the rank. The index j we may call the *symplectic spin*, and m the Seidel index [12]. The third-order aberration sextuplet thus reduces, under linear transformations, into a spin-2 quintuplet and a spin-0 singlet (that is in effect the Petzval itself). Only the $m = 0$ member of the quintuplet and the singlet 'mix' the

RANK:	$k=2$	$k=3$	$k=4$
SPIN:	$j=2,\quad 0$	$j=3,\quad 1$	$j=4,\quad 2,\quad 0$
$m=4$			⊙
3		⊙	O▷
2	⊙	O▷	⋈　X
1	O▷	⋈　X	⊂⊃　▷
0	⊂⊃　│	⊂⊃　▷	⋈　X　│
−1	↵	⊂⊃　│	⊂⊃　▷
−2	•	↵	⊂⊃　│
−3		•	↵
−4			•
ORDER:	THIRD	FIFTH	SEVENTH

FIG. 8. The aberration multiplets to seventh order classified into symplectic harmonics by rank, spin and Seidel index m. The symbols are ⊙: spherical aberration, O▷: circular coma, ⊂⊃, ▷: elliptic comas, ⊂⊃, ⋈: curvatisms, │, ⋈: astigmatures, ↵: distorsion, and •: pocus [12].

curvature of field and astigmatism monomials into aberrations that we could call curvatism (in the quintuplet) and astigmature (the singlet, invariant under paraxial transformations). The scheme appears in Figure 8; this is but the harmonic oscillator ℓ-m spectrum from the $k = 2$ level (s-d shell) up, seen sidewise.

The constant in front of the definition of ${}^k\chi_m^j$ in Eq. (15) has been chosen to avoid square roots, because they are time-consuming for symbolic manipulation. Thus we have defined them starting from the normalization of the highest weight ${}^k\chi_k^k = (p^2)^k$, and their lowering through $\{\frac{1}{2}q^2, \circ\}\,{}^k\chi_m^j = (m+j)\,{}^k\chi_{m-1}^j$. They are

$$
\begin{aligned}
{}^k\chi_m^j(\vec{\xi}) &= (\xi^2)^{(k-j)/2}\sqrt{\frac{4\pi(2j+1)(j+m)!(j-m)!}{(2j-1)!!}}\,y_m^j(\xi)\\[6pt]
&= (\xi^2)^{(k-j)/2}\frac{(j+m)!(j-m)!}{2^{m/2}(2j-1)!!}\sum_n \frac{1}{2^n}\frac{\xi_+^{m+n}}{(m+n)!}\frac{\xi_0^{j-m-2n}}{(j-m-2n)!}\frac{\xi_-^n}{n!}.
\end{aligned}
\tag{16}
$$

We give the symplectic harmonics only for $m \geq 0$, because

$$
{}^k\chi_{-m}^j(\xi_+,\xi_0,\xi_-) = {}^k\chi_m^j(\xi_-,\xi_0,\xi_+).
\tag{17}
$$

The first two ranks ($k = 0,1$) correspond to the scalar and the $sp(2,\Re)$ generator functions:

$$
{}^0\chi_0^0 = 1;\qquad {}^1\chi_{\pm1}^1 = \sqrt{2}\xi_\pm,\qquad {}^1\chi_0^1 = \xi_0.
\tag{18a}
$$

The $k = 2$ basis functions are the generators of **third-order aberrations**:

$$^2\chi_2^2 = 2\xi_+^2,$$
$$^2\chi_1^2 = \sqrt{2}\xi_+\xi_0,$$
$$^2\chi_0^2 = \tfrac{2}{3}(\xi_+\xi_- + \xi_0^2), \qquad ^2\chi_0^0 = 2\xi_+\xi_- - \xi_0^2. \tag{18b}$$

For $k = 3$ we have the generators of **fifth-order aberrations**:

$$^3\chi_3^3 = 2\sqrt{2}\xi_+^3,$$
$$^3\chi_2^3 = 2\xi_+^2\xi_0,$$
$$^3\chi_1^3 = \tfrac{2}{5}\sqrt{2}(\xi_+^2\xi_- + 2\xi_+\xi_0^2), \qquad ^3\chi_1^1 = {}^2\chi_0^0\,{}^1\chi_1^1,$$
$$^3\chi_0^3 = \tfrac{2}{5}(3\xi_+\xi_0\xi_- + \xi_0^3), \qquad ^3\chi_0^1 = {}^2\chi_0^0\,{}^1\chi_0^1. \tag{18c}$$

For $k = 4$ we have those of *seventh*-order aberrations:

$$^4\chi_4^4 = 4\xi_+^4,$$
$$^4\chi_3^4 = 2\sqrt{2}\xi_+^3\xi_0,$$
$$^4\chi_2^4 = \tfrac{4}{7}(\xi_+^3\xi_- + 3\xi_+^2\xi_0^2), \qquad ^4\chi_2^2 = {}^2\chi_0^0\,{}^2\chi_2^2, \tag{18d}$$
$$^4\chi_1^4 = \tfrac{2}{7}\sqrt{2}(3\xi_+^2\xi_0\xi_- + 2\xi_+\xi_0^3), \qquad ^4\chi_1^2 = {}^2\chi_0^0\,{}^2\chi_1^2,$$
$$^4\chi_0^4 = \tfrac{4}{35}(3\xi_+^2\xi_-^2 + 12\xi_+\xi_0^2\xi_- + 2\xi_0^4), \quad ^4\chi_0^2 = {}^2\chi_0^0\,{}^2\chi_0^2, \quad ^4\chi_0^0 = ({}^2\chi_0^0)^2.$$

7 Spot Diagrams of Harmonic Oscillator States

The **spot diagram** of a transformation $\mathcal{M} : (\mathbf{p}, \mathbf{q}) \mapsto \big(\mathbf{p}'(\mathbf{p},\mathbf{q}), \mathbf{q}'(\mathbf{p},\mathbf{q})\big)$ is a projection of $\Re^4$ phase space on the screen plane, that pictures $\mathbf{q}'(\mathbf{p},\mathbf{q})$, the image of a pencil of rays (range of $\mathbf{p}$) diverging from a *fixed* object point $\mathbf{q}$. If we let $\mathbf{p}$ mark a polar coordinate grid around the optical axis, we obtain the spots of Figures 9.1–9.5 [12]. These are the new "faces" of the harmonic oscillator states that we present in aberration optics.

8 Characterization of Optical Elements

Optical elements may be characterized by

$$\mathcal{G}\{A_4, A_3, A_2, \mathbf{M}\} = \exp\{A_4, \circ\}\ \exp\{A_3, \circ\}\ \exp\{A_2, \circ\}\ \exp\{A_1, \circ\}, \tag{19a}$$

with the coefficients their linear action $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det \mathbf{M} = 1$, and the coefficients $A_{k,j,m}$ of its aberration polynomials

$$A_k = \sum_{j=k,k-2,\dots}^{1\ \text{or}\ 0} \sum_{m=j,j-1,\dots}^{-j} A_{k,j,m}\ {}^k\chi_m^j(\vec{\xi}). \tag{19b}$$

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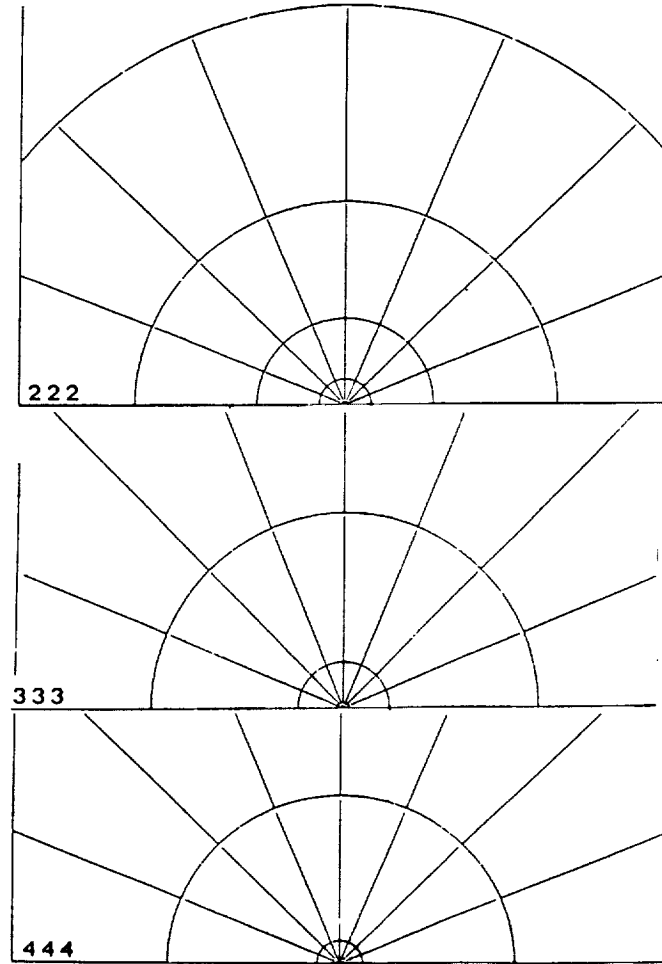
FIG. 9.1. Spherical aberrations $\odot$ generated by ${}^{k}\chi_{k}^{k}$ (at left we indicate the kjm classification of the spot diagram).

Some of the most common optical elements thus represented are the following:

Free propagation: The paraxial part is $\mathbf{F}(z) = \begin{pmatrix} 1 & 0 \\ -z/n & 1 \end{pmatrix}$. The (spherical aberration) coefficients are:

$$f_{2,2,2}^{z} = -z/(8n^{3}), \quad f_{3,3,3}^{z} = -z/(16n^{5}), \quad f_{4,4,4}^{z} = -5z/(128n^{7}).$$

Elliptic-profile fiber free propagation in medium $n = \sqrt{\nu^{2} - \mu^{2}q^{2}}$ is treated in [13]. The
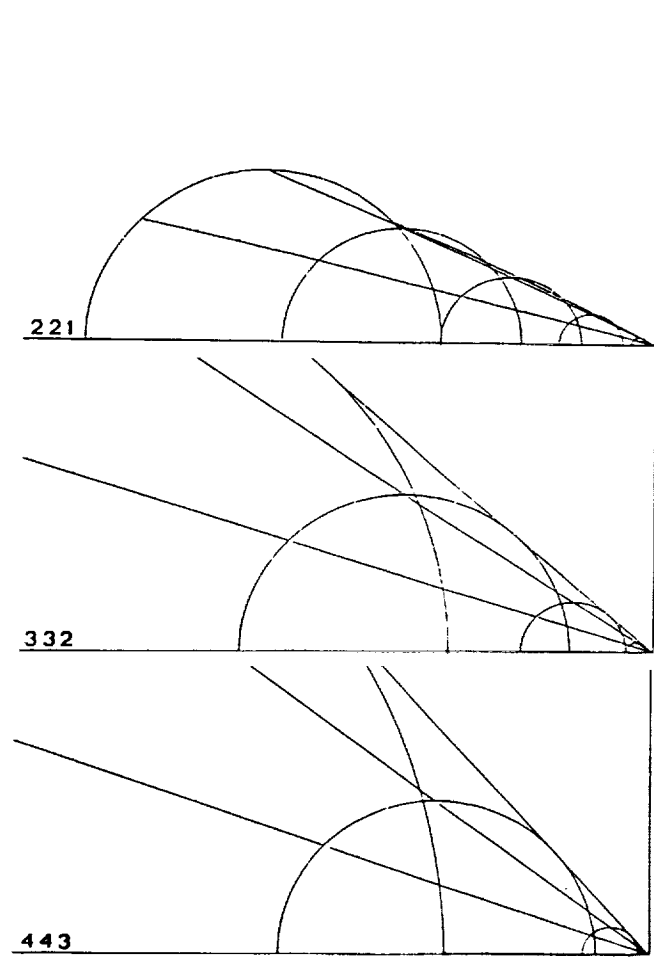
FIG. 9.2. Circular comas ⟳ generated by $^k\chi_{k-1}^k$.

paraxial part is $\mathbf{F}_{n(q)}(z) = \begin{pmatrix} \cos\frac{\mu z}{\nu} & \mu\sin\frac{\mu z}{\nu} \\ -\frac{1}{\mu}\sin\frac{\mu z}{\nu} & \cos\frac{\mu z}{\nu} \end{pmatrix}$. The aberration polynomials are:

$$F_2 = -z(|\mathbf{p}|^2 + \mu^2|\mathbf{q}|^2)^2/8\nu^3,$$
$$F_3 = -z(|\mathbf{p}|^2 + \mu^2|\mathbf{q}|^2)^3/16\nu^5,$$
$$F_4 = -5z(|\mathbf{p}|^2 + \mu^2|\mathbf{q}|^2)^4/128\nu^7.$$

The root transformation [14] indicated by $R_{n;S}$ (that is the root of refraction —see below) in medium n associated to the surface $S(q^2) = \varsigma_2 q^2 + \varsigma_4 q^4 + \varsigma_6 q^6 + \cdots$. The paraxial part is $\mathbf{R}_{n,S} = \begin{pmatrix} 1 & -2n\varsigma_2 \\ 0 & 1 \end{pmatrix}$. The aberration coefficients $R_{k,j,m}$, arranged as row multi-vectors with components numbered by descending values of j and m, are

$$^2\mathbf{r} = \{\{0, 0, -\varsigma_2/(2n), 0, n\varsigma_4\}, \{-\varsigma_2/(3n)\}\},$$
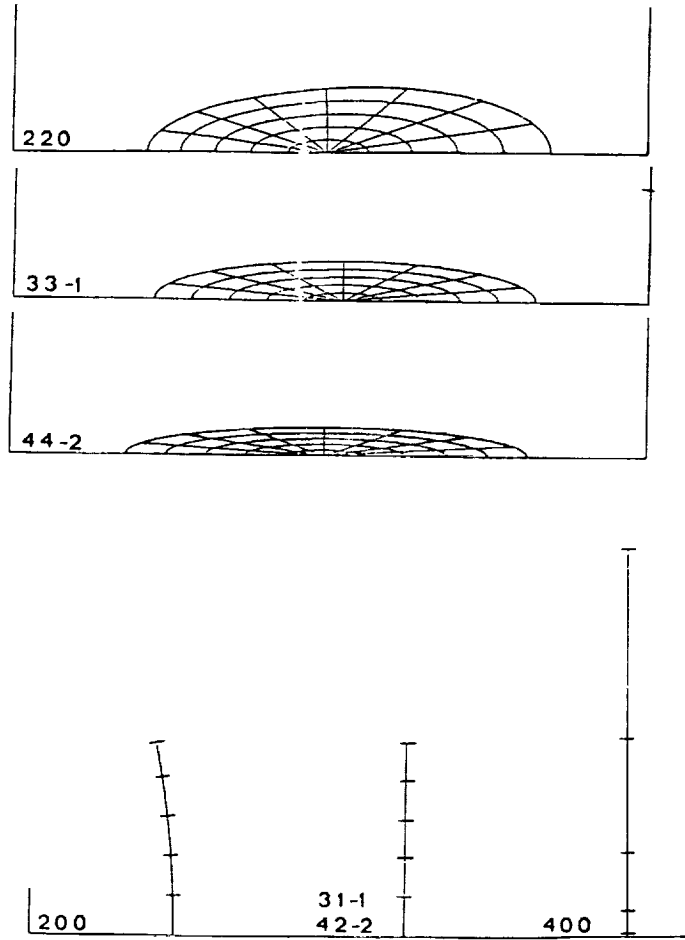
FIG. 9.3. The curvatism-astigmature m-degenerate pairs $\bigcirc\!\!-\!|$, generated by $^k\chi^k_{k-2}$ and $^k\chi^{k-2}_{k-2}$.

$$^3\mathbf{r} = \{\{0,0,-\varsigma_2/(8n^3),-\varsigma_2^2/(2n^2),-\varsigma_4/(2n),2\varsigma_2\varsigma_4,n\varsigma_6\},\{-\varsigma_2/(10n^3),-\varsigma_2^2/(5n^2),-2/5\varsigma_4/n\}\},$$

$$^4\mathbf{r} = \{\{0,0,-\varsigma_2/(16n^5),-\varsigma_2^2/(4n^4),-\varsigma_4/(8n^3)-5/6\varsigma_2^3/n^3,-2\varsigma_2\varsigma_4/n^2,-\varsigma_6/(2n)+$$
$$16/3\varsigma_2^2\varsigma_4/n,6\varsigma_2\varsigma_6,8/3n\varsigma_2\varsigma_4^2+n\varsigma_8\},\{-3/56\varsigma_2/n^5,-\varsigma_2^2/(7n^4),-\varsigma_4/(7n^3)-$$
$$2/7\varsigma_2^3/n^3,-8/7\varsigma_2\varsigma_4/n^2,-3/7\varsigma_6/n-16/21\varsigma_2^2\varsigma_4/n\},\{-\varsigma_4/(15n^3)\}\}.$$

The refracting surface S between medium n and medium m is given by the factorization theorem [14], [15]

$$S_{n,m;S} = \mathcal{R}_{n,S}\,\mathcal{R}_{m,S}^{-1}. \tag{20}$$

The linear part is $\mathbf{S} = \begin{pmatrix} 1 & 2m\varsigma_2 - 2n\varsigma_2 \\ 0 & 1 \end{pmatrix}$ and the aberrations are:
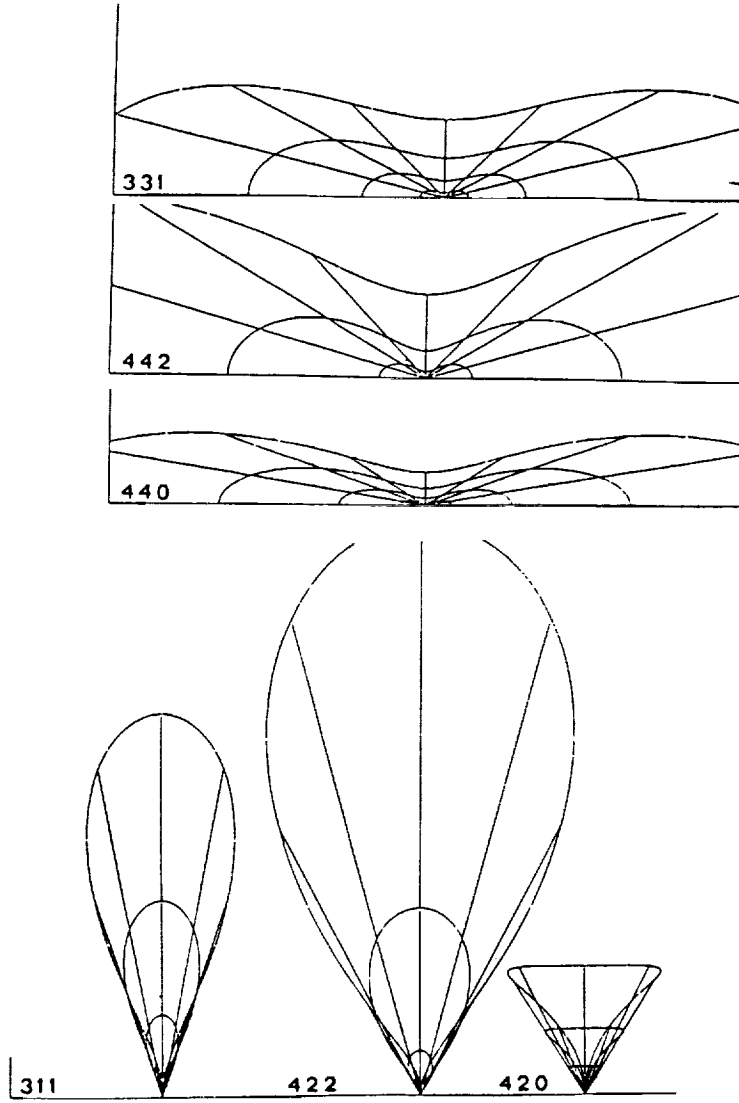
FIG. 9.4. The m-degenerate pairs $\bowtie$—X.

$$^2s = \{\{0,0,\varsigma_2/(2m) - \varsigma_2/(2n), 2n\varsigma_2^2/m - 2\varsigma_2^2, -m\varsigma_4 + 2m\varsigma_2^3 + n\varsigma_4 - 4n\varsigma_2^3 + 2n^2\varsigma_2^3/m\}, \{\varsigma_2/(3m) - \varsigma_2/(3n)\}\},$$

$$^3s = \{\{0,0,\varsigma_2/(8m^3) - \varsigma_2/(8n^3), n\varsigma_2^2/m^3 - \varsigma_2^2/(2m^2) - \varsigma_2^2/(2n^2), 3n^2\varsigma_2^3/m^3 - 3n\varsigma_2^3/m^2 + \varsigma_4/(2m) +$$
$$2\varsigma_2^3/m - \varsigma_4/(2n) - 2\varsigma_2^3/n, -2m\varsigma_2\varsigma_4/n + 4m\varsigma_2^4/n - 2\varsigma_2\varsigma_4 + 4n^3\varsigma_2^4/m^3 - 6n^2\varsigma_2^4/m^2 + 4n\varsigma_2\varsigma_4/m +$$
$$4n\varsigma_2^4/m - 6\varsigma_2^4, -m\varsigma_6 + 6m\varsigma_2^2\varsigma_4 - 2m\varsigma_2^5 + n\varsigma_6 - 12n\varsigma_2^2\varsigma_4 + 4n\varsigma_2^5 + 2n^4\varsigma_2^5/m^3 - 4n^3\varsigma_2^5/m^2 +$$
$$6n^2\varsigma_2^2\varsigma_4/m\}, \{\varsigma_2/(10m^3) - \varsigma_2/(10n^3), 2/5n\varsigma_2^2/m^3 - \varsigma_2^2/(5m^2) - \varsigma_2^2/(5n^2), 2/5n^2\varsigma_2^3/m^3 -$$
$$2/5n\varsigma_2^3/m^2 + 2/5\varsigma_4/m - 2/5\varsigma_2^3/m - 2/5\varsigma_4/n + 2/5\varsigma_2^3/n\}\},$$

$$^4s = \{\{0,0,\varsigma_2/(16m^5) - \varsigma_2/(16n^5), 3/4n\varsigma_2^2/m^5 - \varsigma_2^2/(4m^4) - \varsigma_2^2/(4m^3n) - \varsigma_2^2/(4n^4), 15/4n^2\varsigma_2^3/m^5 -$$
$$5/2n\varsigma_2^3/m^4 + \varsigma_4/(8m^3) - 5/12\varsigma_2^3/m^3 - \varsigma_4/(8n^3) - 5/6\varsigma_2^3/n^3, 10n^3\varsigma_2^4/m^5 - 10n^2\varsigma_2^4/m^4 + 3n\varsigma_2\varsigma_4/m^3 +$$
$$8/3n\varsigma_2^4/m^3 - 2\varsigma_2\varsigma_4/m^2 - 4\varsigma_2^4/m^2 + \varsigma_2\varsigma_4/(mn) + 4\varsigma_2^4/(mn) - 2\varsigma_2\varsigma_4/n^2 - 8/3\varsigma_2^4/n^2, -16/3m\varsigma_2^2\varsigma_4/n^2 +$$
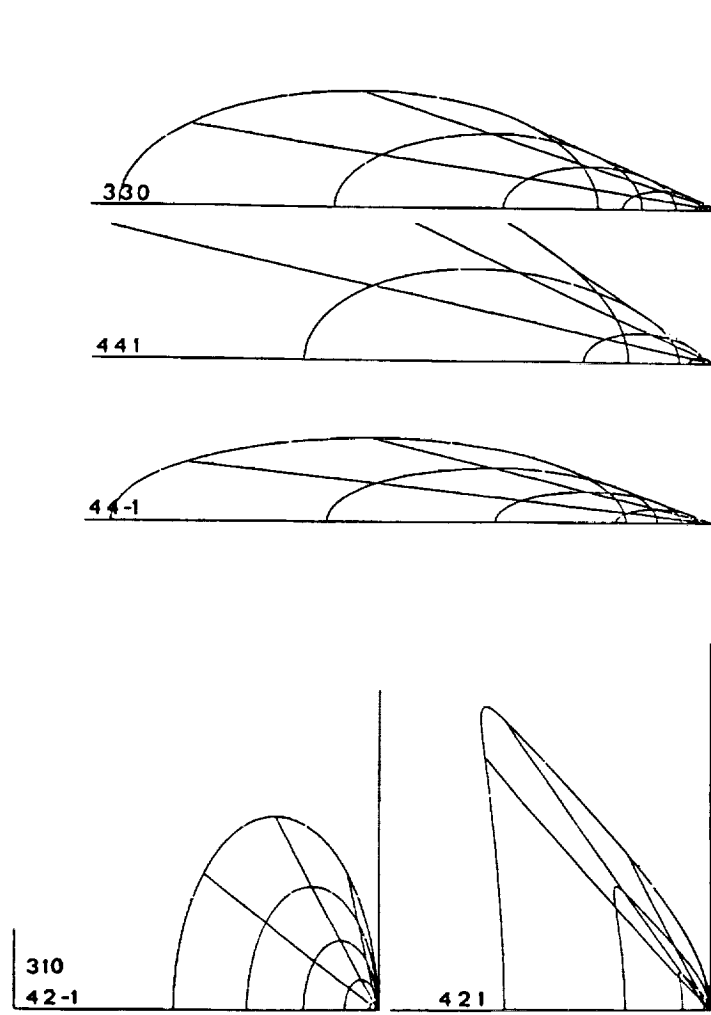
FIG. 9.5.   Elliptical comas and their m-degenerate partners $\bigcirc\!\!-\!\!\triangleright$.

$32/3m\varsigma_2^5/n^2 + 15n^4\varsigma_2^5/m^5 - 20n^3\varsigma_2^5/m^4 + 15n^2\varsigma_2^2\varsigma_4/m^3 + 6n^2\varsigma_2^5/m^3 - 62/3n\varsigma_2^2\varsigma_4/m^2 - 20/3n\varsigma_2^5/m^2 + \varsigma_6/(2m) + 19\varsigma_2^2\varsigma_4/m + 11\varsigma_2^5/m - \varsigma_6/(2n) - 8\varsigma_2^2\varsigma_4/n - 16\varsigma_2^5/n, -6m\varsigma_2\varsigma_6/n + 116/3m\varsigma_2^3\varsigma_4/n - 52/3m\varsigma_2^6/n + 4\varsigma_2\varsigma_6 + 12n^5\varsigma_2^6/m^5 - 20n^4\varsigma_2^6/m^4 + 28n^3\varsigma_2^3\varsigma_4/m^3 - 4/3n^3\varsigma_2^6/m^3 - 176/3n^2\varsigma_2^3\varsigma_4/m^2 + 64/3n^2\varsigma_2^6/m^2 + 2n\varsigma_2\varsigma_6/m + 72n\varsigma_2^3\varsigma_4/m - 36n\varsigma_2^6/m + 4n\varsigma_4^2/m - 80\varsigma_2^3\varsigma_4 + 124/3\varsigma_2^6 - 4\varsigma_4^2, 8m\varsigma_2\varsigma_4^2 - m\varsigma_8 + 2m\varsigma_2^2\varsigma_6 + 86/3m\varsigma_2^4\varsigma_4 - 124/3m\varsigma_2^7 - 24n\varsigma_2\varsigma_4^2 + n\varsigma_8 - 4n\varsigma_2^2\varsigma_6 - 40n\varsigma_2^4\varsigma_4 + 200/3n\varsigma_2^7 + 4n^6\varsigma_2^7/m^5 - 8n^5\varsigma_2^7/m^4 + 18n^4\varsigma_2^4\varsigma_4/m^3 - 20/3n^4\varsigma_2^7/m^3 - 152/3n^3\varsigma_2^4\varsigma_4/m^2 + 112/3n^3\varsigma_2^7/m^2 + 40/3n^2\varsigma_2\varsigma_4^2/m + 2n^2\varsigma_2^2\varsigma_6/m + 164/3n^2\varsigma_2^4\varsigma_4/m - 188/3n^2\varsigma_2^7/m + 8/3m^2\varsigma_2\varsigma_4^2/n - 32/3m^2\varsigma_2^4\varsigma_4/n + 32/3m^2\varsigma_2^7/n\}, \{3/56\varsigma_2/m^5 - 3/56\varsigma_2/n^5, 3/7n\varsigma_2^2/m^5 - \varsigma_2^2/(7m^4) - \varsigma_2^2/(7m^3n) - \varsigma_2^2/(7n^4), 9/7n^2\varsigma_2^3/m^5 - 6/7n\varsigma_2^3/m^4 + \varsigma_4/(7m^3) - \varsigma_2^3/(7m^3) - \varsigma_4/(7n^3) - 2/7\varsigma_2^3/n^3, 12/7n^3\varsigma_2^4/m^5 -$

$12/7n^2\varsigma_2^4/m^4 + 12/7n\varsigma_2\varsigma_4/m^3 - 8/7n\varsigma_2^4/m^3 - 8/7\varsigma_2\varsigma_4/m^2 + 12/7\varsigma_2^4/m^2 + 4/7\varsigma_2\varsigma_4/(mn) -$
$12/7\varsigma_2^4/(mn) - 8/7\varsigma_2\varsigma_4/n^2 + 8/7\varsigma_2^4/n^2, 16/21m\varsigma_2^2\varsigma_4/n^2 - 32/21m\varsigma_2^5/n^2 + 6/7n^4\varsigma_2^5/m^5 -$
$8/7n^3\varsigma_2^5/m^4 + 20/7n^2\varsigma_2^2\varsigma_4/m^3 - 20/7n^2\varsigma_2^5/m^3 - 64/21n\varsigma_2^2\varsigma_4/m^2 + 104/21n\varsigma_2^5/m^2 + 3/7\varsigma_6/m -$
$12/7\varsigma_2^2\varsigma_4/m - 18/7\varsigma_2^5/m - 3/7\varsigma_6/n + 8/7\varsigma_2^2\varsigma_4/n + 16/7\varsigma_2^5/n\}, \{\varsigma_4/(15m^3) - \varsigma_4/(15n^3)\}\}.$

The curved mirror transformation by S can be found from $\mathcal{M}_{n,S} = \mathcal{R}_{n,S}\mathcal{R}_{-n,S}^{-1}$ [16]. The refracting surface between two elliptic-profile fibers has been calculated in Ref. [13].

9 Multiplication Law in the Aberration Group

For aberration orders 1, 3, 5, and 7 (ranks $k = 1, 2, 3,$ and 4), the dimensionality of the basis is, respectively: the 3 generators of $sp(2, \Re)$, the 6 third-order aberrations (separated into a quintuplet and a singlet), the 10 fifth-order aberrations (a septuplet and a triplet), and the 15 seventh-order aberrations (divided into a nonuplet, quintuplet and singlet). The number of parameters of the corresponding aberration group elements in Eqs. (11) and (19) thus accumulates to:

$$
\begin{array}{ll}
3 & \text{SP}(2,\Re) \text{ (ABERRATION ORDER 1)} \\
9 & \text{LINEAR} + \text{ABERRATION ORDER 3} \\
19 & \text{UP TO ABERRATION ORDER 5} \\
34 & \text{UP TO ABERRATION ORDER 7}
\end{array}
$$

If we indicate the seventh-order *pure aberration* group elements by the coefficients of the polynomials in

$$\mathcal{G}\{\mathbf{A_4, A_3, A_2, 1}\} = \exp\{A_4, \circ\}\ \exp\{A_3, \circ\}\ \exp\{A_2, \circ\}, \tag{21}$$

the central problem is to find the **multiplication table** involving the 31 up-to-seventh order aberration coefficients in the product

$$\mathcal{G}\{\mathbf{C_4, C_3, C_2, 1}\} = \mathcal{G}\{\mathbf{A_4, A_3, A_2, 1}\}\mathcal{G}\{\mathbf{B_4, B_3, B_2, 1}\}, \quad i.e. \quad \mathbf{C = A\ \# \ B}. \tag{22}$$

To find explicitly the *gato* operation $\#$ between the individual coefficients, we may use Baker-Campbell-Hausdorff relations. Order 3 is abelian; order 5 is the practical limit for hand calculations, and order 7 is definitely nontrivial and needs symbolic computation [12], [16]. The pure aberration group composition law was calculated once and for all, to find the composition of aberrations. It is:

Aberration order 3:

$$C_{2,j,m} = A_{2,j,m} + B_{2,j,m}, \qquad j = 2, 0, \quad m = j, j-1, \ldots, -j.$$

Aberration order 5:

$$C_{3,3,3} = 2A_{2,2,1}B_{2,2,2} - 2A_{2,2,2}B_{2,2,1} + A_{3,3,3} + B_{3,3,3},$$
$$C_{3,3,2} = 4A_{2,2,0}B_{2,2,2} - 4A_{2,2,2}B_{2,2,0} + A_{3,3,2} + B_{3,3,2},$$

$C_{3,3,1} = 6A_{2,2,-1}B_{2,2,2} + 2A_{2,2,0}B_{2,2,1} - 2A_{2,2,1}B_{2,2,0} - 6A_{2,2,2}B_{2,2,-1} + A_{3,3,1} + B_{3,3,1},$

$C_{3,3,0} = 8A_{2,2,-2}B_{2,2,2} + 4A_{2,2,-1}B_{2,2,1} - 4A_{2,2,1}B_{2,2,-1} - 8A_{2,2,2}B_{2,2,-2} + A_{3,3,0} + B_{3,3,0},$

$C_{3,3,-1} = 6A_{2,2,-2}B_{2,2,1} + 2A_{2,2,-1}B_{2,2,0} - 2A_{2,2,0}B_{2,2,-1} - 6A_{2,2,1}B_{2,2,-2} + A_{3,3,-1} + B_{3,3,-1},$

$C_{3,3,-2} = 4A_{2,2,-2}B_{2,2,0} - 4A_{2,2,0}B_{2,2,-2} + A_{3,3,-2} + B_{3,3,-2},$

$C_{3,3,-3} = 2A_{2,2,-2}B_{2,2,-1} - 2A_{2,2,-1}B_{2,2,-2} + A_{3,3,-3} + B_{3,3,-3};$

$C_{3,1,1} = 4/5A_{2,2,-1}B_{2,2,2} - 2/5A_{2,2,0}B_{2,2,1} + 2/5A_{2,2,1}B_{2,2,0} - 4/5A_{2,2,2}B_{2,2,-1} + A_{3,1,1} + B_{3,1,1},$

$C_{3,1,0} = 16/5A_{2,2,-2}B_{2,2,2} - 2/5A_{2,2,-1}B_{2,2,1} + 2/5A_{2,2,1}B_{2,2,-1} - 16/5A_{2,2,2}B_{2,2,-2} + A_{3,1,0} + B_{3,1,0},$

$C_{3,1,-1} = 4/5A_{2,2,-2}B_{2,2,1} - 2/5A_{2,2,-1}B_{2,2,0} + 2/5A_{2,2,0}B_{2,2,-1} - 4/5A_{2,2,1}B_{2,2,-2} + A_{3,1,-1} + B_{3,1,-1}.$

Aberration order 7:

$C_{4,4,4} = 8A_{2,2,1}^2B_{2,2,2} + 32/3A_{2,2,2}^2B_{2,2,0} - 16/3A_{2,2,0}B_{2,2,2}^2 - 32/3A_{2,2,0}A_{2,2,2}B_{2,2,2} - 8A_{2,2,1}A_{2,2,2}B_{2,2,1} + 4A_{2,2,1}B_{2,2,1}B_{2,2,2} + 6A_{2,2,1}B_{3,3,3} - 4A_{2,2,2}B_{2,2,1}^2 + 16/3A_{2,2,2}B_{2,2,0}B_{2,2,2} - 4A_{2,2,2}B_{3,3,2} + A_{4,4,4} + B_{4,4,4},$

$C_{4,4,3} = 32A_{2,2,2}^2B_{2,2,-1} - 16A_{2,2,-1}B_{2,2,2}^2 - 32A_{2,2,-1}A_{2,2,2}B_{2,2,2} + 64/3A_{2,2,0}A_{2,2,1}B_{2,2,2} - 80/3A_{2,2,0}A_{2,2,2}B_{2,2,1} - 8/3A_{2,2,0}B_{2,2,1}B_{2,2,2} + 12A_{2,2,0}B_{3,3,3} + 16/3A_{2,2,1}A_{2,2,2}B_{2,2,0} + 40/3A_{2,2,1}B_{2,2,0}B_{2,2,2} + 2A_{2,2,1}B_{3,3,2} + 16A_{2,2,2}B_{2,2,-1}B_{2,2,2} - 32/3A_{2,2,2}B_{2,2,0}B_{2,2,1} - 8A_{2,2,2}B_{3,3,1} + A_{4,4,3} + B_{4,4,3},$

$C_{4,4,2} = 64/3A_{2,2,0}^2B_{2,2,2} + 8/3A_{2,2,1}^2B_{2,2,0} + 64A_{2,2,2}^2B_{2,2,-2} - 32A_{2,2,-2}B_{2,2,2}^2 - 64A_{2,2,-2}A_{2,2,2}B_{2,2,2} + 16A_{2,2,-1}A_{2,2,1}B_{2,2,2} - 56A_{2,2,-1}A_{2,2,2}B_{2,2,1} - 20A_{2,2,-1}B_{2,2,1}B_{2,2,2} + 18A_{2,2,-1}B_{3,3,3} - 4/3A_{2,2,0}B_{2,2,1}^2 - 8/3A_{2,2,0}A_{2,2,1}B_{2,2,1} - 64/3A_{2,2,0}A_{2,2,2}B_{2,2,0} + 32/3A_{2,2,0}B_{2,2,0}B_{2,2,2} + 8A_{2,2,0}B_{3,3,2} + 40A_{2,2,1}A_{2,2,2}B_{2,2,-1} + 28A_{2,2,1}B_{2,2,-1}B_{2,2,2} + 4/3A_{2,2,1}B_{2,2,0}B_{2,2,1} - 2A_{2,2,1}B_{3,3,1} - 32/3A_{2,2,2}B_{2,2,0}^2 + 32A_{2,2,2}B_{2,2,-2}B_{2,2,2} - 8A_{2,2,2}B_{2,2,-1}B_{2,2,1} - 12A_{2,2,2}B_{3,3,0} + A_{4,4,2} + B_{4,4,2},$

$C_{4,4,1} = 16/3A_{2,2,0}^2B_{2,2,1} + 16A_{2,2,1}^2B_{2,2,-1} - 96A_{2,2,-2}A_{2,2,2}B_{2,2,1} - 48A_{2,2,-2}B_{2,2,1}B_{2,2,2} + 24A_{2,2,-2}B_{3,3,3} - 8A_{2,2,-1}B_{2,2,1}^2 + 160/3A_{2,2,-1}A_{2,2,0}B_{2,2,2} - 16A_{2,2,-1}A_{2,2,1}B_{2,2,1} - 176/3A_{2,2,-1}A_{2,2,2}B_{2,2,0} - 8/3A_{2,2,-1}B_{2,2,0}B_{2,2,2} + 14A_{2,2,-1}B_{3,3,2} - 16/3A_{2,2,0}A_{2,2,1}B_{2,2,0} + 16/3A_{2,2,0}A_{2,2,2}B_{2,2,-1} + 88/3A_{2,2,0}B_{2,2,-1}B_{2,2,2} + 8/3A_{2,2,0}B_{2,2,0}B_{2,2,1} + 4A_{2,2,0}B_{3,3,1} - 8/3A_{2,2,1}B_{2,2,0}^2 + 96A_{2,2,1}A_{2,2,2}B_{2,2,-2} + 48A_{2,2,1}B_{2,2,-2}B_{2,2,2} + 8A_{2,2,1}B_{2,2,-1}B_{2,2,1} - 6A_{2,2,1}B_{3,3,0} - 80/3A_{2,2,2}B_{2,2,-1}B_{2,2,0} - 16A_{2,2,2}B_{3,3,-1} + A_{4,4,1} + B_{4,4,1},$

$C_{4,4,0} = 40A_{2,2,-1}^2B_{2,2,2} + 40A_{2,2,1}^2B_{2,2,-2} - 20A_{2,2,-2}B_{2,2,1}^2 + 160/3A_{2,2,-2}A_{2,2,0}B_{2,2,2} - 40A_{2,2,-2}A_{2,2,1}B_{2,2,1} - 320/3A_{2,2,-2}A_{2,2,2}B_{2,2,0} - 80/3A_{2,2,-2}B_{2,2,0}B_{2,2,2} + 20A_{2,2,-2}B_{3,3,2} + 40/3A_{2,2,-1}A_{2,2,0}B_{2,2,1} - 80/3A_{2,2,-1}A_{2,2,1}B_{2,2,0} - 40A_{2,2,-1}A_{2,2,2}B_{2,2,-1} + 20A_{2,2,-1}B_{2,2,-1}B_{2,2,2} - 20/3A_{2,2,-1}B_{2,2,0}B_{2,2,1} + 10A_{2,2,-1}B_{3,3,1} + 40/3A_{2,2,0}A_{2,2,1}B_{2,2,-1} + 160/3A_{2,2,0}A_{2,2,2}B_{2,2,-2} + 160/3A_{2,2,0}B_{2,2,-2}B_{2,2,2} + 40/3A_{2,2,0}B_{2,2,-1}B_{2,2,1} + 20A_{2,2,1}B_{2,2,-2}B_{2,2,1} - 20/3A_{2,2,1}B_{2,2,-1}B_{2,2,0} - 10A_{2,2,1}B_{3,3,-1} - 20A_{2,2,2}B_{2,2,-1}^2 - 80/3A_{2,2,2}B_{2,2,-2}B_{2,2,0} - 20A_{2,2,2}B_{3,3,-2} + A_{4,4,0} + B_{4,4,0},$

$C_{4,4,-1} = 16A_{2,2,-1}^2B_{2,2,1} + 16/3A_{2,2,0}^2B_{2,2,-1} + 96A_{2,2,-2}A_{2,2,-1}B_{2,2,2} + 16/3A_{2,2,-2}A_{2,2,0}B_{2,2,1} -$

$$176/3A_{2,2,-2}A_{2,2,1}B_{2,2,0} - 96A_{2,2,-2}A_{2,2,2}B_{2,2,-1} - 80/3A_{2,2,-2}B_{2,2,0}B_{2,2,1} + 16A_{2,2,-2}B_{3,3,1} -$$
$$8/3A_{2,2,-1}B_{2,2,0}^2 - 16/3A_{2,2,-1}A_{2,2,0}B_{2,2,0} - 16A_{2,2,-1}A_{2,2,1}B_{2,2,-1} + 48A_{2,2,-1}B_{2,2,-2}B_{2,2,2} +$$
$$8A_{2,2,-1}B_{2,2,-1}B_{2,2,1} + 6A_{2,2,-1}B_{3,3,0} + 160/3A_{2,2,0}A_{2,2,1}B_{2,2,-2} + 88/3A_{2,2,0}B_{2,2,-2}B_{2,2,1} +$$
$$8/3A_{2,2,0}B_{2,2,-1}B_{2,2,0} - 4A_{2,2,0}B_{3,3,-1} - 8A_{2,2,1}B_{2,2,-1}^2 - 8/3A_{2,2,1}B_{2,2,-2}B_{2,2,0} -$$
$$14A_{2,2,1}B_{3,3,-2} - 48A_{2,2,2}B_{2,2,-2}B_{2,2,-1} - 24A_{2,2,2}B_{3,3,-3} + A_{4,4,-1} + B_{4,4,-1},$$

$$C_{4,4,-2} = 64A_{2,2,-2}^2B_{2,2,2} + 8/3A_{2,2,-1}^2B_{2,2,0} + 64/3A_{2,2,0}^2B_{2,2,-2} - 32/3A_{2,2,-2}B_{2,2,0}^2 +$$
$$40A_{2,2,-2}A_{2,2,-1}B_{2,2,1} - 64/3A_{2,2,-2}A_{2,2,0}B_{2,2,0} - 56A_{2,2,-2}A_{2,2,1}B_{2,2,-1} -$$
$$64A_{2,2,-2}A_{2,2,2}B_{2,2,-2} + 32A_{2,2,-2}B_{2,2,-2}B_{2,2,2} - 8A_{2,2,-2}B_{2,2,-1}B_{2,2,1} + 12A_{2,2,-2}B_{3,3,0} -$$
$$8/3A_{2,2,-1}A_{2,2,0}B_{2,2,-1} + 16A_{2,2,-1}A_{2,2,1}B_{2,2,-2} + 28A_{2,2,-1}B_{2,2,-2}B_{2,2,1} +$$
$$4/3A_{2,2,-1}B_{2,2,-1}B_{2,2,0} + 2A_{2,2,-1}B_{3,3,-1} - 4/3A_{2,2,0}B_{2,2,-1}^2 + 32/3A_{2,2,0}B_{2,2,-2}B_{2,2,0} -$$
$$8A_{2,2,0}B_{3,3,-2} - 20A_{2,2,1}B_{2,2,-2}B_{2,2,-1} - 18A_{2,2,1}B_{3,3,-3} - 32A_{2,2,2}B_{2,2,-2}^2 + A_{4,4,-2} + B_{4,4,-2},$$

$$C_{4,4,-3} = 32A_{2,2,-2}^2B_{2,2,1} + 16/3A_{2,2,-2}A_{2,2,-1}B_{2,2,0} - 80/3A_{2,2,-2}A_{2,2,0}B_{2,2,-1} -$$
$$32A_{2,2,-2}A_{2,2,1}B_{2,2,-2} + 16A_{2,2,-2}B_{2,2,-2}B_{2,2,1} - 32/3A_{2,2,-2}B_{2,2,-1}B_{2,2,0} +$$
$$8A_{2,2,-2}B_{3,3,-1} + 64/3A_{2,2,-1}A_{2,2,0}B_{2,2,-2} + 40/3A_{2,2,-1}B_{2,2,-2}B_{2,2,0} - 2A_{2,2,-1}B_{3,3,-2} -$$
$$8/3A_{2,2,0}B_{2,2,-2}B_{2,2,-1} - 12A_{2,2,0}B_{3,3,-3} - 16A_{2,2,1}B_{2,2,-2}^2 + A_{4,4,-3} + B_{4,4,-3},$$

$$C_{4,4,-4} = 32/3A_{2,2,-2}^2B_{2,2,0} + 8A_{2,2,-1}^2B_{2,2,-2} - 4A_{2,2,-2}B_{2,2,-1}^2 - 8A_{2,2,-2}A_{2,2,-1}B_{2,2,-1} -$$
$$32/3A_{2,2,-2}A_{2,2,0}B_{2,2,-2} + 16/3A_{2,2,-2}B_{2,2,-2}B_{2,2,0} + 4A_{2,2,-2}B_{3,3,-2} +$$
$$4A_{2,2,-1}B_{2,2,-2}B_{2,2,-1} - 6A_{2,2,-1}B_{3,3,-3} - 16/3A_{2,2,0}B_{2,2,-2}^2 + A_{4,4,-4} + B_{4,4,-4};$$

$$C_{4,2,2} = -64/21A_{2,2,0}^2B_{2,2,2} - 8/21A_{2,2,1}^2B_{2,2,0} + 256/21A_{2,2,2}^2B_{2,2,-2} - 128/21A_{2,2,-2}B_{2,2,2}^2 -$$
$$256/21A_{2,2,-2}A_{2,2,2}B_{2,2,2} + 176/21A_{2,2,-1}A_{2,2,1}B_{2,2,2} - 16/3A_{2,2,-1}A_{2,2,2}B_{2,2,1} +$$
$$32/21A_{2,2,-1}B_{2,2,1}B_{2,2,2} + 24/7A_{2,2,-1}B_{3,3,3} + 4/21A_{2,2,0}B_{2,2,1}^2 + 8/21A_{2,2,0}A_{2,2,1}B_{2,2,1} +$$
$$64/21A_{2,2,0}A_{2,2,2}B_{2,2,0} - 32/21A_{2,2,0}B_{2,2,0}B_{2,2,2} - 8/7A_{2,2,0}B_{3,3,2} - 64/21A_{2,2,1}A_{2,2,2}B_{2,2,-1} +$$
$$8/3A_{2,2,1}B_{2,2,-1}B_{2,2,2} - 4/21A_{2,2,1}B_{2,2,0}B_{2,2,1} + 2A_{2,2,1}B_{3,1,1} + 24/35A_{2,2,1}B_{3,3,1} +$$
$$32/21A_{2,2,2}B_{2,2,0}^2 + 128/21A_{2,2,2}B_{2,2,-2}B_{2,2,2} - 88/21A_{2,2,2}B_{2,2,-1}B_{2,2,1} - 4A_{2,2,2}B_{3,1,0} -$$
$$24/35A_{2,2,2}B_{3,3,0} + A_{4,2,2} + B_{4,2,2},$$

$$C_{4,2,1} = -16/7A_{2,2,0}^2B_{2,2,1} - 88/21A_{2,2,1}^2B_{2,2,-1} + 64/3A_{2,2,-2}A_{2,2,1}B_{2,2,2} -$$
$$704/21A_{2,2,-2}A_{2,2,2}B_{2,2,1} - 128/21A_{2,2,-2}B_{2,2,1}B_{2,2,2} + 96/7A_{2,2,-2}B_{3,3,3} +$$
$$44/21A_{2,2,-1}B_{2,2,1}^2 - 32/21A_{2,2,-1}A_{2,2,0}B_{2,2,2} + 88/21A_{2,2,-1}A_{2,2,1}B_{2,2,1} -$$
$$32/21A_{2,2,-1}A_{2,2,2}B_{2,2,0} - 32/21A_{2,2,-1}B_{2,2,0}B_{2,2,2} + 16/7A_{2,2,0}A_{2,2,1}B_{2,2,0} +$$
$$64/21A_{2,2,0}A_{2,2,2}B_{2,2,-1} + 16/21A_{2,2,0}B_{2,2,-1}B_{2,2,2} - 8/7A_{2,2,0}B_{2,2,0}B_{2,2,1} + 4A_{2,2,0}B_{3,1,1} -$$
$$32/35A_{2,2,0}B_{3,3,1} + 8/7A_{2,2,1}B_{2,2,0}^2 + 256/21A_{2,2,1}A_{2,2,2}B_{2,2,-2} + 352/21A_{2,2,1}B_{2,2,-2}B_{2,2,2} -$$
$$44/21A_{2,2,1}B_{2,2,-1}B_{2,2,1} - 2A_{2,2,1}B_{3,1,0} + 48/35A_{2,2,1}B_{3,3,0} - 32/3A_{2,2,2}B_{2,2,-2}B_{2,2,1} +$$
$$16/21A_{2,2,2}B_{2,2,-1}B_{2,2,0} - 8A_{2,2,2}B_{3,1,-1} - 96/35A_{2,2,2}B_{3,3,-1} + A_{4,2,1} + B_{4,2,1},$$

$$C_{4,2,0} = -16/7A_{2,2,-1}^2B_{2,2,2} - 16/7A_{2,2,1}^2B_{2,2,-2} + 8/7A_{2,2,-2}B_{2,2,1}^2 + 128/7A_{2,2,-2}A_{2,2,0}B_{2,2,2} +$$
$$16/7A_{2,2,-2}A_{2,2,1}B_{2,2,1} - 256/7A_{2,2,-2}A_{2,2,2}B_{2,2,0} - 64/7A_{2,2,-2}B_{2,2,0}B_{2,2,2} +$$
$$48/7A_{2,2,-2}B_{3,3,2} - 24/7A_{2,2,-1}A_{2,2,0}B_{2,2,1} + 48/7A_{2,2,-1}A_{2,2,1}B_{2,2,0} +$$
$$16/7A_{2,2,-1}A_{2,2,2}B_{2,2,-1} - 8/7A_{2,2,-1}B_{2,2,-1}B_{2,2,2} + 12/7A_{2,2,-1}B_{2,2,0}B_{2,2,1} +$$
$$6A_{2,2,-1}B_{3,1,1} - 48/35A_{2,2,-1}B_{3,3,1} - 24/7A_{2,2,0}A_{2,2,1}B_{2,2,-1} + 128/7A_{2,2,0}A_{2,2,2}B_{2,2,-2} +$$
$$128/7A_{2,2,0}B_{2,2,-2}B_{2,2,2} - 24/7A_{2,2,0}B_{2,2,-1}B_{2,2,1} - 8/7A_{2,2,1}B_{2,2,-2}B_{2,2,1} +$$
$$12/7A_{2,2,1}B_{2,2,-1}B_{2,2,0} - 6A_{2,2,1}B_{3,1,-1} + 48/35A_{2,2,1}B_{3,3,-1} + 8/7A_{2,2,2}B_{2,2,-1}^2 -$$

$$64/7A_{2,2,2}B_{2,2,-2}B_{2,2,0} - 48/7A_{2,2,2}B_{3,3,-2} + A_{4,2,0} + B_{4,2,0},$$

$$C_{4,2,-1} = -88/21A_{2,2,-1}^2B_{2,2,1} - 16/7A_{2,2,0}^2B_{2,2,-1} + 256/21A_{2,2,-2}A_{2,2,-1}B_{2,2,2} +$$
$$64/21A_{2,2,-2}A_{2,2,0}B_{2,2,1} - 32/21A_{2,2,-2}A_{2,2,1}B_{2,2,0} - 704/21A_{2,2,-2}A_{2,2,2}B_{2,2,-1} -$$
$$32/3A_{2,2,-2}B_{2,2,-1}B_{2,2,2} + 16/21A_{2,2,-2}B_{2,2,0}B_{2,2,1} + 8A_{2,2,-2}B_{3,1,1} + 96/35A_{2,2,-2}B_{3,3,1} +$$
$$8/7A_{2,2,-1}B_{2,2,0}^2 + 16/7A_{2,2,-1}A_{2,2,0}B_{2,2,0} + 88/21A_{2,2,-1}A_{2,2,1}B_{2,2,-1} +$$
$$64/3A_{2,2,-1}A_{2,2,2}B_{2,2,-2} + 352/21A_{2,2,-1}B_{2,2,-2}B_{2,2,2} - 44/21A_{2,2,-1}B_{2,2,-1}B_{2,2,1} +$$
$$2A_{2,2,-1}B_{3,1,0} - 48/35A_{2,2,-1}B_{3,3,0} - 32/21A_{2,2,0}A_{2,2,1}B_{2,2,-2} + 16/21A_{2,2,0}B_{2,2,-2}B_{2,2,1} -$$
$$8/7A_{2,2,0}B_{2,2,-1}B_{2,2,0} - 4A_{2,2,0}B_{3,1,-1} + 32/35A_{2,2,0}B_{3,3,-1} + 44/21A_{2,2,1}B_{2,2,-1}^2 -$$
$$32/21A_{2,2,1}B_{2,2,-2}B_{2,2,0} - 128/21A_{2,2,2}B_{2,2,-2}B_{2,2,-1} - 96/7A_{2,2,2}B_{3,3,-3} + A_{4,2,-1} + B_{4,2,-1},$$

$$C_{4,2,-2} = 256/21A_{2,2,-2}^2B_{2,2,2} - 8/21A_{2,2,-1}^2B_{2,2,0} - 64/21A_{2,2,0}^2B_{2,2,-2} + 32/21A_{2,2,-2}B_{2,2,0}^2 -$$
$$64/21A_{2,2,-2}A_{2,2,-1}B_{2,2,1} + 64/21A_{2,2,-2}A_{2,2,0}B_{2,2,0} - 16/3A_{2,2,-2}A_{2,2,1}B_{2,2,-1} -$$
$$256/21A_{2,2,-2}A_{2,2,2}B_{2,2,-2} + 128/21A_{2,2,-2}B_{2,2,-2}B_{2,2,2} - 88/21A_{2,2,-2}B_{2,2,-1}B_{2,2,1} +$$
$$4A_{2,2,-2}B_{3,1,0} + 24/35A_{2,2,-2}B_{3,3,0} + 8/21A_{2,2,-1}A_{2,2,0}B_{2,2,-1} + 176/21A_{2,2,-1}A_{2,2,1}B_{2,2,-2} +$$
$$8/3A_{2,2,-1}B_{2,2,-2}B_{2,2,1} - 4/21A_{2,2,-1}B_{2,2,-1}B_{2,2,0} - 2A_{2,2,-1}B_{3,1,-1} - 24/35A_{2,2,-1}B_{3,3,-1} +$$
$$4/21A_{2,2,0}B_{2,2,-1}^2 - 32/21A_{2,2,0}B_{2,2,-2}B_{2,2,0} + 8/7A_{2,2,0}B_{3,3,-2} + 32/21A_{2,2,1}B_{2,2,-2}B_{2,2,-1} -$$
$$24/7A_{2,2,1}B_{3,3,-3} - 128/21A_{2,2,2}B_{2,2,-2}^2 + A_{4,2,-2} + B_{4,2,-2};$$

$$C_{4,0,0} = A_{4,0,0} + B_{4,0,0}.$$

This *gato* operation # is a noncommutative product, here expressed in the basis of the three-dimensional harmonic oscillator states. It has several properties that link the to physical properties of the optical elements with mathematical statements on selection rules [10]. But further, if we count the number of terms in the preceding gato operation in the symplectic bases, and a corresponding count in the monomial bases, we find some economy in the symplectic basis [16]:

BASIS	ORDER 3	ORDER 5	ORDER 7
MONOMIAL	12	54	422
SYMPLECTIC	12	52	318

10 Economy in Aberration Calculations

Two general linear (M) and aberration (A) group elements multiply through

$$\mathcal{G}\{A,M\}\ \mathcal{G}\{B,N\} = \mathcal{G}\{A \,\#\, D(M) B\, MN\}. \tag{23}$$

Now, the symplectic basis ${}^k\chi_m^j$ is block-diagonal under paraxial —linear— transformations, with a matrix composed of the (analytically continued) Wigner D-matrices:

$$
\begin{pmatrix}
\boxed{\mathbf{D}^1_{[3\times3]}} & 0 & 0 & 0 \\[4pt]
0 & \begin{array}{cc} \boxed{\mathbf{D}^2_{[5\times5]}} & 0 \\ 0 & \boxed{\mathbf{D}^0_{[1\times1]}} \end{array} & 0 & 0 \\[10pt]
0 & 0 & \begin{array}{cc} \boxed{\mathbf{D}^3_{[7\times7]}} & 0 \\ 0 & \boxed{\mathbf{D}^1_{[3\times3]}} \end{array} & 0 \\[10pt]
0 & 0 & 0 & \begin{array}{ccc} \boxed{\mathbf{D}^4_{[9\times9]}} & 0 & 0 \\ 0 & \boxed{\mathbf{D}^2_{[5\times5]}} & 0 \\ 0 & 0 & \boxed{\mathbf{D}^0_{[1\times1]}} \end{array}
\end{pmatrix}
$$

The number of non-zero matrix elements in the two bases is

BASIS	ORDER 1	ORDER 3	ORDER 5	ORDER 7	$k \to \infty$
MONOMIAL	9	36	100	225	$\sim k^4/4$
SYMPLECTIC	9	26	58	107	$\sim 2k^3/3$

The total number of operations necessary to 7th aberration order, including matrix multiplication by the linear part is, for the two bases,

$$\text{monomial}: 7680, \qquad \text{symplectic}: 3882.$$

For this reason we conclude that the most efficient basis to carry through aberration computations in axis-symmetric systems is the angular momentum basis of the harmonic oscillator states.

11 Outlook

This has been a quick revision of the state of the art in seventh order aberration calculations in axis-symmetric geometric-optics systems. We have seen that the state schemes of the harmonic oscillator provide order and symmetry in the classification of aberrations. In these conference proceedings we cannot go much further, so let me state that *wave* optics is the true objective of this quest. We can design and specify systems in geometric optics; once this is done we would like to predict the imaging behavior of such a system when light of a definite color is used.

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IV. SPECIAL RELATIVITY

POINT FORM RELATIVISTIC QUANTUM MECHANICS AND RELATIVISTIC SU(6)

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Abstract

The point form is used as a framework for formulating a relativistic quantum mechanics, with the mass operator carrying the interactions of underlying constituents. A symplectic Lie algebra of mass operators is introduced from which a relativistic harmonic oscillator mass operator is formed. Mass splittings within the degenerate harmonic oscillator levels arise from relativistically invariant spin-spin, spin-orbit and tensor mass operators. Internal flavor (and color) symmetries are introduced which make it possible to formulate a relativistic $SU(6)$ model of baryons (and mesons). Careful attention is paid to the permutation symmetry properties of the hadronic wave functions, which are written as polynomials in Bargmann spaces.

1 Relativistic Introduction

Despite many successes, one of the main difficulties of the old $SU(6)$ theory [1] was that the underlying quantum mechanics was nonrelativistic. In this paper we combine what Dirac called the point form of relativistic quantum mechanics [2] with an $SU(3)$ flavor internal symmetry to formulate a relativistic $SU(6)$ theory. The goal is to be able to get hadronic bound-state wave functions and then compute form factors, structure functions, decay rates, and even production scattering amplitudes. In this paper we restrict our attention to formulating a relativistic $SU(6)$ theory, and then introduce a harmonic oscillator mass operator to obtain bound-state wave functions.

We will view hadrons as bound states of underlying spin $\frac{1}{2}$ constituents which carry internal $SU(3)$ flavor and $SU(3)$ color degrees of freedom. Combining a (relativistic) $SU(2)$ spin with $SU(3)$ flavor then leads to a (relativistic) $SU(6)$ spin-flavor group. The relativistic $SU(2)$ spin structure comes from properties of the Pauli-Lubanski operator; as will be shown in Section 2 properly chosen sets of four vectors dotted into the Pauli-Lubanski operator generate an $SU(2)$ Lie algebra. Moreover in the point form of relativistic quantum mechanics all Lorentz transformations are kinematic. n-particle constituent states called velocity states have the property that under Lorentz transformations the internal momenta and spins are uniformly rotated by a Wigner notation, meaning that the relativistic $SU(2)$ spin structure can be extended to n-particle systems.

In the point form of relativistic quantum mechanics, the four-momentum operator supplies the dynamical information. The (interacting) four-momentum operator is written as $P^\mu = MV^\mu$, where M is the mass operator and V^μ the four-velocity operator. The Hamiltonian is then $H = P^0 = MV^0$. Since Lorentz transformations are kinematic and the mass operator commutes with all Poincaré operators, the theory is Lorentz covariant. As discussed in Section 3 mass operators are self-adjoint operators on the n-constituent Hilbert space that commute with Lorentz transformations and the velocity operator. Of particular interest for hadron spectroscopy are confining potentials; in Section 3 we will show how to construct relativistic harmonic oscillator potentials algebraically, using generators of an underlying symplectic group.

To obtain a realistic hadronic mass spectrum mass operators that split the degenerate oscillator levels are needed. We show that spin-orbit, spin-spin, and tensor operators are all readily introduced in the context of point form quantum mechanics. Moreover it is straightforward to construct mass operators out of internal symmetry generators; these operators can be used to obtain Gell-Mann-Okubo and Gürsey-Radicati type mass formulae. Such operators have the usual internal symmetry transformation properties, but the mass splittings are not given by Clebsch-Gordan coefficients, but by matrix elements of the appropriate mass operator.

The Hilbert space of n-constituents are tensor products of representation spaces of the Poincaré and internal symmetry groups. The relativistic kinematics of n-particle systems is discussed in Section 2. A hadronic wave function is an appropriately symmetrized wave function containing spatial, spin-flavor, and color pieces. As shown in Section 4 it is convenient to carry out the detailed calculations of the wave functions in Bargmann space, rather than the usual Hilbert space. Thus, the color, spin-flavor, and spatial parts of the wave function are all realized as polynomials in Bargmann spaces; the connection between these polynomials and wave functions in the usual Hilbert space is then given in terms of creation operators acting on a vacuum state.

2   Relativistic Kinematics

The Hilbert space of n-constituent particles is the n-fold tensor product of single-particle spaces which are the representation spaces of the Poincaré group corresponding to particles of mass m and spin $j(j = \frac{1}{2})$. In this paper we take the masses of the constituents to be nonzero; in a later paper we will investigate the properties of hadrons as bound states of massless constituents.

For particles of mass $m > 0$ and spin j the representations of the Poincaré group are well known [3], with the representation space $\mathcal{H} = L^2(\mathbf{R}^3) \times V^j$. The action of unitary operators corresponding to Lorentz transformations and space-time translations is given by

$$U_\Lambda |pj\sigma f\rangle = \sum_{\sigma'} |\Lambda p, j\sigma' f\rangle D^j_{\sigma'\sigma}(p, \Lambda)$$

$$U_a |pj\sigma f\rangle = e^{ip\cdot a}|pj\sigma f\rangle \tag{2.1}$$

where $\Lambda \in SO(1,3)$ is a Lorentz transformation, $a \in \mathbf{R}^4$ is a space-time translation and $p \cdot a := p^T g a$ is the Lorentz invariant inner product with the metric $g = \text{diag}(1, -1, -1, -1)$. σ is a spin projection variable and $(p, \Lambda) \in SO(3)$ is a Wigner rotation defined by

$$(p, \Lambda) := B^{-1}(\Lambda p)\Lambda B(p) \quad \in SO(3) , \tag{2.2}$$

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with $B(p)$ a boost (coset representative of $SO(1,3)/SO(3)$) satisfying $p = B(p)p^{\text{rest}}$, $p^{\text{rest}} = (m, 0, 0, 0)$. $D^j(\ )$ is an $SO(3)$ matrix element and f is an internal symmetry label to be discussed below.

The four-vector momentum p satisfies $p \cdot p = m^2$. There is some ambiguity in specifying p which corresponds roughly to the different forms of relativistic dynamics. In the instant form p is written as $(E, \vec{p})$, with $E = \sqrt{m^2 + \vec{p} \cdot \vec{p}}$ and wave functions are written as $\varphi(\vec{p}, \sigma)$. The kinematic subgroup consists of rotations $R \in SO(3)$ and space translations $\vec{a}$. Interactions are introduced in the Hamiltonian $\mathcal{H}$ and pure Lorentz generators. The instant form of dynamics has been used to obtain hadronic wave functions and form factors by several groups [4,5].

Another possibility is to write p as $p_\perp = p_x + ip_y$, $p_+ = E + p_z$, so that $p_- = [(|p_\perp|^2 + m^2)/p_+]$ and wave functions are written as $\varphi(p_+, p_\perp, \sigma)$. In the front form of relativistic dynamics the kinematic subgroup is the two-dimensional Euclidean subgroup $E(2)$ of the Lorentz group, along with the translations $a_\perp = a_x + ia_y$ and $a_+ = a_0 + a_z$. In this case the dynamics is introduced in the p_+ generators, as seen in Refs. [6] and [7].

In the point form of relativistic dynamics to be used in this paper, p is written as $p = mv$, with v the four-velocity satisfying $v \cdot v = 1$. In this case wave functions are written as $\varphi(v, \sigma)$ and the kinematic subgroup is the full Lorentz group $SO(1,3)$, while the dynamics is introduced in the four-momentum operator P^μ.

Interactions will be introduced in Section 3. To see how they came about, it is first necessary to get the free mass and spin operators. The infinitesimal transformations of Eq. (2.1) generate the operators $J^{\alpha\beta}$ and P_0^μ, the free Lorentz and four-momentum operators. From these it is possible to form the free mass, velocity, and spin operators:

$$M_0^2 := P_0 \cdot P_0 \quad , \quad V_{(0)}^\mu := P_{(0)}^\mu M_0^{-1}$$

$$W_\mu^{(0)} = \tfrac{1}{2} \epsilon_{\mu\nu\alpha\beta} J^{\alpha\beta} P_{(0)}^\nu \qquad \text{(Pauli-Lubanski operator)} \tag{2.3}$$

$$\widetilde{W}_\mu^{(0)} := \tfrac{1}{2} \epsilon_{\mu\nu\alpha\beta} J^{\alpha\beta} V_{(0)}^\nu \qquad \text{(modified Pauli-Lubanski operator)}$$

$V_{(0)}^\mu$ is the free four-velocity operator and will be used extensively in the following sections. Notice that there is no "0" subscript or superscript on the $J^{\alpha\beta}$ operators, because in the point form these operators are not modified in the presence of interactions. The labels p, j, and σ appearing in Eq. (2.1) are now seen to be eigenvalues of the operators $P_{(0)}^\mu = M_0 V_{(0)}^\mu$, $\widetilde{W} \cdot \widetilde{W}$, and $n \cdot \widetilde{W}$, respectively, with n a four vector; $\widetilde{W} \cdot \widetilde{W}$ has eigenvalues $j(j+1)$ with no mass factor (see Ref. [3] for details).

Besides the space-time and Lorentz transformations of single-particle constituents given in Eq. (2.1), there are also internal symmetry transformations which mix charges and other internal symmetry quantum numbers. Let G be an internal symmetry group (such as $SU(2)$ isospin or $SU(3)$ flavor or $SU(3)$ color or a direct product) for which there is a unitary representation operator U_g, $g \in G$, acting on a vector space V with basis $|f\rangle$. Then

$$U_g |pj\sigma f\rangle = \sum_{f'} |p, j\sigma f'\rangle D_{f'f}(g) \tag{2.4}$$

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gives the action of $g \in G$ on the basis state $|pj\sigma f\rangle$; $D_{f'f}(g)$ is a matrix element of G. Thus $|pj\sigma f\rangle$ is a basis state for the one-particle Hilbert space $L^2(\mathbf{R}^3) \times V^j \times V$.

As shown in Ref. [3] (j, σ) are eigenvalues of relativistic operators that form a Lie algebra of $SU(2)$. Let n, $n_\pm$ be four vectors such that $n \cdot \widetilde{W}$, $n_\pm \cdot \widetilde{W}$ form the Lie algebra of $SU(2)$ with σ the eigenvalue of $n \cdot \widetilde{W}$ and $j(j+1)$ the eigenvalue of $\widetilde{W} \cdot \widetilde{W}$. If j is chosen to be $\frac{1}{2}$, the spin of the constituents, and G is chosen to be $SU(3)_{\text{flavor}}$, the two algebras, relativistic $SU(2)$ and $SU(3)_f$, can be combined to give a larger algebra, namely $SU(6)$. The labels (σ, f) in the basis state are jointly transformed under the action of $SU(6)$; that is

$$U_g|p, j = \tfrac{1}{2}, \sigma, f\rangle = \sum_{\sigma', f'} |p, j = \tfrac{1}{2}, \sigma', f'\rangle D_{\sigma'f'\sigma f}(g) , \quad g \in SU(6) , \tag{2.5}$$

so that the group element g of $SU(6)$ mixes the relativistic spin variable σ and the flavor variable f. $D_{\sigma'f'\sigma f}(g)$ is the six-dimensional $SU(6)$ matrix element. It is the infinitesimal actions of U_g that will be used in Section 3 to obtain mass splitting operators.

Hadrons are bound states of confined constituents. An n-particle constituent particle space is defined to be the n-fold tensor product of single-particle constituent spaces, with basis and group actions given by

$$|p_1 j_1 \sigma_1 f_1, \ldots, p_n j_n \sigma_n f_n\rangle := |p_1 j_1 \sigma_1 f_1\rangle \cdots |p_n j_n \sigma_n f_n\rangle$$

$$U_\Lambda |p_1 j_1 \sigma_1 f_1, \ldots, p_n j_n \sigma_n f_n\rangle = \prod_{\alpha=1}^{n} (\Lambda p_1, j_1 \sigma_1' f_1, \ldots, \Lambda p_n, j_n \sigma_n' f_n) D^{j_i}_{\sigma_\alpha' \sigma_\alpha}(p_i, \Lambda)$$

$$U_a |p_1 j_1 \sigma_1 f_1, \ldots, p_n j_n \sigma_n f_n\rangle = e^{i \sum_\alpha p_\alpha \cdot a} |p_1 j_1 \sigma_1 f_1, \ldots, p_n j_n \sigma_n f_n\rangle$$

$$U_\pi |p_1 j_1 \sigma_1 f_1, \ldots, p_n j_n \sigma_n f_n\rangle := |\pi(p_1 j_1 \sigma_1 f_1, \ldots, p_n j_n \sigma_n f_n)\rangle , \tag{2.6}$$

where in the last equation π is an element of the permutation group on n letters, S_n; permutation symmetry will play an important role in the hadronic wave functions to be discussed in Section 4.

To develop the point form of relativistic dynamics, it is useful to define n-particle "velocity states" that are eigenstates of the free velocity operator $V^\mu_{(0)}$. Define

$$|v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha\rangle := U_{B(v)} |k_1 j_1 \mu_1 f_1, \ldots, k_n j_n \mu_n f_n\rangle \tag{2.7}$$

with $\sum_{\alpha=1}^{n} \vec{k}_\alpha = \vec{0}$. We want to show that the spin labels μ_α transform like nonrelativistic variables. To see this, consider a Lorentz transformation Λ acting on the boost $B(v)$ defined in Eq. (2.2)ff:

$$U_\Lambda |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha \rangle = U_\Lambda U_{B(v)} |k_1 j_1 \mu_1 f_1, \ldots, k_n j_n \mu_n f_n \rangle$$

$$= U_{B(\Lambda v)} U_{R_w} |k_1 j_1 \mu_1 f_1, \ldots, k_n j_n \mu_n f_n \rangle$$

$$= U_{B(\Lambda v)} \sum_{\mu'_\alpha} |R_w k_1, j_1 \mu'_1 f_1, \ldots, R_w k_n, j_n \mu'_n f_n \rangle \prod_{\alpha=1}^{n} D^{j_\alpha}_{\mu'_\alpha \mu_\alpha}(k_\alpha, R_w)$$

$$= \sum_{\mu'_\alpha} |\Lambda v, R_w \vec{k}_\alpha, j_\alpha \mu'_\alpha f_\alpha \rangle \prod_{\alpha=1}^{n} D^{j_\alpha}_{\mu'_\alpha \mu_\alpha}(R_w) \tag{2.8}$$

where use has been made of the fact that the boosts defining the Wigner rotation (k_α, R_w) are canonical boosts, so that $(k_\alpha, R_w) = R_w$ (see Ref. [3]). R_w is itself the Wigner rotation $B^{-1}(\Lambda v)\Lambda B(v)$ as defined by Eq. (2.2).

Equation (2.8) states that for velocity states, Eq. (2.7), the internal momenta $\vec{k}_\alpha$ and internal spins $(j_\alpha \mu_\alpha)$, transform like nonrelativistic variables. If they were nonrelativistic variables, the Wigner rotation R_w determined by v and Λ would be replaced by $R \in SO(3)$. But since the Wigner rotations appearing in the velocity state are all the same rotation, the spins $j_1 \cdots j_n$ can all be coupled together to give an overall spin, the internal momenta $\vec{k}_\alpha$ can be replaced by $k_\alpha \ell_\alpha m_{\ell_\alpha}$, the magnitude of $\vec{k}$ and the orbital and orbital projection quantum numbers of the $\alpha^{\underline{th}}$ constituent, and these coupled together to give the overall orbital angular momentum of the n-particle constituents, exactly as is done nonrelativistically. Thus the external variables v, and j, σ (if the spin and orbital angular momentum are coupled to give the total angular momentum of the n-particle system) transform as relativistic variables [see Eq. (2.1)] while the internal variables transform as though they were nonrelativistic variables.

Similarly the action of a space-time translation a on a velocity state gives

$$U_a |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha \rangle = U_a U_{B(v)} |k_1 j_1 \mu_1 f_1, \ldots, k_n j_n \mu_n f_n \rangle$$

$$= e^{iB^{-1}(v)a \cdot \sum k_\alpha} |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha \rangle$$

$$= e^{ia \cdot B(v) \sum k_\alpha} |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha \rangle$$

$$= e^{ia \cdot v m_n} |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha \rangle \tag{2.9}$$

which means that

$$P^\mu_{(0)} |v, \vec{k}_\alpha \mu_\alpha f_\alpha \rangle = v^\mu m_n |v, \vec{k}_\alpha \mu_\alpha f_\alpha \rangle$$

$$V^\mu_{(0)} |v, \vec{k}_\alpha \mu_\alpha f_\alpha \rangle = v^\mu |v, \vec{k}_\alpha \mu_\alpha f_\alpha \rangle$$

$$M_0 |v, \vec{k}_\alpha \mu_\alpha f_\alpha \rangle = m_n |v, \vec{k}_\alpha \mu_\alpha f_\alpha \rangle \tag{2.10}$$

where $\sum_\alpha k_\alpha = \sum (\omega_\alpha, \vec{k}_\alpha) = (\sum \omega_\alpha, \vec{0}) = (m_n, \vec{0})$, with $\omega_\alpha := \sqrt{m^2 + \vec{k}_\alpha \cdot \vec{k}_\alpha}$ and $m_n := \sum \omega_\alpha$. It is the free mass operator M_0 acting on the n-particle space of constituents that will be modified to give the interacting mass operator.

The connection between velocity states, Eq. (2.7) and n-particle constituent states is

$$|v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha\rangle = U_{B(v)}|k_1 j_1 \mu_1 f_1, \ldots, k_n j_n \mu_n f_n\rangle$$

$$= \sum_{\sigma_\alpha} |p_1 j_1 \sigma_1 f_1, \ldots, p_n j_n \sigma_n f_n\rangle \prod_{\alpha=1}^{n} D^{j_\alpha}_{\sigma_\alpha \mu_\alpha}(k_\alpha, B(v)) \qquad (2.11)$$

where $p_\alpha = B(v)k_\alpha$, $\sum_{\alpha=1}^{n} \vec{k}_\alpha = 0$.

In the following sections we will set $j_\alpha = \frac{1}{2}$, and suppress the label j_α in the velocity states. μ_α is the eigenvalue of $n_\alpha \cdot \widetilde{W}_\alpha$ and together with $n_{\pm\alpha} \cdot \widetilde{W}_\alpha$ forms an $SU(2)$ algebra. Hence as seen in Eq. (2.5) for single-particle states, an $SU(6)$ element mixes the (μ_α, f_α) labels:

$$U_g|v, \vec{k}_\alpha \mu_\alpha f_\alpha\rangle = \sum_{\mu'_\alpha f'_\alpha} |v, \vec{k}_\alpha \mu'_\alpha f'_\alpha\rangle \prod_{\alpha=1}^{n} D_{\mu'_\alpha f'_\alpha \mu_\alpha f_\alpha}(g) , \quad g \in SU(6) .$$

3 Relativistic Dynamics

In the point form of relativistic dynamics the free four-momentum operator is modified to include interactions. The six Lorentz generators do not change when interactions are included and hence the unitary operators U_Λ representing Lorentz transformations retain their form as given in Eqs. (2.6) or (2.8). The easiest way to modify the free four-momentum operator, $P^\mu_{(0)} = M_0 V^\mu_{(0)}$, is to change the free mass operator M_0 to the interacting mass operator M while leaving $V^\mu_{(0)}$ unchanged:

$$P^\mu := M V^\mu_{(0)} . \qquad (3.1)$$

As shown in a succeeding paper dealing with electromagnetic currents and form factors, it is also necessary to modify the free velocity operator, but when dealing with hadronic wave functions it suffices to use only $V^\mu_{(0)}$. M must commute with U_Λ and $V^\mu_{(0)}$, for then

$$U_\Lambda P^\mu U_\Lambda^{-1} = U_\Lambda M V^\mu_{(0)} U_\Lambda^{-1}$$

$$= M U_\Lambda V^\mu_{(0)} U_\Lambda^{-1}$$

$$= M(\Lambda^{-1})^\mu{}_\nu V^\nu_{(0)}$$

$$= (\Lambda^{-1})^\mu{}_\nu P^\nu , \qquad (3.2)$$

which along with $[P^\mu, P^\nu] = 0$ guarantees the commutation relations of the Poincaré group.

The condition that M commute with U_Λ and $V^\mu_{(0)}$ is easily satisfied on velocity states. Since U_Λ transforms v to Λv and $\vec{k}_\alpha$ to $R_w \vec{k}_\alpha$ [see Eq. (2.8)], it follows that if the kernel of M on the velocity state is independent of v and rotationally invariant, M will commute with U_Λ and V^μ:

$$\langle v', \vec{k}'_\alpha \mu'_\alpha f'_\alpha | M | v k_\beta \mu_\beta f_\beta\rangle = v^0 \delta^3(\vec{v}' - \vec{v}) K(\vec{k}'_\alpha \mu'_\alpha f'_\alpha, \vec{k}_\beta \mu_\beta f_\beta) , \qquad (3.3)$$

where $K(\quad)$ is rotationally invariant. Since $SU(6)$ spin-flavor transformations can be made rotationally invariant, it is clear that mass operators can be formed out of $SU(6)$ generators, resulting in relativistic $SU(6)$ mass splitting terms.

To analyze mass operators more carefully, it is convenient to make all the internal momentum variables independent. The internal momenta satisfy $\sum_{\alpha=1}^{n} \vec{k}_\alpha = \vec{0}$ and the Hilbert space norm is given from

$$
\sum_{\mu_\alpha f_\alpha} \int \frac{d^3 p_1}{E_1} \cdots \frac{d^3 p_n}{E_n} = \sum_{\mu_\alpha f_\alpha} \int \frac{d^3 v}{\sqrt{1 + \vec{v} \cdot \vec{v}}} \frac{d^3 k_1}{\omega_1} \cdots \frac{d^3 k_n}{\omega_n} m_n^3 \delta^3 \left(\sum \vec{k}_\alpha \right)
$$

$$
= \sum_{\mu_\alpha f_\alpha} \int \frac{d^3 v}{\sqrt{1 + \vec{v} \cdot \vec{v}}} \frac{d^3 k_1}{\omega_1} \cdots \frac{d^3 k_{n-1}}{\omega_{n-1}} m_n^3 \tag{3.4}
$$

where $\vec{k}_n = -\sum_{\alpha=1}^{n-1} \vec{k}_\alpha$. Wave functions are now written in independent variables as $\varphi(\vec{k}_\alpha, \mu_\alpha f_\alpha)$, where it is understood that $\alpha = 1 \ldots n - 1$ for the internal momenta, while for spin and flavor, $\alpha = 1 \ldots n$. With $n - 1$ independent internal momenta, the action of the permutation group S_n changes its form from Eq. (2.6); for transpositions in which the $\alpha^{\underline{th}}$ and $n^{\underline{th}}$ momenta are interchanged the representation matrix is

$$
\pi \begin{pmatrix} \vec{k}_1 \\ \vdots \\ \vec{k}_\alpha \\ \vdots \\ \vec{k}_{n-1} \end{pmatrix} = \begin{pmatrix} \vec{k}_1 \\ \vdots \\ -\vec{k}_1 - \cdots \vec{k}_{n-1} \\ \vdots \\ -\vec{k}_{n-1} \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ -1 \cdots & -1 & \cdots & -1 \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} \vec{k}_1 \\ \vdots \\ \vec{k}_\alpha \\ \vdots \\ \vec{k}_{n-1} \end{pmatrix} = \widetilde{D}(\pi) \begin{pmatrix} \vec{k}_1 \\ \vdots \\ \vec{k}_\alpha \\ \vdots \\ \vec{k}_{n-1} \end{pmatrix} \tag{3.5}
$$

where $\pi = (\alpha, n) \in S_n$. All of the representation matrices $\widetilde{D}(\pi)$ involving the $n^{\underline{th}}$ label are nonorthogonal; nevertheless, they form an irreducible representation of S_n with Young diagram $(n - 1, 1)$ ($n - 1$ boxes in the first row, 1 box in the second row).

In analogy with nonrelativistic Hamiltonians, interacting mass operators can be written as perturbations of the free mass operator:

$$
M = M_0 + V , \tag{3.6}
$$

where the "potential" V satisfies Eq. (3.3). As in the nonrelativistic case there are one-body, two-body, ... n-body interactions. From Eq. (3.6) it is possible to define a relativistic Lippman-Schwinger equation, generated by the time translation operator $H = V^0 M$:

$$e^{-iHt}\psi_{t=0} = \psi_t$$

$$i\frac{\partial \psi_t}{\partial t} = H\psi_t$$

$$= V^0_{(0)} M\psi_t$$

$$\psi = \varphi + G_0 V\psi \tag{3.7}$$

where the free Green function $G_0(z) := (1/z - M_0)$. M_0 is of course more complicated than its nonrelativistic counterpart; in internal momentum variables it is $\sum_\alpha \sqrt{m_\alpha^2 + \vec{k}_\alpha \cdot \vec{k}_\alpha}$.

Because mass operators are any (self-adjoint) operators that commute with $V^\mu_{(0)}$ and are rotationally invariant, spin-spin, spin-orbit, and tensor forces of the kind defined in nonrelativistic quantum mechanics can all be defined in an analogous fashion for relativistic n-body systems. The relative orbital angular momentum operator is

$$\vec{L} = \sum_{\alpha=1}^{n-1} \vec{k}_\alpha \times \frac{1}{i}\frac{\partial}{\partial \vec{k}_\alpha} \tag{3.8}$$

and if generators of $SU(6)$ for the αth particle are written $\lambda_A^{(\alpha)}$, $\vec{\sigma}^{(\alpha)}\lambda_A^{(\alpha)}$, $\vec{\sigma}^{(\alpha)}$, $A = 1\ldots 8$, where $\lambda_A^{(\alpha)}$ are the $SU(3)$ generators and $\vec{\sigma}^{(\alpha)}$ Pauli matrices, then for example spin-orbit mass operators of the form

$$M_{LS} = \vec{L} \cdot \vec{\sigma}^{(\alpha)}\lambda_A^{(\alpha)} \tag{3.9}$$

are rotationally invariant.

Mass operators may also be obtained from Lie algebra elements which commute with the orbital and spin angular momentum. Since one of the goals of this paper is to formulate a relativistic $SU(6)$ model and harmonic oscillator wave functions have been used for the unperturbed energy levels (see Ref. [1]), we wish to obtain relativistic harmonic oscillator mass operators. Consider the operators

$$\vec{k}_\alpha \cdot \vec{k}_\beta \; , \quad \vec{k}_\alpha \cdot \frac{\partial}{\partial \vec{k}_\beta} \; , \quad \frac{\partial}{\partial \vec{k}_\alpha} \cdot \frac{\partial}{\partial \vec{k}_\beta} \tag{3.10}$$

with $\alpha, \beta = 1\ldots n-1$. These operators form a representation of the Lie algebra of $Sp(2(n-1), \mathbf{R})$. The middle operators in Eq. (3.10) come from the action of the general linear group $GL(n-1, \mathbf{R})$, which is a subgroup of $Sp(2(n-1), \mathbf{R})$:

$$(U_g\varphi)(\vec{k}_\alpha\mu_\alpha f_\alpha) = \varphi\big((g^{-1}\vec{k})_\alpha\mu_\alpha f_\alpha\big) \; , \quad g \in GL(n-1, \mathbf{R}) \; ; \tag{3.11}$$

note that though the permutation group representation $\tilde{D}(\pi)$, $\pi \in S_n$ [Eq. (3.5)] is a (nonorthogonal) representation of S_n, the action of S_n on wave functions φ is unitary as seen in Eq. (3.11) with $g = \tilde{D}(\pi)$.

The Lie algebra of $Sp(2(n-1), \mathbf{R})$ is more evident if the creation and annihilation operators

$$\vec{c}_\alpha^\dagger := \frac{1}{\sqrt{2}} \left(\vec{k}_\alpha - \frac{\partial}{\partial \vec{k}_\alpha} \right)$$

$$\vec{c}_\alpha := \frac{1}{\sqrt{2}} \left(\vec{k}_\alpha + \frac{\partial}{\partial \vec{k}_\alpha} \right) \qquad (3.12)$$

replace the $\vec{k}_\alpha$ and $\partial/\partial \vec{k}_\alpha$ operators. Define

$$X^0_{\alpha\beta} := \vec{c}_\alpha^\dagger \cdot \vec{c}_\beta = \frac{1}{2} \left(\vec{k}_\alpha - \frac{\partial}{\partial \vec{k}_\alpha} \right) \cdot \left(\vec{k}_\beta + \frac{\partial}{\partial \vec{k}_\beta} \right)$$

$$X^+_{\alpha\beta} := \vec{c}_\alpha^\dagger \cdot \vec{c}_\beta^\dagger = \frac{1}{2} \left(\vec{k}_\alpha - \frac{\partial}{\partial \vec{k}_\alpha} \right) \cdot \left(\vec{k}_\beta - \frac{\partial}{\partial \vec{k}_\beta} \right)$$

$$X^-_{\alpha\beta} := \vec{c}_\alpha \cdot \vec{c}_\beta = \frac{1}{2} \left(\vec{k}_\alpha + \frac{\partial}{\partial \vec{k}_\alpha} \right) \cdot \left(\vec{k}_\beta + \frac{\partial}{\partial \vec{k}_\beta} \right) . \qquad (3.13)$$

Then $X^0_{\alpha\alpha}$ is a harmonic oscillator operator that commutes with $\vec{L}$ and $\vec{S}$, and hence is a possible mass operator; it is not of the form $M_0 + V$, as is the case nonrelativistically, but nevertheless has an equally spaced discrete spectrum.

With this Lie algebra of mass operators and mass operators of the form Eq. (3.9) breaking the degenerate harmonic oscillator levels, it is possible to formulate a relativistic $SU(6)$ model in which the mass operators are not given just in terms of their transformation properties under $SU(6)$, but as actual mass operators as defined in Eq. (3.3).

4   Relativistic SU(6)

To formulate a relativistic $SU(6)$ theory, it is necessary to pay particular attention to the permutation group properties of the spatial, spin-flavor, and color parts of the overall wave function. A hadronic wave function should be (anti)symmetric under interchange of all constituent particle labels. Though the color degrees of freedom have only been implicitly included in the discussion on internal symmetries, we assume that the color part of a hadronic wave function must be a color singlet under $SU(3)_c$ with a definite permutation symmetry. The possible permutation symmetries for n-body color singlets, labeled by the Young diagram Y_c are given in Ref. [8].

Wave functions in the n-constituent particle Hilbert space can thus be written as $\varphi(v_0; \vec{k}_\alpha, \mu_\alpha f_\alpha, c_\alpha)$, where v_0 is the overall four velocity of the n-constituents, $\vec{k}_\alpha$, $\alpha = 1 \ldots n - 1$ are the internal momenta, $\mu_\alpha f_\alpha$, $\alpha = 1 \ldots n$ the spin and flavor labels transforming under $SU(6)$ transformations, and c_α the color label transforming under $SU(3)$ transformations. Under a Lorentz transformation, v_0 goes to Λv_0, $\vec{k}_\alpha \rightarrow R_w \vec{k}_\alpha$ and $\mu_\alpha \rightarrow \mu'_\alpha$, as seen in Eq. (2.8); thus μ_α, the internal spin label transforms differently under Lorentz and $SU(6)$ transformations, a property which can be used to generate mass splittings for different spin particles with $SU(6)$ multiplets.

We now wish to compute relativistic harmonic oscillator wave functions with the appropriate spin-flavor and color symmetry:

$$|v; \quad \underbrace{NY_s\ell m_\ell}_{Sp(2(n-1),\mathbf{R})\times 0(3)} \quad ; \quad \underbrace{\chi_{SU(6)}Y_f, \chi_{SU(3)}f, ss_3}_{SU(6)\supset SU(3)_f \times SU(2)} \quad ; \quad \underbrace{1Y_c)}_{SU(3)_c}$$

$$= \delta^3(v - v_0)\varphi_{NY_s\ell m_\ell}(\vec{k}_\alpha)\varphi_{\chi_{SU(6)}Y_f \chi_{SU(3)}f ss_3}(\mu_\alpha f_\alpha)\varphi_{1Y_c}(c_\alpha) \qquad (4.1)$$

where v is the four velocity of the hadron, N is the harmonic oscillator eigenvalue label, Y_s is the Young diagram giving the spatial permutation symmetry, and ℓ, m_ℓ are the orbital and orbital projection quantum numbers. Similarly $\chi_{SU(6)}$ are the $SU(6)$ multiplet labels, with basis labels including the flavor $(\chi_{SU(3)}, f)$ and spin (s, s_3) labels. Y_f is the Young diagram giving the spin-flavor permutation symmetry. Finally, "1" designates an $SU(3)$ color singlet, and Y_c is the color permutation symmetry. To obtain an overall antisymmetric (for baryons) or symmetric (for mesons) wave function, the permutation types must be coupled together, $Y_s \otimes Y_f \otimes Y_c \rightarrow A$ (baryons) or S (mesons). Once these wave functions are known, mass operators arising from spin-orbit, spin-spin, tensor and $SU(6)$ type forces of the kind discussed in Section 3 can be introduced to split the degenerate harmonic oscillator mass spectrum.

Though the wave functions described in Eq. (4.1) may seem complicated, we want to show that they can be readily computed when realized as polynomials in Bargmann spaces. Reference [8] shows how to realize the spin-flavor and color parts of the wave function as polynomials in Bargmann spaces. Here we show how to realize harmonic oscillator wave functions as polynomials in a Bargmann space.

The holomorphic Hilbert (or Bargmann) space $\mathcal{B}(\mathbf{C}_{n-1\times 3})$ needed for the spatial part of the wave function consists of holomorphic functions $F(z)$ in $n-1\times 3$ complex variables, $z \in \mathbf{C}_{n-1\times 3}$, with the norm given by

$$\|F\|^2 = F\left(\frac{\partial}{\partial z}\right)\overline{F(\bar{z})}|_{z=0} , \quad F \in \mathcal{B}(\mathbf{C}_{n-1\times 3}) , \qquad (4.2)$$

where $F(\partial/\partial z)$ means replacing the entries in $F(z)$ by the differential operators $\partial/\partial z$. Creation and annihilation operators are particularly simple, in that

$$c_{\alpha i}^\dagger = z_{\alpha i} \qquad , \quad \begin{array}{l} \alpha = 1\cdots n-1 \\ i = 1,2,3 \end{array}$$

$$c_{\alpha i} = \frac{\partial}{\partial z_{\alpha i}} \qquad\qquad (4.3)$$

$$[c_{\alpha i}, c_{\beta j}^\dagger] = \delta_{\alpha\beta}\delta_{ij} .$$

$\mathcal{B}(\mathbf{C}_{n-1\times 3})$ is isomorphic to the Hilbert space of internal momenta $\vec{k}_\alpha$ [see Eq. (3.4)] and the creation and annihilation operators, Eq. (3.12), can be used to transform the polynomial harmonic oscillator wave functions to wave functions in the internal momenta; examples will be given at the end of this section.

There are two natural group actions on elements of $\mathcal{B}$ that will be needed for permutation group and orbital angular momentum operators. Write

$$(R_g F)(z) := F(zg) \quad , \quad g \in U(3) \supset SO(3)$$

$$(L_h F)(z) := F(h^{-1}z) \, , \quad h \in U(n-1)$$

$$[R_g, L_h] = 0 \, . \tag{4.4}$$

That is, $g \in U(3)$ restricted to elements of $SO(3)$ gives the orbital angular momentum operators. Infinitesimal operators coming from L_h give the harmonic oscillator operators defined in Eq. (3.13):

$$X_{\alpha\beta}^0 = \sum_{i=1}^{3} c_{\alpha i}^\dagger c_{\beta i} = \sum_{i=1}^{3} z_{\alpha i} \frac{\partial}{\partial z_{\beta i}} \, , \tag{4.5}$$

which along with the other two sets of operators,

$$X_{\alpha\beta}^+ = \sum_{i=1}^{3} c_{\alpha i}^\dagger c_{\beta i}^\dagger = \sum_{i=1}^{3} z_{\alpha i} z_{\beta i}$$

$$X_{\alpha\beta}^- = \sum_{i=1}^{3} c_{\alpha i} c_{\beta i} = \sum_{i=1}^{3} \frac{\partial}{\partial z_{\alpha i}} \frac{\partial}{\partial z_{\beta i}} \, , \tag{4.6}$$

give the Lie algebra action of $Sp(2(n-1), \mathbf{R})$ on $\mathcal{B}$, and commute with $R_g, g \in SO(3)$, Eq. (4.4).

It is convenient to transform from a Cartesian basis, with $i = 1, 2, 3$ to a spherical basis with $\mu = \pm 1, 0$. The transformation is

$$c_{\alpha\pm} = \frac{1}{\sqrt{2}} \left(c_{\alpha 1} \mp i c_{\alpha 2} \right) \, , \quad c_{\alpha 0} = c_{\alpha 3}$$

$$c_{\alpha\pm}^\dagger = \frac{1}{\sqrt{2}} \left(c_{\alpha 1}^\dagger \pm i c_{\alpha 2}^\dagger \right) \, , \quad c_{\alpha 0}^\dagger = c_{\alpha 3}^\dagger \, . \tag{4.7}$$

Then

$$X_{\alpha\beta}^0 = c_{\alpha+}^\dagger c_{\beta+} + c_{\alpha-}^\dagger c_{\beta-} + c_{\alpha 0}^\dagger c_{\beta 0}$$

$$X_{\alpha\beta}^+ = c_{\alpha+}^\dagger c_{\beta-}^\dagger + c_{\alpha-}^\dagger c_{\beta+}^\dagger + c_{\alpha 0}^\dagger c_{\beta 0}^\dagger$$

$$X_{\alpha\beta}^- = c_{\alpha+} c_{\beta-} + c_{\alpha-} c_{\beta+} + c_{\alpha 0} c_{\beta 0} \, . \tag{4.8}$$

In the spherical basis the orbital angular momentum operators are

$$L_0 = \sum_{\alpha=1}^{n-1} \left(c_{\alpha+}^\dagger c_{\alpha+} - c_{\alpha-}^\dagger c_{\alpha-} \right)$$

$$L_+ = \sqrt{2} \sum_{\alpha=1}^{n-1} \left(c_{\alpha+}^\dagger c_{\alpha 0} - c_{\alpha 0}^\dagger c_{\alpha-} \right)$$

$$L_- = (L_+)^\dagger . \tag{4.9}$$

We now want to construct a relativistic harmonic oscillator mass operator out of the operators in Eq. (4.8) that commutes with the permutation group S_n and the orbital angular momentum operators, Eq. (4.9), for then spatial wave functions will be polynomials in z labeled by N, Y_s, ℓ, and m_ℓ, $p_{|NY_s \ell m_\ell}(z)$, as required from Eq. (4.1). By construction the $X_{\alpha\beta}^0$ commute with the angular momentum operators, Eq. (4.9); we now show that the symmetric group action is a subgroup of $U(n-1)$, so that if the harmonic oscillator mass operator M_{H0} is chosen to be

$$M_{H0} = m X^0$$

$$= m \sum_{\alpha=1}^{n-1} X_{\alpha\alpha}^0 , \tag{4.10}$$

it will automatically commute with S_n. The factor m in Eq. (4.10) is a constant having the dimensions of mass, and sets the mass scale for the hadronic mass spectrum.

As shown in Eq. (3.5), the action of permutation group elements $\pi \in S_n$ on internal momentum vectors $\vec{k}_\alpha$, $\alpha = 1 \ldots n-1$ results in nonorthogonal $n-1$ dimensional representation matrices; nevertheless, as can be ascertained by taking traces of these matrices and using character formulae [9], the S_n representation matrices are irreducible, with Young tableau $(n-1,1)$. The corresponding orthogonal matrices will be denoted by $D(\pi)$, so that

$$\pi \in S_n \to D(\pi) \subset 0(n-1) \subset U(n-1)$$

$$(L_\pi F)(z) = F\left(D^{-1}(\pi)z\right) , \quad F \in \mathcal{B}(\mathbf{C}_{n-1\times 3}) . \tag{4.11}$$

That is, the orthogonal representation matrices of dimension $n-1$ of the group S_n act on elements F in $\mathcal{B}$ via the $U(n-1)$ action defined in Eq. (4.4). Since the L_π action is generated by orthogonal matrices $D(\pi)$, L_π will not only commute with $X^0 = \sum_\alpha X_{\alpha\alpha}^0$, but with $X^+ := \sum_\alpha X_{\alpha\alpha}^+$, and $X^- := \sum_\alpha X_{\alpha\alpha}^-$. Thus, we have $Sp(2,\mathbf{R}) \times S_n$ embedded in $Sp(2(n-1),\mathbf{R})$ in which each eigenvalue of X^0 carries a definite permutation symmetry and $X^\pm$ raise and lower the polynomial eigenfunctions of X^0.

X^- acts as a lowering operator on X^0 eigenfunctions. The simplest polynomial corresponding to $N = 0$, $Y_s = S$, and $\ell = 0$ is $p(z) = 1$:

$$P_{|N=0,Y_s=S,\ell=0)}(z) = 1 (= |0\rangle) . \tag{4.12}$$

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There is then a tower of eigenstates generated by the raising operator $X^+ = \sum_{\alpha,\mu} z_{\alpha\mu} z_{\alpha\mu}$:

$$P_{|2N,S,\ell=0\rangle}(z) = (X^+)^N |0\rangle$$

$$= \left(\sum_{\alpha,\mu} z_{\alpha\mu} z_{\alpha\mu} \right)^N . \qquad (4.13)$$

Similarly there is a tower of $\ell = 1$ states, starting with $N = 1$, given by the polynomials $z_{\alpha\mu}$, with $z_{\alpha+}$ the polynomials with $L_3 = +1$. At higher levels in the angular momentum towers, states cannot be uniquely labeled by Y_s; additional operators commuting with X^0 and S_n must be introduced; the construction of these operators is given in Ref. [10].

Constituent quark models assume that baryons are bound states of three quarks. To conclude this section we exhibit polynomials for baryons consisting of three constituents. The relevant permutation group is S_3, and the representation matrices $D(\pi)$ are given on page 224 of Ref. [9]. There are three types of irreducible representations, $Y_s = S$ (symmetric), A (antisymmetric) or M (mixed, two dimensional). We list here some low N polynomials for $\ell = 0, 1, 2$ and $L_3 = \ell$. (The other angular momenta can be obtained from the lowering operator L_-, Eq. (4.9), which means differentiating the given polynomials in a prescribed way):

$$
\begin{aligned}
P_{|N,Y_s,\ell,\ell\rangle}(z) \qquad\qquad &= c^\dagger \cdots c^\dagger |0\rangle \\
|0,S,0,0\rangle = 1 \qquad\qquad &= |0\rangle \\
|2,S,0,0\rangle = \tfrac{1}{\sqrt{12}}\left(\sum_\mu z_{1\mu}^2 + \sum_\mu z_{2\mu}^2 \right) &= \tfrac{1}{\sqrt{12}} \sum_{\alpha,\mu} c_{\alpha\mu}^\dagger c_{\alpha\mu}^\dagger |0\rangle
\end{aligned}
$$

$$
|2,M,0,0\rangle = \begin{cases} \tfrac{1}{\sqrt{12}}\left(\sum_\mu z_{1\mu}^2 - \sum_\mu z_{2\mu}^2 \right) = \tfrac{1}{\sqrt{12}}\left(\sum_\mu c_{1\mu}^{2\dagger} - c_{2\mu}^{2\dagger} \right)|0\rangle \\[2mm] \tfrac{1}{\sqrt{3}} \sum_\mu z_{1\mu} z_{2\mu} \qquad\qquad = \tfrac{1}{\sqrt{3}} \sum c_{1\mu}^\dagger c_{2\mu}^\dagger |0\rangle \end{cases}
$$

$$
|1,M,1,1\rangle = \begin{cases} z_{1+} \qquad\qquad\qquad\quad = c_{1+}^\dagger |0\rangle \\[2mm] z_{2+} \qquad\qquad\qquad\quad = c_{2+}^\dagger |0\rangle \end{cases}
$$

$$
|2,A,1,1\rangle = \tfrac{1}{\sqrt{2}}\left(z_{10} z_{2+} - z_{1+} z_{20} \right) = \tfrac{1}{\sqrt{2}}\left(c_{10}^\dagger c_{2+}^\dagger - c_{1+}^\dagger c_{20}^\dagger \right)|0\rangle
$$

$$
|2,S,2,2\rangle = \tfrac{1}{2}\left(z_{1+}^2 + z_{2+}^2 \right) = \tfrac{1}{2}\left(c_{1+}^{\dagger 2} + c_{2+}^{\dagger 2} \right)|0\rangle
$$

$$
|2,M,2,2\rangle = \begin{cases} \tfrac{1}{2}\left(z_{1+}^2 - z_{2+}^2 \right) = \tfrac{1}{2}\left(c_{1+}^{\dagger 2} - c_{2+}^{\dagger 2} \right)|0\rangle \\[2mm] \left(z_{1+} z_{2+} \right) \qquad = c_{1+}^\dagger c_{2+}^\dagger |0\rangle \end{cases} \qquad (4.14)
$$

The coefficients appearing in front of the harmonic oscillator polynomials normalize the polynomials to one; these factors are easily computed using the differentiation inner product, Eq. (4.2). Moreover the polynomial eigenfunctions are easily transformed to harmonic oscillator wave

functions in internal momentum variables. In this case the vacuum state $|0\rangle$ is realized as $e^{-\frac{1}{2}\sum_{\alpha,\mu} k^2_{\alpha\mu}}$ and the creation operators in the right-hand column of Eq. (4.14) are given in Eq. (3.12).

When the spatial polynomial wave functions with permutation symmetry Y_s are combined with the spin-flavor and color (for which $Y_c = A$) wave functions, the resulting symmetry type must be antisymmetric. For a given Y_s this fixes Y_f, namely

Y_s	Y_f	dim $SU(6)$
M	M	70
S	S	56
A	A	20

5 Conclusion

We have shown how to construct a relativistic quantum mechanics using Dirac's "point form," in which Lorentz transformations are kinematic and interactions appear in the mass operator. The four-momentum operator is then the product of the mass operator and the four-velocity operator. For eigenstates of the four-velocity operator, mass operators are rotationally invariant self-adjoint operators. Mass operators corresponding to spin-orbit, spin-spin, and tensor forces are readily constructed because the internal coordinates of velocity states transform like nonrelativistic coordinates. Nevertheless, the theory is covariant in that four vectors transform in the usual way under the kinematic Lorentz group. A modified Pauli-Lubanski operator, in which the four-velocity operator replaces the four-momentum operator, when dotted into appropriate four vectors, forms a relativistic $SU(2)$ spin algebra. The eigenvalue of the spin Casimir operator is $j(j + 1)$. Combining this $SU(2)$ algebra with an internal symmetry into a larger symmetry produces mixing between spin and internal symmetry quantum numbers in a relativistically invariant way.

When the internal symmetry is $SU(3)$ flavor, and the spin of the constituents is $\frac{1}{2}$, the result is a relativistic $SU(6)$ theory. In such a theory there are many ways of choosing mass operators (such as QCD inspired mass operators), but the simplest choice is a harmonic oscillator mass operator with equally spaced mass eigenvalues. Such a mass operator is not constructed like its nonrelativistic counterpart, with r^2 potentials between each of the constituents, but rather is constructed algebraically using a symplectic algebra. By using Bargmann spaces it is possible to realize the harmonic oscillator wave functions as polynomials with definite permutation properties. Moreover, the harmonic oscillator mass operator can be modified without changing the polynomial eigenfunctions by adding on the operator X^+X^-, in which case the eigenvalues $N = 0, 1, 2, \ldots$ become $(N - \ell)(N + 3\ell + 1)$, where ℓ is the orbital angular momentum.

Mass operators can also be formed out of $SU(6)$ generators, which then give Gürsey-Radicati type mass formulae [11]. By adding such mass operators to spin-orbit or tensor mass operators, it should be possible to reproduce the observed baryon mass spectrum. And if constituents and their antiparticles are combined into a larger internal symmetry, it should also be possible to fit the meson spectrum, as well as the spectrum of some of the low-mass nuclei.

Once realistic relativistic wave functions for mesons and baryons are available, it should be possible to compute form factors, structure functions, decays, and the like for hadrons viewed as bound states of spin $\frac{1}{2}$ constituents. In a succeeding paper [12] we show how to formulate a point form relativistic quantum mechanical impulse approximation, wherein the electromagnetic properties of the hadrons are determined by the electromagnetic properties of their constituents.

It is possible to generalize the relativistic $SU(6)$ theory to a Fock space theory, where the Fock space is formed by taking the direct sum of the n-constituent Hilbert spaces discussed in this paper from n equals zero to infinity. Such a Fock space is the appropriate space on which to compute decays of excited baryons, such as the $\Delta \to \pi + N$ decay which was forbidden in the old $SU(6)$ theory. Finally, we mention that mass operators need not commute with the number operator; for such mass operators hadrons consist of a direct sum of an indefinite number of constituents and correspond to the current quarks in QCD, in contrast to constituent quarks.

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PHASE SPACE LOCALIZATION FOR ANTI-DE SITTER QUANTUM MECHANICS AND ITS ZERO CURVATURE LIMIT

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Abstract

Using techniques of geometric quantization and $SO_0(3,2)$-coherent states, a notion of optimal localization on phase space is defined for the quantum theory of a massive and spinning particle in anti-de Sitter spacetime. We show that this notion disappears in the zero curvature limit, providing one with a concrete example of the regularizing character of the constant (nonzero) curvature of the anti-de Sitter spacetime. As a byproduct a geometric characterization of masslessness is obtained.

The present contribution is based on a joint work with Stephan De Bièvre (see references quoted below).

1 Introduction

It is a well known fact that the Poincaré group, $\mathcal{P}_+^\uparrow(3,1)$, the kinematical group of Minkowski spacetime, can be obtained by means of a contraction from the anti-de Sitter (AdS) group, $SO_0(3,2)$, the kinematical group of anti-de Sitter spacetime. The contraction parameter is the constant positive curvature κ of the anti-de Sitter spacetime. This contraction procedure is thus nothing but a zero curvature limit. According to this fact, one would like to approximate $\mathcal{P}_+^\uparrow(3,1)$-invariant theories by $SO_0(3,2)$-invariant ones, hoping that such approximations give rise to regularized relativistic theories [1] [2]. Indeed, the nonzero curvature equips the AdS theories with a lengthlike parameter, which is actually the source of the sought regularizations.

Up to now, this very stimulating idea has not been fully exploited, though it has received a large amount of attention for its potential implications in the context of quantum field theories. The main drawback of the known approaches arises from the emphasis made on the spacetime or the momentum space realizations of those theories. Indeed, it is a known fact that such realizations, in both Poincaré and AdS cases, lack of a natural notion of localization. Moreover the modulus of the wave functions corresponding to the one particle quantum states of a Poincaré, as well as an AdS, free massive theory can not be interpreted, in those realizations, as a probability distribution. The regularizing role of κ is thus not effective for such realizations.

In this short contribution we propose the phase space realization as the regularizing alternative. In fact, for the case of a free massive spinning particle in AdS spacetime, the phase space is a Kähler $SO_0(3,2)$ homogeneous space, whose (geometric) quantization gives rise to a discrete series representation of $SO_0(3,2)$. The latter is known to be a square integrable representation, so its Hilbert space contains a particular family of quantum states: the coherent states. A natural notion of localization is attached to these states. They are *optimally localized* states in phase space. Moreover the modulus of the wave functions of the quantum states in this realization can be actually interpreted as a probability distribution.

Here we exhibit the explicit form of these coherent states and we show how their physical interpretation arises. We also stress the disappearance of this notion of localization in the flat space limit, confirming the effectiveness of the regularizing character of κ. We proceed as follows. In section 2 we describe the classical theory, in order to fix both the notations and the physical interpretations. In section 3, the quantum theory is obtained through the application of geometric quantization, then the explicit form and the zero curvature limit of the optimally localized states is given. Section 4 contains a brief discussion of a geometric characterization of masslessness as it arises from the description of section 2. For more details we refer to the papers [3], [4], [5] and [6].

2 The classical theory

The phase space description of the classical theory of a spin s and mass $m \neq 0$ free particle in AdS spacetime finds its best formulation within the scheme developped by Souriau [7]. The latter construction starts with the determination of an *evolution space*, $(E_\kappa^{m,s}, \omega_E)$, which is a presymplectic manifold (ω_E is a closed but degenerate 2-form), with a projection on the AdS spacetime of constant curvature κ, M_κ. The symmetries of M_κ are helpful guides in doing so. In fact, M_κ is just the one sheeted hyperboloïd in $(\mathbf{R}^5, \eta)$, whith diag $\eta = (\overset{5\ 0\ 1\ 2\ 3}{--+++})$,

$$y \cdot y \equiv \eta_{\alpha\beta} y^\alpha y^\beta = -(y^5)^2 - (y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 = -\kappa^{-2}, \tag{2.1}$$

$\alpha, \beta \in \{5, 0, 1, 2, 3\}$. Clearly, $O(3,2)$ is the isometry group of (2.1), its connected component to the identity, $SO_0(3,2)$, is the so-called AdS group.

We choose for $E_\kappa^{m,s}$ the $SO_0(3,2)$-principal homogeneous space, $E_\kappa^{m,s} \cong SO_0(3,2)$, realized through the following $SO_0(3,2)$-invariant constraints in $\mathbf{R}^{25}$ (five copies of $(\mathbf{R}^5, \eta)$),

$$y \cdot y = -\kappa^{-2}, \quad q \cdot q = -m^2, \quad u \cdot u = 1, \quad v \cdot v = 1 \quad \text{and} \quad t \cdot t = m^2 s^2, \tag{2.2a}$$

$$y \cdot q = 0 = \text{all the other scalar products} \tag{2.2b}$$

$$\epsilon_{\alpha\beta\gamma\rho\sigma} y^\alpha q^\beta u^\gamma v^\rho t^\sigma = \frac{m^2 s}{\kappa} \quad \text{and} \quad y^5 q^0 - y^0 q^5 > 0. \tag{2.2c}$$

The physical interpretation of the coordinates (y, q, u, v, t) is then as follows: in (2.2a) y is the position on the hyperboloid (2.1), q is its conjugate momentum, t is what we call the AdS-Pauli-Lubanski vector. The remaining five-vectors u and v are introduced in order to have a covariant description of $E_\kappa^{m,s}$, i.e. $E_\kappa^{m,s} \cong SO_0(3,2)$. They shall represent the spin part in the quantum theory. The two last constraints (2.2b-c) are needed in order to fix an orientation.

The choice of ω_E is constrained by the requirement that the projection on M_κ of each integral curve of the completely integrable distribution generated by $\ker \omega_E$ in $E_\kappa^{m,s}$, results in a time-like geodesic of M_κ, i.e. the dynamic of the theory is obtained from $\ker \omega_E$. Such an ω_E is provided by,

$$\omega_E = dy \wedge dq + s\, du \wedge dv. \tag{2.3}$$

This choice is not unique but it fulfils the above dynamic generating requirement. The phase space of the theory, $(\Sigma_\kappa^{m,s}, \omega_\Sigma)$, is obtained by symplectic reduction of $(E_\kappa^{m,s}, \omega_E)$. It appears, for $\frac{m}{\kappa} \neq s$, to be the $SO_0(3,2)$ symplectic homogeneous space $SO_0(3,2)/SO(2) \times SO(2)$. For symmetry reasons, i.e. obvious action of $SO_0(3,2)$ on $E_\kappa^{m,s}$, we use $(E_\kappa^{m,s}, \omega_E)$ as the arena for the forthcoming constructions. The special case $\frac{m}{\kappa} = s$ is discussed in section 4.

In order to carry out the zero curvature limit in a meaningful way, we introduce a new set of coordinates on $E_\kappa^{m,s}$. This is the set of four-vectors (x, p, a, b, s). Interpreted in the same way as the five-vectors (y, q, u, v, t), they are related to the latters through the following equations,

$$y^5 = Y \cos \kappa x^0, \ y^0 = Y \sin \kappa x^0 \quad \text{and} \quad \vec{y} = \vec{x}, \tag{2.4a}$$

where $-\pi \leq \kappa x^0 \leq \pi$, $\vec{x} \in \mathbf{R}^3$ and $Y = \sqrt{\kappa^{-2} + (\vec{x})^2}$; and

$$q \cdot dy = g_{\mu\nu} p^\mu dx^\nu, \ u \cdot dy = g_{\mu\nu} a^\mu dx^\nu, \ v \cdot dy = g_{\mu\nu} b^\mu dx^\nu \quad \text{and} \quad t \cdot dy = g_{\mu\nu} s^\mu dx^\nu. \tag{2.4b}$$

Here $g_{\mu\nu}$ is the metric of M_κ for the global coordinates $(x^0, \vec{x})$ and $\mu, \nu \in \{0,1,2,3\}$. The zero curvature limit of $g_{\mu\nu}$ is just the flat Minkowski metric. The constraints (2.2a-c) translated in terms of the new coordinates become,

$$g_{\mu\nu} p^\mu p^\nu = -m^2, \ g_{\mu\nu} a^\mu a^\nu = 1, \ g_{\mu\nu} b^\mu b^\nu = 1 \quad \text{and} \quad g_{\mu\nu} s^\mu s^\nu = m^2 s^2, \tag{2.5a}$$

$$g_{\mu\nu} p^\mu s^\nu = 0 = \text{all the other scalar products of the subset } (p, a, b, s), \tag{2.5b}$$

$$\epsilon_{\mu\nu\lambda\delta} p^\mu a^\nu b^\lambda s^\delta = m^2 s \quad \text{and} \quad p^0 > 0. \tag{2.5c}$$

The physical interpretation of the above constraints can now be confirmed by their zero curvature limits.

3 The quantum theory and the optimal localization

The methods of geometric quantization allow one to quantize the classical theory described above [8]. In other words, using those methods one is able to construct the unitary irreducible representation of $SO_0(3,2)$ associated to the coadjoint orbit of $SO_0(3,2)$ for which the phase space $\Sigma_\kappa^{m,s}$ is a covering. Exploiting the principal bundle structure $E_\kappa^{m,s} \cong SO_0(3,2) \rightarrow SO_0(3,2)/SO(2) \times SO(2) \cong \Sigma_\kappa^{m,s}$, the prequantum Hilbert space, $\mathcal{H}$, is realized as follows,

$$\mathcal{H} = \left\{ \psi : E_\kappa^{m,s} \longrightarrow \mathbf{C} \ \Big| \ \int_{E_\kappa^{m,s}} |\psi|^2 d\mu_\kappa^{m,s} < \infty, \ Y_{50}\psi = i\frac{m}{\kappa}\psi \ \text{and} \ Y_{12}\psi = is\,\psi \right\}. \tag{3.1}$$

Here $d\mu_\kappa^{m,s}$ is the invariant measure on $E_\kappa^{m,s}$ and Y_{50} and Y_{12} are the left invariant vector fields generating $\ker \omega_E$. Since $E_\kappa^{m,s} \cong SO_0(3,2)$, there exists a natural action of $SO_0(3,2)$ in $L^2(E_\kappa^{m,s}, d\mu_\kappa^{m,s})$. This yields the left regular representation of $SO_0(3,2)$. The latter restricts

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to a unitary (reducible) representation in $\mathcal{H}$, i.e. the representation of $SO_0(3,2)$ induced by the character $e^{i(\frac{m}{\kappa}\tau + s\tau')}$ of $SO(2) \times SO(2)$. Indeed, this holds provided $\frac{m}{\kappa}$ and s are both integers.

There actually exists a positive invariant Kählerian polarization of $\Sigma_\kappa^{m,s}$ allowing one to select in $\mathcal{H}$ an invariant subspace $\mathcal{H}_\kappa^{m,s}$. The restriction of the previous unitary representation to the latter gives rise to a unitary irreducible representation of $SO_0(3,2)$. Concretely,

$$\mathcal{H}_\kappa^{m,s} = \left\{ \psi \in \mathcal{H} \mid \overline{Z}_i \psi = 0, \; i \in \{1,2,3\} \quad \text{et} \quad \overline{\Xi}\psi = 0 \right\}; \tag{3.2}$$

where $Z_i = Y_{0i} + iY_{i5}$, $i \in \{1,2,3\}$ and $\Xi = Y_{23} + iY_{31}$. The $Y_{\alpha\beta}$'s are the left invariant vector fields. The way one obtains the unitary irreducible representation carried by $\mathcal{H}_\kappa^{m,s}$ is known in the mathematic litterature as the holomorphic induction, it yields the disrete series representation of $SO_0(3,2)$ with highest weight $(\frac{m}{\kappa},s)$. (A necessary condition for the unitarity is $\frac{m}{\kappa} > s$.)

The quantum states of the theory are representated by well defined wave functions belonging to $\mathcal{H}_\kappa^{m,s}$. The physical interpretation of their modulus as probability distributions on $\Sigma_\kappa^{m,s}$ is also well defined. The particular states belonging to the orbit, $\mathcal{O}_{\varphi_0} \subset \mathcal{H}_\kappa^{m,s}$, of the unitary representation of $SO_0(3,2)$ passing through the highest weight state φ_0 possess many interesting properties [9]. These states, which are nothing but the generalized coherent states of $SO_0(3,2)$, are in a natural way optimally localized in phase space. In fact, by construction they are labeled by points $w \in E_\kappa^{m,s}$, specifying them through the equations,

$$\langle \varphi_w \mid \hat{L}_{\alpha\beta} \mid \varphi_w \rangle = L_{\alpha\beta}(w), \quad \forall \alpha,\beta \in \{5,0,1,2,3\}; \tag{3.3}$$

here the $L_{\alpha\beta}$'s are the classical observables and the $\hat{L}_{\alpha\beta}$'s are their quantum counterparts. The determination through (3.3) of the ten $L_{\alpha\beta}(w)$ specifies in fact uniquely the leave of the distribution $\ker \omega_E$ passing through w. Thus by symplectic reduction a unique point $\bar{w} \in \Sigma_\kappa^{m,s}$ is specified by (3.3). The state φ_w is then said to be localized in $\bar{w} \in \Sigma_\kappa^{m,s}$. Moreover, since the coherent states minimize the incertainty relations associated to the commutation relations of the $\hat{L}_{\alpha\beta}$'s, this notion of localization is then optimal.

The optimally localized states are given by the following formula,

$$\varphi_{z',\xi'}(z,\xi) = (-2)^{\frac{m}{\kappa}}(2)^{-s}\left(\bar{z}' \cdot z\right)^{-\frac{m}{\kappa}-s}\left[(\bar{z}' \cdot z)(\bar{\xi}' \cdot \xi) - (\bar{z}' \cdot \xi)(\bar{\xi}' \cdot z)\right]^s. \tag{3.4}$$

Here $(z,\xi) \equiv w$ are the complex coordinates of $E_\kappa^{m,s}$ associated to the Kählerian polarization, they are related to the coordinates given in (2.2) through the transformations $z = \kappa y - im^{-1}q$ and $\xi = u - iv$.

The zero curvature limit of these states is as follows,

$$\lim_{\kappa \to 0} \left(\frac{m}{4\pi\kappa}\right)^{\frac{3}{2}} \varphi_{z',\xi'}(z,\xi) = m^2 p^0 \delta(\vec{p} - \vec{p}') e^{-ip_\mu(x'^\mu - x^\mu)} \left(\frac{\bar{\zeta}' \cdot \zeta}{2}\right)^s. \tag{3.5}$$

where $\zeta_\mu = a_\mu - ib_\mu$, $\mu \in \{0,1,2,3\}$. Clearly, these states are no longer optimally localized. They are completely delocalized in position (x), perfectly localized in momentum (p) and still optimally localized in spin (ζ). This zero curvature behaviour supports the regularization argument stressed in the introduction. In fact one can consider the AdS states in (3.4) as regularizations of the (generalized) Poincaré states in (3.5).

4 Remarks on masslessness

When evaluating $\ker \omega_E$ in section 2 two possibilities actually arises. Either $\frac{m}{\kappa} = s$ or $\frac{m}{\kappa} \neq s$. We have dealt here and in [3], [4] and [5] only with the second case, which corresponds to a massive elementary system. In fact, when $\frac{m}{\kappa} \neq s$ dim $\ker \omega_E = 2$ and then dim $\Sigma_\kappa^{m,s} = 8$, since $\Sigma_\kappa^{m,s} \equiv E_\kappa^{m,s} / \ker \omega_E$. This eight dimensional phase space becomes in the zero curvature limit an eight dimensional Poincaré ($\mathcal{P}_+^\uparrow(3,1)$) phase space [4] [5]. It is well known [7] that only the massive ($m \neq 0$) and spinning ($s \neq 0$) free particles on Minkowski spacetime have their dynamics described by an eight dimensional $\mathcal{P}_+^\uparrow(3,1)$-invariant phase space. This confirms the fact that the AdS systems for which $\frac{m}{\kappa} \neq s$ describe massive AdS elementary systems.

The situation is different when $\frac{m}{\kappa} = s$. Actually, in this case dim $\ker \omega_E = 4$ and then dim $\Sigma_\kappa^{m,s} = 6$. The limiting theory is clearly no longer the ($m \neq 0, s \neq 0$) $\mathcal{P}_+^\uparrow(3,1)$-invariant one (the limiting phase space is also six dimensional). The known six dimensional $\mathcal{P}_+^\uparrow(3,1)$-invariant phase spaces are those associated to the ($m = 0, s \neq 0$) and ($m \neq 0, s = 0$) free particles on Minkowski spacetime. Since the contraction $SO_0(3,2) \rightarrow \mathcal{P}_+^\uparrow(3,1)$ preserves the $SO_0(3,1)$ Lorentz subgroup, the spin part is not altered in the zero curvature limit. Hence, the obvious candidate for the limiting theory is the ($m = 0, s \neq 0$) $\mathcal{P}_+^\uparrow(3,1)$-invariant system. Thus, the AdS systems such that $\frac{m}{\kappa} = s$ describe massless AdS elementary systems. A deeper analysis of this phenomenon is addressed elsewhere [6]. There one can find a more rigorous treatment.

5 Acknowledgments

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PROPER-TIME RELATIVISTIC DYNAMICS

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ABSTRACT

Proper-time relativistic single-particle classical Hamiltonian mechanics is formulated using a transformation from observer time to system proper time which is a canonical contact transformation on extended phase space. It is shown that interaction induces a change in the symmetry structure of the system which can be analyzed in terms of a Lie-isotopic deformation of the algebra of observables.

1. INTRODUCTION

We begin with some historical remarks. In the transition from nonrelativistic to relativistic quantum mechanics, the Hamiltonian $H = \dfrac{(\underset{\sim}{P} - \frac{e}{c}\underset{\sim}{A})^2}{2m} + V$ is replaced by $H = [c^2(\underset{\sim}{P} - \frac{e}{c}\underset{\sim}{A})^2 + m^2 c^4]^{1/2} + V$. It was quite natural to expect that the first choice for a relativistic wave equation would be

$$i\hbar\,\frac{\partial\psi}{\partial t}(\underset{\sim}{x},t) = ([c^2(\underset{\sim}{P} - \frac{e}{c}\underset{\sim}{A})^2 + m^2 c^4]^{1/2} + V)\psi(\underset{\sim}{x},t), \qquad (1.1)$$

where $\underset{\sim}{P} = -i\hbar\underset{\sim}{\nabla}$.

In a survey article on relativistic wave equations, Foldy [1] points out that in the absence of interaction, equation 1.1 gives a perfectly good relativistic wave equation for the description of a (spin zero) free particle. When $\overset{\rightharpoonup}{A}$ is not zero, the non—commutativity of $\overset{\rightharpoonup}{P}$ with $\overset{\rightharpoonup}{A}$ appeared to make it impossible to give an unambiguous meaning to the radical operator. Historically, many authors [2] attempted to circumvent this problem by starting with the relationship $(H - V)^2 = m^2 c^4 + C^2 (\overset{\rightharpoonup}{P} - e/c\, A)^2$ which led to the Klein—Gordon equation. The problems with this equation were so great, that all involved became frustrated and it was dropped from serious consideration for a few years. Dirac [3] argued that the proper equation should be first order in both the space and time variables, in order to be a true relativistic wave equation. This lead to the well—known Dirac equation.

In the same paper that Dirac provided the basic ideas which lead to the Feynman integral [4], he noted that "the Hamiltonian method is essentially non—relativistic in form, since it marks out a particular time variable as the canonical conjugate of the Hamiltonian function."

Dirac's position, that the equation should be first order in the space and time variables, emphasizes the relativistic invariance point of view in the merging of special relativity with quantum mechanics. From the quantum mechanical point of view, one could argue that a proper relativistic wave equation would elevate the time coordinate to the same level as the space coordinates, so that all become operators. In the relativistic quantum theory of the present day, the time coordinate does not have equal status with the space coordinates.

The Proper—Time Problem

If one attempts to implement the successful procedures and methods of nonrelativistic quantum mechanics with the special theory of relativity, it is well—known that problems of

physical interpretation appear. The problems are well–known, and discussed by many writers [5]. In order to clearly see one apparent problem, let us note that the three fundamental relationships of classical special relativity:

$$\frac{d\tau}{dt} = (1 - \frac{v^2}{c^2})^{1/2}, \; E = mc^2(1 - \frac{v^2}{c^2})^{-1/2},$$

$$E = (c^2 \underset{\sim}{P}^2 + m^2 c^4)^{1/2},$$

may be uniquely combined to give $\frac{d\tau}{dt} = mc^2(c^2 \underset{\sim}{P}^2 + m^2 c^4)^{-1/2}$. If we now make the transition to quantum mechanics, $\underset{\sim}{P} \rightarrow -i\hbar \underset{\sim}{\nabla}$, we obtain $\frac{d\tau}{dt} = mc^2(-c^2 \hbar^2 \Delta + m^2 c^4)^{-1/2}$. This result is consistent with quantum mechanics but is inconsistent with the many attempts [5] to treat proper time as a parameter.

The Third Postulate Problem

The two postulates of special relativity are:

1. The physical laws of nature and the results of all experiments are independent of the inertial frame of the observer.

2. The speed of light is independent of the motion of the source.

The first postulate abandons the notion of absolute space, while the second postulate abandons the concept of absolute time. It is of interest to note that another postulate is:

3. The correct implementation of postulates 1 and 2 is to require that time be represented as a fourth coordinate (Minkowski space) and to require that the relativistic laws of physics be invariant or covariant under Lorentz transformations.

This third postulate was proposed by H. Minkowski, a well–known mathematician in the early part of the 20th century. Most of the physics community of the time did not accept it,

regarding it as a mathematical obstruction without physical content.

The inability to obtain an alternate approach dictated by physical considerations forced acceptance of the current implementation. Although the second postulate eliminated absolute time, the transformation theory associated with postulate 3 revealed a new unique time variable associated with the observed system, its proper time. The purpose of the present paper is to show how the use of this variable in place of the observer time variable leads to a conceptually (and technically) much simpler implementation of the special theory of relativity. To be sure, the use of this variable is not new. However, we treat the transformation from observer time to system proper time as a canonical contact transformation on extended phase space. This approach forces the identification of the canonical Hamiltonian which generates the Lie Algebra bracket. The problem of interaction is discussed for two–particle momentum – independent potentials. These include, of course, the important case of the relativistic harmonic oscillator. We confine our study to the single–particle classical theory. The many–particle classical theory and the quantum case will be explored elsewhere.

In Section 2 we formulate proper–time Hamiltonian dynamics for a single classical massive particle and discuss some properties of the group of proper–time transformations on extended phase space. Section 3 is devoted to the discussion of the case of particle interaction for two–body potentials independent of the particle momenta, and Section 4 contains some concluding remarks.

2. SINGLE–PARTICLE FORMULATION

The dynamics of a classical observable can be conveniently studied by Hamiltonian mechanics using the Poisson bracket $\{A(p,q), B(P,q)\} = \frac{\partial A}{\partial P}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial P}$. The Hamilton equations ensure that the time development of an arbitrary classical function $W(q,p,t)$ is given by

$$\frac{dW}{dt}(q,P,t) = \{H, W(q,P,t)\} + \frac{\partial W}{\partial t}(q,P,t).$$

Defining the proper time τ by $dt = \frac{H}{mc^2}\, d\tau$, the proper–time evolution of the function

W is given by the chain rule:

$$\frac{dW}{d\tau} = \frac{dW}{dt}\frac{dt}{d\tau} = \frac{H}{mc^2}\{H,W\} + \frac{\partial W}{\partial t}. \tag{2.1}$$

An energy functional K which is conjugate to the proper–time τ will be defined by

$\{K,W\} = \frac{H}{mc^2}\{H,W\}$ with $K = mc^2$ when $H = mc^2$. If the mass m remains invariant during

the evolution, this functional can be directly determined to be

$$K = \frac{H^2}{2mc^2} + \frac{mc^2}{2}, \tag{2.2}$$

and the evolution of the function W in terms of τ can be expressed as

$\frac{dW}{d\tau} = \{K,W\} + \frac{\partial W}{\partial \tau}.$

Consider the behavior of a single noninteracting particle of mass m, with momentum p

as measured in some inertial frame. The usual form of the Hamiltonian representing this

system is $H = \sqrt{c^2P^2 + (mc^2)^2}$. For this example, the conjugate proper energy is given by

$K = \frac{P^2}{2m} + mc^2$. Several interesting points should be noted:

a. The functional form of the energy K is the same as that of the nonrelativistic energy of the system, even though the system is fully relativistic.

b. The momentum parameter in the functional form of the energy K is the momentum as measured in the original inertial frame, not the proper frame of the particle (which of course would measure zero momentum). This emphasizes the form of the transformation as a canonical time transformation, rather than as a Lorentz transformation.

c. If the particle were to interact with external influences, the proper frame would not be an inertial frame, but the proper time is always defined.

Transformation Group

We noted earlier that the proper time is invariant for all inertial observers. However, different observers will use different Hamiltonians to describe the phase flow of the system. In order to relate the phase flows for different inertial observers, we note that the proper–time transformations form a subgroup of the full group of transformations on the extended phase space which, since they do not transform the time, include the group of symplectic diffeomorphisms.

Consider two inertial observers in frames X, X' with extended phase space coordinates (p,q,t), (p',q',t') respectively. Let L denote the set of Lorentz transformations on space–time reference frames, $L(X,X')$: $X \to X'$, and denote by T the set of canonical proper–time transformations defined on extended phase space. We denote the map $(P,q,t) \longmapsto (P,q,\tau)$ by $T(q,t,\tau)$.

THEOREM. The proper–time coordinates on X are related to those on X' by the transformation:

$$S_m(q',q,\tau) = T(q',t',\tau) \, L_m(X,X') T^{-1}(q,t,\tau).$$

<u>Proof</u>. The proof follows from the commutativity of the following diagram.

$$
\begin{array}{ccc}
X(q,P,t) & \xrightarrow{\;\;L\;\;} & X'(q',P',t') \\
{}_{\sim\;\sim} & & \\
T_X \downarrow & & \downarrow T_{X'} \\
(q,P,\tau) & \text{- - - - -} \xrightarrow{\;\;S\;\;} & (q',P',\tau)
\end{array}
$$

It is easy to prove that, for each fixed system, the set of proper—time transformations between inertial observers is a group which relates the dynamics as viewed by one observer to the dynamics as viewed by any other observer.

We have used the particle mass in the statement of the above theorem to fix the observed system. The group of proper—time transformations depends on 14 parameters (m,P,q,P',q',τ). It follows that the free—particle laws will be the same for all inertial observers and will be form invariant under a similarity group action on the Lorentz group.

<u>COROLLARY</u>. <u>There exist Poincare transformations that preserve the time coordinate</u>.
<u>Proof</u>. We note that, in the proof of the above theorem, both (q,P,τ) and (q',P',τ) are inertial frames in the free—particle case.

Lie—Isotopic Algebras

Prior to studying the case of interactions, we introduce the essential ideas concerning Lie—isotopes and their properties. For a complete review of these objects, we refer to [6]. Let G denote a given Lie algebra with bracket $[A,B] = AB - BA$ and let T be an invertible element in G. A Lie—isotope of G is then defined as G with the bracket $[A,B]^* = A*B - B*A \equiv ATB - BTA$. It is easy to show that $[\,,\,]^*$ is a Lie bracket and that

(G, [,]*) is a Lie algebra. It turns out that two nonisomorphic groups may have isotopic Lie algebras. The standard example concerns the groups SO(3), SO(2,1). These are symmetry groups for the following respective Hamiltonians:

$$H_1(\underline{q},\underline{P}) = \tfrac{1}{2}(P_1^2 + P_2^2 + P_3^2) + \tfrac{1}{2}(q_1^2 + q_2^2 + q_3^2),$$

$$H_2(\underline{q},\underline{P}) = \tfrac{1}{2}(P_1^2 - P_2^2 + P_3^2) + \tfrac{1}{2}(q_1^2 - q_2^2 + q_3^2).$$

These Hamiltonians lead to the same equations of motion and to the same conservation laws (via Noether's theorem) for the components of angular momentum $L_b(b = 1,2,3)$. Using the notation $A\overset{\leftarrow\rightarrow}{T}B = ATB - BTA$, we have

$$[L_b,L_c] = \frac{\partial L_b}{\partial q_i}\overset{\leftarrow\rightarrow}{\delta_j^i}\frac{\partial L_c}{\partial p_j}, \quad [\delta_j^i] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$[L_b,L_c]^* = \frac{\partial L_b}{\partial q_i}\overset{\leftarrow\rightarrow}{\alpha_j^i}\frac{\partial L_c}{\partial p_j}, \quad [\alpha_j^i] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for the respective Lie algebras of the groups SO(3) and SO(2,1). In the latter case we have $T = [\alpha_j^i] = T^{-1}$.

In order to understand the requirement that T be invertible, recall that the group SO(3) leaves the standard inner product $<\underline{a},\underline{b}>_3 = a_1b_1 + a_2b_2 + a_3b_3$ invariant while the group SO(2,1) leaves $<\underline{a},\underline{b}>_{2,1} = a_1b_1 - a_2b_2 + a_3b_3$ invariant. We can write

$<\underline{a},\underline{b}>_3 = (u\underline{a})^t I(u\underline{b}) = (\underline{a})^t I(\underline{b}) = (\underline{a})^t (\underline{b})$ so that $u^t I u = I$ if $u^t = u^{-1}$, $u \in$ SO(3); while

$<\underline{a},\underline{b}>_{2,1} = (\bar{u}\underline{a})^t \bar{I}(\bar{u}\underline{b}) = (\underline{a})^t \bar{I}(\underline{b})$ if $\bar{u}^t \bar{I}\bar{u} = \bar{I}$ for $\bar{u} \in$ SO(2,1) with $\bar{I} = T^{-1}$.

3. Interaction

The question of where to put the potential energy was essentially resolved when it was found to fit perfectly as the scalar component of a four–vector. Since this point of view is being questioned in our approach, we must revisit this issue.

Consider the following Hamiltonians:

Case 1. $H = [c^2 \underset{\sim}{P}^2 + (mc^2 + V)^2]^{1/2}$,

Case 2. $H = [c^2 \underset{\sim}{P}^2 + m^2c^4]^{1/2} + V$,

corresponding to two different ways of describing particle interactions. Here, V is assumed to be independent of the momenta.

In case 1 we obtain

$$\frac{d\underset{\sim}{q}}{dt} = \frac{c^2}{H} \underset{\sim}{P}, \frac{dt}{d\tau} = \frac{H}{mc^2 + V},$$

$$\frac{d}{dt} \underset{\sim}{P} = \frac{(mc^2 + H)}{H} (-\underset{\sim}{\nabla} V),$$

so that

$$\frac{d\underset{\sim}{q}}{d\tau} = (m + \frac{V}{c^2})^{-1} \underset{\sim}{P} \tag{3.1}$$

and $\frac{d\underset{\sim}{P}}{d\tau} = -\underset{\sim}{\nabla} V$. We note from (3.1) that, when $V \ll mc^2$,

$$\frac{d\underset{\sim}{q}}{d\tau} = \frac{\underset{\sim}{P}}{m}, \tag{3.2}$$

the corresponding nonrelativistic form relative to the time τ.

We take K as in (2.2) so that, by an analogue of (2.1), we have

$\frac{dW}{d\tau} = \frac{H}{mc^2 + V} \{H,W\}$ or

$$\frac{dW}{d\tau} = \frac{H}{mc^2}[\frac{\partial H}{\partial \underset{\sim}{P}} (1 + \frac{V}{mc^2})^{-1} \frac{\partial W}{\partial \underset{\sim}{q}} - \frac{\partial H}{\partial \underset{\sim}{q}} (1 + \frac{V}{mc^2})^{-1} \frac{\partial W}{\partial \underset{\sim}{P}}]. \tag{3.3}$$

Thus, if we set $T = (1 + \dfrac{V}{mc^2})^{-1}$, which we note is comparable to unity in the nonrelativistic regime (3.2), we obtain

$$\frac{dW}{d\tau} = \frac{\partial K}{\partial \underset{\sim}{P}} T \frac{\partial W}{\partial \underset{\sim}{q}} - \frac{\partial K}{\partial \underset{\sim}{q}} T \frac{\partial W}{\partial \underset{\sim}{P}} = \{K,W\}^*$$

demonstrating that the proper–time dynamics is described by an isotopic Lie algebra. We infer from the above discussion that *the interaction induces a change in the symmetry structure of the system.*

We can formalize this result as follows. Define $\bar{I} = T^{-1}$ and replace the complex number field C by $\bar{C} = \{c\bar{I}: c \; \epsilon \; C\}$, so that $\bar{C}$ is an example of an *isofield* for which $\bar{I}$ is the unit [6]. For example, the multiplication of two isonumbers is defined as

$$\bar{c} * \bar{b} = (c\bar{I}) T \, (b\bar{I}) = cb\bar{I} = (cb)^- \text{ for c,b } \epsilon \; C.$$

In a similar manner, a Lie algebra G can be "deformed" to obtain a Lie–isotope of G as discussed in Section 2.

For case 2 we obtain

$$\frac{d\underset{\sim}{q}}{dt} = \frac{c^2 \underset{\sim}{P}}{H - V} \, , \frac{d\underset{\sim}{P}}{dt} = -\underset{\sim}{\nabla} V,$$

$$\frac{dt}{d\tau} = \frac{H - V}{mc^2} = \frac{H}{mc^2}(1 - \frac{V}{H}),$$

$$\frac{d\underset{\sim}{q}}{d\tau} = \frac{\underset{\sim}{P}}{m} \, , \frac{d\underset{\sim}{P}}{d\tau} = (\frac{H - V}{mc^2}) \, (-\underset{\sim}{\nabla} V),$$

and the analogue of (3.3):

$$\frac{dW}{d\tau} = \frac{H}{mc^2}[\frac{\partial H}{\partial \underset{\sim}{P}}(1 - \frac{V}{H})\frac{\partial W}{\partial q} - \frac{\partial H}{\partial q}(1 - \frac{V}{H})\frac{\partial W}{\partial \underset{\sim}{P}}].$$

In the present case we set $T = 1 - \frac{V}{H}$ and $\bar{I} = T^{-1}$ so that $T = T^{-1} \cong I$ in the region $V \ll H$. The operator K is again given by (2.4), and we find

$$K = \frac{\underset{\sim}{P}^2}{2m} + mc^2 + V\sqrt{1 + (\frac{\underset{\sim}{P}}{mc})^2} + \frac{V^2}{2mc^2}. \tag{3.4}$$

For purposes of comparison, we note that for case 1 we obtain from (2.4):

$$K = \frac{\underset{\sim}{P}^2}{2m} + mc^2 + V + \frac{V^2}{2mc^2}. \tag{3.5}$$

We note that the two Hamiltonians (3.4), (3.5) agree in the nonrelativistic limit but differ from each other in the ultrarelativistic regime.

4. CONCLUDING REMARKS

We have discussed a formulation of single–particle classical relativistic Hamiltonian mechanics in terms of a proper–time implementation of special relativity using a transformation from observer time to system proper–time which is a canonical contact transformation on extended phase space. The problem of interaction was investigated for two–body potentials independent of the particle momenta. It was shown that the interaction induces a change in the symmetry structure of the system which can be analyzed in terms of a Lie–isotopic deformation of the (Lie) algebra of observables.

In both cases considered in Section 3, the total energy of the system is conserved. In the first case we find an easy physical interpretation; viz., the particle is interacting with a comoving force. The second case does not seem to have a simple interpretation. We infer

from it the possibility that the particle can tell the difference between a change in mass at each point and an external comoving force which does not depend on its clock. We believe that our approach makes the four—vector concept unnecessary and solves the interpretational problems associated with the second case.

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REMARKS ABOUT MASSIVE AND MASSLESS PARTICLES IN SUPERSYMMETRY

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Abstract

The internal space-time symmetry and simple supersymmetry of relativistic particles are briefly discussed in terms of the little group of the Poincaré group. The little group generators in a finite-dimensional matrix representation of the $N = 1$ super-Poincaré algebra are explicitly constructed. The supergeometry of a massive case *continuously* becomes that of a massless case in the infinite-momentum limit. The origin of the gauge transformations associated with the massless supermultiplets becomes transparent in that limit.

1 Introduction

The concept of the little group of the Poincaré group turned out to be very useful in analyzing the internal space-time symmetries of elementary particles and, hence, in assigning quantum numbers for them [1,2]. The internal space-time symmetry groups for massive and massless particles are known to be locally isomorphic to the three-dimensional rotation group $O(3)$ and the two-dimensional Euclidean group $E(2)$, respectively. The little group of the massless particle can also be represented by the cylindrical group, which is isomorphic to the Euclidean group, when the cylindrical axis being parallel to the momentum [3]. The little groups for massive and massless particles are in fact related by the Wigner-Inonü-type group contraction [4]. As was explained recently [5], the little group for massless particles is an infinite-momentum zero-mass limit of the little group for massive particles.

Our purpose is to extend those observations to the case of supersymmetry. Here we will restrict ourselves to the case of simple or $N = 1$ supersymmetry in four space-time dimensions, though the extended supersymmetries with or without central charges [6], as well as higher-dimensional supersymmetries, could also be studied along similar lines. The role of Wigner's little groups in particle theory and supersymmetry is illustrated in Table I.

We denote the generators for translations and Lorentz transformations by P_μ and $M_{\mu\nu}$, respectively, and for global supersymmetry transformations by Q_a. The algebra of global simple supersymmetry is an extension of the ordinary Poincaré algebra, and it is known as the $N = 1$ super-Poincaré algebra [7]. It comprises

$$\frac{1}{i}[M_{\mu\nu}, M_{\lambda\rho}]_- = \eta_{\mu\lambda}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\lambda}M_{\mu\rho} ,$$

$$\frac{1}{i}[M_{\mu\nu}, P_\lambda]_- = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu ,$$

$$[M_{\mu\nu}, Q_a]_- = (\Sigma_{\mu\nu})_a{}^b Q_b ,$$

$$\{Q_a, Q_b\}_+ = (\gamma^\mu C)_{ab} P_\mu ,$$

$$[P_\mu, P_\nu]_- = [P_\mu, Q_a]_- = 0 , \tag{1}$$

where the third line means, in particular, that the Q transforms as a spinor under Lorentz transformations. The most important equation is represented by the fourth line, which allows to interpret the supersymmetry as the square-root of space-time.

We use the conventions in which $x^\mu \equiv (x^i, t) = (x, y, z, t)$ and $\eta = \text{diag}(+ + +-)$. In eq. (1) the $\Sigma_{\mu\nu}$ denote the Lorentz generators in the spinor representation, $\Sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]_-$, the C is the four-dimensional charge conjugation matrix, and the γ^μ are Dirac matrices in four dimensions.

2 Matrix Representation of Supersymmetry

An explicit 5×5 matrix representation of the $N = 1$ super-Poincaré algebra (1) is known due to Ferrara and van Nieuwenhuizen [8]

$$M_{\mu\nu} = \begin{pmatrix} & & \Sigma_{\mu\nu} & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , P_\mu = \begin{pmatrix} & & & & 0 \\ & \gamma_\mu(1 - \gamma_5) & & & 0 \\ & & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$Q_a = \begin{pmatrix} 0 & 0 & 0 & 0 & [(1 + \gamma_5)C]_{1a} \\ 0 & 0 & 0 & 0 & [(1 + \gamma_5)C]_{2a} \\ 0 & 0 & 0 & 0 & [(1 + \gamma_5)C]_{3a} \\ 0 & 0 & 0 & 0 & [(1 + \gamma_5)C]_{4a} \\ (1 - \gamma_5)_{a1} & (1 - \gamma_5)_{a2} & (1 - \gamma_5)_{a3} & (1 - \gamma_5)_{a4} & 0 \end{pmatrix} , \tag{2}$$

In particular, the relation $[P_\mu, P_\nu]_- = 0$ easily follows from the definitions $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0$, $\gamma_5^2 = 1$. All of the momentum-component operators in eq. (2) are in fact nilpotent and, hence, the representation (2) can serve for the massless case only. Clearly, this finite-dimensional representation of the super-Poincaré group is not unitary. Another convenient representation of the generators of the super-Poincaré group in terms of differential operators is provided by the superspace [7] parametrized by $(x^\mu, \vartheta^\alpha, \bar{\vartheta}_{\dot\alpha})$, where ϑ's represent the Grassmannian anticommuting spinor coordinates in the two-component notation:

$$\vartheta_a = \begin{pmatrix} \vartheta_\alpha \\ \bar{\vartheta}^{\dot\alpha} \end{pmatrix} , \gamma_\mu = \begin{pmatrix} 0 & -i\sigma_\mu \\ -i\tilde\sigma_\mu & 0 \end{pmatrix} , \begin{matrix} \sigma_\mu = (-\sigma_i, 1); & \alpha = 1, 2 \\ \tilde\sigma_\mu = (+\sigma_i, 1); & \alpha = 1, 2 \end{matrix} \tag{3}$$

The representation of the super-Poincaré algebra in superspace reads [7]

$$P^\mu = -i\partial^\mu , \quad M^{\mu\nu} = -i(x^\mu\partial^\nu - x^\nu\partial^\mu) + \tfrac{1}{2}(\vartheta\sigma^{\mu\nu}\frac{\partial}{\partial\vartheta} + \bar{\vartheta}\tilde\sigma^{\mu\nu}\frac{\partial}{\partial\bar{\vartheta}}) ,$$

$$Q_\alpha = -i\frac{\partial}{\partial\vartheta^\alpha} - (\sigma^\mu\partial_\mu\bar{\vartheta})_\alpha \ , \ \bar{\vartheta}_{\dot\alpha} = i\frac{\partial}{\partial\bar{\vartheta}^{\dot\alpha}} + (\vartheta\sigma^\mu\partial_\mu)_{\dot\alpha} \ , \tag{4}$$

where the $\frac{\partial}{\partial\vartheta}$ and $\frac{\partial}{\partial\bar{\vartheta}}$ are the Grassmannian left derivatives, and

$$\sigma_{\mu\nu} \equiv \sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu \ , \ \bar{\sigma}_{\mu\nu} \equiv \bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu \ . \tag{5}$$

This representation can be used for both massive and massless cases.

3 Little Group and Wigner-Inonü Contraction

According to Wigner [1], the little group is the maximal subgroup of the Poincaré group whose transformations leave the four-momentum of a given particle invariant. For a massive point particle one can choose a Lorentz frame in which the particle is at rest. In this frame, the little group is clearly the three-dimensional rotation group. The whole group of Lorentz transformations is generated by these three rotation generators J_i and, in addition, three Lorentz boost generators K_i [1,2]. Hence, the little group of the moving (say, along the z direction) massive particle can be obtained by boosting with the operator $B(\eta) = \exp(-\eta K_3)$. Then the little group is generated by

$$J_1' = (\cosh\eta)J_1 + (\sinh\eta)K_2 \ ,$$

$$J_2' = (\cosh\eta)J_2 - (\sinh\eta)K_1 \ , \tag{6}$$

$$J_3' = J_3 \ .$$

The idea is to consider the rapidly moving massive particle for large values of η. Then after renormalizing the generators J_1' and J_2' as $N_1 \equiv -(\cosh\eta)^{-1}J_2'$ and $N_2 \equiv (\cosh\eta)^{-1}J_1'$, in the infinite-$\eta$ limit one obtains

$$N_1 = K_1 - J_2 \ ,$$

$$N_2 = K_2 + J_1 \ . \tag{7}$$

These operators and J_3 satisfy the commutation relations of the $E(2)$-like little group for *massless* particles [1,2,9] and, hence, the massless case is not needed to be considered as independent. The supersymmetry representation theory was usually considered separately for the massive and massless cases, while the Wigner-Inonü group contraction provides a connection between them.

In case of the representation (2) of the super-Poincaré algebra, let $\bar{P} \equiv P_0 + P_3$ be the fixed momentum. Then it is easy to check that the associated little group is generated by the three generators among $M_{\mu\nu}$: $\Sigma_{12} \sim \gamma_1\gamma_2$, $(\Sigma_{01} + \Sigma_{31}) \sim (\gamma_0 + \gamma_3)\gamma_1$ and $(\Sigma_{02} + \Sigma_{32}) \sim (\gamma_0 + \gamma_3)\gamma_2$. Taking the convenient representation of the 4×4 γ-matrices, in which

$$\gamma_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix} \ , \ \gamma_0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \ , \ \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{8}$$

where we have introduced the standard 2×2 Pauli matrices as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{9}$$

we have

$$C = i\gamma_2\gamma_0 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \qquad (10)$$

to satisfy the defining equation $C\gamma_\mu = -\gamma_\mu^T C$ for the charge conjugation matrix C. Therefore, we find

$$\gamma_1 C = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \gamma_2 C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 C = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \gamma_0 C = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \qquad (11)$$

Now it is easy to calculate the square root $\tilde{Q}$ of the given momentum $\tilde{P}$ in the supersymmetry algebra:

$$\tilde{Q} = \sum_{a=1}^{4} d_a Q_a , \{\tilde{Q}, \tilde{Q}\}_+ = \sum_{a,b=1}^{4} d_a d_b (\gamma^\mu C)_{ab} P_\mu = 2\tilde{P} , \qquad (12)$$

i.e. find the appropriate numerical coefficients d_a. The result is given by

$$\tilde{Q} = 2e^{-i\pi/4}(Q_2 + Q_3) = 2e^{-i\pi/4} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad (13)$$

so that $\tilde{Q}^2 = \tilde{P}$ indeed. One should emphasize that no such notion as the little group of $\tilde{Q}$, can be introduced, since the Q-operators are defined in the *spinor* representation space and that group would be trivial. Now it becomes clear why the generators of the little group of $\tilde{P}$ do *not* commute with its square root $\tilde{Q}$.

With each massless particle one can associate a circular cylinder whose axis is parallel to the momentum. Then one can rotate a point on the surface of this cylinder around the axis or translate along the direction of the axis. As is well known, the rotational degree of freedom is associated with the helicity, while the translation corresponds to a gauge transformation [3,9]. This translational degree of freedom is shared by all massless particles. In case of supersymmetry, we can extend the contents of gauge transformations to all massless supermultiplets by considering again the massive supermultiplets in the infinite-momentum limit.

Taking the mass value to be equal to 1 for convenience, the massive particle at rest is characterized by the four-momentum $P_{m,r}^\mu = (0,0,0,1)$. The same particle moving with the momentum p along the z direction, has the four-momentum $P_m^\mu = (0,0,p,\sqrt{p^2+1})$. Renormalizing this operator as $P_m^\mu \to p^{-1}P_m^\mu \equiv P_r^\mu$, we obtain in the infinite-p limit that $P_r^\mu = (0,0,1,\sqrt{1+p^{-2}}) \to P_0^\mu \equiv (0,0,1,1)$, which is just the conventional choice of the four-momentum in the massless case.

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These very simple observations are still very useful in the case of supersymmetry. The supersymmetry algebra can conveniently be represented for our purposes here in the two-dimensional notation of eq. (3) as

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}_+ = \sigma^\mu_{\alpha\dot{\beta}} P_\mu \, ,$$

$$\{Q_\alpha, Q_\beta\}_+ = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \, . \tag{14}$$

Now, on the one hand, we immediately see that in the massive case at rest we obtain a Clifford algebra of the form

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}_+ = \mathbf{1}_{\alpha\dot{\beta}} \, , \tag{15}$$

where all of the Q-operators are active. They can be interpreted as the operators of creation and destruction, and then used to develop the massive supermultiplets structure [6,7].

On the other hand, in the massless case we obtain instead

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}_+ = (1 + \sigma_3)_{\alpha\dot{\beta}} \, ,$$

$$1 + \sigma_3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \, , \tag{16}$$

which means the degeneracy of the supersymmetry algebra. Eq. (16) can be obtained from eq. (15) in the infinite-momentum limit after the renormalization $Q_{\text{mass}} \to Q_{\text{massless}}$ induced by the transition $P^\mu_{m,r} \to P^\mu_0$ discussed above. This gives rise to the reduced supermultiplets structure since only *a half* of the Q-operators are now active. The rest represents the supersymmetric gauge transformations which always accompany the massless supermultiplets containing photino or gravitino in this picture (their role is to kill the redundant degrees of freedom), just like the invariance under the translational gauge symmetry is associated with photons and gravitons [2,10].

The main point of our brief discussion is that the massive and massless cases in supersymmetry should be considered on equal footing, the connection between them being provided by the Wigner-Inonü contraction, which has a clear physical meaning. Of course, this fact is already known and can be read off, in particular, from the contents of Refs. [11,12]. Nevertheless, we would like to stress its conceptual simplicity in this paper, and give it in the most obvious way, which was not presented in the past.

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TABLE I. *Symmetries of massive and massless particles*

The first two rows display the unification of the energy-momentum relations and the internal symmetries of massive and massless particles, as given in Ref. [5]. The third row means that supersymmetry can also be included into this picture.

	Massive or Slow	$\leftarrow$ between $\rightarrow$	Massless or Fast
Energy and Momentum	$E = \frac{p^2}{2m}$	$\leftarrow E = \sqrt{m^2 + p^2} \rightarrow$	$E = p$
Spin, Gauge and Helicity	S_3 $S_1 \quad S_2$	$\leftarrow$ Little Groups $\rightarrow$	S_3 Gauge Transformations
Supersymmetry	$Q_1, \bar{Q}_{\dot{1}}$ $Q_2, \bar{Q}_{\dot{2}}$	Square Root $\leftarrow$ of Space-Time $\rightarrow$ Translations	$Q_1, \bar{Q}_{\dot{1}}$ Non-Active Charges

ON HARMONIC OSCILLATORS
AND THEIR KEMMER RELATIVISTIC FORMS

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Abstract

We show that Dirac (Kemmer) equations are intimately connected with (para)supercharges coming from (para)supersymmetric quantum mechanics, a nonrelativistic theory. The dimensions of the irreducible representations of Clifford (Kemmer) algebras play a fundamental role in such an analysis. These considerations are illustrated through oscillatorlike interactions, leading to (para)relativistic oscillators.

1   Introduction

Supersymmetric quantum mechanics (SSQM) as initiated by Witten [1] is characterized by a superposition of bosonic and fermionic operators, leading from an algebraic point of view to the so-called Lie superalgebras [2]. It is now well known [3] that some of their generators i.e. the supercharges, can be related to Dirac hamiltonians if the spin-orbit coupling procedure [4] is under study. In particular, when oscillatorlike interactions are considered, this connection gives rise to the Dirac oscillator [5] whose nonrelativistic limit corresponds to an ordinary harmonic oscillator with a strong spin-orbit coupling term.

Moreover, extensive studies have recently combined bosons and parafermions [6] leading to parasupersymmetric quantum mechanics (PSSQM) [6,7,8]. The Lie structures subtended by this generalized context are now referred to as parasuperalgebras [9]. We plan to show [10] here that the corresponding parasupercharges can be connected with Kemmer hamiltonians [11] when a specific procedure of parasupersymmetrization, compatible with Kemmer algebras [12], is considered. We also prove [13] that the

nonrelativistic limit of the associated Kemmer oscillator is an ordinary harmonic oscillator still coupled with a spin-orbit term <u>but</u> where the values of the spin involved in this term are now zero or one.

The contents of this communication are then distributed as follows. In Section 2, we point out the connection between SSQM and Dirac hamiltonians through the free case and the harmonic oscillator one. Then, in Section 3, we visit the extension to the parasupersymmetric context by considering again the free and the harmonic oscillator contexts.

2 Supersymmetric Quantum Mechanics and Dirac Formalism

Let us recall that the N=2-supersymmetric quantum mechanics [1] is characterized by the existence of two supercharges Q_1 and Q_2 satisfying the relations

$$\left\{ Q_\alpha , Q_\beta \right\} = 2 \delta_{\alpha\beta} H_{ss} , \tag{2.1}$$

$$\left[H_{ss} , Q_\alpha \right] = 0 , \quad \alpha , \beta = 1 , 2 , \tag{2.2}$$

where H_{ss} is the supersymmetric hamiltonian. In the so-called spin-orbit coupling procedure [4], these supercharges are realized through 4 by 4 matrices associated with the irreducible unitary representations of the unitary Lie superalgebra $su(2|2)$ [14], when the 3-dimensional spatial context is under study. More precisely the free case corresponds to

$$H_{ss} = \frac{\vec{p}^2}{2m} , \tag{2.3}$$

$$Q_\alpha = \frac{1}{\sqrt{2m}} \varphi_\alpha^j p^j , \quad \alpha = 1 , 2 , \tag{2.4}$$

where the sum over repeated indices is understood. As the hermitian matrices φ_α^j generate $su(2|2)$, we can propose the following choice

$$\varphi_1^j = \alpha^j , \quad \varphi_2^j = i \beta_\alpha^j , \tag{2.5}$$

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leading to identify the free Dirac hamiltonian H_D with

$$H_D \equiv \vec{\alpha} \cdot \vec{p} + m \beta = \sqrt{2m}\ Q_1 + m \beta .$$ (2.6)

Here the velocity of light is taken to be one. The second supercharge leads to a new Dirac hamiltonian unitarily equivalent to the usual one given by Eq. (2.6), as it is easily verified through

$$U = \frac{1}{\sqrt{2}} \left(I + i \beta \right).$$ (2.7)

The harmonic oscillator case can be studied in a completely parallel way. Its supersymmetrized version is characterized by

$$H_{ss} = \frac{\vec{p}^2}{2m} + \frac{1}{2} m \omega^2 \vec{x}^2 + \frac{\omega}{2} \left(3 + 2 \vec{L} \cdot \vec{\sigma} \right) \otimes \sigma_3 ,$$ (2.8)

$$Q_1 = \frac{1}{\sqrt{2m}} \left(\vec{\alpha} \cdot \vec{p} - i m \omega \beta \vec{\alpha} \cdot \vec{x} \right) ,$$ (2.9a)

$$Q_2 = \frac{1}{\sqrt{2m}} \left(i \beta \vec{\alpha} \cdot \vec{p} + m \omega \vec{\alpha} \cdot \vec{x} \right).$$ (2.9b)

These two supercharges (2.9) give rise to two (unitarily equivalent) Dirac hamiltonians by using analogous identifications to (2.6), the first one coinciding with the Dirac oscillator proposed by Moshinsky and Szczepaniak [5]. These two operators lead to the same nonrelativistic limit [15] i.e. an ordinary harmonic oscillator with a strong spin-orbit coupling term.

3   Parasupersymmetric Quantum Mechanics and Kemmer Formalism

The relations characterizing the N=2-parasupersymmetric quantum mechanics in terms of the two parasupercharges Q_1 and Q_2 write [10]

$$Q_\alpha^3 - 3 Q_\beta Q_\alpha Q_\beta = Q_\alpha H_{PSS} ,$$ (3.1)

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$C\text{-}4$

$$2 \left\{ Q_\alpha , Q_\beta^2 \right\} - Q_\beta Q_\alpha Q_\beta - Q_\alpha^3 = Q_\alpha H_{PSS} \; , \tag{3.2}$$

$$\left[H_{PSS} , Q_\alpha \right] = 0 \; , \quad \alpha , \beta = 1 , 2 \; , \quad \alpha \neq \beta \; , \tag{3.3}$$

where H_{PSS} is the parasupersymmetric hamiltonian and where there is no summation on repeated indices.

The 3-dimensional free case is still associated with (2.3) and (2.4) but now the matrices φ_α^j correspond to the Kemmer algebra $K(4)$ [12]

$$\varphi_1^j = \left[\beta^0 , \beta^j \right] \; , \quad \varphi_2^j = i \left\{ \beta^0 , \beta^j \right\} . \tag{3.4}$$

Here the matrices β_μ satisfy

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = 2 g_{\mu\nu} \beta_\lambda + 2 g_{\nu\lambda} \beta_\mu \; , \quad g_{00} = - g_{ii} = 1 . \tag{3.5}$$

This realization enables the following identification for the free Kemmer hamiltonian [11]

$$H_\kappa \equiv \left[\beta^0 , \beta^j \right] p^j + m \beta^0 = 2 \sqrt{2 m} \; Q_1 + m \beta^0 \; , \tag{3.6}$$

and it also ensures a new proposal with respect to the second parasupercharge, unitarily equivalent [10] to the one proposed in (3.6).

The harmonic oscillator context is subtended by parallel identifications leading in particular to

$$H_\kappa = \left[\beta^0 , \beta^j \right] p^j + i m \omega \left\{ \beta^0 , \beta^j \right\} x^j . \tag{3.7}$$

The nonrelativistic hamiltonians corresponding to (3.7) in the spin 0 and spin 1 cases are respectively given by [13]

$$H_0 = \frac{\vec{p}^2}{2 m} + \frac{1}{2} m \omega^2 \vec{x}^2 - \frac{3}{2} \omega \; , \tag{3.8}$$

$$H_1 = \frac{\vec{p}^2}{2 m} + \frac{1}{2} m \omega^2 \vec{x}^2 - \frac{\omega}{2} \left(3 + 2 \vec{L} \cdot \vec{S} \right) , \tag{3.9}$$

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where S_j (j = 1, 2, 3) refer to the 3 by 3 spin 1 matrices. Through these results, the system described by the hamiltonian (3.7) is called the Kemmer or pararelativistic oscillator [10].

4 Acknowledgments

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THE ALGEBRA OF SUPERTRACES FOR
2+1 SUPER DE SITTER GRAVITY

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Abstract

We calculate the algebra of the observables for 2+1 super de Sitter gravity, for one genus of the spatial surface. The algebra turns out to be an infinite Lie algebra subject to non-linear constraints. We solve the constraints explicitly in terms of five independent complex supertraces. These variables are the true degrees of freedom of the system and their quantized algebra generates a new structure which we refer to as a "central extension" of the quantum algebra $SU(2)_q$.

1 Introduction

The discovery by Witten that many gravity theories in $2 + 1$ dimensions are equivalent to Chern Simons theories, and are in principle exactly quantizable, has sparked a great deal of interest in their study [1]. Perhaps the key obstacle in carrying out this quantization explicitly has been our poor understanding of the observable phase space. Pure Chern-Simons theories in vacuum are locally trivial and interesting situations arise either in the presence of sources or when the topology of the space-time manifold is non-trivial. In either case, the observable degrees of freedom for the field theory are the traces of the holonomies (alternatively called the integrated connections) associated to non-contractible loops of the space-time manifold M, which are classified by the fundamental group $\pi_1(M)$. These traces span the reduced phase space of the theory in a highly redundant way. Indeed, the group $\pi_1(M)$ is infinite, while the dimension of the reduced phase space is known to be $(2g - 2) \times dim(\mathcal{A})$, where $\mathcal{A}$ is the Lie algebra considered in the Chern-Simons action. The traces are subject to non-linear constraints (NLC) which depend on the characteristic equation for the matrices in the defining representation. Our approach here is to first reduce the classical system to a finite-dimensional observable phase space and then quantize. Unfortunately, this is an extremely difficult task, which has only recently been solved for arbitrary genus in de Sitter gravity [2].

The reduced phase space is well-understood for any genus in Poincaré gravity [3], but in terms of inhomogeneous variables which have not yet been generalized to curved spacetimes. The purpose of this contribution is to provide the reduced phase space for one genus in 2+1 super de Sitter gravity.

2   The Algebra of Supertraces

Following previous discussions, we will consider the case when the space-time manifold has the topology $M = \Sigma \times R$, where R is the time and Σ is an arbitrary closed, orientable two-dimensional surface of genus g. Also we will restrict the discussion to only one genus of such a surface. The Poincaré [4,5], de Sitter [6], and conformal [7] cases have been previously discussed along these lines and the Poisson bracket algebra of the traces calculated. The quantized version of the algebra of observables for the de Sitter case provides a realization of a pair of commuting $SU(2)_q$ quantum algebras [8].

Witten's formulation of 2+1 dimensional gravity theories as Chern-Simons theories has been extended to the supersymmetric case in Ref. [9], where the super de Sitter case is studied by considering the orthosymplectic group $OSp(1|2; \mathbb{C})$ as the gauge group. The system is described by the Chern Simons action [10],

$$I = \frac{1}{2} Tr \int_M \left(dA - \frac{2}{3} A \wedge A \right) \wedge A, \tag{1}$$

where $A = A_\mu dx^\mu$ ($\mu = 0, 1, 2$) is the superconnection

$$A = A^A T_A = e^a P_a + \omega^a J_a + \chi^\alpha U_\alpha + \Theta^\alpha V_\alpha, \tag{2}$$

which takes values on the Lie algebra of $OSp(1|2; \mathbb{C})$. Here $T_A = (P_a, J_a, U_\alpha, V_\alpha)$, where $P_a, J_a (a = 0, 1, 2)$ are the bosonic generators and $U_\alpha, V_\alpha (\alpha = 1, 2)$ are the fermionic ones. The fields $\chi^\alpha, \Theta^\alpha$ are spinors whose components are odd Grassmann numbers. The trace in Eq.(1) is defined in terms of the group-invariant non-degenerate bilinear tensor

$$Tr(T_A T_B) = D_{AB} = \begin{bmatrix} 0 & \eta_{ab} & 0 & 0 \\ \eta_{ab} & 0 & 0 & 0 \\ 0 & 0 & -2\epsilon_{\alpha\beta} & 0 \\ 0 & 0 & 0 & 2\epsilon_{\alpha\beta} \end{bmatrix}, \tag{3}$$

where $\eta_{ab} = diag(-1, 1, 1), \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, with $\epsilon_{12} = +1$ and $\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha^\beta$. The generators satisfy the superalgebra of $OSp(1 \mid 2); \mathbb{C}$) which is given in Ref. [9].

The constraints equations that follow from (1) imply that A is a pure gauge, that is $A = d\psi\psi^{-1}$ where $\psi \in OSp(1|2; \mathbb{C})$. The Poisson brackets of A are easily calculated from the action (1) [11],

$$\{A_i(x), A_j(y)\}_{P.B.} = -2\epsilon_{ij} M \delta^2(x - y), \tag{4a}$$

where x, y are generic points on the Σ, $i, j = 1, 2$ are spatial vector indices on Σ, $\epsilon_{ij} = -\epsilon_{ji}$ with $\epsilon_{12} = +1$ and

$$M \equiv D^{AB}T_A \otimes T_B = P_a \otimes J^a + J_a \otimes P^a + \frac{1}{2}\left(U_\alpha \otimes U^\alpha - V_\alpha \otimes V^\alpha\right),\qquad (4b)$$

with $D_{AC}D^{CB} = \delta_A^B$.

Let us consider two generic points P, Q on Σ and a path ρ joining them, parametrized by $x(t)$, $t \in [0,1]$ with $x(0) = P$ and $x(1) = Q$. The solution to the differential equation

$$\frac{d\psi}{dt} = A_t \psi \qquad (5)$$

subject to the boundary condition $\psi(0) = 1$, where $A_t \equiv A_\alpha T^\alpha$ is a tangent vector along ρ, will depend only on the homotopy class of ρ and it is denoted by $\psi(\rho)$ (see Ref. [6] for details). For a second path ρ' with end points Q, R we have the solution $\psi(\rho')$ of (5). The solution for the path $\rho'\rho$, with end points P, R is then

$$\psi(\rho'\rho) = \psi(\rho')\psi(\rho). \qquad (6)$$

By restricting to closed paths, this equation defines a group homomorphism $\psi : \pi_1(\Sigma) \to OSp(1|2;\mathbb{C})$. The fundamental group of the surface Σ based on the point B, $\pi_1(\Sigma, B)$, is presented via 2g generators u_i, v_i $i = 1,...,g$ which satisfy the relation $u_1 v_1 u_1^{-1} v_1^{-1}...u_g v_g u_g^{-1} v_g^{-1} = 1$.

Let ψ, ϕ be generic elements of $OSp(1|2;\mathbb{C})$. The Poisson brackets of the integrated connections $\psi(\rho)$, $\psi(\sigma)$ of two elements of $\pi_1(\Sigma)$, with base points P, Q respectively, which have a single intersection may be calculated from (4) by a procedure already established in Refs. [4,6]. The result is

$$\left\{\psi_\alpha{}^\beta(\rho),\ \psi_\mu{}^\nu(\sigma)\right\}_{P.B.} = 2s\,(-1)^{[(g(\beta)-g(\eta))(g(\alpha)-g(\eta))+(g(\mu)-g(\theta))(g(\nu)-g(\theta))]}$$
$$M_{\xi\theta}{}^{\eta\varphi}\,\psi_\eta{}^\beta(\rho_i)\psi_\alpha{}^\xi(\rho_f)\psi_\varphi{}^\nu(\sigma_i)\psi_\mu{}^\theta(\sigma_f) \qquad (7)$$

where $M_{\xi\theta}{}^{\eta\varphi} = D^{AB}(T_A)_\xi{}^\eta (T_B)_\theta{}^\varphi$. The subindex i (f) labels that part of the path before (after) the intersection and $s = s(\rho,\sigma) = -s(\sigma,\rho) = \pm 1$ is called the intersection number.

The integrated connection $\psi(\rho)$ is not gauge invariant, but the supertrace $C(\rho) \equiv Str\psi(\rho) = (-1)^{g(\alpha)}\psi_\alpha{}^\alpha$ is, namely:

$$C(\rho) = C(\nu\rho\nu^{-1}) \qquad (8)$$

with $\rho \in \pi_1(\sigma)$ and ν being any open path. Equation (8) expresses the invariance of $C(\rho)$ under a change of the base point of $\pi_1(\sigma)$. Thus, one can calculate the Poisson bracket of two closed paths ρ and σ based on two different points P and Q respectively and make $P = Q$ after the calculation, so that ρ, σ become elements of $\pi_1(\sigma; Q)$. By supertracing (7) one obtains [13]

$$\left\{C(\rho),\ C(\sigma)\right\}_{P.B.} = is\sqrt{\Lambda}\left(C(\rho\sigma) - C(\rho\sigma^{-1})\right), \qquad (9)$$

for paths with a single intersection or with no intersection ($s = 0$). This result is the same that has been obtained for the de Sitter, Poincaré and conformal groups [4-7].

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By repeated use of (7) and with the help of (6) we obtain the following general formula for the Poisson brackets of elements ρ, σ of $\pi_1(\Sigma; Q)$ with n intersections

$$\{ C(\rho), C(\sigma) \}_{P.B.} = i\sqrt{\Lambda} \sum_{k=1}^{n} s_k \left(C(\rho_k \sigma_k) - C(\sigma_k \rho_k^{-1}) \right), \tag{10}$$

where s_k is the intersection number of the k-th intersection and the subindex k on each path means that the product of them is constructed by taking the k-th intersection point as the base point, instead of the point Q.

Any matrix $\psi(\rho)$ which is an element of $OSp(1|2; \mathbb{C})$ satisfies the generalized Cayley-Hamilton identity

$$\psi(\rho^3) - (C(\rho) + 2)\left(\psi(\rho^2) - \psi(\rho)\right) - 1 = 0. \tag{11}$$

Multiplying (11) by $\psi(\sigma\rho^{-1})$ and supertracing one obtains the non linear constraint

$$R(\rho, \sigma) \equiv C(\rho)C(\rho\sigma) - C(\rho)C(\sigma) - C(\rho^2\sigma) + C(\rho\sigma^{-1}) + 2C(\rho\sigma) - 2C(\sigma) = 0. \tag{12}$$

In order to obtain the algebra of observables we must take into account the relation (12). This relation appears to be an ideal of the traces algebra. Although we were not able to obtain an algebraic proof, computer calculations in various examples indicate that R has zero Poisson bracket with the traces, as in the ordinary de Sitter case [6]. This implies that the relations (12) hold "strongly", i.e. that they can be used within the Poisson brackets (10).

Fortunately, it is possible to solve the relations $R(u,v) = 0$ explicitly, by expressing all traces on one genus in terms of five fundamental ones, which can be chosen as $C(u) = A$, $C(v) = B$, $C(uv) = C$, $C(uv^2) = D$ and $C(uvu^2v^2) = E$. This property can be shown to be a direct consequence of the identity (12).

Finally, we can calculate the algebra satisfied by these variables. To this end it is more convenient to define the following combinations of the basic traces previously introduced

$$X = \frac{1+A}{2}, \quad Y = \frac{1+B}{2}, \quad Z = \frac{1+C}{2},$$
$$V = \frac{1}{2}\left(\frac{1+D}{2} + X - 2YZ\right),$$
$$U = \frac{1+E}{2} - Z(1 + 8XYZ - 4X^2 - 4Y^2 \tag{13}$$
$$+ 8(X+Y)V + 4V).$$

This choice is dictated by the property that in the de Sitter limit (fermionic variables equal to zero) X, Y and Z go into the variables used in Ref. [6], while U and V go to zero. The Poisson brackets of these variables can be computed with the help of (10), assuming that the relations (12) are indeed an ideal of the algebra. We find

$$\{X,Y\}_{P.B.} = is\sqrt{\Lambda}(Z - XY - V),$$
$$\{X,V\}_{P.B.} = \{X,U\}_{P.B.} = \{U,V\}_{P.B.} = 0, \tag{14}$$

plus cyclical permutations of X, Y, Z.

We quantize the above system using the correspondence principle $XY - YX = i\hbar \times \{X, Y\}_{P.B.}$ and symmetrising the XY product. The result can be written as

$$e^{i\theta/2}XY - e^{-i\theta/2}YX = 2i\sin\theta/2 \ (Z - V), \tag{15}$$

and cyclical, where $\tan\theta/2 \equiv \frac{i\hbar s\sqrt{\Lambda}}{2}$ and U, V are central elements. The de Sitter limit ($U = V = 0$) on Ref. [6] is clearly recovered from Eqs. (15) now in terms of arbitrary complex variables X, Y and Z. The algebra (15) provides a central extension of $SU(2)_q$ [14], with V being the central charge.

3 The NLC Constraints

These are relations among the supertraces (see for example Eq. (12)), which constitute the basic tool for reducing the original infinite dimensional supertraces algebra to a finite one. A general way of obtaining such relations is starting from a Cayley-Hamilton type identity satisfied by the matrix. In the case of a supermatrix M, the characteristic polynomial is not given by $p(x) = Sdet(M - xI)$, and the problem of constructing such polynomial in the general case seems to be still an open one. The basic definition is $p(x) = \Pi_i(\lambda_i - x)$, where λ_i are the eigenvalues of M and the idea is to translate this into "simpler" operations which would bypass the explicit calculation of the eigenvalues. In the case of an arbitrary 2×2 supermatrix with entries $M_{12} = a$, $M_{12} = \alpha$, $M_{21} = \beta$, $M_{22} = b$, where a, b (α, β) are even (odd) Grassmann numbers, the characteristic polynomial is

$$p(x) = (a - b)x^2 - (a^2 - b^2 + 2\alpha\beta)x + (ab(a - b) + (a + b)\alpha\beta), \tag{16}$$

and one can verify that $p(M) \equiv 0$ as a matrix identity. Another explicit example of such polynomials is Eq. (11) which corresponds to a particular case of a 3×3 supermatrix.

4 Acknowledgments

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QUANTUM WORMHOLES
AND
HARMONIC OSCILLATORS

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Abstract

The quantum state of a wormhole can be represented by a path integral over all asymptotically Euclidean four–geometries and all matter fields which have prescribed values, the arguments of the wave function, on a three–surface which divides the spacetime manifold into two disconnected parts. Minisuperspace models which consist of a homogeneous massless scalar field coupled to a Friedmann–Robertson–Walker spacetime are considered. Once the path integral over the lapse function is performed, the requirement that the spacetime be asymptotically Euclidean can be accomplished by fixing the asymptotic gravitational momentum in the remaining path integral. It is argued that there does not exist any wave function which corresponds to asymptotic field configurations such that the effective gravitational constant is negative in the asymptotic region. Then, the wormhole wave functions can be written as linear combinations of harmonic oscillator wave functions.

1   Introduction

Wormholes have been considered as instantons, solutions of the Euclidean Einstein equations, which consist of two asymptotically Euclidean regions connected by a throat [1,2]. These classical wormholes are saddle points of the Euclidean action and therefore, they allow the Euclidean path integral to be approximated semiclassically. One makes the dilute wormhole approximation in which the wormhole ends are far apart from each other, so that one can consider that wormholes do not interact and then, they can be treated separately. Wormholes on the Planck scale may affect the constants of nature and, in particular, may provide a mechanism for the vanishing of the cosmological constant [3,4]. Wormholes may play an important role in solving problems associated with the complete evaporation and disappearance of black holes [2]. However, classical wormholes may only exist for very special types of matter, those which allow the Ricci tensor to have negative eigenvalues [1,5,6]. This may place a strong restriction on the possibility that wormholes have any role in these processes.

However, we need not restrict ourselves to such a semiclassical treatment and the special types of matter that it requires. More generally, one can regard wormholes as solutions of the quantum Wheeler–DeWitt equation with some suitable boundary conditions. Hawking and Page [7] have

proposed that the boundary conditions are as follows. The wormhole wave function can be represented as the path integral over all asymptotically Euclidean four-geometries that match a given compact three-geometry which is the argument of the wave function. This can be interpreted as saying that there are no gravitational excitations at infinity (*i.e.*, at large distance compared with the characteristic scale of the wormhole). An extra surface term which eliminates the infinite contribution that comes from the asymptotically Euclidean region should be added to the action. It will be seen that, once the path integrals over the lapse and shift functions have been performed, the requirement that the spacetime is asymptotically Euclidean is equivalent to the fixation of the asymptotic gravitational momentum [8]. Also, one must integrate over all matter fields which have no sources in the asymptotic region. This reflects the fact that there are no matter excitations at infinity. This is accomplished by requiring that the matter Hamiltonian must vanish asymptotically. If one considers a coupled scalar field with a potential as the matter content, this means that, at infinity, the field must approach an homogeneous configuration at which the potential has a vanishing minimum [8].

In what follows, we shall study a massless homogeneous scalar field minimally and conformally coupled to a Friedmann–Robertson–Walker (FRW) spacetime. It turns out that, as in many other physical theories, harmonic oscillators are present in wormhole physics. In fact the wormhole wave functions, in these models can be written in terms of harmonic oscillators wave functions.

2 Minimal scalar field

We shall consider a massless homogeneous scalar field $\phi(\tau)$ minimally coupled to a FRW whose metric can be written

$$ds^2 = \frac{2G}{3\pi} \left(\frac{N^2(\tau)}{2q(\tau)} d\tau^2 + 2q(\tau) d\Omega_3^2 \right), \tag{1}$$

where $N(\tau)$ is the lapse function which measures the proper time separation between two neighbouring three-spheres whose radii are represented by $\sqrt{2q(\tau)}$. The action for this system takes the simple form

$$\tilde{I} = \int_0^\infty d\tau \left(\pi_q \dot{q} \pi_\phi \dot{\phi} - NH \right), \qquad H = \frac{1}{2} \left(-\pi_q^2 + 1 \right) + \frac{1}{8} q^{-2} \pi_\phi^2, \tag{2}$$

whre H is the Hamiltonian and the relations between the canonical momenta π_q, π_ϕ, and the time derivatives of the variables are given by $\pi_q = -\dot{q}/N$, $\pi_\phi = q^2 \dot{\phi}/4N$. The wormhole wave function is defined by the path integral over all asymptotically Euclidean four-metrics and over all scalar fields whose asymptotic configuration is given by a constant value ϕ_0. If the four-metric is going to be asymptotically Euclidean, then the variable q must have the behaviour $q(\tau) \sim \int^\tau d\tau N(\tau)$ when $\tau \to \infty$, as can be seen from the expression for the line element. This means that the asymptotic condition must be $\pi_q(\infty) = -1$. Since we have to fix the asymptotic momentum rather than the canonical variable q, it will be necessary to introduce the term $\pi_q q|_{\tau=\infty}$ in the action, so that the variational problem associated to it, subjected to the boundary conditions mentioned above, be well posed. Therefore, the action will have the form $I = \tilde{I} - \pi_q q|_{\tau=\infty}$.

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Under time reparametrizations defined by the transformations

$$\delta N(\tau) = \dot{\epsilon}(\tau), \qquad \delta\zeta = \{\zeta, \epsilon H\}, \qquad \delta\pi_\zeta = \{\pi_\zeta, \epsilon H\}, \qquad (\zeta = q, \phi) \qquad (3)$$

the variation of the action is $\delta I = -\epsilon(0)\left[\sum_{\zeta=q,\phi} \pi_\zeta \{\zeta, H\} - H\right]_{\tau=0} + \frac{1}{2}\epsilon(\infty)H\Big|_{\tau=\infty}$. The boundary conditions ensure that $H|_{\tau=0} = 0$ and, therefore, the action will be invariant under transformations such that $\epsilon(0) = 0$, $i.e.$, that do not change the arguments of the wave function, which are defined at $\tau = 0$: $q(0) = q'$, $\phi(0) = \phi'$. The wave function can, then, be written as

$$\Psi(q', \phi') = \int \mathcal{D}N \int_{\mathcal{C}_{wh}} \prod_{\zeta=q,\phi} \mathcal{D}\zeta \mathcal{D}\pi_\zeta \delta(F)\Delta_{FP}e^{-I[\zeta,\pi_\zeta,N]}, \qquad (4)$$

where $\mathcal{C}_{wh}$ is the set of histories which satisfy the boundary conditions, F is the gauge fixing condition that singles out a representant of each equivalence class under time reparametrizations and Δ_{FP} is the Fadeev–Popov determinant which ensures that the path integral does not depend on this choice. It is easily seen that the simplest admissible [9] gauge fixing condition is $N = 1$. Any other history $N(\tau)$ can be obtained from this one by means of the time reparametrization whose coefficient $\epsilon(\tau)$ is the only solution to the equation $N(\tau) - 1 = \dot{\epsilon}(\tau)$, subjected to the condition $\epsilon(0) = 0$. This solution is $\epsilon(\tau) = -\tau + \int_0^\tau d\tau N(\tau)$. The Fadeev–Popov determinant is independent of the integration variables and, therefore, the wave function acquires the form [8]

$$\Psi(q', \phi') = \int_{\mathcal{C}_{wh}} \mathcal{D}q\mathcal{D}\pi_q\mathcal{D}\phi\mathcal{D}\pi_\phi e^{-I[q,\phi]}, \qquad (5)$$

where $I[q, \phi] = \int_0^\infty d\tau \left(\pi_q \dot{q} + \pi_\phi \dot{\phi} - H\right) - q\pi_q|_{\tau=\infty}$, and $\mathcal{C}_{wh}$ is the set of histories such that

$$q(0) = q', \quad \phi(0) = \phi', \quad \pi_q(\infty) = -1, \quad \phi(\infty) = \phi_0. \qquad (6)$$

With the change of variables [10]

$$q = \left(x^2 - t^2\right)^{\frac{1}{2}}, \quad \phi = \frac{1}{2}\tanh^{-1}\frac{t}{x}, \quad q\pi_q = x\pi_x + t\pi_t, \quad \pi_\phi = 2(x\pi_t + t\pi_x), \qquad (7)$$

the action takes the simple form

$$I = \int_0^\infty d\tau \left(\pi_x \dot{x} + \pi_t \dot{t} - H\right) - (x\pi_x + t\pi_t)|_{\tau=\infty}, \qquad H = \frac{1}{2}\left(-\pi_x^2 + \pi_t^2 + 1\right). \qquad (8)$$

The boundary conditions (6) transform into

$$x(0) = q'\cosh 2\phi', \quad t(0) = q'\sinh 2\phi', \quad \pi_x(\infty) = -\cosh 2\phi_0, \quad \pi_t(\infty) = \sinh 2\phi_0. \qquad (9)$$

and the measure into $\mathcal{D}x\mathcal{D}t\mathcal{D}\pi_x\mathcal{D}\pi_t$, because the Jacobian is equal to one.

To do the path integral

$$\Psi_{\phi_0}(q', \phi') = \int \mathcal{D}x\mathcal{D}t\mathcal{D}\pi_x\mathcal{D}\pi_t e^{-I[x,t]}, \qquad (10)$$

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with the boundary conditions (9), it is convenient to shift the integration variables

$$x(\tau) = \bar{x}(\tau) + X(\tau), \quad t(\tau) = \bar{t}(\tau) + T(\tau),$$ (11)

where $\bar{x}$ and $\bar{t}$ correspond to the classical solution of the variational problem associated with (8) which satisfies the boundary conditions (9). Explicitly, this solution is

$$\bar{x}(\tau) = \tau \cosh 2\phi_0 + q' \cosh 2\phi', \quad \bar{t}(\tau) = \tau \sinh 2\phi_0 + q' \sinh 2\phi'.$$ (12)

The new variables X and T satisfy the conditions: $X(0) = T(0) = 0$, $P_X(\infty) = P_T(\infty) = 0$. Then, the action (8) splits into two parts

$$I[x,t] = I_0[\bar{x},\bar{t}] + I_2[X,T],$$ (13)

where $I_0[\bar{x},\bar{t}] = q' \cosh(\phi' - \phi_0)$ is the action for the classical solution (12) and

$$I_2[X,T] = \int_0^\infty d\tau \left\{ P_X \dot{X} + P_T \dot{T} - \frac{1}{2}\left(-P_X^2 + P_T^2\right) \right\}.$$ (14)

The linear term $I_1[\bar{x},\bar{t},X,T]$ vanishes identically due to the fact that (12) is a saddle point of the action (8). The measure in the path integral (10) is directly changed into $\mathcal{D}X\mathcal{D}P_X\mathcal{D}T\mathcal{D}P_T$. I_0 does not depend on X or T and then, can be taken out of the integral. The remaining path integral becomes $\int \mathcal{D}X\mathcal{D}T \exp\left\{-\frac{1}{2}\int_0^\infty d\tau \left(-\dot{X}^2 + \dot{T}^2\right)\right\}$ which is independent of q' and ϕ' and therefore, it turns out to be a numerical factor [10]. Thus, up to numerical prefactors,

$$\Psi_{\phi_0}(q,\phi) = \exp\left\{-q \cosh 2(\phi - \phi_0)\right\},$$ (15)

which had already been found as a solution of the Wheeler–DeWitt equation [11]. This wave function can also be written as a linear combination of harmonic oscillator wave functions

$$\Psi_{\phi_0}(q,\phi) = \sum_{n=0}^\infty c_n(\phi_0)\Psi_n(q,\phi), \qquad c_n(\phi_0) = \frac{\sinh^n \phi_0}{\cosh^{n+1} \phi_0}$$ (16)

where $\Psi_n(q,\phi) = \psi_n\left(\sqrt{2q}\cosh\phi\right)\psi_n\left(\sqrt{2q}\sinh\phi\right)$ and $\psi_n(x)$ are the harmonic oscillator eigenfunctions.

This wave function behaves in a regular way when the three–geometry degenerates. However, the linear combination

$$\Psi_k(q,\phi) = \int_{-\infty}^{+\infty} d\phi_0 e^{-ik\phi_0} \Psi_{\phi_0}(q,\phi)$$ (17)

is an eigenfunction of the operator π_ϕ with eigenvalue k and therefore, has non zero flux through each three–surface. This wave function cannot close off with a four–geometry and will oscillate an infinite number of times when the three–geometry collapses to zero. In fact, once the integral over ϕ_0 is performed, the wave function (17) takes the form

$$\Psi_k(q,\phi) = K_{i\frac{k}{2}}(q) e^{-ik\phi},$$ (18)

where $K_{ik/2}(q)$ is a modified Bessel function of imaginary order. This wave function will oscillate for $q < |k/2|$, while for $q > |k/2|$, it will decrease exponentially. Thus, $|k/2|$ can be considered as the throat radius of the wormhole [7]. Since (17) is a Fourier transform, it suggests a kind of uncertainty relation between the asymptotic field and the wormhole throat radius. This could be expected because k does not only represent the throat size but it is also the eigenvalue of the momentum conjugate to the scalar field and therefore, k and ϕ_0 naturally satisfy an uncertainty relation [11].

3 Conformal scalar field

When an homogeneous scalar field $\varphi(\tau)$ is conformally coupled to a FRW spacetime, the path integral which defines the wormhole wave function can also be done explicitly in a similar way. The FRW metric can be written in the convenient form

$$ds^2 = \frac{2G}{3\pi} a^2(\tau) \left(d\tau^2 + d\Omega_3^2 \right).$$ (19)

With the field redefinition $\varphi(\tau) = \chi(\tau) a^{-1}(\tau)$, the Euclidean action for this system becomes

$$I = \frac{1}{2} \int_0^\infty d\tau \left(-\dot{a}^2 - a^2 + \dot{\chi}^2 + \chi^2 \right) + \frac{1}{2} \frac{\dot{a}}{a} \left(a^2 - \chi^2 \right) |_{\tau=\infty} .$$ (20)

The boundary conditions are, in this case,

$$a(0) = a', \quad \varphi(0) = \varphi' = \frac{\chi'}{a'}, \quad \frac{\dot{a}}{a}(\infty) = 1, \quad \varphi(\infty) = \frac{\chi}{a}(\infty) = \varphi_0.$$ (21)

The conditions at $\tau = 0$ indicate which are the arguments of the wave function. The conditions at $\tau \to \infty$ mean that the spacetime is asymptotically Euclidean and that the field φ takes the value φ_0 at infinity. The wormhole wave function will be labeled by the asymptotic field value φ_0:

$$\Psi_{\varphi_0}(a', \chi') = \int_{\mathcal{C}_{wh}} \mathcal{D}a \mathcal{D}\chi e^{-I[a,x]},$$ (22)

where $\mathcal{C}_{wh}$ is the class of histories that satisfy (21).

Let

$$\bar{a}(\tau) = a' \cosh \tau + \frac{1 + \varphi_0^2}{1 - \varphi_0^2} \left(a' - \frac{2\varphi_0}{1 + \varphi_0^2} \chi' \right) \sinh \tau,$$

$$\bar{\chi}(\tau) = \chi' \cosh \tau + \frac{1 + \varphi_0^2}{1 - \varphi_0^2} \left(\frac{2\varphi_0}{1 + \varphi_0^2} a' - \chi' \right) \sinh \tau$$ (23)

be the solution of the classical equations of motion that satisfies (21). Then, as in the case of a minimal coupling and due to the fact that the action is also quadratic, under the shift of the integration variables $a = \bar{a} + A, \quad \chi = \bar{\chi} + X$, where the new variables A and X are such that

$$A(0) = X(0) = 0, \quad \frac{\dot{A}}{A}(\infty) = 1, \quad \frac{X}{A}(\infty) = \varphi_0,$$ (24)

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the path integral (22) transforms into

$$\Psi_{\varphi_0}(a', \chi') = e^{-I_0(a', \chi')} \int \mathcal{D}A\mathcal{D}X e^{-I_2[A,X]}, \tag{25}$$

where $I_2[A, X]$ does not depend on a' or χ' and $I_0(a', \chi')$ is the action of the classical solution (23). Then, up to numerical factors,

$$\Psi_{\varphi_0}(a, \chi) = \exp\left\{-\frac{1}{2}\left(\frac{1 + \varphi_0^2}{1 - \varphi_0^2}\right)\left(a^2 + \chi^2 - \frac{4\varphi_0}{1 + \varphi_0^2}a\chi\right)\right\}. \tag{26}$$

As it was expected, this wave function is also a solution of the Wheeler–DeWitt equation. The condition

$$\varphi_0^2 < 1 \tag{27}$$

must be fulfilled by the asymptotic field in order to have a positive effective gravitational constant in the asymptotic region [8]. Then, (26) will represent the wormhole wave function. Condition (27) will also allow us to write this wave function as a linear combination of those obtained in [2] and [7], as it happens in the previous section:

$$\Psi_{\varphi_0}(a, \chi) = \sum_{n=0}^{\infty} \varphi_0^n \left(1 - \varphi_0^2\right)^{\frac{1}{2}} \Psi_n(a, \chi), \tag{28}$$

where $\Psi_n(a, \chi) = \psi_n(a)\psi_n(\chi)$.

4   Summary and conclusions

In the context of the minisuperspace models which consist of a homogeneous massless scalar field minimally and conformally coupled to a FRW spacetime, it has been performed the path integral that defines the wormhole wave function. The wormhole boundary conditions require that the spacetime be asymptotically Euclidean. This implies that the asymptotic gravitational momentum must have a given value at infinity. Therefore, a surface term which takes this fact into account must be added to the action. The path integral over the lapse function can be interpreted as a sum over all proper time separations between the asymptotic region and the surface in which the arguments of the wave function are defined. This separation is infinite and, therefore, the whole sum is trivially reduced to a single contribution. The action is quadratic and this allowed us to perform the whole path integral. The problem of the unboundedness from below of the action is harmless in these models, since the unbounded part can be isolated in a term which does not depend on the arguments of the wave function and can, therefore, be factored out of the wave function. In the case of a conformal scalar field, one must also impose the condition that the asymptotic gravitational constant be positive. When this physical requirement is satisfied, the wormhole wave functions can be written as linear combinations of harmonic oscillator wave functions.

5 Acknowledgements

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ON INDUCING FINITE DIMENSIONAL PHYSICAL FIELD REPRESENTATIONS FOR MASSLESS PARTICLES IN EVEN DIMENSIONS

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Abstract

Assuming trivial action of Euclidean translations, the method of induced representations is used to derive a correspondence between massless field representations transforming under the full generalized even dimensional Lorentz group, and highest weight states of the relevant little group. This gives a connection between 'helicity' and 'chirality' in all dimensions. We also state restrictions on 'gauge independent' representations for physical particles that this induction imposes.

1. Introduction

For $d = 2n + 2$, the generalized Lorentz group $SO(2n + 1, 1)$ commutation relations are

$$[J_{AB}, J_{CD}] = i(\delta_{AC} J_{BD} + \delta_{BD} J_{AC} - \delta_{AD} J_{BC} - \delta_{BC} J_{AD}) \tag{1.1}$$

with $A, B = 0, \ldots (d-1)$ where δ is the d dimensional Kronecker symbol ($\delta_{AB} = 1$ if $A = B$; 0 otherwise), the boosts in the ith direction are defined as

$$K_i = i J_{i0} = -i J_{0i} \tag{1.2}$$

and for $i, j \neq 0$, the rotation generator in the i, j hyperplane is $J_{ij} = -J_{ji}$. Alternatively, the commutation relations may be written

$$[J_{AB}, J_{CD}] = i(g_{AC} J_{BD} + g_{BD} J_{AC} - g_{AD} J_{BC} - g_{BC} J_{AD}) \tag{1.3}$$

where $g = \text{diag}(-1, 1, \ldots, 1, 1)$ with the boosts generated by

$$K_i = J_{i0} = -J_{0i} \tag{1.4}$$

and the rotations generated by $J_{ij} = -J_{ji}$, for $i \neq j$. The -1 in the metric in (1.3) arises from the i on the boost generators in (1.1) ; to 'Wick' rotate the noncompact algebra to the compact algebra of the special orthogonal algebra $SO(2n + 2)$, or vice versa,

$$SO(2n + 1, 1) \leftrightarrow SO(2n + 2)$$

$$i J_{i0} \leftrightarrow J_{i,d} \tag{1.5}$$

$$g \leftrightarrow \delta.$$

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$SO(2n+2) \equiv D_{n+1}$ has rank $= n+1$, dimension $= (n+1)(2n+1)$, # of roots $= 2n(n+1)$, # of positive roots $= n(n+1)$, # of simple positive roots $= (n+1)$, where the roots are all the raising and lowering operators in the algebra, the simple roots are the linearly independent raising or lowering operators, and the positive simple roots are just the linearly independent raising operators in the algebra.

The relevant *little group* of Wigner is defined to be the maximal subgroup of the Lorentz group that leaves invariant the 'standard' momentum of a particle in $d = 2n+2$ dimensions. We choose our *massless* particle to be moving in the $d-1$ direction with d momentum $k_\mu = (k, 0, \ldots, k); \mu = 0, \ldots, d-1$. For $d = 2n+2$ the little group is generated by the n commuting rotations $J_{12}, J_{34}, \ldots, J_{(2n-1,2n)}$ which will be the Cartan generators (and hereafter the 'weight notation', notated by a subscript W, will refer to the specification of a state in terms of the eigenvalues of this ordered set of Cartan generators), *and* the $2n$ 'translations'

$$L_i^+ \equiv J_{i,2n+1} + iJ_{i,0}, \ i = 1, \ldots 2n \tag{1.6}$$

which, using (1.1) can be seen to form an Abelian subalgebra

$$[L_i^+, L_j^+] = 0. \tag{1.7}$$

Each translation indexed by i is a sum of a boost in the ith direction and a rotation in the $i, d-1$ plane. The commutation relations of the little group are

$$[L_i^+, L_j^+] = 0$$
$$[J_{ij}, L_k^+] = i(\delta_{ik} L_j^+ - \delta_{jk} L_i^+) \tag{1.8}$$
$$[J_{ij}, J_{kl}] = i(\delta_{ik} J_{jl} - \text{permutations}).$$

The little group (1.8) is not semi-simple (it has an Abelian subalgebra of translations) and is isomorphic to $E(d-2)$, the Euclidean group in $d-2$ dimensions.

$$L_i^- \equiv J_{i,2n+1} - iJ_{i,0} = L_i^+ - 2iJ_{i,0} \tag{1.9}$$

also form an Abelian subalgebra (but they do not belong to the little group for above choice of standard momentum), since

$$[L_i^-, L_j^-] = 0. \tag{1.10}$$

Note also that

$$[L_i^+, L_j^-] = 2iJ_{ij}, i \neq j$$
$$= 2J_{2n+1,0}, i = j \tag{1.11}$$

and under complex-conjugation

$$(L_i^+)^* = -L_i^-$$
$$(L_i^-)^* = -L_i^+. \tag{1.12}$$

Much will be said about the significance of $L^\pm$ shortly.

We now state a key assumption: **the translation generators annihilate physical states.** There are a number of reasons for this: (1) *Finite dimensional representations only:* The group transformation corresponding to the translation L_i^+, is written as $D_i^+(\chi_i) = e^{-i\chi_i L_i^+}$; the dimensionality of the representation is characterized by the length of the translation vector $\sum_i \chi_i^2$. For finite dimensional representations the translation parameter $\chi_i = 0, \forall i$. (2) *Gauge-independence:* In general, an eigenstate of the Cartan generators of the little group $J_{12}, \ldots J_{(2n-1,2n)}$ is not an eigenstate of the translation generators L_i^+ since they do not commute. The eigenstates of the Cartan generators can be written as the d dimensional polarization vectors (e.g. in four dimensions the generator J_{12} which generates z-rotations has the eigenvectors $\epsilon^\mu = (0, 1, \pm i, 0)$). It can be checked, for instance, that the transversality condition required for the photon vector potential to be a Cartan eigenstate is not invariant under finite translations. In fact, the translations generate effects identical to Abelian 'gauge' transformations [1] [2] . Thus the requirement of trivial translations is the requirement that only 'gauge independent' objects be considered. We believe that these two consequences of trivial translations are desirable from the point of view of making a scattering theory for a finite number of physical degrees of freedom without auxiliary conditions, as in [3] [4]. (3) *'Factorization' of invariant operators:* It can also be shown as a rather nontrivial consequence [5] that the eigenvalue of an invariant operator in the enveloping algebra of the higher dimensional Lorentz group factorizes into the eigenvalue of a generalized Pauli-Lubanski pseudovector and and a simple factor related to the boost generator.

2. The Main Theorem

Only the main results are highlighted here and the interested reader is referred for further details to [6]. Define a *physical field* Λ as a representation transforming under the full higher dimensional Lorentz group and obeying the condition of trivial translations $L_i^+ \Lambda = 0$, $i = 1, 2 \ldots 2n$.

Main Theorem: *A physical field Λ is a highest weight of the Lorentz group.*

Proof: Take the full group to be $SO(2n + 1, 1)$. The little group is then $E(2n)$. By definition, a physical field is annihilated by all the translations $L_1^+, \ldots L_{2n}^+$. Now, a highest weight is by definition the state annihilated by *all* the linearly independent raising operators. For $SO(2n + 1, 1)$, which is rank $n + 1$, we need thus to find the $n + 1$ positive simple roots and show that they all annihilate the physical field. To this end, we first want to prove the following fact:

Lemma: *All the raising operators can be made using linear combinations of the translation generators.*

Proof of Lemma: Since there are $2n$ available translations, and $n + 1$ required linearly independent raising operators, for $n \geq 1$, i.e. for four dimensions or more, there are certainly enough translations available to make all the linearly independent raising operators.

We choose the Cartan generators in $SO(2n+1,1)$ to be $J_{12}, \ldots, J_{2n-3,2n-2}, J_{2n+1,0}$. We claim that the complete set of linearly independent raising operators for $n > 1$ is

$$
\begin{aligned}
&L_1^+ \pm iL_2^+ \\
&L_3^+ + iL_4^+ \\
&\quad \vdots \\
&L_{2n-1}^+ + iL_{2n}^+.
\end{aligned}
\tag{2.1}
$$

For instance, for the four dimensional Lorentz group $(n = 1)$ the two linearly independent raising operators are $L_1^+ \pm iL_2^+$. To check, in any dimension that (2.1) is indeed the complete set of linearly independent raising operators, commute each member of the set with the Cartan generators using (1.1) and (1.8) to obtain the coordinates of the positive simple roots in the Cartan basis. Translating this to the Dynkin basis [6] [7] obtain the rows of the Cartan matrix. But since the rows of the Cartan matrix are defined to be the coordinates of the linearly independent raising operators under commutation with the Cartan generators, (2.1) is in fact all of them. With the physical field condition and (2.1) the theorem is proved. **QED**

Note that using (1.11) we obtain

$$
[L_i^+ + iL_{i+1}^+, L_i^- - iL_{i+1}^-] = 4(J_{2n+1,0} + J_{i,i+1}), \quad i = 1, 3, 5, 2n-1
\tag{2.2}
$$

and

$$
[L_1^+ - iL_2^+, L_1^- + iL_2^-] = 4(J_{12} - J_{30}).
\tag{2.3}
$$

Since the right hand side lies in the Cartan subalgebra, the lowering operator corresponding to each of the raising operators is obtained by replacing $L_i^+ \to L_i^-$, $i \to -i$ in (2.1). With the definition

$$
\begin{aligned}
E_{1,1,0,\ldots,0}^+ &= A_+^+ \equiv L_1^+ + iL_2^+ \\
E_{-1,1,0,\ldots,0}^+ &= A_-^+ \equiv L_1^+ - iL_2^+ \\
E_{1,-1,0,\ldots,0}^- &= A_-^- \equiv L_1^- - iL_2^- \\
E_{-1,-1,0,\ldots,0}^- &= A_+^- \equiv L_1^- + iL_2^-
\end{aligned}
\tag{2.4}
$$

where the subscripts on the roots E denote the eigenvalue under commutation with the Cartan generators,

$$
[A_+^+, A_-^+] = [A_+^+, A_+^-] = [A_+^-, A_-^-] = [A_-^+, A_-^-] = 0
\tag{2.5}
$$

we note that A_+^+ and A_-^+ are raising operators in two orthogonal directions (with A_-^- and A_+^- the orthogonal lowering operators). In four dimensions, A_+^+, A_-^+ are the two linearly independent raising operators, and A_-^-, A_+^- are the corresponding lowering operators. From the physical field condition

$$
L_1^+ \Lambda = L_2^+ \Lambda = 0 \Leftrightarrow A_+^+ \Lambda = A_-^+ \Lambda = 0
\tag{2.6}
$$

which means that the physical field is a highest weight, and complex conjugating the last two equations with the use of (1.12)

$$A_-^- \Lambda^* = A_+^- \Lambda^* = 0 \tag{2.7}$$

which shows that the complex conjugate is the lowest weight. Looking, for instance, at the weight diagram for the left and right handed spinors in four dimensions (in Dynkin notation)

$$
\begin{array}{cc}
\text{Left} & \text{Right} \\
(+1 \quad 0) & (0 \quad +1) \\
(-1 \quad 0) & (0 \quad -1)
\end{array} \tag{2.8}
$$

we confirm the result of (2.7) that indeed the spinors are inequivalent and self-conjugate (i.e. the weight diagrams reflect to minus themselves).

3. Helicity-Chirality Correlation in Higher Dimensions

In one of his classic papers on 'Feynman Rules for Any Spin', Weinberg [8] proves that the annihilation operator for a massless particle of helicity λ and the creation operator for the antiparticle with helicity $-\lambda$ can only be used to form a field transforming as $U[\Lambda]\psi_n(x)U[\Lambda]^{-1} = \sum_m D_{nm}[\Lambda^{-1}]\psi_m(\Lambda x)$ under those representations $[A,B]$ of $SO(3,1)$ such that $\lambda = B - A$[1]. This restriction arises solely due to the non-semi-simplicity of the little group for massless particles; in particular due to the requirement that the Euclidean 'translations' of the little group act trivially on the physical Hilbert space. As a direct consequence of Weinberg's result it is observed that in four-dimensions a physical left-handed, helicity $-j$ particle can only correspond to a representation $[j + n, n]$, (n is an integral multiple of $\frac{1}{2}$) whereas a physical right-handed, helicity j particle can only correspond to the representation $[n, j + n]$. The generalization of the statement to higher even dimensions will be stated as the following corollary to the main theorem:

Helicity-Chirality Correlation: *A physical field of the full group corresponds to a highest weight state of the little group (given trivial action of translations), and the eigenvalue of each generator common to the full group and the little group remains unchanged under the projection of a representation of the full group to a representation of the little group.*

Proof: Since the little group with trivial translations is just the orthogonal group in two lower dimensions, a subset of the linearly independent raising operators of the full group is exactly the complete set of linearly independent raising operators of the little group, with the caveat that they have to be appropriately Wick rotated to obtain the compact form. For $SO(2n + 1, 1)$ ($n > 1$) as the full group, the subset of (2.1) without the last raising operator, $L_{2n-1} + iL_{2n}$, upon Wick rotation, is the complete set of linearly independent raising operators for $SO(2n)$. For example, for $SO(5, 1)$ the raising operators are $L_1^+ \pm iL_2^+, L_3^+ + iL_4^+$, whereas for the little group of trivial translations $\sim SO(4)$ they are $L_1^+ \pm L_2^+$. Since the full set of raising operators annihilates the full group highest weight,

---

[1] *A and B correspond to independent $SU(2)'s$.*

and the projection from the full group to the little group is an orthogonal projection (since the boost generator is the only non-common generator and it commutes with the Cartan generators of the little group), it follows that the little group state is a highest weight state of the little group, and most importantly, the eigenvalue of the Cartan generators of the little group is invariant under the projection (hence the consequences of the corollary are most explicit in the weight notation). To recapitulate, the helicity-chirality correlation in higher dimensions is nothing but the fact that under projection of the field representation from the full Lorentz group to a little group state, the eigenvalue of the Cartan generators remains unchanged.

In conclusion, we demonstrate that our main theorem and corollaries reproduce some familiar results of four dimensions:

•In weight notation, the last result shows why a chiral left-handed field transforming as $(-\frac{1}{2}, \frac{1}{2})_W$ ($[\frac{1}{2}, 0]$ in conventional $SU(2) \times SU(2)$ notation) corresponds to a helicity $-\frac{1}{2}$ particle and a chiral right-handed field transforming as $(\frac{1}{2}, \frac{1}{2})_W$ ($[0, \frac{1}{2}]$ in conventional notation) corresponds to a helicity $\frac{1}{2}$ particle.

•Using $\frac{1}{2}(\mathbf{J} \pm i\mathbf{K}) = \mathbf{A}, \mathbf{B}$ and the definition $J_3 \Lambda \equiv J_{12} \Lambda = \lambda \Lambda$, with λ the helicity, we get, on using the physical field condition and (2.1)

$$
\begin{aligned}
(L_1^+ - iL_2^+)\Lambda = 0 \rightarrow (A_1 - iA_2)\Lambda = 0 \rightarrow A_3 = -A \\
(L_1^+ + iL_2^+)\Lambda = 0 \rightarrow (B_1 + iB_2)\Lambda = 0 \rightarrow B_3 = B
\end{aligned}
\tag{3.1}
$$

since $\mathbf{A}, \mathbf{B}$ generate independent $SU(2)$ algebras and $A_1 - iA_2$ is the lowering operator for one and $B_1 + iB_2$ is the raising operator for the other. Then, by definition $J_3 = A_3 + B_3 = B - A = \lambda$ which is Weinberg's condition [8] .

•As mentioned earlier, the translations are also generators of Abelian gauge transformations. Requiring them to be trivial restricts us to gauge independent, finite dimensional repesentations of the full group. Our theorem and the corollary then tell us what is the little group representation corresponding to this gauge independent full group representation. For example, the representation corresponding to the field strength tensor in the conventional ($SU(2) \times SU(2)$), Dynkin and weight basis is:

$$
F^{\mu\nu} = [1, 0] + [0, 1] = (2, 0)_D + (0, 2)_D = (-1, 1)_W + (1, 1)_W
\tag{3.2}
$$

which has J_{12} eigenvalues ± 1 and so is admissible as the J_{12} eigenvalue remains unchanged and corresponds to the correct helicity of the photon as we project to the little group state. However, the vector potential corresponds to

$$
A^\mu = [\frac{1}{2}, \frac{1}{2}] = (1, 1)_D = (0, 1)_W
\tag{3.3}
$$

which has a J_{12} eigenvalue of 0 which does not correspond to a transversely polarized helicity ± 1 photon.

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NONLINEAR MODES OF THE TENSOR DIRAC EQUATION AND CPT VIOLATION

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Abstract

Recently, it has been shown that Dirac's bispinor equation can be expressed, in an equivalent tensor form, as a constrained Yang-Mills equation in the limit of an infinitely large coupling constant. It was also shown that the free tensor Dirac equation is a completely integrable Hamiltonian system with Lie algebra type Poisson brackets, from which Fermi quantization can be derived directly without using bispinors. In this paper we investigate the Yang-Mills equation for a finite coupling constant. We show that the nonlinear Yang-Mills equation has exact plane wave solutions in one-to-one correspondence with the plane wave solutions of Dirac's bispinor equation. We apply the theory of nonlinear dispersive waves to establish the existence of wave packets. We investigate the CPT violation of these nonlinear wave packets, which could lead to new observable effects consistent with current experimental bounds.

1 Introduction

In a recent paper [1] it was shown that square-integrable positive energy bispinor fields in a Minkowski space-time cannot be physically distinguished from constrained tensor fields. It was also shown [1], [2] that the free tensor Dirac equation is a completely integrable Hamiltonian system with (non-canonical) Lie algebra type Poisson brackets, from which Fermi quantization can be derived directly without using bispinors.

Also, it was shown [1] that the tensor Dirac Lagrangian may be derived from the following Yang-Mills Lagrangian for $SL(2,C) \times U(1)$ gauge potentials A_α^K and complex scalar field ρ:

$$L = -\tfrac{1}{4} \, \text{Re} \left[A_K^{\alpha\beta} \, A_{\alpha\beta}^K \right] + \overline{(D_\alpha \sigma)} \, (D^\alpha \sigma) - V(|\rho|^2) \tag{1}$$

where $\sigma = \rho + c$ where c is a constant, and $V = V(|\rho|^2)$ is a smooth (at least twice differentiable) function of $|\rho|^2$. The gauge potentials A_α^K satisfy the orthogonal constraint:

$$A_\alpha^K \, A_{K\beta} = -|\rho|^2 \, g_{\alpha\beta} \tag{2}$$

where $g_{\alpha\beta}$ is the metric tensor. More detailed discussion of formulas (1) and (2) is given in Section 2. With the further condition:

$$\lim_{g \to \infty} g^{-2} V = \tfrac{1}{2} |\rho|^4 \tag{3}$$

where g is the Yang-Mills coupling constant, and setting $c = 2m_e/g$ where m_e is the fermion mass, then the Dirac Lagrangian equals the Yang-Mills Lagrangian (1) in the limit that the coupling constant g tends to infinity.

The fact that the free Dirac equation is a constrained Yang-Mills equation in the limit of a large coupling constant is significant for both classical and quantized theories. The classicial theory, which we regard as the first quantized theory, is characterized by the classical observables given by the electric current J^α, the energy-momentum tensor $T^{\alpha\beta}$, and the spin-polarization tensor $S^{\alpha\beta\gamma}$. These classical observables are sufficient to describe many experiments with electron beams [3]. The present paper addresses these classical observables.

In Section 2, we consider the Lagrangian (1) for the case of a finite coupling constant g. We show that for any smooth self-interaction $V(|\rho|^2)$, the constrained Euler-Lagrange equations have *exact* plane wave solutions in one-to-one correspondence with the plane wave solutions of Dirac's equation. We apply the theory of non-linear dispersive waves [4] to establish the existence of wave packets.

For the special case $V = \frac{1}{2} g^2 |\rho|^4$ we show that:

a) The mass, m, of each plane wave is independent of amplitude and equals $\frac{1}{2} g |c|$.

b) The wave packets are identical with bispinor wave packets.

c) The wave packets are covariant under the CPT operation (defined in Section 3).

When $V \neq \frac{1}{2} g^2 |\rho|^4$, the properties (a), (b), and (c) are all violated. However, we show in Section 3 that these violations could lead to new experimental observations consistent with present bounds for CPT violations.

2 Plane Waves and Wave Packets

In [1] we showed that Dirac's bispinor Lagrangian equals the Yang-Mills Lagrangian (1) in the limit of an infinitely large coupling constant g. In the remainder of this paper we will investigate the possibly observable effects of a finite coupling g.

First, we show in this section that the Euler-Lagrange equations for (1) and (2) have exact plane wave solutions in one-to-one correspondence with the plane wave solutions of Dirac's equation. For finite coupling g, and general V, the mass of each plane wave depends on its amplitude given in formula (2) as $|\rho|$. However, for the special case in which $V = \frac{1}{2} g^2 |\rho|^4$, we will see that the mass equals the constant $\frac{1}{2} g |c|$, and hence is independent of amplitude.

Second, we will establish the existence of wave packets using results from the theory of nonlinear dispersive waves [4]. The most significant departure from linearity is the splitting of the group velocity for finite g and general V. However, again for the special case in which $V = \frac{1}{2} g^2 |\rho|^4$, the velocity splitting does not occur, and the wave packets are identical to the bispinor wave packets which are derived from Dirac's equation.

The Euler-Lagrange equations for (1) and (2) are given by:

$$D^\alpha \overline{A}_{\alpha\beta} = -2 \lambda_\beta^\alpha \overline{A}_\alpha$$
$$D^\alpha A_{\alpha\beta}^0 - 2g \operatorname{Re} \left[i \, \sigma D_\beta \sigma \right] = -2 \operatorname{Re} \left[\lambda_\beta^\alpha \right] A_\alpha^0$$
$$D^\alpha D_\alpha \, \sigma + V'\rho = \operatorname{Re} \left[\lambda_\alpha^\alpha \right] \rho \tag{4}$$

where $\lambda^{\alpha\beta} = \lambda^{\beta\alpha}$ are the Lagrange multipliers for the constraint (2), where V' denotes the derivative of V with respect to $|\rho|^2$, and where the Yang-Mills covariant derivatives D_γ acting on $A_{\alpha\beta}^K$ and σ are given by:

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$$D_\gamma A^0_{\alpha\beta} = \nabla_\gamma A^0_{\alpha\beta}$$

$$D_\gamma \vec{A}_{\alpha\beta} = \nabla_\gamma \vec{A}_{\alpha\beta} - g\,\vec{A}_\gamma \times \vec{A}_{\alpha\beta}$$

$$D_\gamma \sigma = \nabla_\gamma \sigma + ig\, A^0_\gamma \sigma \tag{5}$$

Again, $\sigma = \rho + c$ and solutions A^K_α and ρ of equations (4) are required to satisfy the constraint (2).

Plane wave solutions of equations (4) are defined by:

$$A^0_\alpha(x^\beta) = A^0_\alpha(0)$$

$$\vec{A}_\alpha(x^\beta) = e^{2i\theta(x^\beta)T}\,\vec{A}_\alpha(0)$$

$$\rho(x^\beta) = \rho(0) \tag{6}$$

where $x^\beta \varepsilon\ R^4$ denotes the space-time coordinates, T generates a one-parameter subgroup of SL(2, C) gauge transformations, and $\theta(x^\beta) = p_\beta\, x^\beta$ where $p_\beta \varepsilon\ R^4$ denotes the momentum variables. Note that if $A^K_\alpha(0)$ and $\rho(0)$ satisfy the orthogonal constraint (2), then the same is true for $A^K_\alpha(x^\beta)$ and $\rho(x^\beta)$ for all $x^\beta \varepsilon\ R^4$, since in formula (6), the SL(2, C) gauge transformations generated by T preserve the orthogonal constraint. Note also that

$$T\,(\vec{A}_\alpha) = i\,\vec{\omega}\ \times \vec{A}_\alpha \tag{7}$$

for some $\vec{\omega}\ \varepsilon\ C^3$ satisfying $\vec{\omega}\ \cdot \vec{\omega} = 1$. (The reader is reminded that SL(2, C) is the complexification of SU(2) for which we can take $\vec{\omega}\ \varepsilon\ R^3$.)

On differentiating formula (6) we get using (7):

$$\nabla_\beta A^0_\alpha = 0$$

$$\nabla_\beta \vec{A}_\alpha = -2p_\beta\,\vec{\omega} \times \vec{A}_\alpha$$

$$\nabla_\beta \rho = 0 \tag{8}$$

Note in formula (8) that the $\vec{A}_\alpha$ have twice the rotation rate of bispinors, and $p^\alpha p_\alpha = m^2$ where m is the mass in Dirac's equation. Suppose that the plane waves (6) satisfy the same conditions which are satisfied by bispinor plane waves, given as follows:

$$p^\alpha A^0_\alpha = 0$$

$$p^\alpha \vec{A}_\alpha = \pm\, m\, |\rho|\ \vec{\omega} \tag{9}$$

where the positive sign is used for particles and the negative sign for antiparticles. Since ρ is constant by (8), formula (9) can be regarded as the initial conditions for the fields A^K_α. Note that formula (9) is consistent with $p^\alpha p_\alpha = m^2$, $\vec{\omega} \cdot \vec{\omega} = 1$, and the constraint (2), and moreover, p_α for particles becomes $-p_\alpha$ for antiparticles. Conversely, with p_α so defined, formula (9) defines $\vec{\omega}$ and hence the gauge generator T in formulas (6) and (7).

Substituting (8) into the first two equations (4), using (2), (5), and (9), we get:

$$\lambda_{\alpha\beta} = -2\, p_\alpha\, p_\beta + g^2\,\frac{A^0_\alpha A^0_\beta}{|\rho|^2}|\sigma|^2 + \frac{\vec{A}_\alpha \cdot \vec{A}_\beta}{|\rho|^2}\,(2m^2 \pm 2\,mg|\rho| + g^2|\rho|^2) \tag{10}$$

Note that $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}$ and by the constraint (2), $\lambda_{\alpha\beta}$ is real. (Recall that A^0_α is real and $\vec{A}_\alpha$ is complex.)

Now substituting (10) into the last equation (4), we get:

$$4m^2 \pm 6 \, mg \, |\rho| \, = V' + g^2 \, c\tilde{\rho} + g^2 \, (|\rho{+}c|^2 - 2|\rho|^2) \tag{11}$$

Clearly, since all other terms are real, $c\tilde{\rho}$ is also real. Without loss of generality, we assume that $c \geq 0$, hence ρ is real. Choosing $\rho \geq 0$ for particle plane waves and $\rho \leq 0$ for antiparticle plane waves, formula (11) becomes:

$$4 \left(m^2 - \frac{g^2 c^2}{4} \right) + 6g \left(m - \frac{gc}{2} \right) \rho = V' - g^2 \rho^2 \tag{12}$$

with the obvious solution:

$$V = \frac{g^2}{2} \, |\rho|^4$$

$$m = \frac{gc}{2} \tag{13}$$

We see in this case that the mass $m = gc/2$ is independent of amplitude.

Wave packets are defined to be plane waves with slowly changing parameters (e.g., amplitude, spin, and momentum). To describe such wave packets we introduce "slow" coordinates $y^\beta = \varepsilon x^\beta$, where ε is a small parameter, into formula (6) as follows [4]:

$$A_\alpha^0 \, (x^\beta) = \underset{\sim}{A}_\alpha^0 \, (y^\beta)$$

$$\vec{A}_\alpha \, (x^\beta) = e^{2i\varepsilon^{-1}\theta \, (y^\beta)T} \, \vec{\underset{\sim}{A}}_\alpha \, (y^\beta)$$

$$\rho \, (x^\beta) = \rho(y^\beta) \tag{14}$$

where $\theta \, (x^\beta) = \underset{\sim}{\theta}(\varepsilon \, x^\beta)$, etc. The resulting equations governing the wave packets [4] are given by:

$$p^\alpha p_\alpha = m^2$$

$$\nabla_\alpha \, p_\beta = \nabla_\beta \, p_\alpha$$

$$\nabla_\alpha \, J^\alpha = 0 \tag{15}$$

where now $p_\alpha = \nabla_\alpha \, \theta$, where $m = m(\rho)$ is given in formula (12), and

$$J_\alpha = F \, v_\alpha, \quad v_\alpha = p_\alpha/m, \quad F = \frac{8m}{g} \rho^2 + 4 \, \rho^3 \tag{16}$$

To analyze equations (15) we now consider a space-time with one space dimension. Then $v_\alpha = (v_0, v_1)$ and the group velocity is $v = v_1/v_0$.

That is,

$$v_0 = \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$v_1 = v \, \gamma \tag{17}$$

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Similarly, $x^\alpha = (t, x)$. Formula (15) becomes:

$$\frac{\partial}{\partial t}(F\gamma) + \frac{\partial}{\partial x}(Fv\gamma) = 0$$

$$\frac{\partial}{\partial t}(mv\gamma) + \frac{\partial}{\partial x}(m\gamma) = 0 \tag{18}$$

The equations for the characteristic curves for (18) are easily derived [4], and are given by:

$$\frac{dx}{dt} = \frac{v \pm \delta}{1 \pm v\delta} \tag{19}$$

where

$$\delta = \sqrt{\frac{Fm'}{F'm}} \tag{20}$$

where F' and m' denote the derivatives of F and m with respect to ρ. On the curves (19) we have:

$$\gamma^2 \, dv = \pm \frac{F'\delta}{F} \, d\rho \tag{21}$$

When $V = \frac{1}{2} g^2 |\rho|^4$, $m' = 0$ by formula (13) so that by (21), $dv = 0$, that is, v is constant on the characteristic curves (19). Since then $\delta = 0$, the curves (19) are straight lines. It is then straightforward to show that the wave packets are identical to bispinor wave packets.

In general, for wave packets to exist, δ, in formula (20) must be real. If $V \neq \frac{1}{2} g^2 |\rho|^4$, a general wave packet will split into two wave packets that propagate along the characteristic curves (19).

3   CPT and Velocity Splitting

By the Cartan map, the CPT operation which, for bispinors, is given by [5]:

$$\psi(x^\beta) \longrightarrow i\gamma_5 \psi(-x^\beta) \tag{22}$$

becomes for the tensor fields A_α^K and ρ:

$$A_\alpha^K(x^\beta) \longrightarrow A_\alpha^K(-x^\beta)$$
$$\rho(x^\beta) \longrightarrow -\rho(-x^\beta) \tag{23}$$

Note that because of the constant c, the Yang-Mills Lagrangian L in formula (1) is not covariant under the CPT operation (23). Nevertheless in the limiting case that the coupling constant g tends to infinity, the Euler-Lagrange equations commute with CPT. In this section we examine the question of CPT violation for finite coupling g.

One of the main tests for CPT covariance is the equality of particle and antiparticle masses [6]. According to formula (13), when $V = \frac{1}{2} g^2 |\rho|^4$ the masses are equal. Therefore, suppose instead that

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$$V = \frac{g^2}{2} |\rho|^4 + \varepsilon |\rho|^4 \tag{24}$$

where ε is a small parameter. Then to first order in ε, formula (12) gives:

$$m = \frac{gc}{2} + \frac{\varepsilon}{3g} \rho \tag{25}$$

Since $\rho \geq 0$ for particle plane waves and $\rho \leq 0$ for antiparticle plane waves, the mass difference Δm is:

$$\Delta m = \frac{2\varepsilon}{3g} |\rho| \tag{26}$$

On substituting formula (25) into (20), the velocity splitting 2δ becomes, to lowest order in ε and g^{-1} (ignoring factors close to one; i.e., $\sqrt{2/3}$):

$$2\delta = \sqrt{\frac{\Delta m}{m}} \tag{27}$$

Assuming a fractional mass difference for electrons and positrons of one part in a million, the velocity splitting would be $2\delta = 10^{-3}$ or 3×10^5 meters per second, which should be observable in experiments that measure the spreading of low energy electron wavepackets. CPT violations of 10^{-6} are consistent with current observations of particle-antiparticle mass difference and suggest new experiments to observe velocity splittings of 3×10^5 meters per second, or less [6].

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GALILEAN COVARIANT HARMONIC OSCILLATOR

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Abstract

A Galilean covariant approach to classical mechanics of a single particle is described. Within the proposed formalism we reject all non-covariant force laws defining acting forces which become to be defined covariantly by some differential equations. Such an approach leads out of the standard classical mechanics and gives an example of non-Newtonian mechanics. It is shown that the exactly solvable linear system of differential equations defining forces contains the Galilean covariant description of harmonic oscillator as its particular case. Additionally we demonstrate that in Galilean covariant classical mechanics the validity of the second Newton law of dynamics implies the Hooke law and vice versa. We show that the kinetic and total energies transform differently with respect to the Galilean transformations.

1 Introduction

Recently we have proposed a new approach to classical mechanics which leads to a manifestly Galilean covariant models of mechanics for a single interacting particle [1]. Our main goal was to construct a self-consistent and complete scheme avoiding all relations of standard classical mechanics which break the Galilean covariance. It is easy to see that all such relations belong to the class of the so-called "constitutive relations" [2] and in order to achieve our goal we had to reexamine the role of these relations in mechanics. The relation between momentum and velocity is an example of the Galilean covariant constitutive relation [3] while all explicit expressions of the mechanical forces in terms of positions and velocities, called force laws, obviously break this covariance. Hence, in a Galilean covariant formulation of classical mechanics of a single particle we have to reject all known force laws. To keep the formalism as predictive as the usual one we propose to determine all mechanical quantities from the set of differential equations of the evolution type.

Our program leads us to a broader than Newtonian formalism model of classical mechanics in which more than one vector-valued measure of mechanical interaction is introduced. The time evolution of these measures is described by a set of differential equations called the equations of the environment which are used to determine the interaction of the particle with its environment in a fully covariant way. The simplest version of such a scheme contains two measures of interaction:

the customary force $\vec{F}(t)$ measuring the momentum non-conservation and a new quantity which we have called the influence $\vec{I}(t)$ governing the time evolution of the acceleration. We do not assume *a priori* the Galilean covariant Newton's second law of dynamics in the form

$$M\vec{a}(t) = \vec{F}(t) \tag{1.1}$$

where M denotes the inertial mass of the particle because this equation is not of the evolution type for the acceleration and contains a physical constant. According to our general philosophy [2] we avoid to use any such constants unless we really need to introduce them as phenomenological parameters. In our case this will happen only for the equations of the environment for which without any doubt we are forced to use in the theory some information of the phenomenological character. All the remaining equations describing the particle are universal, interrelate only basic theoretical concepts and do not contain any phenomenological constant. In our theory the experimental input is used therefore only for the description of the environment and we consider this fact as a big advantage of our formalism. The relation between classical Newtonian mechanics based on the equation (1.1) and our scheme is established using (1.1) as a constraint put on the set of solutions of the differential equations. It is also a constraint put on solutions of the equation

$$\frac{d\vec{F}(t)}{dt} = \frac{1}{M}\vec{I}(t) \tag{1.2}$$

which in the framework of the Newton's mechanics follows from the definition of $\vec{I}(t)$. The solutions of our model which satisfy (1.1) we shall call Newtonian solutions while solutions satisfying the relation (1.2) only will be called the generalized Newtonian solutions.

2 Linear model

The aim of this talk is to illustrate our approach on a simple example of linear evolution equations for the force and influence. We shall show that such a model includes, as its particular case, the Newtonian mechanics of the material point which motion is defined by the force provided by a linear in position and velocity force law.

In the case under consideration the complete set of differential equations describing the system consists of two purely kinematical equations of motion

$$\frac{d\vec{x}(t)}{dt} = \vec{v}(t) \tag{2.1}$$

$$\frac{d\vec{v}(t)}{dt} = \vec{a}(t) \tag{2.2}$$

one dynamical equation of motion

$$\frac{d\vec{a}(t)}{dt} = \vec{I}(t) \tag{2.3}$$

one equation of balance

$$\frac{d\vec{p}(t)}{dt} = \vec{F}(t) \tag{2.4}$$

and the system of two equations of environment

$$\frac{d\vec{F}(t)}{dt} = \alpha \vec{F}(t) + \beta \vec{I}(t)$$

$$\frac{d\vec{I}(t)}{dt} = \gamma \vec{F}(t) + \delta \vec{I}(t) \tag{2.5}$$

where $\vec{x}(t), \vec{v}(t), \vec{a}(t)$ and $\vec{p}(t)$ are the trajectory function of the particle, its velocity, acceleration and momentum, respectively. The meaning of $\vec{F}(t)$ and $\vec{I}(t)$ has been explained above and the parameters α, β, γ and δ represent dimensional coupling constants specifying the model.

The model is covariant with respect to the Galilean transformations parametrized by a rotation R, a boost $\vec{u}$, and a space-time translation $(\vec{a}, b)$ if all mechanical quantities used obey the following transformation rules

$$\vec{x}(t) \rightarrow \vec{x}'(t') = R\vec{x}(t) + \vec{u}t + \vec{a} \tag{2.6}$$

$$\vec{v}(t) \rightarrow \vec{v}'(t') = R\vec{v}(t) + \vec{u} \tag{2.7}$$

$$\vec{a}(t) \rightarrow \vec{a}'(t') = R\vec{a}(t) \tag{2.8}$$

$$\vec{p}(t) \rightarrow \vec{p}'(t') = R\vec{p}(t) + m\,\vec{u} \tag{2.9}$$

$$\vec{F}(t) \rightarrow \vec{F}'(t') = R\vec{F}(t) \tag{2.10}$$

$$\vec{I}(t) \rightarrow \vec{I}'(t') = R\vec{I}(t) \tag{2.11}$$

where

$$t \rightarrow t' = t + b \tag{2.12}$$

and m is the Galilean mass of the particle [3] which we shall not assume to be equal to the inertial mass.

As we stressed in the Introduction the only external parameters characterizing the model are coupling constants in the equation of the environment (2.5). The mutual relation between them defines the shape of general solution of (2.1) - (2.5). Denoting by A the matrix of coupling constants

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2.13}$$

and their following combinations by $\lambda_\pm$

$$\lambda_\pm = \frac{1}{2}\left[trA \pm \sqrt{(TrA)^2 - 4detA}\right] \tag{2.14}$$

we may write down for $4detA > (TrA)^2$ the general solution of the equations (2.1) - (2.5) in the form

$$\vec{x}(t) = \vec{A} + \vec{B}t + \vec{C}t^2 + \vec{D}\,exp(\lambda_+ t) + \vec{E}\,exp(\lambda_- t) \tag{2.15}$$

$$\vec{v}(t) = \vec{B} + 2\vec{C}t + \vec{D}\,\lambda_+\,exp(\lambda_+ t) + \vec{E}\,\lambda_-\,exp(\lambda_- t) \tag{2.16}$$

$$\vec{a}(t) = 2\vec{C} + \vec{D}\,\lambda_+^2\,exp(\lambda_+ t) + \vec{E}\,\lambda_-^2\,exp(\lambda_- t) \tag{2.17}$$

$$\vec{p}(t) = \vec{P} - \frac{\delta - \lambda_+}{\gamma}\,\lambda_+^2\,\vec{D}\,exp(\lambda_+ t) - \frac{\delta - \lambda_-}{\gamma}\,\lambda_-^2\,\vec{E}\,exp(\lambda_- t) \tag{2.18}$$

$$\vec{F}(t) = -\frac{\delta - \lambda_+}{\gamma} \lambda_+^3 \; \vec{D} \; exp(\lambda_+ t) - \frac{\delta - \lambda_-}{\gamma} \lambda_-^3 \; \vec{E} \; exp(\lambda_- t) \tag{2.19}$$

$$\vec{f}(t) = \lambda_+^3 \; \vec{D} \; exp(\lambda_+ t) + \lambda_-^3 \; \vec{E} \; exp(\lambda_- t) \tag{2.20}$$

Vector-valued constants $\vec{A}, \vec{B}, \vec{C}, \vec{D}, \vec{E}$ and $\vec{P}$ are the integration constants of the system of differential equations (2.1) - (2.5) and in order to satisfy the transformation rules (2.6) - (2.11) they have to transform in the following way

$$\vec{A} \rightarrow \vec{A}' = R\vec{A} - bR\vec{B} + b^2 R\vec{C} - b\vec{u} + \vec{a} \tag{2.21}$$

$$\vec{B} \rightarrow \vec{B}' = R\vec{B} - 2bR\vec{C} + \vec{u} \tag{2.22}$$

$$\vec{C} \rightarrow \vec{C}' = R\vec{C} \tag{2.25}$$

$$\vec{D} \rightarrow \vec{D}' = R\vec{D} \; exp(-\lambda_+ b) \tag{2.26}$$

$$\vec{E} \rightarrow \vec{E}' = R\vec{E} \; exp(\lambda_- b) \tag{2.27}$$

$$\vec{P} \rightarrow \vec{P}' = R\vec{P} + m\vec{u} \tag{2.28}$$

which explicitly show how their values depend on the choice of the reference frame.

Here we should like to stress the difference between our approach, demanding the Galilean covariance as the most fundamental feature of the theory and standard expositions of mechanics which treat it almost always as a branch of the theory of ordinary differential equations. There is no principle of relativity in the theory of differential equations and, consequently, there is no problem of transformation properties of the solutions and integration constants. In contradistinction to mathematics, this subject is of primary interest to physics and we have to realize that the integration constants take the whole responsibility for the transformation properties of all physical quantities. This means that the original preparation of physical system already contains almost the whole information on the symmetries of this system. The time evolution of the system has only to preserve the original symmetries. It should not be unexpected that in our scheme which is an example of a non-Newtonian mechanics (and, as a matter of fact, its generalization) the careful analysis of the properties of integration constants and their relation to the initial conditions may lead out of the framework of standard classical mechanics.

There is a lot of different initial conditions which may be imposed on the solution (2.13 - 2.18). For instance, we may use the values of the first four derivatives of the function $\vec{x}(t)$ at the same instant of time t_0 to fix the values of the constants $\vec{A}$ to $\vec{E}$. It remains in obvious contradiction to the widely spread opinion that in mechanics only the initial position and velocity are needed for the unique determination of the trajectory. This is the property of Newtonian mechanics only in which the relation (1.1) is always satisfied. In our formalism the acceleration $\vec{a}(t)$ and the force $\vec{F}(t)$ are a priori independent as determined from independent equations and the relation (1.1) imposed on these quantities reduces the number of degrees of freedom for initial conditions. It enables us to calculate some parameters of a model in terms of the other. We shall see below that it may be used for determination of the inertial mass M in terms of the coupling constants given by elements of the matrix A.

The analysis of the model depends on the mutual relation between TrA and $detA$. In order to concentrate the attention on the harmonic oscillator problem we shall omit the case $TrA \geq 4detA$ because it does not describe oscillatory motion. The complete analysis of the problem will be found in [4].

3 Oscillatory motion

It is immediately seen from (2.14) and (2.15) that the trajectory (2.15) oscillates if the inequality $(TrA)^2 < 4detA$ holds and the oscillations may be damped or not depending on the value of $Re\lambda_\pm$. The reality of all mechanical quantities requires that the constants $\vec{D}$ and $\vec{E}$ are complex valued and they must be complex conjugated

$$\vec{D} = \vec{E}^* \tag{3.1}$$

which is the first condition restricting the arbitrariness of integration constants $\vec{A}, \vec{B}, \vec{C}, \vec{D}, \vec{E}$. The Newtonian condition (1.1), as well as its generalization of the form

$$\vec{F}(t) = M [\vec{a}(t) - 2\vec{C}] \tag{3.2}$$

are satisfied provided

$$M = -\frac{\delta - \lambda_+}{\gamma} \lambda_+ = -\frac{\delta - \lambda_-}{\gamma} \lambda_- \tag{3.3}$$

supplemented additionally in the Newtonian case by the Galilean invariant relation

$$\vec{C} = 0 \tag{3.4}$$

Relation (3.4) fixes invariantly one of the parameters of the solution and we shall use it as a criterion of the Newtonian character of the solution considered.

Substituting in (3.3) the values $\lambda_\pm$ from (2.14) we come to the conclusion that the equality (3.3) may be satisfied only for $\alpha = 0$ which gives

$$M = \beta \tag{3.5}$$

The value of the Galilean mass m remains arbitrary because it is a parameter which identifies the particle and has nothing to do with its possible interactions.

Taking into account (2.15) and (2.19) it is easy to see that the famous Hooke force law

$$\vec{F}(t) = -k\vec{x}(t) \tag{3.6}$$

may be satisfied in a selected reference frame for which $\vec{A} = \vec{B} = \vec{C} = 0$ i.e. only for reference frames satisfying the criterion (3.3) of the Newtonian character of mechanics. This means, because of the invariance of this relation, that the Newtonian condition (1.1) is equivalent to the requirement of the existence of the Hooke law. This fact has a far going consequences because in all treatments of the foundations of mechanics the forces are measured by dynamometers which operate on the principle of the Hooke law. Therefore any mechanics using such an operational definition of forces must be Newtonian. The Newton laws of mechanics follow thus from the adopted operational definition of force. In order to detect any violation of these laws we should first invent a new operational definition of the force not based on the Hooke law. It is indeed a very surprising conclusion which however uniquely follows from our more general approach to mechanics.

The above conclusion is less surprising after observing that the linear relation between momentum and velocity

$$\vec{p}(t) = M\vec{v}(t) \tag{3.7}$$

is possible also in the Newtonian mechanics only. Therefore the almost always assumed relation (3.7) prevent to observe any deviation from the Newton's laws. We have to conclude that many fundamental assumptions of standard mechanics are interrelated and their possible interrelations may be found only if the analysis is performed in the framework of the approach to mechanics more general than the standard one. Our method is just an example of such a scheme.

For the non-Newtonian mechanics we may replace the Hooke law by the relation

$$\vec{F}(t) = -k\left[\vec{x}(t) - \vec{A} - \vec{B}t - \vec{C}t^2\right] \tag{3.8}$$

which for $\vec{A} = \vec{B} = \vec{C} = 0$ reduces to (3.6). Using again the solutions (2.15) and (2.19) we come to the conclusion that (3.8) may be satisfied only if

$$k = \frac{\delta - \lambda_+}{\gamma}\lambda_+^3 = \frac{\delta - \lambda_-}{\gamma}\lambda_-^3 \tag{3.9}$$

Together with (3.3) and (2.14) it implies that

$$(TrA)\sqrt{(TrA)^2 - 4detA} = 0 \tag{3.10}$$

The square root must be different from 0 due to the condition $4detA > (TrA)^2$ assumed and therefore we must have

$$TrA = 0 \tag{3.11}$$

Since we already have got $\alpha = 0$ this condition gives $\delta = 0$. The frequency of oscillations is given by

$$\omega^2 = detA \tag{3.12}$$

and because of (3.5) and $\alpha = \delta = 0$ we have

$$\omega^2 = -\beta\gamma = -M\gamma \tag{3.13}$$

We may therefore conclude that in the framework of Galilean covariant approach to classical mechanics the non-Newtonian generalization of the standard harmonic oscillator is given by linear evolution equations for the force and the influence and that the matrix of the coupling constants has the form

$$A = \begin{pmatrix} 0, & M \\ -\dfrac{\omega^2}{M}, & 0 \end{pmatrix} \tag{3.14}$$

The most general Galilean covariant linear relation between the force, the position and the velocity is the non-Newtonian generalization of the superposition of Hooke and linear friction ($\eta < 0$) forces

$$\vec{F}(t) = \eta\left(\vec{v}(t) - \vec{B} - \vec{C}t\right) - \kappa\left(\vec{x}(t) - \vec{A} - \vec{B}t - \vec{C}t^2\right) \tag{3.15}$$

which, after substitution of (2.15), (2.16) and (2.19) into it leads to the following relations between parameters of the model

$$M\lambda_+^2 = \eta\lambda_+ - \kappa$$

$$M\lambda_-^2 = \eta\lambda_- - \kappa \tag{3.16}$$

if additionally the generalized Newtonian condition (3.2) is demanded. We are still restricted to the case $4detA > (TrA)^2$ which gives the only solution of (3.16) in the form

$$\delta = \frac{\eta}{M} \tag{3.17}$$

$$\gamma = -\frac{\kappa}{M^2} \tag{3.18}$$

and it immediately follows from it and (2.15) that the matrix of coupling constants

$$A = \begin{pmatrix} 0, & M \\ -\dfrac{\kappa}{M^2}, & \dfrac{\eta}{M} \end{pmatrix} \tag{3.19}$$

describes damped oscillatory motion with frequency given by

$$\omega = \frac{1}{2M}\sqrt{\eta^2 - 4\kappa M} \tag{3.20}$$

and an amplitude damping exponentially according to the factor $exp\dfrac{\eta}{2M}t$.

4   Kinetic and total energies

In the standard approach to classical mechanics the kinetic energy is defined by one of the equivalent expressions

$$k(t) = \frac{\vec{p}^2(t)}{2M} = \frac{M\,\vec{v}^2(t)}{2} = \frac{1}{2}\,\vec{p}(t)\cdot\vec{v}(t) \tag{4.1}$$

where M is the inertial mass of the particle. Relations (4.1) are a straightforward consequence of the Newtonian relation between momentum and velocity (3.7) which in Galilean covariant scheme proposed should be treated as additional assumption only. Discarding (3.7) as a *priori* valid we cannot identify the inertial mass present in second law of dynamics and the mass parameter appearing in the momentum transformation rule (2.9). The general relation between momentum and velocity written down with Galilean mass introduced into it has now, according to [3], the form

$$\vec{p}(t) = (m - M)\vec{v}(t_0) + M\vec{v}(t) \tag{4.2}$$

where $\vec{v}(t_0)$ is an integration constant having the meaning of an initial velocity which has to be specified from initial conditions.

We define the kinetic energy as bilinear form of momentum and velocity satisfying two fundamental conditions put on it:

i.) the balance equation

$$\frac{dk(t)}{dt} = \vec{F}(t)\cdot\vec{v}(t) \tag{4.3}$$

and

ii.) the Galilean transformation rule

$$k(t) \rightarrow k'(t') = k(t) + R\vec{p}(t)\cdot\vec{u} + \frac{1}{2}m\vec{u}^2 \tag{4.4}$$

According to these conditions [3] the kinetic energy is given by

$$k(t) = \frac{m - M}{2} \vec{v}^{\,2}(t_0) + \frac{M}{2} \vec{v}^{\,2}(t) \tag{4.5}$$

which, in notation introduced by (2.15) - (2.20) and in Newtonian regime $\vec{C} = 0$, may be written down as

$$k(t) = \frac{\left(\vec{P} - M\vec{B}\right)^2}{2(m - M)} + \frac{M}{2} \vec{v}^{\,2}(t) \tag{4.6}$$

To obtain the correct formula for the kinetic energy in non-Newtonian regime we shall start with the general expression

$$k(t) = A\vec{p}^{\,2}(t) + B\vec{v}^{\,2}(t) + C\vec{p}(t) \cdot \vec{v}(t) +$$
$$+ \vec{\lambda} \cdot \vec{x}(t) + \mu t + \nu t^2 + \Delta \tag{4.7}$$

The transformation rule (4.4) implies the following conditions and transformation properties which parameters in (4.7) have to obey

$$A' = A, \qquad\qquad B' = B, \qquad\qquad C' = C$$

$$B = -\frac{m}{2}(1 - 2mA)$$

$$C = 1 - 2mA$$

$$\nu' = \nu, \tag{4.8}$$

$$\vec{\lambda}' = R\vec{\lambda}$$

$$\mu' = \mu - 2\nu b - R\vec{\lambda} \cdot \vec{u}$$

$$\Delta' = \Delta + \nu b^2 - \mu b - R\vec{\lambda} \cdot \vec{a} + R\left(\vec{\lambda} \cdot \vec{u}\right) b$$

while the balance equation (4.3) gives

$$A = \frac{1}{2(m - M)}, \qquad B = \frac{mM}{2(m - M)}, \qquad C = -\frac{M}{m - M}$$

$$\vec{\lambda} = -2M\vec{C}$$

$$\mu = \frac{2M}{m - M}\left(\vec{P} - M\vec{B}\right) \cdot \vec{C} \tag{4.9}$$

$$\nu = -\frac{2M^2}{m - M}\vec{C}^{\,2}$$

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It is obvious that the balance equation (4.3) cannot fix the value of the constant Δ in (4.7) which remains arbitrary but has to satisfy the transformation rule listed in (4.8) as the last. For example, we may represent Δ in the following form

$$\Delta = -\vec{\lambda} \cdot \vec{x}_0 - \mu t_0 - \nu t_0^2 \tag{4.10}$$

with $(\vec{x}_0, t_0)$ denoting the space-time coordinates of an arbitrary event. They may be chosen as coordinates of an event for which the momentum and the velocity of the particle simultaneously vanish. Such a choice guarantees that the kinetic energy also vanishes at this point which we consider the most natural condition possible to demand.

Substituting all values of coefficients (4.9) into (4.7) we obtain

$$k(t) = \frac{(\vec{P} - M\vec{B})^2}{2(m - M)} + \frac{M}{2}\vec{v}^2(t) - 2M\vec{C} \cdot [\vec{x}(t) - \vec{x}_0(t)] -$$

$$- \frac{2M}{m - M}\left(\vec{P} - M\vec{B}\right) \cdot \vec{C} \, t_0 + \frac{2M^2}{m - M}\vec{C}^2 t_0^2 \tag{4.11}$$

and comparing it with the expression obtained for the Newtonian case (4.5) we see that the only parameter which controls the Newtonian character of mechanics is $\vec{C}$ the vanishing of which is equivalent to vanishing of $\vec{\lambda}$, μ, ν and Δ in any reference frame.

In contradistinction to the kinetic energy the definition of the total energy E for conservative system cannot be based on the above listed basic properties of the kinetic energy. The balance equation for the total energy

$$\frac{dE}{dt} = 0 \tag{4.12}$$

does not give any hint on the transformation rule of E. This rule cannot be of the same shape as for the kinetic energy since this immediately leads to a contradiction. Indeed, if we suppose

$$E \rightarrow E' = E + R\vec{p}(t) \cdot \vec{u} + \frac{1}{2}m\vec{u}^2 \tag{4.13}$$

the conservation law (4.12) implies that

$$\frac{d\vec{p}(t)}{dt} = 0 \tag{4.14}$$

which is true for free particles only. To construct the correct expression for the total energy we shall start from the general bilinear form of $\vec{v}, \vec{p}, \vec{x}, t, \vec{F}, \vec{I}$ which satisfies the following two conditions:

 i.) it reduces to the expression for $k(t)$ if $\vec{F} = \vec{I} = 0$,
 ii.) it satisfies the conservation law (4.12).

After straightforward but tedious calculations it can be shown that the only form which obeys these two conditions is given by

$$E = k(t) + \frac{\gamma(\delta^2 - \gamma M)}{2(\gamma M)^3}\vec{F}^2 + \frac{1}{2(\gamma^2}\vec{I}^2 - \frac{\delta}{(\gamma M)^2}\vec{F} \cdot \vec{I} +$$

$$+ \frac{\delta}{\gamma M}\vec{F} \cdot \vec{v} - \frac{1}{\gamma}\vec{I} \cdot \vec{v} + \frac{2}{\gamma M}\vec{F} \cdot \vec{C} \tag{4.15}$$

and consequently we shall take it as the definition of the total energy for the Galilean covariant harmonic oscillator.

It is now easy to see that under Galilean transformations the total energy changes according to the following rule

$$E \rightarrow E' = E + R\left(\vec{p}(t) + \frac{1}{\gamma M}\left(\delta\vec{F} - M\vec{I}\right)\right) \cdot \vec{u} + \frac{1}{2}m\vec{u}^2 \qquad (4.16)$$

The most important point is connected with the quantity in the second term. It shows that the momentum associated with the total energy E is not the momentum $\vec{p}(t)$ but

$$P = \vec{p}(t) + \frac{1}{\gamma M}\left(\delta\vec{F}(t) - M\vec{I}(t)\right) \qquad (4.17)$$

which is conserved in time because of the fundamental equations (2.4) - (2.5). The difference in the transformation rules for the kinetic and total energies is a new fact in mechanics which without our Galilean covariant approach to mechanics could not be derived. Here we would like to remark this so important fact is not specific for the non-Newtonian case only. As we have mentioned several times the Newtonian case which is equivalent to the Galilean invariant choice $\vec{C} = 0, \delta = 0$. However it must not be taken directly by putting these values into (4.15) because such a choice corresponds to the singular system of algebraic equations used to determine coefficients in (4.15). The correct result is given by

$$E_N = \frac{M}{2}\left(\vec{v}^2 + \frac{2\vec{I}\cdot\vec{v}}{\omega^2} + \frac{\vec{I}^2}{\omega^4}\right) + \chi\left(\vec{F}^2 + \left(\frac{M}{\omega}\vec{I}\right)^2\right) \qquad (4.18)$$

where $\omega^2 = -\gamma M$ according to (3.13) and χ is arbitrary parameter. It remains in full agreement with our previous result obtained in [1] within less general approach and gives for the total energy the Galilean transformation rule

$$E_N \rightarrow E'_N = E_N + R\left(\vec{p} + \frac{M}{\omega^2}\vec{I}\right) \cdot \vec{u} + \frac{1}{2}m\vec{u}^2 \qquad (4.19)$$

which means again that the total energy transforms differently from the kinetic energy and that its transformation properties are associated with a conserved quantity

$$P = \vec{p}(t) + \frac{M}{\omega^2}\vec{I}(t) \qquad (4.20)$$

and not with ordinary momentum $\vec{p}(t)$.

5 Conclusions

We have demonstrated that the requirement of the Galilean covariance of classical mechanics leads to a formalism broader than the standard Newtonian one. The new formalism enlarges the class of mechanical systems including those with some unusual properties. In particular, in the next talk we shall discuss the application of the formalism obtained to description of the so-called confined systems.

6 Acknowledgments

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V. THERMODYNAMICS AND STATISTICAL MECHANICS

DOUBLE SIMPLE-HARMONIC-OSCILLATOR FORMULATION OF THE THERMAL EQUILIBRIUM OF A FLUID INTERACTING WITH A COHERENT SOURCE OF PHONONS

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ABSTRACT

A formulation is given for a collection of phonons (sound) in a fluid at a non-zero temperature which uses the simple harmonic oscillator twice; one to give a stochastic thermal "noise" process and the other which generates a coherent Glauber state of phonons. Simple thermodynamic observables are calculated and the acoustic two point function, "contrast" is presented. The role of "coherence" in an equilibrium system is clarified by these results and the simple harmonic oscillator is a key structure in both the formulation and the calculations.

1. Introduction

The problem of understanding the thermal properties of a radiation field in a finite volume is both old and subtle.[1-3] Here a sound wave propagating in a water will be studied and the key issue will be the interaction of the sound radiation with the fluid matter and with the walls of the container. The time scales of sound waves, $\nu = 20 - 2 \times 10^9 Hz$, and those of the water molecules lead to adiabatic (isentropic, if approximately reversible) thermodynamic processes rather than constant temperature transitions.[4] This work is a special case of a project by one of us (AVN) which addresses the full nonlinear problem of bubble formation by sound waves. The linearized problem will be studied here, where the fluid has a coherent interaction with the sound radiation and an incoherent, or stochastic, interaction with the reservoir.

The harmonic oscillator has played a central role in the coherent states[6-11] and will be used for the coherent (Poisson) process describing the phonon radiation the sound field. Since the reservoir is incoherent the total interaction with the fluid is partially coherent.[14,15] The reservoir is analogous to Feynman's rest of the universe[12] and Han, Kim and Noz[13] have shown the relation of this idea to quantum squeezed states and time-uncertainty.

The model presented here will use the simple-harmonic-oscillator twice: first to generate "stochastic or chaotic" noise and second to generate a Glauber coherent state of scalar, longitudinal phonons. This is a more realistic model of noise, in that it has both coherent and a random components. It will be called partially coherent following a standard useage in quantum optics. The density, entropy free energy and a two-point function which gives the acoustic contrast are all calculated. The reason that the SHO is so useful is that since their Gaussian functions are dense in L^2, quantum mechanics guarantees that the crucial interaction between the fluid and the sound radiation can be approximated by an infinite collection of oscillators. This is why the approach of Planck[2] was correct even though quantum mechanics was not yet created. Also, finite energy classical solutions will lie in L^2 or at least in the Soboler space $H^1 = L^{2,1}$, which is the space where the "function"

and its "gradient" are square integrable.

In Sec. 2 the model will be presented, and the density ρ, the entropy S, the free energy F and the pair correlation function $g^{(2)}$ are calculated for both single and N-mode partially coherent states. In Sec. 3 the Conclusions and Outlook are presented.

2. The Model

In Fig. 1, a schematic is given which shows a source of sound S_0 (treated as a coherent state of phonons) a fluid F in thermal contact with the reservoir R, which is much larger than S_0 or F. In general, phonons can enter the fluid from S_0 and the fluid and reservoir can exchange particles as well as heat but all other exchanges are negligible.
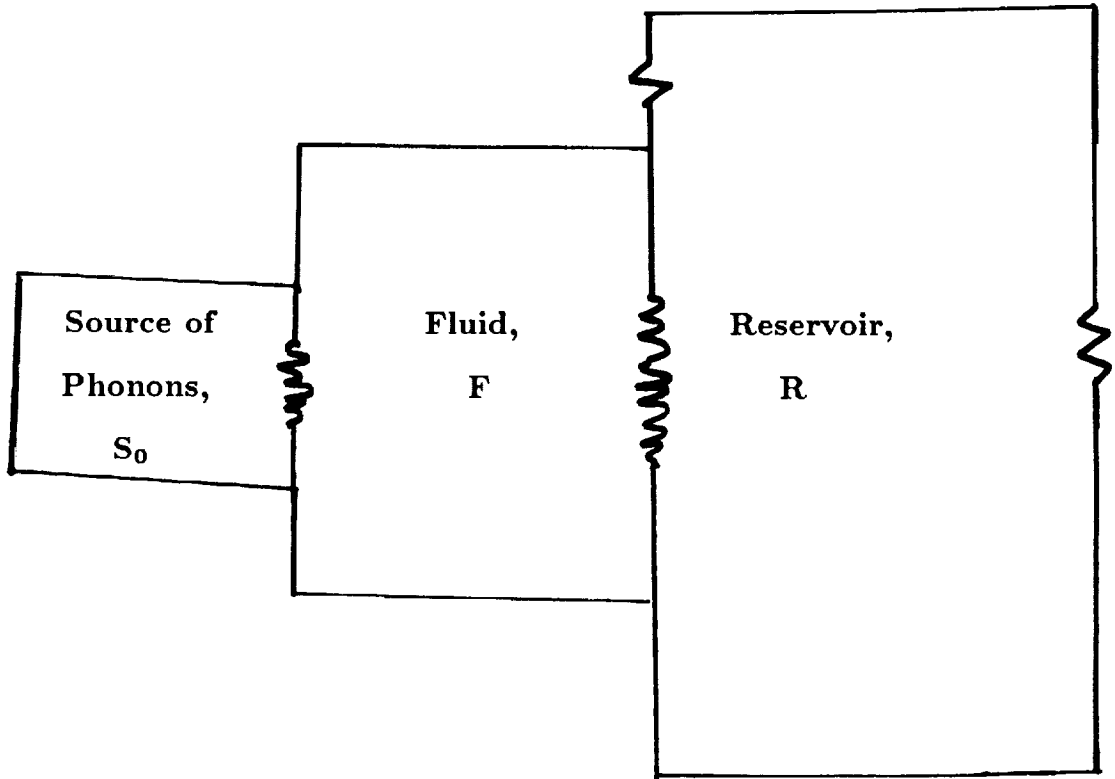
Fig. 1. Schematic of the system modeled. The source of a coherent state of phonons is S_0, F is the stationary fluid volume and R is the reservoir which is much larger than the sum of S_0 and F. The wavy lines indicate boundaries which allow particles and energy to pass.

The idea is a modification of one due to Kaup[16,17] that cavitons (here bubbles) are solitons (here solitary waves). In other cases, Williamsson and Wieland[18], Glimm[19], and others have shown that many physically interesting model solutions for plasmas and classical fluids are nonlinear, coherent exitations of the medium. The formulation given can easily be generalized to M_1 independent random components of noise and M_2 independent, coherent components. Thus, solitons and some other nonlinear modes could easily be added to the analysis.

The phonons are bosons so that their creation and destruction operators a^*, a satisfy the canonical commutation relations,

$$[a, a] = 0 = [a^*, a^*], \quad [a, a^*] = 1 \tag{1}$$

and a unique, translationally invariant Fock vacuum $|0 >$ exist s.t.

$$a(\vec{x}, t)|0 >= 0 \quad . \tag{2}$$

The astersik power of an operator is its adjoint (a^* and a are not self-adjoint) and on a complex number is its complex conjugate. Physically, the Fock vacuum is a quantum state with no phonons. The number operator N is defined as

$$N = a^* a \tag{3}$$

and a number or Fock states is given by

$$|n >= \frac{(a^*)^n}{n!} 10 >$$

for each $n \epsilon Z_+$, the positive integers including zero. They are eigenfunctions of the number operator with eigenvalues $n \epsilon z_+$. The Fock representation $\mathcal{H}_F$ of the quantum Hilbert space $\mathcal{H}$ is the L^2 closure of the linear span of the $|n >$ states. The inner-product of the Hilbert space will be written as $< \cdot, \cdot >$ and the inner-product compatible norm is written as

$\| \cdot \| = [< \cdot, \cdot >]^{\frac{1}{2}}$. For any complex valued $z \epsilon C^1$ the unitary displacement operator, $U(z)$, acting on the Fock vacuum yields the minimum uncertainty coherent state $|z >$ given by

$$|z >= U(z)|0 >= e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n > \quad . \tag{4}$$

The Fock vacuum is the ground state of the SHO for the minimum uncertainty coherent states which will be used here. In terms of c-number coördinate q and momentum p the complex number z is written as

$$z = (q, p) = q + ip \tag{5}$$

so that C^1 corresponds to the phase-space of the $(1 - d)$ system. Two properties of the coherent states are that

$$a|z >= z|z > \tag{6}$$

and

$$< z_1, z_2 >= e^{-\frac{1}{2}(z_1^* - z_2^*)(z_1 - z_2)} e^{-\frac{1}{2}(z_1^* z_2 - z_1 z_2^*)} \quad . \tag{7}$$

From eq. (7) it is clear that the coherent are continuous in the label z and therefore are an overcomplete family of states, OFS. The L^2-closure of the linear span of the coherent states $|z >$ provides a continuous representation of the physical Hilbert space which will be written as $\mathcal{H}_{cs}$.

A density operator is a positive, self-adjoint operator which satisfies

$$\rho^2 = \rho = \rho^* \quad . \tag{8}$$

The expected value of an observable $A = A^*$ in a state $\psi \epsilon \mathcal{H}$ with corresponding density operator ρ_ψ can be expressed as

$$< A >_\psi = < \psi, A\psi >= Tr(\rho_\psi A) \quad . \tag{9}$$

And additive thermal noise can be added "by hand." The entropy of the system, S, is given by

$$S = -k_B Tr[\rho \, ln\rho] \tag{10}$$

where k_B is the Boltzman constant. In information theory, one can set $k_B = 1/ln2$ and still use eq. (10). The entropy is obtained from maximizing eq. (10) subject to the constraints

$$Tr(\rho) = 1 \qquad (11)$$

and

$$Tr(\rho N) = c \qquad (12)$$

where $c \epsilon R^1$ is a parameter which labels the strength of the thermal state. If the thermal noise is Gaussian, its density operator can be expressed as

$$\rho(c) = \frac{1}{\pi c} \int d^2 z e^{-|z|^2/c} |z><z| \quad . \qquad (13)$$

In $\mathcal{H}_F$ this can be re-written as

$$\rho(c) = \frac{1}{\pi c} \int d^2 z e^{-|z|^2/c} e^{-|z|^2} \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} |n><n| \quad . \qquad (14)$$

Using Fubini's theorem to interchange the integral and the infinite sum and then expressing z in plane polar coordinates gives

$$\rho(c) = \sum_n \left[\frac{1}{(c+1)(1+1/c)^{n+1}} |n><n| \right] \quad , \qquad (15)$$

in the Fock representation. A similar calculation of the entropy gives

$$S = k_B[(c+1)ln(c+1) - c\, ln(c)] \qquad (16)$$

This sort of calculation was given by Glauber[10], Wolf[14], Sudarshan[15] and probably by others. The nth-order correlation function, $g^{(n)}(X_1, \ldots, X_n)$, with Glauber's normalization convention is

$$g^{(n)}(X_1, \ldots, X_n) := \frac{G^{(n,n)}(X_1, \ldots X_{2n})}{\prod_{k=1}^{2n} \left[G^{(1,1)}(X_k, X_k) \right]^{1/2}} \quad , \qquad (17)$$

where the $G^{(i,j)}$'s are Green's functions or correlation functions. This is a quantum generalization of the classical coherence degree γ of Born and Wolf.[20] The visibility v of a two-slit interference pattern is related to γ_{12} by

$$v = 1 \pm |\gamma_{12}(\varphi)| \tag{18}$$

which physically represents the extremes in intensity. For the two-point function, $g^{(2)}(\cdot)$, the $G^{(nm)}$'s are chosen s.t

$$G^{(2)}(\rho) = \frac{Tr[\rho N(N-1)]}{Tr(\rho N)]^2} - 1 \quad . \tag{19}$$

This object is proportional to Glauber's $g^{(2)}$ in ref. (10) but is not equal to his function because of different normalizations..

In the case of the thermal states, the Fock representation of $g^{(2)}$ for $n \neq 0$ is given by

$$g_F^{(2)}(|n><n|) = \frac{Tr\left[|n><n|N(N-1)\right]}{\left[Tr(|n><n|N)\right]^2} - 1 = \frac{n^2 - n}{n^2} - 1 = -\frac{1}{n} \quad . \tag{20}$$

A coherent state is very different from eq. (20) since

$$g_{cs}^{(2)}(|z><z|) = \frac{Tr\left[|z><z|N(N-1)\right]}{\left[Tr(|n><n|N)\right]^2} - 1 = 0 \quad , \tag{21}$$

i.e. the Glauber state is *perfectly coherent.*

For the thermal states, not surprisingly, $g^{(2)}$ has the opposite behavior from eq. (21) because

$$g_{Th}^{(2)}[\rho_{Th}(c)] = \frac{Tr\left[\sum_n \left(\frac{1}{c+1}\right)\left(\frac{1}{1+1/c}\right)^{n+1}|n><n N(N-1)\right]}{\left\{Tr\left[\sum_n \left(\frac{1}{c+1}\right)\left(\frac{1}{1+1/c}\right)^{n+1}|n><n|N\right]\right\}^2}$$

$$= \frac{\sum_n \left[\left(\frac{1}{c+1}\right)\left(\frac{1}{1+1/c}\right)^{n+1} n(n-1)\right]}{\left[\sum_n \left(\frac{1}{(c+1)}\right)\left(\frac{1}{1+1/c}\right)^{n+1} n\right]^2} - 1 \quad . \tag{22}$$

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Defining

$$f_n(c) := \frac{1}{(c+1)}\left(\frac{1}{1+1/c}\right)^{n+1} \quad ,$$

eq. (22) becomes

$$g_{Th}^{(2)}(c) = \frac{\sum\limits_n f_n(c)(n^2 - n) - \sum f_n(c)n}{\left[\sum\limits_n f_n(c)n\right]^2} = 1 \quad , \qquad (23)$$

which is *perfectly incoherent*. Now a partially coherent state has a density operator which is given by

$$\rho(z;c) = U(z)\rho_{Th}(c)U^*(z) = \frac{1}{\pi c}\int d^2x e^{-|x|^2/c}|z+x><z+x| \quad . \qquad (24)$$

A calculation similar to the one outlined between eqs. (14) and (15) gives

$$Tr[\rho(z;c)N] = Tr\left\{\frac{1}{c\pi}\int d^2 \times e^{-|\times|2/c}\sum_m, n\left[\frac{(z+x)^n(z^*+x^*)^m}{n!m!}e^{-|z+z|2/2}|n><n|Nf_{mn}]\right\}$$

$$= |z|^2 + c \quad . \qquad (25)$$

It is straightfoward to calculate the entropy of the partially coherent state from

$$S(z;c) = -k_B Tr\left[\rho(z;c)ln\big(\rho z;c)\big)\right]$$

$$= -k_B Tr\left[\rho(c)ln\big(\rho(c)\big)\right]$$

$$= S(c)$$

$$= k_B\big[(c+1)ln(c+1) - c\, ln(c)\big] \quad , \qquad (26)$$

where eq, (16) was used in the last two equalities. The pair correlation function, $g^{(2)}$, of a partially coherent state is given by

$$g_{cs}^{(2)}\big[\rho(z;c)\big] = 1 - \left(\frac{|z|^2}{c+|z|^2}\right)^2 \quad , \qquad (27)$$

in the coherent state basis. Let $m \geq n$ with $m, n \epsilon Z_+$ and form the nm^th matrix element of the partially coherent density operator as

$$<n, \rho(z;c)m> = e^{-|z|^2/(1+c)}\frac{1}{1+c}\frac{1}{(1+1/c)^{(m+n/2)}}$$

$$\cdot \left(\frac{n!}{m!}\right)^{1/2}\left(\frac{z}{\sqrt{c(1+c)}}\right)^{n-m}P_{n-m}^m\left(\frac{|z|^2}{c(1+c)}\right) \quad , \qquad (28)$$

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where $P_l^m(r)$ is the polynomial given by

$$P_l^m(r) = \sum_{k=0}^m \frac{m!}{k!(l+k)!(m-k)!} r^k \quad .$$

The thermal or noise density can be found from the $c \to 0$ limit of the previous equation and is given by the familiar expression

$$< n, \rho(z;0)m > = e^{-|z|^2} \frac{z^n(z^*)^m}{\sqrt{n!m!}} \quad . \tag{29}$$

By calculating another Gaussian integral, an overlap of two partially coherent states is found to be

$$Tr\left[\rho(z_1;c_1)\rho(z_2;c_2)\right] = \frac{exp\left[\frac{|z_1-z_2|^2}{1+c_1+c_2}\right]}{(1+c_1+c_2)} \quad . \tag{30}$$

Clearly as $c_1 \to c_2 = c$, eq. (30) reduces to

$$Tr\left[\rho(0;c)\rho(0;c)\right] = \frac{1}{1+2c} \quad . \tag{31}$$

Remark: The non-zero part of the entropy $S(c) = S(z;c)$ and the non-unit part of $Tr\left[(\rho(z,c))^2\right]$ give a measure of the nagnitude of the departure from a pure coherent state. The generalization to M modes, with M a positive, finite integer is straightforward. Let (a,z) be $M \times 1$ matrices, let (a^*, z^*) be $1 \times M$ matrices, 1 the $M \times M$ unit matrix where $k = 1,2,\ldots,M$ labels which mode. Let $\mathcal{B}$ be a given non-singular, Hermitian, $M \times M$, positive, covariance matrix and express the M-mode Fock state as

$$|n_1,\ldots,n_m > = \prod_{k=1}^M \frac{(a^*)^{n_k}}{\sqrt{n_k!}}|0 > \quad , \tag{32}$$

where n_k is the occupation number of the kth-mode. The M-mode thermal density operator is given by

$$\rho_{Th}(\mathcal{B}) = \frac{e^{-a^* ln(1+\mathcal{B}^{-1})a}}{det(1+\mathcal{B})} \tag{33}$$

$$= \frac{1}{det(\mathcal{B})} \int \prod_{z=1} d^2x \cdot e^{-x^* \cdot \mathcal{B}^{-1} \cdot x}|x >< x| \quad , \tag{34}$$

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where $x = (x_1, \ldots, x_M)$ is the M-component, complex, coherent state amplitude. The entropy of an M-mode phonon packet is given by

$$S(\mathcal{B}) = k_B Tr\big[(1 + \mathcal{B})ln(1 + \mathcal{B}) - \mathcal{B}ln\mathcal{B}\big] = S(x; \mathcal{B}) \quad . \tag{35}$$

for both the incoherent and the partially coherent cases. The pair correlation function $g^{(2)}$ for an M-mode partially coherent system is

$$g^{(2)}(z; \mathcal{B}) = \frac{Tr[\mathcal{B} \cdot \mathcal{B} + 2z^* \cdot \mathcal{B}z]}{(Tr(\mathcal{B}) + |z|^2)^2} \tag{36}$$

which has the limits

$$g^{(2)}(0, \mathcal{B}) = 1 \tag{37a}$$

and

$$g^{(2)}(z, 0) = 0 \tag{37b}$$

To find the partition function for the partially coherent system one needs the mode energy e_k for the kth-mode which is real

$$e_k = e_k^* \quad , \tag{38}$$

and the M-mode vector

$$\overrightarrow{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_M \end{pmatrix} \tag{39}$$

which is a vector-valued, intensive variable. The mode energies for the kth mode is

$$E_k = n_k e_k = E_k^* \tag{40}$$

where n_k is the occupation number for the kth mode and

$$\overrightarrow{\mathcal{E}} = (E_1, \ldots, E_m) \tag{41}$$

is a real, vector-valued, extensive variable dual to eq. (39). In this more general case the density operator $\rho(z; \mathcal{B})$ which maximizes the entropy

$$S = -k_B Tr[\rho \, ln\rho]$$

is subject to the constraints

$$Tr(\rho) = 1 \quad , \tag{42a}$$

$$Tr(\rho a) = z \quad , \tag{42b}$$

$$Tr(\rho a^*) = z^* \quad , \tag{42c}$$

and

$$Tr(\rho a^* \overrightarrow{e} a) = z^* \overrightarrow{e} z + \overrightarrow{e} \quad . \tag{42d}$$

Let $\overrightarrow{\beta}$ be an m-vector with the value $\beta = 1/k_B T$ for each component and express the *given* covariance matrix for bosons as

$$\mathcal{B} = \mathcal{B}(\beta) = (e^{-\overrightarrow{\beta} \cdot \overrightarrow{\mathcal{E}}} - 1)^{-1} = (e^{-\beta E} - 1)^{-1} \quad . \tag{43}$$

(A similar argument for Fermi-Dirac particles would replace -1 by $+1$ in eq. (43) but this is not needed here.) The partition function $Z(\beta, V)$ is now

$$Z(\beta, V) = det\left(\frac{1}{1 - e^{-\beta E}}\right) \tag{44}$$

at thermal equilibrium and V is the volume of the fluid plus radiation. From eq. (44) all of the equilibrium thermodynamic quantities can be calculated, for example the average energy $\bar{\mathcal{E}}$ is

$$\bar{\mathcal{E}} = \frac{\sum_r E_r e^{-\beta E_r}}{z} = -\frac{\partial ln(Z)}{\partial \beta} \tag{45a}$$

when classically

$$Z(\beta, V) = \sum_r e^{-\beta E_r} \tag{45b}$$

and in quantum statistical mechanics a Trace over matrix elements of $e^{-\beta H}$ is taken. The pressure p is

$$p = \frac{1}{\beta} \frac{\partial ln(Z)}{\partial V} \tag{45c}$$

and the (reversible) differential work dW is

$$dW = -pdV \qquad (45d)$$

and the previous equation can be used to obtain

$$dW = -\frac{1}{\beta}\frac{\partial ln(Z)}{\partial V}dV \quad .$$

The entropy of eq. (10) can also be written as

$$S = k_B\big[ln(Z) + \beta\bar{\mathcal{E}}\big] \qquad (45e)$$

and the Helmholtz free energy is

$$F(\beta, V) = -\frac{1}{\beta}ln\big[Z(\beta, V)\big] \quad . \qquad (45f)$$

The coherent state density operator can be given as

$$\rho(z) = det\left[(1 - e^{-\beta E})e^{-(a^* - z^*)\beta E(a-z)}\right] \quad . \qquad (46)$$

Near equilibrium, a Kubo linear response theory[21] can be established by studying small deviations from equilibrium where the fluctuations will be equal to the dissipations. This exercise will be left to a future project.

3. Conclusions and Outlook

A two-component thermodynamics was formulated for a fluid in thermal equilibrium with a reservoir radiated by a coherent state of phonons. One future project will be to derive the Kubo fluctuation-dissipation theorem for this system, another will be to compare and contrast these results with both the Langevin equation and the stochastic quantization approaches. These future studies should illuminate (or simplify) the lattice Monte Carlo methods.

Much remains to learned from harmonic, SHO, systems.

4. Acknowledgements

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Wigner Expansions for Partition Functions of Nonrelativistic and Relativistic Oscillator Systems

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Abstract

The equilibrium quantum statistics of various anharmonic oscillator systems including relativistic systems is considered within the Wigner phase space formalism. For this purpose the Wigner series expansion for the partition function is generalised to include relativistic corrections. The new series for partition functions and all thermodynamic potentials yield quantum corrections in terms of powers of $\hbar^2$ and relativistic corrections given by Kelvin functions (modified Hankel funktions) $K_\nu(mc^2/kT)$. As applications are treated the symmetric Toda oscillator, isotonic and singular anharmonic oscillators and hindered rotators, i.e. oscillators with cosine potential.

1 Introduction

In recent years, the Wigner formalism as a phase space representation of quantum mechanics, quantum field theory and quantum statistics has found growing interest [1,2]. One of the main fields is the theory of anharmonic oscillator systems modelling various quantum systems, e.g. solitonic systems, quantum field theories, and transport processes in more complicated systems. The purpose of this paper is twofold,

1. to generalize the Wigner series expansion of equilibrium phase space quantum statistics in order to include special relativistic systems,

2. to show applicability and utility of the formalism by means of various examples.

Up to now a small number of papers on relativistic quantum theory in the Wigner formalism have appeared for equilibrium as well as for nonequilibrium systems. The topics treated for nonequilibrium processes range from multiparticle production and kinetic theory [3] to cosmology [4]. In the realm of relativistic phase space quantum mechanics there exist only a few papers. Janussis et al. [5] starting from the Dirac Hamiltonian introduce a Wigner function with 4x4 spinor components. Ali [6] discusses the quantization of relativistic systems on phase space whereas Kim and Wigner [7] describe a covariant phase space representation for harmonic oscillators. A relativistic Fermi gas is treated in the frame work of the grand canonical ensemble by Greiner et al.

[8]. A pseudo-classical phase space description of the relativistic electron in terms of Grassmann variables was given in several versions [9]. There are also papers on relativistic quantum kinematics [10,11].

Over the years, fundamental problems regarding the correct description of simple systems such as the classical relativistic harmonic oscillator and its quantization have been treated in a number of publications, see f.i. [12-15]. Important papers on the covariance problem are [7,16] ; a symplectic formulation of relativistic quantum mechanics has been given in [17,18]. We develop a Wigner formalism which is frame dependent. The focus of our attention is the comoving frame, i.e., we use the so-called synchronous gauge [19].

2 Relativistic Wigner Formalism

One of the advantages of the phase space methods of quantum statistics - Wigner-Weyl formalism [1], coherent state methods [20], Husimi transforms [21] or Bargmann representation [22] - is the possibility to evaluate exactly the partition function by means of a phase space integration. In practice it is convenient to expand the phase space integral into the Wigner series in powers of $\hbar^2$ [23] as a basis of semiclassical quantum statistics.

The (non-relativistic) Wigner function $f_W(q,p)$ can be defined as the Fourier transform of the off-diagonal elements of the density operator $\langle q \mid \rho \mid q' \rangle$ (for systems without spin) [1]:

$$f_W(q,p) = \frac{1}{(2\pi\hbar)^f} \int dz \exp{(ipz/\hbar)} \langle q - \frac{z}{2} \mid \rho \mid q + \frac{z}{2} \rangle \qquad (1)$$

Here f is the number of the degrees of freedom of the system considered. The density operator is given in the coordinate representation by

$$\langle q \mid \rho \mid q' \rangle = \sum_n \psi_n^*(q') w_n \psi_n(q) \qquad (2)$$

with $w_n = \exp{(-\beta E_n)}/Z$, where $\hat{H}\psi_n = E_n\psi_n$, $\beta = 1/kT$, and the partition function is

$$Z = Tr \exp{(-\beta\hat{H})} = \sum_n \exp(-\beta E_n) \qquad (3)$$

Within the frame of the Wigner formalism the partition function can be expressed as

$$Z = \int\int dq\,dp\, \Omega_W(q,p;\beta) \quad , \qquad (4)$$

where $\Omega_W = (\exp(-\beta\hat{H}))_W$ is the Wigner equivalent of the operator $\hat{\Omega} = \exp(-\beta\hat{H})$ defined as

$$\Omega_W(q,p;\beta) = \int dz \exp{(ipz/\hbar)} \langle q - \frac{z}{2} \mid \hat{\Omega} \mid q + \frac{z}{2} \rangle \qquad (5)$$

The Bloch equation for the calculation of Ω_W within the Wigner formalisms reads

$$\frac{\partial\Omega_W(q,p;\beta)}{\partial\beta} = -H_W(q,p)\cos(\frac{\hbar}{2}\Lambda)\Omega_W(q,p;\beta) \quad , \qquad (6)$$

here $H_W(q,p)$ is the Wigner equivalent of the Hamiltonian $\hat{H}$, and Λ denotes the Poisson Bracket operator or symplectic differential operator

$$\Lambda = \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} \tag{7}$$

acting in (6) to the left and right hand sides. Up to now the formalism is completely exact. The partition function (4) can be calculated by means of a phase space integration without any knowledge of the energy levels of the system considered and without the necessity to sum up an infinite series of Boltzmann terms $\exp(-\beta E_n)$, the zero-point energy beeing already included.

In order to elucidate the principle of our formalism, we consider a system with a single degree of freedom characterized by a Hamiltonian $\hat{H} = H_0(\hat{p}) + V(\hat{q})$ with

$$H_0(\hat{p}) = \sqrt{m^2c^4 + \hat{p}^2c^2} - mc^2 \tag{8}$$

and the potential energy $V(q)$. One has $H_W(q,p) = H_0(p) + V(q)$. The Wigner series [17] for the partition function Z is given by

$$Z = \frac{1}{2\pi\hbar}\iint dqdp \; \exp\{-\beta(H_0(p) + V(q))\}\sum_{n=0}^{+\infty}\hbar^{2n}\;\Phi_n(q,p;\beta) \tag{9}$$

with $\Phi_0 = 1$, Φ_n : n-th quantum correction. Solving the Bloch equation (6) for our relativistic quantum system yields

$$\Phi_1(q,p;\beta) = -\frac{1}{8}\beta^2 H_0'' \; V'' \; + \; \frac{1}{24}\beta^3(H_0'' \; V'^2 + H_0'^2 \; V'') \tag{10}$$

where the primes denote differentiation with respect to the corresponding variables. We restrict ourselves to the first quantum correction.

We will evaluate (9) with (10), this approximative partition function is denoted by Z_{01} . We begin with the p-integration:

$$Z_{01(kin)} = Z_{0(kin)} + Z_{1(kin)} \tag{11}$$

$$\text{with} \quad Z_{0(kin)} = \frac{1}{2\pi\hbar}\int e^{-\beta H_0(p)}dp$$

$$\text{and} \quad Z_{1(kin)} = \frac{1}{2\pi\hbar}\int e^{-\beta H_0(p)}\hbar^2[-\frac{1}{8}\beta^2 H_0'' \; V'' \; + \; \frac{1}{24}\beta^3(H_0'' \; V'^2 + H_0'^2 \; V'')]dp \quad .$$

Substituting $p/mc = \sinh u$ and using [43]

$$\int\limits_0^\infty e^{b\cosh u} \; \cosh \nu u \; du = K_\nu(b) \tag{12}$$

where the K_ν denote the Kelvin functions (modified Hankel functions) one gets

$$Z_{0(kin)} = 2\frac{1}{2\pi\hbar}e^{+\beta mc^2}mcK_1(\beta mc^2) \tag{13}$$

and after some algebra

$$Z_{1(kin)} = 2\frac{1}{2\pi\hbar}e^{+\beta mc^2}c\hbar^2[-\frac{1}{8}\beta^2 V'' + \frac{1}{24}\beta^3 V'^2] \times$$
$$\times \; [\frac{206}{128}K_1(\beta mc^2) - \frac{93}{128}K_3(\beta mc^2) + \frac{15}{128}K_5(\beta mc^2)] \;+$$
$$+ \; 2\frac{1}{2\pi\hbar}e^{+\beta mc^2}mc\hbar^2\frac{1}{24}\beta^3 V'' c^2 \times$$
$$\times \; [-\frac{48}{128}K_1(\beta mc^2) + \frac{56}{128}K_3(\beta mc^2) - \frac{8}{128}K_5(\beta mc^2)] \tag{14}$$

where the series expansions

$$[1 + \frac{p^2}{m^2c^2}]^{-3/2} = [1 + \sinh^2 u]^{-3/2} \simeq 1 - \frac{3}{2}\sinh^2 u + \frac{15}{8}\sinh^4 u \tag{15}$$

and

$$\frac{p^2}{m^2c^2}[1 + \frac{p^2}{m^2c^2}]^{-1} = \sinh^2 u[1 + \sinh^2 u]^{-1} \simeq \sinh^2 u - \sinh^4 u \tag{16}$$

have been used.

Now the integration over the momentum space is done, and (13), (14) represent, as a main result, the partition function up to the first quantum correction, proportional to $\hbar^2$, and relativistic corrections up to second and fourth order in p/mc. To proceed further one has to take into account the potential $V(q)$ of the system considered. Then the full partition function (in our approximation) is given by

$$Z_{01} = Z_0 + Z_1 = \int dq \; \exp(-\beta V(q))Z_{01(kin)}(q) \tag{17}$$

Here we only mention that with

$$K_1(z) \simeq \sqrt{\frac{\pi}{2z}}e^{-z}$$

in $Z_{0(kin)}$ and

$$K_1(z) \simeq \sqrt{\frac{\pi}{2z}} \, e^{-z} \, (1 + \frac{3}{8z}) \quad , \quad K_3(z) = \sqrt{\frac{\pi}{2z}} \, e^{-z} \, (1 + \frac{35}{8z}) \quad , \quad K_5(z) = \sqrt{\frac{\pi}{2z}} \, e^{-z} \, (1 + \frac{99}{8z})$$

in $Z_{1(kin)}$ the non-relativistic limit is reproduced correctly.

3 Toda oscillator

As a first example we choose the symmetric relativistic quantum Toda oscillator with the potential

$$V(q) = V_0 \, (\cosh aq - 1) \tag{18}$$

Its eigenvalues are not exactly known. If we compare (18) with the potential $V(q) = m\,\omega_0^2\,q^2\,/\,2$ of the harmonic oscillator we find

$$a = \sqrt{m\omega_0^2/V_0} \tag{19}$$

where ω_0 is the (circular) frequency for harmonic (i.e. very small) oscillations. Toda molecules are discussed in [24], generalizations thereof in [25]. There exist a few papers on quantum statistics of toda chains [26] and Toda fields [27]. The non-relativistic phase space quantum statistics of the symmetric Toda oscillator with (18) is treated in [28]. The dynamics of non quantum relativistic Toda lattices is the topic of [29-31].

The part Z_0 of (17) becomes

$$Z_0 = 2 \frac{1}{2\pi\hbar} e^{\beta mc^2} mc \, K_1(\beta mc^2) e^{\beta V_0} \frac{2}{a} K_0(\beta V_0) \tag{20}$$

and the part Z_1 can be expressed as

$$
\begin{aligned}
Z_1 =\ & 2\hbar^2 \frac{1}{2\pi\hbar} c \, e^{\beta mc^2} K_1(\beta V_0) \beta^2 V_0 e^{\beta V_0} \times \\
& \times \{ -\frac{1}{6} [\frac{206}{128} K_1(\beta mc^2) - \frac{93}{128} K_3(\beta mc^2) + \frac{15}{128} K_5(\beta mc^2)] + \\
& + \frac{mc^2\beta}{12} [-\frac{48}{128} K_1(\beta mc^2) + \frac{56}{128} K_3(\beta mc^2) - \frac{8}{128} K_5(\beta mc^2)] \}
\end{aligned} \tag{21}
$$

The formulae (20) and (21) yield in the nonrelativistic limit $kT \ll mc^2$ the correct partition function of [28]. The evaluation of higher order relativistic and quantum corrections is straight forward. From the Wigner series for the partition function, corresponding series expressions for the thermodynamic potentials follow in the well-known manner. These results are published in [32].

4 Isotonic or Singular Oscillator

Our next example is the quantum oscillator with the potential energy

$$V(q) = V_0 \, [\frac{q}{a} - \frac{a}{q}]^2 \tag{22}$$

(normalized so that the potential minimum at $q_{min} = a$ gives $V(q_{min}) = 0$). This asymmetric anharmonic oscillator appears in the literature under various headings in two different interpretations:

1. isotonic oscillator [33-36], radial oscillator (with centripetal barrier) [37-39] or inverse quadratic oscillator [40],

2. singular oscillator [41] or nonpolynomial oscillator [42].

Interesting subject for oscillators of this type are the energy spectrum [38], canonical transformations [37], ladder operators and coherent states [34-36], phase space path integrals and the dynamics and symplectic groups [37,41]. The connection with three-dimensional potentials was discussed thoroughly [39]. Here we add the second quantum correction to the partition function:

$$Z(\hbar^4) = \frac{1}{\hbar} \sqrt{\frac{m}{2\pi\beta}} \int e^{-\beta V(q)} \frac{\hbar^4 \beta^3}{240 m^2} Q(q) \, dq \tag{23}$$

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where

$$Q(q) = -V^{(4)} + \beta(2V'V^{(3)} + \tfrac{3}{2}V''^2) - \tfrac{11}{6}\beta^2 V'^2 V'' + \tfrac{5}{24}\beta^3 V'^4 \qquad (24)$$

here is $V' = \partial V(q)/\partial q$, $V^{(3)} = V'''$, and so forth. For the isotonic oscillator potential (22) all integrals to be calculated are of the type [43]:

$$\int_0^\infty x^{n-1} e^{-(Aq^2 + Bq^{-2})} \, dx = [\tfrac{B}{A}]^{n/4} K_{n/2}(2\sqrt{AB}) \qquad (25)$$

where the K-functions are again the modified Hankel functions (Kelvin functions) but here only of half integer index, n is an odd integer. These functions have a simple analytical shape, in particular one has

$$
\begin{aligned}
K_{\pm 1/2}(z) &= \sqrt{\frac{\pi}{2z}}\, e^{-z} \\
K_{\pm 3/2}(z) &= \sqrt{\frac{\pi}{2z}}\, e^{-z}(1 + \tfrac{1}{z}) \\
K_{\pm 5/2}(z) &= \sqrt{\frac{\pi}{2z}}\, e^{-z}(1 + \tfrac{3}{z} + \tfrac{3}{z^2})
\end{aligned} \qquad (26)
$$

$$\vdots$$

In (25) one has for the isotonic oscillator

$$A = \beta V_0/a^2 \ , \ B = \beta V_0 a^2 \ , \ z = 2\sqrt{AB} = 2\beta V_0 \ . \qquad (27)$$

The exact classical (i.e. semiclassical) partition function becomes

$$Z_{cl} = kT/\hbar\omega \qquad (28)$$

where

$$\omega = \sqrt{8V_0/ma^2} \qquad (29)$$

is the classical (angular) frequency of our oscillator in the limit of very small (i.e. harmonic) oscillations. The partition function (28) is formally the same as the well-known expression for the classical harmonic oscillator with (angular) frequency ω.

A simple but somewhat lengthy calculation then yields the first two terms of the partition function

$$Z = \frac{kT}{\hbar\omega}\{1 - [\tfrac{1}{24}(\tfrac{\hbar\omega}{kT})^2(1 + \tfrac{3}{8}\tfrac{kT}{V_0})]\} \qquad (30)$$

where the expression in the square brackets being the famous $\hbar^2/24$ quantum correction. The higher order quantum corrections can be calculated in a similar way using (25). Again all integrals lead only to the handy Kelvin functions with half-integer index. As is well known one gets the thermodynamic functions for a system of N oscillators from the partition function. The

(Helmholtz) free energy (here identical to the free enthalpy G) , the internal energy U(here identical to the enthalpy H), the entropy S and the heat capacity $C = \partial U/\partial T$ are given, respectively, by

$$F = -NkT \ln Z$$

$$U = NkT^2 \frac{\partial}{\partial T} \ln Z$$

$$S = Nk(\ln Z + T\frac{\partial}{\partial T} \ln Z)$$

$$C = Nk\frac{\partial}{\partial T}(T^2\frac{\partial}{\partial T} \ln Z) \quad . \tag{31}$$

Starting from (30) these functions can be written in the form of classical contribution plus quantum correction up to terms with $\hbar^2$ using the series expansion of the logarithm function:

$$F = -NkT\{\ln\frac{kT}{\hbar\omega} - \frac{1}{24}(\frac{\hbar\omega}{kT})^2(1 + \frac{3}{8}\frac{kT}{V_0})\} \tag{32}$$

$$U = NkT\{1 + \frac{1}{12}(\frac{\hbar\omega}{kT})^2 + \frac{1}{64}\frac{(\hbar\omega)^2}{V_0 kT}\} \tag{33}$$

$$S = Nk\ln\frac{kT}{\hbar\omega} + Nk\{1 + \frac{1}{24}(\frac{\hbar\omega}{kT})^2\} \tag{34}$$

$$C = Nk\{1 - \frac{1}{12}(\frac{\hbar\omega}{kT})^2\} \tag{35}$$

More detailed estimates show that these semiclassical expressions are of considerable accuracy for atomic oscillators above $T = 100$ K, the errors being only a few percent and enlarging in the order F, U, S, C.

5 Hindered Rotators

The internal rotation of molecular groups (f.i. the methyl group) within molecules (f.i. ethane, polymers) or molecular complexes (f.i. $C_3H_6 OH_2$) is a topic of large interest since many years [44,45]. Related motions are the rotation of molecules adsorbed on solid surfaces [46,47] and the torsional oscillations of molecules or ions in the interior of molecular crystals [48,49]. The contributions of the corresponding degrees of freedom to the thermodynamic functions and to the equilibrium and rate constants are, generally, important. In calculations in the frame of statistical thermodynamics these rotators were mostly treated as (classical) free rotators [50] or sometimes as (harmonic) torsional oscillators [51].

However the rotators have to be classified, in most cases, as hindered rotators or strongly anharmonic torsional oscillators, the more the lower the temperature is. There exists only a few number of papers on the quantum statistics of hindered rotators [52-56]whereas the literature on the quantum statistics of various types of anharmonic oscillators is abundant. The reason is that the Schrödinger equation for a hindered rotator is of the Mathieu-Hill type whose energy eigenvalues can be calculated only approximately. Therefore, the evaluation of the partition function and the thermodynamic functions is possible only with numerical methods; closed general formulae cannot be derived from this approach [52,53]. Here , the Wigner formalism shows its advantages.

The main quantities characterising a hindered rotator are the shape and the strength of the potential barrier. For a symmetric rotator, the hindering potential can be given by

$$V(\phi) = \frac{1}{2}V_0(1 - \cos n\phi) \tag{36}$$

where n is the symmetry number (f.i. $n = 3$ for the methyl group) or, more generally, by Fourier series [57]

$$V(\phi) = \frac{1}{2}V_0 (1 - \sum_k a_k \cos kn\phi) \tag{37}$$

In the most simple case $V(\phi)$ is dominated by the repulsive interaction with a single near-neighbour atom within the molecule (or crystal) considered; then all coefficients a_k for $k \geq 2$ are zero. The presumption is also that the cosine series converges rather rapidly. A reasonable representation of the barrier can then made with the first two terms. For the important case of $n = 3$ one has

$$V(\phi) = \frac{1}{2}[V_3(1 - \cos 3\phi) - V_6 \cos 6\phi] \tag{38}$$

The effect of adding a small sixfold potential $V_6 \cos 6\phi$ to the threefold main potential $V_3 \cos 3\phi$ is to broaden the potential wells at the expanse of the barriers. (If V_6 is negative then the wells are narrowed and the barriers broadened.) For many acetates and other molecules and complexes containing the methyl group we have $V_6 \ll V_3$ (f.i. $V_6/V_3 \approx 0.15$ for acetates) [57].

For a hindered rotator with a simple cosine potential the canonically conjugate coordinates are given by the rotation angle ϕ and the angular momentum L, respectively; and the mass is to be replaced by the moment of inertia I.

In order to have a convenient comparison with the limiting case of a harmonic oscillator ($V_0 \to \infty$) we introduce the angular frequency Ω of of the harmonically oscillating torsional vibrator by

$$\Omega = \sqrt{\frac{V_0}{2I}}\, n \tag{39}$$

With (36) the classical partition function is written as

$$Z_{cl} = \frac{\sqrt{2\pi I/\beta}}{hn} e^{-x} \int_0^{2\pi} e^{x \cos n\phi}\, d\phi \tag{40}$$

where $x = \beta V_0/2$. We get the well-known expression

$$Z_{cl} = Z_f\, e^{-x}\, I_0(x) \tag{41}$$

where

$$Z_f = \frac{\sqrt{2\pi I/\beta}}{h} \frac{2\pi}{n} \tag{42}$$

is the classical is the classical partition function of a free internal rotator of symmetry number n, and $I_0(x)$ is the modified Bessel function (or the Bessel function $J_0(ix)$ of the purely imaginary argument ix) of order zero [43].

The evaluation of the quantum corrections $Z(\hbar^2)$ and $Z(\hbar^4)$ yields with $u = \beta\hbar\Omega$ at first

$$Z(\hbar^2) = Z_f e^{-x} \frac{u^2}{2\pi} \int_0^{2\pi} e^{x \cos n\phi} \left(-\frac{1}{12} \cos n\phi + \frac{1}{24} x \sin^2 n\phi\right) d\phi \tag{43}$$

$$Z(\hbar^4) = Z_f e^{-x} \frac{u^4}{2\pi} \int_0^{2\pi} e^{x \cos n\phi} \frac{1}{240} \left(\frac{1}{x} \cos n\phi - 2\sin^2 n\phi \right. +$$

$$\left. + \frac{3}{2} \cos^2 n\phi - \frac{11}{6} x \sin^2 n\phi \cos n\phi + \frac{5}{24} x^2 \sin^4 n\phi\right) d\phi \tag{44}$$

All integrals in (43) and (44) can be expressed by Bessel functions $J_\nu(z)$ of purely imaginary argument $z = i\xi$, where ν is a positive integer index [43],

$$J_\nu(ix) = \frac{i^{-\nu}}{2\pi} \int_0^{2\pi} e^{ix \cos n\phi} \cos \nu n\phi \, d\phi \tag{45}$$

or by the modified Bessel functions

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \quad . \tag{46}$$

One gets

$$Z(\hbar^2) = -Z_f e^{-x} \frac{1}{24} \beta^2 (\hbar\omega)^2 I_1(x) \tag{47}$$

$$Z(\hbar^4) = +Z_f e^{-x} \frac{1}{240} \beta^4 (\hbar\omega)^4 \{-\frac{1}{\xi} I_1(x) + \frac{3}{2} [I_0(x) - \frac{1}{\xi} I_1(x)] \quad -$$

$$- \frac{11}{24} \xi [I_1(x) - I_3(x)] + \frac{5}{8} I_2(x)\} \quad . \tag{48}$$

By means of the recursive relations [43]

$$I_{\nu-1}(x) - I_{\nu+1}(x) = 2\nu I_\nu(x)/x \tag{49}$$

we have finally the total partition function as a compact expression

$$Z = Z_f e^{-x} I_0(x) \{1 - \frac{\beta^2 (\hbar\omega)^2}{24} \frac{I_1(x)}{I_0(x)} + \frac{\beta^4 (\hbar\omega)^4}{5760} (7 - \frac{2}{\xi} \frac{I_1(x)}{I_0(x)})\} \quad . \tag{50}$$

The factor in curly brackets represents the quantum correction to the classical partition function (41).

In the case of a modulated cosine potential given by

$$V(\phi) = \frac{1}{2} V_0 (1 - \cos n\phi - \varepsilon \cos 2n\phi) \tag{51}$$

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where $\varepsilon \approx 0.1$ for most molecules or molecuar complexes, at first we calculate the classical partition function written in the shape

$$Z_{cl} = Z_f e^{-x} \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos n\phi} [1 + \sum_{\nu=1}^{\infty} \frac{x^\nu \varepsilon^\nu}{\nu!} \cos^\nu 2n\phi] \, d\phi \quad . \tag{52}$$

With [43]

$$\cos^{2\mu-1} 2n\phi = \frac{1}{2^{2\mu-2}} \sum_{k=0}^{\mu-1} \binom{2\mu-1}{k} \cos(2\mu - 2k - 1)2n\phi \tag{53}$$

$$\cos^{2\mu} 2n\phi = \frac{1}{2^{2\mu}} \{ \sum_{k=0}^{\mu-1} 2 \binom{2\mu}{k} \cos 2(\mu - k)2n\phi + \binom{2\mu}{\mu} \} \tag{54}$$

one gets for Z_{cl} a series expansion in terms of Bessel functions $I_{2(2\mu-2k-1)}(x)$ and $I_{4(\mu-k)}(x)$, respectively. Clearly this series converges rather rapidly, and in practice one needs only terms up to the order ε or ε^2. Therefore we use the approximation

$$Z_{cl} = Z_f e^{-x} \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos n\phi} [1 + \varepsilon x \cos 2n\phi + \varepsilon^2 \frac{x^2}{2} \cos^2 2n\phi] \, d\phi \tag{55}$$

and get, by use of (45), after elementary transformations

$$Z = Z_f e^{-x} I_0(x)[1 + \varepsilon x \frac{I_2(x)}{I_0(x)} + \varepsilon^2 \frac{x^2}{4} (1 + \frac{I_4(x)}{I_0(x)}] \quad . \tag{56}$$

Comparing this expression with the classical partition function (41) without the potential proportional to $\varepsilon \cos 2n\phi$, we identify the factor in square brackets in (56) as the potential modulation correction factor in Z_{cl}. With the help of the recursive relations (49) it is again possible to express the whole correction factor in terms of the Bessel functions $I_0(\xi)$ and $I_1(\xi)$ only.

We content ourselves with the first quantum correction $Z(\hbar^2)$ and write

$$Z(\hbar^2) = Z_f e^{-x} \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos n\phi} [1 + \varepsilon R_1 + \varepsilon^2 R_2] \frac{\hbar^2 \beta^2}{12I} \{-V'' + \frac{\beta}{2} V'^2\} \, d\phi \tag{57}$$

where

$$R_1 = x \cos 2n\phi \tag{58}$$

$$R_2 = \frac{x^2}{2} \cos^2 2n\phi = \frac{x^2}{4}(1 + \cos 4n\phi) \tag{59}$$

$$V' = \frac{1}{2} V_0(n \sin n\phi + \varepsilon 2n \sin 2n\phi) \tag{60}$$

$$V'' = \frac{1}{2} V_0(n^2 \cos n\phi + \varepsilon 4n^2 \cos 2n\phi) \tag{61}$$

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After somewhat lengthy calculations using elementary formulae for trigonometric functions of multiple arcs and (45), (46), one ends with

$$Z(\hbar^2) = -\frac{(\hbar\omega)^2\beta^2}{24} Z_f e^{-x}\{I_1(x) + \epsilon x(\frac{3}{2}I_1(x) + I_3(x)) - \epsilon^2 a\}$$

(62)

with

$$a = x(6I_0(x) - \frac{3}{2}I_1(x) + 4I_2(x) - I_3(x) - 2I_4(x)) + x^2(I_3(x) - I_5(x))$$

(63)

This quantum correction again contains terms proportional to ϵ and ϵ^2 due to the potential modulation. From now we neglect the term $\epsilon^2 a$. For the free energy of this simple hindered rotator we get:

$$F = F_{cl} + F(\hbar^2)$$

(64)

where

$$F_{cl} = -RT \ln Z_f - RT(\ln I_0(x) - x)$$

(65)

is the free energy of the semiclassical approximation [56], and

$$F(\hbar^2) = RT \frac{(\hbar\omega)^2\beta^2}{24} \frac{I_1(x)}{I_0(x)}$$

(66)

Now we come to the hindered rotator with the modulated cosine potential. First we write the full partition function according to (56)

$$
\begin{aligned}
Z &= Z_{cl} + Z(\hbar^2) \\
&= Z_f e^{-x} I_0(x)[1 + \epsilon x \frac{I_2(x)}{I_0(x)} + \epsilon^2 \frac{x^2}{4}(1 + \frac{I_4(x)}{I_0(x)})] - \\
&\quad - Z_f e^{-x} I_0(x) \frac{(\hbar\omega)^2\beta^2}{24}[\frac{I_1(x)}{I_0(x)} + \epsilon x(\frac{3}{2}\frac{I_1(x)}{I_0(x)} + \frac{I_3(x)}{I_0(x)})] \\
&= Z_f e^{-x} I_0(x)[1 + A - \frac{(\hbar\omega)^2\beta^2}{24} B]
\end{aligned}
$$

(67)

where A, B are temporary abbreviations (notice that $I_0(x) \geq 1$, $x = \beta V_0/2$). The free energy now becomes

$$F = -RT \ln(Z_f e^{-x} I_0) - RT\{A - \frac{(\hbar\omega)^2\beta^2}{24} B - \frac{1}{2}(A^2 - \frac{\beta^2(\hbar\omega)^2}{12} AB)\}$$

(68)

Here again we take the quantum correction up to terms in $\hbar^2$ and consider the potential modulation correction up to terms in ϵ^2 in the semiclassical contributions and up to terms in ϵ within the quantum correction. With these approximations we get finally:

$$
\begin{aligned}
F &= -RT \ln Z_f - RT(\ln I_0(x) - x) - RT[\epsilon x \frac{I_2(x)}{I_0(x)} + \epsilon^2 \frac{x^2}{4}(1 + \frac{I_4(x)}{I_0(x)} - 2(\frac{I_1(x)}{I_0(x)})^2)] + \\
&\quad + RT \frac{\beta^2(\hbar\omega)^2}{24}[\frac{I_1(x)}{I_0(x)} + \epsilon x(\frac{3}{2}\frac{I_1(x)}{I_0(x)} + \frac{I_3(x)}{I_0(x)} - \frac{I_1(x)I_2(x)}{I_0^2(x)})]
\end{aligned}
$$

(69)

6   Acknowledgements

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Decoherence of
Multimode Thermal Squeezed Coherent States [*]

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Abstract

It is well known that any multimode positive definite quadratic Hamiltonian can be transformed into a Hamiltonian of uncoupled harmonic oscillators. Based on this theorem, the multimode thermal squeezed coherent states are constructed in terms of density operators. Decoherence of multimode thermal squeezed coherent states is investigated via the characteristic function and it is shown that the decohered (reduced) states are still thermal squeezed coherent states in general.

1 Introduction

In the past three decades, decoherence (reduction) has become more and more widely recognized, being studied in many fields of physics from non–equilibrium statistical mechanics [1] to quantum measurement and quantum cosmology [2, 3]. The purpose of this paper is to investigate decoherence of multimode thermal Squeezed Coherent States (SqCS's).

In the literature there are many equivalent definitions for one–mode, two–mode [4, 5, 6, 7] and multimode SqCS's [8]. However, thermal SqCS's are usually defined for one–mode [9]. (Two–mode thermal SqCS in thermo–field formalism is effectively one–mode.) Therefore, in this paper we first introduce a general definition of the multimode thermal SqCS's in terms of density operators. Then we will discuss two related representations—the Wigner function and the characteristic function—and will show that the latter is the better representation for decoherence problems. Finally we will use the characteristic function to study the decoherence of multimode thermal SqCS's.

This paper is organized as follows: In Sec. 2 notations, conventions and a lemma on matrix are introduced for the mathematics used in this paper. In Sec. 3 a unified definition of multimode SqCS's with the aid of a special kind of Hamiltonian is presented. In Sec. 4 the multimode thermal SqCS is constructed by thermalizing the multimode SqCS defined in Sec. 3. In Sec. 5 we calculate the Wigner functions and the characteristic functions of some multimode thermal SqCS's. In Sec. 6 the decoherence of multimode thermal SqCS is effectuated and it is shown that the decohered state is still a thermal SqCS.

2 Mathematical Preliminaries

Throughout this paper, $\hbar$ is set equal to 1; "$\dagger$" denotes hermitian conjugate and "t" denotes the transpose of a matrix. The physical system under consideration is of n degrees of freedom, hence the dummy indices run from 1 to n unless otherwise specified.

We use $\vec{x} = < x_1, x_2, \ldots, x_n >$ and $\vec{k} = < k_1, k_2, \ldots, k_n >$ for the n–dimensional canonical coordinate and momentum respectively. Thus $< \vec{x}; \vec{k} >$ is a vector in

$2n$–dimensional phase space. $\vec{q}$ and $\vec{p}$ denote the n–dimensional position and momentum operators corresponding to the canonical variables $\vec{x}$ and $\vec{k}$. The Canonical Commutation Relations (CCR's) are:

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{q}_i, \hat{p}_j] = i\delta_{ij}. \tag{1}$$

$|0\rangle$ denotes the n–mode Fock vacuum state, i.e., the ground state of an n–dimensional harmonic oscillator with unit mass and frequency:

$$\langle \vec{x}|0\rangle = \pi^{-\frac{n}{4}} \exp[-\frac{1}{2}(\vec{x})^2]. \tag{2}$$

The number operators are defined in the ordinary way:

$$\hat{N}_i = \frac{1}{2}(\hat{p}_i^2 + \hat{q}_i^2 - 1). \tag{3}$$

The (phase space) displacement operator $\hat{D}(< \vec{x}; \vec{k} >)$ is defined as:

$$\hat{D}(< \vec{x}; \vec{k} >) = \exp[i(\vec{k} \cdot \vec{q} - \vec{x} \cdot \vec{p})]. \tag{4}$$

$\hat{D}(< \vec{x}; \vec{k} >)$ is unitary and has the following properties:

$$\hat{D}^\dagger(< \vec{x}; \vec{k} >) = \hat{D}^{-1}(< \vec{x}; \vec{k} >) = \hat{D}(- < \vec{x}; \vec{k} >), \tag{5}$$

$$\hat{D}(< \vec{x}; \vec{k} >) < \vec{q}; \vec{p} > \hat{D}^{-1}(< \vec{x}; \vec{k} >) = < (\vec{q} - \vec{x}); (\vec{p} - \vec{k}) > . \tag{6}$$

The coherent state is defined as [4]:

$$|\vec{x}, \vec{k}\rangle = \hat{D}(< \vec{x}, \vec{k} >)|0\rangle. \tag{7}$$

Another kind of unitary operator we will use in this paper are the elements of metaplectic group **Mp(2n,R)**—the quantum analogue of symplectic group **Sp(2n,R)**. **Mp(2n,R)** is an $n(2n + 1)$–dimensional Lie group with its algebra spanned by $\{\hat{q}_i\hat{q}_j, \hat{p}_i\hat{p}_j, \hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i\}$. The elements of the Lie algebra of **Mp(2n,R)** can be organized as anti–hermitian operators in the following form:

$$\begin{aligned}
\hat{\Phi}(m) &= \frac{i}{2} \sum_{i,j=1}^{n} [\alpha_{ij}\hat{q}_i\hat{q}_j + \beta_{ij}\hat{p}_i\hat{p}_j + \gamma_{ij}(\hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i)] \\
&= \frac{i}{2} < \vec{q}; \vec{p} > \begin{pmatrix} \alpha & \gamma \\ \gamma^t & \beta \end{pmatrix} < \vec{q}; \vec{p} >^t \\
&= \frac{i}{2} < \vec{q}; \vec{p} > Jm < \vec{q}; \vec{p} >^t, \tag{8}
\end{aligned}$$

where $\alpha_{ij} = \alpha_{ji}$, $\beta_{ij} = \beta_{ji}$ and

$$m = \begin{pmatrix} -\gamma^t & -\beta \\ \alpha & \gamma \end{pmatrix} \in \text{sp}(2\text{n},\text{r}) \tag{9}$$

is a $2n \times 2n$ real matrix [10], while

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad 1 = n \times n \text{ unit matrix.} \tag{10}$$

From CCR's, we have:

$$[\hat{\Phi}(m), < \vec{q}; \vec{p} >^t] = \begin{pmatrix} \gamma^t & \beta \\ -\alpha & -\gamma \end{pmatrix} < \vec{q}; \vec{p} >^t = -m < \vec{q}; \vec{p} >^t, \tag{11}$$

and

$$[\hat{\Phi}(m_1), \hat{\Phi}(m_2)] = \hat{\Phi}([m_1, m_2]). \tag{12}$$

Therefore the Lie algebra of **Mp(2n,R)** is isomorphic to **sp(2n,r)**—the Lie algebra of **Sp(2n,R)**

The action of $\exp[\hat{\Phi}(m)] \in$ **Mp(2n,R)** on $< \vec{q}; \vec{p} >$ can be defined and calculated from (11):

$$\exp[\hat{\Phi}(m)] < \vec{q}; \vec{p} >^t \exp[-\hat{\Phi}(m)] = \exp(-m) < \vec{q}; \vec{p} >^t, \tag{13}$$

where $\exp(-m) \in$ **Sp(2n,R)**.

Now we replace $\exp(-m)$ in (13) by a general element $S \in$ **Sp(2n,R)** and try to find a unitary operator $\hat{U}(S) \in$ **Mp(2n,R)** such that

$$\hat{U}(S) < \vec{q}; \vec{p} >^t \hat{U}(S)^{-1} = S < \vec{q}; \vec{p} >^t. \tag{14}$$

From linear algebra and group theory, we know that there is a unique polar decomposition $S = RP$ for any element S in **Sp(2n,R)**, where R is orthogonal, P is symmetric and positive definite, and both R and P are in **Sp(2n,R)**. Therefore we can always put $S = \exp(m_R)\exp(m_P)$, where $R = \exp(m_R)$, $P = \exp(m_P)$, and both m_R and m_S are elements in **sp(2n,r)** (m_P is symmetric and unique, while m_R is anti–symmetric and not unique) [11]. The element $\hat{U}(S) \in$ **Mp(2n,R)** which is unitary and satisfies (14) can be constructed as follows:

$$\hat{U}(S) = \exp[\hat{\Phi}(-m_P)]\exp[\hat{\Phi}(-m_R)], \tag{15}$$

where $\exp[\hat{\tilde{\Phi}}(-m_P)]$ corresponds to a generalized squeezing and $\exp[\hat{\tilde{\Phi}}(-m_R)]$ to a rotation in $2n$–dimensional phase space. This decomposition is crucial in the construction of multimode SqCS since the Fock vacuum is an eigenstate of $\exp[\hat{\tilde{\Phi}}(-m_R)]$, hence only the degrees of freedom in $\exp[\hat{\tilde{\Phi}}(-m_P)]$ are effective in the SqCS constructed as $\hat{U}(S)|0\rangle$ [8].

Lemma [12]

If M is a symmetric and positive definite $2n \times 2n$ matrix, then there exists a matrix $S \in \mathbf{Sp(2n, R)}$ (but not unique), such that

$$M = S^t \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} S, \tag{16}$$

where $\omega = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_n)$, $\omega_j > 0$ for all j.

Remarks:

(1) $S \in \mathbf{Sp(2n, R)}$ if and only if $S^t J S = J$ by definition.

(2) ω_j is not an eigenvalue of M in general.

(3) The eigenvalues of JM are $\pm i\omega_j$'s, hence we can calculate ω_j's from JM as an ordinary eigenvalue problem.

(4) If the matrix C_j corresponds to a 2–dimensional rotation on the $< x_j, k_j >$ plane, then

$$C_j^t \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} C_j = C_j^t C_j \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}. \tag{17}$$

Therefore S in (16) can be replaced by $C_j S$ and hence is not unique.

3 Unified Definition of Multimode Squeezed Coherent States

The Hamiltonian we will use in this section is of n–mode, inhomogeneously quadratic, time–independent, and with its quadratic part positive definite:

$$\hat{H} = \frac{1}{2} < \vec{q}; \vec{p} > M < \vec{q}; \vec{p} >^t + V < \vec{q}; \vec{p} >^t, \tag{18}$$

where M is a $2n \times 2n$, symmetric and positive definite (hence invertible) real matrix and V is a $1 \times 2n$ real vector. This kind of Hamiltonian can be transformed into the "standard form":

$$\hat{H} = \hat{D}_0 \hat{U}_0 \hat{H}_0 \hat{U}_0^{-1} \hat{D}_0^{-1} + \text{constant}, \tag{19}$$

where $\hat{D}_0$ is a displacement operator, $\hat{U}_0$ is an operator in $\text{Mp}(2n, \mathbf{R})$ and

$$\hat{H}_0 = \sum_{i=1}^n \omega_i \hat{N}_i, \qquad \omega_i > 0. \tag{20}$$

Using the formulas discussed in last section, the derivation of (19) is straightforward:

$$
\begin{aligned}
\hat{H} &= \frac{1}{2} < \vec{q}; \vec{p} > M < \vec{q}; \vec{p} >^t + V < \vec{q}; \vec{p} >^t \\
&= \frac{1}{2}[< \vec{q}; \vec{p} > + V M^{-1}] M [< \vec{q}; \vec{p} >^t + M^{-1} V^t] + \text{constant} \\
&= \frac{1}{2} < (\vec{q} - \vec{x}_0); (\vec{p} - \vec{k}_0) > M < (\vec{q} - \vec{x}_0); (\vec{p} - \vec{k}_0) >^t + \text{constant} \\
&= \frac{1}{2} \hat{D}_0 < \vec{q}; \vec{p} > \hat{D}_0^{-1} M \hat{D}_0 < \vec{q}; \vec{p} >^t \hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0 [\frac{1}{2} < \vec{q}; \vec{p} > M < \vec{q}; \vec{p} >^t] \hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0 [\frac{1}{2} < \vec{q}; \vec{p} > S^t \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} S < \vec{q}; \vec{p} >^t] \hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0 [\frac{1}{2} \hat{U}_0 < \vec{q}; \vec{p} > \hat{U}_0^{-1} \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \hat{U}_0 < \vec{q}; \vec{p} >^t \hat{U}_0^{-1}] \hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0 \hat{U}_0 [\frac{1}{2} < \vec{q}; \vec{p} > \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} < \vec{q}; \vec{p} >^t] \hat{U}_0^{-1} \hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0 \hat{U}_0 [\frac{1}{2} \sum_{i=1}^n \omega_i (\hat{q}_i^2 + \hat{p}_i^2)] \hat{U}_0^{-1} \hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0 \hat{U}_0 [\sum_{i=1}^n \omega_i \hat{N}_i] \hat{U}_0^{-1} \hat{D}_0^{-1} + \text{constant} \\
&= \hat{D}_0 \hat{U}_0 \hat{H}_0 \hat{U}_0^{-1} \hat{D}_0^{-1} + \text{constant}, \tag{21}
\end{aligned}
$$

where $V M^{-1} = - < \vec{x}_0; \vec{k}_0 >$, $\hat{D}_0 = \hat{D}(< \hat{x}_0; \hat{k}_0 >)$ and $\hat{U}_0 = \hat{U}(S)$.

Without loss of generality, we can always drop the constant term and consider

$$\hat{H} = \hat{D}_0 \hat{U}_0 \hat{H}_0 \hat{U}_0^{-1} \hat{D}_0^{-1}. \tag{22}$$

It is easy to see that the normalized ground state of this Hamiltonian is:

$$\hat{D}_0 \hat{U}_0 |0\rangle \propto \hat{D}_0 \exp[\hat{\Phi}(-m_P)]|0\rangle, \tag{23}$$

which is a SqCS in general, it contains the coherent state ($U_0 = 1$, $D_0 \neq 1$) and the squeezed state ($U_0 \neq 1$, $D_0 = 1$) as two special cases.

Therefore we can take (23) as a unified definition of the multimode SqCS. However, since those ω_i's in $\hat{H}_0$ do not appear in (23), the correspondence between (22) and (23) is many–to–one. The non–uniqueness of S, hence $\hat{U}_0$, will not cause any trouble, because we have shown that S is unique up to some 2–dimensional rotations in phase space, and rotations correspond to $\exp[\hat{\Phi}(-m_R)]$ in $\hat{U}(S)$ which will not appear in (23).

4 Multimode Thermal Squeezed Coherent States

Consider immersing a physical system described by the Hamiltonian (22) in a heat bath of temperature T. This constitutes a canonical ensemble and the density operator of this system is:

$$\begin{aligned}
\hat{\rho} &= Z^{-1} \exp(-\beta \hat{H}) \\
&= Z^{-1} \exp[-\beta(\hat{D}_0 \hat{U}_0 \hat{H}_0 \hat{U}_0^{-1} \hat{D}_0^{-1})] \\
&= Z^{-1} \hat{D}_0 \hat{U}_0 \exp(-\beta \hat{H}_0) \hat{U}_0^{-1} \hat{D}_0^{-1}, \tag{24}
\end{aligned}$$

where

$$\beta = \frac{1}{kT}, \qquad Z = Tr[\exp(-\beta \hat{H})] = Tr[\exp(-\beta \hat{H}_0)]. \tag{25}$$

This density operator $\hat{\rho}$ describes a mixed state unless $T = 0$. In the limit as $T \to 0$, since

$$\lim_{\beta \to \infty} \exp(-\beta \hat{H}_0) = |0\rangle\langle 0|, \tag{26}$$

we have

$$\hat{\rho} = \hat{D}_0 \hat{U}_0 |0\rangle\langle 0| \hat{U}_0^{-1} \hat{D}_0^{-1}, \tag{27}$$

which corresponds to the pure SqCS (23), hence (23) is a special case of (24) and (24) is a "thermalized state" of (23). Therefore we can take (24) as the definition of multimode thermal SqCS.

5 Representations of Multimode Thermal Squeezed Coherent States

There are many equivalent representations of the density operator $\hat{\rho}$, e.g., the coordinate representation, P–representation, Q–representation, Fock space representation, Wigner function and characteristic function, etc. In this paper we will discuss the last two representations.

5.1 Wigner Function

The Wigner function of a density operator $\hat{\rho}$ is defined as [13, 14]:

$$W(\vec{x}; \vec{k}) = \pi^{-n} \int_{-\infty}^{\infty} d\vec{y} \exp(2i\vec{k} \cdot \vec{y}) \rho(\vec{x} - \vec{y}, \vec{x} + \vec{y}), \qquad (28)$$

where $\rho(\vec{x}, \vec{x'})$ is the coordinate representation of the density operator $\hat{\rho}$.

The Wigner function can also be put into the following form [15]:

$$W(\vec{x}; \vec{k}) = Tr[\hat{\rho} \hat{\Delta}_W(< \vec{x}; \vec{k} >)], \qquad (29)$$

where the "Wigner operator" $\hat{\Delta}_W(< \vec{x}; \vec{k} >) = \pi^{-n} \hat{D}(2 < \vec{x}; \vec{k} >) \exp(i\pi \sum_{i=1}^{n} \hat{N}_i)$ is a well–defined hermitian operator with $< \vec{x}; \vec{k} >$ as its parameters.

The Wigner function is normalized by definition:

$$\int_{-\infty}^{\infty} d\vec{x} d\vec{k} W(\vec{x}; \vec{k}) = 1, \qquad (30)$$

and it is real because the Wigner operator is hermitian. However, the Wigner function is not always positive–definite and it is thus called the quasi–probability distribution function over the "phase space" $(\vec{x}; \vec{k})$.

In the following, we will calculate the Wigner functions for some thermal SqCS's. First let us consider the simplest one–mode case, i.e.,

$$\hat{H} = \frac{1}{2}\omega(\hat{p}^2 + \hat{q}^2 - 1) = \omega\hat{N}, \qquad (31)$$

the density operator is:

$$\hat{\rho} = Z^{-1}\exp(-\beta\omega\hat{N}), \tag{32}$$

and the Wigner function takes the form [14]:

$$W(x,k) = Tr[\hat{\rho}\hat{\Delta}_W(x,k)] = \frac{1}{\pi}\tanh(\frac{\beta\omega}{2})\exp[-\tanh(\frac{\beta\omega}{2})(x^2 + k^2)]. \tag{33}$$

In the limit as $T \to 0$, (33) becomes

$$W(x,k) = \frac{1}{\pi}\exp[-(x^2 + k^2)], \tag{34}$$

which is exactly the Wigner function of the vacuum state [14].

Noticing that the Wigner function (33) is a Gaussian distribution function in (x,k), we can use the exponent of (33) to define the "Wigner ellipse" in the phase space (x,k) as:

$$\tanh(\frac{\beta\omega}{2})(x^2 + k^2) = 1. \tag{35}$$

The area of the Wigner ellipse represents the range of uncertainty of the corresponding state. In this simplest case, the Wigner ellipse is a circle with radius $\sqrt{\coth(\frac{1}{2}\beta\omega)} \geq 1$ and with its center at the origin.

Next we consider the general one–mode Hamiltonian:

$$\hat{H} = \omega\hat{D}_0\hat{U}_0\hat{N}\hat{U}_0^{-1}\hat{D}_0^{-1}, \tag{36}$$

the density operator is:

$$\hat{\rho} = Z^{-1}\hat{D}_0\hat{U}_0\exp(-\beta\omega\hat{N})\hat{U}_0^{-1}\hat{D}_0^{-1}. \tag{37}$$

The Wigner function takes the form [16]:

$$\begin{aligned}
W(x,k) &= Z^{-1}Tr[\hat{D}_0\hat{U}_0\exp(-\beta\omega\hat{N})\hat{U}_0^{-1}\hat{D}_0^{-1}\hat{\Delta}_W(x,k)] \\
&= Z^{-1}Tr[\exp(-\beta\omega\hat{N})\hat{U}_0^{-1}\hat{D}_0^{-1}\hat{\Delta}_W(x,k)\hat{D}_0\hat{U}_0] \\
&= Z^{-1}Tr[\exp(-\beta\omega\hat{N})\hat{\Delta}_W(x',k')] \\
&= \frac{1}{\pi}\tanh(\frac{\beta\omega}{2})\exp[-\tanh(\frac{\beta\omega}{2})(x'^2 + k'^2)], \tag{38}
\end{aligned}$$

where

$$\begin{pmatrix} x' \\ k' \end{pmatrix} = S\begin{pmatrix} x - x_0 \\ k - k_0 \end{pmatrix}, \tag{39}$$

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and the Wigner ellipse is:

$$\tanh(\frac{\beta\omega}{2})(x - x_0 \, , \, k - k_0)S^t S \begin{pmatrix} x - x_0 \\ k - k_0 \end{pmatrix} = 1. \tag{40}$$

For the n–mode cases we first consider the uncoupled Hamiltonian, i.e., $\hat{H} = \hat{H}_0$. The Wigner function in this case is a product of each individual one–mode Wigner function:

$$W(\vec{x}; \vec{k}) = \pi^{-n}[\prod_{i=1}^{n} \tanh(\frac{\beta\omega_i}{2})] \exp[-\sum_{i=1}^{n} \tanh(\frac{\beta\omega_i}{2})(x_i^2 + k_i^2)]. \tag{41}$$

In this multimode case, we can define the "Wigner ellipsoid" in $2n$–dimensional phase space as:

$$\sum_{i=1}^{n} \tanh(\frac{\beta\omega_i}{2})(x_i^2 + k_i^2) = 1, \tag{42}$$

or equivalently,

$$(\vec{x}; \vec{k}) \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} (\vec{x}; \vec{k})^t = 1, \tag{43}$$

where $T = \mathrm{diag}(\tanh(\frac{1}{2}\beta\omega_1), \tanh(\frac{1}{2}\beta\omega_2), \ldots, \tanh(\frac{1}{2}\beta\omega_n))$. Analogously, the $2n$–dimensional volume of the Wigner ellipsoid represents the range of uncertainty.

For the most general Hamiltonian $\hat{H} = \hat{D}_0 \hat{U}_0 \hat{H}_0 \hat{U}_0^{-1} \hat{D}_0^{-1}$, analogue to (38), the Wigner function is:

$$\begin{aligned}
W(\vec{x}; \vec{k}) &= \pi^{-n}[\prod_{i=1}^{n} \tanh(\frac{\beta\omega_i}{2})] \\
&\quad \exp[-(\vec{x} - \vec{x}_0; \vec{k} - \vec{k})S^t \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} S(\vec{x} - \vec{x}_0; \vec{k} - \vec{k}_0)^t],
\end{aligned} \tag{44}$$

and the Wigner ellipsoid becomes:

$$(\vec{x} - \vec{x}_0; \vec{k} - \vec{k})S^t \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} S(\vec{x} - \vec{x}_0; \vec{k} - \vec{k}_0)^t = 1. \tag{45}$$

5.2 Characteristic Function

The characteristic function of a density operator $\hat{\rho}$ is defined as:

$$X(\vec{x}; \vec{k}) = Tr[\hat{\rho}\hat{D}(-\vec{x}; -\vec{k})], \tag{46}$$

From the symplectic Fourier transformation of the Wigner operator $\hat{\Delta}_W(\vec{x}; \vec{k})$:

$$\mathbf{F}[\hat{\Delta}_W(\vec{x}; \vec{k})] = \int_{-\infty}^{\infty} d\vec{x}' d\vec{k}' \hat{\Delta}_W(\vec{x}'; \vec{k}') \exp[-i(\vec{x}' \cdot \vec{k} - \vec{k}' \cdot \vec{x})]$$

$$= \hat{D}(-\vec{x}; -\vec{k}), \tag{47}$$

we can see that the characteristic function is the symplectic Fourier transformation of the Wigner function:

$$\mathbf{F}[W(\vec{x}; \vec{k})] = X(\vec{x}; \vec{k}). \tag{48}$$

The normalization condition of the Wigner function corresponds to $X(\vec{0}; \vec{0}) = 1$ in the characteristic function. Since the operator $\hat{D}(-\vec{x}; -\vec{k})$ is unitary instead of hermitian, $X(\vec{x}; \vec{k})$ is complex in general.

The characteristic function of the general n-mode thermal SqCS, which corresponds to the Wigner function (44), is:

$$X(\vec{x}; \vec{k}) = \exp[-\frac{1}{4}(\vec{x}; \vec{k}) S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S(\vec{x}; \vec{k})^t + i(\vec{x} \cdot \vec{k}_0 - \vec{k} \cdot \vec{x}_0)]. \tag{49}$$

5.3 Covariance Matrix

For an n-mode (mixed) state with density operator $\hat{\rho}$, the covariance matrix is a $2n \times 2n$ matrix of the form:

$$\begin{pmatrix} U & Q \\ Q^t & V \end{pmatrix}, \tag{50}$$

$$U_{ij} \equiv \langle (\hat{q}_i - \langle \hat{q}_i \rangle)(\hat{q}_j - \langle \hat{q}_j \rangle) \rangle = \langle \hat{q}_i \hat{q}_j \rangle - \langle \hat{q}_i \rangle \langle \hat{q}_j \rangle, \tag{51}$$

$$V_{ij} \equiv \langle (\hat{p}_i - \langle \hat{p}_i \rangle)(\hat{p}_j - \langle \hat{p}_j \rangle) \rangle = \langle \hat{p}_i \hat{p}_j \rangle - \langle \hat{p}_i \rangle \langle \hat{p}_j \rangle, \tag{52}$$

$$Q_{ij} \equiv \frac{1}{2} \langle (\hat{q}_i - \langle \hat{q}_i \rangle)(\hat{p}_j - \langle \hat{p}_j \rangle) + (\hat{p}_j - \langle \hat{p}_j \rangle)(\hat{q}_i - \langle \hat{q}_i \rangle) \rangle$$

$$= \langle \frac{1}{2}(\hat{q}_i \hat{p}_j + \hat{p}_j \hat{q}_i) \rangle - \langle \hat{q}_i \rangle \langle \hat{p}_j \rangle, \tag{53}$$

where $\langle \hat{q}_i \rangle \equiv Tr(\hat{\rho} \hat{q}_i)$, etc.

For the thermal SqCS which corresponds to the Wigner function (44) or the characteristic function (49), it can be proved that the covariance matrix is

$$\frac{1}{2} S^{-1} \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} (S^{-1})^t. \tag{54}$$

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6   Decoherence Problems

6.1   General Theory of Decoherence

Consider a quantum system which contains two subsystems (A) and (B) with the density operator $\hat{\rho}_{AB}$. Any (monomial) operator $\hat{O}$ which corresponds to a measurement on the system can be decomposed into $\hat{O}_A \otimes \hat{O}_B$, where $\hat{O}_A$ corresponds to a measurement on and only on (A) and $\hat{O}_B$ correspondingly on (B). If we decohere this system by ignoring (B), i.e., not making any measurement on (B), then the operator $\hat{O}$ will be reduced to $\hat{O}_A \otimes \hat{1}$ and the expectation value of $\hat{O}_A$ will become:

$$
\begin{aligned}
\langle \hat{O}_A \rangle &= Tr[\hat{\rho}_{AB}(\hat{O}_A \otimes \hat{1})] \\
&= Tr_{(A)}Tr_{(B)}[\hat{\rho}_{AB}(\hat{O}_A \otimes \hat{1})] \\
&= Tr_{(A)}[(Tr_{(B)}(\hat{\rho}_{AB}))\hat{O}_A] \\
&= Tr[\hat{\rho}_A \hat{O}_A],
\end{aligned}
\tag{55}
$$

where $Tr_{(A)}/\ Tr_{(B)}$ represents the "partial trace" which only takes trace with respect to the degrees of freedom of (A)/(B), and $\hat{\rho}_A = Tr_{(B)}(\hat{\rho}_{AB})$ is a well-defined reduced density operator.

If the Wigner function $W(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B)$ corresponds to the original density operator $\hat{\rho}_{AB}$, then the reduced Wigner function corresponding to $\hat{\rho}_A$ is [14]:

$$
W_A(\vec{x}_A; \vec{k}_A) = \int_{-\infty}^{\infty} d\vec{x}_B d\vec{k}_B W(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B).
\tag{56}
$$

As for the characteristic function, if $X(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B)$ corresponds to $\hat{\rho}_{AB}$, the reduced characteristic function corresponding to $\hat{\rho}_A$ will take the form:

$$
X_A(\vec{x}_A; \vec{k}_A) = X(\vec{x}_A, \vec{0}; \vec{k}_A, \vec{0}),
\tag{57}
$$

which is a restriction of the original $X(\vec{x}_A; \vec{x}_B, \vec{k}_A; \vec{k}_B)$ to a subspace in the $2n$–dimensional phase space. From the mathematical point of view, it is easier to use the characteristic function to study decoherence problems.

6.2 Decoherence of a Thermal Squeezed Coherent State From n–Mode to m–Mode

For a given characteristic function of an n–mode thermal SqCS:

$$
\begin{aligned}
& X(x_1, x_2, \ldots, x_m, \ldots, x_n; k_1, k_2, \ldots, k_m, \ldots, k_n) \\
= & \exp\{-\frac{1}{4}(x_1, x_2, \ldots, x_m, \ldots, x_n; k_1, k_2, \ldots, k_m, \ldots, k_n) \\
& S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S(x_1, x_2, \ldots, x_m, \ldots, x_n; k_1, k_2, \ldots, k_m, \ldots, k_n)^t \\
& +i\sum_{j=1}^{n}(x_j k_{0j} - k_j x_{0j})\}.
\end{aligned}
\tag{58}
$$

The reduced characteristic function is:

$$
\begin{aligned}
& X(x_1, x_2, \ldots, x_m, \vec{0}; k_1, k_2, \ldots, k_m, \vec{0}) \\
= & \exp\{-\frac{1}{4}(x_1, x_2, \ldots, x_m, \vec{0}; k_1, k_2, \ldots, k_m, \vec{0}) \\
& S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S(x_1, x_2, \ldots, x_m, \vec{0}; k_1, k_2, \ldots, k_m, \vec{0})^t \\
& +i\sum_{j=1}^{m}(x_j k_{0j} - k_j x_{0j})\} \\
= & \exp\{-\frac{1}{4}(x_1, x_2, \ldots, x_m; k_1, k_2, \ldots, k_m,)K(x_1, x_2, \ldots, x_m; k_1, k_2, \ldots, k_m)^t \\
& +i\sum_{i=1}^{m}(x_i k_{0i} - k_i x_{0i})\},
\end{aligned}
\tag{59}
$$

where the matrix K is $2m \times 2m$ and still symmetric and positive–definite, its elements are a subset of the the elements of $S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S$:

$$
K_{i,j} = [S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S]_{i,j},
\tag{60}
$$

$$
K_{i+m,j} = [S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S]_{i+n,j},
\tag{61}
$$

$$
K_{i,j+m} = [S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S]_{i,j+n},
\tag{62}
$$

$$K_{i+m,j+m} = [S^t \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} S]_{i+n,j+n},$$ (63)

where $1 \le i, j, \le m$.

From the Lemma in Sec. 1, we can find a $2m \times 2m$ symplectic matrix σ such that:

$$K = \sigma^t \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} \sigma,$$ (64)

where $\tau = \text{diag}(\tau_1, \tau_2, \ldots, \tau_m)$, $\tau_i > 0$, for all $i = 1, 2, \ldots, m$.

We can make a further restriction on τ_i from the following physical consideration: Since the reduced density operator $\hat{\rho}_A = Tr_{(B)}(\hat{\rho}_{AB})$ is well–defined, it will never correspond to any non–physical state. Noticing that (59) is of the same form as (49), comparison with (54) shows the covariance matrix of this decohered state to be:

$$K = \frac{1}{2}\sigma^{-1} \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} (\sigma^{-1})^t.$$ (65)

Since σ^{-1} corresponds to a symplectic (hence canonical) transformation on the canonical coordinates,

$$\frac{1}{2} \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix}$$ (66)

is also a covariance matrix for the same state in another canonical coordinates. This guarantees that $\tau_i \ge 1$ for all $i = 1, 2, \ldots, m$, otherwise (59) will give a state that violates the uncertainty principle. Therefore we conclude that the reduced characteristic function (59) corresponds to an m–mode thermal SqCS.

7 Conclusion

The results of this paper are threefold (1) A unified construction of multimode (thermal) SqCS's. (2) Proof of the statement: The decohered multimode thermal SqCS is still a (multimode) thermal SqCS. (3) Introduction of the decohering technique via characteristic function, which is very efficient and can be applied to many related problems.

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Quantum Harmonic Oscillator In A Thermal Bath

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ABSTRACT

In this talk, we briefly review the influence functional path-integral treatment of quantum Brownian motion. We report on a newly derived exact master equation of a quantum harmonic oscillator coupled to a general environment at arbitrary temperature. We apply it to the problem of loss of quantum coherence.

INTRODUCTION

Recently there has been considerable interest in quantum Brownian motion. It was motivated by possible observation of macroscopic effects in quantum systems. Among them are quantum tunneling with dissipation [1], loss of quantum coherence due to system-environment interaction [2], just name a few. The newest application of quantum Brownian motion is in quantum cosmology, where the issue of quantum-to-classical transition of an open system is very important [3]. These issues also appear in semiclassical theory of early universe in which noise, fluctuation and dissipation play important roles in particle production, back reaction, phase transition, inflation and galaxies formation [4]. In these problems, the interaction between a system and its environment is quite complicated giving rise to nonlocal dissipation and colored noise.

The effect of nonlocal dissipation and colored noise in quantum Brownian motion is an outstanding problem, which has been studied only to a limited extent. In some limiting cases, the quantum master equation (the time evolution equation) for the reduced density matrix of the Brownian motion has been derived before by different authors with different methods. These cases are all in the class of ohmic environment, for which the dissipation is always local [5]. It corresponds to having a linear damping force proportional to the velocity of the Brownian particle classically. The noise associated with the dissipation is colored at low temperature.

Our contribution reported in this talk is the derivation of an exact master equation for the reduced density matrix of a Brownian harmonic oscillator linearly coupled to a general environment (with a general thermal bath spectral density) at arbitrary temperature [6]. In our model, the environment is a set of bath harmonic oscillators with different natural frequencies. The environment is at a thermal equilibrium state. The system (Brownian particle) is brought to contact with this thermal bath. The derivation is done from first principles of statistical and quantum physics with Feynman path-integral

method and Feynman-Vernon influence functional formalism [7]. This master equation can accommodate all possible forms of the nonlocal dissipation kernel and nonlocal noise kernel. It is a linear partial differential equation with time dependent coefficients. The non-Markovian character resides in these coefficients. In particular we examine the cases of ohmic, subohmic and superohmic environment and compute these time dependent coefficients numerically. We show that all the previous master equations obtained otherwise are just special examples of our master equation.

INFLUENCE FUNCTIONAL

Let us briefly review the Feynman-Vernon influence functional formalism of quantum open system. Consider a Brownian particle with mass $M = 1$ and natural (bare) frequency Ω. The environment is modeled by a set of harmonic oscillators with mass m_n and natural frequency ω_n. The Brownian particle is coupled linearly to each bath oscillator with strength C_n. The total action of the combined system plus environment is

$$
\begin{aligned}
S[x, q] &= S[x] + S_E[q] + S_{int}[x, q] \\
&= \int_0^t ds \left\{ \frac{1}{2}\dot{x}^2 - \frac{1}{2}\Omega^2 x^2 \right\} + \int_0^t ds \sum_n \left\{ \frac{1}{2}m_n \dot{q}_n^2 - \frac{1}{2}m_n \omega_n^2 q_n^2 \right\} \\
&\quad + \int_0^t ds \sum_n \left\{ C_n x q_n \right\}
\end{aligned}
\tag{1}
$$

where x and q_n are the coordinates of the particle and the n-th bath oscillators.

It is well known that the time evolution of the total density matrix of the system plus environment $\hat{\rho}(t)$ is governed by the following quantum Liouville equation

$$
i\hbar \frac{d}{dt}\hat{\rho}(t) = \left[\hat{H}, \ \hat{\rho}(t) \right]
\tag{2}
$$

In the coordinate representation, the solution of the above quantum Liouville equation can

be written as

$$\rho(x_f, q_f; x'_f, q'_f; t) = \int\limits_{-\infty}^{+\infty} dx_i \int\limits_{-\infty}^{+\infty} dx'_i \int\limits_{-\infty}^{+\infty} dq_i \int\limits_{-\infty}^{+\infty} dq'_i$$

$$\times J(x_f, q_f, x'_f, q'_f, t \mid x_i, q_i, x'_i, q'_i, 0) \, \rho(x_i, q_i; x'_i, q'_i; 0) \tag{3}$$

where

$$J(x_f, q_f, x'_f, q'_f, t \mid x_i, q_i, x'_i, q'_i, 0)$$

$$= \int\limits_{x_i}^{x_f} Dx \int\limits_{x'_i}^{x'_f} Dx' \int\limits_{q_{ni}}^{q_{nf}} Dq \int\limits_{q'_{ni}}^{q'_{nf}} Dq \, \exp\frac{i}{\hbar}\Big\{ S[x, q] - S[x', q'] \Big\} \tag{4}$$

is the propagator of the total density matrix in path-integral form. Here q represents the full set of bath oscillator coordinates and the subscript i and f denote the initial and final variables.

We are only interested in how the dynamics of the system (the Brownian particle) under the influence of the environment (all bath oscillators). The quantity containing this information is the reduced density matrix of the system

$$\rho_r(x, x') = \int\limits_{-\infty}^{+\infty} dq \int\limits_{-\infty}^{+\infty} dq' \, \rho(x, q; x', q')\delta(q - q') \tag{5}$$

which is propagated in time by the the evolution operator

$$\rho_r(x_f, x'_f, t) = \int\limits_{-\infty}^{+\infty} dx_i \int\limits_{-\infty}^{+\infty} dx'_i \, J_r(x_f, x'_f, t \mid x_i, x'_i, 0) \, \rho_r(x_i, x'_i, 0) \tag{6}$$

If we assume that at $t = 0$ the system and the environment are uncorrelated

$$\hat{\rho}(t = 0) = \hat{\rho}_s(0) \times \hat{\rho}_e(0), \tag{7}$$

then

$$J_r(x_f, x'_f, t \mid x_i, x'_i, 0) = \int\limits_{x_i}^{x_f} Dx \int\limits_{x'_i}^{x'_f} Dx' \, \exp\frac{i}{\hbar}\Big\{ S[x] - S[x'] \Big\} \, F[x, x'] \tag{8}$$

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The functional factor $F[x, x']$ in (8), called influence functional, is defined as

$$F[x, x'] = \int\limits_{-\infty}^{+\infty} dq_f \int\limits_{-\infty}^{+\infty} dq_i \int\limits_{-\infty}^{+\infty} dq_i' \int\limits_{q_i}^{q_f} Dq \int\limits_{q_i'}^{q_f} Dq'$$

$$\times \exp \frac{i}{\hbar} \Big\{ S_b[q] + S_{int}[x, q] - S_b[q'] - S_{int}[x', q'] \Big\} \rho_b(q_i, q_i', 0) \tag{9}$$

It is first introduced by Feynman and Vernon [7].

For the problem described by (1), the influence functional can be computed exactly. The result is:

$$F[x, x'] = \exp \Big\{ -\frac{i}{\hbar} \int\limits_0^t ds_1 \int\limits_0^{s_1} ds_2 \Big[x(s_1) - x'(s_1) \Big] \eta(s_1 - s_2) \Big[x(s_2) + x'(s_2) \Big]$$

$$-\frac{1}{\hbar} \int\limits_0^t ds_1 \int\limits_0^{s_1} ds_2 \Big[x(s_1) - x'(s_1) \Big] \nu(s_1 - s_2) \Big[x(s_2) - x'(s_2) \Big] \Big\} \tag{10}$$

where

$$\nu(s) = \int\limits_0^{+\infty} d\omega \, I(\omega) \coth \frac{1}{2}\beta\hbar\omega \, \cos \omega s \tag{11}$$

is the noise kernel, also

$$\eta(s) = \frac{d}{ds} \gamma(s) \tag{12}$$

and

$$\gamma(s) = \int\limits_0^{+\infty} d\omega \, \frac{I(\omega)}{\omega} \, \cos \omega s \tag{13}$$

is dissipation kernel. Here $I(\omega)$ is the dissipation spectral density defined as

$$I(\omega) = \sum_n \delta(\omega - \omega_n) \frac{C_n^2}{2m_n\omega_n} \tag{14}$$

The kernels $\eta(s)$ and $\nu(s)$ are generally non-local. There exists an important relation between the noise and dissipation kernels, known as the fluctuation–dissipation relation. It can be written as

$$\nu(s) = \int\limits_{-\infty}^{+\infty} ds' \, K(s - s') \, \gamma(s') \tag{15}$$

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where the kernel $K(s)$ is

$$K(s) = \int\limits_{0}^{+\infty} \frac{d\omega}{\pi} \; \omega \coth \frac{1}{2} \beta \hbar \omega \; \cos \omega s \qquad (16)$$

which is independent of the dissipation spectral density $I(\omega)$

EXACT MASTER EQUATION

The detailed derivation of the exact master equation of a quantum harmonic oscillator with influence functional (10) has been published in Ref. [6]. Here we just give that equation below

$$\begin{aligned}
i\hbar \frac{\partial}{\partial t} \; \rho_r(x, x', t) = & \left\{ -\frac{\hbar^2}{2} \Big(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \Big) + \frac{1}{2} \Omega^2 \big(x^2 - x'^2 \big) \right\} \; \rho_r(x, x', t) \\
& + \frac{1}{2} \delta \Omega^2(t) \big(x^2 - x'^2 \big) \; \rho_r(x, x', t) \\
& - i\hbar \Gamma(t)(x - x') \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \Big) \; \rho_r(x, x', t) \\
& - i\Gamma(t) h(t)(x - x')^2 \; \rho_r(x, x', t) \\
& + \hbar \Gamma(t) f(t)(x - x') \Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \Big) \; \rho_r(x, x', t)
\end{aligned} \qquad (17)$$

where the time dependent coefficients are

$$\Gamma(t) = \frac{d_1(t)}{2\dot{u}_1(t)} \qquad (18)$$

$$\delta \Omega^2(t) = d_2(t) - 2\Gamma(t)\dot{u}_2(t) \qquad (19)$$

$$f(t) = 2\frac{a_{12}(t)}{\dot{u}_2(0)} + \frac{e_2(t) - c_1(t)}{2\Gamma(t)\dot{u}_2(0)} \qquad (20)$$

$$h(t) = \dot{u}_2(t) \; f(t) + 8 \; a_{11}(t) + \frac{e_1(t) - c_2(t)}{\Gamma(t)} \qquad (21)$$

The time dependent functions $c_i(t)$, $d_i(s)$ and $e_i(t)$ in (18) to (21) are

$$c_i(t) = \int\limits_0^t ds_1 \int\limits_0^t ds_2 \int\limits_0^t ds_3 \; \eta(t - s_1) \Big[G_{12}(s_1, s_2) + G_{21}(s_2, s_1) \Big] \nu(s_2 - s_3) u_i(s_3) \qquad (22)$$

$$d_i(t) = 2 \int\limits_0^t ds \; \eta(t-s)u_i(s) \tag{23}$$

$$e_i(t) = \int\limits_0^t ds \; \nu(s)u_i(s) \tag{24}$$

The elementary functions $u_i(s)$ satisfy the following boundary value problem

$$\begin{cases} \dfrac{d^2 u_i(s)}{ds^2} + 2 \int\limits_0^s ds' \eta(s-s')u_i(s') + \omega_0^2 u_i(s) = 0 \\[2mm] u_1(0) = 1, \quad u_1(t) = 0 \qquad u_2(0) = 0, \quad u_2(t) = 1 \end{cases} \tag{25}$$

and

$$v_1(s) = u_2(t-s) \qquad v_2(s) = u_1(t-s) \tag{26}$$

The Green function $G_{12}(s_1, s_2)$ in (22) is

$$G_{12}(s_1, s_2) = \frac{u_1(s_1)u_2(s_2)\theta(s_1-s_2) - u_2(s_1)u_1(s_2)\theta(s_2-s_1)}{\dot{u}_1(s_2)u_2(s_2) - u_1(s_2)\dot{u}_2(s_2)} \tag{27}$$

A similar expression for $G_{21}(s_1, s_2)$ can be written in terms of $v_i(s)$. In all the above equations, the index i runs from 1 to 2.

Let us take a closer look of this master equation. The first line corresponds to the usual unitary Liouvillian evolution, which is independent of the system-bath interaction. The second line corresponds to a time-dependent frequency shift (frequency renormalization). The third line contains a dissipative term with a time-dependent dissipative coefficient $\Gamma(t)$. The last two lines contain two diffusive terms with time-dependent coefficients. All of these terms depend on the system-bath coupling. Further, one can see that all these time-dependent coefficients vanish at $t = 0$, when the initial uncorrelated condition is assumed valid. The frequency shift and the dissipation coefficient depend only on the dissipation kernel while the diffusion coefficients depend on the noise kernel. From (11), (12) and (13), we find all these time dependent coefficients are determined by the dissipation spectral density (14).

A simpler closed formula for the time-dependent coefficients in the master equation can be found in the weak coupling limit (up to the first order in the coupling constant between the system and the bath),

$$\delta\Omega^2(t) = 2 \int_0^t ds \ \eta(s) \ \cos\Omega s \tag{28}$$

$$\Gamma(t) = -\frac{1}{\Omega} \int_0^t ds \ \eta(s) \ \sin\Omega s \tag{29}$$

$$\Gamma(t)f(t) = \frac{1}{\Omega} \int_0^t ds \ \nu(s) \ \sin\Omega s \tag{30}$$

$$\Gamma(t)h(t) = \int_0^t ds \ \nu(s) \ \cos\Omega s \tag{31}$$

EXAMPLES

An important class of dissipation spectral density is

$$I(\omega) = \frac{2}{\pi}\gamma_0\omega(\frac{\omega}{\tilde{\omega}})^{n-1}e^{-\frac{\omega^2}{\Lambda^2}} \tag{32}$$

where Λ is the physical cutoff frequency and $\tilde{\omega}$ is another frequency scale usually taken to be Λ. The environment is classified as ohmic if $n = 1$, as supra-ohmic if $n > 1$ or as sub-ohmic if $0 < n < 1$. It is important to introduce the physical cutoff frequency because, on physical grounds, one expects the spectral density to go to zero for very high frequencies. It is clear to see that after introducing the physical cutoff frequency, the dissipation kernel (13) is a non-local kernel even for the ohmic environment.

We have numerically computed the coefficients of the master equation given by (18) to (21) for three different environments, namely, ohmic ($n = 1$), subohmic ($n = 0.5$) and superohmic ($n = 3$). The damping constant is $\gamma_0 = 0.3$ and the cutoff frequency

is $\Lambda = 2000$. The bare frequency Ω is determined from the renormalized one, namely, $\Omega_p^2 = \Omega^2 + \delta\Omega^2 \to \Omega_r^2 = 1$ (see explanation in Ref. [6]). Both high temperature region $(T = 10^5)$ and low temperature region $(T = 10)$ have been studied. The numerical plots of these time dependent coefficients and detail analysis could be found in Ref. [6].

APPLICATION: QUANTUM DECOHERENCE

As a simple application, we discuss the damping of the interference between two Gaussian wave packets [8].

Let $\psi_{1,2}(x,t)$ be the wave functions of Gaussian wave packets located initially $(t = 0)$ at $x = \pm x_0$ respectively with the same initial spread σ

$$\psi_{1,2}(x,0) = Ne^{-\frac{(x \pm x_0)^2}{4\sigma^2}} \tag{33}$$

Let $\psi(x,t)$ be the wave function of a system consisting of the superposition of these two wave packets,

$$\psi(x,t) = \psi_1(x,t) + \psi_2(x,t) \tag{34}$$

The density matrix of the system can be written as the sum of three parts

$$\rho(x,x',t) = \psi(x,t)\psi^\dagger(x',t) = \rho_1(x,x',t) + \rho_2(x,x',t) + \rho_{int}(x,x',t) \tag{35}$$

The probability density function

$$P(x,t) = |\psi(x,t)|^2 = \rho(x,x,t) \tag{36}$$

can also be written as the sum of three parts

$$P_1(x,t) = P_1(x,t) + P_2(x,t) + P_{int}(x,t) \tag{37}$$

By using the influence functional (10) and the master equation (17), we get

$$P_{1,2}(x,t) = \tilde{N}(t)\exp\left[-\frac{(x \pm x_0(t))^2}{2\sigma^2(t)}\right] \tag{38}$$

and

$$P_{int}(x,t) = 2\sqrt{P_1(x,t)}\sqrt{P_2(x,t)}e^{-D(t)}\cos\phi(t) \tag{39}$$

where

$$x_0(t) = -\frac{\dot{u}_1(0)}{\dot{u}_2(0)}x_0 \tag{40}$$

and

$$\sigma(t) = \left[\frac{\dot{u}_1^2(0)}{\dot{u}_2^2(0)} + \frac{2a_{11}(t)}{\sigma^2\dot{u}_2^2(0)} + \frac{1}{4\sigma^4\dot{u}_2^2(0)}\right]^{1/2} \tag{41}$$

are respectively the position and spread of the wave packet at time t,

$$\phi(t) = \frac{2x_0[\dot{u}_2(0)x + \dot{u}_1(0)x_0]}{1 + 8a_{11}(t)\sigma^2 + 4\dot{u}_1^2(0)\sigma^4} \tag{42}$$

is the oscillatory angle (which is present even in the absence of the environment) and

$$D(t) = \frac{4a_{11}(t)x_0^2}{1 + 8a_{11}(t)\sigma^2 + 4\dot{u}_1^2(0)\sigma^4} \tag{43}$$

is the decay factor (which is present only because of the environment). It is this last term depicting the decay of interference between the two wave packets which is usually regarded as providing a measure of decoherence.

We have numerically computed the decay factor $e^{-D(t)}$ for all the cases described in the previous section. The results could be found in Ref. [6].

ACKNOWLEDGEMENT:

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A STUDY OF ELECTRON TRANSFER USING A THREE-LEVEL SYSTEM COUPLED TO AN OHMIC BATH

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Abstract

Electron transfer is studied using a multi-level system coupled to a bosonic bath. Two-body correlation functions are obtained using both exact enumeration of spin paths and Monte Carlo simulation. We find that the phase boundary for the coherent-incoherent transition lies at a smaller friction in the asymmetric two-level model than in the symmetric two-level model. A similar coherent-incoherent transition is observed for three-level system.

1 Introduction

Electron transfer in liquids is an important phenomenon in chemistry and physics. Following Marcus' picture [1], we use the spin boson model, which is a multi-level system coupled to a harmonic bath. The case of symmetric two-level model has been studied by many people, in the context of the Kondo problem and electron transfer in liquids[2]. It is not an exactly solvable model, except for the case of an adiabatic bath. In Section 3, we analyze the model with Feynman path integral which we evaluate with the exact enumeration of spin paths and Monte Carlo simulation. In Section 4, we present the results for an asymmetric two-level system and compare the results with the symmetric two-level system [3, 4, 5]. We also discuss the results for a three-level system. Finally, in Section 5, we summarize our results.

2 Model Hamiltonian

The Hamiltonian which defines our system is given by:

$$H = H_0 + H_B + H_{int}. \tag{1}$$

Here H_0 is the the Hamiltonian for the free three-level system,

$$H_0 = \begin{pmatrix} E_1 & J_{12} & J_{13} \\ J_{12} & E_2 & J_{23} \\ J_{13} & J_{23} & E_3 \end{pmatrix}, \tag{2}$$

where E_i gives the energy for an electron localized on site i, and J_{ij} gives the electronic coupling between sites i and j. H_B is the Hamiltonian of the harmonic bath. H_{int} is the coupling between the three-level system and the bath, and is given by

$$H_{int} = \mathcal{E}_{12}\sigma_{12} + \mathcal{E}_{23}\sigma_{23} + \mathcal{E}_{13}\sigma_{13}. \tag{3}$$

Here σ_k ($k = (12), (23), (13)$) is defined by

$$\sigma_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \sigma_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \sigma_{13} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

and the field $\mathcal{E}_k$ is a linear combination of the bath modes, $\mathcal{E}_k = \sum_j c_j^k x_j$. Harmonic baths linearly coupled to spin systems can be defined by the spectral density:

$$J^{kl}(\omega) = \frac{\pi}{2} \sum_j \frac{c_j^k c_j^l}{m_j \omega_j} \delta(\omega - \omega_j). \tag{5}$$

Here, k and l are the pairs, (12), (23) or (13). For electron transfer in a liquid, an Ohmic spectral density gives a good model for the environment surrounding the electron. [2, 4]

$$J^{kl}(\omega) = \eta^{kl}\omega \exp(-\omega/\omega_c^{kl}) \tag{6}$$

3   Numerical Methods

The quantity we are interested in is the following two-body time correlation function.

$$< n_1(0)n_1(t) > = \frac{1}{Z} Tr \exp(-\beta H)n_1 \exp(iHt/\hbar)n_1 \exp(-iHt/\hbar), \tag{7}$$

where n_1 is the population operator for being on site 1. Since H_{int} does not commute with H_0 and H_B, we use the following Suzuki-Trotter formula to do the path integral evaluations.

$$\exp(-\beta H) = \lim_{p \to \infty} \{\exp(-\beta H_0/p) \exp(-\beta H_B/p) \exp(-\beta H_{int}/p)\}^p. \tag{8}$$

Expanding this equation in a path integral representation and performing the Gaussian bath integrals analytically leads to:

$$< n_1(0)n_1(t) > = \frac{1}{Z} \sum_{\{\sigma_i\}} \exp(\varphi(\{\sigma_i\}))n_1(\sigma_{p+1})n_1(\sigma_{p+q+1}), \tag{9}$$

where p and q, sometimes called Trotter numbers, are the numbers of partitions the path has been divided into the thermal and real time part of the path. The sum goes over 3^N states of S=1 Ising system, where N is the total Trotter number, $N = p + 2q$. The action φ in Eq. 9 consists of two parts,

$$\varphi = \varphi_0 + \varphi_{int}. \tag{10}$$

The first part, φ_0, comes from the free three-level system Hamiltonian, and second part, φ_{int}, results from the interaction between the system and the bath which we have already integrated out. The action φ_{int} is given by:

$$\varphi_{int} = \frac{1}{2} \sum_{ij} \sum_{kl} f_k(\sigma^i) f_l(\sigma^j) \chi_{ij}^{kl}, \tag{11}$$

where $f_k(\sigma^i)$ is a quadratic function of σ^i, and the χ_{ij}^{kl} is expressed by the spectral density $J^{kl}(\omega)$ as follows.

$$\chi_{ij}^{kl} = \frac{\hbar}{\pi} \int_0^\infty J^{kl}(\omega) \cosh \frac{\omega}{2} (\Delta_{ij} - \Delta_{ji}). \tag{12}$$

Here,

$$\Delta_{ij} = \sum_{k=i}^{j-1} \Delta_k, \tag{13}$$

where $\Delta_k = \beta/p$, $-it/q$ or it/q, depending on the location of σ^k on the path.

Thus, the original three-level system coupled to a bath is now transformed into a S=1 Ising model with infinite-range interactions. This model is exactly solvable for the case of an adiabatic bath[6]. This case corresponds to Ising magnets in a very slow Gaussian field. In the general case, the analytic solution of this model is unknown.

To do the numerical calculation on this model for small Trotter number, we used the exact enumeration of the path integral by adding all the 3^N states of the Ising spin system. For example, exact enumeration of N=17 spins (1.29×10^8 states) takes 3 minutes on a Cray X-MP.

We also used Monte Carlo simulation. Some filtering method[5] was necessary to overcome the so-called sign-problems [7, 8, 9, 10, 11, 12], which is often seen in quantum calculations of fermionic systems or spin models.

4   Results

To check the validity of our calculations, we have studied a free three-level system (i.e., coupling between the spin system and the bath is turned off), and a three-level system coupled to an adiabatic bath. For these systems, we computed the following correlation functions.

$$C_1(t) = Re < n_1(0)n_1(t) >, C_2(t) = Re < n_1(0)n_2(t) >, C_3(t) = Re < n_1(0)n_3(t) > \tag{14}$$

Our results give good agreement with analytic results in these limits.

For the special case of an asymmetric two-level system, we have calculated the time-correlation functions, and compared the results with the symmetric case[4]. Figure 1 shows the results for an asymmetric case with the parameters $E_1 = 0$, $E_2 = 2K$, $E_3 = \infty$, and $J_{12} = -K$, $J_{13} = J_{23} = 0$. We find from Figure 1 that the coherent-incoherent transition occurs around

$\eta/\hbar = 0.4$, or $\alpha = 2\eta/\hbar\pi = 0.25$. For a symmetric two-level system[4], the phase boundary for this temperature was at $\alpha = 0.4$. Thus, the whole phase boundary is expected to lie at smaller α for the asymmetric case than for the symmetric case. In the asymmetric case, the symmetry is already broken, thus the coherence is easier to break than in the symmetric case.
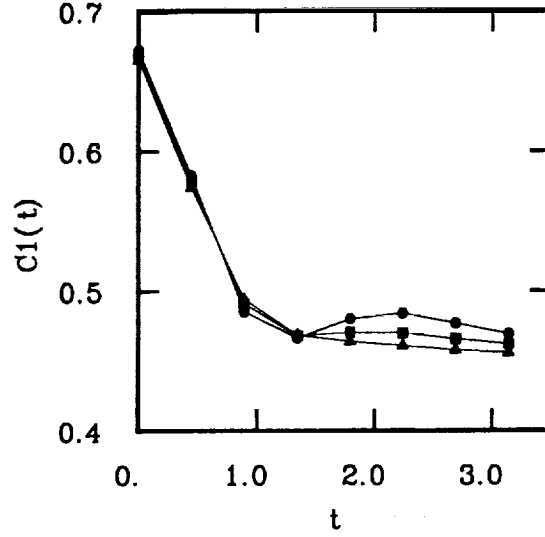

FIG. 1 The correlation function $C_1(t)$ for an asymmetric two-level model of $E_2 = 2K$, $\beta K = 2.5$, with $\eta/\hbar = 0.3$ (circles), 0.4(squares), 0.5 (triangles) obtained by exact enumeration of spin paths with p=2 and q=7.
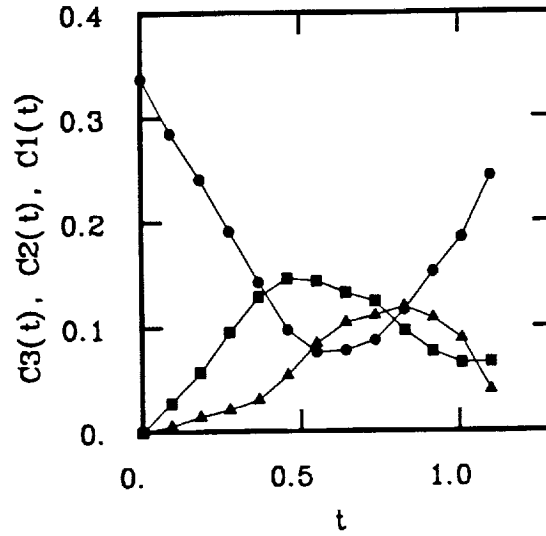

FIG. 2 The correlation functions $C_1(t)$ (circles), $C_2(t)$ (squares) and $C_3(t)$ (triangles) for a three-level system in the coherent region, $J_{12} = J_{23} = -2K$, $J_{13} = -K$, $E_1 = 0.5K$, $E_2 = K$, $E_3 = 0$, $\beta K = 0.25$, $t_c^{13} = 2$, $t_c^{12} = t_c^{23} = 0$, $(t_c = K/\omega_c)$, $\eta^{13}/\hbar = 1$, $\eta^{12}/\hbar = \eta^{23}/\hbar = 0$.

Figures 2 and 3 show results for a three-level system. The calculations were done by Monte Carlo simulation, with the Trotter numbers p=2 and q=10. As has been found in quantum Monte

Carlo simulation of spin systems[8], for systems having sign problems, it is usually more efficient to define 1 Monte Carlo step (MCS) to be a small subset of all the possible flips than to define it as all the possible flips. In our simulation, we define 1 MCS as 1 single spin-flip, 1 double spin-flip, 1 global spin-flip (i.e., flips all the spins), and 1 spin-flip of random length. We determine whether the spins should be flipped by the standard Metropolis algorithm, using the modulus of the complex weight $\exp(\varphi)$ for the transition probability. In this way, we have carried out simulation of 10^6 MCS, taking about 14 minutes on the Cray X-MP. To estimate the degree of sign cancellation, we measured the quantity r, the remaining ratio (related to the negative ratio defined in [8]).

$$r = \frac{Z_+ - Z_-}{Z_+ + Z_-} \tag{15}$$

Here Z_+ denotes the sampled sum of the positive real parts of the weights, and Z_- denotes the same for absolute values of negative real parts of the weights. If r is small, the cancellation of the signs is large, leading to inaccuracy in the data. If r is large, the cancellation is small, thus giving more accurate results. In this definition, we are ignoring the effect of the cancellation due to the imaginary parts of the weights.
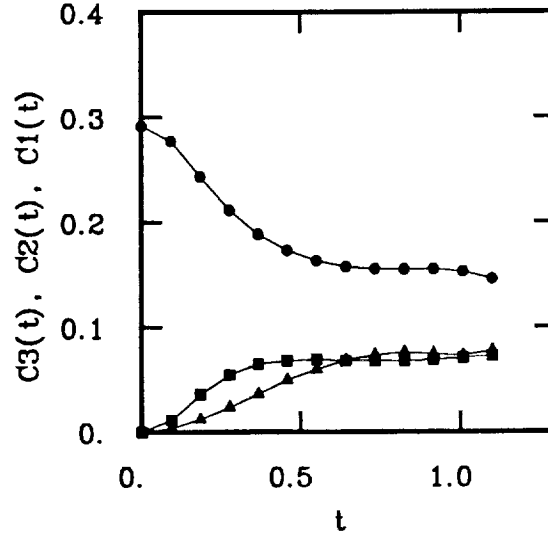

FIG. 3 The correlation functions $C_1(t)$, $C_2(t)$ and $C_3(t)$ for a three-level system in the incoherent region. J_j, E_j and β are same as in Fig. 2. $t_c^{13} = 2$, $t_c^{12} = t_c^{23} = 1$, $\eta^{13}/\hbar = 1$, $\eta^{12}/\hbar = \eta^{23}/\hbar = 2$.

For the free three-level Hamiltonian H_0, we assume the parameters, $J_{12} = J_{23} = -2K$, $J_{13} = -K$, $E_1 = 0.5K$, $E_2 = K$, $E_3 = 0$. This could correspond to a system of redox-sites 1, 2 and 3, where distance between states 1 and 2, or 2 and 3 is shorter than the distance between 1 and 3. The correlation function $< n_1(0)n_2(t) >$ approximately tells the rate of the electron transfer starting at state 1 and reaching 2 after time t. The energy of the state 2, E_2, is assumed to be highest, followed by the energy of the state 1, E_1. Starting from the state 1, the electron moves to state 2, since the exchange J_{12} is stronger than J_{13}, then gradually goes to state 3. In Fig. 3, the bath has the role of dephasing the coherence, preventing the electron population from going back to the

original state. This is a very brief picture of electron transfer over 3 states, with the intermediate state strongly coupled to the initial and terminal states.

As for the effect of the sign cancellations, the remaining ratio r defined in Eq. 15 is 3% for Fig.2 and 13% for Fig.3. The magnitude of error is about 0.1 in Fig.2 and 0.02 in Fig.3 The incoherent case has less effect of the exchange K, thus leading to less sign cancellations.

5 Summary

We have briefly described the numerical calculations of the time-correlation functions of an asymmetric two-level system and a three-level system. For an asymmetric two-level system, we find that the coherent-incoherent transition occurs at smaller friction η than for the symmetric case. For a three-level system, we calculated the population transfer of the electron when there is an intermediate high-energy state. We observed a coherent-incoherent transition similar to the two-level system. Further application of this model will be discussed elsewhere.

6 Acknowledgments

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VI.   GROUP REPRESENTATIONS

SYMMETRY ALGEBRA OF A GENERALIZED ANISOTROPIC HARMONIC OSCILLATOR

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Abstract

It is shown that the symmetry Lie algebra of a quantum system with accidental degeneracy can be obtained by means of the Noether's theorem. The procedure is illustrated by considering a generalized anisotropic two dimensional harmonic oscillator, which can have an infinite set of states with the same energy characterized by an $u(1,1)$ Lie algebra.

1 Introduction

We are going to study the accidental degeneracy [1,2] of the Hamiltonian

$$H = \frac{1}{2}\sum_i (p_i^2 + x_i^2) + \lambda M \tag{1.1}$$

which is a two dimensional harmonic oscillator plus the projection of the angular momentum in the z direction, M. We use atomic units in which $\hbar = m = e = 1$ and λ is a constant parameter. This quantum system, for $\lambda = 1$, describes the motion of an electron in a constant magnetic field [3, 4] and its corresponding symmetry Lie algebra has been discussed by Moshinsky *et al* [4]. A procedure that use the Noether's theorem [5] is established to get the symmetry algebra of the hamiltonian systems (1.1), for rational values of the parameter lambda. We show that (1.1) represents a generalization of the degeneracies present in the anisotropic two dimensional harmonic oscillator [6,7].

For the purpose of the paper it is convenient to introduce appropriate combinations of the creation η_i and annihilation ξ_i operators, with $i = 1, 2$, *i.e.*

$$\eta_\pm = \frac{1}{\sqrt{2}}(\eta_1 \pm i\eta_2), \quad \xi_\pm = \frac{1}{\sqrt{2}}(\xi_1 \mp i\xi_2), \tag{1.2}$$

with the properties

$$[\xi_a, \xi_b] = [\eta_a, \eta_b] = 0; \quad [\xi_a, \eta_b] = \delta_{ab}, \quad (\eta_a)^\dagger = \xi_a, \quad a = +, - . \tag{1.3}$$

It is straightforward to find the expression of the hamiltonian (1.1) in terms of these operators

$$H = (1 + \lambda)N_+ + (1 - \lambda)N_-, \tag{1.4}$$

where a constant term was neglected and N_a, denotes the number of quanta in direction a. The eigenstates of (1.4) are well known [4] and its eigenvalues are given by $E_{\nu m} = \nu + \lambda m$; with $|m| = \nu, \nu - 2 \ldots 1$ or 0 and ν denoting the total number of quanta. From this expression, it is immediate that there is degeneracy for rational values of λ, which can be defined as follows

$$\lambda = -\frac{\Delta \nu}{\Delta m} = -\frac{\nu_f - \nu_i}{m_f - m_i} \ . \tag{1.5}$$

Thus the accidental degeneracy associated to the hamiltonian (1.4) can be classified according to the strength of the parameter λ in three groups

$$\{\lambda = \pm 1\} \ , \quad \{\lambda > 1, \ \lambda < -1\} \ , \quad \{-1 < \lambda < 1\} \ . \tag{1.6a, b, c}$$

For the cases (1.6a, b), there are an infinite number of levels with the same energy, while for the case (1.6c), there is a finite number of levels with the same energy .

In the second section, we find the classical symmetry Lie algebra of the generalized two–dimensional anisotropic harmonic oscillator. In the section three, we discuss for all the cases of λ the corresponding symmetry Lie algebras which are responsible of the accidental degeneracy of the hamiltonian (1.4). Finally some conclusions and remarks are made.

2 Classical Symmetry Lie Algebra for the Hamiltonian

In this section we apply Noether's theorem in its active version [8] to the system described by (1.4), its corresponding lagrangian is given by

$$L = \frac{1}{2\lambda_a}(\dot{x}_a^2 - \lambda_a^2 x_a^2) \ , \tag{2.1}$$

where we associate indices 1 and 2 to the labels + and −, and we define $\lambda_1 = 1 + \lambda$ and $\lambda_2 = 1 - \lambda$. From now onwards we adopt the convention: repeated indices are summed except when one of them appears with λ_a. Let us propose a symmetry transformation in terms of an arbitrary function of coordinates and velocities, $\delta x_a = F_a(x_b, \dot{x}_b)$. The corresponding variation of the lagrangian (2.1) is given by

$$\delta L = (\ddot{x}_b \frac{\partial F_a}{\partial \dot{x}_b} + \dot{x}_b \frac{\partial F_a}{\partial x_b}) \frac{1}{\lambda_a} \dot{x}_a - F_a \lambda_a x_a \ . \tag{2.2}$$

Because δx_a is a symmetry transformation, (2.2) must be a total time derivative of a function Ω. This implies that the following system of equations must be satisfied

$$\frac{\partial \Omega}{\partial \dot{x}_a} = \frac{1}{\lambda_b} \dot{x}_b \frac{\partial F_b}{\partial \dot{x}_a} \ , \quad \dot{x}_a \frac{\partial \Omega}{\partial x_a} = \frac{1}{\lambda_b} \dot{x}_b \dot{x}_a \frac{\partial F_b}{\partial x_a} - F_a \lambda_a x_a \ . \tag{2.3a, b}$$

In order to establish the integrability conditions for this system, we derive (2.3a) with respect to x_c and (2.3b) with respect to $\dot{x}_c$, and compare the results. Thus we get

$$\frac{\partial \Omega}{\partial x_b} = \frac{1}{\lambda_b} \frac{\partial F_b}{\partial x_c} \dot{x}_c + \frac{1}{\lambda_c} \frac{\partial F_c}{\partial x_b} \dot{x}_c - \lambda_c x_c \frac{\partial F_c}{\partial \dot{x}_b} \ . \tag{2.4}$$

Now we set up the equality between the five crossed partial derivatives of Ω, and give rise to the following system of second order partial differential equations

$$\frac{\partial F_a}{\partial p_b} = \frac{\partial F_b}{\partial p_a}, \tag{2.5a}$$

$$\frac{1}{2}\mathcal{O}\left(\frac{\partial F_a}{\partial p_b} + \frac{\partial F_b}{\partial p_a}\right) - \left(\lambda_a \frac{\partial F_b}{\partial x_a} + \lambda_b \frac{\partial F_a}{\partial x_b}\right) = 0, \tag{2.5b}$$

$$\mathcal{O}\left(\lambda_a \frac{\partial F_b}{\partial x_a} - \lambda_b \frac{\partial F_a}{\partial x_b}\right) - \left(\lambda_a^2 \frac{\partial F_a}{\partial p_b} - \lambda_b^2 \frac{\partial F_b}{\partial p_a}\right) = 0, \tag{2.5c}$$

where the change from velocities to momenta $\dot{x}_a = \lambda_a p_a$ was made, and have defined the differential operator

$$\mathcal{O} = \lambda_c \left(x_c \frac{\partial}{\partial p_c} - p_c \frac{\partial}{\partial x_c}\right).$$

From Eq. (2.5a) it is immediate that $F_k = \frac{\partial G}{\partial p_k}$, which means that G is the generator of the symmetry transformation. Through the change of variables $z_k = \frac{1}{\sqrt{2}}(x_k + ip_k)$, and its complex conjugate, z_k^*, it is straightforward to show that the operator $\mathcal{O} = i(\mathcal{N} - \mathcal{N}^*)$, with $\mathcal{N} = \lambda_k z_k \frac{\partial}{\partial z_k}$. Using these results, we arrive to a set of partial differential equations which has a solution of the form

$$G(z_k, z_k^*) = z_1^{n_1} z_2^{n_2} z_1^{*n_3} z_2^{*n_4}, \tag{2.6}$$

if the n_i are integer numbers and satisfy that $n_1 = n_3$ and $n_2 = n_4$ or the condition

$$\frac{(n_1 - n_3)}{(n_2 - n_4)} = -\frac{\lambda_2}{\lambda_1} = \frac{\Delta m + \Delta \nu}{-\Delta m + \Delta \nu} \equiv \frac{*k_1}{\epsilon k_2}, \tag{2.7}$$

where the Eq. (1.5) was used. The integers k_1 and k_2 are relatively prime integers, and the parameter ϵ takes the value 1 or -1. It takes the value 1 when $\Delta m + \Delta \nu$ and $\Delta \nu - \Delta m$ have the same sign, and -1 otherwise. Thus we get, besides the trivial solution, six fundamental solutions, although only three of them are independent. Then the corresponding conserved quantities are given by

$$N_1 = z_1 z_1^*, \qquad K_3 = z_1^{*k_1} z_2^{-\epsilon k_2}, \qquad K_5 = z_1^{*k_1} z_2^{*\epsilon k_2},$$

$$N_2 = z_2 z_2^*, \qquad K_4 = z_1^{k_1} z_2^{*-\epsilon k_2}, \qquad K_6 = z_1^{k_1} z_2^{\epsilon k_2}. \tag{2.8}$$

From this set we must find a symmetry algebra for the classical system. It is important to realize that to build the algebra once we select a conserved quantity its complex conjugate must be included. To do this, we find separately for the cases indicated in Eqs. (1.6) the corresponding expressions for the constants of the motion and from them select the independent ones which allows its extension to the quantum case.

For $\lambda = 1$ and $\lambda = -1$ the sets are given by $\{1, N_1, z_2, z_2^*\}$ and $\{1, N_2, z_1, z_1^*\}$, respectively. In order to identify the symmetry algebra, we calculate its Poisson brackets, and clearly they correspond to the direct sum of one-dimensional Weyl and unitary algebras, $w(1) \oplus u(1)$.

For $\lambda > 1$ and $\lambda < -1$, the constants of the motion are identical and we choose the set

$$h_1 = \frac{1}{k_1}N_1 - \frac{1}{k_2}N_2 \ , \quad m_1 = \frac{1}{k_1 - k_2}(N_1 - N_2) \ , \qquad (2.9a, b)$$

$$K_5 = F_5(N_1, N_2)z_1^{*k_1} z_2^{*k_2} \ , \quad K_6 = F_6(N_1, N_2)z_1^{k_1} z_2^{k_2} \ . \qquad (2.9c, d)$$

The F_5 and F_6 functions are defined in such a way to obtain that the Poisson bracket

$$\{K_5, K_6\} = iCm_1, \qquad (2.10)$$

where C is a constant that can be ± 1. This condition implies that

$$F_5 F_6 = \frac{C}{2(k_1 - k_2)^2}(N_1 - N_2)^2 N_1^{-k_1} N_2^{-k_2}. \qquad (2.11)$$

Then it is easy to prove that the set of constants of the motion $\{h_1, m_1, K_5, K_6\}$ constitute the classical symmetry Lie algebra which, depending on the value of C, can be identified with an $u(2)$ or $u(1, 1)$ algebra.

For $-1 < \lambda < 1$ we select the following independent constants of the motion:

$$h_2 = \frac{1}{k_1}N_1 + \frac{1}{k_2}N_2 \ , \quad m_2 = \frac{1}{k_1 + k_2}(N_1 - N_2) \ , \qquad (2.12a, b)$$

$$K_3 = F_3(N_1, N_2) z_1^{*k_1} z_2^{k_2} \ , \quad K_4 = F_4(N_1, N_2) z_1^{k_1} z_2^{*k_2}, \qquad (2.12c, d)$$

where as in the previous case the F_3 and F_4 functions are defined to give the Poisson bracket

$$\{K_3, K_4\} = iCm_2, \qquad (2.13)$$

with C equal to ± 1. This condition implies that

$$F_3 F_4 = \frac{C}{2(k_1 + k_2)^2}(N_1 - N_2)^2 N_1^{-k_1} N_2^{-k_2}. \qquad (2.14)$$

Therefore the set of constants of the motion $\{h_2, m_2, K_3, K_4\}$ generates the classical symmetry Lie algebras $u(2)$ or $u(1, 1)$, depending if the value of C is $+1$ or -1, respectively.

3 Quantum Symmetry Lie Algebra for the Hamiltonian

To quantize the system we replace the classical variables x and p by the corresponding quantum operators in definitions (2.8), and Poisson brackets by commutators, i.e., $\{\} \rightarrow \frac{1}{i}[]$. Then the classical variables z_k and z_k^* are replaced by the operators

$$\hat{z}_k = \frac{1}{\sqrt{2}}(\hat{x}_k + i\hat{p}_k) \ , \quad \hat{z}_k^\dagger = \frac{1}{\sqrt{2}}(\hat{x}_k - i\hat{p}_k) \ , \qquad (3.1)$$

which satisfy the standard commutation relations of creation and annihilation operators.

We choose as a base for the physical space the simultaneous eigenstates of $\{N_1, N_2\}$, which we label as $|n_1, n_2 >$, because they form a complete set of commuting operators. This let us see that not all operators in (2.8) make sense all the time. According to the previous section we consider three cases:

(i) For $\lambda = \pm 1$, we have two sets of operators, $\{I, \hat{N}_1, \hat{z}_2, \hat{z}_2^\dagger\}$ and $\{I, \hat{N}_2, \hat{z}_1, \hat{z}_1^\dagger\}$, whose commutation relations correspond to the direct sum $w(1) \oplus u(1)$.

(ii) When $\lambda > 1$ and $\lambda < -1$, the set of constants of the motion (2.9), must be replaced by its quantum version, however this is ambiguous for the constants (2.9c,d) and so we eliminate from them the F_5 and F_6 functions. It is easy to evaluate their commutators and get an algebra but to identify a Lie algebra a redefinition of the constants of the motion must be done. This is achieved by constructing the new operators [7]

$$\tilde{z}_i^\dagger = \left(\lfloor \frac{\hat{N}_i}{k_i} \rfloor \frac{(\hat{N}_i - k_i)!}{(N_i)!} \right)^{\frac{1}{2}} (\hat{z}_i^\dagger)^{k_i}, \tag{3.2a}$$

$$\tilde{z}_i = (\hat{z}_i)^{k_i} \left(\lfloor \frac{\hat{N}_i}{k_i} \rfloor \frac{(\hat{N}_i - k_i)!}{(N_i)!} \right)^{\frac{1}{2}}. \tag{3.2b}$$

where $\lfloor x \rfloor$ denotes the largest integer $\leq x$. From (3.2) it is easy to check that

$$\tilde{N}_i = \tilde{z}_i^\dagger \tilde{z}_i = \lfloor \frac{\hat{N}_i}{k_i} \rfloor. \tag{3.3}$$

Then the Lie algebra is identified by considering the following operators

$$\hat{h}_1 = \tilde{N}_1 - \tilde{N}_2 , \quad \tilde{K}_5 = \tilde{z}_1^\dagger \tilde{z}_2^\dagger , \quad \tilde{K}_6 = \tilde{z}_1 \tilde{z}_2 , \quad \hat{C}_1 = \frac{1}{2}(\tilde{N}_1 + \tilde{N}_2 + 1) . \tag{3.4}$$

that satisfy the commutation relations

$$[\hat{C}_1, \tilde{K}_5] = \tilde{K}_5 , \quad [\hat{C}_1, \tilde{K}_6] = -\tilde{K}_6 , \quad [\tilde{K}_5, \tilde{K}_6] = -2\hat{C}_1 . \tag{3.5}$$

These were evaluated by using that $[\tilde{z}_i, \tilde{z}_j^\dagger] = \delta_{ij}$, which is valid for any state $|n_1, n_2 >$ of the Hilbert space of the system, and they are the generators of a $u(1,1)$ Lie algebra, with $\hat{h}_1$ generating the invariant subalgebra.

(iii) Finally for $-1 < \lambda < 1$, the symmetry algebra can be found by considering the operators

$$\tilde{K}_3 = \tilde{z}_1^\dagger \tilde{z}_2, \quad \tilde{K}_4 = \tilde{z}_2^\dagger \tilde{z}_1 , \quad \hat{h}_2 = \tilde{N}_1 + \tilde{N}_2 , \quad \hat{C}_2 = \frac{1}{2}(\tilde{N}_1 - \tilde{N}_2) . \tag{3.6}$$

Evaluating the commutation relations between these operators we have

$$[\hat{C}_2, \tilde{K}_3] = \tilde{K}_3, \quad [\hat{C}_2, \tilde{K}_4] = -\tilde{K}_4, \quad [\tilde{K}_3, \tilde{K}_4] = 2\hat{C}_2, \tag{3.7}$$

and the operator $\hat{h}_2$ is the ideal of the algebra. Thus we get for this case a $u(2)$ symmetry Lie algebra.

4 Conclusions

We have established a procedure that uses Noether's theorem to find the symmetry Lie algebra of a quantum system with accidental degeneracy. First, we solve the differential equations that determine the constants of the motion. Second, once we have chosen the minimal set of constants of the motion that close under Poisson brackets, to identify the classical Lie algebra we need in general to form combinations of the selected Noether charges. And third, to find the corresponding quantum counterparts. Afterwards, the identification of the quantum symmetry Lie algebra can be done immediately by making the standard replacement of Poisson brackets by commutators. However, this is true if there are not ambiguities in establishing the associated quantum operators for the constants of motion which form a Lie algebra under the Poisson bracket operation. If this is not the case, it is more convenient to choose the minimal set of constants of the motion that allows a quantum extension, and make the necessary redefinitions to build the associated Lie algebra of the system. Following this procedure we get for the generalized anisotropic two dimensional harmonic oscillator (1.4) the symmetry algebra which determine the degeneracy of the system. The symmetry Lie algebras are, depending on the value for λ, $w(1) \oplus u(1), u(2)$, and $u(1,1)$. However with the generators of the first one a Holstein–Primakoff realization [4] of a $u(1,1)$ Lie algebra can be obtained.

5 Acknowledgment

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FERMION REALIZATION OF EXCEPTIONAL LIE ALGEBRAS FROM MAXIMAL UNITARY SUBALGEBRAS

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Abstract

From the decomposition of the exceptional Lie algebras (ELAs) under a maximal unitary subalgebra a realization of the ELAs is obtained in terms of fermionic oscillators.

1   Introduction

Realizations of classical Lie algebras (LAs) in terms of bosonic and/or fermionic oscillators are known long since and are very useful in several physical contexts. Via the embedding of $SO(8) \oplus SO(8) \subset E_8$ a realization of ELAs in terms of fermionic oscillators has been obtained by the author [1]. However it is more convenient to dispose of several different realizations of ELAs which allow to describe in a more appropriate way different subalgebras embedding chains. Moreover, e.g., the embeddings $G_2 \subset SO(7)$ and $F_4 \subset E_6$ are not "deformable", while the embeddings $SU(3) \subset G_2$ and $SO(9) \subset F_4$ are "deformable". The proposal of this contribution is to present a realization of ELAs in terms of multilinears in fermionic oscillators via the embedding of a maximal unitary subalgebra. It should be quoted that constructions of ELAs as bilinears in fermionic fields in the basis $SU(9)$ and $SU(3)^4$ has been obtained by Koca [2]. While Koca's approach makes a more evident connection with physical applications in a GUT framework, the multilinear approach keeps a closer connection with the algebraic structure of LAs (roots, weights, etc.). Moreover this formalism allows to obtain multilinear realizations for all the fundamental representations and for generators and vector spaces of all maximal embeddings of ELAs [3].

2   Composition law for fermionic multilinears

Let us introduce a set of N fermionic oscillators a_i^+, a_i satisfying: $(i,j = 1,2,...,N)$

$$\{a_i^+, a_j^+\} = \{a_i, a_j\} = 0 \qquad \{a_i, a_j^+\} = \delta_{ij} \tag{1}$$

A fermionic multilinear (f.m.) X is defined by the following formula:
$(f_i = a_i^+, f_{-i} = a_i, i > 0)$

$$X = \prod_i f_i \qquad i \in I \subset Z^* \tag{2}$$

The number of f_i will be called the order of X.

We define the **contraction of two bilinears** X and Y of, resp., order N and N' as a operation giving a f.m. $(\widehat{XY})$ obtained from the m. XY by deleting the couples (if any) (f_i, f_{-i}) with f_i "in" X and f_{-i} "in" Y, multiplied by a factor $(-1)^n$, n being the number of transpositions necessary to obtain all the f_i near to f_{-i} in XY, and by a rational coefficent $C(N, N', Z)$, Z being the number of **contractions**.

We define a **composition law** $(X \circ Y)$ of two f.m. by the following equation $(i_k \in I, j_l \in J)$

$$X \circ Y = \frac{1}{2} \times (\widehat{XY} - \widehat{YX}) + \delta_{I,-J} \times \frac{1}{N} \sum_k \sum_l (f_{i_k} f_{j_l} - f_{j_l} f_{i_k}) \times (-1)^{k-1} \delta_{i_k j_l} \tag{3}$$

We remark:

• $X \circ Y = - (Y \circ X)$

• $X \circ Y = [X, Y]$ \qquad (N, N'$\in$ 1,2)

We put (N,N' = 1, 2, 3, 6; N_T = order of $\widehat{XY}$):

• C(N, N', 0) = 1

• C(N, N', 1) = $\delta_{N_T,N}$ or $\delta_{N_T,N'}$

• C(N, N, N-1) = $\frac{2}{N}$

• C(N, 2N, N) = $\frac{1}{2}$ \quad ($N > 1$)

• C(N, N, $\frac{N}{2}$) = -1 \quad (N even)

3 Realization of E_8

We consider the embedding SU(9) $\subset E_8$. The adjoint representation of E_8 decomposes as :

$$248 \Longrightarrow 80 + 84 + \overline{84} \tag{4}$$

Introducing a set of 9 fermionic creation and annihilation operators and we can write (i,j = 1,2,..,9):

$$80 \equiv \{ \{a_i^+ a_j\} \quad (i \neq j), \quad a_k^+ a_k - a_{k+1}^+ a_{k+1} = h_k - h_{k+1} \quad (k \neq 9)\} \tag{5}$$

$$84 \equiv \{a_i^+ a_j^+ a_k^+ + \frac{1}{6!} \, \varepsilon_{ijklmnpqr} a_l a_m a_n a_p a_q a_r\} \tag{6}$$

$$\overline{84} \equiv \{a_i a_j a_k + \frac{1}{6!} \, \varepsilon_{ijklmnpqr} a_l^+ a_m^+ a_n^+ a_p^+ a_q^+ a_r^+\} \tag{7}$$

In the following we call:

- a_i^+ "hermitian coniugate" (h.c.) of a_i;

- $\varepsilon_{ijklmnpqr} a_l^+ a_m^+ a_n^+ a_p^+ a_q^+ a_r^+$ "dual coniugate" (d.c.) of $a_i a_j a_k$.

Proposition 1 *The above set of bilinears and trilinears in the fermionic oscillators closes and satisfies the Jacobi identity under the composition law ($\circ$) defined in Sec. 2.*

The generators corresponding to the simple roots are:

$$\alpha_1 \to a_1^+ a_2, \quad \alpha_2 \to a_1 a_2 a_3 + d.c., \quad \alpha_k \to a_{k-1}^+ a_k \quad (3 \le k \le 8) \tag{8}$$

The generator corresponding to the highest root is $a_8^+ a_9$.

4 Realization of E_7

In the embedding $SU(8) \subset E_7$ the adjoint representation decomposes as:

$$133 \Longrightarrow 63 + 70 \tag{9}$$

The SU(8) $\subset E_7$ is not contained in the SU(9) $\subset E_8$, Exploiting the property that the two unitary algebras have a common maximal subalgebra SU(6), the following realization of E_7 is obtained (i,j,k = 1,2,..6 ; r = 1,2,...,5):

$$
\begin{aligned}
63 \equiv \{ & a_i^+ a_j, \quad a_7^+ a_8^+ a_9^+ + d.c., \quad a_i^+ a_7, \quad a_i^+ a_8^+ a_9^+ + d.c., \quad h.c. \\
& h_r - h_{r+1}, \quad \frac{2}{3}(h_7 + h_8 + h_9) - \frac{1}{3}\sum_i h_i, \quad 2h_7 - h_8 - h_9 \}
\end{aligned}
\tag{10}
$$

$$70 \equiv \{ a_i a_j a_7 + d.c., \quad a_i a_j a_k + d.c., \quad h.c. \} \tag{11}$$

5 Realization of E_6

In the embedding $SU(6) \oplus SU(2) \subset E_6$ the adjoint representation decomposes as:

$$78 \Longrightarrow (35,1) + (1,3) + (20,2) \tag{12}$$

We have (i,j,k = 1,2,..6; r = 1,2,..5):

$$(35,1) \equiv \{ a_i^+ a_j, \quad h_r - h_{r+1} \} \tag{13}$$

$$(1,3) \equiv \{ a_7^+ a_8^+ a_9^+ + d.c., \quad h.c., \quad \frac{2}{3}(h_7 + h_8 + h_9) - \frac{1}{3}\sum h_i \} \tag{14}$$

$$(20,2) \equiv \{ a_i a_j a_k + d.c., \quad h.c. \} \tag{15}$$

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6 Realization of F_4

In the embedding $SU(4) \oplus SU(2)' \subset F_4$ the adjoint representation decomposes as:

$$52 \Longrightarrow (15,1) + (1,3) + (4,2) + (\bar{4},2) + (6,3) \tag{16}$$

The most convenient way to identify the elements of F_4 is the following:

i) draw the Dynkin diagram of E_6;

ii) from i) draw, by folding, the Dynkin diagram of F_4, identify the corresponding simple roots and the highest root;

iii) draw the extended Dynkin diagram of F_4 and then, by deleting a dot, identify $SU(4) \oplus SU(2)'$.

We get for the 52 (i,j,k = 1,2,..6):

$$a_7^+ a_8^+ a_9^+ + d.c., \quad a_i^+ a_j \quad (i+j=7), \quad h.c.$$

$$a_j^+ a_j + (-1)^{k+l-1} a_k^+ a_l \quad (i \neq j \neq k \neq l; i+j \leq k+l; i+j+k+l = 14), \quad h.c.$$

$$a_i a_j a_k + d.c. \quad (i < j < k; i+j+k = M; M = 6,7,9,10,..14), \quad h.c.$$

$$a_i a_j a_t + a_k a_l a_t + d.c. \quad (t = 1,3.4; i \neq j \neq k \neq l; i+j+k+l = 7), \quad h.c.$$

$$-\frac{2}{3}(h_1 + h_2 + h_3) + \frac{1}{3}\sum_i h_i, \quad h_3 + h_4$$

$$h_1 + h_5 - h_2 - h_6, \quad h_2 + h_4 - h_3 - h_5 \tag{17}$$

7 Realization of G_2

In the emdedding $SU(3) \subset G_2$ the adjoint representation decomposes as

$$14 \Longrightarrow 8 + 3 + \bar{3} \tag{18}$$

where (i = 1,2,..9; j = 1,2...6):

$$8 \equiv \{a_1 a_2 a_3 + d.c., \quad a_7 a_8 a_9 + d.c., \quad a_4^+ a_3, \quad h.c.$$

$$-\frac{2}{3}(h_1 + h_2 + h_3) + \frac{1}{3}\sum_{i>3} h_i \quad -\frac{2}{3}(h_7 + h_8 + h_9) + \frac{1}{3}\sum_j h_j\} \tag{19}$$

$$3 + \bar{3} \equiv \{a_1^+ a_6, \quad a_2 a_3 a_6 + d.c., \quad a_1^+ a_4^+ a_5^+ + d.c., \quad h.c.\} \tag{20}$$

8 Conclusions

One of the advantages of the oscillators construction of LAs is the knowledge of the Fock space which becomes the carrier space of irrep. of the the LAs.

In the case of construction of LAs SU(N) by using fermionic oscillators it is well known that the carrier space of antisymmetric irreps. can be realized on the Fock space. As the fundamental irreps. of G_2, E_6, E_7, of dimension, resp. $7, 27, 56$, decompose under the maximal unitary subalgebras as a sum of antisymmetric irreps. as:

$$
\begin{aligned}
7 &\implies 3 + \bar{3} + 1 \\
27 &\implies (15, 1) + (\bar{6}, 2) \\
56 &\implies 28 + \overline{28}
\end{aligned}
\tag{21}
$$

one can think that on the Fock space of the fermionic oscillators it is possible build up the fundamental representations, at least, of these ELAs.

Indeed for G_2 this has already been obtained [4].

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Quantized Discrete Space Oscillators

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A quasi-canonical sequence of finite dimensional quantizations has been found which has canonical quantization as its limit. In order to demonstrate its practical utility and its numerical convergence, this formalism is applied to the eigenvalue and "eigenfunction" problem of several harmonic and anharmonic oscillators.

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I. Introduction

Harmonic and anharmonic oscillators have long been used to illustrate new approximation techniques. Here they are used to demonstrate the application of an approximation procedure based upon the approach of a sequence of discrete non-canonical quantizations to the standard canonical quantization limit .

Consider a discrete one dimensional space in which the coordinates of allowed positions are interger multiples of a fundamental scale parameter a having dimensions of length. Although it is well known that the canonical commutation relation

$$[Q, P] = i\hbar I \tag{1}$$

does not admit finite dimensional matrix representations,[1] one can ask whether a limit procedure exists such that sequences of matrices $\{Q_N, P_N\}$ satisfy in some way

$$\lim_{N \to \infty} [Q_N, P_N] = i\hbar I \tag{2}$$

in the weak sense. We have found the answer to be in the affirmative[2] and therefore briefly sketch the theoretical analysis and apply the formalism to the numerical eigenvalue and eigenstate problems of harmonic and anharmonic oscillators.

II. Brief Analysis

The quantum mechanical scalar product of two wave functions in the Schroedinger representation can be written in the form

$$(\Psi, \Phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(q)\delta(q - q')\Phi(q')dqdq', \tag{3}$$

a form equivalent to the traditional one. With respect to this scalar product the Schroedinger coordinate and momentum representatives have the form

$$Q(q, q') = q\delta(q - q'), \tag{4}$$

$$P(q, q') = -i\hbar \frac{\partial}{\partial q}\delta(q - q'). \tag{5}$$

On the other hand in a finite discrete space representation of the canonical coordinates and momenta one defines the Schroedinger representative of the coordinate operator by the diagonal matrix

$$
Q_N = a \begin{pmatrix}
(N-1)/2 & 0 & . & . & & . & . & . & & 0 \\
0 & & . & & & & & . & & . \\
. & & & . & & & & . & & . \\
. & & & & 1/2 & 0 & & & & . \\
. & & & & 0 & . & -1/2 & & & . \\
. & & & & & & . & & & . \\
. & & & & & & & . & & 0 \\
0 & & . & . & . & & . & . & 0 & -(N-1)/2
\end{pmatrix} , \qquad (6)
$$

or equivalently in component notation by

$$
Q_{N_s}^r = ra\delta_s^r . \qquad (7)
$$

Here a defines the distance between neighboring points in space and N is a power of 2.

The components of the momentum operator are taken from the discrete Fourier transform of the diagonal part of Q_N,

$$
P_{N_s}^r = \frac{\hbar^2}{N} \sum_{k=-(N-1)/2}^{(N-1)/2} \frac{2\pi k}{Na} \exp\left\{ -\frac{2\pi i k}{N}(r-s) \right\} . \qquad (8)
$$

Thus P_N is a Toeplitz matrix. The expressions (4) and (5) follow from a proper definition of the state space scalar product and limiting procedure.

The state space scalar product is defined by

$$
(\Phi, \Psi) = \frac{1}{a} \sum_{k=-(N-1)/2}^{(N-1)/2} \overline{\Phi^k}\Psi^k \qquad (9)
$$

where the bar denotes complex conjugation. With respect to this scalar product it is convenient for the sake of the weak limiting procedure to write the rth eigenvector of Q_N as either

$$
q_r^s = \delta_r^s , \qquad (10)
$$

or equivenently expressed in terms of the finite discrete Fourier transform as

$$q'_r = \frac{1}{N} \sum_{k=-(N-1)/2}^{(N-1)/2} \exp\left\{-\frac{2\pi i k}{N}(r-s)\right\}.$$ (11)

Using the latter one can verify that

$$(q_r, q_s) = \frac{1}{a}\,\delta_{rs} = \frac{1}{Na} \sum_{k=-(N-1)/2}^{(N-1)/2} \exp\left\{-\frac{2\pi i k}{N}(r-s)\right\}$$ (12)

One can now define the refinement or weak limit as that in which

$$Na \to \infty, a \to 0, ra \to q, sa \to q',$$ (13)

so that the matrix product summations carry over into integrals (in the same way that a Fourier series can be carried over into a Fourier integral). Note that this approach differs from those of others in which the domain of the right hand side of (2) is restricted to a subspace of the Hilbert space.[3]

For a small enough and N, r, s large enough one can get as close as desired to any real valued q or q'. In this case using the right hand side of (12) one can see that in the refinement or weak limit

$$(q_r, q_s) \to \delta(q - q').$$ (14)

Similarly one can see that (2)and (7) and (8) have the weak limits

$$(q_r, [Q, P]\, q_s) \to i\hbar q \frac{\partial}{\partial q}\delta(q - q'),$$ (15)

$$(q_r, Q\, q_s) \to \delta(q - q'),$$ (16)

$$(q_r, P\, q_s) \to i\hbar \frac{\partial}{\partial q}\delta(q - q').$$ (17)

Note that with all of these weak limits the factor $\frac{1}{a}$ associated with the implied metric tensor in the scalar product (9) is essential.

The existence of these limits is considered sufficient justification for investigating the practical utility of this finite dimensional quasi-canonical quantization. Hence we investigate the eigenvalue problem of several of the oscillators described by the Hamiltonian

$$H = \frac{P^2}{2m} + FQ + m\omega^2\frac{Q^2}{2} + \lambda m\frac{Q^4}{4}.$$ (18)

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III. Oscillator Eigenstates and Eigenvalues

Given (8) and (7) the Hamiltonian (18) can be written in the form

$$H^r_{Ns} = \frac{1}{N} \sum_{k=-(N-1)/2}^{(N-1)/2} \left(\frac{(2\pi\hbar k)^2}{2m(Na)^2} + Fra + \frac{m\omega^2}{2}(ra)^2 + \frac{m\lambda}{4}(ra)^4 \right) \exp\left\{ -2\pi k \frac{(r-s)}{N} \right\}.$$

(19)

It now remains to make proper choices of N and a and to carry out the numerical calculations in a manner compatible with a refinement process.

Define exponents n and l such that

$$N = 2^n, a = a_0 2^{-\alpha}, L = a_0 2^{n-\alpha}, n - \alpha > 0,$$

(20)

where a_0 is a length scale appropriate to the problem and where L gives the physical size of our one dimensional space. Clearly n sets the dimensionality of the matrix and α the refinement. The size of the space is L, while the rth eigenvector q_r corresponds to the physical coordinate $r a_0 2^{n-\alpha}$.

A. The Harmonic Oscillator

For the case of the harmonic oscillator ($F = \lambda = 0$) it is convenient to choose

$$a_0 = \left(\frac{\hbar}{m\omega} \right)^{1/2}.$$

(21)

With the choices (19),(20), and (21) the discretized dimensionless form for the Hamiltonian is

$$\frac{H}{\hbar\omega} = \sum_{k=-(N-1)/2}^{(N-1)/2} \left(\frac{(2\pi k)^2}{2^{2\alpha-3n+1}} + \frac{2^{2\alpha-n}}{2} r^2 \right) \exp\left\{ -2\pi k \left(\frac{r-s}{N} \right) \right\}.,$$

(22)

One has different approximations for different choices of n and α. Figure I. illustrates the first 4 normalized "wave" eigenvectors for the discrete Harmonic Oscillator ($F = \lambda = 0$) with $n = 8, \alpha = 1$. Amazingly, for the first two eigenvectors of $\frac{H}{\hbar\omega}$ the absolute value of the error between a component of the eigenvector and the exact corresponding eigenfunction solution to the Schroedinger equation is less that 10^{-3} for the ground state and 10^{-2} for the first excited state at any of the allowed positions in the discrete space.
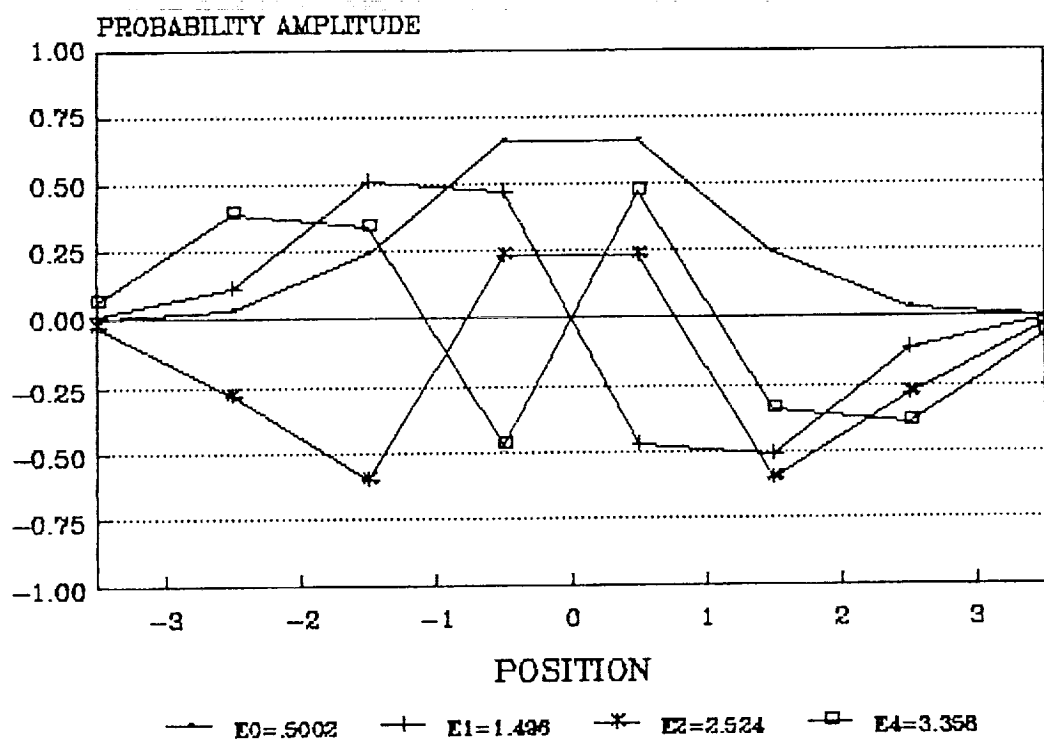
FIGURE I. HARMONIC OSC WAVEVECTORS (8,0)

Table I illustrates the numerical convergence of the 1st four harmonic oscillator eigenvalues as a function of matrix dimensionality N and α.

| | Table I | | H.O. | |
	$N = 2, \alpha = 0$	$N = 4, \alpha = 0$	$N = 8, \alpha = 0$	$N = 16, \alpha = 1$
E_0	1.3587	.5410	.50018	.5000001
E_1	1.3587	1.2186	1.4961	1.499984
E_2	–	3.1156	2.5241	2.500205
E_3	–	3.7933	3.3582	3.497943

B. The Asymmetric Oscillator

For the case of the asymmetric oscillator we take the choice

$$a_0 = \left(\frac{2\hbar^2}{Fm}\right)^{\frac{1}{4}}, \omega^2 = -\frac{8F}{ma_0}, \lambda = \frac{F}{ma_0^3}. \qquad (23)$$

The components of the dimensionless Hamiltonian now have the form

$$\frac{H_{N_s}^r}{Fa_0} = \sum_{k=-(N-1)/2}^{(N-1)/2} \left(\frac{(2\pi k)^2}{2^{2\alpha - 3n + 1}} + 2^{\alpha - n}r - 8 \cdot 2^{2\alpha - n}r^2 + \frac{2^{4\alpha - n}r^4}{4}\right) \exp\left\{-2\pi k \frac{(r-s)}{N}\right\}$$

Figure II illustrates the asymmetric potential while Figure III illustrates the solutions to the eigenvector problem for this asymmetric oscillator. Of interest is the "trapping" in the virtual potential well occuring with the 4th energy level, an effect not easily accounted for with other approximation techniques.

Table II illustrates the numerical convergence of the first four eigenvalues using several choices of N and α.

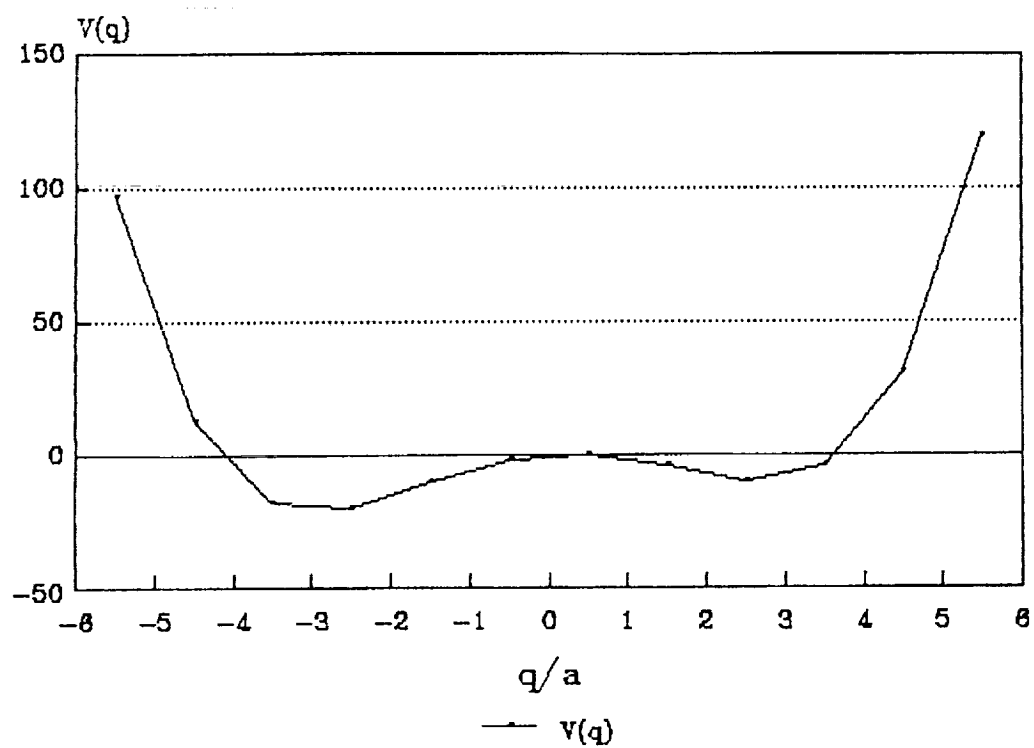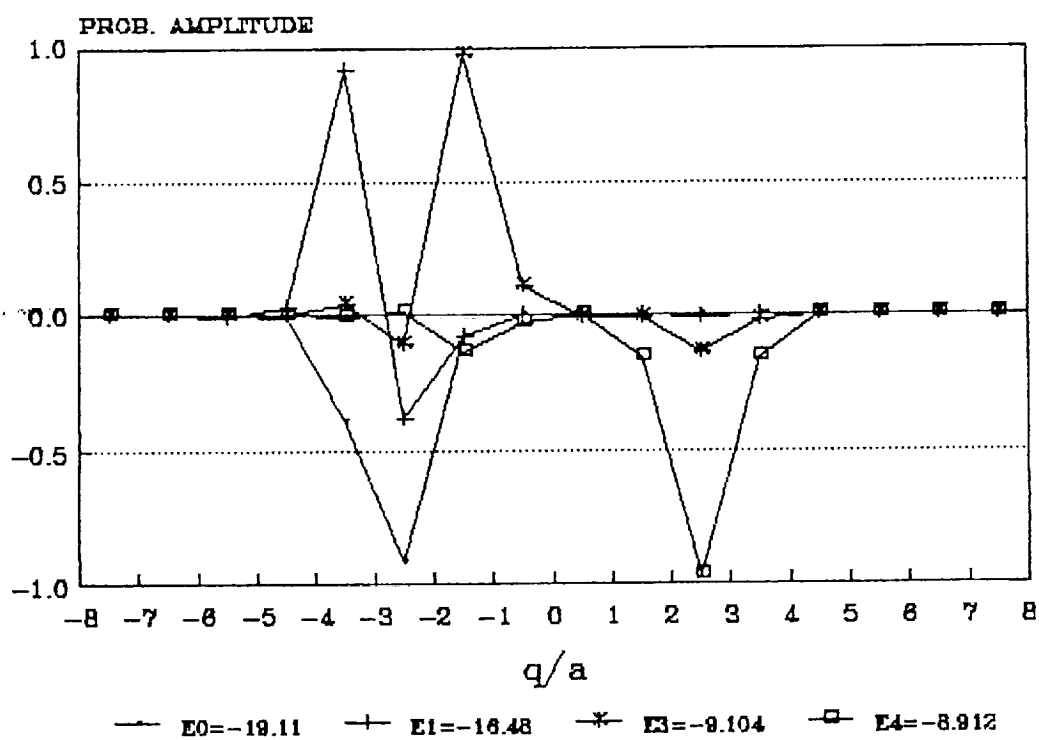| | Table II | | ASYM OSC. | | |
	$N = 2, \alpha = 0$	$N = 4, \alpha = 0$	$N = 8, \alpha = 0$	$N = 16, \alpha = 1$	$N = 64, \alpha = 2$
E_0	−.7507	−9.3999	−19.1162	−19.6787	−19.6803
E_1	−1.2493	−3.2519	−16.5510	−15.6227	−15.6025
E_2	–	−0.6331	−9.1040	−11.7496	−11.7147
E_3	–	2.0159	−8.9120	−8.6628	−8.65986

FIGURE II. ASYMMETRIC OSC POTENTIAL

FIGURE III. ASYM OSC WAVEVECTORS (8,0)

C. The Quartic Oscillator

In the case of the quartic oscillator ($F=\omega = 0$) the choice

$$a_0 = \left(\frac{2\pi\hbar}{m\lambda^{1/2}}\right)^{\frac{1}{5}} \tag{24}$$

suffices. In this case the components of the dimensionless Hamiltonian have the form

$$\frac{H_{N_s}^r}{\lambda a_0^4} = \sum_{k=-(N-1)/2}^{(N-1)/2} \left(\frac{(2\pi k)^2}{2^{2\alpha-3n+1}} + \frac{2^{4\alpha-n} r^4}{4}\right) \exp\left\{-2\pi k \frac{(r-s)}{N}\right\}$$

The quartic oscillator is included here because it has been a frequent subject of study and its eigenvalues are numerically established. Table III illustrates the numerical convergence for the first four eigenvalues and gives the "exact" eigenvalues as numerically established by others[4,5,6] using distinct techniques.

Table III   QUARTIC OSC

	$N = 2, \alpha = 0$	$N = 4, \alpha = 0$	$N = 8, \alpha = 0$	$N = 16, \alpha = 1$	"exact"
E_0	1.2493	.4498	.4217	.420805	.420804
E_1	1.2493	1.2760	1.5587	1.50790	1.50790
E_2	—	3.0895	2.9422	2.95880	2.95880
E_3	—	3.9157	3.8709	4.62128	4.62122

IV. Summary.

The purpose here has been to demonstrate the numerical application of the quasi-canonical quantization procedure to 1 dimensional oscillatory systems. With this application it is clear that numerical convergence occurs for each of the potentials.

The calculation procedure is computationally straightforward since the momentum matrix (8) can be conveniently calculated using well established signal processing techniques.[7] In particular, fast Fourier transform (FFT)[8] techniques allow for rapid determination of "state" vectors and quantum numbers.

The approximation procedure utilized here is based not upon the truncation of the wave functions but rather on how well the canonical commutation relation is approximated. In contrast to usual perturbation theories the number of allowed states in each order is thesame as the number of allowed positions. The convergence of this discrete quantization to a canonical one and detailed derivations of results quoted in the Section II will be discussed elsewhere. However it is useful to note two important facts about the procedure. First, the use of the discrete fourier transform (and numerical use of FFT's[8]) does not implies the periodicity of space, rather the quantization is carried out over a finite region of space. Secondly, the matrix Q_N as a $N=2^n$ dimensional matrix insures not only the faster speed of the FFT's utilized but also that on the discrete scale that Q_N is an invertible matrix for which the eigenvalue zero exists only as a refinement or weak limit. Thus one should not expect numerical problems in dealing with coulomb like potentials.

Finally, with the refinement limit we note that one can get as close as one wishes to canonical quantization with discrete space, suggesting that quantum theory cannot readily distinguish between discrete and continuous space time. In addition, please note that the combination (9) and (11) allows a sequence of matrices to serve as the definition of required generalized functions, as in (14,15,16,17), somewhat in analogy to the good functions of Lighthill[9].[10]

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ORBIT-ORBIT BRANCHING RULES

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Abstract

We show how the derivation of group-subgroup (IR to IR) branching rules is facilitated by the use of the much simpler (Weyl) orbit-orbit branching rules.

1 Introduction

We must begin by relating our subject to harmonic oscillators. This is easy if we use as basis states for an IR polynomials in the states of the fundamental IR's of the group under consideration. The variables representing the fundamental states can be replaced by creation operators for similarly labeled harmonic oscillators—the Schwinger boson calculus.

The audience does not need to be convinced of the utility of "building blocks" of larger objects of interest in physics. The role of (Weyl) orbits as constituents of IR's (we use IR as an abbreviation for basis of an irreducible representation) has not been exploited much. Their use simplifies considerably the derivation of group-subgroup (IR) branching rules.

The problem is broken into three steps: I decomposition of the group IR into group orbits, II decomposition of each group orbit into subgroup orbits and III the assembly of subgroup orbits into subgroup IR's.

In § 2 we discuss steps I and III; in § 3 we discuss step II. Most of the material presented here appears in articles by Patera and Sharp[1] and by Gingras, Patera and Sharp[2].

2 Orbits expanded in IR's and vice versa.

Let λ_n be a set of weights of a semisimple algebra and c_n the multiplicity of the weight λ_n; we suppose the weights have Weyl symmetry. Then the weights can be written as a superposition of weights of IR's:

$$\sum_n \Lambda^{\lambda_n} c_n = \sum_a \chi_a g_a.$$ (1)

χ_a is the character of the IR (a) and g_a is an integer which we call the multiplicity of (a) even though it may now take negative as well as positive values. The dummies Λ_i carry weight components λ_i as exponents: $(\Lambda^\lambda \equiv \prod_i \Lambda_i^{\lambda_i})$.

To find g_a use Weyl's character formula

$$\chi_a = \xi_a/\xi_0$$ (2)

where ξ_a is the Weyl characteristic function

$$\xi_a = \sum_W (-1)^W \Lambda^{W(a+R)}.$$ (3)

The sum is over Weyl reflections, $(-1)^W$ is the determinant of the matrix of W, i.e., ± 1 according to whether W is a product of an even or odd number of relections and R is half the sum of the positive roots, or the sum of the fundamental weights; ξ_0 is the characteristic of the scalar IR $(a = 0)$. Then

$$\sum_n \Lambda^{\lambda_n} c_n \xi_0 = \sum_a \xi_a g_a$$ (4)

Now ξ_a has just one term Λ^{a+R} in the dominant Weyl sector, so g_a is the coefficient of Λ^a in $\sum_n \Lambda^{\lambda_n - R} c_n \xi_0$. We take the λ_n in (1) to be the weights of the Weyl orbit $[\lambda]$. Then g_a is the multiplicity of the IR (a) in the expansion of the orbit $[\lambda]$.

The sum in (4) can be visualized graphically in the spirit of a Speiser[3] diagram. But since Speiser's methodology is impracticable for rank higher than 2 we prefer a numerical approach, effecting Weyl reflections with the help of the Cartan matrix. For illustration we use the G_2 orbit [2,1]. The instructions given in the caption of Table I apply unchanged for any simple group. An orbit weight is recognized as lying on a reflection hyperplane if any of its weight components is zero at any stage of the reflection process (e.g. in Column 2 or 3). This completes our description of step III, in which subgroup orbits are converted to subgroup IR's.

TABLE I. The G_2 orbit [2,1] expanded in modules.

1	2	3	4
$\{2,1\}$	$\{3,2\}$	$\{3,2\}$	$(2,1)$
$\{\bar{2},7\}$	$\{\bar{1},8\}$	$-\{1,5\}$	$-(0,4)$
$\{\bar{3},8\}$	$\{\bar{2},9\}$	$-\{2,3\}$	$-(1,2)$
$\{\bar{5},8\}$	$\{\bar{4},9\}$	$\{1,3\}$	$(0,2)$
$\{5,\bar{8}\}$	$\{6,\bar{7}\}$	$\{1,4\}$	$(0,3)$
$\{3,\bar{8}\}$	$\{4,\bar{7}\}$	$-\{1,2\}$	$-(0,1)$

Column 1 contains those orbit weights which do not lie on a reflection line. Column 2 contains the weights of Column 1 augmented by R. Column 3 contains the weights of Column 2 reflected to the dominant sector with sign ± 1 according to whether an even or odd member of reflections is involved. Column 4 contains the IR's in the expansion of the orbit [2,1], obtained from Column 3 by subtracting R.

We now turn to step I, in which group IR's are decomposed in group orbits. We first carry out step III for all group orbits no further from the origin than the highest weight of the IR under discussion. Only orbits of the same congruence class as the IR in question need to be considered. These orbit → IR expansions define the triangular orbit-IR matrix which is easily inverted to give the IR-orbit matrix; it gives the orbit content of IR's. This procedure is simpler to implement than other methods such as Freudenthal's recursion formula for weight multiplicities, or the character formulas of Weyl or Demazure.

3 Orbit-orbit branching rules.

Orbit-orbit branching rules are always much simpler to derive and to describe than the usual IR-IR branching rules. One approach which has general applicability makes use of the orbit-weight generating function $F(M, \Lambda)$. Its power-expansion

$$F(M, \Lambda) = \sum_{\mu, \lambda} M^\mu \Lambda^\lambda c_{\mu\lambda} \tag{5}$$

gives the multiplicity $c_{\mu\lambda}$ of the weight $\{\lambda\}$ in the orbit $[\mu]$. As a simple example the SU(3) orbit-weight generating function is

$$\begin{aligned}
F(M_1, M_2; \Lambda_1, \Lambda_2) = {} & \frac{1}{(1 - M_1\Lambda_1)(1 - M_2\Lambda_2)} \tag{6} \\
& + \frac{M_1\Lambda_1^{-1}\Lambda_2}{(1 - M_2\Lambda_2)(1 - M_1\Lambda_1^{-1}\Lambda_2)} + \frac{M_2\Lambda_1^{-1}}{(1 - M_1\Lambda_1^{-1}\Lambda_2)(1 - M_2\Lambda_1^{-1})} \\
& + \frac{M_1\Lambda_2^{-1}}{(1 - M_2\Lambda_1^{-1})(1 - M_1\Lambda_2^{-1})} + \frac{M_2\Lambda_1\Lambda_2^{-1}}{(1 - M_1\Lambda_2^{-1})(1 - M_2\Lambda_1\Lambda_2^{-1})} \\
& + \frac{M_1 M_2 \Lambda_1^2 \Lambda_2^{-1}}{(1 - M_2\Lambda_1\Lambda_2^{-1})(1 - M_1\Lambda_1)} .
\end{aligned}$$

Λ_1, Λ_2 carry weight components in a fundamental weights basis. To convert the orbit-weight generating function to an orbit-orbit branching rules generating function it is necessary only to replace the dummies Λ which carry weight components with new dummies carrying subgroup weight components and then retain the part that contains only non-negative powers of the new dummies. For example for SU(3)⊃SU(2)×U(1) the replacements are $\Lambda_1 \to NY^{\frac{1}{3}}, \Lambda_2 \to Y^{\frac{2}{3}}$. We remark that for a dummy like Y carrying a U(1) label one should retain both negative and positive powers. For SU(3)⊃SO(3) the replacements are $\Lambda_1 \to N^2, \Lambda_2 \to 1$.

The method described in the preceding paragraph becomes laborious for higher rank groups because of the large order of the Weyl group. It is usually simpler to determine an integrity basis for subgroup orbits by examining low

group orbits. The integrity basis consists of "elementary" orbits, from which all can be obtained as stretched products (orbit labels additive).

Two types of subgroup may be distinguished according to whether the Weyl sectors of group and subgroup do or do not "line up." When we compare regions of group and subgroup weight space, a region of subgroup weight space, say a Weyl chamber, means the region of group weight space which projects into the subgroup region in question. The simpler situation is that in which group and subgroup chambers line up, i.e., each subgroup chamber contains only complete group chambers, N/N' of them, where N and N' are the orders of the group and subgroup Weyl groups.

Consider the lining up case. Let W be one of the N/N' group Weyl elements which carry the dominant group chamber within the dominant subgroup chamber. Then a group-subgroup orbit pair (a,b) corresponding to the terms $A^a B^b$ in the orbit-orbit generating function can be written (a,PWa) corresponding to $\prod_i \left(A_i \prod_j B_j^{(PWM_i)_j} \right)^{a_i}$ where P is the projection onto subgroup weight space. Thus the elementary orbits correspond to $A_i \prod_j B_j^{(PWM_i)_j}$, i.e., they are the subgroup orbits contained in the fundamental group orbits. The compatibility rules for elementary orbits can be stated as follows: two elementary orbits are compatible if and only if the two weights WM_i and WM_k can be obtained by the same Weyl element W; in particular two subalgebra orbits belonging to the same fundamental group orbit are incompatible. $SU(3) \supset SO(3)$ and $SU(4) \supset SU(2) \times SU(2) \times U(1)$ are examples of group and subgroup chambers' lining up. A sufficient but not necessary condition for the lining up is that group and subgroup have equal ranks; for all known maximal subjoint algebras that is always the case. The Weyl chambers line up for a regular subgroup.

Examples of cases where the chambers do not line up are $SO(5) \supset SU(2)$ and $SU(4) \supset SU(2) \times SU(2)$ (Wigner supermultiplet). When a dominant subgroup weight lies inside a chamber of group weight space that is only partly in the dominant subalgebra sector, it cannot be compounded from elementary orbits belonging to fundamental algebra orbits; hence composite elementary orbits (more than one algebra label nonzero) arise.

We conclude with an example where group and subgroup have equal rank, $F_4 \supset SO(9)$.

The decomposition of the fundamental group orbits is as follows:

$$
\begin{array}{ccc}
& a & b \\
[1000] \supset & [0100]\,, & [0100] \supset [1010]\,, \\
& c \quad\quad d & \\
[0010] \supset & [1001] + [0010]\,, & \\
& e \quad\quad f & \\
[0001] \supset & [1000] + [0001]\,. &
\end{array}
$$

"Names" for the elementary orbits have been written above them. Compatibility rules are found by looking at orbits with two labels non zero; in this case we

need examine only the composite F_4 orbit [0011] (since a and b are the only subgroup orbits in their respective group orbits they are compatible with all other elementary orbits).

$$[0011] \supset \overset{ce}{[2001]} + \overset{cf}{[1002]} + \overset{df}{[0011]}.$$

The interpretation of subalgebra orbits as products of elementary ones gives us the compatibility rules. There are three $(= N/N')$ sets of mutually compatible orbits, $abce$, $abcf$ and $abdf$. Hence the general F_4 orbit $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ decomposes into SO(9) orbits $[\lambda_2 + \lambda_3 + \lambda_4, \lambda_1, \lambda_2, \lambda_3]$, $[\lambda_2 + \lambda_3, \lambda_1, \lambda_2, \lambda_3 + \lambda_4]$ and $[\lambda_2, \lambda_1, \lambda_2 + \lambda_3, \lambda_4]$.

We remark that the methods and results here apply equally to Kac–Moody algebras.

Acknowledgments

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VII. ATOMIC, NUCLEAR, AND PARTICLE PHYSICS

MASS SPECTRA OF THE PARTICLE-ANTIPARTICLE SYSTEM WITH A DIRAC OSCILLATOR INTERACTION

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Abstract

The present view about the structure of measons is that they are a quark-antiquark system. The mass spectrum corresponding to this system should, in principle, be given by chromodynamics, but this turns out to be a complex affair. Thus it is of some interest to consider relativistic systems of particle-antiparticle, with a simple type of interaction, which could give some insight on the spectra we can expect for mesons. In the present paper we carry this analysis when the interaction is of the Dirac oscillator type. We show that the Dirac equation of the antiparticle can be obtained from that of the particle by just changing the frequency ω into $-\omega$. Following a procedure suggested by Barut we derive the equation for the particle-antiparticle system and solve it by a perturbation procedure. We thus obtain explicit expressions for the square of the mass spectra and discuss its implications in the meson case.

1 Introduction and summary

It is well known[1] that mesons are considered as formed by a quark-antiquark systems where, in many cases,[1] the particle and antiparticle are of the same mass *i.e.* $u\bar{u}, u\bar{d}, d\bar{u}, d\bar{d}; s\bar{s}$ etc.

The calculation of the mass spectra of mesons within the framework of quantum chromodynamics would be a complex affair[2]. Thus it is of some interest to consider relativistic systems of particle-antiparticle, with a simple type of interaction, that could give us some insight in the type of spectra that we can expect for mesons.

In the present paper we intend to carry this analysis when the interaction is of a Dirac oscillator type[3,4]. We begin in section 2 by considering the positive and negative energy solutions of the one particle Dirac oscillator problem[3,4], and show that the equation for the anti-particle can be derived from that of the particle if we change the frequency ω of the oscillator to $-\omega$.

In section 3 we consider the Barut[5] procedure for deriving a single equation for n-free relativistic particles of spin 1/2, and generalize it to n particles with Dirac oscillator interactions with different frequencies $\omega_s, s = 1, 2, ...n$. We then apply it to the particle-antiparticle case where $n = 2$ and $\omega_1 = -\omega_2 \equiv \omega$.

In section 4 we reduce our equation, which has four components, to just a single one, and proceed to show how to solve the latter by perturbation theory.

In section 5 we derive explicitly the square of the mass spectra of our particle-antiparticle system to first order perturbation theory, and proceed to draw the square of the mass level scheme as function of the total angular momentum j, of the parity $(-1)^j$ or $-(-1)^j$ as well as of the number of quanta N of the oscillator, for different values of ω.

Finally, in the concluding section, we also give the square of the mass spectra of the mesons and show that, while quite different from our present theoretical analysis, it could, as in the three quark case of baryons[6,7], give a better agreement if other interactions are also considered.

2 The Dirac oscillator equation for a particle and for an antiparticle

The single particle Dirac oscillator equation was suggested by the replacement[3]

$$\mathbf{p} \rightarrow \mathbf{p} - i\omega\mathbf{x}\beta, \tag{2.1}$$

in the Dirac free particle expression[8] giving rise to

$$i(\partial\psi/\partial x^0) = \left[\boldsymbol{\alpha} \cdot (\mathbf{p} - i\omega\mathbf{x}\beta) + \beta\right]\psi, \tag{2.2}$$

where x^0 is the time and ω the frequency of the oscillator, all in the units

$$\hbar = m = c = 1 \tag{2.3}$$

where m is the mass of the particle and c the velocity of light. Note furthermore that

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2.4a, b}$$

where $\boldsymbol{\sigma}$ is the Pauli spin vector.

We require now the solutions of (2.2) both for positive and negative energy where E will denote its absolute value. For positive energy we can write

$$\psi^+ = \begin{pmatrix} \psi_1^+ \\ \psi_2^+ \end{pmatrix} \exp(-iEx^0), \tag{2.5}$$

where ψ_1^+, ψ_2^+ are the large and small components depending only on the coordinates and, from (2.4a,b), we obtain

$$(E - 1)\psi_1^+ = \left[\boldsymbol{\sigma} \cdot (\mathbf{p} + i\omega\mathbf{x})\right]\psi_2^+, \tag{2.6a}$$

$$(E + 1)\psi_2^+ = \left[\boldsymbol{\sigma} \cdot (\mathbf{p} - i\omega\mathbf{x})\right]\psi_1^+, \tag{2.6b}$$

so that from the second equation

$$\psi_2^+ = (E + 1)^{-1}[\boldsymbol{\sigma} \cdot (\mathbf{p} - i\omega\mathbf{x})]\psi_1^+, \tag{2.7}$$

and substituting in the first we get

$$(E^2 - 1)\psi_1^+ = [\mathbf{p}^2 + \omega^2\mathbf{x}^2 - 3\omega - 4\omega\mathbf{L}\cdot\mathbf{S}]\psi_1^+, \tag{2.8}$$

where

$$\mathbf{L} = \mathbf{x}\times\mathbf{p} \quad, \quad \mathbf{S} = \boldsymbol{\sigma}/2. \tag{2.9a, b}$$

Clearly then ψ_1^+ is given by

$$\psi_1^+ \equiv |N\left(\ell,\frac{1}{2}\right)jm\rangle \equiv \phi = R_{N\ell}(r) \sum_{\sigma=-1/2}^{1/2}\left[\langle \ell m-\sigma,\frac{1}{2}\sigma|jm\rangle Y_{\ell m-\sigma}(\theta,\varphi)\chi_{1/2\sigma}\right], \tag{2.10}$$

while to obtain ψ_2^+ we have to carry out the operation (2.7). In (2.10) $R_{N\ell}(r)$ is the radial function given in terms of Laguerre polynomials, $< \mid >$ a Clebsch-Gordan coefficient, $Y_{\ell m-\sigma}(\theta,\varphi)$ a spherical harmonic, and $\chi_{1/2\sigma}$ a spin 1/2 state with projection σ. The N indicates the total number of quanta while $\ell, j = \ell \pm \frac{1}{2}$ are the orbital and total angular momentum.

Now we turn our attention to the negative energy states where we can take

$$\psi^- = \begin{pmatrix} \psi_1^- \\ \psi_2^- \end{pmatrix}\exp(iEx^0), \tag{2.11}$$

from which

$$-(E+1)\psi_1^- = \left[\boldsymbol{\sigma}\cdot(\mathbf{p}+i\omega\mathbf{x})\right]\psi_2^-, \tag{2.12a}$$

$$(-E+1)\psi_2^- = \left[\boldsymbol{\sigma}\cdot(\mathbf{p}-i\omega\mathbf{x})\right]\psi_1^-, \tag{2.12b}$$

so that

$$\psi_1^- = -(E+1)^{-1}\left[\boldsymbol{\sigma}\cdot(\mathbf{p}+i\omega\mathbf{x})\right]\psi_2^-, \tag{2.13}$$

while ψ_2^- satisfies

$$(E^2 - 1)\psi_2^- = \left[\mathbf{p}^2 + \omega^2\mathbf{x}^2 + 3\omega + 4\omega\mathbf{L}\cdot\mathbf{S}\right]\psi_2^-, \tag{2.14}$$

so it is again given by the ket $|N(\ell,\frac{1}{2})jm\rangle \equiv \phi$ of (2.10).

The particle state ψ_p is the positive energy one ψ^+ which from (2.5) can be written as[10]

$$\psi_p \equiv \psi^+ = \left[\begin{matrix} \phi \\ (E+1)^{-1}[\boldsymbol{\sigma}\cdot(\mathbf{p}-i\omega\mathbf{x})]\phi \end{matrix}\right]\exp(-iEx^0), \tag{2.15}$$

with ϕ given by (2.10).

For the antiparticle state ψ_a we follow Bjorken and Drell[9] by taking the conjugate of ψ^- of (2.11) and apply to it

$$i\gamma^2 = i\beta\alpha_2 = i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \tag{2.16}$$

thus getting[10]

$$\psi_a \equiv i\gamma_2\psi^{-*} = \begin{bmatrix} i\sigma_2\phi^* \\ (E+1)^{-1}\sigma \cdot (\mathbf{p} + i\omega\mathbf{x})(i\sigma_2\phi^*) \end{bmatrix} \exp(-iEx^0)$$

(2.17)

as

$$(-i\sigma_2)\sigma^*(-i\sigma_2) = \sigma \quad , \quad (-i\sigma_2)(i\sigma_2) = I.$$

(2.18)

Furthermore as

$$i\sigma_2\chi_{\frac{1}{2}\frac{1}{2}} = -\chi_{\frac{1}{2}-\frac{1}{2}} \quad , \quad i\sigma_2\chi_{\frac{1}{2}-\frac{1}{2}} = \chi_{\frac{1}{2}\frac{1}{2}}$$

(2.19a, b)

we see that

$$i\sigma_2\phi^* = R_{N\ell}(r) \sum_{\sigma=-1/2}^{1/2} \left[\langle \ell m - \sigma, \frac{1}{2}\sigma | jm \rangle (-1)^{m+\frac{1}{2}} Y_{\ell,-m+\sigma}(\theta,\varphi)\chi_{\frac{1}{2}-\sigma} \right]$$

(2.20)

so changing σ, m into $-\sigma, -m$ and using properties of the Clebsch-Gordan coefficients[11] we obtain

$$i\sigma_2\phi^* = (-1)^{m+j-\ell}\phi.$$

(2.21)

Thus, (except for the phase factor $(-1)^{m+j-\ell}$ which is irrelevant) the state ψ_a of the antiparticle is the solution of the Dirac oscillator equation (2.2) when we change ω by $-\omega$.

In the next section we consider a Poincaré invariant equation for the two body system of particle and antiparticle.

3 Equation for the particle-antiparticle system with Dirac oscillator interaction

As in previous publications[4,10] we start from the Dirac equation for n-free particles

$$\sum_{s=1}^{n}(\alpha_s \cdot \mathbf{p}_s + \beta_s)\psi = E\psi,$$

(3.1)

where E is the total energy for the system, α_s, β_s are direct products such as

$$\beta_s = I \otimes I...I \otimes \beta \otimes I... \otimes I,$$

(3.2)

with β in s^{th} position, while $\mathbf{p}_s$ is the momentum of the sth particle.

Following the analysis of Barut[5], we showed that the Poincaré invariant form of the equation (3.1) is[4,10]

$$\sum_{s=1}^{n} \left[\Gamma_s(\gamma_s^\mu p_{\mu s} + 1) \right] \psi = 0.$$

(3.3)

We first explain all the symbols appearing in (3.3). The index μ takes now the values $\mu = 0, 1, 2, 3,$ and

$$\gamma^0 = \beta, \gamma^i = \beta\alpha_i, i = 1, 2, 3,$$

(3.4a, b)

with β, α_i given by (2.4a,b). For n particles we have $\gamma_s^\mu, s = 1, 2, ...n$, given by direct products such as (3.2).

We also introduce the concept[5] of unit time like four vector $(u_\mu) = (u_0, u_1, u_2, u_3)$, which means that in some reference frame it can tkae the value $(1,0,0,0)$. With the help of u_μ we define the Lorentz scalars

$$\Gamma = \prod_{r=1}^{n}(\gamma_r^\mu u_\mu), \tag{3.5a}$$

$$\Gamma_s = (\gamma_s^\mu u_\mu)^{-1}\Gamma, \tag{3.5b}$$

where repeated indeces are summed over $\mu = 0, 1, 2, 3$. Note that $(\gamma_s^\mu u_\mu)^{-1}$ in (3.5b) just eliminates the corresponding term in Γ of (3.5a) so Γ_s is still in product form.

The terms in our equation (3.3) are then fully defined and we proceed now to look at it in the frame of reference where $(u_\mu) = (1, 0, 0, 0)$, where it takes the form

$$\left\{\Gamma^0 \sum_{s=1}^{n} p_{0s} + \sum_{s=1}^{n}\left[\Gamma_s^0(\boldsymbol{\gamma}_s \cdot \mathbf{p}_s + 1)\right]\right\}\psi = 0, \tag{3.6}$$

where

$$\Gamma^0 = \prod_{r=1}^{n} \gamma_r^0 = \beta \otimes \beta \otimes .... \otimes \beta \equiv B, \tag{3.7a}$$

$$\Gamma_s^0 = (\gamma_s^0)^{-1}\Gamma^0 = \beta \otimes \beta .... \beta \otimes I \otimes \beta .... \otimes \beta, \tag{3.7b}$$

Multiplying (3.6) by Γ^0 and using $\beta^2 = I, \gamma_i = \beta\alpha_i$ and $(\Gamma^0)^2 = I$ we obtain

$$\left[-P^0 + \sum_{s=1}^{n}(\boldsymbol{\alpha}_s \cdot \mathbf{p}_s + \beta_s)\right]\psi = 0, \tag{3.8}$$

where we put the time like component P_0 of the four momentum

$$P_\mu = p_{\mu 1} + p_{\mu 2} + ... + p_{\mu n}, \tag{3.9}$$

in its contravariant form $-P^0$ as our metric tensor is

$$g_{\mu\nu} = 0 \text{ if } \mu \neq \nu, \ g_{11} = g_{22} = g_{33} = -g_{00} = 1. \tag{3.10}$$

Clearly we then recover equation (3.1) if we interpret P^0 as E, as is usually done.

The Barut equation (3.3) will then provide the starting point for the one involving n-particles with Dirac oscillator interactions of frequency $\omega_s, s = 1, 2, ...n$. To proceed in this direction we could replace $p_{\mu s}; \mu = 0, 1, 2, 3; s = 1, 2, ...n$ in (3.3) by a linear function of $p_{\mu s}$ and $x_{\mu s}$ as was done in (2.1) for the one particle problem. We note though that while $p_{\mu s}$ commutes with the total four momentum P_ν of (3.9), which is a genereator of the Poincaré group, $x_{\mu s}$ does not. Thus it is more suggestive[4] to use the translationally invariant coordinates $x'_{\mu s}$ defined by

$$x'_{\mu s} = x_{\mu s} - X_\mu, \tag{3.11a}$$

where

$$X_\mu = n^{-1}(x_{\mu 1} + x_{\mu 2} + ... + x_{\mu n}).$$ (3.11b)

Furthermore we would like that the resulting equation, in the frame of reference where the center of mass is at rest *i.e.* $P_i = 0, i = 1, 2, 3$, should depend only on a single time as is the case in Eq. (3.8). Thus it is convenient to use the transverse coordinates $\hat{x}'_{\mu s}$ defined by

$$\hat{x}'_{\mu s} = x'_{\mu s} - (P^\nu x'_{\nu s})P_\mu (P_\tau P^\tau)^{-1/2},$$ (3.12)

which has the property that in the center of mass frame, where $P_i = 0, i = 1, 2, 3, \hat{x}'_{0s} = 0$.

With the above restrictions we could obtain from (3.3) a Poincaré invariant equation with a Dirac oscillator interaction if we make in it the replacement

$$p_{\mu s} \to p_{\mu s} - i\omega_s \hat{x}'_{\mu s}\Gamma,$$ (3.13)

where Γ is defined as in (3.5a), and we assignate a different frequency ω_s for each particle. We then arrive at the equation

$$\sum_{s=1}^{n} \left\{ \Gamma_s \left[\gamma_s^\mu (p_{\mu s} - i\omega_s \hat{x}'_{\mu s}\Gamma) + 1 \right] \right\}\psi = 0,$$ (3.14)

where we make the choice[4] for the unit time like vector u_μ in Γ and Γ_s of (3.5) as

$$u_\mu = P_\mu (-P_\nu P^\nu)^{-1/2}$$ (3.15)

In the center of mass frame of reference, where $P_i = 0, i = 1, 2, 3$, we have $\{u_\mu\} = (1000)$ and so by a reasoning similar to the one that leads from (3.3) to (3.8) we obtain the equation

$$\left\{ -P^0 + \sum_{s=1}^{n} \left[\boldsymbol{\alpha}_s \cdot (\mathbf{p}'_s - i\omega_s \mathbf{x}'_s B) + \beta_s \right] \right\}\psi = 0,$$ (3.16)

where B is given by (3.7a) while $\boldsymbol{\alpha}_s, \beta_s$ are direct products of the form (3.2) and

$$\mathbf{p}'_s \equiv \mathbf{p}_s - n^{-1}\mathbf{P},$$ (3.17)

becomes identical to $\mathbf{p}_s$ in the center of mass frame.

As P^0 is the total energy of the system, the rest of the expression (3.16) is then the mass operator[4], which we will designate by $\mathcal{M}$, for the n particles interacting with Dirac oscillators of frequencies $\omega_s, s = 1, 2, ...n$. If we are dealing with the particle-antiparticle system $n = 2$ and, from the discussion of the previous section $\omega_1 = -\omega_2 \equiv \omega$, so we get[4]

$$\mathcal{M} = (1/\sqrt{2})\left\{ (\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot \mathbf{p} - i\omega \left[(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2) \cdot \mathbf{x} \right] B \right\} + (\beta_1 + \beta_2)$$ (3.18)

where[4]

$$\mathbf{p} = (1/\sqrt{2})(\mathbf{p}_1 - \mathbf{p}_2), \mathbf{x} = (1/\sqrt{2})(\mathbf{x}_1 - \mathbf{x}_2),$$ (3.19a, b)

$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1, & 0 & 0 \\ \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1 \\ 0 & 0 & \sigma_1 & 0 \end{pmatrix}, \tag{3.20a}$$

$$\alpha_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \\ \sigma_2 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \end{pmatrix}, \tag{3.20b}$$

$$\beta_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}, \tag{3.21a}$$

$$\beta_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}, \tag{3.21b}$$

$$B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \tag{3.22}$$

and the wave function can be written as[4]

$$\psi = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{12} \\ \psi_{22} \end{pmatrix} \tag{3.23}$$

4   Solution of the eq. (3.18) by a perturbative procedure

Denoting by μ the eigenvalue of the mass operator of (3.18) and making use of (3.19-3.23) we obtain the equation

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_1 \cdot (\mathbf{p} + i\omega\mathbf{x}) & -\sigma_2 \cdot (\mathbf{p} - i\omega\mathbf{x}) & 0 \\ \sigma_1 \cdot (\mathbf{p} - i\omega\mathbf{x}) & 0 & 0 & -\sigma_2 \cdot (\mathbf{p} + i\omega\mathbf{x}) \\ -\sigma_2 \cdot (\mathbf{p} + i\omega\mathbf{x}) & 0 & 0 & \sigma_1 \cdot (\mathbf{p} - i\omega\mathbf{x}) \\ 0 & -\sigma_2 \cdot (\mathbf{p} - i\omega\mathbf{x}) & \sigma_1 \cdot (\mathbf{p} + i\omega\mathbf{x}) & 0 \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{12} \\ \psi_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (\mu - 2) & \psi_{11} \\ \mu & \psi_{21} \\ \mu & \psi_{12} \\ (\mu + 2) & \psi_{22} \end{pmatrix}, \tag{4.1}$$

where $\mathbf{p} = -i\nabla$.

Introducing now the creation and annihilation operators by the definitions

$$\boldsymbol{\eta} = (1/\sqrt{2})(\omega^{1/2}\mathbf{x} - i\omega^{-1/2}\mathbf{p}), \boldsymbol{\xi} = (1/\sqrt{2})(\omega^{1/2}\mathbf{x} + i\omega^{-1/2}\mathbf{p}), \tag{4.2a, b}$$

we get the equations

$$i\omega^{1/2}\begin{pmatrix} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\eta} & \boldsymbol{\sigma}_2 \cdot \boldsymbol{\xi} \\ \boldsymbol{\sigma}_2 \cdot \boldsymbol{\xi} & \boldsymbol{\sigma}_1 \cdot \boldsymbol{\eta} \end{pmatrix}\begin{pmatrix} \psi_{21} \\ \psi_{12} \end{pmatrix} = \begin{pmatrix} (\mu - 2) & \psi_{11} \\ (\mu + 2) & \psi_{22} \end{pmatrix}, \tag{4.3a}$$

$$-i\omega^{1/2}\begin{pmatrix} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\xi} & \boldsymbol{\sigma}_2 \cdot \boldsymbol{\eta} \\ \boldsymbol{\sigma}_2 \cdot \boldsymbol{\eta} & \boldsymbol{\sigma}_1 \cdot \boldsymbol{\xi} \end{pmatrix}\begin{pmatrix} \psi_{11} \\ \psi_{22} \end{pmatrix} = \mu\begin{pmatrix} \psi_{21} \\ \psi_{12} \end{pmatrix}, \tag{4.3b}$$

Multiplying (4.3a) by μ and substituting in it (4.3b), we obtain, after some straightforward algebra, that

$$\omega\begin{bmatrix} A & D \\ D & A \end{bmatrix}\begin{bmatrix} \psi_{11} \\ \psi_{22} \end{bmatrix} = \begin{bmatrix} \mu^2 - 2\mu & 0 \\ 0 & \mu^2 + 2\mu \end{bmatrix}\begin{bmatrix} \psi_{11} \\ \psi_{22} \end{bmatrix}, \tag{4.4}$$

where

$$A \equiv 2\boldsymbol{\eta} \cdot \boldsymbol{\xi} + 3 - \mathbf{L} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2), \tag{4.5a}$$

$$D \equiv 2(\mathbf{S} \cdot \boldsymbol{\eta})^2 + 2(\mathbf{S} \cdot \boldsymbol{\xi})^2 - (\boldsymbol{\eta} \cdot \boldsymbol{\eta}) - (\boldsymbol{\xi} \cdot \boldsymbol{\xi}), \tag{4.5b}$$

while

$$\mathbf{L} = \mathbf{x} \times \mathbf{p} = -i(\boldsymbol{\eta} \times \boldsymbol{\xi}), \mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2), \tag{4.6a, b}$$

and extensive use was made of the relation between Pauli spin matrices *i.e.*

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k. \tag{4.7}$$

It is convenient to substitute ψ_{11}, ψ_{22} by ϕ_+, ϕ_-, through the relation

$$\begin{bmatrix} \psi_{11} \\ \psi_{22} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}, \tag{4.8}$$

so that equation (4.4) becomes

$$\omega\begin{bmatrix} A - D & 0 \\ 0 & A + D \end{bmatrix}\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = \begin{bmatrix} \mu^2 & -2\mu \\ -2\mu & \mu^2 \end{bmatrix}\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}. \tag{4.9}$$

Writing the two equations in ϕ_+, ϕ_- explicitly and eliminating ϕ_- between them we obtain for ϕ_+, which from now on we denote simply by ϕ, the equation

$$\left[\mu^4 - (4 + 2A\omega)\mu^2 + \omega^2(A^2 - D^2 - AD + DA)\right]\phi = 0 \tag{4.10}$$

Unfortunately, because of the term D of (4.5b), this problem is not exactly soluble as was the case of two particles, *i.e.* $\omega_1 = \omega_2 = \omega$, discussed in reference 4. We note though that the operator in (4.10) contains the frequency ω as a parameter. As this frequency is given in units

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of the rest mass 1 of the particle, we expect ω to be small as compared with 1, as is the case in nuclear physics. We can then begin by disregarding the term in ω^2 and so our equation becomes

$$\mu^2 \left[\mu^2 - (4 + 2A\omega) \right] \phi = 0, \tag{4.11}$$

so our first objective will be to find the eigenstates and eigenvalues of the operator A given by (4.5a).

To achieve our purpose we start by introducing the ket

$$|N(\ell, s)jm\rangle = R_{N\ell}(r) \sum_\sigma \langle \ell m - \sigma, s\sigma | jm \rangle Y_{\ell m - \sigma}(\theta, \varphi) \chi_{s\sigma}, \tag{4.12}$$

where all the functions and symbols are defined as in the paragraph following (2.10), except that now the spin function $\chi_{s\sigma}$ corresponds to $s = 0$ or 1 and not the 1/2 appearing in (2.10).

We note from (4.5) that the operators in (4.10a,b) are invariant under reflections, *i.e.* change of η, ξ into $-\eta, -\xi$ and thus the parity of the states (4.12), which is $(-1)^\ell$, is a good quantum number. Considering then separately the states (4.12) in which $\ell = j$, and those in which $\ell = j \pm 1$, we find by straightforward Racah algebra[11] that the eigenstates of A of (4.5a), which we shall denote by ϕ_0, are

$$\phi_0 \equiv (1/\sqrt{2}) \left[|N(j, 0)jm\rangle \pm |N(j, 1)jm\rangle \right] \text{ for parity } (-1)^j, \tag{4.13a}$$

$$\phi_0 \equiv |N(j \pm 1, 1)jm\rangle \quad \text{for parity } -(-1)^j, \tag{4.13b}$$

The corresponding eigenvalues of μ^2, which we denote by μ_0^2, are given by

$$\mu_0^2 = 4 + 2\omega \left\{ (2N + 3) \pm [j(j + 1)]^{1/2} \right\}, \tag{4.14a}$$

for parity $(-1)^j$, while for parity $-(-1)^j$ we get

$$\mu_0^2 = 4 + 2\omega(2N + 3), \tag{4.14b}$$

and thus we have a complete solution of the problem (4.11).

Our interest though is in the equation (4.13) which we can solve by a perturbation procedure. We first define

$$W \equiv \mu^2, \tag{4.15a}$$

$$H_0 = 4 + 2A\omega, \tag{4.15b}$$

$$H' = \omega^2(A^2 - D^2 - AD + DA), \tag{4.15c}$$

so equation (4.10) becomes

$$(W^2 - WH_0 + H')\phi = 0. \tag{4.16}$$

We then, as for example in Schiff book[12], replace H' by $\lambda H'$ where λ is a parameter and write

$$W = W_0 + \lambda W_1 + \lambda^2 W_2 + ..., \tag{4.17a}$$

$$\phi = \phi_0 + \lambda \phi_1 + \lambda^2 \phi_2 + ..., \tag{4.17b}$$

where $W_0 = \mu_0^2$ of (4.14) and ϕ_0 is given by (4.13).

From (4.17a) we obtain

$$W^2 = W_0^2 + \lambda(2W_0W_1) + \lambda^2(2W_0W_2 + W_1^2) + ..., \qquad (4.17c)$$

so that using (4.17) we see that, up to first order in λ, (4.16) takes the form

$$\left[W_0(W_0 - H_0)\phi_0\right] + \lambda\left[(2W_0W_1 - W_1H_0 + H')\phi_0 + W_0(W_0 - H_0)\phi_1\right] + ... = 0, \qquad (4.18)$$

where each of the square brackets must vanish[12]. For the first one this is automatic as from (4.11) we have

$$H_0\phi_0 = W_0\phi_0. \qquad (4.19)$$

From the second square bracket, when we take its scalar product[12] with ϕ_0, we obtain

$$W_1 = -W_0^{-1}(\phi_0, H'\phi_0), \qquad (4.20)$$

where we made use of the hermitian character of H_0 and of Eq. (4.19).

Thus to first order in perturbation theory, when we take, as usual[12], $\lambda = 1$, we have that

$$\mu^2 = \mu_0^2 - \mu_0^{-2}(\phi_0, H'\phi_0) + ... \qquad (4.21)$$

where μ_0 is given by (4.14), ϕ_0 by (4.13) and H' is (4.15c). In the next section we calculate this μ^2 explicitly.

5 Square of the mass spectra of the particle-antiparticle system

To determine the square of the mass μ^2, given to first order perturbation theory by (4.21), we need to calculate the scalar product $(\phi_0, H'\phi_0)$. As ϕ_0 has a definite number of quanta N, which is indicated in (4.13), we need only to consider that part of H' in (4.15c) that does not change the number of quanta. The terms AD, DA in (4.15) change the number of quanta by ± 2, as indicated in (4.5), so we can disregard them. The A^2 is diagonal in the basis ϕ_0 of (4.13) and its contribution to the scalar product in (4.20) is

$$(\phi_0, A^2\phi_0) = \left\{(2N + 3) \pm [j(j + 1)]^{1/2}\right\}^2 \quad \text{for parity}(-1)^j \qquad (5.1a)$$

$$(\phi_0, A^2\phi_0) = (2N + 3)^2 \quad \text{for parity} - (-1)^j \qquad (5.1b)$$

For the D^2 operator, where D is given by (4.5b), the only terms that contribute to its expectation value with respect to ϕ_0 i.e. that do not change the number of quanta N, are

$$4(\mathbf{S} \cdot \boldsymbol{\eta})^2(\mathbf{S} \cdot \boldsymbol{\xi})^2 + 4(\mathbf{S} \cdot \boldsymbol{\xi})^2(\mathbf{S} \cdot \boldsymbol{\eta})^2 - 2(\mathbf{S} \cdot \boldsymbol{\eta})^2(\boldsymbol{\xi} \cdot \boldsymbol{\xi}) - 2(\mathbf{S} \cdot \boldsymbol{\xi})^2(\boldsymbol{\eta} \cdot \boldsymbol{\eta})$$
$$- 2(\boldsymbol{\eta} \cdot \boldsymbol{\eta})(\mathbf{S} \cdot \boldsymbol{\xi})^2 - 2(\boldsymbol{\xi} \cdot \boldsymbol{\xi})(\mathbf{S} \cdot \boldsymbol{\eta})^2 + (\boldsymbol{\eta} \cdot \boldsymbol{\eta})(\boldsymbol{\xi} \cdot \boldsymbol{\xi}) + (\boldsymbol{\xi} \cdot \boldsymbol{\xi})(\boldsymbol{\eta} \cdot \boldsymbol{\eta}). \qquad (5.2)$$

To evaluate the matrix element $(\phi_0, H'\phi_0)$ we need then to determine the matrix elemtns of $(S \cdot \eta)^2, (S \cdot \xi)^2, (\eta \cdot \eta), (\xi \cdot \xi)$ with respect to states of the form $|N(\ell, s)jm\rangle$ of (4.12). This is done in the Appendix using results for the matrix elements of $(S \cdot \eta), (S \cdot \xi)$ given in reference 13 as well as the operator form of the harmonic oscillator states of reference 14.

We finally arrive at the following results for the μ^2 of (4.21): For the states of parity $(-1)^j, \mu^2$ can be expressed as a function of N, j, ω, using both signs $\pm$ in the ϕ_0 of (4.13a) i.e.

$$\mu_{\pm}^2(N, j, \omega) = 4 + 2\omega\left\{(2N + 3) \pm [j(j + 1)]^{1/2}\right\}$$

$$-(\omega^2/4)\left\{[2N + 3) \pm j^{1/2}(j + 1)^{1/2}]^2 - 2[N(N + 3) - j(j + 1) + 3]\right\} + \dots \quad (5.3)$$

For the states of parity $-(-1)^j$, the two cases of orbital angular momentum $\ell = j + 1$ or $\ell = j - 1$, have to be written separately. We shall distinguish them from (5.3) by putting a bar above the μ^2 and an index $+$ or $-$ when ℓ is respectively $j + 1$ or $j - 1$. Thus we obtain

$$\bar{\mu}_{+}^2 = 4 + 2\omega(2N + 3) - (\omega^2/4)\left\{(2N + 3)^2 - 2[j(j - 1) + N(N + 3) + 1]\right\} + \dots$$

$$(5.4a)$$

$$\bar{\mu}_{-}^2 = 4 + 2\omega(2N + 3) - (\omega^2/4)\left\{(2N + 3)^2 - 2[j(j + 3) + N(N + 3) + 3]\right\} + \dots$$

$$(5.4b)$$

In all of these cases we keep only terms up to ω^2 so that μ_0^{-2} appearing as a coefficient of the scalar product in (4.21) is replaced by $(1/4)$. Note furthermore that as $N = 2n + \ell$ where ℓ is the *orbital* angular momentum, we see that for $\mu_{\pm}^2(N, j, \omega)$ of (5.3) the N takes the values $N = j, j+2, j+4, \dots$. On the other hand for $\bar{\mu}_{+}^2(N, j, \omega)$ of (5.4a) N takes the values $N = j + 1, j + 3, j + 5, \dots$, while for $\bar{\mu}_{-}^2(N, j, \omega)$ of (5.4b) we have $N = j - 1, j + 1, j + 3, j + 5, \dots$.

In Fig. 1 we graph $\mu_{\pm}^2$ of (5.3) for $\omega = 0.1$ as function of j, indicating the sign $\pm$ to which the level corresponds on its left hand side while on the right hand side we give the value of N. Note that when $j = 0$ the state $|N(0, 1)00\rangle$ does not exist so that the normalized ϕ_0 of (4.13a) reduces to $|N(0, 0)00\rangle$. In this case the spin is 0 as indicated int the left hand side of the level $j = 0$, and not the mixture $\pm$ in (4.13a).

In Fig. 2 we graph $\bar{\mu}_{\pm}^2$ of (5.4) for $\omega = 0.1$ as function of j. The orbital angular momentum $\ell = j \pm 1$ is indicated on the left of the levels and the total number of quanta N is given on the right. Note that for $j = 0$, $|N(-1, 1)00\rangle$ does not exist so that we have only $|N(1, 1)00\rangle$ corresponding to $\ell = 1$ indicated on the left hand side of the levels with $j = 0$. Also the levels with $N = \ell = j - 1$ are unique as indicated in the corresponding left hand side of the levels. For the other states $|N(j \pm 1, 1)jm\rangle$ both values $\ell = j + 1$ and $j - 1$ are possible, and because of the first order corrections, i.e. the term in ω^2 of (5.4) they are separated by $\omega^2(2j + 1)$. In Fig. 2 we give on the left hand side the values $\ell = j + 1$ and $\ell = j - 1$ of the paired levels.

The parity denoted by a script $\mathcal{P}$ is given in all three figures i.e. $\mathcal{P} = (-1)^j$ or $\mathcal{P} = -(-1)^j$.

In the next section we discuss the comparison of our reuslts with squares of the mass spectra for non-strange mesons[1].

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6 Comparison with the meson spectra

In page 37 of reference 1 there is Meson Summary Table in which, in the first two columns, are given the non-strange mesons with the code name (a single letter with an index indicating the total angular momentum j), mass μ in MeV, j, isospin I, and parity defined there[1] as $P = -(-1)^\ell$, as well as a charge conjugation number $C = (-1)^{\ell+s}$, where ℓ is the orbital angular momentunm and s the total spin of the quark-antiquark system.

From the above information we can get the square of the mass μ^2 in units $(GeV)^2$, the j, s, I as well as $(-1)^\ell$, so that, $\ell = j$ or $\ell = j \pm 1$. As mentioned before our notation for parity will be the script $\mathcal{P} = (-1)^\ell$ i.e. $\mathcal{P} = (-1)^j$ or $-(-1)^j$.

The information given in the previous paragraph is summarized in Figs. 3 to 6, where in the abscissa we have $j = 0, 1, 2, 3, 4, 5$ and μ^2 in the ordinate. We note first that for comparison with the particle-antiparticle system, with a Dirac oscillator interaction, we have divided the information according to parity $\mathcal{P} = (-1)^j$ or $-(-1)^j$ in Figs. 3,4 or Figs. 5,6 as $\mathcal{P}$ is an integral of motion of our problem. Furthermore the isospin $I = 0, 1$ is completely independent from the Poincaré group and thus of the analysis previous sections. We could then consider that in our theoretical $\mu_\pm^2$ of (5.3) or $\bar{\mu}_\pm$ of (5.4), ω is a function of I and another one could be added to these $\mu_\pm, \bar{\mu}_\pm$ variables. Thus it is convenient to graph separately the levels for $I = 1, \mathcal{P} = (-1)^j$ (Fig. 3); $I = 0, \mathcal{P} = (-1)^j$ (Fig. 4); $I = 1, \mathcal{P} = -(-1)^j$ (Fig. 5); and $I = 0, \mathcal{P} = -(-1)^j$ (Fig. 6).

In figures 3 to 6 we put on the left hand side the name of the meson and where there are several of the same name we distinguish them by primes i.e. π_j, π'_j, π''_j etc. In Figs. 3,4 we put on the right hand side the total spin $s = 0$ or 1 and furthermore we differentiate the two values by using either a full or dashed line. In Figs. 5,6 only the names of the mesons appear on the levels as the spin s is always 1.

We now wish to make a qualitative comparison between the theoretical figures 1 to 2 and the experimental ones for mesons in figures 3 to 6.

We begin with parity $\mathcal{P} = (-1)^j$ where we should compare Fig. 1 with Figs. 3,4. For angular momentum $j = 0$ the comparison in Figs. 1 and 3 is quite good and the spin is $s = 0$. For all other levels we cannot compare because in Fig. 3 they have definite spin, while in Fig. 1 we have 50% each of admixtures of $s = 0$ and $s = 1$, in or out of phase, as shown in (4.13a). This clearly shows the need to add other interactions in our equation (3.14) that are Poincaré invariant, which we shall discuss below. Note also that even for $j = 0$ the comparison between Fig. 1 and Fig. 4 is not good, even if we change the scale in Fig. 1, but this can be attributed to the fact that the η_0 mesons may have and admixture of $s\bar{s}$ where s has different mass from u, d.

Turning now our attention to parity $\mathcal{P} = -(-1)^j$ we could compare Fig. 2 with Fig. 5 or 6. In Fig. 5 i.e. $I = 1$ the information is sparse though for $j = 1$ we see what may be a pair ρ', ρ'' which, with a change of scale, could resemble the pair for $j = 1, N = 2$ in Fig. 2. This is more noticeable when we compare, for $j = 2$, Fig. 2 with Fig. 6, where we see a number of pairs i.e. $N = 3$ corresponding f'_2, f''_2; $N = 5$ to f'''_2, f_2^{IV}; and if we had graphed it, $N = 7$ to f_2^V, f_2^{VI}; $N = 9$ to f_2^{VII}, f_2^{VIII}. Note also that for parity $\mathcal{P} = -(-1)^j$ the lowest level in Figs. 5, 6 appears for $j = 1$ and not $j = 0$, and this is also true in the theoretical calculations of Fig. 2.

Clearly though in all cases we would have to modify our starting hypothesis in a similar way as we did in the baryon example[7]. We want to do this in a Poincaré invariant way and thus it is

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convenient to introduce the four vector[7)]

$$W_\mu = \frac{1}{2}\mathcal{E}_{\mu\nu\sigma\tau}P^\nu K^{\sigma\tau},$$ (6.1)

where $K^{\sigma\tau}$ could be either $J^{\sigma\tau}, L^{\sigma\tau}, S^{\sigma\tau}$ defined respectively by

$$J^{\sigma\tau} = L^{\sigma\tau} + S^{\sigma\tau},$$ (6.2a)

$$L^{\sigma\tau} = \sum_{s=1}^{2}(x_s^\sigma p_s^\tau - x_s^\tau p_s^\sigma),$$ (6.2b)

$$S^{\sigma\tau} = (i/4)\sum_{s=1}^{2}(\gamma_s^\sigma \gamma_s^\tau - \gamma_s^\tau \gamma_s^\sigma),$$ (6.2c)

where $x_s^\tau, p_s^\tau, \gamma_s^\tau; \tau = 0,1,2,3; s = 1,2$ are the ones in section 3 of this paper with $n = 2$.

We consider now the Poincaré invariants

$$W^2 = W^\mu W_\mu \quad , \quad P^2 = -P_\mu P^\mu,$$ (6.3a, b)

as well as the Γ of (3.5a), which we combine in the form

$$\Gamma(W^2/P^2),$$ (6.4)

and in the center of mass frame $i.e.$ when $P_i = 0, i = 1,2,3$, it reduces to[7)]

$$J^2, L^2, S^2,$$ (6.5a, b, c)

depending on whether $K^{\sigma\tau}$ is equal to $J^{\sigma\tau}, L^{\sigma\tau}, S^{\sigma\tau}$.

Thus, as indicated in Eq. (2.9a) of reference 7, our equation (3.16) could be modified to

$$\left\{ -P^0 + \mathcal{M} + aJ^2 + bL^2 + cS^2 \right\}\psi = 0$$ (6.6)

where $\mathcal{M}$ is given by (3.18) and a, b, c are, so far, arbitrary real constants.

If $b = c = 0$, as J^2 is an integral of motion of the operator $\mathcal{M}$, we have that the new mass, which we call μ', is related with the old one by

$$\left[\mu' - aj(j+1)\right]^2 = \mu^2,$$ (6.7)

and thus the new mass spectra is

$$\mu' = \mu + aj(j+1),$$ (6.8)

where we have at our disposal the parameter a with which we can adjust the spectra corresponding to different $j's$.

When we have b, c also different from zero, as L^2, S^2 are not integrals of motion, our only way to proceed is by considering the matrix of the operator

$$\mathcal{M} + aJ^2 + bL^2 + cS^2,$$ (6.9)

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with respect to the states $|N(\ell,s)jm\rangle$, where j is fixed, and diagonalizing this matrix up to certain maximum number of quanta N_{max}.

Calculations of this type, done by Luis Benet, when $a = b = 0$, *i.e.* only with an S^2 term, allow us to break the wave functions of (4.13a) into states of definite spin *i.e.* $s = 0$ or 1, but keeping some of the ordering as in Fig. 1, so that its more comparable with the meson spectra in which the spin is given.

We do not wish though to consider the more general operator (6.9) in this paper. To begin with, when dealing only with the operator $\mathcal{M}$ of (3.18), we have a single parameter, the frequency ω, and also as a scale in our calcualtions the mass m of the quarks. If we go to the operator (6.9) we have the parameters ω, a, b, c, plus the m. Thus we can of course adjust the meson spectra better, but it may be meaningless.

Thus we conclude by stating that the particle-antiparticle system with a Dirac oscillator interaction, may give some insight on the meson spectra, and that is all we aspire to achieve in the present paper.

7   Appendix

To obtain the matrix elements of $(\phi_0, D^2\phi_0)$ we need first those of the factors in each product appearing in (5.2). Those of $(\mathbf{S}\cdot\boldsymbol{\eta})^2$ can be obtained from the ones of $(\mathbf{S}\cdot\boldsymbol{\eta})$ given in (3.20) and (3.24) of reference 13, and thus we have

$$\langle N + 2(j,1)jm|(\mathbf{S}\cdot\boldsymbol{\eta})^2|N(j,1)jm\rangle = -\left[(N+j+3)(N+2-j)\right]^{1/2}$$
$$(A.1a)$$

$$\langle N + 2(j+1,1)jm|(\mathbf{S}\cdot\boldsymbol{\eta})^2|N(j+1,1)jm\rangle = -\left[(N+j+4)(N+1-j)\right]^{1/2}\left[j/(2j+1)\right]$$
$$(A.1b)$$

$$\langle N + 2(j-1,1)jm|(\mathbf{S}\cdot\boldsymbol{\eta})^2|N(j+1,1)jm\rangle = \left[(N-j+3)(N+1-j)\right]^{1/2}\left[j(j+1)\right]^{1/2}(2j+1)^{-1}$$
$$(A.1c)$$

$$\langle N + 2(j+1,1)jm|(\mathbf{S}\cdot\boldsymbol{\eta})^2|N(j-1,1)jm\rangle = \left[(N+j+2)(N+j+4)\right]^{1/2}\left[j(j+1)\right]^{1/2}(2j+1)^{-1}$$
$$(A.1d)$$

$$\langle N + 2(j-1,1)jm|(\mathbf{S}\cdot\boldsymbol{\eta})^2|N(j-1,1)jm\rangle = -\left[(N+j+2)(N-j+3)\right]^{1/2}\left[(j+1)/(2j+1)\right].$$
$$(A.1e)$$

From the hermitian conjugates of (5.1) we obtain those of $(\mathbf{S}\cdot\boldsymbol{\xi})^2$ i.e.

$$\langle N - 2(j,1)jm|(\mathbf{S}\cdot\boldsymbol{\xi})^2|N(j,1)jm\rangle = -\left[(N+j+1)(N-j)\right]^{1/2}$$
$$(A.2a)$$

$$\langle N - 2(j+1,1)jm|(\mathbf{S}\cdot\boldsymbol{\xi})^2|N(j+1,1)jm\rangle = -\left[(N+j+2)(N-j-1)\right]^{1/2}\left[j/(2j+1)\right]$$
$$(A.2b)$$

$$\langle N - 2(j+1,1)jm|(\mathbf{S}\cdot\boldsymbol{\xi})^2|N(j-1,1)jm\rangle = \left[(N-j+1)(N-j-1)\right]^{1/2}\left[j(j+1)\right]^{1/2}(2j+1)^{-1}$$
$$(A.2c)$$

$$\langle N - 2(j-1,1)jm|(\mathbf{S}\cdot\boldsymbol{\xi})^2|N(j+1,1)jm\rangle = \left[(N+j)(N+j+2)\right]^{1/2}\left[j(j+1)\right]^{1/2}(2j+1)^{-1}$$
$$(A.2d)$$

$$\langle N - 2(j-1,1)jm|(\mathbf{S}\cdot\boldsymbol{\xi})^2|N(j-1,1)jm\rangle = -\left[(N+j)(N-j+1)\right]^{1/2}\left[(j+1)/(2j+1)\right].$$
$$(A.2e)$$

Finally, from the operator form of the harmonic oscillator states, given in reference 14, we have that

$$\langle N + 2(\ell,s)jm|\boldsymbol{\eta}\cdot\boldsymbol{\eta}|N(\ell,s)jm\rangle = -\left[(N+\ell+3)(N-\ell+2)\right]^{1/2}, \qquad (A.3a)$$

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and from its hermitian conjugate we get

$$\langle N - 2(\ell, s)jm|\boldsymbol{\xi} \cdot \boldsymbol{\xi}|N(\ell, s)jm\rangle = -\left[(N + \ell + 1)(N - \ell)\right]^{1/2}. \qquad (A.3b)$$

With the help of these expressions we obtain straighforwardly the $\mu_\pm^2$ of (5.3) and $\bar{\mu}_\pm^2$ of (5.4).
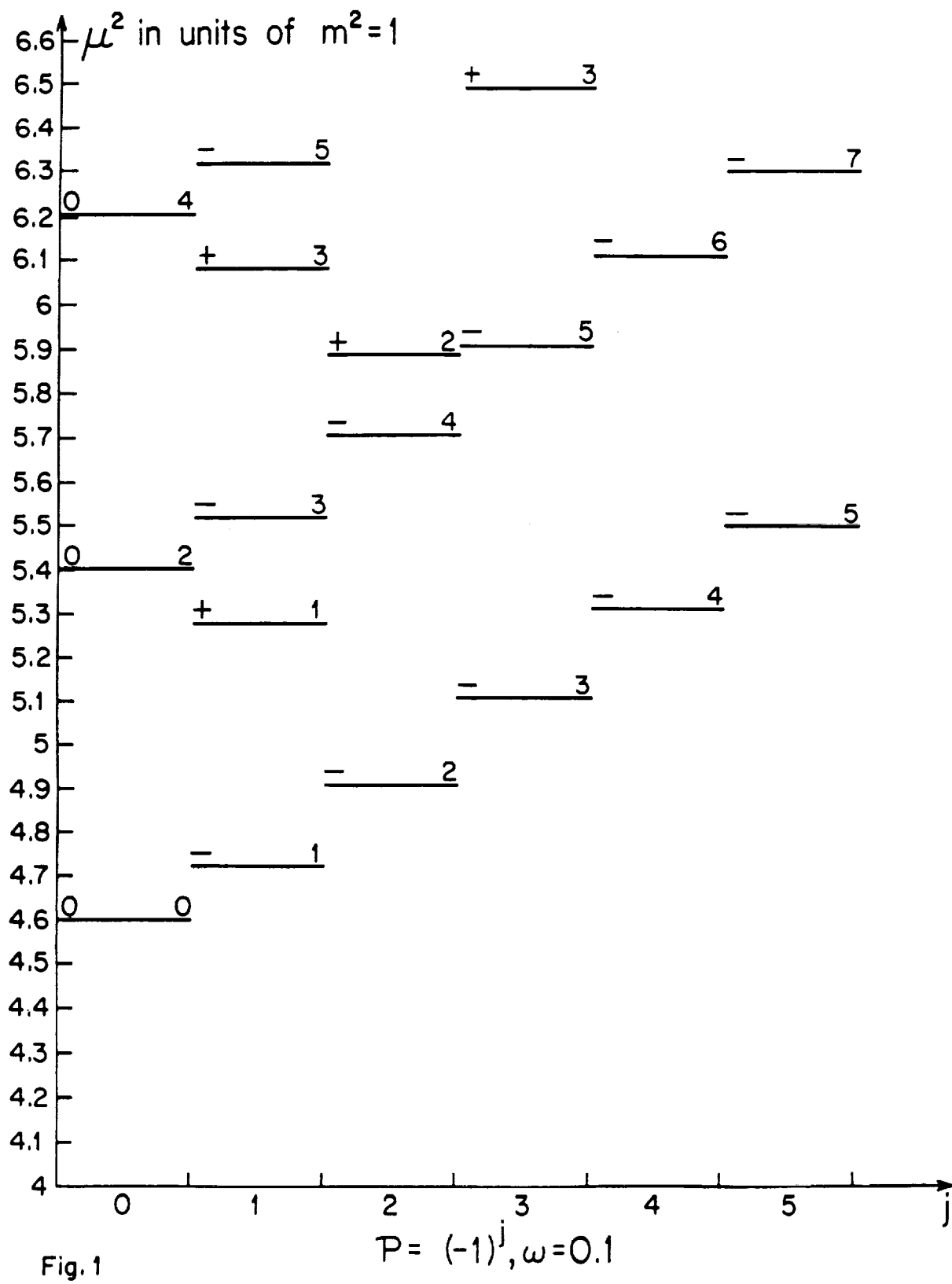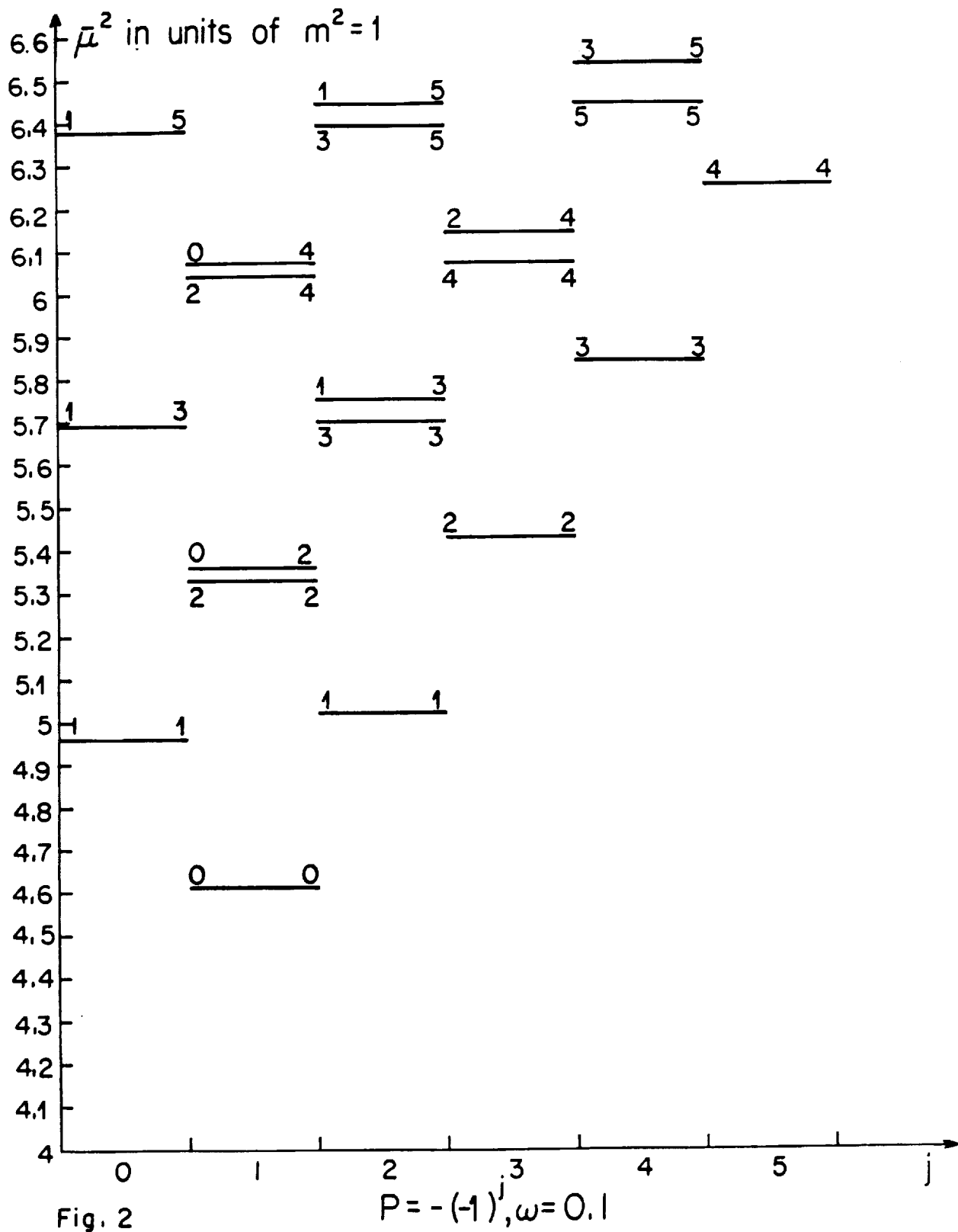
8 Acknowledgements

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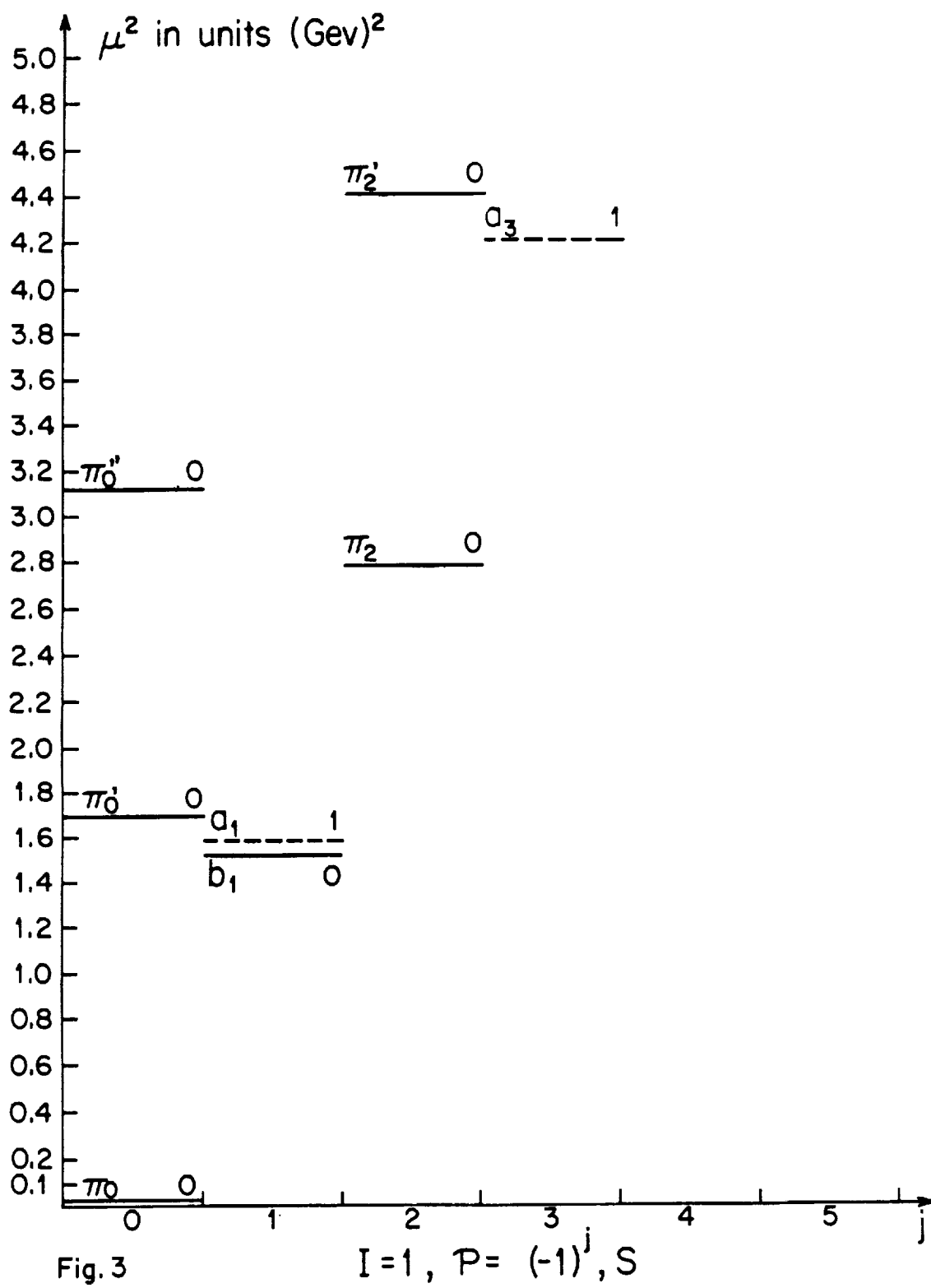
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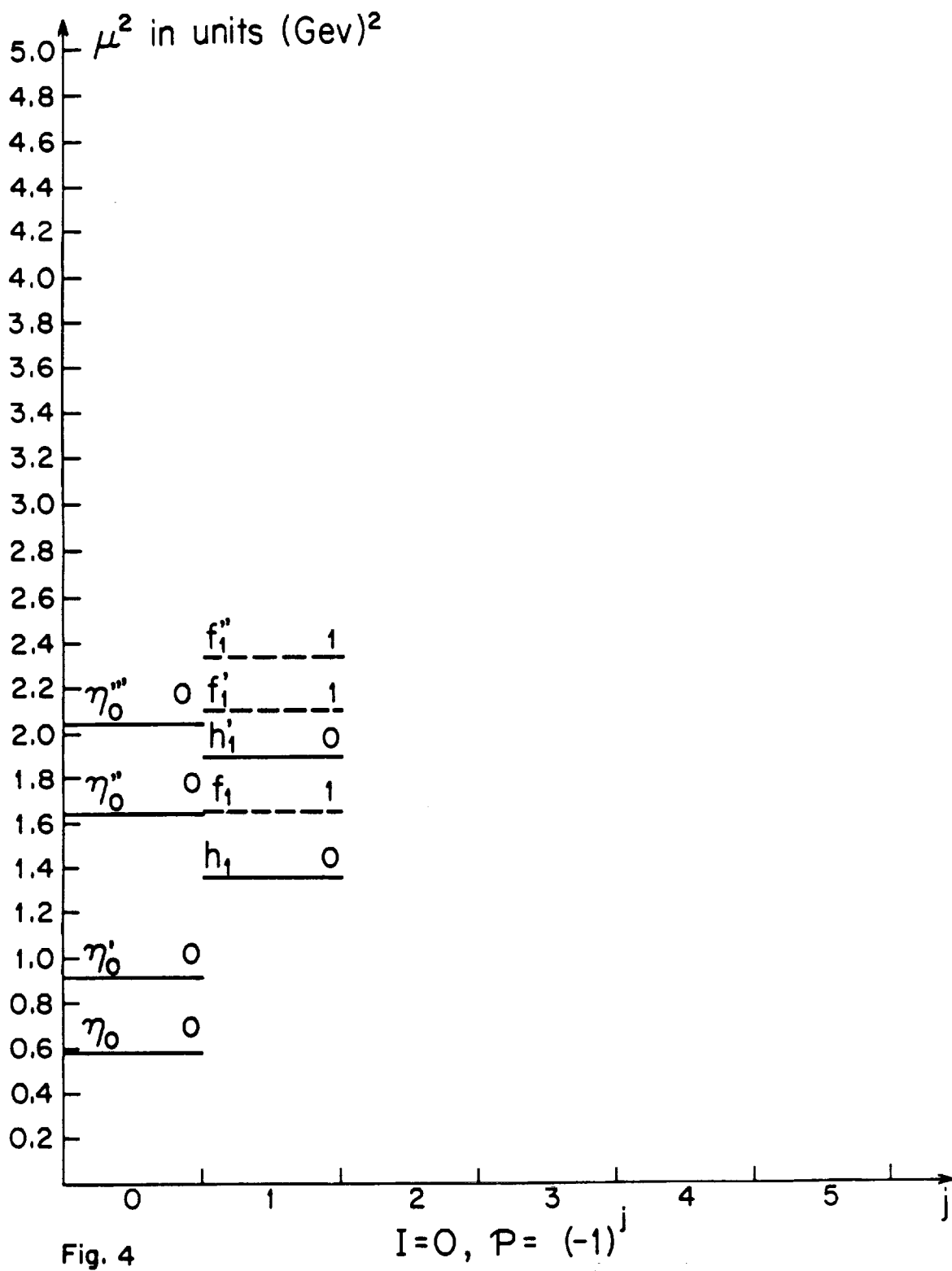
9 Figure Captions

- Fig. 1. We graph $\mu_{\pm}^2$ of (5.3) for $\omega = 0.1$ as function of j, indicating the sign $\pm$ to which the level corresponds on its left hand side while on the right hand side we give the value of N. Note that when $j = 0$ the state $|N(0,1)00 >$ does not exist so that the normalized ϕ_o of (4.13a) reduces to $|N(0,0)00 >$. In this case the spin is 0 as indicated in the left hand side of the level $j = 0$, and not the mixture $\pm$ in (4.13a).

- Fig. 2. We graph $\bar{\mu}_{\pm}^2$ of (5.4) for $\omega = 0.1$ as function of j. The orbital angular momentum $\ell = j \pm 1$ is indicated on the left of the levels and the total number of quanta N is given on the right. Note that for $j = 0, |N(-1,1)00 >$ does not exist so that we have only $|N(1,1)00 >$ that corresponds to $\ell = 1$ indicated on the left hand side of the levels with $j = 0$. Also the levels with $N = \ell = j - 1$ are unique as indicated in the corresponding left hand side of the levels. For the other states $|N(j \pm 1,1)jm >$ both values $\ell = j + 1$ and $j - 1$ are possible, and their separation is given by $\omega^2(2j + 1)$, thus both values $j - 1, j + 1$ appear on the left hand side of those levels.

- Fig. 3. The square of the masses of the mesons are given as functions of j for isospin $I = 1$ and parity $\mathcal{P} = (-1)^j$. The name of the meson is given on the left hand side with an index j and upper primes if there are several of them. Full lines correspond to spin 0 and dashed to spin 1 as indicated on the right hand side.

- Fig. 4. The square of the masses of the mesons are given as function of j for isospin $I = 0$ and parity $\mathcal{P} = (-1)^j$. The name of the meson is given on the left hand side with an index j and upper primes if there are several of them. Full lines correspond to spin 0 and dashed to spin 1 as indicated on the right hand side.

- Fig. 5. The square of the masses of the mesons are given as function of j for isospin $I = 1$ and parity $\mathcal{P} = -(-1)^j$. The name of the meson is given on the left hand side with an index j and upper primes if there are several of them. The spin is always 1.

- Fig. 6. The square of the masses of the mesons are given as function of j for isospin $I = 0$ and parity $\mathcal{P} = -(-1)^j$. The name of the meson is given on the left hand side with an index j and upper primes if there are several of them. The spin is always 1.

Fig. 1

$\mathcal{P} = (-1)^j, \omega = 0.1$

μ^2 in units of $m^2 = 1$

Fig. 2

$P = - (-1)^{j}, \omega = 0.1$

Fig. 3 $I = 1$, $\mathcal{P} = (-1)^j$, S

Fig. 4

$I = 0$, $\mathcal{P} = (-1)^j$

Fig. 5 $\quad I = 1 , \quad \mathcal{P} = -(-1)^{j} \quad\quad S = 1 , \quad \ell = j \pm 1$
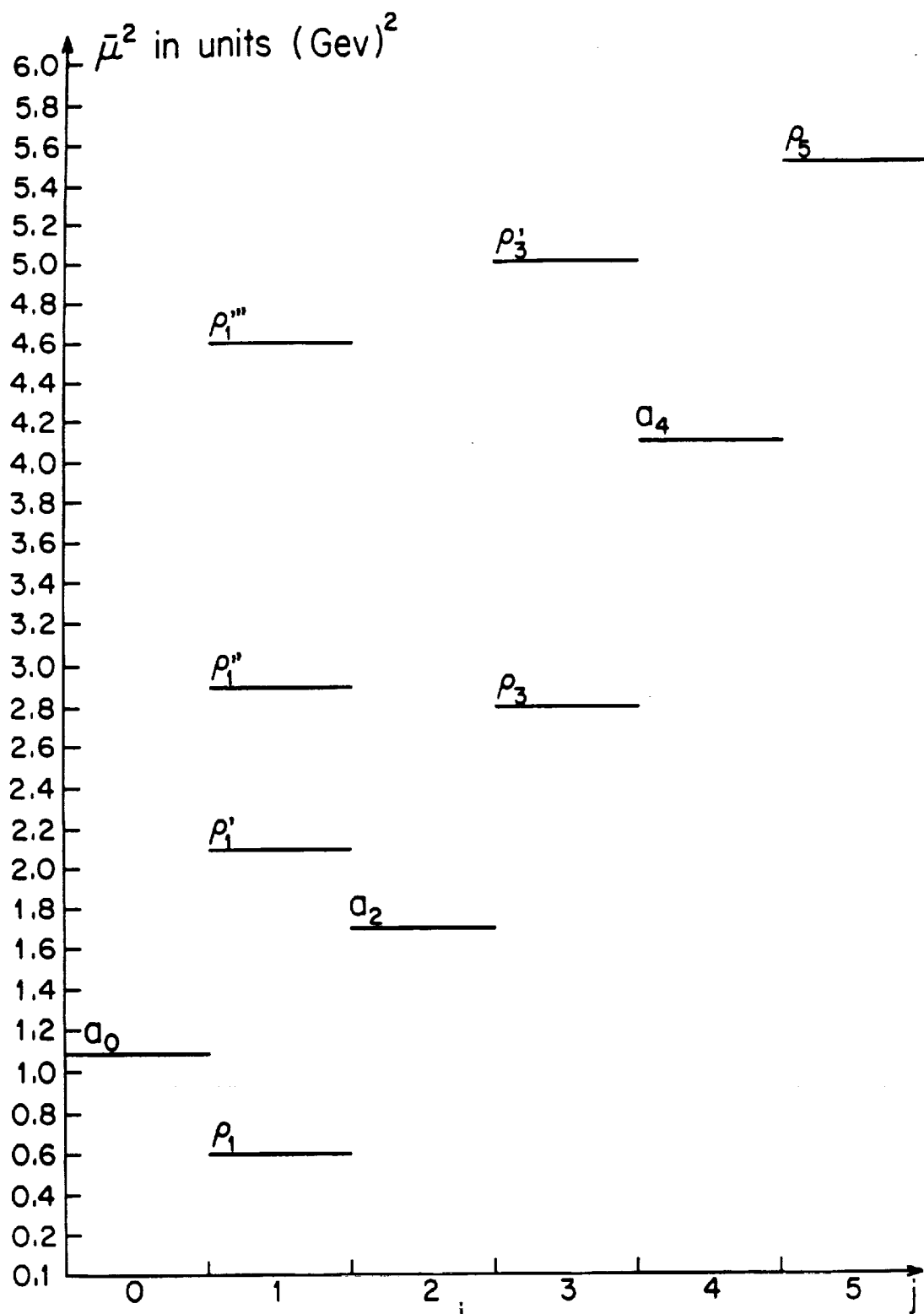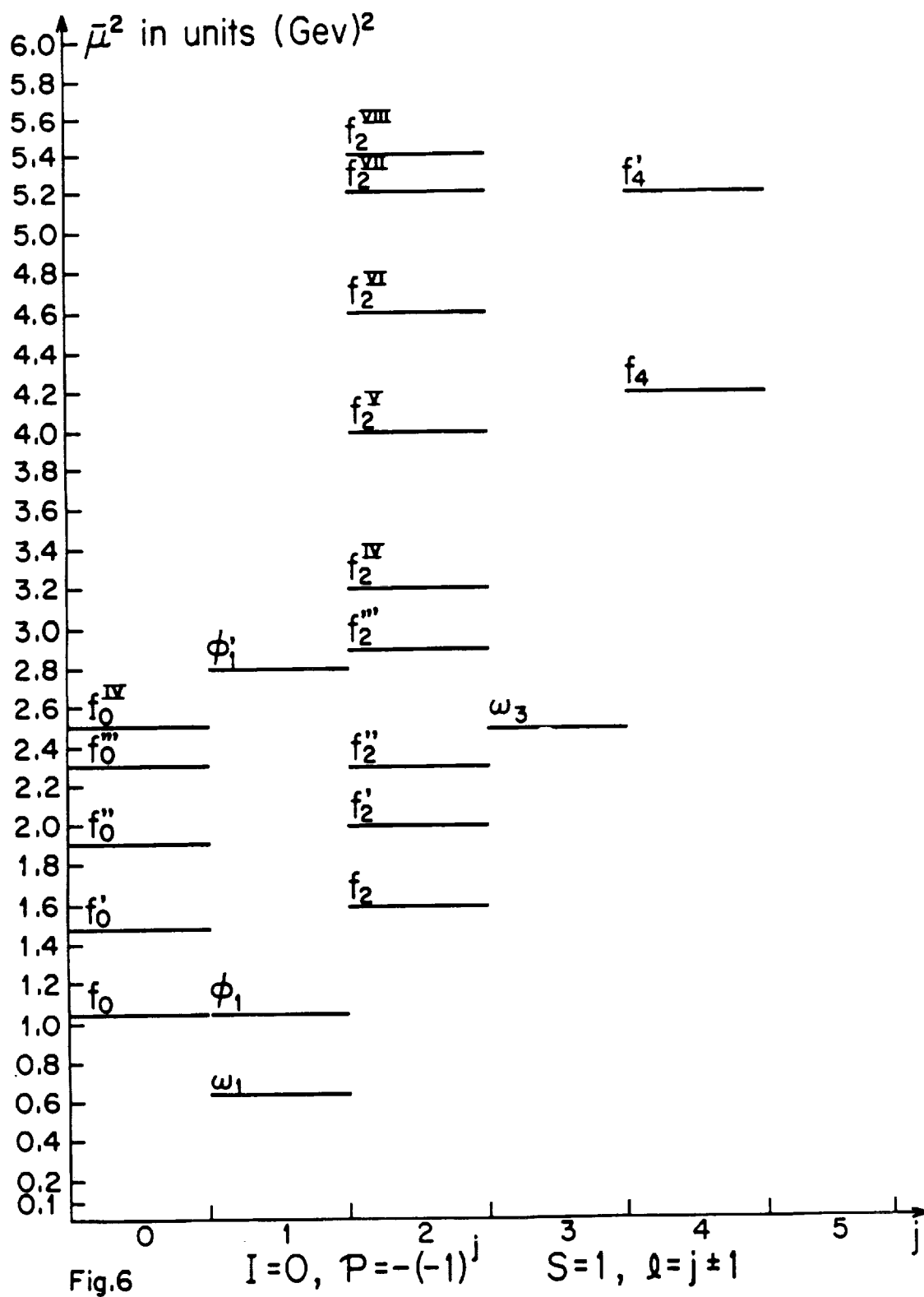
Fig.6 $I = 0$, $\mathcal{P} = -(-1)^j$ $S = 1$, $\ell = j \pm 1$

FROM HARMONIC TO ANHARMONIC OSCILLATORS

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Abstract

The algebraic approach to quantum mechanics is briefly reviewed. The role of oscillator realizations is discussed. Applications to vibrations of complex molecules are presented.

1 Introduction

In recent years, a formulation of quantum mechanics, called algebraic theory, has been put forward, in which any quantum mechanical problem is mapped onto an algebraic structure following the logic scheme shown in Fig. 1.

Quantum Mechanical System

⇓

Algebraic structure
$\begin{cases} \text{Lie algebras} \\ \text{Graded Lie algebras} \\ \text{Infinite dimensional (Kac-Moody) algebras} \\ \text{q-deformed (Hopf) algebras} \\ \dots \end{cases}$

⇓

Observables
$\begin{cases} \text{Energies} \\ \text{Transition rates} \\ \dots \end{cases}$

⇓

Experiment

Fig.1. Logic scheme of algebraic theory.

In implementing algebraic theory, it has been found to be very useful to make

use of oscillator representations. In this contribution, I will briefly review the use of oscillators in algebraic theory.

2   Oscillators in ν dimensions

I begin with the (trivial) example of the one-dimensional harmonic oscillator. In the algebraic theory this case is described by the introduction of the Heisenberg algebra [1]

$$H(2) \; : \; a, \; a^\dagger, \; 1, \; a^\dagger a \tag{2.1}$$

Table I shows the parallelism between the usual treatment in terms of differential operators (Schrödinger equation) and the algebraic approach. This case is well known and does not require further explanation.

I consider instead the (non-trivial) example of the one-dimensional anharmonic Morse oscillator. The differential approach requires the solution of the eigenvalue problem

$$H \psi = E \psi \quad ,$$

$$H = - \frac{\hbar^2}{2\mu} \frac{d^2}{dx^2} + V(x) \quad ,$$

$$V(x) = D[1 - \exp(-\beta x)]^2 \quad . \tag{2.2}$$

The solution of the eigenvalue problem produces wave functions

$$\psi_v(x) = N_v \, z^{\eta - v} \, e^{-\frac{z}{2} + \frac{1}{2} x\beta} \, L_v^{2\eta - 2v - 1}(z) \tag{2.3}$$

where N_v is a normalization and $L(z)$ denotes a Laguerre polynomial. Also

Table I. Differential and algebraic treatment of the one dimensional harmonic oscillator.

Differential approach	Algebraic approach
$H = \frac{1}{2} (p^2 + x^2) =$	$a = \frac{1}{\sqrt{2}} (x + \frac{d}{dx})$
$= \frac{1}{2} \left[-\frac{d^2}{dx^2} + x^2 \right]$	$a^\dagger = \frac{1}{\sqrt{2}} (x - \frac{d}{dx})$
$H \psi_n = E_n \psi_n$	$[a, a^\dagger] = 1$
	$H = (a^\dagger a + \frac{1}{2})$
$E_n = (n + \frac{1}{2})$	$E_n = (n + \frac{1}{2})$
$u_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2} x^2}$	$\|0>$
$u_n(x) = \left[\pi^{-\frac{1}{2}} 2^n n! \right]^{-\frac{1}{2}} (x - \frac{d}{dx})^n e^{-\frac{1}{2} x^2}$	$\|n> = (n!)^{-\frac{1}{2}} (a^\dagger)^n \|0>$
$I_{nn'} = \int_{-\infty}^{+\infty} u_{n'}(x) \, f(x, \frac{d}{dx}) \, u_n(x) \, dx$	$I_{nn'} = <n' \| f(a, a^\dagger) \| n>$

$$z = 2\eta \, e^{-\beta x} \quad ; \quad \eta = \frac{1}{\hbar \beta} \sqrt{2\mu D} \quad ; \quad v = 0, 1, \ldots, \eta - \frac{1}{2} \qquad (2.4)$$

The eigenvalues are

$$E(v) = 2\hbar\beta\sqrt{\frac{D}{2\mu}} \ (v + \frac{1}{2}) - \frac{1}{2} \frac{\hbar^2\beta^2}{\mu} \ (v + \frac{1}{2})^2 \quad . \tag{2.5}$$

The mass μ, strength of interaction D and range β have been put explicitly in Eqs. (2.2)-(2.5), while they were deleted in Table I.

In algebraic theory, the one-dimensional Morse oscillator can be dealt with by introducing [2] the Lie algebra U(2), composed of four elements F_+, F_-, F_0, N. The Hamiltonian can be written as

$$H = AC \quad ; \quad C = F_0^2 - N^2 \quad , \tag{2.6}$$

where C is the Casimir operator of the O(2) subalgebra of U(2). The eigenvalues are

$$E(m) = A(m^2 - N^2) \quad , \ m = N, N-2, \ldots, 1 \text{ or } 0 \ (N\text{=odd or even}). \tag{2.7}$$

With the change of variable $v = (N-m)/2$ one has

$$E(v) = - 4A(Nv - v^2) \quad ,$$

$$v = 0, 1, \ldots, \frac{N}{2} \text{ or } \frac{N}{2} - \frac{1}{2} \qquad (N\text{= even or odd}) \quad , \tag{2.8}$$

which are the eigenvalues of the Morse oscillator, Eq. (2.5). The eigenstates can be written as

$$\left| \begin{array}{ccc} U(2) & \supset & O(2) \\ \downarrow & & \downarrow \\ N & & v \end{array} \right\rangle \tag{2.9}$$

and intensities can be computed by taking matrix elements of operators

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$$\langle N,v'|\hat{T}|N,v\rangle \qquad\qquad (2.10)$$

As a result, all calculations for anharmonic oscillators can be done very easily.

An oscillator realization of U(2) is provided by the Jordan-Schwinger construction in terms of two boson operators σ, τ and their adjoints $\sigma^\dagger, \tau^\dagger$. The algebra is

$$U(2) : \tau^\dagger\sigma, \ \sigma^\dagger\tau, \ \tau^\dagger\tau, \ \sigma^\dagger\sigma \quad . \qquad\qquad (2.11)$$

Incidentally, in the oscillator realization the harmonic oscillator appears as a contraction of the anharmonic oscillator, obtained by letting

$$N \to \infty \qquad , \qquad N_\sigma \to \infty \quad ,$$

$$\tau^\dagger\sigma \to \sqrt{N_\sigma}\ \tau^\dagger \quad , \qquad \sigma^\dagger\tau \to \sqrt{N_\sigma}\ \tau \quad ,$$

$$\tau^\dagger\tau \to \tau^\dagger\tau \quad , \qquad \sigma^\dagger\sigma \to N_\sigma \quad . \qquad\qquad (2.12)$$

Thus, by adding one extra dimension (with the constraint N=const) one can treat, within the same framework, both harmonic and anharmonic oscillators. The anharmonic Morse oscillator in one dimension is related to the harmonic oscillator in two dimensions.

The same situation occurs in any number of dimensions. For example, in three dimensions, one introduces four boson operators [3,4]

$$b_\alpha^\dagger = \sigma^\dagger \ , \ \pi_\mu^\dagger \ (\mu=0,\pm1) \quad ; \quad \alpha=1,\ldots,4 \quad ;$$

$$b_\alpha = \sigma \ , \ \pi_\mu \ (\mu=0,\pm1) \quad ; \quad \alpha=1,\ldots,4 \quad ; \qquad (2.13)$$

divided into a scalar σ and a vector π_μ. The bilinear products $b_\alpha^\dagger b_\beta$ generate the Lie algebra U(4)

$$\mathcal{G} \ : \qquad G_{\alpha\beta} = b_\alpha^\dagger \, b_\beta \quad . \qquad\qquad\qquad (2.14)$$

The contracted form of U(4) is the oscillator algebra in three dimensions, H(4).

In three dimensions the situation is even richer than in one dimension, since the algebra of U(4) can be reduced in two ways:

$$
\text{U(4)}
\begin{cases}
\nearrow & \text{U(3)} \supset \text{O(3)} \supset \text{O(2)} \quad , \quad \text{(I)} \quad , \\
\searrow & \text{O(4)} \supset \text{O(3)} \supset \text{O(2)} \quad , \quad \text{(II)} \quad ,
\end{cases}
\qquad (2.15)
$$

corresponding to spherical (I) and deformed situations (II). In one dimension we have

$$
\text{U(2)}
\begin{cases}
\nearrow & \text{U(1)} \\
\searrow & \text{O(2)}
\end{cases}
\qquad\qquad\qquad (2.16)
$$

but U(1)≈O(2) and therefore the spherical and deformed coincide.

It has been suggested [3] that in general any quantum mechanical problem in ν space dimensions can be written in terms of the unitary algebra U(ν+1). The harmonic oscillator in ν dimensions can be obtained from U(ν+1) by a limiting procedure leading to H(ν+1). The Heisenberg algebra H(ν+1) contains U(ν), the degeneracy algebra of the ν dimensional harmonic oscillator. The anharmonic oscillator and the deformed anahrmonic roto-oscillator can all be obtained from U(ν+1). These results allow one to do <u>anharmonic</u> analysis of spectral problems in a way as simple as that of <u>harmonic</u> analysis.

3 Coupled oscillators in ν dimensions

In most problems in physics, one often encounters coupled systems. In algebraic theory, the coupling of physical systems corresponds to the coupling of algebras. Oscillator realizations are particularly useful here and, as a simple example, I will discuss the case of coupled one-dimensional anharmonic oscillators. The algebraic structure of the system is the direct sum of the individual U(2) algebras

$$\mathcal{G} = \sum_{i=1}^{n} \oplus U_i(2) \quad , \tag{3.1}$$

where the sum extends over the number of oscillators, n. An oscillator realization can be done in terms of boson operators $\tau^\dagger_i$, $\sigma^\dagger_i$, τ_i, σ_i. Each algebra $U_i(2)$ is

$$U_i(2) : \tau^\dagger_i \sigma_i, \ \sigma^\dagger_i \tau_i, \ \tau^\dagger_i \tau_i, \ \sigma^\dagger_i \sigma_i \quad . \tag{3.2}$$

Coupled harmonic oscillators can be obtained as before by eliminating the σ_i bosons, as in Eq. (2.12).

In the last year, algebraic models of coupled anharmonic oscillators have been used extensively in order to provide a realistic description of the vibrations of complex molecules [5]. In general, the algebraic Hamiltonian of coupled oscillators is written as

$$H = \sum_{i=1}^{n} h_i + \sum_{i<j=1}^{n} V_{ij} \tag{3.3}$$

where

$$h_i = A_i \ C_i \tag{3.4}$$

The operators C_i are the Casimir operators of the $O_i(2)$ algebras and h_i has eigenvalues

$$\epsilon_i = A_i(m_i^2 - N_i^2) = -4A_i(N_i v_i - v_i^2) \tag{3.5}$$

The couplings V_{ij} depend on the problem under consideration. Two types of

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couplings are usually considered: (i) diagonal couplings (Casimir couplings) and (ii) non-diagonal couplings (Majorana couplings). In the product basis, labelled by the quantum numbers of each $U_i(2) \supset O_i(2)$ algebra, the matrix elements of the Casimir couplings are given by

$$< N_i v_i; N_j, v_j \mid C_{ij} \mid N_i, v_i; N_j, v_j > =$$

$$= 4\left[(v_i + v_j)^2 - (v_i + v_j)(N_i + N_j)\right] - \left(\frac{N_i + N_j}{N_i}\right)4\left[v_i^2 - v_i N_i\right] - \left(\frac{N_i + N_j}{N_j}\right)4\left[v_j^2 - v_j N_j\right] ,$$

$$(3.6)$$

while the matrix elements of the Majorana couplings are given by:

$$<N_i, v_i; N_j, v_j \mid M_{ij} \mid N_i, v_i; N_j, v_j > = v_i N_j + v_j N_i - 2v_i v_j ,$$

$$<N_i, v_i+1; N_j, v_j-1 \mid M_{ij} \mid N_i, v_i; N_j, v_j > = - \sqrt{v_j(v_i+1)(N_i - v_i)(N_j - v_j + 1)} ,$$

$$<N_i, v_i-1; N_j, v_j+1 \mid M_{ij} \mid N_i v_i; N_j v_j > = - \sqrt{v_i(v_j+1)(N_j - v_j)(N_i - v_i + 1)} .$$

$$(3.7)$$

As an example of application of these models consider the case of the benzene molecule, C_6H_6, (Fig.2).

This molecule has 12 atoms and thus 36-6=30 independent vibrations. A conventional treatments of this molecule in terms of coupled differential equations is rather complicated. On the other side, an algebraic treatment is feasible, since the Hamiltonian, expressed in terms of algebraic operators, can be easily diagonalized.

In view of the hexagonal geometry of benzene, in the coupling terms $\sum_{ij} V_{ij}$, one can have three types of couplings: (I) first neighbor couplings, (II) second neighbor couplings and (III) third neighbor couplings. The algebraic Hamiltonian appropriate to benzene can be written as
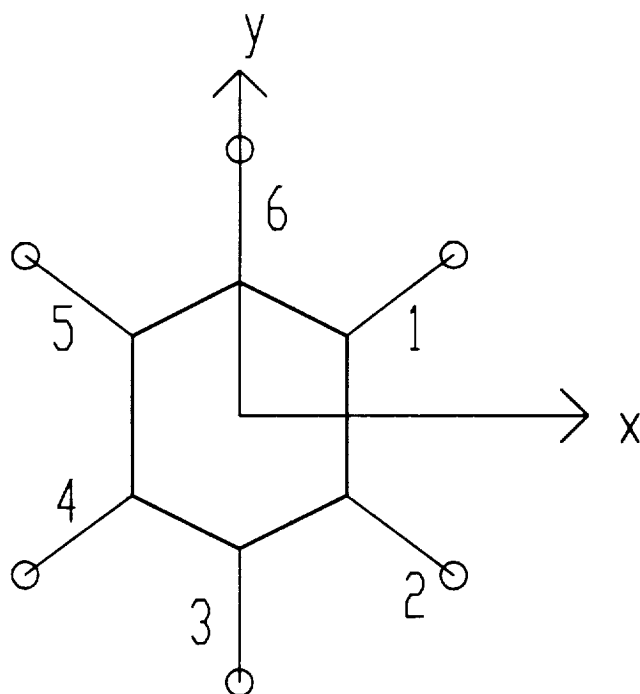
Fig.2. The benzene molecule.

$$H = A_H C + A_{HH} \ C' + \lambda_{HH}^{(I)} \ S^{(I)} + \lambda_{HH}^{(II)} \ S^{(II)} + \lambda_{HH}^{(III)} \ S^{(III)} \quad , \quad (3.8)$$

where

$$A_i = A_{HH} \quad ; \quad A_{ij} = A_{HH} \quad ; \quad \lambda_{ij} = \lambda_{HH} \quad ,$$

$$C = \sum_i C_i \quad , \quad C' = \sum_{i<j} C_{ij} \quad , \qquad (3.9)$$

and the three operators $S^{(I)}$, $S^{(II)}$ and $S^{(III)}$ (called symmetry adapter operators) are given by:

$$S^{(I)} = \sum_{i<j} c'_{ij} M_{ij} \quad , \quad S^{(II)} = \sum_{i<j} c''_{ij} \ M_{ij} \quad , \quad S^{(III)} = \sum_{i<j} c'''_{ij} M_{ij} \quad ,$$

$$c'_{12} = c'_{23} = c'_{34} = c'_{45} = c'_{56} = c'_{16} \quad , \qquad c'_{13} = c'_{24} = c'_{35} = c'_{46} = c'_{15} = c'_{26} = 0 \quad , \qquad c'_{14} = c'_{25} = c'_{36} = 0;$$

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$$c''_{12} - c''_{23} - c''_{34} - c''_{45} - c''_{56} - c''_{16} = 0, \quad c''_{13} - c''_{24} - c''_{35} - c''_{46} - c''_{15} - c''_{26} = 0 \quad, \quad c''_{14} - c''_{25} - c''_{36} = 0;$$

$$c'''_{12} - c'''_{23} - c'''_{34} - c'''_{45} - c'''_{56} - c'''_{16} = 0 \quad, \quad c'''_{13} - c'''_{24} - c'''_{35} - c'''_{46} - c'''_{15} - c'''_{26} = 0 \quad,$$

$$c'''_{14} - c'''_{25} - c'''_{36} = 1 \qquad . \tag{3.10}$$

It is important to note that the use of algebraic oscillator realizations solves another crucial problem in the theory of molecules, that is the construction of states that transform according to irreducible representations of point groups (For C_6H_6, the point group is D_{6h}). For this reason the operators $S^{(I)}$, $S^{(II)}$ and $S^{(III)}$, whose diagonalization produces states that transform as irreducible representations, have been called symmetry adapter operators [5,6]. The role of these operators in the representation theory of finite groups will be discussed elsewhere [7]. They can be constructed for any finite group and provide an oscillator (or boson) realization of finite groups (a new and very important mathematical result).

The algebraic Hamiltonians (3.8) allow one to do anharmonic analysis of molecular vibration spectra. One determines the coefficients $A_H, A_{HH}, \lambda^{(I)}_{HH}, \lambda^{(II)}_{HH}$ and $\lambda^{(III)}_{HH}$ from some known energies and then computes all the others. This procedure can be applied not only to the molecule C_6H_6, but also to all other molecules obtained by replacing the hydrogen atoms with deuterium atoms [8]. Table II shows some calculated frequencies and infrared intensities in C_6H_6 and C_6D_6. This Table, reproduced from Ref. [8], is shown here as an example of the power of the method which allows a simultaneous calculation of all frequencies and infrared intensities of many molecules. One must note that this is still a small portion of the complete spectrum of benzene, since it describes only the so-called stretching vibrations in which the hydrogen atoms move in a radial direction relative to the carbon skeleton. (Vibrational modes v_7, v_{13}, v_{20} and v_2 in Wilson notation)[8]. A calculation of all the other modes and their combinations has been performed and will appear soon [9].

Table III shows a partial comparison of the calculation with experiment. One may note the close agreement not only for the fundamental vibrations (n=1) but also for the overtones (n=2, n=3). This agreement originates from the use of anharmonic oscillators. Had one used harmonic oscillators the expected frequencies of the n=2 and n=3 modes would have been respectively twice and three

TABLE II. Calculated frequencies[a] and infrared intensities[b] in C_6H_6 and C_6D_6.

	symm	computed energy	computed IR int.	symm	computed energy	computed IR int.
	\multicolumn{3}{}{C_6H_6}		\multicolumn{3}{}{C_6D_6}	

	\multicolumn{3}{c}{C_6H_6}	\multicolumn{3}{c}{C_6D_6}
	symm	computed energy	computed IR int.	symm	computed energy	computed IR int.
n=1	$E_{2g}(\nu_7)$	3056.91		$E_{2g}(\nu_7)$	2272.03	
	$B_{1u}(\nu_{13})$	3057.51		$B1_u(\nu_{13})$	2284.62	
	$E_{1u}(\nu_{20})$	3065.13	0.16(+2)	$E1u(\nu_{20})$	2288.94	0.64(+1)
	$A_{1g}(\nu_2)$	3073.96		$A_{1g}(\nu_2)$	2303.20	
	A_{1g}	6004.04		A_{1g}	4494.02	
	E_{1u}	6004.40	0.10(+1)	E_{1u}	4497.19	0.42(+0)
	E_{2g}	6004.92		E_{2g}	4497.55	
	B_{1u}	6005.10		B_{1u}	4498.21	
	A_{1g}	6109.49		E_{2g}	4550.04	
	E_{2g}	6110.93		A_{1g}	4556.93	
n=2	E_{1u}	6113.02	0.17(-2)	E_{1u}	4557.10	0.65(-3)
	B_{2u}	6117.07		B_{2u}	4559.26	
	A_{1g}	6118.94		A_{1g}	4571.15	
	E_{2g}	6121.24		E_{2g}	4572.49	
	B_{1u}	6123.40		E_{2g}	4575.46	
	E_{2g}	6125.35		B_{1u}	4579.11	
	E_{1u}	6128.15	0.60(-2)	E_{1u}	4582.29	0.15(-1)
	A_{1g}	6139.40		A_{1g}	4600.31	

[a]All values in cm^{-1}; [b]All values in 10^6 barns/cm.

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TABLE III. Experimental frequencies[a] and infrared intensities[b] in C_6H_6 and C_6D_6.

C_6H_6

| | | Energy | | | IR Intensity | |
	symm.	calc.	obs.	obs-calc	calc.	obs.
	$E_{2g}(\nu_7)$	3056.91	3056.6	-0.31		
	$B_{1u}(\nu_{13})$	3057.51	3057	-0.51		
n=1	$E_{1u}(\nu_{20})$	3065.13	3064.367[c]	-0.763	0.16(+2)	0.16(+2)
	$A_{1g}(\nu_2)$	3073.96	3073.94	-0.02		
n=2	E_{1u}	6004.40	6006	1.60	0.10(+1)	0.58(+0)
n=3	E_{1u}	8827.53	8827	-0.53	0.51(-1)	0.35(-1)

C_6D_6

| | | Energy | | | IR Intensity | |
	symm.	calc.	obs.	obs-calc	calc.	obs.
	$E_{2g}(\nu_7)$	2272.24	2272.5	0.26		
n=1	$B_{1u}(\nu_{13})$	2284.83	2285	0.17		
	$E_{1u}(\nu_{20})$	2289.14	2289.3	0.16	0.64(+1)	0.64(+1)
	$A_{1g}(\nu_2)$	2303.41	2303.44	0.03		
n=2	E_{1u}	4497.60	4497	-0.60	0.42(+0)	0.42(+0)
n=3	E_{1u}	6643.81	6644	0.19	0.22(-1)	0.22(-1)

[a]All values in cm^{-1}; [b]All values in 10^6 barns/cm; [c]deperturbed value.

times those of the fundamental, in disagreement with experiment.

4 Summary

In summary, algebraic theory is an expansion of all operators of physical interest into elements of an algebra, $\mathcal{G}$. For example, the Hamiltonian H can be expanded as

$$H = E_0 + \sum_{\alpha\beta} \epsilon_{\alpha\beta} \, G_{\alpha\beta} + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} u_{\alpha\beta\gamma\delta} \, G_{\alpha\beta} \, G_{\gamma\delta} + \cdots$$

$$G_{\alpha\beta} \in \mathcal{G} \quad . \tag{4.1}$$

In implementing algebraic theory an oscillator realization is often useful. The elements of $\mathcal{G}$ are then constructed from boson creation and annihilation operators

$$b_\alpha^\dagger \quad , \quad b_\alpha \qquad , \qquad \alpha = 1,\ldots,\nu+1 \quad . \tag{4.2}$$

The bilinear products

$$G_{\alpha\beta} = b_\alpha^\dagger \, b_\beta \quad , \tag{4.3}$$

generate the Lie algebras $U(\nu+1)$. Within this algebra one can describe both harmonic and anharmonic situations (and isotropic and anisotropic situations).

The oscillators (and algebras) can be coupled. The expansion of the operators is now in terms of the direct sum of algebras $\mathcal{G}_i$,

$$\mathcal{G} = \sum_i \oplus \, \mathcal{G}_i \quad . \tag{4.4}$$

The oscillator realization is in terms of boson operators

$$b_{\alpha i}^{\dagger} \quad , \quad b_{\alpha i} \qquad ; \; \alpha = 1,\ldots,\nu+1 \quad ; \quad i = 1,\ldots,n \quad . \qquad (4.5)$$

The index α provides a treatment of <u>continuous</u> symmetries (space index). The index i provides a treatment of <u>discrete</u> symmetries (oscillator index).

5 Conclusions

Algebraic theory is a powerful tool to deal with complex spectroscopic problems. Oscillator realizations of this theory have proven to be very useful in the analysis of several physical situations. In particular, the extension of harmonic to anharmonic analysis has led to a new and deeper understanding of the spectra of complex molecules.

Aknowledgements

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ATOMIC SUPERSYMMETRY

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Atomic supersymmetry is a quantum-mechanical supersymmetry connecting the properties of different atoms and ions. In this talk, I provide a short description of some established results in the subject and discuss a few recent developments, including the extension to parabolic coordinates and the calculation of Stark maps using supersymmetry-based models.

1. Introduction

It can often be impractical to find exact solutions to the equations determining the properties of a physical system. The identification and use of symmetries is one method that can be useful in the search for a mathematically simpler but physically sufficient description.

An example is provided by the behavior of the valence electron in alkali-metal atoms, which is governed in detail by the solution to an involved many-body problem. The essential physics of this situation can largely be contained in a single-electron model with an effective central potential [1]. This talk concerns symmetry issues involved in this approach, in particular, the role played by supersymmetry in a realistic central-potential approximation.

The first part of this talk (sections 2-5) provides some background and an overview of established results in atomic supersymmetry using spherical coordinates. The second part (sections 6-8) discusses recently developed extensions of these ideas to other coordinate systems and to the supersymmetry-based description of the Stark effect. More information on the approach used, its relation to other methods, and recent developments can be found in the references cited.

2. Supersymmetric Quantum Mechanics

A quantum-mechanical system is called supersymmetric [2] if its hamiltonian H commutes with N supersymmetry generators Q_j satisfying $\{Q_j, Q_k\} = \delta_{jk} H$. The superalgebra generated by H and Q_j is denoted sqm(N). Of interest here is the

special case sqm(2), for which it is convenient to consider the linear combination $Q = (Q_1 + iQ_2)/\sqrt{2}$ satisfying $H = \{Q, Q^\dagger\}$.

A simple representation of sqm(2) sets

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad , \quad H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \quad , \tag{2.1}$$

where $A = -i\partial_x - iU'/2$ with $U' = dU/dx$ for some function $U = U(x)$. The components $H_\pm$ satisfy

$$H_\pm \Psi_{\pm n} \equiv \left[-\frac{d^2}{dx^2} + V_\pm(x)\right]\Psi_{\pm n} = \epsilon_n \Psi_{\pm n} \, , \tag{2.2}$$

where $V_\pm(x) = U'^2/4 \mp U''/2$.

If the supersymmetry is unbroken, the energy of the ground state is zero. The eigenspectra of $H_\pm$ are degenerate except for this state, which is associated with H_+. The spectra of H_+ and H_- are called, respectively, the bosonic and fermionic stacks. Degenerate states in the bosonic and fermionic stacks are mapped into one another by the supersymmetry generators $Q, Q^\dagger$.

3. Atomic Supersymmetry

The Schrödinger equation for the hydrogen atom separates in spherical polar coordinates. The radial piece can be written in atomic units as

$$\left[-\frac{d^2}{dy^2} - \frac{1}{y} + \frac{l(l+1)}{y^2} - \frac{1}{2}E_n\right]\chi_{nl}(y) = 0 \, , \tag{3.1}$$

where $y = 2r$, $E_n = -1/2n^2$, and $\chi_{nl}(2r) = rR_{nl}(r)$. The radial wave functions are

$$R_{nl}(r) = \frac{2}{n^2}\left[\frac{\Gamma(n-l)}{\Gamma(n+l+1)}\right]^{\frac{1}{2}}\left(\frac{2r}{n}\right)^l \exp\left(-\frac{r}{n}\right)L_{n-l-1}^{(2l+1)}\left(\frac{2r}{n}\right) \, , \tag{3.2}$$

where the Sonine-Laguerre polynomials are defined by

$$L_n^{(\alpha)}(x) = \sum_{p=0}^{n}(-x)^p \frac{\Gamma(n+\alpha+1)}{p!\,(n-p)!\,\Gamma(p+\alpha+1)} \, . \tag{3.3}$$

The idea is to identify the radial equation (3.1) for fixed l with the hamiltonian H_+ of a supersymmetric quantum mechanics, as in Eq. (2.2). One can then

construct the supersymmetry generator Q, determine H_-, and seek a physical interpretation [3].

The supersymmetry is determined by specifying the function U. For the Coulomb problem, it is $U(y) = y/(l + 1) - 2(l + 1) \ln y$. The partner hamiltonian H_- can be shown to have the same form as H_+ except that l is replaced with $(l + 1)$ in the $1/y^2$ term. This means that the eigenfunctions of H_- at fixed l are $R_{n,l+1}$ with $n \geq 2$. These form a complete and orthonormal set.

To understand the physical interpretation, recall that H_- has the same spectrum as H_+, excluding the lowest state. This suggests H_- describes a physical system that looks hydrogenic except that the ground state is inaccessible. The idea is to attribute this to the Pauli principle. The partner system is then an atom with the lowest state filled with electrons, and H_- is an effective hamiltonian for the valence electron. For example, if $l = 0$ the spectrum of H_+ spans the s orbitals of hydrogen and the spectrum of H_- represents the s orbitals accessible to the valence electron in lithium. Evidently, the supersymmetry holds only if the non-hydrogenic electron interactions are disregarded. The incorporation of these symmetry-breaking effects is described in section 5.

In the exact-symmetry limit, the supersymmetry can be extended. For example, the s orbitals of lithium and sodium can also be viewed as supersymmetric partners. There are supersymmetric connections of this sort among atoms and ions across the periodic table, all physically incorporating the effects of the Pauli principle and mathematically implemented by integer shifts in the angular quantum number l.

Ref. [3] presents more details of the construction and discusses the experimental support for this atomic supersymmetry.

4. Oscillator Formulation

The oscillator formulation of atomic supersymmetry outlined in section 3 is closely related to harmonic oscillators. Indeed, connections exist in arbitrary dimensions among the radial equations for the Coulomb problem, the harmonic oscillator, and the two sqm(2) supersymmetric partners of these systems [4].

The radial equation for the d-dimensional Coulomb problem can be written in atomic units as

$$\left[-\frac{d^2}{dy^2} - \frac{1}{y} + \frac{(l+\gamma)(l+\gamma+1)}{y^2} - \frac{1}{2}E_{dn}\right] v_{dnl}(y) = 0 \ , \qquad (4.1)$$

where $y = 2r$, $E_{dn} = -1/2(n + \gamma)^2$, $\gamma = (d - 3)/2$, and the radial wave functions are

$$v_{dnl}(y) = c_{dnl}\, y^{l+\gamma+1} \exp\left(-y/2(n + \gamma)\right) L_{n-l-1}^{(2l+2\gamma+1)}\left(y/(n + \gamma)\right) \ , \qquad (4.2)$$

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with c_{dnl} a normalization constant. If the operator in Eq. (4.1) is interpreted as H_+ in an sqm(2) supersymmetry, the partner H_- can be shown to have the same form but with l replaced by $l + 1$.

The radial equation for the D-dimensional harmonic oscillator with unit angular frequency in atomic units is

$$\left[-\frac{d^2}{dY^2} - \frac{1}{Y} + \frac{(L+\Gamma)(L+\Gamma+1)}{Y^2} - E_{DN}\right]V_{DNL}(Y) = 0 \ , \qquad (4.3)$$

where Y is the radial variable, $E_{DN} = 2N + 2\Gamma + 3$, $\Gamma = (D-3)/2$, and the radial wave functions are

$$V_{DNL}(Y) = C_{DNL} \ Y^{L+\Gamma+1} \exp(-Y^2/2) L_{N/2-L/2}^{(L+\Gamma+1/2)}(Y^2) \ , \qquad (4.4)$$

where C_{DNL} is a normalization constant. The sqm(2) partner H_- to the operator H_+ in Eq. (4.3) again has the same form but with L replaced by $L + 1$.

A state in the d-dimensional Coulomb problem can be related by a one-parameter mapping to a state in a D-dimensional oscillator [4]. The eigenfunctions v_{dnl} and V_{DNL} are related by $v_{dnl}\big((n+\gamma)Y^2\big) \propto Y^{1/2}V_{DNL}(Y)$ with $N = 2n-2+\lambda$, $L = 2l + \lambda$, and even $D = 2d - 2 - 2\lambda$. The parameter λ is an integer. Restrictions on the possible choices of the parameter, the dimensionality, and the quantum numbers can arise if several Coulomb-problem states from one system are required to be mapped to a specified harmonic oscillator. The map can be combined with the supersymmetric maps shifting the angular quantum numbers, resulting in a commutative diagram between states of the four systems. The reader is referred to [4] for further details.

5. Supersymmetry-Based Quantum-Defect Theory

The energy eigenvalues of the valence electron in an alkali-metal atom are shifted relative to the hydrogenic values by interactions with the core. The eigenvalues can be expressed as the Rydberg series [5] $E_{n^*} = -1/2n^{*2}$ with $n^* = n - \delta(n,l)$, where for given l the quantum defect $\delta(n,l)$ rapidly approaches an asymptotic value $\delta(l)$ as n increases. These shifts break the supersymmetries described in section 3.

Despite this, it is possible to develop a central-potential model incorporating supersymmetric features that has analytical wavefunctions as solutions and eigenenergies given by the Rydberg series [6]. The theory is defined by a radial equation obtained from Eq. (3.1) by replacing n, l, E_n with n^*, l^*, E_{n^*}. Here, l^* is a new angular quantum number given by $l^* = l - \delta(l) + I(l)$, where $I(l)$ is an integer shift characteristic of supersymmetry. The eigenfunctions are given by

the replacement of n and l in Eq. (3.2) with n^* and l^*. The Sonine-Laguerre polynomials still appear because $n^* - l^* - 1 = n - l - I(l) - 1$ remains an integer. For asymptotic quantum defects $\delta(l)$, these eigenfunctions form an orthonormal and complete set.

Ref. [6] contains more details about this model and demonstrates that the analytical eigenfunctions yield transition probabilities that are in agreement with experiment and with accepted values [7]. Related developments, other than those to be discussed in the remainder of this paper, include the following.

- The eigenfunctions of the model have been used as trial wavefunctions for detailed atomic calculations [8].

- The validity of the theory at short distances has been examined by investigating predictions for the fine structure of alkali-metal atoms [9]. Certain key features observed experimentally are reproduced and the Landé semiempirical formula naturally appears in the model.

- The model has been used to examine transition probabilities for other atoms, notably alkaline-earth ions [10].

- The mathematical structure of the model has been shown to be connected with parastatistics [11].

6. Separable-Coordinate Supersymmetries

The Schrödinger equation with a Coulomb potential $V(r) = -1/r$ separates directly (no modulation factor) in four different coordinate systems: spherical, conical, prolate-spheroidal, and parabolic. Each separation results in three ordinary differential equations, so *a priori* there are twelve candidate equations for the role of a supersymmetric partner hamiltonian in an sqm(2) realization. However, additional constraints arise from the structure of Eq. (2.2): to play the role of H_+, an equation must be expressed in the appropriate form and the eigenspectrum must involve a tower of states labeled by the appropriate eigenvalue. Some results of an examination of possible sqm(2) supersymmetries involving the twelve candidate equations are described in this section. More details are provided in ref. [12].

First, consider spherical coordinates. The eigenfunctions for the Coulomb problem separate into a product of radial wave functions given in Eq. (3.2) and the spherical harmonics, which in turn decompose into products of associated Legendre functions and exponentials. As discussed in section 3, the radial equation does admit a supersymmetric partner. The equation for the associated Legendre functions can be rewritten in the form of Eq. (2.2) by admitting a modulation factor, but the ensuing equation lacks an eigenvalue representing an infinite tower

of states. The remaining azimuthal equation is uninteresting because it has zero potential. Therefore, the only interesting sqm(2) supersymmetry in spherical coordinates appears in the radial equation.

In conical coordinates, the coordinate surfaces are spheres and cones of elliptic cross section along the z and x axes. One coordinate is just the radial variable, and the eigenfunctions for hydrogen are products of Lamé functions with the same radial wavefunctions as in spherical coordinates. The radial equation has the supersymmetry described in section 3 and the same physical interpretation applies, although the complete eigenfunctions for the supersymmetric partner states involve Lamé functions rather than the spherical harmonics. Each of the other two separated equations can be rewritten without a linear derivative term as in Eq. (2.2), but no interesting supersymmetry exists.

The Schrödinger equation for the Coulomb problem also separates in prolate-spheroidal coordinates. It can be shown that none of the three resulting equations admits an interesting sqm(2) structure.

Finally, consider parabolic coordinates. These are defined by $\rho = r + z$, $\sigma = r - z$, $\phi = \tan^{-1}(y/x)$, where r is the radial coordinate. The hydrogen eigenfunctions in parabolic coordinates can be written as the product $u(\rho)v(\sigma)\Phi(\phi)$. The equation determining $\Phi(\phi)$ is uninteresting in the present context. The equation for $u(\rho)$ can be expressed in the form of an H_+ by extracting a modulation factor $u(\rho) = \rho^{-1/2}\chi_1(\rho)$. The ensuing equation for χ_1 is

$$
\left[\frac{d^2}{d\rho^2} + \frac{1 - m^2}{4\rho^2} + \frac{Z_1}{\rho} + \frac{E}{2}\right]\chi_1(\rho) = 0, \tag{6.1}
$$

where E is the energy and m is the magnetic quantum number. A similar analysis can be carried out for $v(\sigma)$, resulting in an identical equation but with χ_1, ρ, and Z_1 replaced by χ_2, σ, and Z_2. The separation constants Z_1 and Z_2 are required to satisfy $Z_1 + Z_2 = 1$. The eigensolutions have the form

$$
u_{n_1 n_2 m}(\rho) = \left(\frac{1}{\pi}\right)^{\frac{1}{4}}\varepsilon^{\frac{m}{2}+1}\left[\frac{\Gamma(n_1 + 1)}{\Gamma(n_1 + m + 1)}\right]^{\frac{1}{2}}e^{-\frac{1}{2}\varepsilon\rho}\rho^{\frac{1}{2}m}L_{n_1}^{(m)}(\varepsilon\rho), \tag{6.2}
$$

with a similar expression for $v_{n_1 n_2 m}(\sigma)$. Here, $n_1 = Z_1/\varepsilon - (m + 1)/2$, $n_2 = Z_2/\varepsilon - (m + 1)/2$ are nonnegative integers, and $\varepsilon = \sqrt{-2E}$ for $E_n = -1/2n^2$ with $n = n_1 + n_2 + m + 1$.

The equations for χ_1, χ_2 admit supersymmetric partners. The functions $U(\rho)$, $U(\sigma)$ determining the supersymmetry are $U(\rho) = 2Z_1\rho/(m+1) - (m+1)\ln\rho$ and $U(\sigma) = 2Z_2\sigma/(m + 1) - (m + 1)\ln\sigma$. Ref. [12] shows that the superpartner equations and eigensolutions have the same form as Eqs. (6.1), (6.2) and their analogues for the σ coordinate, but with the replacements $n_1 \rightarrow n_1 - 1$, $n_2 \rightarrow$

$n_2 - 1$, $m \to m + 2$. The supersymmetry leaves invariant the principal quantum number n, as needed for a mapping between degenerate states.

The basic approach to a physical interpretation is analogous to the spherical-coordinate case but the existence of two supersymmetries permits more possibilities. Only two (natural) cases are mentioned here; see ref. [12] for more details.

The valence-electron states for light alkali-metal atoms can be labeled using a spectroscopic notation based on the parabolic quantum numbers n_1, n_2, m. For convenience, the lowest- and highest-energy sublevels in a manifold of states with fixed $m \geq 0$ are called the red and blue levels, respectively. It turns out that a natural choice for the bosonic stack is the set of all blue states of hydrogen. Interpreting the absence of a fermionic-stack partner for the ground state as a consequence of the Pauli principle and treating electron-electron interactions as supersymmetry-breaking effects, the blue lines of hydrogen can be shown to be partnered by the blue lines of lithium. A similar construction can be made for the red lines of hydrogen and lithium. These supersymmetries extend to other alkali-metal atoms in a manner analogous to the spherical-coordinate case.

7. Quantum-Defect Theory in Parabolic Coordinates

The interactions breaking the supersymmetries in parabolic coordinates can be incorporated using notions from quantum-defect theory. For this case, the asymptotic quantum defects δ entering the Rydberg expression for the eigenenergies depend on the two parabolic quantum numbers n_1 and m. It can be shown that an effective potential exists for the two-dimensional separated equation in ρ and σ that has analytical eigenfunctions of the same form as in section 6 with the Rydberg series as eigenvalues, while incorporating the integer shifts characteristic of supersymmetry [12].

In separated form, the model is specified by

$$[\frac{d^2}{d\rho^2} + \frac{1 - m_1^{*2}}{4\rho^2} + \frac{Z_1^*}{\rho} + \frac{E^*}{2}]\chi_1^*(\rho) = 0 \tag{7.1}$$

and a similar expression for the equation in σ with subscripts 1 replaced by subscripts 2. The functions $\chi_1^*(\rho)$ play a role analogous to $\chi_1(\rho)$ in section 6, and the separation constants Z_1^* and Z_2^* satisfy $Z_1^* + Z_2^* = 1$.

Solutions to these equations exist only if $n_1^* = Z_1^*/\sqrt{-2E^*} - (m_1^* + 1)/2$ and $n_2^* = Z_2^*/\sqrt{-2E^*} - (m_2^* + 1)/2$ are nonnegative integers. Note that $n^* \equiv n - \delta = n_1^* + n_2^* + (m_1^* + m_2^*)/2 + 1$. The wavefunctions are

$$u_{n_1^* n_2^* m_1^* m_2^*}(\rho) = (\frac{1}{\pi})^{\frac{1}{4}} (n^*)^{-(\frac{m_1^*}{2}+1)} [\frac{\Gamma(n_1^* + 1)}{\Gamma(n_1^* + m_1^* + 1)}]^{\frac{1}{2}} e^{-\frac{\rho}{2n^*}} \rho^{\frac{1}{2}m_1^*} L_{n_1^*}^{(m_1^*)}(\frac{\rho}{n^*}) , \tag{7.2}$$

with an analogous result for the σ-dependent equation. It can be shown that these model wavefunctions are not fully orthogonal for physical values of the asymptotic quantum defect, although the off-diagonal matrix elements are numerically small.

The four quantum numbers that appear depend on the parabolic quantum numbers, the quantum defect, and the supersymmetry integers. Define $n_1^* = n_1 - I_1$, $n_2^* = n_2 - I_2$, where I_1 and I_2 are integers, and $m_1^* = m + \alpha_1$, $m_2^* = m + \alpha_2$, with the constraint $\alpha_1 + \alpha_2 = 2I_1 + 2I_2 - 2\delta$. If δ is set to zero, the constraint reduces the four variables I_1, I_2, α_1, α_2 to three and the exact supersymmetry limit of section 6 can be recovered.

8. The Stark Effect

In an external electric field of strength F, the energy levels of the valence electron in an alkali-metal atom are shifted by amounts depending on F. A plot of the eigenenergies as a function of F is called a Stark map. An exact theoretical treatment for Stark maps of alkali-metal atoms is impractical. However, numerical approximation methods yield Stark maps in good agreement with experiment [13].

The analytical eigenfunctions obtained from supersymmetry-based quantum-defect theories in parabolic and spherical coordinates provide another approach to obtaining Stark maps [12]. In parabolic coordinates, the nonorthogonality of the eigenfunctions is a barrier to exact analysis. One possible approximation is to neglect the small off-diagonal components of the zero-field energies. This calculation yields Stark maps with curves that are largely linear, in disagreement with experimental results.

This difficulty is absent in spherical coordinates. The zero-field wavefunctions of the model described in section 5 can be used to calculate Stark matrix elements, and the ensuing energy matrix can be diagonalized for a subset of the perturbation basis. Ref. [12] contains detailed results for the $n = 15$ Stark states of lithium and sodium for $m = 0$ and $m = 1$, evaluated over the ranges used in ref. [13] for the energies and electric field strengths. The non-hydrogenic structure is clearly reproduced, demonstrating the expected sizable anticrossings and small-field quadratic Stark effects for the s and p levels. All the Stark maps generated for the $n = 15$ lines of lithium and sodium by the supersymmetry-based theory are indistinguishable from the numerical and experimental results of ref. [13].

$C-\varphi$

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THE HARMONIC OSCILLATOR
AND NUCLEAR PHYSICS

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Abstract

The three-dimensional harmonic oscillator plays a central role in nuclear physics. It provides the underlying structure of the independent-particle shell model and gives rise to the dynamical group structures on which models of nuclear collective motion are based. It will be shown that the three-dimensional harmonic oscillator features a rich variety of coherent states, including vibrations of the monopole, dipole and quadrupole types, and rotations of the rigid flow, vortex flow and irrotational flow types. Nuclear collective states exhibit all of these flows. It will also be shown that the coherent state representations, which have their origins in applications to the dynamical groups of the simple harmonic oscillator, can be extended to vector coherent state representations with a much wider range of applicability. As a result, coherent state theory and vector coherent state theory become powerful tools in the application of algebraic methods in physics.

1 Introduction

Harmonic oscillators are important in many-body physics for many reasons. The following are some of the reasons:

(i) Small amplitude normal modes of a system about a configuration of stable equilibrium are harmonic both in classical and quantum mechanics.

(ii) Harmonic oscillators have non-dispersive coherent states. Thus, they exhibit a perfect correspondence between classical and quantum mechanics.

(iii) The boson quanta of the harmonic oscillator provide important building blocks for the boson second quantization of the observables of a large number of systems; e.g., the Hamiltonian is often usefully expressed in the second quantized form

$$h = \sum \hbar\omega_\nu a_\nu^\dagger a_\nu + \sum \left(V^{\mu\nu\mu'\nu'} a_\mu^\dagger a_\nu^\dagger a_{\mu'}^\dagger a_{\nu'}^\dagger + V_\nu^{\mu\nu\mu'} a_\mu^\dagger a_\nu^\dagger a_{\mu'}^\dagger a_{\nu'} \ldots \right). \tag{1}$$

(iv) The classical Lie algebras all have boson (Weil) representations; e.g., an abstract element X of GL(n,C) can be realised as a matrix (X_{ij}) or as the boson operator

$$X = \sum_{ij} X_{ij} a_i^\dagger a_j. \tag{2}$$

It is shown in the following that the three-dimensional harmonic oscillator has a rich variety of symmetries and coherent states all of which feature in the theory of nuclear collective motions.

2 Symmetries and coherent states of the simple harmonic oscillator

The symmetries and dynamical group structures of the harmonic oscillator are most easily recognized when the latter is expressed in terms of the Heisenberg-Weyl algebra. The Heisenberg-Weyl algebra is a Lie algebra spanned by harmonic oscillator raising and lowering operators (also called boson operators) and the identity operator; i.e.,

$$hw(1) = \langle a^\dagger, a, I \rangle .$$
(3)

The elements of hw(1) satisfy the boson commutation relations

$$[a, a^\dagger] = I, \quad [a, I] = [a^\dagger, I] = 0 .$$
(4)

The simple harmonic oscillator Hamiltonian is given by

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) .$$
(5)

One finds that the symmetry group of the Hamiltonian H is the one-dimensional unitary group U(1) and that H has three dynamical groups: viz., the Heisenberg-Weyl group HW(1), the symplectic group Sp(1,R) (strictly the metaplectic group) and the inhomogeneous symplectic group ISp(1,R). The Lie algebra, u(1), of U(1) is spanned by a single element

$$u(1) = \langle a^\dagger a \rangle .$$
(6)

The Lie algebras of the symplectic and inhomogeneous symplectic groups are given, respectively, by

$$
\begin{aligned}
sp(1, R) &= \langle a^\dagger a^\dagger, aa, a^\dagger a + aa^\dagger \rangle \\
isp(1, R) &= \langle a^\dagger a^\dagger, aa, a^\dagger a + aa^\dagger, a^\dagger, a, I \rangle .
\end{aligned}
$$
(7)

The coherent states of the dynamical groups are all of considerable interest. First recall that a coherent state of a group is, by definition [1, 2] a state obtained by applying a group transformation to a particular state of a (usually irreducible) representation space on which the group acts. The standard (Glauber) coherent states, for example, are obtained by applying elements of the Heisenberg-Weyl group to the harmonic oscillator ground state [3].

An arbitrary element g of the Heisenberg-Weyl group is represented as the operator

$$T(g) = e^{\alpha a^\dagger - \alpha^* a + i\varphi I} ,$$
(8)

where α and φ are complex and real parameters, respectively, and I is the identity operator. Thus, we have

$$T(g)|0\rangle = e^{\alpha a^\dagger - \alpha^* a + i\varphi I}|0\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle e^{i\varphi} .$$
(9)

The state

$$|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle$$
(10)

is well known to have a wave function that is of the same form as the ground state wave function but with its centroid displaced from the harmonic oscillator equilibrium position and given some non-zero mean value of momentum. A HW(1) coherent state for a real value of α is illustrated, for example, in Fig. 1. The phase factor $e^{i\varphi}$ can be regarded as a U(1) gauge factor.
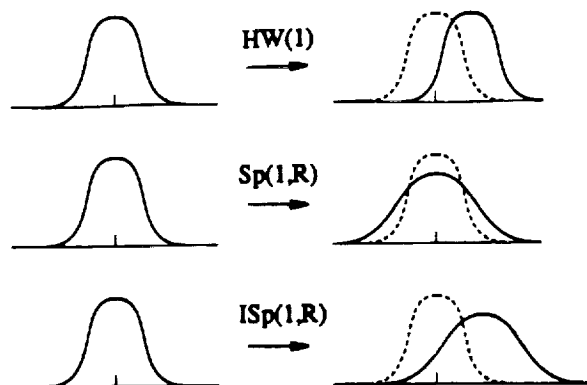
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Fig. 1. Harmonic oscillator coherent states induced by HW(1), Sp(1,R) and ISp(1,R) transformations, respectively.

Coherent states of the symplectic group are constructed in a similar way; i.e., a Sp(1,R) group element is represented in factored form

$$T(g) = e^{za^\dagger a^\dagger - z^* aa} e^{i\varphi(a^\dagger a + aa^\dagger)}, \quad g \in \text{Sp}(1,R) \tag{11}$$

and symplectic coherent states are given by

$$T(g)|0\rangle = e^{za^\dagger a^\dagger - z^* aa}|0\rangle e^{i\varphi}. \tag{12}$$

Again there is a U(1) gauge factor. Coherent states of the symplectic group [4, 5, 6] are often described as *squeezed states*.

There has been much interest in squeezed states in optics in recent years. One of their predicted properties [7] which, as far as I know, has never been investigated is that they should exhibit enhanced non-linear phenomena. This is expected because non-linear properties require the simultaneous presence of two photons of light and one may anticipate, therefore, that the squeezed light emitted, for example, from a two-photon laser should be particularly effective at providing photons in pairs. Squeezed coherent states are also of paramount importance in nuclear physics as we discuss in the following.

3 The three-dimensional harmonic oscillator

The Heisenberg-Weyl algebra

$$\text{hw}(3) = \langle a_i^\dagger, a_i, I; i = 1, 2, 3 \rangle \tag{13}$$

satisfies the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij} I, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = [a_i, I] = [a_i^\dagger, I] = 0 \tag{14}$$

and the Hamiltonian of the three-dimensional harmonic oscillator is given by

$$h = \sum_i \hbar \omega_i a_i^\dagger a_i. \tag{15}$$

This Hamiltonian has an energy spectrum as shown in Fig. 2, where the standard spectroscopic notation is used to label the radial and angular momentum quantum numbers of the energy levels.

Fig. 2. The energy level spectrum for a single particle in a three-dimensional harmonic oscillator potential. Each energy levels consists of a degenerate multiplet of states which span an irrep of the symmetry group U(3).

The three-dimensional harmonic oscillator underlies the shell-model theory of nuclear physics. One finds that if one were to assume that the neutrons and protons inside a nucleus obeyed an independent-particle Schrödinger equation in which the potential energy is that of a harmonic oscillator then, because of the Pauli exclusion principle, one can assign a given harmonic oscillator wave function to at most two neutrons, one with spin up and one with spin down. The same is true for the protons. Thus, one constructs the ground state of a nucleus by *filling* the harmonic oscillator energy levels starting from the bottom and progressing upwards. Because of the multiplicity of harmonic oscillator states of a given energy, the harmonic oscillator energy of a many-nucleon state increases discontinuously as each level is filled and the next level starts to be populated. The nucleon numbers at which this happens are called *magic numbers* and the nuclei at which it happens are called *closed-shell* nuclei. Such nuclei are expected to be particularly stable, like the inert gases of atomic physics. Now, experimentally observed magic numbers differ from those of the harmonic oscillator Hamiltonian. However, if one adjusts the harmonic oscillator Hamiltonian by the addition of an angular momentum term proportional to the square of the orbital angular momentum, to simulate the effects of a more realistic shell model potential, and includes a spin-orbit interaction, then the single-particle levels become of the type shown in Fig. 3 and the experimentally observed magic numbers are reproduced [8].

A typical shell model Hamiltonian is therefore of the form

$$H = \sum_n h_n + V , \tag{16}$$

where h_n is a single-particle Hamiltonian for the n'th nucleon of the form

$$h = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 r^2 + C\mathbf{l} \cdot \mathbf{s} + Dl^2 \tag{17}$$

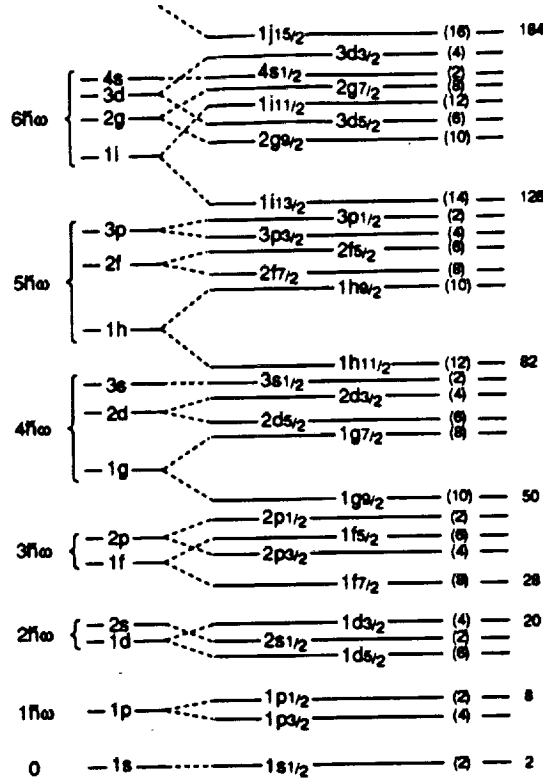and V is the residual interaction between the nucleons.

Fig. 3. The single-particle energy level spectrum for a shell model potential with an l^2 term and a spin-orbit interaction. The numbers in parenthesis indicate the multiplicity of states of a given energy level. The numbers to the right of an energy level indicates the cumulative number of states up to that energy. These numbers correspond to the experimentally observed magic numbers.

The degeneracies, and hence the symmetries, of the harmonic oscillator are broken in nuclear physics. Nevertheless, the states of the harmonic oscillator provide a basis in which realistic Hamiltonians can be diagonalized and in which the symmetry breaking effects can be described.

It will be shown in the following that the symmetries and dynamical group structures of the harmonic oscillator Hamiltonian are also of vital importance for identifying and separating the important collective degrees of freedom of a strongly interacting system of nucleons.

The symmetry group of the three-dimensional harmonic oscillator is the unitary group U(3) whose Lie algebra is spanned by the bilinear combinations of boson operators $\{a_i^\dagger a_j; i, j = 1, 2, 3\}$. The dynamical groups of the three-dimensional harmonic oscillator are the Heisenberg-Weyl group, the symplectic group Sp(3,R) and the inhomogeneous symplectic group ISp(3,R). The Lie algebras of these groups are given by the natural extensions of their one-dimensional counterparts; e.g.,

$$\text{sp}(3, R) = \langle a_i^\dagger a_j^\dagger, a_i a_j, a_i^\dagger a_j + a_j a_i^\dagger \rangle. \tag{18}$$

The coherent states of a single particle in a three-dimensional harmonic oscillator potential will not be discussed here. Instead, we proceed directly to the coherent states of a many-particle nucleus. It will be shown that different kinds of coherent states are generated depending on the symmetries of the initial (undisplaced) state. In particular, the coherent states of open-shell nuclei have a richer structure than those of closed-shell nuclei.

4 Heisenberg-Weyl coherent states of a nucleus

The degenerate (equal energy) states of a three-dimensional harmonic oscillator energy level span an irreducible representation of the symmetry group U(3). The same is true of the SU(3) ⊂ U(3) subgroup; i.e., the states of the N'th harmonic oscillator level span an SU(3) irrep $(N, 0)$ as shown in Fig. 3. It follows that the equal energy states of a nucleus with an independent-particle harmonic oscillator Hamiltonian span the reducible representation of SU(3) given by the Kronecker product of all the $(N, 0)$ irreps to which the nucleons separately belong.

4.1 Closed-shell nuclei

The ground state of a closed shell nucleus is characterized by a single closed-shell state which must, therefore, span the one-dimensional identity SU(3) representation $(0, 0)$. Nuclei for which such harmonic oscillator closed-shell states are believed to provide a good approximation to their ground states are the light nuclei ^{16}O and ^{40}Ca which, respectively, close the $N = 1$ and $N = 2$ harmonic oscillator shells.

We now consider coherent states of the Heisenberg-Weyl group obtained by applying a group transformation to a closed-shell state. Note, however, that a straightforward Heisenberg-Weyl transformation simply displaces the centre-of-mass of the whole nucleus or gives the whole nucleus centre-of-mass momentum without exciting it or changing its intrinsic structure in any way. Thus it is not very interesting. However, there is another representation of the Heisenberg-Weyl Lie algebra that is interesting. It is the representation in which the neutrons are displaced in one direction while the protons are displaced in the opposite direction in such a way that the centre-of-mass position and momentum remain fixed. In this representation the boson operators of the Heisenberg-Weyl Lie algebra are the linear combinations of the elementary (harmonic oscillator) boson operators for neutrons and protons

$$a_i^\dagger = \frac{1}{\sqrt{2Z}} \sum_{p=1}^{Z} a_{pi}^\dagger - \frac{1}{\sqrt{2N}} \sum_{n=1}^{N} a_{ni}^\dagger , \qquad (19)$$

where n indexes the neutrons and p indexes the protons. These combinations satisfy the commutation relations of Eq. (14) and, therefore, belong to a Heisenberg-Weyl Lie algebra.

The coherent states of this representation of the Heisenberg-Weyl Lie group are of the form

$$T(g)|0\rangle = \exp\left[\sum_i \alpha_i \left(\frac{1}{\sqrt{2Z}} \sum_p a_{pi}^\dagger - \frac{1}{\sqrt{2N}} \sum_n a_{ni}^\dagger\right) - \text{h.c.}\right] |0\rangle e^{i\varphi} . \qquad (20)$$

Thus, the group transformation $T(g)$ is seen to displace the ground state distributions of neutrons and protons in opposite directions as illustrated in Fig. 4.

Such a coherent state is of major interest in nuclear physics. It corresponds to a coherent collective motion of the nucleus in which neutrons and protons oscillate in antiphase and thereby generate an oscillating electric dipole moment. It is the so-called *Goldhaber-Teller mode* [9] of the *giant dipole resonance*. The mass associated with this mode is given by the reduced mass of the separate neutron and proton centres of mass. The restoring force for a dipole displacement can be estimated from the nuclear symmetry energy (i.e., the energy associated with a neutron-proton
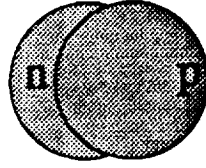
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Fig. 4. The displaced neutron and proton density distributions for a coherent state of the Heisenberg-Weyl group in which neutrons are displaced one way while protons are displaced the opposite way.

density difference). Thus, one deduces the frequency and the rate of electric dipole radiation for such a coherent oscillation. The requantization of this mode leads to a relatively high energy (one quantum) excited state which decays rapidly to the ground state by electric dipole radiation.

4.2 Open-shell nuclei

An open-shell nucleus does not have a unique lowest harmonic oscillator energy state; there is a multiplicity of lowest energy states. As already observed, such a multiplet of states spans a generally reducible representation of the harmonic oscillator symmetry group U(3).

Consider, for example, the nucleus ^{20}Ne. The first 16 nucleons form an ^{16}O closed-shell core and combine to give a $(0,0)$ SU(3) (identity) irrep. Thus, the lowest energy states of ^{20}Ne have four nucleons, two neutrons and two protons, in the $N = 2$ shell (the so-called sd shell). They span a reducible SU(3) representation given by the Kronecker product of four copies of the $(2,0)$ representation; i.e.,

$$(2,0) \times (2,0) \times (2,0) \times (2,0) = (8,0) + (4,2) + \ldots \qquad (21)$$

The residual interaction of the shell-model Hamiltonian (16) will cause the states constructed in this way to be mixed and the SU(3) symmetry to be broken. However, although the energies of states become non-degenerate, the states retain their SU(3) quantum numbers to a first approximation. One says that SU(3) is an approximate dynamical symmetry for light nuclei.

In the Elliott model [10], it is assumed that states of different SU(3) irreps do not mix and that the energies of states are given by a Hamiltonian which is a sum of SU(3) and SO(3) Casimir invariants

$$H = AC_2 + BL^2 ; \qquad (22)$$

L^2 is the square of the angular momentum of the SO(3) subgroup of SU(3). This Hamiltonian has an energy spectrum characteristic of a rotor

$$E = A(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) + BL(L+1) , \qquad (23)$$

where (λ, μ) are the labels of an SU(3) irrep. The coefficient A of C_2 is assumed to be negative so that the states of the SU(3) irrep with largest value of the Casimir invariant lie lowest in energy. The parameter B is then adjusted to give the lowest band of energy levels for ^{20}Ne as shown in comparison to the experimentally observed energy spectrum in Fig. 5. The agreement is far from
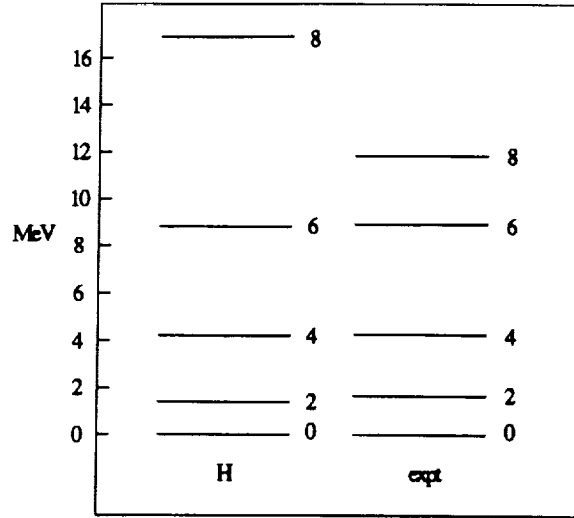
Fig. 5. The low energy spectrum for ^{20}Ne as calculated in the Elliott model and as observed experimentally.

perfect but it does describe the rotational properties of ^{20}Ne to a first approximation. Moreover, from the SU(3) quantum numbers, one can infer the shape of the deformed ^{20}Ne nucleus which gives rise to the observed rotational band.

We now consider the coherent states corresponding to the giant dipole vibrations of such a deformed nucleus. Since there are many degenerate harmonic oscillator ground states for an open-shell nucleus, it is appropriate to consider coherent states of the semi-direct product group comprising both the Heisenberg-Weyl group and the SU(3) group. For this group, there is a uniquely defined lowest weight state for each irrep. We can then construct simultaneous coherent states of the combined groups in which we have both rotations and dipole vibrations of the type illustrated in Fig. 6. As one can see, the energy level spectrum of such coupled rotations and vibrations can be relatively complex. Nevertheless, it is simply described in terms of the SU(3) and Heisenberg-Weyl dynamical groups.

5 Symplectic coherent states in nuclear physics

A many-particle representation of an element of the symplectic group Sp(3,R) can be expressed in the form

$$T(g) = \exp\left[\sum_{nij} z_{ij} a_{ni}^\dagger a_{nj}^\dagger - \text{h.c.}\right] \exp\left[i \sum_{nij} \alpha_{ij} \left(a_{ni}^\dagger a_{nj} + a_{nj} a_{ni}^\dagger\right)\right]. \tag{24}$$

5.1 Closed-shell nuclei

When acting on a closed-shell state $|0\rangle$, such an operator generates the coherent state

$$T(g)|0\rangle = \exp\left[\sum_{nij} z_{ij} a_{ni}^\dagger a_{nj}^\dagger - \text{h.c.}\right]|0\rangle e^{i\varphi}, \tag{25}$$
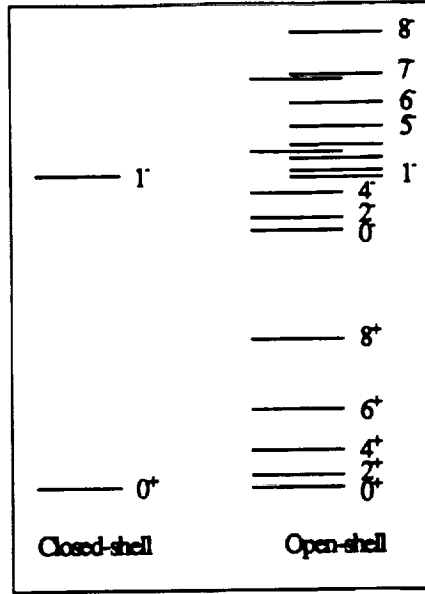
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Fig. 6. The low energy spectrum rotational spectrum and the combined rotation-giant resonance vibration of a nucleus like ^{20}Ne.

where φ is given by

$$\sum_{nij} \alpha_{ij} \left(a_{ni}^{\dagger} a_{nj} + a_{nj} a_{ni}^{\dagger} \right) |0\rangle = \varphi |0\rangle . \tag{26}$$

Thus, as for the one-dimensional case, there is a gauge angle φ associated with such coherent states. However, the phase factor $\exp(i\varphi)$ now comes from a one-dimensional representation of the symmetry group U(3) of the three-dimensional harmonic oscillator.

The coherent states of the symplectic group Sp(3,R) constructed in this way are natural extensions of the squeezed states of the one-dimensional harmonic oscillator. The difference is that there are now three different directions and the squeezing or dilation in the three directions need not all be the same. Fig. 7(a) shows a coherent state deformation of the ground state density distribution of a nucleus in which the squeezing/dilation is the same in all directions. Such a coherent state deformation preserves the spherical symmetry of the density distribution of the (closed-shell) nucleus and is described as a monopole or breathing mode vibration. Fig. 7(b) shows a coherent state deformation in which the nucleus is squeezed in one direction and dilated in another. The result is an ellipsoidal (i.e., quadrupole) deformation.

5.2 Open-shell nuclei

For an open-shell nucleus, we may construct symplectic coherent states by applying symplectic group transformations to the lowest weight state of an Sp(3,R) irrep. It can be shown that an Sp(3,R) lowest weight state is also an SU(3) lowest weight state for the SU(3) $\subset$ Sp(3,R) subgroup discussed in the previous section and, like it, has a non-spherical density distribution. Application of a symplectic transformation to such a state can effect the changes shown in Fig. 8. It can cause
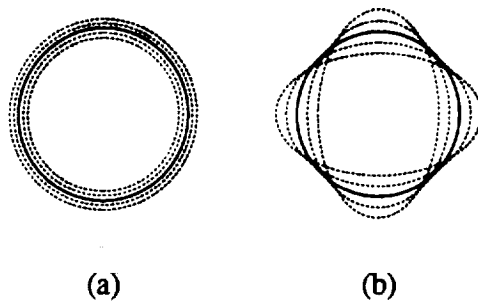
Fig. 7. Coherent state deformations of a spherical closed-shell density distribution: (a) shows a monopole (breathing mode) deformation and (b) shows a quadrupole deformation.
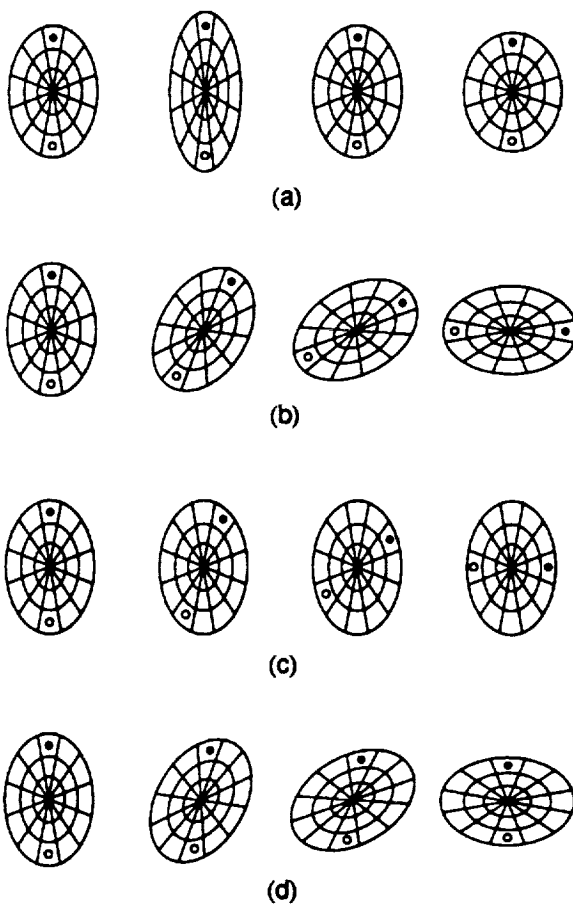

Fig. 8. Coherent states of a deformed (open-shell) nucleus: (a) shows a vibration; (b) a rigid rotation; (c) a vortex rotation; and (d) an irrotational-flow rotation.

squeezing and dilation of the nucleus thereby changing the deformation of the density distribution as illustrated in Fig. 8(a). Since the symplectic group Sp(3,R) contains the group SO(3) as a subgroup, symplectic group transformations include pure rotations, i.e., rigid body rotations of the type illustrated in Fig. 8(b). Figs. 8(c) and (d) show other kinds of rotations that are possible for a fluid but would not be possible for a rigid body. Fig. 8(c) shows a flow, called *vortex rotation*, in which the fluid circulates but leaves the quadrupole shape of the deformed nucleus fixed. Fig. 8(d) is a combination of rigid and vortex flow, called *irrotational flow*, in which the shape rotates but elements of the fluid move as little as possible consistent with the rotating shape.

It is of interest to note that a pure vortex rotation is not observable if one looks only at the density distribution of the nucleus. Thus, the vortex degree of freedom is naturally regarded as a non-abelian gauge degree of freedom.

To observe vortex flows, one needs a probe of the nucleus that is sensitive to currents and not just densities. Electron scattering is a natural choice. One is acustomed to think of an electron microscope as giving snapshots of the density distribution of a microscopic object. However, electrons are also sensitive to charge currents. Thus, electron scattering experiments have the potential for probing the contributions of vortex flows in nuclear (and perhaps other) collective motions. Some experimental beginnings have already been made in this direction. One thing is clear. The moments of inertia needed to describe the rotational energy spectra of nuclei are smaller, by approximately a factor of two, than those for rigid-body rotations and larger, by approximately a factor of five, than for irrotational flow. This suggests that nuclear rotational flows have a vortex component similar to that of a slightly viscous fluid.

We next consider the construction of a model of nuclear vibrations and rotations that admits rotational flows with arbitrary amounts of vorticity, ranging from none, for a viscous-free fluid, up to that of a rigid body. The fact that the coherent states of the symplectic group span the full range of possibilities, suggests that the symplectic group is a suitable dynamical group for such a model.

6   The nuclear symplectic model

The nuclear symplectic model [11, 12, 13] is based on the observation that the symplectic group Sp(3,R) is a suitable dynamical group for a microscopic theory of nuclear collective rotations, vibrations and vortex rotations. The important feature of a dynamical group is that a Hamiltonian which can be expressed as a polynomial in the Lie algebra of the dynamical group cannot mix states belonging to different irreps of the group.

A suitable Hamiltonian for the symplectic model is of the form given by Eq. (16) with the parameters C and D of Eq. (17) set equal to zero and

$$V = c_2 \mathrm{Tr}Q^2 + c_3 \mathrm{Tr}Q^3 + \ldots ; \tag{27}$$

Q is the Cartesian quadrupole tensor with components given in terms of the nucleon coordinates $\{x_{ni}\}$ by

$$Q_{ij} = \sum_n x_{ni} x_{nj}. \tag{28}$$

Note that by taking traces of powers of Q, we construct rotationally invariant polynomials. Note

also that the trace of Q itself is the sum of the squared nuclear radii

$$\text{Tr}Q = \sum_{ni} x_{ni}^2 = \sum_n r_n^2.$$ (29)

Thus, the harmonic oscillator Hamiltonian

$$\sum_n h_n = \sum_n \left[\frac{1}{2m} p_n^2 + \frac{1}{2} m\omega^2 r_n^2 \right]$$ (30)

already contains a term in $\text{Tr}Q$.

By a suitable choice of the parameters, c_2, c_3, ..., one can construct physically relevant potential energy functions for a deformed (rotational) nucleus. An important point is that polynomials in the traces of powers of Q are functions of three observable quantities which characterize the shape of the nucleus: the three quantities can be denoted as α, which measures the mean square radius of the nucleus, β which, measure the magnitude of the quadrupole moment of a nucleus, and γ, which measures the magnitude of the axial asymmetry of the deformation. Thus, for a spherical nucleus, β and γ would be zero. For a spheroidal (axially symmetric) nucleus, γ would vanish but β would be non-zero. But, for a generic ellipsoidal nuclear shape, both β and γ would be non-zero. It can be shown that

$$\text{Tr}Q \propto \alpha$$
$$\text{Tr}Q^2 \propto \beta^2$$
$$\text{Tr}Q^3 \propto \beta^3 \cos 3\gamma.$$ (31)

A simple two-parameter potential of this kind, with only c_2 and c_3 non-zero, is shown in Fig. 9. The potential shown has a minimum at a non-zero, but axially symmetric, deformation.

A calculation within the shell model space for an Sp(3,R) irrep was carried out for the low energy states of each of four heavy nuclei by Park et al. [14]. Their results for the energy levels and electric quadrupole (E2) radiative transition rates is shown in Fig. 10 in comparison with experimentally measured results. The agreement with experiment is not perfect but it is remarkably good considering that there is very little flexibility in the choice of the two parameters of the potential. The minimum of the potential is fixed at the known experimental deformation of the nucleus and the strength of the potential is fixed such that the potential is just strong enough to ensure that the wave function has the same deformation as the potential minimum. The most remarkable feature of the results is that one gets the correct moment of inertia (i.e., the energy level spacing comes out correctly) even though there is no adjustable momentum of inertia in the Hamiltonian; the kinetic energy of the Hamiltonian is the known microscopic kinetic energy for a system of nucleons. This is a major success of the model because it suggests that the amount of vorticity predicted by the symplectic model is just about right.

7 Coherent state and vector coherent state representations

In the application of algebraic models in physics, like the symplectic model, it is necessary to construct a basis for an appropriate irrep of the dynamical algebra and calculate the matrix
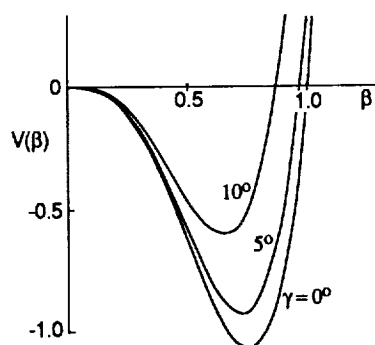
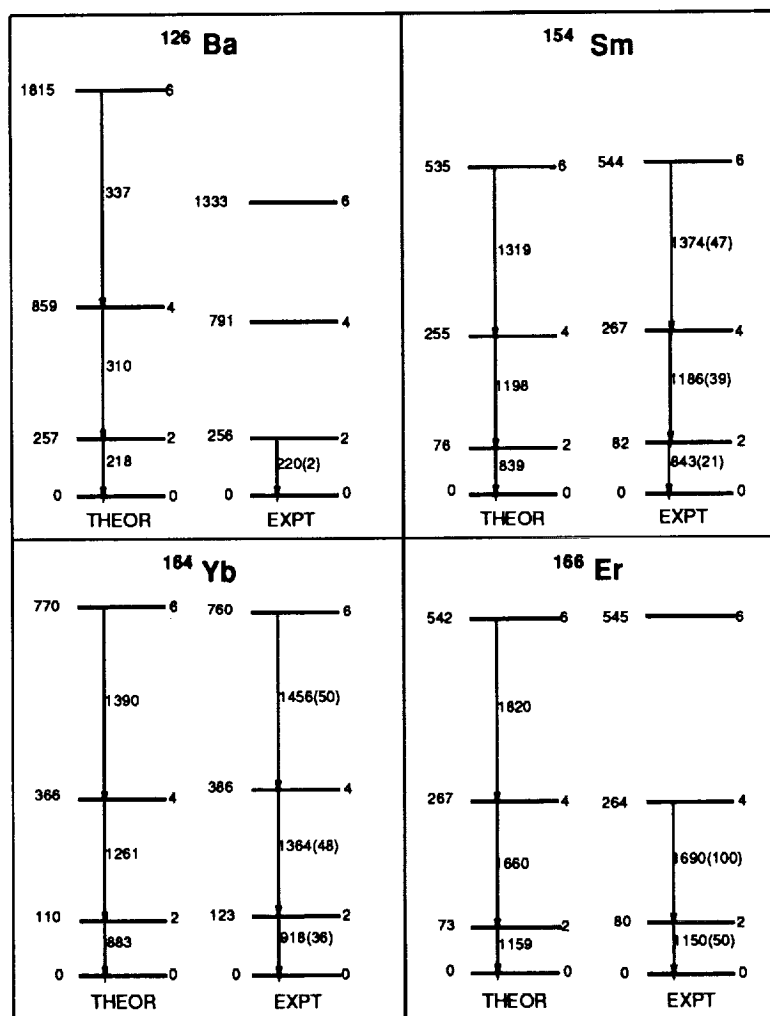Fig. 9. A two-parameter potential for the symplectic model.

Fig. 10. The low energy spectrum and E2 transition rates between states as calculated by Park *et al.* [14] in the symplectic model and as determined experimentally.

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elements of observable quantities. I do not wish to go into the technology of this subject here. However, I do wish to point out that the solution to the problem is given by a straightforward application of coherent-state and more genereally, vector-coherent-state representation theory.

7.1 Coherent state representations

A coherent state representation of a group G can be defined, following Perelomov [1] and Onofri [2] as follows.

Let T be the desired representation of the group G and suppose it is carried by a module (carrier space) V. Let $|0\rangle$ be a particular reference state in the space V; usually we choose V to be the lowest (or highest) weight state if such exists but it can also be chosen in other ways. Then, an arbitrary state $|\Psi\rangle \in V$ can be represented as a function ψ over the group, with ψ defined by

$$\psi(g) = \langle 0|T(g)|\Psi\rangle, \quad g \in G. \tag{32}$$

The remarkable fact is that without actually knowing the representation T, i.e., knowing only some properties of the special state $|0\rangle$, one can determine the kinds of coherent state functions that can occur. Furthermore, one is able to construct an explicit *coherent state* realization of the desired representation.

The standard example is the Bargmann representation of the Heisenberg-Weyl group. If the state $|0\rangle$ is the lowest weight state of the Heisenberg-Weyl Lie algebra, i.e., it satisfies

$$a|0\rangle = 0, \quad I|0\rangle = |0\rangle, \tag{33}$$

then

$$\psi(g) = \langle 0|e^{za - z^* a^\dagger + i\varphi I}|\Psi\rangle = \langle 0|e^{za}|\Psi\rangle e^{-\frac{1}{2}|z|^2} e^{i\varphi}. \tag{34}$$

Dropping the factor $e^{-\frac{1}{2}|z|^2} e^{i\varphi}$, one obtains the familiar Bargmann representation [15] of a harmonic oscillator state $|\Psi\rangle$ by the holomorhpic function ϕ of a complex variable z with

$$\phi(z) = \langle 0|e^{za}|\Psi\rangle. \tag{35}$$

For example, a harmonic oscillator state

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \tag{36}$$

is represented by the function

$$\phi_n(z) = \frac{1}{\sqrt{n!}} z^n. \tag{37}$$

In the Bargmann representation, the harmonic oscillator raising and lowering operators are simply the differential operators

$$a^\dagger = z, \quad a = \frac{\partial}{\partial z}, \quad I = 1. \tag{38}$$

Coherent state representations of the symplectic groups Sp(1,R) and Sp(3,R) can be constructed in a similar way. Consider, for example, the representation of Sp(3,R) with lowest weight

state given by a closed-shell state; i.e., a state which satisfies

$$\sum_{nij} z_{ij} a_{ni} a_{nj} |0\rangle = 0$$

$$\sum_{nij} \alpha_{ij} \left(a_{ni}^{\dagger} a_{nj} + a_{nj} a_{ni}^{\dagger} \right) |0\rangle = \varphi |0\rangle \, . \tag{39}$$

A coherent state representation of a state $|\Psi\rangle$ is then given by a function ψ over Sp(3,R) with

$$\psi(g) == \langle 0| \exp \sum_{nij} \left[z_{ij} a_{ni} a_{nj} - z_{ij}^{*} a_{ni}^{\dagger} a_{nj}^{\dagger} \right] |\Psi\rangle e^{i\varphi} \, , \tag{40}$$

which is proportional to the holomorphic function of six complex variables

$$\phi(z) = \langle 0| \exp \sum_{nij} z_{ij} a_{ni} a_{nj} |\Psi\rangle \, . \tag{41}$$

The expression of symplectic operators in this representation is simple and enables one to calculate their matrix elements in analytic form.

Such a construction of coherent state representations is an explicit realization of the Borel-Weil theory (see, for example, ref. [16]) of the representations of semi-simple Lie groups.

7.2 Vector coherent state representations

The direct application of the above construction of a general representation, i.e., one whose lowest weight state does not span a trivial one-dimensional representation of the SU(3) $\subset$ Sp(3,R) subgroup, is much more complicated and, therefore, not so useful. However, it is possible to construct a so-called *vector coherent state* representation which is simple.

First observe, from eq. (39), that the gauge factor $e^{i\varphi}$ is a representation of a U(3) transformation; i.e.,

$$\exp \sum_{nij} \alpha_{ij} \left(a_{ni}^{\dagger} a_{nj} + a_{nj} a_{ni}^{\dagger} \right) \rightarrow e^{i\varphi} \, . \tag{42}$$

Now, the lowest weight state of a generic Sp(3,R) irrep does not by itself span an irrep of the U(3) subgroup of Sp(3,R). However, it is one state of a multidimensional irrep. This suggests that more general Sp(3,R) irreps can be constructed in which the one-dimensional U(3) irrep of Eq. (42) is replaced by a general multidimensional U(3) irrep. This is correct and one finds that a state Ψ of any discrete series representation of Sp(3,R) can be realized as a holomorphic vector-valued wave function ψ with

$$\psi(z) = \sum_{\nu} |\nu\rangle \langle \nu| \exp \sum_{nij} z_{ij} a_{ni} a_{nj} |\Psi\rangle \, , \tag{43}$$

where $\{|\nu\rangle\}$ is a basis for a lowest weight irrep of the subgroup U(3) $\subset$ Sp(3,R).

The calculation of matrix elements of the sp(3,R) Lie algebra in such a representation is a simple task. When there are no missing quantum numbers, one obtains analytic expressions for the matrix elements. When there are missing quantum numbers, which is the generic situation, one has to do relatively small numerical calculations to construct orthonormal basis states.

The vector coherent state techniques apply to all the semi-simple Lie groups. They are, in fact, an explicit realization of the Harish-Chandra theory [17] of induced holomorphic representations.

8 Concluding remarks

I hope to have shown that the harmonic oscillator in three dimensions has a rich structure and that its many-particle representations and coherent states provide the framework for both independent-particle and collective models of nuclear states. Moreover, the coherent state and vector coherent state representations, which originated in applications of the dynamical groups of the harmonic oscillator, have much wider applicability and are now essential tools in the hands of those who use algebraic methods in physics.

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ON THE SPRING AND MASS OF THE
DIRAC OSCILLATOR

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Abstract

The Dirac oscillator is a relativistic generalization of the quantum harmonic oscillator. In particular, the square of the hamiltonian for the Dirac oscillator yields the Klein-Gordon equation with a potential of the form: $(ar^2 + b\mathbf{L} \cdot \mathbf{S})$, where a and b are constants. To obtain the Dirac oscillator, a "minimal substitution" is made in the Dirac equation, where the ordinary derivative is replaced with a covariant derivative. However, an unusual feature of the covariant derivative in this case is that the potential is a non-trivial element of the Clifford algebra. A theory which naturally gives rise to gauge potentials which are non-trivial elements of the Clifford algebra is that based on local automorphism invariance. I present an exact solution of the automorphism gauge field equations which reproduces both the potential term and the mass term of the Dirac oscillator.

1 Introduction

The Dirac oscillator exhibits many interesting features. It is the relativistic generalization of the classic non-relativistic harmonic oscillator Schrödinger equation to the Dirac equation in the sense that the square of the Dirac hamiltonian yields the relativistic Klein-Gordon equation for the spinor fields with a potential of the form: $(ar^2 + b\mathbf{L} \cdot \mathbf{S})$ where a and b are constants [1,2,3]. The equation is exactly solvable as in the non-relativistic case [4], and exhibits a hidden supersymmetry [4,5]. In addition, this particular form of potential has been used to model the inter-quark interactions in the hope of obtaining a realistic model of the hadrons [6,7]. Finally, an interesting version involving a "scalar" coupling has been investigated [8].

A highly unusual feature of the Dirac oscillator is that the potential which is introduced as a "minimal substitution" is a non-trivial element of the Clifford algebra. This is to be contrasted with all "usual" gauge theories where the potentials are Clifford scalars (that is, the potentials multiply the unit element of the algebra). A theory which naturally incorporates gauge potentials which are general elements of the Clifford algebra is that based upon local automorphism invariance [9,10]. The basic idea behind automorphism gauge theory is the observation that the particular matrix representation chosen for the Clifford algebra generators should not effect the physical predictions of the theory. If we then demand that this freedom of choice be allowed locally we obtain automorphism gauge theory.

In this paper I present a set of exact "chiral" solutions of the automorphism gauge field equations which reproduces both the potential term and the mass term of the Dirac oscillator as a special case. Additional details and further discussion of these topics may be found in reference [11], upon which this paper is based.

2 The Dirac Oscillator

The connection between the Dirac oscillator and the automorphism gauge theory is most easily seen by considering the "minimal substitution" that is made to obtain the Dirac oscillator [3,4]:

$$\mathbf{p} \rightarrow \mathbf{p} - im\omega\beta\mathbf{r} \tag{1}$$

This "minimal substitution" has the interesting property of being dependent upon the Clifford algebra generators (β is the Dirac matrix), and suggests that the theory can be derived from the automorphism gauge theory, since in this case the gauge potentials naturally occur as general elements of the Clifford algebra

To obtain the covariant form of the Dirac oscillator equation we introduce a unit timelike fourvector u_μ and an antisymmetric tensor $r_{\mu\nu}$ formed from the timelike unit vector and the spacetime coordinate vector:

$$u^\mu u_\mu = 1 \quad , \quad r_{\mu\nu} \equiv (u_\mu x_\nu - u_\nu x_\mu) \tag{2}$$

In the "rest frame " these take the form:

$$u_\mu = (1,0,0,0) \quad , \quad r_{0i} = x_i \quad , \quad r_{ij} = 0 \tag{3}$$

Now the covariant Dirac equation may be written as:

$$\left(\gamma^\mu p_\mu - m + \tfrac{1}{2}m\omega r_{\mu\nu}\gamma^{\mu\nu} \right)\Psi = 0 \tag{4}$$

where the matrices $\gamma_{\mu\nu}$ are the bivector elements of the Clifford algebra basis [12]. This equation has an electromagnetic interpretation as a particle with **zero charge** interacting via a **magnetic dipole moment** with a **radial electric field**. In this case the vector u_μ may be considered the four-velocity of the center of the electric field. Note that the electromagnetic interpretation is valid as long as we take equation (4) as our starting point. However, if we wish to view this equation as arising from a minimal substitution of a covariant derivative for an ordinary derivative, then the electromagnetic interpretation is untenable.

3 Local Automorphism Invariance

I now approach the problem from the point of view of local automorphism invariance [9,10]. Although the theory may be developed in spaces of arbitrary dimension and signature, we will restrict our attention to the case of four-dimensional spacetime. If we assume that the particular matrix representation of the Clifford algebra generators may be chosen arbitrarily **at each point in space**, then we obtain a gauge theory based on the automorphism group U(2,2). To incorporate this local invariance into the theory, the ordinary derivative must be replaced with the covariant derivative:

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + igA_\mu \tag{5}$$

where the gauge potential is given by [9,12]:

$$A_\mu = a_\mu 1 + a^\rho{}_\mu \gamma_\rho + \tfrac{1}{2} a^{\rho\sigma}{}_\mu \gamma_{\rho\sigma} - b^\rho{}_\mu \tilde{\gamma}_\rho - b_\mu \tilde{\gamma} \tag{6}$$

and for the field strength tensor we find:

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \\
&= f_{\mu\nu} 1 + f^\rho{}_{\mu\nu} \gamma_\rho + \tfrac{1}{2} f^{\rho\sigma}{}_{\mu\nu} \gamma_{\rho\sigma} - h^\rho{}_{\mu\nu} \tilde{\gamma}_\rho - h_{\mu\nu} \tilde{\gamma}
\end{aligned} \tag{7}$$

Making the minimal substitution into the Dirac lagrangian we find:

$$\begin{aligned}
L_\Psi &= \tfrac{i}{2} \overline{\Psi} \left(\gamma^\mu \overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu} \gamma^\mu \right) \Psi \\
&\quad - g\overline{\Psi} \left(\Phi 1 + a^\mu \gamma_\mu + 3\tilde{a}^\mu \tilde{\gamma}_\mu - \tilde{b}^{\mu\nu} \gamma_{\mu\nu} \right) \Psi
\end{aligned} \tag{8}$$

where we have made the definitions:

$$\Phi \equiv a^\rho{}_\rho \quad , \quad \tilde{a}_\mu \equiv \tfrac{1}{3!} \varepsilon_{\mu\nu\rho\sigma} a^{\nu\rho\sigma} \quad , \quad \tilde{b}_{\mu\nu} \equiv \tfrac{1}{2} \varepsilon_{\mu\nu\rho\sigma} b^{\rho\sigma} \tag{9}$$

and we see that the automorphism gauge fields couple to the fermion field through scalar, vector, pseudovector, and bivector (spin) interactions. Notice that we have not included the mass term in the basic lagrangian since the scalar coupling (Yukawa interaction) will give rise to mass. The explicit form for the field strength tensor in terms of the gauge potentials and a more detailed derivation of the Dirac part of the lagrangian can be found in my notes on local automorphism invariance [9].

The equations for the gauge fields in the absence of sources is found in the usual manner by demanding stationary action with respect to arbitrary variations of the fields. We find:

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + ig[A_\mu, F^{\mu\nu}] = 0 \tag{10}$$

Notice that we have not included the fermion source term in this equation. This is in concordance with the original approach to the Dirac oscillator in that the potential is introduced as an **external field**. To be entirely consistent, however, we must demand that the gauge potentials satisfy equation (10).

We now make the observation that this interacting lagrangian density will yield the equation for the Dirac oscillator (equation (4)) if the following conditions are met:

$$g\Phi = m \quad , \quad g\tilde{b}_{\mu\nu} = \tfrac{1}{2}m\omega r_{\mu\nu} \tag{11a}$$

$$a_\mu = 0 \quad , \quad \tilde{a}_\mu = 0 \tag{11b}$$

It is remarkable that these particular expressions for the potentials form a subset of exact "chiral" solutions to the pure gauge field equations.

4 Chiral Solutions

We now consider a special subset of solutions to equation (13). Consider the "chiral ansatz" defined as:

$$a_\mu = 0 \quad , \quad a^{\rho\sigma}{}_\mu = 0 \quad , \quad b_\mu = 0$$
$$b^\rho{}_\mu = \pm a^\rho{}_\mu \tag{12}$$

and the field equations become:

$$f^\kappa{}_{\mu\nu} = \partial_\mu a^\kappa{}_\nu - \partial_\nu a^\kappa{}_\mu = h^\kappa{}_{\mu\nu} \quad , \quad \partial_\mu f_\kappa{}^{\mu\nu} = 0 \tag{13a,b}$$

There are, of course, many solutions to equations (13), but consider the special case in which the field strength tensor is constant and uniform. In this case equation (13b) is clearly satisfied. To satisfy equation (13a) an obvious choice is to assume that the potential is linear in the spacetime coordinate. Therefore we write:

$$a^\kappa{}_\mu = c_1 m(1 + d_1 m(u \cdot x))g^\kappa{}_\mu + c_2 m(1 + d_2 m(u \cdot x))u^\kappa u_\mu$$
$$+ c_3 m^2 r^\kappa{}_\mu + c_4 m^2 \tilde{r}^\kappa{}_\mu + c_5 m^2 s^\kappa{}_\mu \tag{14}$$

where the coefficients c_i and d_i are arbitrary dimensionless constants, and we have defined:

$$\tilde{r}_{\mu\nu} \equiv \tfrac{1}{2}\varepsilon_{\mu\nu\kappa\tau} r^{\kappa\tau} \quad , \quad s_{\mu\nu} \equiv (u_\mu x_\nu + u_\nu x_\mu) \tag{15}$$

The parameter m is a quantity with the dimension of mass and it may be conveniently chosen to be the mass appearing in the Dirac oscillator. Finally note that equation (14) is the most general form which is both linear in the spacetime coordinate and which involves only one arbitrary constant vector.

The field strength tensor may be calculated directly from equation (13a) and we find:

$$f^\kappa{}_{\mu\nu} = m^2(c_1 d_1 + c_3 - c_5)(u_\mu g^\kappa{}_\nu - u_\nu g^\kappa{}_\mu) + 2m^2 c_4 \varepsilon^\kappa{}_{\mu\nu\tau} u^\tau \tag{16}$$

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As expected, the field strength tensor is constant and uniform, and therefore trivially satisfies equation (13b). For the part of the gauge potential which interacts directly with the fermion (see equations (8) and (9)) we find:

$$\Phi = (4c_1 + c_2)m + (4c_1 d_1 + c_2 d_2 + 2c_5)m^2(u \cdot x) \tag{17a}$$

$$\tilde{b}_{\mu\nu} = -c_3 m^2 \tilde{r}_{\mu\nu} + c_4 m^2 r_{\mu\nu} \tag{17b}$$

$$a_\mu = 0 \quad , \quad \tilde{a}_\mu = 0 \tag{17c}$$

Notice that only the symmetric part of the gauge potential contributes to the scalar interaction (equation (17a)), and only the antisymmetric part of the gauge potential contributes to the spin interaction (equation (17b)). This statement is generally true as may be seen by inspection of equation (9).

We may now recover the Dirac oscillator interactions as a special case if we make the following choices (compare equations (11) with equations (17)):

$$g(4c_1 + c_2) = 1 \quad , \quad 2gmc_4 = \omega \tag{18a}$$

$$c_3 = 0 \quad , \quad (4c_1 d_1 + c_2 d_2 + 2c_5) = 0 \tag{18b}$$

Now we state without proof (see [11]) that equations (18) can always be satisfied by an appropriate choice of gauge. In other words, the potentials appearing in the Dirac oscillator (where we are including the mass as a constant potential) are essentially unique chiral solutions of the automorphism gauge field equations, since the potentials may always be brought into this form (equations (11)) by a chiral gauge transformation.

5 Summary and Conclusions

I have shown that both the mass and the potential introduced into the Dirac equation to produce the Dirac oscillator may be viewed as a special case of a chiral solution to the automorphism gauge field equations. In addition, this chiral solution is essentially unique in that a gauge transformation can always be found which puts the potential in the form displayed in the Dirac oscillator.

To gain insight into the physical interpretation of this system consider the more familiar situation of an electron interacting with a constant magnetic field. In this case, since the field strength is constant and uniform, the electromagnetic potential will be a linear function of the spacetime coordinate. As is well known [13], this system exhibits harmonic oscillator behavior in the two spatial directions perpendicular to the magnetic field. Therefore, the point of view considered in this paper is actually more like this situation in that there is a constant and uniform field strength and a corresponding linear potential. This should be contrasted with the direct electromagnetic interpretation of a particle with zero charge and non-zero magnetic moment interacting with a linear electric field. Notice, however, that the case of a constant automorphism

field strength does not lend itself easily to the construction of hadrons as advocated by Moshinsky et al [5,6] since we do not view each particle as giving rise to the automorphism field (though they certainly must contribute to the automorphism field as does the electron to the magnetic field, but this is taken here to be a "higher order effect"). In other words, to build the mesons (for example) we may consider a linearly rising potential between the quark-antiquark pair, but the situation with a constant automorphism field is more akin to putting several electrons in a constant magnetic field and neglecting the interactions between them. Each electron undergoes cyclotron motion but the centers of the individual cyclotron orbits are uncorrelated. These two pictures are clearly at odds, and we therefore do not necessarily expect local automorphism invariant gauge field theory to lend itself easily to a model of the hadrons. In fact, the motivation for considering local automorphism invariance is more one of aesthetics in that it may be considered to arise from a generalization of the Principle of Equivalence, and an important anticipated goal is a truly unified approach to the electroweak and gravitational interactions, but these issues will be discussed elsewhere [14].

The approach to the Dirac oscillator discussed in this paper naturally generalizes to spacetimes of arbitrary dimension and signature. In particular, the specific cases of two, three, five, and six dimensions are likely to generate interesting results. In addition, as this application shows, the general theory of local automorphism invariance should be a worthwhile and interesting avenue of exploration.

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CLASSICAL CONFINED PARTICLES

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Abstract

An alternative picture of classical many body mechanics is proposed. In this picture particles possess individual kinematics but are deprived from individual dynamics. Dynamics exists only for the many particle system as a whole. The theory is complete and allows to determine the trajectories of each particle. We propose to use our picture as a classical prototype for a realistic theory of confined particles.

1 Basic Concepts

In modern elementary particle physics, the problem of quark confinement is one among the fundamental and unsolved troubles [1]. All attempts to explain the strange behaviour of quarks which never appear as free particles failed. Usually, the problem of quark confinement is attacked in the framework of quantum mechanics or quantum field theory because quarks are quantum objects. In spite of that in the present paper we propose a new purely classical approach to this problem. We start with the very foundation of classical mechanics because we strongly believe that the heart of the problem is here. We shall show that there exists a particular kind of classical mechanics of N -particle systems in which the constituent particles cannot realize themselves as individual mechanical objects. We are strongly convinced that this kind of classical mechanics is a right prototype for quantum theory of confined particles. Our approach to classical mechanics of confined particles is based on a particular extension to many-particle systems of the recently developed new mathematical formalism of the Galilean covariant dynamics of a single particle [2]. The essence of this new formalism may be recapitulated in two items:

i.) all mechanical quantities which describe motion of particles satisfy simple evolution equations;

ii.) the interaction of each particle with its environment is described by two vector-valued measures which also should be determined from their own evolution equations.

It is to be noticed that the property listed in item i) allows one to divide all mechanical equations into kinematical and dynamical equations of motion, equations of balance and equations of the environment which replace the customary force laws. Among these equations the kinematical

equations of motion and the equations of balance have universal and standard forms while the dynamical equations of motion and the equations of the environment are new in treating mechanics. The latter have a particular meaning for each type of mechanical interaction. They are necessary to complete the theory and to guarantee its Galilean covariance. The equations of environment acquire their beautiful form only after increasing the number of vector-valued measures of mechanical interaction mentioned in item ii). Apart from the usual measure called the force, the new formalism uses a second measure called the influence. The introduction of influences opens new possibilities of extending the range of applicability of classical mechanics to new hypothetical types of systems. For all necessary details we send the interested readers to [2].

2 Many Particle Systems

It is straightforward to extend the new Galilean covariant dynamics described in [2] to mechanical systems which contain more than one particle. We shall describe this extension in two steps. The first step is common to all passages from one-particle to many-particle systems and consists in introducing separate, individual kinematics for each particle of the considered system since otherwise we could not speak about many-particle systems at all. This means that for N-particle systems we have to consider $6N$ kinematical equations of motion

$$\dot{\vec{x}}_j(t) = \vec{v}_j(t) \tag{2.1}$$

$$\dot{\vec{v}}_j(t) = \vec{a}_j(t) \tag{2.2}$$

and $3N$ dynamical equations of motion:

$$\dot{\vec{a}}_j(t) = \vec{I}_j(t) \tag{2.3}$$

where $j = 1, 2, ... N$ and $\vec{x}_j(t), \vec{v}_j(t)$ and $\vec{a}_j(t)$ are the usual trajectory function, velocity function and the acceleration function of the j-th particle, respectively. The quantities $\vec{I}_j(t)$, called influences, represent one of the vector-valued measures of mechanical interaction of the j-th particle with its environment. We would like to stress here the conceptual difference between the new notion of the influence $\vec{I}_j(t)$ and the standard notion of the force $\vec{F}_j(t)$. The former is a measure of non-uniformity of the motion while the latter measures the violation of conservation laws during the motion. Both $\vec{I}_j(t)$ and $\vec{F}_j(t)$ provide the complete description of the dynamical action of the environment of the particle on the particle itself. The second step in the passing from one-particle to many-particle systems consists in introducing a corresponding dynamics. The new formalism described in [2], allows one to implement this step either in a standard or a nonstandard way and this is one of the advantages of the new formalism. In the standard way we introduce for each particle the notion of its momentum $\vec{p}_j(t)$ and the notion of the force $\vec{F}_j(t)$ acting on the j-th particle. The dynamics of particle is then described by the Newton equations balance

$$\dot{\vec{p}}_j(t) = \vec{F}_j(t) \tag{2.4}$$

The inertial properties of each particle appear in double face [3] since the Galilean transformation rules for the momenta

$$\vec{p}_j(t) \rightarrow \vec{p}_j{}'(t') = R\vec{p}_j(t) + m_j\vec{u} \tag{2.5}$$

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introduces the Galilean masses m_j of the particles while the second law of dynamics

$$\vec{F}_j(t) = M_j\, \vec{a}_j(t) \tag{2.6}$$

introduces the inertial masses M_j of the particles. The standard mechanics works with the additional assumption that

$$m_j = M_j \tag{2.7}$$

but an alternative case of violation of equalities (2.7) may be considered as well [4].

The nonstandard approach to many-particle systems we shall follow below consists in the idea that in the following, in contradistinction to kinematics, we do not introduce separate dynamical characteristics of individual particles at all and consider only global dynamical description of the system as a whole. This means that the dynamics of the N-particle system is collectively described by one total momentum $\vec{P}(t)$ which cannot be represented as sum of individual momenta of particle because the latter do not exists and by one total force $\vec{F}(t)$ which also cannot be treated as a resulting force of individual forces $\vec{F}_j(t)$ because the particles do not feel any force at all. The total momentum $\vec{P}(t)$ and the total force $\vec{F}(t)$ satisfy the Newton equation of balance

$$\dot{\vec{P}}(t) = \vec{F}(t) \tag{2.8}$$

and the motions of individual particles are solely governed by the dynamical equations of motions (2.3). The individual influences are related in a certain way, we shall discuss below, to the total force $\vec{F}(t)$. We shall not visualize the total force $\vec{F}(t)$ as a three-dimensional vector acting on a particular point in space and therefore we should not answer the usual question which asks at which point the force is applied. We restrict ourselves solely to the interpretation of the force $\vec{F}(t)$ as a vector-valued measure of interaction of the system with its environment which violates the conservation law of the total momentum. Under Galilean transformations the force $\vec{F}(t)$ transforms in the usual way

$$\vec{F}(t) \rightarrow \vec{F}'(t') = R\vec{F}(t) \tag{2.9}$$

Similarly, the transformation rule for the momentum $\vec{P}(t)$ must be of the form

$$\vec{P}(t) \rightarrow \vec{P}'(t') = R\vec{P}(t) + \mathcal{M}\vec{u} \tag{2.10}$$

and introduces the notion of the Galilean mass $\mathcal{M}$ of the system of N particles as a whole. The notion of its inertial mass will be introduced below. We do not introduce the notion of individual masses of the particles because for them we have neither individual momenta nor individual forces. Therefore, for the considered particles we can write neither equations (2.5) nor equations (2.6). The particles of the considered systems have only individual kinematics but are deprived from individual dynamics. In this sense the particles are "confined in a system" because they cannot realize themselves as individual mechanical entities. They may exist only as constituents of collections and there is no possibility to extract from these collections any individual constituent or a cluster of constituents.

We are fully aware that our point of view on the nature of confined particles is very radical and it may be rejected by the argument that in the domain of classical mechanics all particles should be treated in a classical way. Confined particles observed in Nature are objects quite different

from the usual particles just due to their being absent in a free state. They cannot therefore have all the attributes of usual particles and any theory of confined particles has to take this fact into account from the very beginning. Otherwise, irrespective of any eventual partial success, a theory of that type is physically wrong. Realistic confined particles are quantum objects and any realistic consistent theory of such particles must be of a quantum type. Unfortunately, all present quantum theories are based on classical prototypes in which the confinement of particles is excluded. To have physically well motivated theory of confined particles, we should construct it from an assumption different than the usual theories have incorporated in. It is therefore highly necessary to study alternative classical prototypes for new quantum theories and this was the main motivation of writing the present paper.

In addition, it should be taken into account that all detection methods in particle physics are based on the principles of classical physics because the interaction of quantum particles with measurement apparatus is always classical. Therefore, independently from any quantum theory which may explain the confinement any complete theory of confined particles has to look for alternative approaches already on the classical level.

One crucial novelty of our approach to Galilean covariant mechanics [2] consists in rejecting all known force laws and in replacing them by a system of differential equations from which the influences and forces may be determined. In Newtonian mechanics the influences $\vec{I}_j(t)$ and forces $\vec{F}_j(t)$ are always related by simple relations

$$\dot{\vec{F}}_j(t) = M_j\,\vec{I}_j(t) \tag{2.11}$$

In more sophisticated forms of mechanics the differential equations which relate $\vec{I}_j(t)$ and $\vec{F}_j(t)$ are of the form

$$\Phi_l\left(\vec{F}(t), \dot{\vec{F}}(t), \ddot{\vec{F}}(t); \vec{I}(t), \dot{\vec{I}}(t), \ddot{\vec{I}}(t)\right) = 0 \tag{2.12}$$

where Φ_l is a function of its arguments collectively denoted by symbols without the label j and the label l runs over a set which is sufficient to produce a complete system of differential equations for specification of all $\vec{I}_j(t)$ and $\vec{F}_j(t)$. In any mechanics of confined particles we have only one force $\vec{F}(t)$ and N influences $\vec{I}_j(t)$, and consequently we should modify the standard way of writing the differential equations (2.11) and (2.12). To do this, let us observe that in the standard mechanics the force $\vec{F}(t)$ may be represented in the form

$$\vec{F}(t) = \sum_{j=1}^{N} \vec{F}_j(t) \tag{2.13}$$

where $\vec{F}_j(t)$ are the forces acting on the j-th particle. From this representation and (2.11) we find the equation

$$\dot{\vec{F}}(t) = \sum_{j=1}^{N} M_j\vec{I}_j(t) \tag{2.14}$$

which apart from the undetermined inertial masses contains quantities used also in the mechanics of confined particles. We cannot however take this equation as one of the equations of the environment because confined particles have no inertial masses and the meaning of the coefficients in (2.14) is not clear. We have argued in [2] that all constants that appear in the equations

for influences and forces may be interpreted as coupling constants of the model of the particle environment , and following this line of reasoning we could take instead of (2.14) the equation

$$\vec{F}(t) = \sum_{j=1}^{N} \alpha_j \vec{I}_j(t) \tag{2.15}$$

where α_j $(j = 1,2...N)$ are fixed coupling constants. However, such an approach leaves too much arbitrariness. The situation is much better for identical particles because for such particles all relations should be symmetric under all permutations of quantities ascribed to individual particles. This means that for identical particles instead of (2.15) we will have

$$\dot{\vec{F}}(t) = \alpha \sum_{j=1}^{N} \vec{I}_j(t) \tag{2.16}$$

and we have to do only with one coupling constant the meaning of which is easily established. To find the interpretation of α, let us observe that the case of identical particles is the only case for which the usual notion of the center of mass has a purely geometrical meaning independent of the values of masses of particles. In fact, for a system of identical particles the usual definition of the center of mass leads to the following identification of its position vector

$$\vec{X}(t) = \frac{1}{N} \sum_{j=1}^{N} \vec{x}_j(t) \tag{2.17}$$

its velocity

$$\vec{V}(t) = \frac{1}{N} \sum_{j=1}^{N} \vec{v}_j(t) \tag{2.18}$$

and its acceleration

$$\vec{A}(t) = \frac{1}{N} \sum_{j=1}^{N} \vec{a}_j(t) \tag{2.19}$$

Clearly, all these concepts have a pure geometrical and kinematical meaning as the average position, velocity and acceleration, respectively. We may therefore use these notions also for systems of confined particles for which the notion of mass of the individual particles is not defined. The equation of motion for a whole system of confined particles will then be of the form

$$\vec{F}(t) = \mathbf{M}\vec{A}(t) \tag{2.20}$$

where $\mathbf{M}$ is the inertial mass of the system as a whole. Differentiating now this equation we get

$$\dot{\vec{F}}(t) = \frac{\mathbf{M}}{N} \sum_{j=1}^{N} \dot{\vec{a}}_j(t) = \frac{\mathbf{M}}{N} \sum_{j=1}^{N} \vec{I}_j(t) \tag{2.21}$$

and comparing this result with equation (2.16) we come to the identification

$$\alpha = \frac{\mathbf{M}}{N} \tag{2.22}$$

which completes the interpretation of the coupling constant α.

Clearly, equation (2.16) is only one of equations which should determine the individual influences of each particle. We still need more equations to determine the individual influences of each particle. Whatever their form may be, they should not contain then total force $\vec{F}(t)$ because the behaviour of an individual confined particle should be independent of the interaction of the system as a whole with its environment. Consequently, the environment should never be able to disjoin the particles into separate clusters or constituents. This may be achieved if we postulate that the only equation in which participate $\vec{F}(t)$ is equation (2.16). The eventual equations for determining $\vec{F}(t)$ should arise from treating at least two many-particle systems because the force $\vec{F}(t)$ is connected with the interaction of the system as a whole and not with the interaction of its individual constituents.

Since the remaining equations which contain individual influences should determine only relative motion of the confined particles we must, instead of (2.12), postulate equations of type

$$\Phi_l\left(\vec{I}_{jk}, \dot{\vec{I}}_{jk}, \ddot{\vec{I}}_{jk}\right) = 0 \tag{2.23}$$

where

$$\vec{I}_{jk} \equiv \vec{I}_j - \vec{I}_k \tag{2.24}$$

are the relative influences.

Solving equations (2.23) together with (2.16) we will find all individual influences $\vec{I}_j(t)$ and we may integrate all kinematical equations (2.1)-(2.3). Each equation needs however some initial condition in order to specify the physical quantity determined by this equation, and the overall picture of the motion of particles will crucially depend on the choice of initial conditions. To keep the confined particles together in some finite regions of space, we must additionally assume the condition

$$\sup_{-\infty < t < \infty} |\vec{x}_j(t) - \vec{x}_k(t)| < \infty \tag{2.25}$$

This is a condition on the initial preparation of the system and may not be satisfied. Note, however, that (2.25) is not a defining property of confined particles but only one of their particular features.

3 Example

To illustrate the proposed approach, we complete the paper with a simple example of the system of two confined particles for which equations (2.23) are of the oscillator type

$$\ddot{\vec{I}}_{12} + \omega^2 \vec{I}_{12} = 0 \tag{3.1}$$

and for which the external force $\vec{F}(t)$ is constant in time. Clearly, equation (26) has the solution

$$\vec{I}_{12}(t) = \vec{I}\cos\omega t + \vec{J}\sin\omega t \tag{3.2}$$

where $\vec{I}$ and $\vec{J}$ are two integration constants which describe the internal structure of the system and which have to be determined from some kind of initial conditions. From (2.24) and (3.2) we

get then the individual influences in the form

$$\vec{I}_1(t) = \frac{1}{2}\vec{I}\cos\omega t + \frac{1}{2}\vec{J}\sin\omega t$$

$$\vec{I}_2(t) = -\frac{1}{2}\vec{I}\cos\omega t - \frac{1}{2}\vec{J}\sin\omega t$$

(3.3)

and integrating equations (2.1)-(2.3) we get the trajectories of particles in the form

$$\vec{x}_1(t) = \vec{\alpha}_1 + \vec{\beta}_1 t + \vec{\gamma}_1 t^2 - \frac{1}{2\omega^3}\vec{I}\sin\omega t + \frac{1}{2\omega^3}\vec{J}\cos\omega t$$

$$\vec{x}_2(t) = \vec{\alpha}_2 + \vec{\beta}_2 t + \vec{\gamma}_2 t^2 + \frac{1}{2\omega^3}\vec{I}\sin\omega t - \frac{1}{2\omega^3}\vec{J}\cos\omega t$$

(3.4)

where α_j, β_j and γ_j for $j = 1, 2$ are integration constants to be determined from initial conditions. Condition (2.25) requires that

$$\beta_1 = \beta_2$$

(3.4)

$$\gamma_1 = \gamma_2$$

Imposing now the initial conditions

$$\vec{x}_j(t_0) = \vec{x}_{j0}$$

$$\vec{v}_j(t_0) = \vec{v}_{j0}$$

(3.5)

$$\vec{a}_1(t_0) - \vec{a}_2(t_0) = \vec{a}_0$$

and using equation (2.20) we get the solution (3.4) in the form

$$\vec{x}_1(t) = \vec{x}_{10} + \frac{\vec{v}_{10} + \vec{v}_{20}}{2}(t - t_0) + \frac{\vec{F}}{2M}(t - t_0)^2 +$$

$$+ \frac{1}{2\omega^2}\vec{a}_0\left[1 - \cos\omega(t - t_0)\right] + \frac{\vec{v}_{10} + \vec{v}_{20}}{2\omega}\sin\omega(t - t_0)$$

(3.6)

$$\vec{x}_2(t) = \vec{x}_{20} + \frac{\vec{v}_{10} + \vec{v}_{20}}{2}(t - t_0) + \frac{\vec{F}}{2M}(t - t_0)^2 -$$

$$- \frac{1}{2\omega^2}\vec{a}_0\left[1 - \cos\omega(t - t_0)\right] - \frac{\vec{v}_{10} - \vec{v}_{20}}{2\omega}\sin\omega(t - t_0)$$

It is easy to see that the relative motion of the particles is independent of the force $\vec{F}$ and therefore no external force can disjoin the particles.

4   Acknowledgments

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SECOND QUANTIZATION IN BIT-STRING PHYSICS

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Abstract

Using a new fundamental theory based on bit-strings we derive a finite and discrete version of the solutions of the free one particle Dirac equation as segmented trajectories with steps of length h/mc along the forward and backward light cones executed at velocity $\pm c$. Interpreting the statistical fluctuations which cause the bends in these segmented trajectories as emission and absorption of radiation, these solutions are analagous to a fermion propagator in a second quantized theory. This allows us to interpret the mass parameter in the step length as the *physical* mass of the free particle. The radiation in interaction with it has the usual harmonic oscillator structure of a second quantized theory. We sketch how these free particle masses can be generated gravitationally using the *combinatorial hierarchy* sequence $(3, 10, 137, 2^{127} + 136)$, and some of the predictive consequences.

1 Bit-String Paths and Trajectories

Bit-String Physics, which we have also called *Discrete Physics*, [1, 2] grew out of the discovery of the *combinatorial hierarchy* by A.F. Parker-Rhodes in 1961. [3] A convenient introduction is provided by the Proceedings of the 9th meeting of the Alternative Natural Philosophy Association. [4] Recent work is summarized at the end of this paper.

In a technical sense, about all we need from the theory for this paper is the fact that we employ a universe of bit-strings generated by the algorithm called *program universe* in DP. Define a *bit-string* a containing W ordered bits by its sequentially ordered elements $a_w \in 0, 1$, $w \in 1, 2, 3, ..., W$, and its *Hamming measure* a by $a = \Sigma_{w=1}^{W} a_w := |\mathbf{a}(W)|$. Define *discrimination*, symbolized by "$\oplus$", between two bit-strings by the ordered elements $(a \oplus b)_w = (a_w - b_w)^2$; this is 1 when $a_w \neq b_w$ and 0 when $a_w = b_w$. Starting from a universe of strings of length W, all that program universe does is to pick two strings arbitrarily and discriminate them. If the result is non-null (i.e. the two strings differ), the discriminant is adjoined to the universe and the process begins again. If the two strings discriminate to the *null string* (i.e $0_w = 0$ for all w), we concatenate an arbitrary bit to the growing end of each string (i.e. $W \rightarrow W + 1$) and the process begins again.

We consider two strings **a**, **b** and their discriminant $\mathbf{a} \oplus \mathbf{b}$. Given no further information, we now show that the situation can be described by four integers which are invariant under any permutation of the ordering parameter w applied simultaneously to all three strings. Let n_{10} be the number of positions where $a_w = 1$, $b_w = 0$, n_{01} the number of positions where $a_w = 0$, $b_w = 1$, n_{11} the number of positions where $a_w = 1$, $b_w = 1$, and n_{00} the number of positions where $a_w = 0$, $b_w = 0$. Then

$$a = n_{10} + n_{11}; \quad b = n_{01} + n_{11}; \quad |\mathbf{a} \oplus \mathbf{b}| = n_{10} + n_{01} \tag{1}$$

$$n_{10} + n_{01} + n_{11} + n_{00} = W \tag{2}$$

Note that the three non-null Hamming measures a, b, $|\mathbf{a} \oplus \mathbf{b}|$ are independent of both n_{00} and W. Only one of those two parameters can be chosen arbitrarily, subject to the constraint that $W \geq n_{10} + n_{01} + n_{11}$, or $n_{00} \geq 0$. It is the *independence* of our result from both string length and permutation of the order parameter which allows the statistics of the bit-strings generated by program universe to differ from the binomial distribution usually assigned to Bernoulli sequences, or "random walks".

The "random walk" with which we will be concerned is obtained from our more general model by defining a single, shorter string of length $n_{10} + n_{01}$ by $c_w = 1$ if $a_w = 1$, $b_w = 0$, and $c_w = 0$ if $a_w = 0$, $b_w = 1$. Then $r = n_{10}$ is the number of 1's and $\ell = n_{01}$ is the number of 0's in $\mathbf{c}(r + \ell)$, We now view this situation as describing the "motion" of a "particle" which is taking discrete steps of length h/mc in space and h/mc^2 in time at velocity $\pm c$ along the light cones. This is the starting point suggested by Feynman[5] and articulated, for example, by Jacobson and Schulman[6] for a derivation of the Dirac equation in 1+1 dimensions. If the particle is assumed to start at the origin $(0,0)$ in the x, ct plane, their boundary condition on the trajectories connecting two events at $(0,0)$ and (x, ct) is $x = (r - \ell)(h/mc)$, $ct = (r + \ell)(h/mc)$. We tie our model to this same space-time *trajectory*, but as noted above include an additional degree of freedom.

We now classify any string $\mathbf{c}$ by the number of *bends* $k(\mathbf{c})$, which counts the number of times a sequence of 1's changes to a sequence of 0's or visa versa. As McGoveran discovered, this number is simply computed from the elements of $\mathbf{c}$ by $k(\mathbf{c}) = \Sigma_{w=1}^{W-1}(c_{w+1} - c_w)^2$. We are interested here in the number of bends in the trajectory string of length $r + \ell = n_{10} + n_{01}$ These strings fall into four classes: RR, LL, RL and LR. For class RR the first and last steps are to the right; it has $k + 1$ right-moving segments, k left-moving segments and k bends; note that $k = 0$ corresponds to the forward light cone. Similarly LL has $k + 1$ left-moving segments, k right-moving segments and k bends. RL and LR cannot have $k = 0$ and have k right-moving segments, left-moving segments *and* bends. This classification is the same as in Jacobson and Schulman, but our statistical treatment is different.

In order to distinguish the connectivity we make between the two events from the space-time trajectories considered by Feynman, we call them *paths*. It is the interpretation of the additional two parameters n_{11} and n_{00} that allows us to extend our single particle treatment to an interpretation that has features in common with second quantized relativistic field theory. In the case of a *statistically causal* trajectory, time ticks ahead at a constant rate. If the particle does not take a step to the right, it *must* take a step to the left. Although our particle follows the same trajectory in space, if we encounter an example of w corresponding to either n_{11} or n_{00} it *does not move* in the single particle configuration space that is all the Feynman approach contains. We interpret this as representing background processes going on in program universe which do not directly affect the particle. In a second quantized relativistic field theory these "disconnected diagrams" are the first to be removed in a renormalization program. Although conceptually crucial to the way we count numbers of paths, they do not enter directly into our calculations.

Using light cone coordinates, a bend can be specified by any one of the r positions on the forward light cone and by any one of the ℓ positions on the backward light cone. However, because of the greater freedom in our string generation, there is no statistical correlation between them. There are r^k ways we can pick a position on the forward light cone and ℓ^k on the left. All we need do is insure that the restrictions imposed by the four classes of trajectories given above

are met. Further, the order in which we make the choices is irrelevant, so we must divide each of these factors in the relative probability by $k!$. Since they are independent we must multiply $r^k/k!$ by $\ell^k/k!$ to get the (unnormalized) probability that both will occur in an ensemble of strings characterized by k bends and meeting our boundary conditions.

We conclude that the relative frequency of *paths* in the space of bit-strings of length $W > r + \ell$ which meet our space-time boundary conditions will have the values

$$P_k^{RL}(r, \ell) = [\frac{r^k}{k!}][\frac{\ell^k}{k!}] = P_k^{LR} \tag{3}$$

$$P_k^{RR}(r, \ell) = [\frac{r^{k+1}}{(k+1)!}][\frac{\ell^k}{k!}] \tag{4}$$

$$P_k^{LL}(r, \ell) = [\frac{r^k}{k!}][\frac{\ell^{k+1}}{(k+1)!}] \tag{5}$$

2 Formal Derivation of the Dirac Equation

Write the Dirac Equation in 1+1 dimensions with $h = 1 = c = 1 = m$ as

$$\psi_1 = (\partial/\partial t - \partial/\partial x)\psi_2; \quad \psi_2 = -(\partial/\partial t + \partial/\partial x)\psi_1 \tag{6}$$

With $z^2 = t^2 - x^2 = 4r\ell$, this equation is solved by

$$\psi_1 = J_0(z) + \frac{2r}{z}J_1(z); \quad \psi_2 = J_0(z) - \frac{2\ell}{z}J_1(z) \tag{7}$$

where J_0 and J_1 are the standard, real Bessel functions. We note that

$$J_0(z) = \Sigma_{j=0}(-1)^j(z/2)^{2j}/(j!)^2 = \Sigma_{j=0}(-1)^j[\frac{r^j}{j!}][\frac{\ell^j}{j!}] \tag{8}$$

Further

$$J_1 = -J_0' = \Sigma_{j=1}j(-1)^{j+1}(z/2)^{2j-1}/(j!)^2 \tag{9}$$

Hence

$$\frac{2r}{z}J_1 = \Sigma_{k=0}(-1)^k\frac{r^{k+1}}{k!}\frac{\ell^k}{k!} \tag{10}$$

$$\frac{2\ell}{z}J_1 = \Sigma_{k=0}(-1)^k\frac{r^k}{k!}\frac{\ell^{k+1}}{(k+1)!} \tag{11}$$

Since

$$J_1' = J_0 - \frac{1}{z}J_1 \tag{12}$$

we can now relate the solution of the differential equation to our relative frequency counts, as we now demonstrate.

We must now interpret the index 1, 2 in the Dirac equation, where it refers to the two spin-states, in the context of our bit-string model. We assume (since there is no coulomb interaction in the problem) that the bends in the trajectories correspond to the emission or absorption of

a γ-ray, and hence to a spin flip. We connect ψ_1 and ψ_2 with the two (global) laboratory spin projection states and the four classes of trajectories as follows. Let ψ_1 correspond to correspond to the wave function for which the laboratory spin projection is $+\frac{1}{2}\hbar$. Consider first the RR trajectories with $k+1$ right moving segments k left moving segments and k bends. For k *even* the relative frequency of such trajectories is $P_k^{RR}(r,\ell) = [\frac{r^{k+1}}{(k+1)!}][\frac{\ell^k}{k!}]$ as we have already seen. Assume that the particle starts moving to the right with positive spin; since it experiences an even number of spin flips,it will have at the end points spin projection $+\frac{1}{2}$ as desired. However, if it started to the left with the same positive spin projection, it would have to take an odd number of bends to end up moving to the right. But then it has an odd number of spin-flips. Since what is conserved is *global* rather than local spin, these cases must be subtracted from the first to get the net number of relative cases with positive spin projection. Consequently, for these two classes taken together, the contribution to ψ_1 of trajectories which end with a step to the right is

$$\psi_1^R = \Sigma_{k=0}(-1)^k[\frac{r^{k+1}}{(k+1)!}][\frac{\ell^k}{k!}] = \frac{2r}{z}J_1(z) \qquad (13)$$

Note that by including the $k=0$ case we have normalized the sum to the forward light cone; this we can do because only relative frequencies and no absolute probabilities are involved.

Note also that *negative frequencies* simply mean that we have a preponderance of cases with the *wrong* helicity compared to that specified by the label. Similarly, if we construct the relative frequencies of trajectories which end up moving to the left and contribute to ψ_1 we find that

$$\psi_1^L = \Sigma_{k=0}(-1)^k[\frac{r^k}{k!}][\frac{\ell^k}{k!}] = J_0(z) \qquad (14)$$

So

$$\psi_1 = \psi_1^R + \psi_1^L = J_0(s) + \frac{2r}{z}J_1(z). \qquad (15)$$

Similarly

$$\psi_2 = \psi_2^R + \psi_2^L = J_0(s) - \frac{2\ell}{z}J_1(z). \qquad (16)$$

Thus, *by imposing the spin projection conservation law* on our relative frequency counts, we arrive at the same formal expression that is obtained by the series solution of the free particle Dirac equation in 1+1 dimensions. Since, for either derivation, the truncation of the series is a practical necessity in any application to laboratory data, we have achieved our formal goal.

3 Second Quantized Interpretation

In our formal derivation, we avoided introducing a "free particle Hamiltonian"; we took our time evolution from the program universe generation of bit-strings. But the labeling of the two spin components ψ_1, ψ_2, was *ad hoc*. In a more detailed treatment, we would develop the spin, angular momentum, energy, momentum, and space-time discrete coordinates consistently from bit-strings. We will present this full discussion elsewhere. [7]

Here we must content ourselves with supplying a *label* to each of the three strings already invoked in our generation process. This can be simply the first two bits in the string. The system

we model consists of fermions labeled by $\mathbf{f} = (10)$, antifermions $\bar{\mathbf{f}} = (01)$, and bosons $\mathbf{b} = (11)$. To these labels we concatenate bit-strings representing the propagation of the three types of particle in either space-time or momentum-energy space. The general connection between the three when there is an interaction, corresponding roughly to a vertex in a Feynman Diagram, is

$$\mathbf{f} \oplus \bar{\mathbf{f}} \oplus \mathbf{b} = \mathbf{0} \tag{17}$$

In this broader context, the single particle trajectory we have been following can be thought of as a particle moving forward in time or an antiparticle moving backward in time, and the two events as space-like rather than time-like separated. This replaces left-right motion in space with forward-backward motion in time, and the spin conservation we invoked with fermion number minus antifermion number conservation. This not only extends our derivation to the full x, ct plane rather than confining it to the forward light cone, but also shows that the single particle "wave functions" we derived have the appropriate CPT symmetry for use as basis functions in a second quantized relativistic field theory.

Once we have accepted this extended context, we can interpret the bends in the single particle trajectory we used above as due to the emission and absorption of quanta with probability $r^k/k!$, etc. Thus the bends in the trajectory are analogous to the states of the atoms in the walls of a black body enclosure invoked by Planck in his derivation of the black body spectrum. Because of the connection to radiation we have established (elsewhere) in our theory, the appearance of the usual statistical factor makes contact with conventional theory. Derivation of the usual connection between radiation states and harmonic oscillator models can proceed in a normal fashion. This was one reason for presenting this new result at this Workshop. Rather than go on translating familiar results into unfamiliar language, or visa versa, we hope you will find it of more interest to hear where this new approach to fundamental theory leads.

To conclude this section, we emphasis that *any* free particle which satisfies the Dirac equation can be thought of as interacting with the radiation background, while retaining the same mass. The fact that our space-time propagation comes from *program universe* rather than from a Hamiltonian allows us to remove the major "self-energy" contribution which occurs in a second quantized field theory by symmetry and equate it to zero. Once we include interactions, there will be finite changes in the effective mass, but no infinite mass renormalization. Our theory is "born renormalized". For us the mass in the free particle Dirac equation a finite first approximation of the *physical* mass; it is *not* the "infinite bare mass" of renormalization theory.

4 A New Fundamental Theory

"Bit-string physics" is a new, fundamental theory based on information theoretic concepts derived primarily from recent work in computer science. This theory has already achieved considerable conceptual clarity and quantitative success. In this section we present an outline of the underlying concepts and how they find physical application, following closely an earlier summary. [8]

We start from sequential counter firings with space interval $L \pm \Delta L$ and time interval $T \pm \Delta T$. We base our theory on invariant squared-intervals $c^2 T^2 - L^2$ between counter firings. We model event intervals by *bit-strings* [i.e. finite ordered sequences of 0's and 1's] with N_1 1's and N_0 0's. We connect our model to laboratory events by taking $L = (N_1 - N_0)(h/mc)$, $T = (N_1 + $

$N_0)(h/mc^2)$. Calling any conceptual carrier of conserved quantum numbers between two distinct events a "particle", the velocity v of the particle is then given by $v = [(N_1 - N_0)/(N_1 + N_0)]c$.

If we now consider three counters, with associated clocks synchronized by limiting velocity signals, we can model the system by three bit-strings of the same length which add (using XOR, i.e. addition modulo 2) to the null string. The number of 1's in the strings satisfy the triangle inequalities, and hence can be used to define the angles between the lines connecting the counters. It also follows that the velocities as defined above satisfy the usual relativistic velocity addition law; this shows that our integer theory is "Lorentz invariant" for finite and discrete rotations and boosts. We prove that the usual position, momentum and angular momentum commutation relations follow from the fact that finite rotations in three dimensions do not commute.

In order to identify particles within the model we attach *labels* to the *content* strings which describe the (finite and discrete) space-time structure. Using 16 bits, the label gives us the 6 quarks, 3 neutrinos, $W^\pm, Z_0, \gamma$ and colored gluons of the standard model. Three strings which add to the null string map onto a Feynman diagram vertex. Baryon number, lepton number, charge and color are conserved; color is necessarily confined.

Mapping the $(2, 4, 16)$ decomposition of the labels onto $2^2 - 1 = 3$; $2^3 - 1 = 7$; $2^7 - 1 = 127$ we obtain the cumulative cardinals $(3, 10, 137)$, which are the first three levels of the four level *combinatorial hierarchy*, discovered by A.F. Parker-Rhodes in 1961. The first level describes chiral neutrinos, the second charged leptons and the third colored quarks. We justify the identification of the 137 as a first approximation to $\hbar c/e^2$ by correctly modeling the relativistic Bohr hydrogen atom, and improve on this result by deriving both the Sommerfeld formula and a logically consistent correction factor: $\hbar c/e^2 = 137/(1 - \frac{1}{30 \times 127}) = 137.0359\ 674$. [9] Weak-electromagnetic unification at the "tree level" comes about by using the same geometrical argument to calculate the electron mass in ratio to the proton mass either from the weak or the electromagnetic interaction and equating the two results. Predictions from the theory are given in Table I.

Extending our label length and mapping from 16 to 256 we get the fourth (terminal) cardinal of the *combinatorial hierarchy*: $2^{127} + 136 \approx 1.7 \times 10^{38} \approx \hbar c/Gm_p^2$, suggesting gravitational closure. Since we have baryon number conservation, we can consider an assemblage of nucleons and anti-nucleons with baryon number $+1$, charge $+e$, spin $\frac{1}{2}\hbar$ containing $N = \hbar c/Gm_p^2$ pairs with average separation $\hbar/m_p c$. Since the escape velocity for a massive particle from this assemblage exceeds c, it is gravitostatically stable against particle emission, but is unstable to energy loss due to Hawking radiation. Thanks to our baryon number conservation it ends up as a rotating, charged black hole with Beckenstein number $\hbar c/Gm_p^2$ [i.e. the number of bits of information lost in its formation [10]] which is indistinguishable from a (stable) proton. This extends Wheeler's "it from bit" [11] to particle physics. It *also* provides us with a *non-perturbative* mass scale relative to which mass ratios of particles which satisfy the free particle Dirac equation derived above can be measured.

Table I: **Coupling constants and mass ratios** predicted by the finite and discrete unification of quantum mechanics and relativity. Empirical Input: $c, \hbar$ and m_p as understood in the "Review of Particle Properties", Particle Data Group, *Physics Letters*, **B 239**, 12 April 1990.

COUPLING CONSTANTS

Coupling Constant	Calculated	Observed
$G^{-1}\frac{\hbar c}{m_p^2}$	$[2^{127}+136]\times[1-\frac{1}{3\cdot7\cdot10}]=1.693\,37\ldots\times10^{38}$	$[1.69358(21)\times10^{38}]$
$G_F m_p^2/\hbar c$	$[256^2\sqrt{2}]^{-1}\times[1-\frac{1}{3\cdot7}]=1.02\,758\ldots\times10^{-5}$	$[1.02\,682(2)\times10^{-5}]$
$\sin^2\theta_{Weak}$	$0.25[1-\frac{1}{3\cdot7}]^2=0.2267\ldots$	$[0.2259(46)]$
$\alpha^{-1}(m_e)$	$137\times[1-\frac{1}{30\times127}]^{-1}=137.0359\,674\ldots$	$[137.0359\,895(61)]$
$G^2_{\pi NN}$	$[(\frac{2M_N}{m_\pi})^2-1]^{\frac12}=[195]^{\frac12}=13.96..$	$^a[13,3(3),>13.9?]$

MASS RATIOS

Mass ratio	Calculated	Observed
m_p/m_e	$\dfrac{137\pi}{\frac{3}{14}\left(1+\frac27+\frac{4}{49}\right)\frac45}=1836.15\,1497\ldots$	$[1836.15\,2701(37)]$
$m_\pi^\pm/m_e$	$275[1-\frac{2}{2\cdot3\cdot7\cdot7}]=273.12\,92\ldots$	$[273.12\,67(4)]$
m_{π^0}/m_e	$274[1-\frac{3}{2\cdot3\cdot7\cdot2}]=264.2\,143\ldots$	$[264.1\,373(6)]$
m_μ/m_e	$3\cdot7\cdot10[1-\frac{3}{3\cdot7\cdot10}]=207$	$[206.768\,26(13)]$

$^a[G^2_{\pi N}=13.3(3)$ from R.A.Arndt *et.al.*, *Phys. Rev. Lett.*, **65**, 157 (1990). F.Sammarruca and R.Machleit (*Bull. Amer. Phys. Soc.*, **36**, No. 4 (1991)) note most modern models for the nuclear force use the strong empirical ρ coupling and therefore require $G^2_{\pi N}>13.9$; the smaller vector-meson-dominance-model value for ρ is compatible with the Arndt value.] Table I. **Coupling constants and mass ratios** predicted by the finite and discrete unification of quantum mechanics and relativity. Empirical Input: $c, \hbar$ and m_p as understood in the "Review of Particle Properties", Particle Data Group, *Physics Letters*, **B 239**, 12 April 1990.

5 Fundamental Principles

The theory has grown from results that many physicists rejected as "numerological" to a framework that provides a consistent way to compute several fundamental constants of physical interest. It is based on fundamental principles that we believe should appeal to physicists who are sympathetic to the operational approach of Bridgman and the early work of Heisenberg. These principles are finiteness, discreteness, finite computability, absolute non-uniqueness [eg. In the absence of further information, all members of a (necessarily finite) collection must be given equal weight.] and our procedures must be strictly constructive. For us, the mathematics in which the Book of Nature is written is finite *and* discrete. We model nature by *context sensitive* bits of information. In this sense we are participant observers.

Physics, as a science of measurement, can expect that at least some of the structures uncovered in nature could result from the way we perform experiments. For example, Stillman Drake [12] has discovered that Galileo measured the ratio of the time it takes for a pendulum to swing to the vertical through a small arc to the time it takes a body to fall from rest through an equal distance as $948/850 = 1.108\ 2....$ We now compute this ratio as $\pi/2\sqrt{2} = 1.110\ 7....$ Thus Galileo *measured* this constant to about 0.3 % accuracy. [13] We now believe that this constant will be the same "anywhere that bodies fall and pendulums oscillate" independent of the units of length and time.

In any theory satisfying our principles which counts events by a single sequence of integers, any metric when extended to large counts can have at most *three* homogeneous and isotropic dimensions in our finite and discrete sense synchronized by one universal ordering operator. [14] More complex degrees of freedom, indirectly inferred to be present at "short distance" automatically "compactify". Hence we can expect to observe at most three absolutely conserved quantum numbers at macroscopic distances and times. Guided by current experience, we can take these to be lepton number, charge and baryon number, connected to the z-component of weak isospin by the extended Gell-Mann Nishijima rule. These are reflected in the experimentally uncontroverted stability of the proton, electron and electron-type neutrino. This choice is empirical but not arbitrary, since structures with appropriate conservation laws isomorphic with this interpretation arise in our construction.

Take the chiral neutrino as specifying two states with lepton number ± 1 and no charge. They couple to the neutral vector boson Z_0. In the absence of additional information, these states *close*. The 4 electron states couple to two helical gamma's and the coulomb interaction. These seven states can be generated by any 3-vertex which includes two electron states and an appropriate gamma. These $3 + 7 = 10$ states when considered together then generate the $W^{\pm}$. This completes the leptonic sector in the first generation of the standard model of quarks and leptons. Bit-strings of length 6 provide a compact representation of these states which *closes* under *discrimination* (exclusive-or), and conserves both lepton number and the z component of weak isospin at each vertex. No unobserved states are predicted at this level of complexity, and no observed states are missing.

Two flavors of quarks and three colored gluons provide the seven elements of the baryonic sector which generate the inferred 127 quark-antiquark, 3 quark, 3 antiquark, 8 gluon ... states (16 fermions times a color octet minus the state with no quantum numbers) needed for the "valence level" description of the quark model. Bit-strings of length 8 provide a compact model using seven

discriminately independent basis strings and again close producing only the appropriate states at this level of complexity. Combining them with the leptonic states allows the strings representing the vector bosons to be extended to length 14, producing all the vertices and only the vertices which occur in the standard weak-electromagnetic unification of the first generation of the standard model. Extending the whole scheme to strings of length 16 we get the three generations which are observed experimentally (and a slot with the quantum numbers of the top quark). The quarks have baryon number 1/3 and charges $\pm 1/3, \pm 2/3$ as required. The $0 \leftrightarrow 1$ bit-string symmetry makes CPT invariance automatic. As already noted, if we have only three large distance quantum numbers, color (although conserved) is confined, and generation number is not conserved in flavor changing decays.

We are now in a position to talk about why we obtain the value of 137 in our first contact between the hierarchical structure generated by program universe and experimental numbers. *Empirically* only one of the 137 states required by the standard model of quarks and leptons corresponds to the coulomb interaction. Hence, by our principle of absolute non-uniqueness, the probability of this interaction occurring is 1/137 in the absence of further information.

Our basic quantum mechanical postulates are that (a) the square of the invariant interval between two events connected by a "particle" which carries conserved quantum numbers and conserved 3-momentum between them, is the product of two integers times $(h/mc)^2$ and that (b) space-like correlations for particle states with the same constant velocity can occur only an integer number n_λ of deBroglie wavelengths ($\lambda = h/p$) apart. These give us relativistic kinematics and the usual commutation relations for position, momentum and angular momentum.

If we model the hydrogen atom by events a distance r from a center we must have $n_\lambda \lambda = 2\pi r$. This interpretation is supported by noting that if the radius vector sweeps out equal areas in equal times, $\Delta A/\lambda^2 = (n_\lambda^2 - 1/4)(1/2\pi)^2$ and with $\ell = n_\lambda - 1/2$, the angular momentum is $\ell(\ell + 1)\hbar^2$. Since these events occur with probability $1/137n_\lambda$, we get the relativistic Bohr formula [15] for the hydrogen spectrum. When we include a second degree of freedom, and take proper account of the ambiguities in counting, we get not only the Sommerfeld formula but the formula for α given above. Similarly, the fact that the basic Fermi interaction involves 16 possible states of four fermions gives us $\sqrt{2}G_F = (256m_p)^{-2}$ where the square root comes from the conventional interaction Lagrangian to which experimental numbers are compared, and m_p comes from the stability of the proton.

Our critics sometimes compare the constants we compute with a calculation of the dielectric constant of diamond as an analogy to how complicated the number $\hbar c/e^2$ must be from their point of view. We accept the challenge. When they assert that the dielectric constant of diamond can be calculated from first principles, they must assume that they already *know* a number of physical constants. Of course one can relate the standards of mass, length and time as measured in the laboratory to three dimensional constants (which could be $c, \hbar$ and G) that occur, self-consistently, in several structures derived from "first principles". But to get to diamond they will also need α, m_e, and M_C in well defined relation to those units, as well as the fact that the carbon nucleus has charge +6 in units of e. Otherwise their calculation has no potential empirical test.

We claim that within their framework, these three numbers are too complicated to calculate from first principles. In fact, when Weinberg discusses *how* a finite coupling constant might emerge from currently acceptable theory, his errors are so large that he cannot even contemplate a quantitative prediction that can be confronted by experiment. In contrast my values for α,

and m_e are good to six or seven significant figures, and I can argue that my "first principles" allow me to predict that the common isotopes of carbon will have masses of approximately 12 and 13 proton masses. I have systematic ways of improving these estimates, and also— thanks to my physical cosmology — of estimating the relative abundance of these two isotopes on a terrestrial-type planet with an age of 4.5×10^9 years in a solar system of the kind in which we are conducting experiments. Somewhere along this line my calculation from "first principles" would find empirical supplements useful, but I believe no where near as soon as theirs.

I would locate the difference in point of view between us as coming from our different views of "space-time". If the "quantum vacuum" (which I would prefer to call a "quantum plenum") of renormalized second quantized relativistic field theory is the underlying concept, its properties certainly change as you "squeeze" it. The received wisdom today is that if the squeezing produces an energy density something like 10^{16} times that of the proton the "strong", "electromagnetic" and "weak" interactions come together (one basic "coupling constant" — grand unification) and that if one can extend the theory another three orders of magnitude, gravitation will find its appropriate place in the scheme. It seems to me that adopting "principles", however beautiful, that force one to go thirteen orders of magnitude beyond currently possible experimental tests to define fundamental parameters is — to say the least — a peculiar methodology for a physicist.

On the other hand, if one starts here and now with separated charges and massive particles and "empty" or "constructed" space as the first approximation, one can *measure* masses and coupling constants in a well defined way. If one can — as we claim — get good approximations for these values from "first principles" and systematically improve the predictions, I fail to see why such values cannot be considered "primordial". After the universe becomes optically thin, we predict about 2×10^{-10} baryons per photon. This both is in agreement with observation and supports our "empty space" philosophy.

6 Acknowledgements

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Covariant Harmonic Oscillators - 1973 Revisited

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Abstract

Using the relativistc harmonic oscillator, we give a physical basis to the phenomenological wave function of Yukawa which is covariant and normalizable. We show that this wave function can be interpreted in terms of the unitary irreducible representations of the Poincaré group. The transformation properties of these covariant wave functions are also demonstrated.

1   Introduction

Because wave functions play a central role in nonrelativistic quantum mechanics, one method of combining quantum mechanics and special relativity takes the form of efforts to construct relativistic wave functions with an approrpriate probability interpretation. The harmonic oscillator, which has the useful property of mathematical simplicity, has served as the first concrete solution to many new physical theories. It played a key role in the developing stages of nonrelativistc quantum mechanics, statistical mechanics, theory of specific heat, molecular theory, quantum field theory, theory of superconductivity, theory of coherent light, and many others. It is, therefore, quite natural to expect that the first nontrivial relativistic wave function would be a relativistic harmonic oscillator wave function[1, 2].

In connection with relativistic particles with internal space-time structure, Yukawa attempted to construct relativistic oscillator wave functions in 1953[3]. Yukawa observed that an attempt to solve a relativistic oscillator wave equation in general leads to infinite-component wave functions, and that finite-component wave functions may be chosen if a subsidiary condition involving the four-momentum of the particle is considered. This proposal of Yukawa was further developed by Markov,[4] Takabayasi,[5, 6] Sogami[7] and Ishida.[8]

The effectiveness of Yukawa's oscillator wave function in the relativistic quark model was first demonstrated by Fujimura et al.[9] who showed that the Yukawa wave function leads to the

correct high-energy asymptotic behavior of the nucleon form factor. The harmonic oscillator wave function was also rediscovered by Feynman *et al.*[10] who advocated the use of relativistic oscillators instead of Feynman diagrams for studying hadronic structures and interactions. The paper of Feynman *et al.* contains all the troubles expected from relativistic wave equations, and the authors of this paper did not make any attempt to hide those troubles.

The basic problem facing any relativistic harmonic oscillator equation is the negative-energy spectrum due to time-like excitations. It had once been widely believed that any attempt to obtain finite- component wave functions by eliminating time-like excitations would lead to a violation of probability conservation. This belief did not turn out to be true. It is now possible to construct harmonic oscillator wave functions without time-like wave functions which form the vector spaces for unitary irreducible representations of the Poincaré group.

In Section 2, we formulate the problem by writing down the relativistically invariant differential equation which leads to the covariant harmonic oscillator formalism. In Section 3, we study solutions of the oscillator differential equation which are normalizable in the four-dimensional x, y, z, t space. In Section 4, representations of the Poincaré group for massive hadrons are constructed from the normalizable harmonic oscillator wave functions. It is shown that they form the basis for unitary irreducible representations of the Poincaré group, as well as that for the O(3)-like little group for massive particles. In Section 5, Lorentz transformation properties of the harmonic oscillator wave functions are studied. The linear unitary representation of Lorentz transformation is provided for the harmonic oscillator wave functions.

2 Covariant Harmonic Oscillator Differential Equations

We first consider the differential equation of Feynman *et al.*[10] for a hadron consisting of two quarks bound together by a harmonic oscillator potential of unit strength:

$$\left\{ -2\left[\left(\frac{\partial}{\partial x_a^\mu} \right)^2 + \left(\frac{\partial}{\partial x_b^\mu} \right)^2 \right] + \left(\frac{1}{16} \right)(x_a^\mu - x_b^\mu)^2 + m_0^2 \right\} \phi(x_a, x_b) = 0, \tag{1}$$

where x_a and x_b are space-time coordinates for the first and second quarks respectively. This partial differential equation has many different solutions depending on the choice of variables and boundary conditions.

In order to simplify the above differential equation, we introduce new coordinate variables:

$$\begin{aligned} X &= (x_a + x_b)/2, \\ x &= (x_a - x_b)/2. \end{aligned} \tag{2}$$

The four-vector X specifies where the hadron is located in space-time, while the variable x measures the space-time separation between the quarks. In terms of these variables, Eq. (1) can be

written as

$$\left(\frac{\partial^2}{\partial X_\mu^2} - m_0^2 + \frac{1}{2}\left[\frac{\partial^2}{\partial x_\mu^2}\right] \right) \phi(X, x) = 0. \tag{3}$$

This equation is separable in the X and x variables. Thus

$$\phi(X, x) = f(X)\psi(x), \tag{4}$$

and $f(X)$ and $\psi(x)$ satisfy the following differential equations respectively:

$$\left(\frac{\partial^2}{\partial X_\mu^2} - m_0^2 - (\lambda + 1) \right) f(X) = 0, \tag{5}$$

$$\frac{1}{2}\left(-\frac{\partial^2}{\partial x_\mu^2} + x_\mu^2 \right)\psi(x) = (\lambda + 1)\psi(x). \tag{6}$$

Eq. (6) is a Klein-Gordon equation, and its solution takes the form

$$f(X) = \exp\left[\pm i p_\mu X^\mu\right], \tag{7}$$

with

$$-P^2 = -P_\mu P^\mu = M^2 = m_0^2 + (\lambda + 1).$$

where M and P are the mass and four-momentum of the hadron respectively. The eigenvalue λ is determined from the solution of Eq. (7). We are using the same notation for the operator and eigenvalue for the hadronic four-momentum. This should not cause any confusion since we are dealing only with free hadronic states with a definite four-momentum.

As for the four-momenta of the quarks p_a and p_b, we can combine them into the total four-momentum and momentum-energy separation between the quarks:

$$\begin{aligned} P &= p_a + p_b, \\ q &= (p_a - p_b). \end{aligned} \tag{8}$$

P is the hadronic four-momentum conjugate to X. The internal momentum- energy separation q is conjugate to x provided that there exist wave functions which can be Fourier-transformed. If the momentum-energy wave functions can be obtained from the Fourier transformation of the space-time wave function, the differential equation in the q space is identical to the harmonic oscillator equation for the x space given in Eq. (7)

3 Normalizable Solutions of the Relativistic Oscillator Equation

Since we are quite familiar with the three-dimensional harmonic oscillator equation from nonrelativistic quantum mechanics, we are naturally led to consider the separation of the space and time

variables and write the four-dimensional harmonic oscillator equation of Eq.(1.6) as

$$\left(-\nabla^2 + \frac{\partial^2}{\partial t^2} + \left[\mathbf{x}^2 - t^2\right]\right)\psi(x) = (\lambda + 1)\psi(x). \tag{9}$$

However, the xt system is not the only coordinate system in which the differential equation takes the above form.

If the hadron moves along the Z direction which is also the z direction, then the hadronic factor $f(X)$ of Eq. (8) is Lorentz-transformed in the same manner as the scalar particles are transformed. The Lorentz transformation of the internal coordinates from the laboratory frame to the hadronic rest frame takes the form

$$
\begin{aligned}
x' &= x, \quad y' = y, \\
z' &= (z - \beta t)/(1 - \beta^2)^{1/2}, \\
t' &= (t - \beta z)/(1 - \beta^2)^{1/2},
\end{aligned}
\tag{10}
$$

where β is the velocity of the hadron moving along the z direction. The primed quantities are the coordinate variables in the hadronic rest frame. In terms of the primed variables, the oscillator differential equation is

$$\left(-\nabla'^2 + \frac{\partial^2}{\partial t'^2} + \left[\mathbf{x}'^2 - t'^2\right]\right)\psi(x) = (\lambda + 1)\psi(x). \tag{11}$$

This form is identical to that of Eq. (10) , due to the fact that the oscillator differential equation is Lorentz-invariant.[1]

Among many possible solutions of the above differential equation, let us consider the form

$$
\begin{aligned}
\psi_\beta &= \left(\frac{1}{\pi}\right)\left(\frac{1}{2}\right)^{(a+b+n+k)/2}\left(\frac{1}{a!b!n!k!}\right)^{1/2} H_a(x')H_b(y')H_n(z')H_k(t') \\
&\times \exp\left[-\frac{1}{2}(x'^2 + y'^2 + z'^2 + t'^2)\right],
\end{aligned}
\tag{12}
$$

where a, b, n and k are integers, and $H_a(x')$, $H_b(y')$ ...are the Hermite polynomials. This wave function is normalizable, but the eigenvalues are:

$$\lambda = (a + b + n - k) \tag{13}$$

Thus for a given finite value of λ, there are infinitely many possible combinations of a, b, n and k. The most general solution of the oscillator differential equation is infinitely degenerate.[3]

Because the wave functions are normalizable, all the generators of the Lorentz transformations are Hermitian operators. The Lorentz transformation applicable to this function space is therefore a *unitary* transformation. Indeed, we can write any function of the coordinate variables x, y, z and t as a linear combination of the above solutions. In particular, a solution of the oscillator equation

with a given set of quantum numbers in the hadronic rest frame can be written as a linear sum of infinitely many solutions in the hadronic rest frame as we shall see in Section 4.

It is very difficult, if not impossible, to give physical interpretations to infinite-component wave functions. For this reason, it is quite natural to seek a finite set from the infinite number of wave functions at least in one Lorentz frame. The simplest way to obtain such a finite set of wave functions is to invoke the restriction that there be no time-like oscillations in the Lorentz frame in which the hadron is at rest and that the integer k in Eqs. (13) and (14) be zero. In doing so, we are led to the following two fundamental questions:

(a). Is it possible to give physical interpretations to the wave functions belonging to the resulting finite set?

(b). Is it still possible to maintain Lorentz covariance with this condition?

Let us examine question (b) closely.

When the hadron moves along the z axis, the $k = 0$ condition is equivalent to

$$\left(t' + \frac{\partial}{\partial t'}\right)\psi_\beta(x) = 0. \tag{14}$$

The most general form of the above condition is

$$p_\mu \left(x^\mu - \frac{\partial}{\partial x_\mu}\right)\psi_\beta(x) = 0. \tag{15}$$

Thus the $k = 0$ condition is covariant. Once this condition is set, we can write the wave function belonging to this finite set as

$$\psi_\beta(x) = (1/\pi)[1/(2^a 2^b 2^n a!b!n!)]^{1/2} H_a(x') H_b(y') H_n(z') \exp\left[-\frac{1}{2}(x'^2 + y'^2 + z'^2 + t'^2)\right]. \tag{16}$$

Except for the Gaussian factor in the t' variable, the above expression is the wave function for the three-dimensional isotropic harmonic oscillator. This means that we can use the spherical coordinate system for the x', y' and z' variables. We shall see in Section 3 how these ideas form the basis for constructing representations of the Poincaré group.

Since the above oscillator wave functions are separable in the Cartesian coordinate system, and since the transverse coordinate variables are not affected by the boost along the z direction, we can omit the factors depending on the x and y variables when studying their Lorentz transformation properties. The most general form of the wave function given in Eq. (13) becomes

$$\psi_\beta^{n,k}(z', t') = \left[1/(\pi 2^n 2^k n!k!)\right] H_n z' H_k(t') \exp\left[-(\frac{1}{2})(z'^2 + t'^2)\right], \tag{17}$$

with

$$\lambda = (n - k)$$

The wave functions satisfying the subsidary condition of Eq. (16) take the simple form

$$\psi_\beta^n(x,t) = [1/(\pi 2^n n!)]^{1/2} H_n(z') \exp[-(1/2)(z'^2 + t'^2)], \qquad (18)$$

with

$$\lambda = n$$

. This normalizable wave function without excitations along the t' axis describes the internal space-time structure of the hadron moving along the z direction with the velocity parameter β. If $\beta = 0$, then the wave function becomes

$$\psi_0^n(x,t) = [1/(\pi 2^n n!)]^{1/2} H_n(z) \exp[-(1/2)(z^2 + t^2)], \qquad (19)$$

Thus

$$\psi_\beta^n(z,t) = \psi_0^n(z',t')$$

. We have therefore obtained the Lorentz-boosted wave function by making a passive coordinate transformation on the z and t coordinate variables.

Let us next study the orthogonality relations of the wave functions. Since the volume element is Lorentz-invariant:

$$dzdt = dz'dt', \qquad (20)$$

there is no difficulty in understanding the orthogonality relation:

$$\int \psi_\beta^{n'}(z,t)\psi_\beta^n(z,t)dzdt = \int \psi_0^{n'}(z,t)\psi_0^n(z,t) = \delta_{n',n}. \qquad (21)$$

However, a more interesting problem is the inner product of two wave functions belonging to different Lorentz frames. This inner product becomes

$$\int \psi_0^{n'}(z,t)\psi_\beta^n(z,t)dzdt = \delta_{n',n} \left[1 - \beta^2\right]^{(n+1)/2}. \qquad (22)$$

The remarkable fact is that the orthogonality in the quantum number n is still preserved because of the Lorentz invariance of the harmonic oscillator differential equation. The oscillator equation does not depend on the velocity parameter β. As for the factor $[1 - \beta^2]^{(n+1)/2}$ in Eq. (23), we note first that, when the oscillator is in the ground state, it becomes like a Lorentz contraction of a rigid rod by $[1 - \beta^2]^{1/2}$. Excited-state wave functions are obtained from the ground state wave function through repeated applications of the step operator:

$$\mid n, \beta >= \sqrt{1/n!} \left(z' - \frac{\partial}{\partial z'}\right)^n \mid 0, \beta > . \qquad (23)$$

The transformation property of each step-up operator is like that of z. Therefore, if the ground-state wave function is like a rigid rod along the z direction, the $n^{\underline{th}}$ excited state should behave like a multiplication of $(n + 1)$ rigid rods.[11, 2]

4 Irreducible Unitary Representations of the Poincaré Group

The Poincaré group consists of space-time translations and Lorentz transformations. Let us go back to the quark coordinates x_a and x_b in Eq. (1) and consider performing Poincaré transformations on the quarks. The same Lorentz transformation matrix is applicable to x_a, x_b, x as well as X. However, under the space-time translation which changes x_a and x_b to $x_a + a$ and $x_b + b$ respectively,

$$X \rightarrow X + a,$$
$$x \rightarrow x. \tag{24}$$

The quark separation coordinate x is not affected by translations. For this reason, the generators of translations for this system are

$$P_\mu = -i \frac{\partial}{\partial X^\mu}, \tag{25}$$

while the generators of Lorentz transformations are

$$M_{\mu\nu} = L^*_{\mu\nu} + L_{\mu\nu}, \tag{26}$$

where

$$L^*_{\mu\nu} = i\left(X_\mu \frac{\partial}{\partial X^\nu} - X_\nu \frac{\partial}{\partial X^\mu} \right),$$
$$L_{\mu\nu} = i\left(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} \right).$$

It is straight-forward to check that the ten generators defined in Eqs. (26) and (27) satisfy the commutation relations of the Poincaré group. We are interested in constructing normalizable wave functions which are diagonal in the Casimir operators P^2 and W^2:

$$P^2 = -\left(\frac{\partial}{\partial X^\mu} \right)^2,$$
$$= \frac{1}{2}\left(-\frac{\partial^2}{\partial x_\mu^2} + x_\mu^2 \right) + m_0^2, \tag{27}$$

$$W^2 = M^2 (\mathbf{L}')^2, \tag{28}$$

where

$$L'_i = -i\varepsilon_{ijk} x'_j \frac{\partial}{\partial x'_k}.$$

The eigenvalue of P^2 is $M^2 = m_0^2 + (\lambda + 1)$, and that for W^2 is $M^2 \ell(\ell + 1)$. M is the hadronic mass, and ℓ is the total intrinsic angular momentum of the hadron due to internal motion of the

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spinless quarks.[12] In addition, we can choose the solutions to be diagonal in the component of the intrinsic angular momentum along the direction of the motion. This component is often called the helicity. If the hadron moves along the Z direction, the helicity operator is L_3.

Because the spatial part of the harmonic oscillator equation in Eq. (12) is separable also in the spherical coordinate system, we can write its solution using spherical variables in the hadronic rest frame space spanned by x', y' and z'. The most general form of the solution is

$$\psi_{\beta\lambda\ell}^{k,m}(x) = R_\mu^\ell(r')Y^\mu\ell(\theta', \phi')[1/([\sqrt{\pi}2^k k!])]^{1/2}H_k(t')e^{-t'^2/2}, \tag{29}$$

where

$$
\begin{aligned}
r' &= [x'^2 + y'^2 + z'^2]^{1/2}, \\
\cos\theta' &= z'/r', \\
\tan\phi &= y'/x',
\end{aligned}
$$

and

$$\lambda = 2\mu + \ell - k. \tag{30}$$

$R_\mu^\ell(r')$ is the normalized radial wave function for the three-dimensional harmonic oscillator:

$$R_\mu^\ell(r) = (2(\mu!)/[\Gamma(\mu + \ell + 3/2)]^3)^{1/2}r^\ell L_\mu^{\ell+1/2}(r^2)e^{-r^2/2}, \tag{31}$$

where $L_\mu^{\ell+1/2}(r^2$ is the associated Laguerre function.[13] The above radial wave function satisfies the orthonormality condition:[14]

$$\int_0^\infty nr^2 R_\mu^\ell(r)R_\nu^\ell(r)dr = \delta_{\mu\nu}. \tag{32}$$

The spherical form given in Eq. (30) can of course be expressed as a linear combination of the wave functions in the Cartesian coordinate system given in Eq. (17).

The wave function of Eq. (30) is diagonal in the Casimir operators of Eqs. (28) and (29), as well as in L^3. It indeed forms a vector space for the $O(3)$-like little group.[15, 16] However, the system is infinitely degenerate due to excitations along the t' axis. As we did in Section 2, we can suppress the time-like oscillation by imposing the subsidiary condition of Eq. (16), or by restricting k to be zero in Eq. (31). The solution then takes the form

$$\psi_{\beta\lambda\ell}^m(x) = R_\mu^\ell(r')Y_\ell^m(\theta', \phi')[(1/\pi)^{1/4}\exp(-t'^2/2)], \tag{33}$$

with

$$\lambda = 2\mu + \ell.$$

Thus for a given λ, there are only a finite number of solutions. The above spherical form can be expressed as a linear combination of the solutions without time-like excitations in the Cartesian coordinate system given in Eq. (17).

We can now write the solution of the differential equation of Eq. (1) as

$$\phi(X, x) = e^{\pm iP \cdot X} \phi^m_{\beta \lambda \ell}(x). \tag{34}$$

This wave function describes a free hadron with a definite four-momentum having an internal space-time structure which can be described by an irreducible unitary representation of the Poincaré group. The representation is unitary because the portion of the wave function depending on the internal variable x is square-integrable, and all the generators of Lorentz transformations are Hermitian operators. We shall study in the next section how these wave functions are Lorentz transformed.

5 Transformation Properties of the Harmonic Oscillator Wave Functions

If the hadronic velocity is zero, then its rest frame coincides with the laboratory frame. The wave function then is

$$\psi_0(x) = R^\ell_\mu(r) Y^m_\ell(\theta, \phi) [(1/\pi)^{1/4} \exp(-t^2/2)]. \tag{35}$$

The simplest way to obtain the wave function for the moving hadron is to replace the r, θ and ϕ variables in the above expression by their primed counterparts. This produces Eq. (30). However, we are interested in obtaining the wave function for a moving hadron as a linear combination of the wave functions for the rest frame. If we apply the boost operator to the wave function for the hadron at rest,

$$\psi^{\ell m}_{\beta \lambda}(x) = [e^{-i\eta K_3}] \psi^{\ell m}_0 \lambda(x), \tag{36}$$

where K_3 is the boost generator along the z axis, its form is

$$K_3 = -i \left(z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z} \right), \tag{37}$$

and η is related to velocity parameter β by

$$\sinh \eta = \beta/(1 - \beta^2)^{1/2}.$$

Both the rest-frame and moving-frame wave functions have the same set of eigenvalues for the Casimir operators P^2 and W^2 of the Poincaré group.

These eigenstate wave functions are linear combinations of the Cartesian forms in their respective coordinate systems. If the hadron moves along the z direction, the x and y variables remain invariant. Therefore, we use the wave function of Eq. (19) with $\beta = 0$:

$$\psi^{n,0}_0 = [1/(\pi 2^n n!)]^{1/2} H_n(z) \exp[-(1/2)(z^2 + t^2)]. \tag{38}$$

The superscript 0 indicates that there are no time-like excitations: $k = 0$. We are now led to consider the transformation

$$
\begin{aligned}
\psi_\beta^{n,0}(z,t) &= [\exp(-i\eta K_3)]\psi_0^{n,0}(z,t) \\
&= \psi_0^{n,0}(z',t'),
\end{aligned}
\tag{39}
$$

and ask what the boost operator $\exp(-i\eta K_3)$ does to $\psi_0^{n,0}(z,t)$.

This boost operator of course changes z and t to z' and t' respectively as is indicated above. However, we are interested in whether the transformation can take the linear form

$$
\psi_\beta^{n,0}(z,t) = \sum_{n',k'} A_{n'k'}^{n,0}(\beta)\psi_0^{n'k'}(z,t).
\tag{40}
$$

Because the oscillator differential equation is Lorentz invariant, the eigenvalue λ of Eq. (18) remains invariant, and only the terms which satisfy the condition

$$
n = (n' - k')
\tag{41}
$$

make non-zero contributions in the sum. Thus the above expression can be simplified to

$$
\psi_\beta^{n,0}(z,t) = \sum_{k=0}^{\infty} A_k^n(\beta)\psi_0^{n+k,k}(z,t).
\tag{42}
$$

This is indeed a *linear unitary representation of the Lorentz group*. The representation is infinte-dimensional because the sum over k is extended from zero to infinity.[17]

The remaining problem is to determine the coefficient $A_k^n(\beta)$. Using the orthogonality relation, we can write

$$
\begin{aligned}
A_k^n(\beta) &= \int dz dt \psi_0^{n+k,k}(z,t)\psi_\beta^{n,0}(z,t) \\
&= \frac{1}{\pi}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{1/2}\left(\frac{1}{n!(n+k)!}\right)^{1/2} \\
&\quad \times \int dz dt H_{n+k}(z)H_k(t)H_n(z') \\
&\quad \times \exp\left(-\frac{1}{2}(z^2 + z'^2 + t^2 + t'^2)\right).
\end{aligned}
\tag{43}
$$

In this integral, the Hermite polynomials and the Gaussian form are mixed with the kinematics of Lorentz transformation. However, if we use the generating function for the Hermite polynomial, the evaluation of the integral is straightforward, and the result is

$$
A_k^n(\beta) = (1 - \beta^2)^{(1+n)/2}\beta^k\left(\frac{(n+k)!}{n!k!}\right)^{1/2}.
\tag{44}
$$

Thus the linear expansion given in Eq. (41) can be written as

$$
\begin{aligned}
\psi_\beta^{n,0}(z,t) &= [1/(2^n\pi)]^{1/2}(1-\beta)^{(n+1)/2}(\exp[-(z^2+t^2)/2]) \\
&\quad \times \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\left(\frac{\beta}{2}\right)^k H_{n+k}(z)H_k(t).
\end{aligned}
\tag{45}
$$

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As was indicated with respect to Eq. (20), this linear transformation has to be unitary. Let us check this by calculating the sum

$$S = \sum_{k=0}^{\infty} |A_k^n(\beta)|^2 .$$

(46)

According to Eq. (45), this sum is

$$S = [1 - \beta^2]^{(n+1)} \sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!}(\beta^2)^k.$$

(47)

On the other hand, the binomial expansion of $[1 - \beta^2]^{-(n+1)}$ takes the form

$$[1 - \beta^2]^{-(n+1)} = \sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!}\beta^{2k}.$$

(48)

Therefore the sum S is equal to one. The linear transformation of Eq. (43) is indeed a unitary transformation.

It is also of interest to see how this transformation can be achieved directly in terms of solutions which are eigenstates of the Casimir operators. For this purpose we construct the solutions in terms of the spherical coordinate variables for the three-dimensional (x, y, z) space and treat t separately. If the hadron is at rest,

$$\psi_{0\lambda\ell}^{k,m}(x) = R_\mu^\ell(r)Y_\ell^m(\theta, \phi)[1/(\sqrt{\pi}2^k k!)]^{1/2}H_k(t)e^{-t^2/2}.$$

(49)

Thus we have to write the generators of Lorentz transformations in terms of these variables. The three rotation generators can be written as[13]

$$
\begin{aligned}
L_3 &= -i\frac{\partial}{\partial\phi}, \\
L_\pm &= L_1 \pm L_2 \\
&= \pm e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right).
\end{aligned}
$$

(50)

It is not difficult to calculate the three boost generators. They take the form

$$
\begin{aligned}
iK_3 &= \cos\theta\left(r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}\right) - \frac{t}{r}\sin\theta\frac{\partial}{\partial\theta}, \\
iK_\pm &= K_1 \pm iK_2 \\
&= e^{\pm i\phi}\left(r\frac{\partial}{\partial t} + t\sin\theta\frac{\partial}{\partial r} - \frac{t}{r}\cos\theta\frac{\partial}{\partial\theta} \pm \frac{t}{r\sin\theta}\frac{\partial}{\partial\theta}\right).
\end{aligned}
$$

(51)

The rotation generators affect only the spherical harmonics in the wave function of Eq. (50). Thus

$$
\begin{aligned}
L_3\psi_{0\lambda\ell}^{k,m} &= m\psi_{0\lambda\ell}^{k,m}, \\
L_\pm\psi_{0\lambda\ell}^{k,m} &= \sqrt{(\ell \mp m)(\ell \pm m + 1)}\psi_{0\lambda\ell}^{k,m\pm 1}.
\end{aligned}
$$

(52)

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The above relations mean that rotations do not change the quantum numbers λ, ℓ and k. They only change m. Eq. (53) indeed corresponds to the fact that the little group for massive hadrons is like $SO(3)$.

On the other hand, if we apply the boost generators, we end up with somewhat complicated formulas:

$$
\begin{aligned}
iK_3\psi_{\lambda\ell}^{km} &= \left[\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y_{\ell+1}^m(\theta,\phi)Q_{-t}F_{\lambda\ell}^k(r,t) \\
&+ \left[\frac{(\ell+1)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y_{\ell-1}^m(\theta,\phi)Q_{\ell+1}F_{\lambda\ell}^k, \\
iK_\pm &= \left[\frac{(\ell\pm m+1)(\ell\pm m+2)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y_{\ell+1}^{m\pm1}(\theta,\phi)Q_{\pm(\ell}F_{\lambda\ell}^k \\
&+ \left[\frac{(\ell\mp m)(\ell\mp m-1)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y_{\ell+1}^{m\pm1}(\theta,\phi)Q\pm(\ell+1)F_{\lambda\ell}^k(r,t). \quad (53)
\end{aligned}
$$

where

$$
Q_t = \left(t\frac{\partial}{\partial r} + r\frac{\partial}{\partial t} + \ell\frac{t}{r}\right),
$$

and

$$
F_{\lambda\ell}^k(r,t) = R_\mu^\ell(r)[1/(\sqrt{\pi}2^k k!)]^{1/2}H_k(t)\exp(-t^2/2).
$$

K_3 does not change the value of m, while K_+ and K_- change m by $+1$ and -1 respectively. In addition, unlike the rotation operators, the boost generators change λ, ℓ and k. This is a manifestation of the fact that the unitary representation is infinite-dimensional as is indicated in Eq. (43).

It is possible to finish the calculation by explicitly carrying out the differentiations contained in the Q_t operators. However, this does not appear necessary, because we already know what the answer should be from our experience with the Cartesian coordinate sytem.

6   Conclusion

The harmonic oscillator applied to the symmetric quark model has withstood the test of time. The work of Karr[18, 19] has fully integrated the field theorectic aspects of this work. Below we present the experimental present status of the non-strange baryon in relation to the harmonic oscillator.

TABLE I. Mass spectrum of nonstrange baryons. The calculated masses based on Eqs. (9.1) and (9.2) in Kim and Noz,[2] **Theory and Applications of the Poincaré Group**. The experimental masses are from "Physical Review D" **45**, No. 11, (June, 1992). The last column contains the identification code of the pion-nucleon resonance used in Particle Data Group. For N = 0 and N = 1, the quark model multiplet scheme is in excellent agreement with the experimental world. For N = 2, the model seems to work well, but more work is needed on both the theoretical and experimental fronts. There are still very few particles in N = 3. Baryonic masses are measured in MeV.

N	L	SU(6)	SU(3)	Spin	J	Calculated Mass	Experimental Mass	PDG-ID
0	0	56	8	1/2	1/2	940	939	P_{11}****
			10	3/2	3/2	1240	1232	P_{33}****
1	1	70	8	1/2	1/2	1520	1535	S_{11}****
					3/2	1520	1520	D_{13} ****
			8	3/2	1/2	1688	1650	S_{11} ****
					3/2	1688	1700	D_{13} ***
					5/2	1688	1675	D_{15} ****
			10	1/2	1/2	1652	1620	S_{31} ****
					3/2	1652	1700	D_{33} ****
2	0	56	8	1/2	1/2	1480	1440	P_{11} ****
			10	3/2	3/2	1780	1600	P_{33} **
		70	8	1/2	1/2	1730	1710	P_{11} ***
			8	3/2	3/2	1898	1900	P_{13} *
			10	1/2	1/2	1862	1750	P_{31} *
2	2	56	8	1/2	3/2	1660	1720	P_{13} ****
					5/2	1660	1680	F_{15} ****
			10	3/2	1/2	1960	1910	P_{31} ****
					3/2	1960	1920	P_{33} ***
					5/2	1960	1905	F_{35} ****
					7/2	1960	1950	F_{37} ****
	2	70	8	1/2	3/2	1900		
					5/2	1900	2000	F_{15} **
			8	3/2	1/2	2078	2100	P_{11} *
					3/2	2078		
					5/2	2078		
					7/2	2078	1990	F_{17} **
			10	1/2	3/2	2030		
					5/2	2030	2000	F_{35} *

Table I. Mass spectrum of nonstrange baryons continued.

N	L	SU(6)	SU(3)	Spin	J	Calculated Mass	Experimental Mass	PDG-ID
3	1	70	8	1/2	1/2	2060		
					3/2	2060	2080	D_{13} **
			8	3/2	1/2	2228	2090	S_{11} *
					3/2	2228		
					5/2	2228		
			10	1/2	1/2	2192	1900	S_{31} ***
					3/2	2192		
		70	8	1/2	1/2	2060		
					3/2	2060		
			8	3/2	1/2	2228		
					3/2	2228		
					5/2	2228		
			10	1/2	1/2	2192		
			10	1/2	1/2	2192		
					3/2	2192		
		56	8	1/2	1/2	1810		
					3/2	1810		
			10	3/2	1/2	2110	2150	S_{31} *
					3/2	2110	1940	D_{33} *
					5/2	2110	1930	D_{35} ***
	2	70	8	1/2	3/2	2180		
					5/2	2180		
			8	3/2	1/2	2348		
					3/2	2348		
					5/2	2348		
					7/2	2348	2190	G_{17} ****
			10	1/2	3/2	2312		
					5/2	2360		
	3	70	8	1/2	5/2	2528		
					7/2	2528		
			8	3/2	3/2	2528		
					5/2	2528	2200	D_{15} **
					7/2	2528		
					9/2	2528	2250	G_{19} ****
			10	1/2	5/2	2492		
					7/2	2492	2200	G_{37} *
		56	8	1/2	5/2	2110		
					7/2	2110		
			10	3/2	3/3	2410		
					5/2	2410	2350	D_{35} *
					7/2	2410	2390	F_{37} *
					9/2	2410	2400	G_{39} **

TABLE II. In addition, there are resonances which do not fit in this table. Since most of these resonaces correspond to even- parity baryons, they presumably belong to $N = 4$ multiplet.

SU(3)	J	Mass	PDG-ID
8	3/2	1540	P_{13} *
8	9/2	2220	H_{19} ****
8	11/2	2600	$I_{1,11}$ ***
8	13/2	2700	$K_{1,13}$ **
10	1/2	1550	P_{31} *
10	9/2	2300	H_{39} **
10	7/2	2390	F_{37} *
10	11/2	2420	$H_{3,11}$ ****
10	13/2	2750	$I_{3,13}$ **
10	15/2	2950	$K_{3,15}$ **

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CALCULATION OF THE NUCLEON STRUCTURE FUNCTION FROM THE NUCLEON WAVE FUNCTION

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Abstract

Harmonic oscillator wave functions have played an historically important role in our understanding of the structure of the nucleon, most notably by providing insight into the mass spectra of the low-lying states. High energy scattering experiments are known to give us a picture of the nucleon wave function at high-momentum transfer and in a frame in which the nucleon is travelling fast. This paper presents a simple model that crosses the twin bridges of momentum scale and Lorentz frame that separate the pictures of the nucleon wave function provided by the deep inelastic scattering data and by the oscillator model.

1 Introduction

While a prediction for the nucleon structure functions from first principles in quantum chromodynamics (QCD) seems, even now, a remote prospect, if we are ever to claim a deep understanding of the structure of the nucleon, a clear interpretation of such gross properties as the neutron-proton structure function ratio (R^{np}) and the polarization asymmetry ($A^{\gamma p}$) of the proton structure function is essential. A notable attempt to relate these features to the nucleon rest-frame wave function was made by Le Yaouanc *et al.* [1, 2, 3], who employed non-relativistic harmonic oscillator spatial wave functions and SU(6) mixing in an attempt to formulate predictions both about the structure functions and the nucleon form factors. While both the large-x behavior of R^{np} and the initial slope of the neutron electric form factor were well accounted for by the inclusion of an admixture $\underline{70}$ excited state in the nucleon wave function, the signs of the mixing angles obtained in the two cases were observed to disagree.

Against the structure-functions calculation of Le Yaouanc *et al.* may be raised the objection that there is no clear prescription for Lorentz-transforming a non-relativistic wave function. It is this concern that will be addressed in this paper. Less widely recognized is an objection that can be raised against the treatment of the form factors by Le Yaouanc *et al.* The latter calculation

involves the assumption that the nucleon's charge (magnetization) density and electric (magnetic) form factor are related by Fourier transformation. The Fourier relationship holds only when the Lorentz transformation of the spatial wave functions is ignored. There are several models which are known to predict a non-zero neutron electric form factor in the absence of SU(6) mixing[4, 5, 6]. In such models, which employ plausible relativistic spin wave functions, the matrix elements involved in the determination of the form factors cannot be factorized into a product of spin, isospin, and spatial matrix elements. Since the spin wave functions play no role in the structure function calculation, the possibility must be considered that the structure functions provide the correct value for the mixing angle.

The spatial wave functions that shall be considered are the "definite metric"[7] solutions to the relativistic harmonic oscillator equation of Feynman *et al.*[8] In their original work, Feynman *et al.* used the non-normalizable "indefinite metric" solutions of their wave equation. These solutions yield divergent form factors as $-q^2$ increases. To remedy this, they multiplied all matrix elements by an ad hoc factor. The "definite metric" solutions are normalizable and, when used to calculate nucleon form factors, yield the proper q^2 behavior, a dipole fall-off for large $-q^2$, without any adjustments. These solutions also help to illuminate features of the structure functions and the parton model, as will be seen later on.

In Section 2 the relativistic harmonic oscillator equation and its normalizable solutions are reviewed. The behavior of these solutions under Lorentz's transformation is discussed, and their form in the infinite momentum frame is exhibited. In Section 3 the infinite-momentum-frame relativistic-oscillator nucleon wave function is combined with QCD momentum scaling incorporated via the valon model of Hwa.[9] The proton and neutron structure functions are considered within the context of the resulting model, and a value for the mixing angle for an admixture of $\underline{70}$ excited state is calculated. In Section4, the significance of this calculation is reviewed.

2 The Relativistic Oscillator Model

For simplicity of discussion, the relativistic oscillator model is introduced for the two particle case first. This model describes the binding of a pair of quarks to from a meson via the differential equation

$$\left\{ 2 \left[\partial_1^2 + \partial_2^2 \right] - \left(\omega^2/16 \right) (x_1 - x_2)^2 - m_0^2 \right\} \Psi(x_1, x_2) = 0 \tag{1}$$

where x_1 and x_2 represent the space-time coordinates of the two constituent quarks, and the metric convention is defined by $-g_{00} = g_{ii} = 1$. The quark spin will be ignored here, though versions of the relativistic oscillator model have been formulated to include spin 1/2 quarks.[10, 11] Eq. (1) is readily solved via separation of variables in terms of the coordinates

$$X_\mu = 1/2\,(x_{1\mu} + x_{2\mu})$$
$$x_\mu = 1/2(x_{1\mu} - x_{2\mu}) \tag{2}$$

where the X_μ are the space-time coordinates of the meson center of momentum and the x_μ determine the space-time separation of the quarks. The separated equations are

$$\left(\partial_X^2 - m^2\right)\psi(X) = 0 \tag{3}$$

and

$$\left(-\partial_x^2 + \omega^2/4x^2 + m_0^2\right)\varphi(x) = m^2\varphi(x) \tag{4}$$

where $\Psi(x_1, x_2) = \psi(X)\varphi(x)$. Eq. (3) is the Klein-Gordon equation for a meson of mass m, while Eq. (4) describes a four-dimensional harmonic oscillator.

Eq. (4) is itself separable in terms of the space-time components x_μ, while the eigenvalue m^2 is given by a linear combination of the eigenvalues corresponding to each of the component equations. In the timelike direction, an increase in the excitation quantum number corresponds to a more negative contribution to the mass squared. To eliminate a degree of freedom which is not observed in nature, and to eliminate, as well, the unphysical possibility of imaginary mass, the oscillator-model solutions are required to obey the subsidiary condition

$$\frac{\partial}{\partial X_\mu}\left(\frac{1}{\sqrt{\omega}}\frac{\partial}{\partial x^\mu} + \frac{\sqrt{\omega}}{2}x_\mu\right)\Psi(X, x) = 0. \tag{5}$$

This condition suppresses timelike excitations in the meson rest frame.

The solutions to Eq. (3) have the familiar form $\exp(iP^\mu X_\mu)$ where P_μ is the four-momentum corresponding to the meson center-of-momentum coordinates X_μ. The solutions to Eq. (4) are products of oscillator solutions in each of the space-time components, with the solution in the timelike coordinate in the restframe being restricted to the fundamental mode via Eq. (5). Such solutions can be written as

$$\begin{aligned}
\varphi_n^{b,k,w}(x') &= \omega/2\pi\left(2^{b+k+w}b!k!w!\right)^{-1/2}H_b[x_r\sqrt{\omega/2}]H_k[y_r\sqrt{\omega/2}]H_w[z_r\sqrt{\omega/2}] \\
&\quad \times \exp\left[-\omega/4\left(x_r^2 + y_r^2 + z_r^2 + t_r^2\right)\right]
\end{aligned} \tag{6}$$

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where H denotes a Hermite polynomial and x' denotes the four vector x, represented in terms of its components, x_r, y_r, z_r and t_r in the meson rest frame. The invariant meson square mass corresponding to P^μ is required by Eq. (3) to be equal to m^2, while Eq. (4) determines that $m^2 = \omega(b + k + w + 1) + m_0^2$.

The above solutions form a complete set of normalized rest-frame solutions. The wave function in a frame in which the meson is not at rest is specified by the Lorentz transformation between the meson rest frame and the frame in which it is moving. For example, the ground state in an arbitrary frame can be written as

$$
\begin{aligned}
\Psi(X, x) &= \omega/(2\pi) \left(2^{b+k+w} b! k! w! \right)^{-1/2} e^{iP^\mu X_\mu} \\
&\quad \times \exp\left\{ -\omega/4[x_\mu^2 - 2(P \cdot x)^2 / P^2] \right\}.
\end{aligned}
\tag{7}
$$

The construction of relativistic-oscillator momentum-space wave functions in arbitrary frames is equally straightforward. Figure 1 provides a pictorial view of the effect of the Lorentz transformation on the rest-frame wave function, both in coordinate space and in momentum space. The bound state quarks are seen to acquire lightlike momenta in the frame where the meson is moving rapidly. The success of the parton model tells us that this should be the case.

Modelling of the nucleon requires that a three particle version of the relativistic oscillator be considered. A harmonic interaction between each pair of quarks is assumed, and the governing differential equation takes the form

$$
\left\{ 3 \left[\partial_1^2 + \partial_2^2 + \partial_3^2 \right] - \omega^2/36 \left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \right] - U^0 \right\} \Psi(x_1, x_2, x_3) = 0
\tag{8}
$$

where x_1, x_2 and x_3 are the space-time coordinates of three constituent quarks.

Separation of variables can be implemented in terms of the coordinates X, r and s, defined as

$$
\begin{aligned}
X_\mu &= 1/3 \left(x_{1\mu} + x_{2\mu} + x_{3\mu} \right) \\
r_\mu &= 1/6 (x_{1\mu} + x_{2\mu} - 2x_{3\mu}) \\
s_\mu &= -1/(2\sqrt{3})(x_{1\mu} - x_{2\mu}).
\end{aligned}
\tag{9}
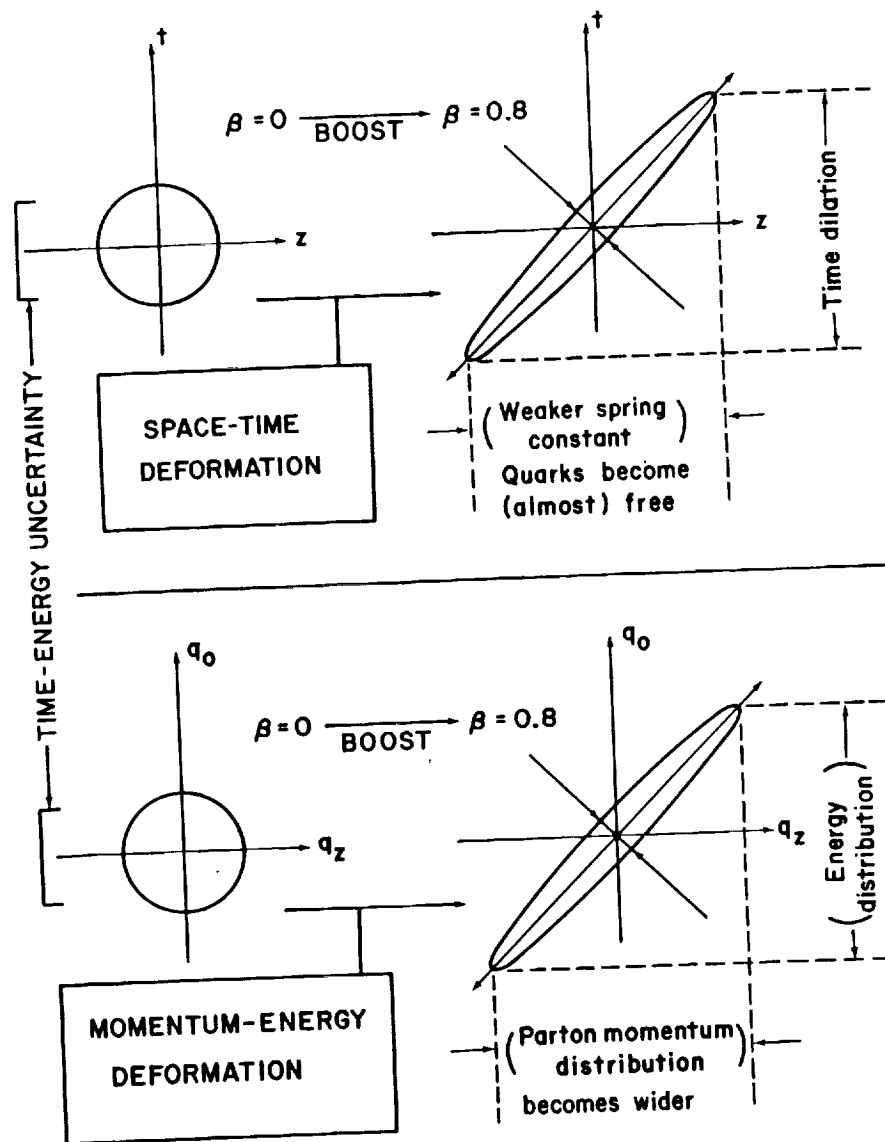$$

QUARKS ⟶ PARTONS

FIG. 1 The Lorentz deformation properties of the relativistic oscillator in coordinate space and in momentum space.

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The wave function $\Psi(x_1, x_2, x_3)$ can be written in terms of these variables as $\psi(X)\varphi(r)\theta(s)$ where $\psi(X)$ satisfies Eq. (3) while $\phi(r)$ and $\theta(s)$ satisfy respectively

$$-1/2(\partial_r^2 - \omega^2 r^2)\varphi(r) = \lambda_r \varphi(r)$$
$$-1/2(\partial_s^2 - \omega^2 s^2)\theta(s) = \lambda_s \theta(s). \tag{10}$$

The square mass is Eq. (3) in the nucleon case is then given by $U^0 + \lambda_r + \lambda_s$. To remove the unphysical timelike degree of freedom from the nucleon spectrum of states determined by Eqs. (10), the three-particle relativistic oscillator equation is supplemented by a pair of subsidiary conditions that suppress such excitations in the nucleon rest frame.

Application of the relativistic oscillator model to the structure functions requires construction of the momentum-space wave function in the infinite momentum frame. In a frame in which a meson whose rest-frame wave function is given by (6) is travelling with velocity parameter β along the z-direction, the internal momentum-space wave function is

$$\varphi^w(p, \beta) = \left(\frac{2/\omega}{\pi 2^n w!}\right)^{1/2} H_w\left[\left(\frac{2}{\omega}\right)^{1/2}\left(\frac{p_z - \beta p_0}{(1 - \beta^2)}\right)^{1/2}\right]$$
$$\times \exp\left[-1/\omega((p_z - \beta p_0)^2 + (p_0 - \beta p_z)^2)/(1 - \beta^2)\right] \tag{11}$$

where p represents the momentum conjugate to the internal separation coordinate x, and where transverse degrees of freedom have been neglected. As $\beta \to 1$, the square magnitude of $\varphi(p, \beta)$ becomes singular along the forward light cone, while vanishing everywhere else. Integrating along the direction perpendicular to the forward light cone results in a distribution for the internal light-cone momentum $p_+(= p_0 + p_z)$ given by

$$\rho(p_+) = \lim_{\beta \to 1} \int dp_- \mid \varphi^w(p, \beta) \mid^2 \tag{12}$$

where $p_- = p_0 - p_z$. The distribution $\rho(p_+)$ is converted into a distribution in Feynman x by setting $p_{1+} = xP$ and requiring $\rho(x)dx = \rho(p_+)dp_+$.

A similar procedure may be followed in the three-particle case. For three particles the result is[12]

$$\rho(x) = 3m/(2\pi\omega)^{1/2} \sum_{i=1}^{w} \binom{w}{i} (1/i!)(1/2)^i H_i^2[(m/\sqrt{2\omega})(1 - 3x)]$$
$$\times \exp[-(m^2/2\omega)(1 - 3x)^2] \tag{13}$$

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in general, and

$$\rho_0(x) = 3m/(2\pi\omega)^{1/2} \cdot \exp[-(m^2/2\omega)(1 - 3x)^2] \tag{14}$$

if the nucleon is assumed to be described by the oscillator ground state wave function. The variable x in Eqs. (13) and (14) is the momentum fraction variable. A calculation of the proton charge structure function $F_2^{ep}(x)$, can, for example, be based directly on Eq. (14). The result is

$$F_2^{ep}(x) = <e_i^2> mx/(2\pi\omega)^{1/2} \cdot \exp[(-9m^2/2\omega)(x - 1/3)^2] \tag{15}$$

where the average of the charge e_i is taken over the three valence quarks. This calculation ignores scaling effects predicted by QCD and yields only qualitative agreement with experiment.

3 Structure Functions

A valon is a bound state or constituent quark whose internal structure is probed in high energy interactions. To be completely general, valons of different spin as well as flavor should be differentiated. Let $G_{v/N}(x)$ represent a momentum-fraction probability distribution for a valon of type v (v representing spin and flavor) in the nucleon N. A nucleonic structure function $F^N(x, Q^2)$ is expressed in terms of convolutions of $G_{v/N}(x)$ with corresponding structure functions for the valons:

$$F^N(x, Q^2) = \sum_v \int_x^1 dx' G_{v/N}(x') F^v(x/x', Q^2). \tag{16}$$

The Q^2 dependence of the structure functions appears only in $F^v(x, Q^2)$. QCD evolution Eq. (13) for the moments of the struct ure functions are used to express this dependence. According to Eq. (16), the moments of a nucleon structure function are given by a sum of products of moments:

$$M^N(n, Q^2) = \sum_v M_{v/N}(n) M^v(n, Q^2) \tag{17}$$

where

$$M^{N,v}(n, Q^2) = \int_0^1 dx \, x^{n-2} F^{N,v}(x, Q^2) \tag{18}$$

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and

$$M_{v/N}(n) = \int_0^1 dx\, x^{n-1} G_{v/N}(x). \tag{19}$$

The evolution equations are the basis for assuming a form for the moments $M^v(n, Q^2)$ of the structure functions $F^v(x, Q^2)$. The $F^v(x, Q^2)$ are understood to be determined by the quark distributions within the valon, v, which distributions can be broken up into components that behave as singlets and as nonsinglets under flavor transformation. The moments $M^v(n, Q^2)$ are correspondingly expressed in terms of singlet and nonsinglet moments, which are defined to be the scaling factors governing the evolution of the moments of such quark distributions in lowest-order, twist-2 QCD. The nonsinglet moments are given by

$$M_{NS}(n, Q^2) = \exp(-d_{NS}^n s) \tag{20}$$

while the singlet moments are

$$M_s(n, Q^2) = 1/2(1 + \rho_n)\exp(-d_+^n s) + 1/2(1 - \rho_n)\exp(-d_-^n s) \tag{21}$$

where

$$s = \ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right). \tag{22}$$

The coefficients d_{NS}^n, d_+^n, d_-^n and ρ_n come from the renormalization group analysis.[13] The constant, Λ, is the usual scaling param eter while Q_0 represents the "starting point" of the evolution.

Since valons of different helicity as well as flavor are to be distinguished, four separate valon distributions will be required to characterize the nucleon. The corresponding moments are denoted as

$$\begin{aligned}
U_\uparrow(n) &= M_{U\uparrow/p}(n) = M_{D\uparrow/n}(n) \\
D_\uparrow(n) &= M_{D\uparrow/p}(n) = M_{U\uparrow/n}(n) \\
U_\downarrow(n) &= M_{U\downarrow/p}(n) = M_{D\downarrow/n}(n) \\
D_\downarrow(n) &= M_{D\downarrow/p}(n) = M_{U\downarrow/n}(n)
\end{aligned} \tag{23}$$

where the symbol ↑ (↓) denotes that the valon's helicity is par allel (antiparallel) to that of the nucleon, and where the identification of valon distributions within the neutron with the corresponding isospin-reversed distributions in the proton follows from charge symmetry. In terms of the singlet and nonsinglet moments Eqs. (20) and (21) and the valon moments Eq. (23), the moments of the nucleon structure functions $F^{2p}(x, Q^2)$ and $F^{2n}(x, Q^2)$ are given by

$$M^{2p}(n, Q^2) = 2/9[2U(n) + D(n)]M_s(n, Q^2) + 1/9[4U(n) - D(n)]M_{NS}(n, Q^2) \qquad (24)$$

$$M^{2n}(n, Q^2) = 2/9[2U(n) + D(n)]M_s(n, Q^2) - 2/9[U(n) - D(n)]M_{NS}(n, Q^2) \qquad (25)$$

with

$$\begin{aligned} U(n) &= U_\uparrow(n) + U_\downarrow(n) \\ D(n) &= D_\uparrow(n) + D_\downarrow(n). \end{aligned} \qquad (26)$$

It is easily verified that these equations describe the lowest-order twist-2 QCD evolution of the moments of F^{2p} and F^{2n} from a starting point at which the nucleon is viewed as consisting of its three bound-state quarks.

Eqs. (24) and (25) were used by Hwa[9] in conjuncti on with experimental moments of F^{2p} and F^{2n} to obtain fitted values for the parameters Q_0 and Λ. These equations are first order, and will therefore not be accurate for low Q^2. Ideally we would like to evolve the bound-state momentum-space wave function from the energy scale Q_0^2 at which the nucleon is describable as a bound state of its three constituent quarks (with, perhaps, an oscillator-like momentum distribution), out to high Q^2 where the structure functions are observed. The fitted parameter Q_0^2 is an approximation for Q_0^2 in the sense that the lowest-order evolution equations are used. This approximation is a key feature of the valon model and is discussed in detail in.[14] The goal of Hwa's fitting procedure was to obtain estimates for the functions $G_{v/N}(x)$. In Figure 2, an "average" valon distribution obtained in[9] by neglecting spin and flavor dependence is compared with $\rho_0(x)$ given by Eq. (14)

Let us now introduce a 70 component of SU(6) into the nucleon wave function in the form

$$\Psi = [\cos\theta\phi_0 \mid 56 >_s + (\sin\theta/\sqrt{2})(\phi_\alpha \mid 70 >_\alpha + \phi_\beta \mid 70 >_\beta)] \cdot \exp(-iP \cdot X). \qquad (27)$$

The ϕ's represent the spatial wave functions; ϕ_0 is the harmonic oscillator ground state, while ϕ_α and ϕ_β are taken to b e excited states with total harmonic oscillator quantum number $n = 2$ and zero orbital angular momentum. The subscripts α and β refer to the two possible types of mixed symmetry which are characterized by the behavior of the (three quark) wave function under exchange of the first and second quarks. The form of the excited-state component is uniquely determined in the oscillator model. The 70 state that involves $n = 1$ oscillator wave functions is disallowed because it is of the wrong parity. No other $n = 2$ state with the same quantum

numbers as the nucleon interferes with the ground state to produce the SU(6) breaking effects that are observed in the structure functions. The wavefunction Eq. (27) leads to spin-and-flavor dependent valon distributions of the form
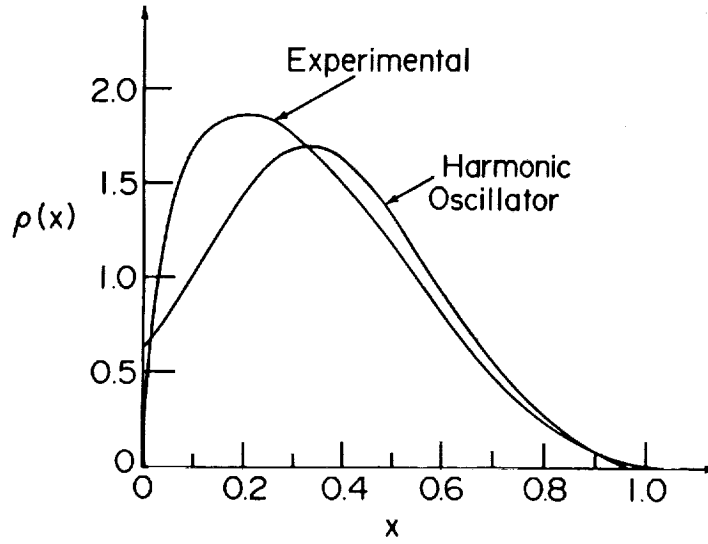

FIG. 2 A comparison of Hwa's "average" valon distribution with $\rho_0(x)$ defined by the infinite-momentum-frame relativistic-oscillator momentum-space wavefunction.

$$
\begin{aligned}
G_{U\uparrow/p}(x) &= (3m/\sqrt{2\pi\omega})\left[5/6\cos^2\theta + \sin^2\theta\left\{5/36h(x) + 1/3i(x)\right\} - 2\sqrt{6}/9\cos\theta\sin\theta j(x)\right] \\
&\quad \times \exp\left[-(m^2/2\omega)(1-3x)^2\right] \\
G_{U\downarrow/p}(x) &= (3m/\sqrt{2\pi\omega})\left[1/6\cos^2\theta + \sin^2\theta\left\{1/36h(x) + 1/3i(x)\right\} + \sqrt{6}/18\cos\theta\sin\theta j(x)\right] \\
&\quad \times \exp\left[-(m^2/2\omega)(1-3x)^2)\right] \\
G_{D\uparrow/p}(x) &= (3m/\sqrt{2\pi\omega})\left[1/3\cos^2\theta + \sin^2\theta\left\{1/18h(x) + 2/3i(x)\right\} + \sqrt{6}/9\cos\theta\sin\theta j(x)\right] \\
&\quad \times \exp\left[-(m^2/2\omega)(1-3x)^2)\right] \\
G_{D\downarrow/p}(x) &= (3m/\sqrt{2\pi\omega})\left[2/3\cos^2\theta + 1/9\sin^2\theta + 2\sqrt{6}/9\cos\theta\sin\theta j(x)\right] \\
&\quad \times \exp\left[-(m^2/2\omega)(1-3x)^2)\right]
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
h(x) &= 43/16 + m^2/8\omega(1-3x)^2 + m^4/16\omega^2(1-3x)^4 \\
i(x) &= 5/8 + m^2/8\omega(1-3x)^2 \\
j(x) &= 1/4 - m^2/4\omega(1-3x)^2.
\end{aligned}
\tag{29}
$$

Moments $U(n)$ and $D(n)$ determined from the above distributions were used in Eqs. (24) and (25) to obtain fits for experimental moments[15] of $F^{2p}(x)$ and $F^{2n}(x)$ derived from the CHIO

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muon data[16] and SLAC electron data[17] at $Q^2 = 22.5$ GeV^2. A somewhat large value of Q^2 was chosen to minimize target mass and higher twist effects that may be present in the data. The ratios $R^{np}(x)$ and $A^{\gamma p}(x)$ do not app ear to show any appreciable Q^2 dependence. The extension of the tails of the distributions into the unphysical regions $x < 0$ and $x > 1$ was ignored for purposes of computing the moments. The resulting small deviation from the Adler sum rule does not appear to lead to noticeable discrepancies.
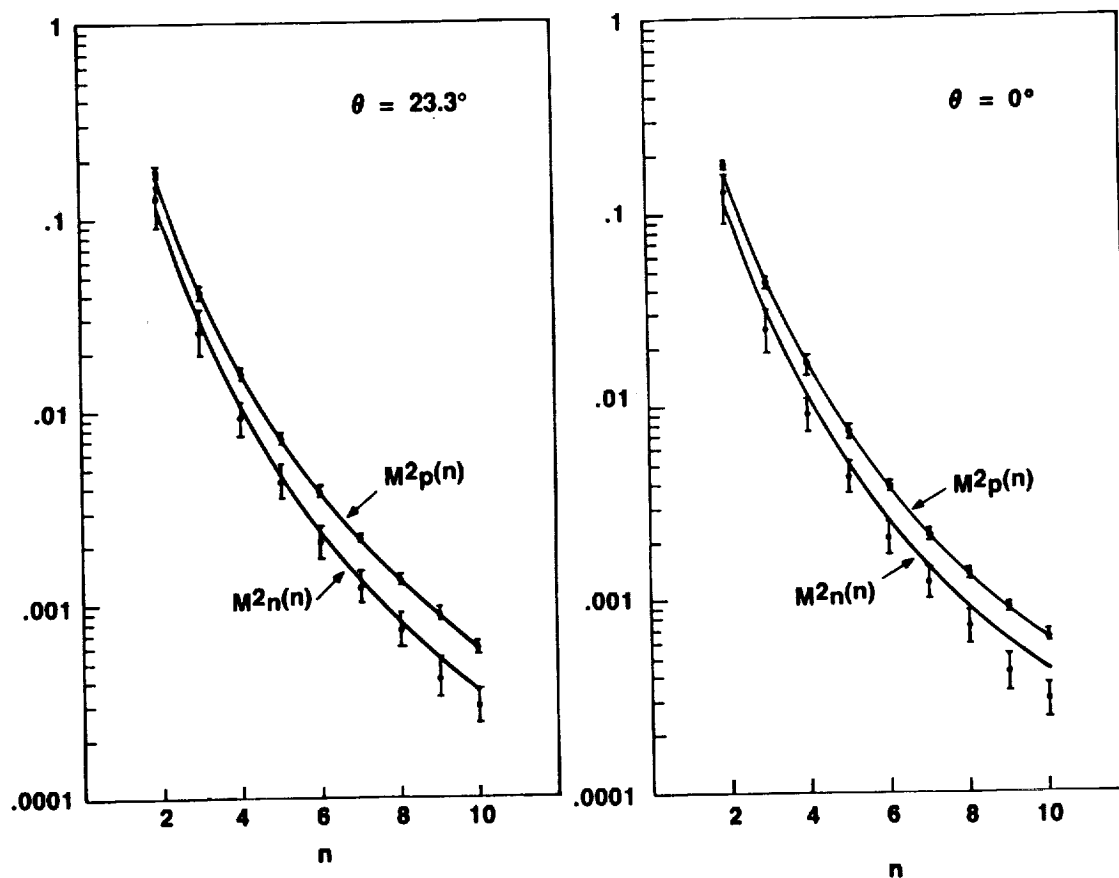
FIG. 3 The moments of the nucleon structure functions vs. n as fitted by Eqs. (24) - (26) in conjunction with Eq. (29). Fitted moments at $\theta = 0°$ and at $\theta = 23.3°$ are represented by the solid curves and are compared with data from[15].

The fitted moments were functions of two parameters - the mixing angle θ and the scaling variable s defined in Eq. (22). χ^2 minimization was used to determine the best fit. The χ^2 function in this case cannot be taken as an absolute indication of the quality of the fit due to the statistical interdependence among the moments of F^{2p} and F^{2n}. χi^2 was used, rather, as a relative determinant of merit, so that the quality of the fit as a function of θ could be evaluated. The minimum of χ^2 occurs at $\theta = 23.3°$, and a positive mixing angle is clearly preferred. Figure 3 compares the best-obtainable predicted moments from Eqs. (24) and (25) for $\theta = 0°$ and for $\theta = 23.3°$ with

the experimental moments. At $\theta = 0°$, the fitted moments of F^{2n} fall outside the error limits for large n. With the inclusion of the $\underline{70}$ state in the wave function at a mixing angle of $23.3°$, a simultaneous fit to the moments of F^{2p} and F^{2n} appears more reasonable, although the fitted moments of F^{2n} remain somewhat large for large n.

4 Conclusion

The simple model presented in this paper falls short of providing us with the ability to draw precise numerical correspondences between nucleonic bound state properties and the structure function data. The model does, however, address the crucial questions of Lorentz transformation and momentum scaling that must be considered if such corespondences are ever to be drawn. The approximate agreement between $\rho_0(x)$ and Hwa's phenomenologically-determined valon distribution (see Figure 2) allows us to believe that some of the essential physics is being captured. The value of the SU(6) $\underline{70}$ state mixing coefficien t obtained in this model via a simultaneous fit to proton and neutron structure function moments is very close to the original value determined by Le Yaouanc *et al.* This fact, together with the dependence of the form factors on the nucleon spin wavefunction, lends credence to the idea that the observed behavior of R^{np} and $A^{\gamma p}$ can be reliably interpreted as evidence of SU(6) mixing in the nucleon wavefunction.

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13. ABSTRACT *(Maximum 200 words)*

This conference publication contains the proceedings of a workshop on harmonic oscillators held at the College Park Campus of the University of Maryland on March 25-28, 1992.

The harmonic oscillator formalism has been and still is playing an important role in many branches of physics. This is the simplest mathematical device which can connect the basic principle of physics with what we observe in the real world. The oscillator formalism is, therefore, a very useful language in establishing communications among the physicists interested in fundamental principles and those interested in describing what we observe in laboratories. Researchers in different branches of physics, such as atomic, nuclear and particle physics, quantum optics, statistical and thermal physics, foundations of quantum mechanics and quantum field theory, and group representations, are developing possible future theories. The harmonic oscillator is the bridge between pure and applied physics.

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