SYMMETRIES OF COUPLED HARMONIC OSCILLATORS

D. Han

National Aeronautics and Space Administration,
Goddard Space Flight Center, Code 902, Greenbelt, Maryland 20771

Y. S. Kim

Department of Physics, University of Maryland, College Park, Maryland 20742

Abstract

It is shown that the system of two coupled harmonic oscillators possesses many interesting symmetries. It is noted that the symmetry of a single oscillator is that of the three-parameter group $Sp(2)$. Thus two uncoupled oscillator exhibits a direct product of two $Sp(2)$ groups, with six parameters. The coupling can be achieved through a rotation in the two–dimensional space of two oscillator coordinates. The closure of the commutation relations for the generators leads to the ten–parameter group $Sp(4)$ which is locally isomorphic to the deSitter group $O(3,2)$.

1   Introduction

Since the classical mechanics of two coupled harmonic oscillators is known to every physicist, there is a tendency to believe that this oscillator problem is completely understood and that nothing new can be learned from it. We are writing this note because there are so many new lessons to learn from the system of coupled oscillators. The system shares symmetries with a number of physical models of current interest, such as the Lee model in quantum field theory [1], the Bogoliubov transformation in superconductivity [2, 3], two-mode squeezed states of light [4, 5, 6], the covariant harmonic oscillator model for the parton picture [7], and models in molecular physics [8]. There are also models of current interest in which one of the variables is not observed, including thermo-field dynamics [9], two-mode squeezed states [10, 11], the hadronic temperature [12].

From the mathematical point of view, the standard approach is to construct a suitable representation of the symmetry group after writing down its generators. The symmetry group in the present case is $Sp(4)$ with ten generators [4, 6]. However, it is extremely difficult to study physics in terms of ten parameters. We should somehow start with a smaller number of parameters.

For example, let us consider the three–dimensional rotation group with three generators. We need only two generators to describe rotations [13]. The third generator is produced during the process of constructing a closed set of commutation relations. For the coupled oscillators, a reasonable approach is to start with simpler groups describing two uncoupled oscillators. We can then introduce an additional generator to couple the two oscillators. The number of generators of the resulting group may be larger than the sum of those for the two starting groups plus the additional generator to couple them.

The process of constructing a larger group from two smaller groups is quite common in physics. We are quite familiar with the "direct product" and "semi–direct product," where the number of generators is the sum of those for the two smaller groups. We shall use the word "construction of the group by soldering two subgroups," when the resulting group has more generators than all those for the starting groups and the additional generator(s) to couple them. We need this new word "soldering" in order to reduce the number of input parameters in physics.

The soldering process takes different forms. We can construct the three dimensional rotation group by soldering the one-parameter rotation group around the x axis and another rotation group around the y axis. The resulting group has three generators. As we shall see in this paper, we can construct the $O(3, 2)$ deSitter group by soldering two $O(3, 1)$ Lorentz groups. In this case, we solder them by observing that the two $O(3, 1)$ groups share the same rotation group. We start with nine generators, but the resulting $O(3, 2)$ deSitter group has ten generators.

Since the symmetry group of each uncoupled oscillator is the three-parameter $Sp(2)$ group, and since it is likely that one more group operation is needed to couple the system, we start here with seven generators. We shall see in this paper that the resulting group is $Sp(4)$ with ten generators. It is easier to study physics with seven generators than with ten.

It is also shown in this paper that the $Sp(4)$ symmetry does not exhaust all possible symmetries of the coupled oscillator system. It is noted that the group $Sp(4)$ is a subgroup of a lager group $SL(4, r)$. Possible physical implications of this larger symmetry group are discussed.

In Sec. 2, we shall study linear canonical transformations in the four–dimensional phase space consisting of two pairs of canonical variables. It is noted that the symmetry group is $Sp(4)$ which is locally isomorphic to the $O(3, 2)$ deSitter group. In Sec. 3, we shall see how the $O(3, 2)$ group can be constructed from two $(3 + 1)$–dimensional Lorentz groups. In Sec. 3, we shall construct the symmetry group of two coupled oscillators from the symmetry group of each oscillator.

Section 5 contains a new parametrization of the coupled oscillator system which is consistent with that of the symmetry group. In Sec. 6, we discuss the quantum mechanics of the oscillator system and the unitary transformations which correspond to canonical transformations in classical mechanics. In Sec. 7, we discuss physical applications of the formalism developed in this note. Finally, in Sec. 8, we discuss scale transformations in phase space and their implications in measurement theory.

2 Linear Canonical Transformations in Classical Mechanics

For a dynamical system consisting of two pairs of canonical variables x_1, p_1 and x_2, p_2, we can introduce the four-dimensional coordinate system:

$$(\eta_1, \eta_2, \eta_3, \eta_4) = (x_1, x_2, p_1, p_2).$$
(1)

Then the transformation of the variables from η_i to ξ_i is canonical if

$$MJ\tilde{M} = J,$$
(2)

where

$$M_{ij} = \frac{\partial}{\partial \eta_j}\xi_i,$$

and

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

For linear canonical transformations, we can work with the group of four-by-four real matrices satisfying the condition of Eq.(2). This group is called the four-dimensional symplectic group or $Sp(4)$. While there are many physical applications of this group, we are interested here in constructing the representations relevant to the study of two coupled harmonic oscillators.

It is more convenient to discuss this group in terms of its generators G, defined as

$$M = \exp(-i\alpha G), \tag{3}$$

where G represents a set of purely imaginary four-by-four matrices. The symplectic condition of Eq.(2) dictates that G be symmetric and anticommute with J or be antisymmetric and commute with J.

In terms of the Pauli spin matrices and the two-by-two identity matrix, we can construct the following four antisymmetric matrices which commute with J of Eq.(2).

$$J_1 = \frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \qquad J_2 = \frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

$$J_3 = \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \qquad J_0 = \frac{i}{2} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{4}$$

The following six symmetric generators anticommute with J.

$$K_1 = \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad K_2 = \frac{i}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad K_3 = -\frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

and

$$Q_1 = \frac{i}{2} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad Q_2 = \frac{i}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad Q_3 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \tag{5}$$

These generators satisfy the commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = [Q_i, Q_j] = -i\epsilon_{ijk}J_k,$$

$$[J_i, J_0] = 0, \qquad [K_i, Q_j] = i\delta_{ij}J_o,$$

$$[J_i, Q_j] = i\epsilon_{ijk}Q_k, \qquad [K_i, J_o] = iQ_i, \qquad [Q_i, J_o] = -iK_i. \tag{6}$$

The group of homogeneous linear transformations with this closed set of generators is called the symplectic group $Sp(4)$. The J matrices are known to generate rotations while K and Q matrices generate squeezes [6]

It is often more convenient to study the physics of four-dimensional phase space using the coordinate system

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (x_1, p_1, x_2, p_2). \tag{7}$$

The transformation from $(\eta_1, \eta_2, \eta_3, \eta_4)$ is

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_{3,} \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix}, \tag{8}$$

and the J matrix becomes

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & .0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{9}$$

In this new coordinate system, the rotation generators take the form

$$J_1 = \frac{-1}{2} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad J_2 = \frac{i}{2} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

$$J_3 = \frac{-1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \qquad J_0 = \frac{-1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{10}$$

The squeeze generators become

$$K_1 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \qquad K_2 = \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad K_3 = -\frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

$$Q_1 = \frac{i}{2} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad Q_2 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad Q_3 = \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{11}$$

When we deal with canonical transformations of functions of the coordinate variables, we have to use the differential operators. The rotation generators are [6]

$$J_1 = +\frac{i}{2} \left\{ \left(x_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial x_1} \right) + \left(x_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial x_2} \right) \right\},$$

$$J_2 = -\frac{i}{2} \left\{ \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) + \left(\frac{p_1 \partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1} \right) \right\},$$

$$J_3 = +\frac{i}{2} \left\{ \left(x_1 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial x_1} \right) - \left(x_2 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial x_2} \right) \right\},$$

$$J_0 = +\frac{i}{2} \left\{ \left(x_1 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial x_1} \right) + \left(x_2 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial x_2} \right) \right\}, \tag{12}$$

and the six squeeze generators are

$$K_1 = -\frac{i}{2} \left\{ \left(x_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial x_1} \right) - \left(x_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial x_2} \right) \right\},$$

$$K_2 = -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial x_1} - p_1\frac{\partial}{\partial p_1}\right) + \left(x_2\frac{\partial}{\partial x_2} - p_2\frac{\partial}{\partial p_2}\right)\right\},$$

$$K_3 = +\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial x_2}\right)\right\},$$

$$Q_1 = +\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial x_1} - p_1\frac{\partial}{\partial p_1}\right) - \left(x_2\frac{\partial}{\partial x_2} - p_2\frac{\partial}{\partial p_2}\right)\right\},$$

$$Q_2 = -\frac{i}{2}\left\{\left(x_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial x_2}\right)\right\},$$

$$Q_3 = -\frac{i}{2}\left\{\left(x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right) - \left(p_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial p_2}\right)\right\}. \tag{13}$$

It was noted that there are two convenient coordinate systems in phase space, namely those of Eq.(1) and Eq.(7). The above differential forms are applicable to both coordinate systems. They of course satisfy the commutation relations given in Eq.(6).

It is remarkable that these operators are also applicable to the Wigner phase–space distribution function which is constructed from the Schrödinger wave function [6]. It is also remarkable that there are unitary transformations on the wave function which lead to canonical transformations of the Wigner function in phase space [6].

3 Construction of the O(3,2) deSitter Group by Soldering Two Lorentz Groups

In Sec. 2, we constructed the ten generators of canonical transformations acting on two pairs of canonical variables. The mathematics is straight-forward, but it is not too comfortable to study physics with ten independent parameters. We can have a better physical picture if we can study the problem in terms of concrete physical examples with smaller symmetries.

The deSitter group $O(3,2)$ is known to be locally isomorphic to the group $Sp(4)$. Indeed, as we shall see in this section, the notations for the generators of $Sp(4)$ given in Sec. 2 are the natural notations for the deSitter group. Thus, one way to study $Sp(4)$ is to study $O(3,2)$. In this section, we shall study $O(3,2)$ by constructing it by soldering two $O(3,1)$ Lorentz groups.

In the space-time of (x, y, z, t, s), where x, y, z are three space-like variables and t and s are two time-like variables, we can consider two $O(3,1)$-like Lorentz groups in the spaces of (x, y, z, t) and (x, y, z, s) respectively. The generators of rotations applicable to the three-dimensional space of x, y and z are

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

57

The Lorentz boosts in the subspace of (x, y, z, t) are generated by

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (15)$$

These three boost generators, together with the rotation generators of Eq.(14), form a closed Lie algebra for the Lorentz group applicable to the four-dimensional Minkowski space of (x, y, z, t). The same is true for the space of (x, y, z, s) with the boost generators:

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}. \quad (16)$$

The above two Lorentz groups have nine generators. If we attempt to form a closed set of commutation relations, we end up with an additional generator

$$J_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}, \quad (17)$$

which will generate rotations in the two–dimensional space of s and t. These ten generators form a closed set of commutations relations.

We started with two $O(3, 1)$ Lorentz groups. Each Lorentz group has its own rotation subgroup. In the present case, both Lorentz groups share the same rotation subgroup. This is how these two groups are soldered.

It is remarkable that this set of commutation relations is identical to that of Eq.(6). The group $O(3, 2)$ is locally isomorphic to $Sp(4)$. The group $O(3, 2)$ occupies a very important place in relativity and elementary particle physics simply because it contains two Lorentz groups as its subgroups. The local isomorphism between $O(3, 2)$ and $Sp(4)$ enables us to study this group in terms of linear canonical transformations in classical mechanics or in the Wigner phase–space picture of quantum mechanics.

4 Construction of the Sp(4) Symmetry Group for Coupled Oscillators by Soldering two Sp(2) Groups

For two uncoupled oscillators, we can start with the coordinate system:

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (x_1, p_1, x_2, p_2). \quad (18)$$

Since the two oscillators are independent, it is possible to perform linear canonical transformations on each coordinate separately. The canonical transformation in the first coordinate system is generated by

$$A_1 = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_1 = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \qquad C_1 = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}. \qquad (19)$$

These generators satisfy the well–known commutation relations:

$$[A_1, B_1] = iC_1, \qquad [B_1, C_1] = -iA_1, \qquad [C_1, A_1] = iB_1. \qquad (20)$$

It is also well known that this set of commutation relations is identical to that for the (2 + 1)–dimensional Lorentz group. Linear canonical transformations on the second coordinate are generated by

$$A_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad B_2 = \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad C_2 = \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \qquad (21)$$

These generators also satisfy the commutation relations of Eq.(20). We are interested here in constructing the symmetry group for the coupled oscillators by soldering two $Sp(2)$ groups generated by A_1, B_1, C_1 and A_2, B_2, C_2 respectively.

It will be more convenient to use the linear combinations:

$$A_+ = A_1 + A_2, \qquad B_+ = B_1 + B_2, \qquad C_+ = C_1 + C_2,$$

$$A_- = A_1 - A_2, \qquad B_- = B_1 - B_2, \qquad C_- = C_1 - C_2, \qquad (22)$$

These matrices take the form

$$A_+ = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad B_+ = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad C_+ = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

$$A_- = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \qquad B_- = \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \qquad C_- = \frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \qquad (23)$$

The sets (A_+, B_+, C_+) and (A_+, B_-, C_-) satisfy the commutation relations of Eq.(20). The same is true for (A_-, B_+, C_-) and (A_-, B_-, C_+).

Next, let us couple the oscillators through a rotation generated by

$$A_0 = \frac{i}{2}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \qquad (24)$$

Then, A_0 commutes with A_+, B_+, C_+, and the following commutation relations generate new operators A_3, B_3 and C_3:

$$[A_0, A_-] = iA_3, \qquad [A_0, B_-] = iB_3, \qquad [A_0, C_-] = iB_3, \qquad (25)$$

where

$$A_3 = \frac{1}{2}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad B_3 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad C_3 = \frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \qquad (26)$$

There are now ten generators. They form the closed set of commutation relations of Eq.(6), if we identify these matrices as

$$A_+ = -J_0, \qquad A_- = -J_3, \qquad A_3 = -J_1, \qquad A_0 = J_2,$$

$$B_+ = K_2, \qquad B_- = -Q_1, \qquad B_3 = Q_3,$$

$$C_+ = Q_2, \qquad C_- = K_1, \qquad C_3 = -K_3, \tag{27}$$

where the J, K and Q matrices are given in Eq.(10) and Eq.(11).

In this section, we started with the generators of the symmetry groups for two independent oscillators. They are A_1, B_1, C_1 and A_1, B_1, C_1. We then introduced another generator A_0 to solder them up. This processes produced three additional generators A_3, B_3, C_3 which are $-J_1, Q_3, and - K_3$ respectively. It is remarkable that K_3, Q_3 and J_0 form the set of generators for another $Sp(2)$ group. They satisfy the commutation relations

$$[Q_3, K_3] = -iJ_0, \qquad [K_3, J_0] = -iQ_3, \qquad [Q_3, J_0] = iK_3 \tag{28}$$

This symmetry group will play the major role in decoupling the coupled oscillator problem.

5 Reparametrization of Coupled Oscillators

Let us consider a system of two coupled harmonic oscillators. The Hamiltonian for this system is

$$H = \frac{1}{2} \left\{ \frac{1}{m_1} p_1^2 + \frac{1}{m_2} p_2^2 + A' x_1^2 + B' x_2^2 + C' x_1 x_2 \right\}. \tag{29}$$

where

$$A' > 0, \qquad B' > 0, \qquad 4A'B' - C'^2 > 0. \tag{30}$$

By making scale changes of x_1 and x_2 to $(m_1/m_2)^{1/4} x_1$ and $(m_2/m_1)^{1/4} x_2$ respectively, it is possible to make a canonical transformation of the above Hamiltonian to the form [14, 15]

$$H = \frac{1}{2m} \left\{ p_1^2 + p_2^2 \right\} + \frac{1}{2} \left\{ A x_1^2 + B x_2^2 + C x_1 x_2 \right\}, \tag{31}$$

with $m = (m_1 m_2)^{1/2}$. This transformation is generated We can decouple this Hamiltonian by making the coordinate transformation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{32}$$

Under this rotation, the kinetic energy portion of the Hamiltonian in Eq.(31) remains invariant. Thus we can achieve the decoupling by diagonalizing the potential energy. Indeed, the system becomes diagonal if the angle α becomes

$$\tan \alpha = \frac{C}{B - A}. \tag{33}$$

This diagonalization procedure is well known.

We now introduce the new parameters K and η defined as

$$K = \sqrt{AB - C^2/4}, \qquad \exp(-2\eta) = \frac{A + B + \sqrt{(A - B)^2 + C^2}}{\sqrt{4AB - C^2}}, \qquad (34)$$

in addition to the rotation angle α. In terms of this new set of variables, A, B and C take the form

$$A = K\left(e^{2\eta}\cos^2\frac{\alpha}{2} + e^{-2\eta}\sin^2\frac{\alpha}{2}\right),$$

$$B = K\left(e^{2\eta}\sin^2\frac{\alpha}{2} + e^{-2\eta}\cos^2\frac{\alpha}{2}\right),$$

$$A = K\left(e^{-2\eta} - e^{2\eta}\right)\sin\alpha. \qquad (35)$$

the Hamiltonian can be written as

$$H = \frac{1}{2m}\left\{q_1^2 + q_2^2\right\} + \frac{K}{2}\left\{e^{2\eta}y_1^2 + e^{-2\eta}y_2^2\right\}, \qquad (36)$$

where y_1 and y_2 are defined in Eq.(32), and

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \qquad (37)$$

This form will be our starting point. The above rotation together with that of Eq.(32) is generated by J_0.

If we measure the coordinate variable in units of $(mK)^{1/4}$, and use $(mK)^{-1/4}$ for the momentum variables, the Hamiltonian takes the form

$$H = \frac{\omega}{2}e^{\eta}\left(e^{-\eta}q_1^2 + e^{\eta}y_1^2\right) + \frac{\omega}{2}e^{-\eta}\left(e^{\eta}q_2^2 + e^{-\eta}y_2^2\right), \qquad (38)$$

where $\omega = \sqrt{K/m}$. If $\eta = 0$, the system becomes decoupled, and the Hamiltonian becomes

$$H = \frac{\omega}{2}\left(p_1^2 + x_1^2\right) + \frac{\omega}{2}\left(p_2^2 + x_2^2\right). \qquad (39)$$

In Sec. 8, we will be dealing with the problem of what happens when no observations are made on the second coordinate. If the system is decoupled, as the above Hamiltonian indicates, the physics in the first coordinate is solely dictated by the Hamiltonian

$$H_1 = \frac{\omega}{2}\left(p_1^2 + x_1^2\right). \qquad (40)$$

It is important to note that the Hamiltonian of Eq.(39) cannot be obtained from Eq.(38) by canonical transformation. For this reason, the Hamiltonian of the form

$$H' = \frac{\omega}{2}\left(e^{-\eta}q_1^2 + e^{\eta}y_1^2\right) + \frac{\omega}{2}\left(e^{\eta}q_2^2 + e^{-\eta}y_2^2\right) \qquad (41)$$

may play a useful role in our discussion. This Hamiltonian can be transformed into the decoupled form of Eq.(39) through a canonical transformation.

6 Quantum Mechanics of Coupled Oscillators

It is remarkable that both the Hamiltonian H of Eq.(38) and H' of Eq.(41) lead to the same Schrödinger wave function. If y_1 and y_2 are measured in units of $(mK)^{1/4}$, the ground-state wave function for this oscillator system is

$$\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{ -\frac{1}{2}(e^\eta y_1^2 + e^{-\eta} y_2^2) \right\}. \tag{42}$$

The wave function is separable in the y_1 and y_2 variables. However, for the variables x_1 and x_2, the story is quite different. If we write this wave function in terms of x_1 and x_2, then

$$\psi(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{ -\frac{1}{2} \left[e^\eta (x_1 \cos\frac{\alpha}{2} - x_2 \sin\frac{\alpha}{2})^2 \right.\right.$$

$$\left.\left. + e^{-\eta}(x_1 \sin\frac{\alpha}{2} + x_2 \cos\frac{\alpha}{2})^2 \right] \right\}. \tag{43}$$

If $\eta = 0$, this wave function becomes

$$\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{ -\frac{1}{2}(x_1^2 + x_2^2) \right\}. \tag{44}$$

For other values of η, the wave function of Eq.(43) can be obtained from the above expression by a unitary transformation.

$$\sum_{m_1 m_2} A_{m_1 m_2}(\alpha, \eta) \psi_{m_1}(x_1) \psi_{m_2}(x_2), \tag{45}$$

where $\psi_m(x)$ is the mth excited state wave function. The coefficients $A_{m_1 m_2}(\eta)$ satisfy the unitarity condition

$$\sum_{m_1 m_2} |A_{m_1 m_2}(\alpha, \eta)|^2 = 1. \tag{46}$$

It is possible to carry out a similar expansion in the case of excited states [16].

As for unitary transformations applicable to wave functions, let us go back the generators of canonical transformations in classical mechanics in Eq.(12) and Eq.(13). As was stated before, they are also applicable to the Wigner phase–space distribution function. The canonical transformation of the Wigner function is translated into a unitary transformation of the Schrödinger wave function. There are therefore ten generators of unitary transformations applicable to Schrödinger wave functions [6, 4]. They are

$$\hat{J}_1 = \frac{1}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \qquad \hat{J}_2 = \frac{1}{2i}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right),$$

$$\hat{J}_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \qquad \hat{J}_0 = \frac{1}{2}\left(a_1^\dagger a_1 + a_2 a_2^\dagger\right),$$

$$\hat{K}_1 = -\frac{1}{4}\left(a_1^\dagger a_1^\dagger + a_1 a_1 - a_2^\dagger a_2^\dagger - a_2 a_2\right),$$

$$\hat{K}_2 = \frac{i}{4}\left(a_1^\dagger a_1^\dagger - a_1 a_1 + a_2^\dagger a_2^\dagger - a_2 a_2\right),$$

$$\hat{K}_3 = \frac{1}{2}\left(a_1^\dagger a_2^\dagger + a_1 a_2\right),$$

$$\hat{Q}_1 = -\frac{i}{4}\left(a_1^\dagger a_1^\dagger - a_1 a_1 - a_2^\dagger a_2^\dagger + a_2 a_2\right),$$

$$\hat{Q}_2 = -\frac{1}{4}\left(a_1^\dagger a_1^\dagger + a_1 a_1 + a_2^\dagger a_2^\dagger + a_2 a_2\right),$$

$$\hat{Q}_3 = \frac{i}{2}\left(a_1^\dagger a_2^\dagger - a_1 a_2\right). \tag{47}$$

where $a^\dagger$ and a are the step-up and step-down operators applicable to harmonic oscillator wave functions. The above operators also satisfy the commutation relations given in Eq.(6).

7 Wigner Functions and Uncertainty Relations

The Wigner phase–space picture of quantum mechanics is often more convenient for studying the uncertainty relations. Unitary transformations in the Schrödinger picture can be achieved through canonical transformations in phase space. It has been known that canonical transformations are uncertainty–preserving transformations. They are also entropy–preserving transformations [17]. Are there then non-canonical transformations in quantum mechanics?

In his book on statistical mechanics [18], Feynman raises the issue of the rest of the universe in connection with the density matrix. Feynman divides the universe into two parts. We make measurements in the first part, but are not able to measure anything in the second part. The second part is Feynman's rest of the universe. Indeed, the density matrix plays the essential role when we are not able to measure all the variables in quantum mechanics [19, 20].

In the present case of coupled harmonic oscillators, we assume that we are not able to measure the x_2 coordinate. It is often more convenient to use the Wigner phase-space distribution function to study the density matrix, especially when we want to study the uncertainty products in detail [15, 18].

For two coordinate variables, the Wigner function is defined as [15]

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \int \exp\left\{-2i(p_1 y_1 + p_2 y_2)\right\}$$

$$\times \psi^*(x_1 + y_1, x_2 + y_2)\psi(x_1 - y_1, x_2 - y_2)dy_1 dy_2. \tag{48}$$

The Wigner function corresponding to the oscillator wave function of Eq.(43) is

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \exp\left\{-e^\eta(x_1 \cos\frac{\alpha}{2} - x_2 \sin\frac{\alpha}{2})^2\right.$$

$$-e^{-\eta}(x_1 \sin \frac{\alpha}{2} + x_2 \cos \frac{\alpha}{2})^2 - e^{-\eta}(p_1 \cos \frac{\alpha}{2} - p_2 \sin \frac{\alpha}{2})^2$$

$$-e^{\eta}(p_1 \sin \frac{\alpha}{2} + p_2 \cos \frac{\alpha}{2})^2 \Big\} . \tag{49}$$

If we do not make observations in the $x_2 p_2$ coordinates, the Wigner function becomes

$$W(x_1, p_1) = \int W(x_1, x_2; p_1, p_2) dx_2 dp_2. \tag{50}$$

The evaluation of the integral leads to

$$W(x_1, x_2; p_1, p_2) = \left\{ \frac{1}{\pi^2(1 + \sinh^2 \eta \sin^2 \alpha)} \right\}^{1/2}$$

$$\times \exp \left\{ - \left(\frac{x_1^2}{\cosh \eta - \sin \eta \cos \alpha} + \frac{p_1^2}{\cosh \eta + \sin \eta \cos \alpha} \right) \right\} . \tag{51}$$

This Wigner function gives an elliptic distribution in the phase space of x_1 and p_1. This distribution gives the uncertainty product of

$$(\Delta x)^2 (\Delta p)^2 = \frac{1}{4}(1 + \sinh^2 \eta \sin^2 \alpha). \tag{52}$$

This expression becomes $1/4$ if the oscillator system becomes uncoupled with $\alpha = 0$. Because x_1 is coupled with x_2, our ignorance about the x_2 coordinate, which in this case acts as Feynman's rest of the universe, increases the uncertainty in the x_1 world which, in Feynman's words, is the system in which we are interested.

In the Wigner phase–space picture, the uncertainty is measured in terms of the area in phase space where the Wigner function is sufficiently different from zero. According to the Wigner function for a thermally excited oscillator state, the temperature and entropy are also determined by the degree of the spread of the Wigner function phase space.

8   Scale Transformations in Phase Space

In addition to the ten generators given in Eq.(10) and also in Eq.(11), we can consider the scale transformation in which both the position and momentum of the first coordinate are expanded and those of the second coordinate contracted. The Hamiltonian given in Eq.(38) suggests such a transformation, and the transformation can be generated by

$$S_0 = \frac{i}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} . \tag{53}$$

This matrix generates scale transformations in phase space. The transformation leads to a radial expansion of the phase space of the first coordinate [21] and contracts the phase space of the second coordinate. What is the physical significance of this operation? As we discussed in Sec. 7, the expansion of phase space leads to an increase in uncertainty and entropy. Mathematically

64

speaking, the contraction of the second coordinate should cause a decrease in uncertainty and entropy. Can this happen? The answer is clearly No, because it will violate the uncertainty principle. This question will be addressed in future publications.

In the meantime, let us study what happens when the matrix S_0 is introduced into the set of matrices given in Eq.(10) and Eq.(11). It commutes with J_0, J_3, K_1, K_2, Q_1, and Q_2. However, its commutators with the rest of the matrices produce four more generators:

$$[S_0, J_1] = \frac{i}{2} \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad [S_0, J_2] = \frac{1}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$[S_0, K_3] = \frac{1}{2} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \qquad [S_0, Q_3] = \frac{1}{2} \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{54}$$

If we take into account the above five generators in addition to the sixteen generators of $Sp(4)$, there are fifteen generators. They form the closed set of commutation relations for the the group $SL(4, r)$. This $SL(4, r)$ symmetry of the coupled oscillator system may have interesting physical implications.

References

[1] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row–Peterson, Elmsford, New York, 1961).

[2] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw–Hill, New York, 1971); M. Tinkham, *Introduction to Superconductivity* (Krieger, Malabar, Florida, 1975).

[3] S. K. Kim and J. L. Birman, Phys. Rev. B *38*, 4291 (1988).

[4] P. A. M. Dirac, J. Math. Phys. **4**, 901 (1963).

[5] C. M. Caves and B. L. Schumaker, Phys. Rev. A **31**, 3068 (1985); B. L. Schumaker and C. M. Caves, Phys. Rev. A **31**, 3093 (1985); B. L. Schumaker, Phys. Rep. **135**, 317 (1986). See also Fan, Hong–Yi and J. Vander Linder, Phys. Rev. A **39**, 2987 (1989).

[6] D. Han, Y. S. Kim, and M. E. Noz, Phys. Rev. A **41**, 6233 (1990).

[7] Y. S. Kim, Phys. Rev. Lett. **63**, 348 (1989).

[8] F. Iachello and S. Oss, Phys. Rev. Lett. **66**, 2976 (1991).

[9] H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North–Holland, Amsterdam, 1982).

[10] B. Yurke and M. Potasek, Phys. Rev. A **36**, 3464 (1987).

[11] A. K. Ekert and P. L. Knight, Am. J. Phys. **57**, 692 (1989). For an earlier paper, see S. M. Barnett and P. L. Knight, J. Opt. Soc. of Amer. B **2** 467 (1985).

[12] D. Han, Y. S. Kim, and M. E. Noz, Phys. Lett. A **144**, 111 (1990).

[13] H. Goldstein, *Classical Mechanics*, Second Edition (Addison–Wesley, Reading, MA, 1980).

[14] P. K. Aravind, Am. J. Phys. **57**, 309 (1989).

[15] Y. S. Kim and M. E. Noz, *Phase Space Picture of Quantum Mechanics* (World Scientific, Singapore, 1991).

[16] Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986).

[17] D. Han, Y. S. Kim, M. E. Noz, and L. Yeh, Univ. of Maryland Physics Paper 93-23 (1992).

[18] R. P. Feynman, *Statistical Mechanics* (Benjamin/Cummings, Reading, MA, 1972).

[19] J. von Neumann, *Mathematical Foundation of Quantum Mechanics* (Princeton Univ. Press, Princeton, 1955).

[20] E. P. Wigner and M. M. Yanase, Proc. National Academy of Sciences (U.S.A.) **49**, 910 (1963).

[21] Y. S. Kim and M. Li, Phys. Lett. A **139** 445 (1989).

II. QUANTUM GROUPS